Lecture Notes of Axiomatic Set Theory

Puzzle:

For each $X \subseteq \omega_1$,the two player game, G_X, is defined as follows:

$$I : \alpha_0 \quad\quad \alpha_1 \quad\quad\quad ...$$
$$II : \quad\quad \beta_0 \quad\quad \beta_1 \quad\quad ...$$

Player I wins iff $\sup_{n<\omega} \alpha_n \in X$; if Player I fails, then Player II wins.

G_X is determined if one of the player has wining strategy.

For example: a wining strategy for I is a function:

$$\sigma : \omega_1^{<\omega} \to \omega_1,$$

such that if I plays $\alpha_0 = \sigma(\phi); \alpha_{n+1} = \sigma\{\alpha_0, \beta_0, ..., \beta_n\}$, then I always wins.

Question: On which $X \subseteq \omega_1$ is the game G_X determined?

Definition (Clubs)**.** Let κ be a regular uncountable cardinal:

1. A subset $C \subseteq \kappa$ is closed if every limit ordinal $\delta < \kappa$, and every sequence: $\alpha_0 < \alpha_1 < ...(\xi < \delta)$, with every $\alpha_\xi < C$, we have $\sup_{\xi<\delta} \alpha_\xi \in C$;
2. C is unbounded if for all $\alpha < \kappa$, exists $\alpha < \beta < \kappa$ such that $\beta \in C$;
3. C is club if C is unbounded and closed (in κ).

Definition.

1. $\mathcal{F}$ is a finite-ary function on A if $\mathcal{F} : A^n \to A$ for some $n \geq 0$;
2. if $\mathcal{F} : A^n \to A$ and $B \subseteq A$, then B is closed under $\mathcal{F}$ if $\mathcal{F}[B^n] \subseteq B$.

Theorem . *If κ is regular uncountable and $\mathcal{F} : \kappa^n \to \kappa$, then*

$$C = \{\alpha < \kappa \mid \alpha \textit{ is closed under } \mathcal{F}\}$$

is a club.

Proof. Clearly C is closed. For unbounded: let $\beta < \kappa$ since $|\beta^n| < \kappa$, it follows that there exists $\beta < \alpha_0 < \kappa$ such that $f[\beta^n] \subseteq \alpha_0$. Continuing in this fashion: we can find

$$\beta < \alpha_0 < \alpha_1 < ... < \alpha_l < ...$$

such that $f[\alpha_l^n] \subseteq \alpha_{l+1}$. Clearly $\sup_{l<\omega} \alpha_l$ is closed under f.

Lemma . *If κ is regular uncountable ordinal, $\lambda < \kappa$ and $\{C_\alpha \mid \alpha < \lambda\}$ are clubs of κ, then $C = \bigcap_{\alpha<\lambda} C_\alpha$ is a club.*

Proof. Clearly C is closed. To see C is unbounded: For each $\alpha < \lambda$, let $\mathcal{F}_\alpha : \kappa \to \kappa$ defined by:

$$\mathcal{F}_\alpha(\beta) = \text{the least } \gamma \in C_\alpha \text{ such that } \beta < \gamma.$$

Next define $g : \kappa \to \kappa$ by $g(\beta) = \sup_{\alpha<\lambda} \mathcal{F} + \alpha(\beta)$.

(Since κ is regular, then g is well-defined.) For each $n \geq 1$, let:

$$g^n(\beta) = (g \circ ... \circ g)(\beta).$$

and let

$$g^\omega(\beta) = \sup_{n\geq 1} g^n(\beta).$$

Then $\beta < g^\omega(\beta) < \kappa$, and we claim that $g^\omega(\beta) \in C = \bigcap_{\alpha<\lambda} C_\alpha$, let $\alpha < \lambda$. Then $\mathcal{F}_\alpha(\beta) = g(\beta) < \mathcal{F}_\alpha(g(\beta)) \leq g^2(\beta) < ...$

Since each $\mathcal{F}_\alpha(g^n(\beta)) \in C_\alpha$, it follows that

$$g^\omega(\beta) = \sup_{n\geq 1} \mathcal{F}_\alpha(g^n(\beta)) \in C_\alpha.$$

Observation.

1. If there exists a club $C \subseteq \omega_1$ such that $C \cap X = \emptyset$, then II has a wining strategy in G_X;
2. If there exists a club $C \subseteq \omega_1$ such that $C \cap (\omega_1 - X) = \emptyset$, then I has a wining strategy in G_X.

Definition. Let κ be a regular uncountable ordinal. A subset $S \subseteq \kappa$ is stationary if $S \cap C = \emptyset$.

For every club $C \subseteq \kappa$:

$$\text{Stationary} \equiv \text{Positive Measure} \equiv \text{Significant}$$

Observation. If S is stationary and C is a club, then $S \cap C$ is stationary.

Examples:
$S = \{\alpha \in \omega_2 \mid cf(\alpha) = \omega\}; T = \{\alpha < \omega_2 \mid cf(\alpha) = \omega_1\}; S = \{\alpha < \omega_1 \mid \lim \alpha\}$ are stationary sets. You CAN'T find a stationary set include ω_1 that doesn't contains a club.

We will next attempt to construct nonisomorphic DLOs(Dense Linear Orderings) without end-points of size ω_1.

For each $A \subseteq \omega_1$, we construct a DLO:

$$D^A = \bigcup_{\alpha<\omega_1} D^A_\alpha$$

as the smooth increasing union of countable DLOs. D^A_α Here "smooth" means if $\lim \alpha$ exists, then

$$D^A_\alpha = \bigcup_{\beta<\alpha} D^A_\beta.$$

At successor stages $D^A_\alpha \subset D^A_{\alpha+1}$, we make use of just 2 embeddings.

Definition. An embedding $D_1 \subset D_2$ of countable DLOs us rational if

1. D_1 is an initial segment of D_2;
2. D_1 has a least upper bound in D_2.

Thus, $(-\infty, 1) \cap \mathbb{Q} \subset \mathbb{Q} \cong D_1 \subset D_2$.

Definition. An embedding $D_1 \subset D_2$ of countable DLOs is irrational if

1. D_1 is an initial segment of D_2;
2. D_1 does not have a least upper bound in D_2.

Thus $(-\infty, \sqrt{2}) \cap \mathbb{Q} \subset \mathbb{Q} \cong D_1 \subset D_2$.

For each $A \subseteq \omega_1$, let

$$D_A = \bigcap_{\alpha<\omega_1} D_\alpha^A$$

be a smooth increasing union of countable DLOs such that

$$D_\alpha^A \subseteq D_{\alpha+1}^A \begin{cases} \text{is irrational if } \alpha \notin A. \\ \text{is rational if } \alpha \in A. \end{cases}$$

It is reasonable to expect that if A, B are "sufficiently different", then $D^A \ncong D^B$.

Propersition . *D^A is not isomorphic to D^B ($D^A \ncong D^B$) iff the symmetric difference $S = A \triangle B$ is stationary.*

Proof. First suppose that $S = A \triangle B$ is stationary. Suppose that $\mathcal{F} : D^A \to D^B$ is an isomorphism. Arguing as is the above proof,

$$C = \{\alpha < \omega_1 \mid f[D^A] = D_\alpha^B\}$$

is a club. Hence there exists $\alpha \in C \cap S$.

Wlog(Without loss of generality) $\alpha \in A\ B$. Then D_α^A has a least upper bound in D^A. While D_α^B has no least upper bound in D^B, there is a contradiction.

Next suppose $S = A \triangle B$ isn't stationary. Then there exists a bound in $D_{\alpha+1}^A$ such that $S \cap C = \emptyset$. Then for each $\alpha \in C$, D_α^A has a least upper bound in $D_{\alpha+1}^A$ iff D_α^B has a least upper bound in $D_{\alpha+1}^B$.

Let $C = \{\alpha_\xi \mid \xi < \omega_1\}$ be the increasing sequence, then we can inductively construct isomorphisms:

$$\phi_\xi : D_{\alpha_\xi}^A \to D_{\alpha_\xi}^B$$

such that if $\xi < \tau < \omega_1$, then $\phi_\xi < \phi_\tau$. Then $\phi = \bigcup_{\xi<\omega_1} \phi_\xi$ is an isomorphism from D^A to D^B.

Propersition (Ulam). *We can express*

$$\omega_1 = \bigsqcup_{\alpha<\omega_1} S_\alpha$$

as the disjoint union of ω_1 *stationary sets.*

Corollary. There exists 2^{ω_1} non-isomorphic DLOs if size ω_1.

Proof. For each $A \subseteq \omega_1$, let $S_A = \bigsqcup_{\alpha\in A} S_\alpha$. If $A \neq B$, then $S_A \triangle S_B$ is stationary and so $D^{S_A} \cong D^{S_B}$.

Homework:

Definition. An infinite graph Γ is random if whenever A, B are nonempty disjoint finite subsets, there exists $v \in \Gamma$ $(A \cup B)$ such that v is adjacent to every $\alpha \in A$ and nonadjacent to every $b \in B$.

Show there exist 2^{ω_1} pairwise non-isomorphic random graphs of size ω_1.

Lecture Notes #2 of Axiomatic Set Theory

Definition. A filter on a set S is a collection $\mathcal{F} \subseteq \mathcal{P}(S)$ such that

(i) $S \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$;

(ii) If $X, Y \in F$ then $X \cap Y \in F$;

(iii) If $X \subseteq Y \subseteq S$ and $X \in \mathcal{F}$ then Y is large.

A filter $\mathcal{F}$ is $\sigma-$complete if whether $\{x_n \mid n < \omega\} \subseteq F$ then $\bigcap\limits_n X_n \in F$.

Example. $\mathcal{F} = \{x \subseteq \omega_1 \mid X \text{ contains a club}\}$ is a $\sigma-$filter on ω_1.

When we call a family of sets an *ideal*, we mean a "complement of filter".

Example. $\mathcal{N} = \{X \subseteq \omega_1 \mid X \text{ isn't stationary}\}$ is a $\sigma-$ideal on ω_1.

Theorem *(Ulam)*. *We can write ω_1 as $\omega_1 = \bigsqcup\limits_{\alpha<\omega_1} S_\alpha$, where S_α are disjoint stationary sets.*

Proof. It is enough to find ω_1 pairwise disjoint stationary sets:

$$\{X_\alpha \mid \alpha < \omega_1\}.$$

Since then we can let:

- $S_0 = X_0 \cup (\omega_1 \setminus \bigcup\limits_{\alpha<\omega_1} X_\alpha)$;
- $S_\alpha = X_\alpha$ for $\alpha > 0$.

For each $\tau < \omega_1$, let $F_\tau : \tau \to \omega$ be an injection; and for each $\alpha < \omega_1$, and $n < \omega$, let

$$X_\alpha^n = \{\tau > \alpha \mid F_\tau(\alpha) = n\}.$$

Clearly: if $\alpha \neq \beta$ and $n < \omega$, then $X_\alpha^n \cap X_\beta^n = \emptyset$. Also, for each $\alpha < \omega_1$, $\omega_1 \setminus (\alpha+1) = \bigcap\limits_{n\in\omega} X_\alpha^n$. Since $\omega_1 \setminus (\alpha+1)$ is a club, there exists $h(\alpha) < \omega$ such that $X_\alpha^{h(\alpha)}$ is stationary. Since

$$h : \omega_1 \to \omega,$$

there exists a fixed $n_0 \in \omega$ such that $|\{\alpha < \omega_1 \mid h(\alpha) = n_0\}| = \omega_1$. Then $\{X_\alpha^{n_0} h(\alpha) = n_0\}$ satisfies our requirements. □

Theorem . *Let $X \subseteq \omega_1$.*

(i) If X contains a club, then Player I has a wining strategy in G_X;

(ii) If $\omega_1 \setminus X$ contains a club, then Player II has a wining strategy in G_X;

(iii) If X is stationary as well as $\omega_1 \setminus X$, then G_X is not determined.

Proof. (i)+(ii) are obvious.

(iii) Suppose for example that $\sigma : \omega_1^\omega \to \omega_1$ is a wining strategy for I. Then there exists a club C which is closed under σ. Let D be the $\gamma \in C$ such that there exists a sequence $\gamma_0 < \gamma_1 < ... < \gamma_n < ... < \gamma$, where $n < \omega$, with each $\gamma_n \in C$ and $\gamma = \sup_n \gamma_n$. Then D is also a club. Hence there exists $\gamma \in D \cap (\omega_1 \setminus X)$ such that $\gamma_{>\alpha_0} = \sigma(\emptyset)$. It follows that there exists $\gamma_n \in C$ such that $\alpha_0 < \gamma_0 < \gamma_1 < ... < \gamma_n < ...\gamma$, where $\gamma = \sup_n \gamma_n$. Since each γ_n is closed, we see inductively that

$$\gamma_n < \alpha_{n+1} = \sigma(\alpha_0, \gamma_0, ..., \gamma_n) < \gamma_{n+1}.$$

Hence $\lim_n \alpha_n = \lim_n \gamma_n = \gamma \notin X$. □

Chapter 2: Trees.

Definition.

(i) A *Tree* is a poset $\langle T, < \rangle$ such that for each $t \in T$, $pred_T(t) := \{s \in T \mid s < t\}$ is well-ordered by $<$.

(ii) If $t \in T$, then the *height* $ht_T(t)$ is the order type of $pred_T(t)$.

(iii) For each ordinal α, the α^{th} *level* of T is

$$T_\alpha = \{t \in T \mid ht_T(t) = \alpha\}.$$

(iv) The *height* of T is

$$ht(T) = \min\{\alpha \mid T_\alpha = \emptyset\}.$$

(v) A *branch/chain* B of T is a maximal linearly ordered subset.

(vi) An *antichain* A of T is a set of pairwise incomparable elements.

Example. If $\alpha < ht(T)$, then T_α is an antichain.

Definition. Let κ be an infinite cardinal. T is a $\kappa-$tree if:

- $ht(T) = \kappa$.
- $|T_\alpha| < \kappa$ for all $\alpha < \kappa$.

Definition. A $\kappa-$Aronszajn tree is a $\kappa-$tree with no branches of size κ.

Theorem *(König)*. *There are no $\omega-$Aronszajn tree.*

Theorem *(Aronszajn,1934,ZFC)*. *There exists an ω_1-Aronszajn tree. (Proof Delayed)*

Theorem *(CH)*. *There exists an ω_2-Aronszajn tree. (Proof Delayed)*

Question.(ZFC) Does there exists an ω_2-Aronszajn tree?

Definition. Let κ be an infinite cardinal. A $\kappa-$Suslin tree is a $\kappa-$Aronszajn tree with no antichains of size κ.

Motivation.

Theorem *(Folklore)*. *Suppose $\langle X, < \rangle$ is a linear ordering satisfying:*

(i) X is a DLO;

(ii) X is complete, ie, each non-empty subset which is bounded has a least upper bound;(This is also called the Least Upper Bound Property.)

(iii) X is separable, ie, X has a countable dense subset.

Then $\langle X, < \rangle \cong \langle \mathbb{R}, < \rangle$.

The Suslin Hypothesis.(SH) The result above remains true if (iii) is replaced by the following:

(iii)' There doesn't exist an uncountable collection of pairwise disjoint non-empty open sets.

Clearly $(iii) \Rightarrow (iii')$.

Theorem *(Kurepa, 1935)*. *The following are equivalent:*

(a) SH.

(b) There are no ω_1-Suslin trees.

Theorem *2.1(ZFC)*. *There exists an ω_1-Aronszajn tree.*

Proof. Let T consists of all functions $t : \alpha \to \omega$ such that:

(i) $\alpha < \omega_1$;

(ii) t is injective;

(iii) $|\omega \setminus ran(t)| = \omega$.

We partially order T by
$$t_1 < t_2 \Leftrightarrow t_1 \subset t_2.$$

Clearly T is a tree of height ω_1. In fact, T_α is the set of $t : \alpha \to \omega$ satisfying (i), (ii) and (iii). Suppose that $C \subseteq T$ is an uncountable branch. Then $f = \bigcap C$ is an injection from ω_1 into ω, which is a contradiction.

Thus T has no uncountable branches. Unfortunately, if $\omega \leq \alpha < \omega_1$, then $|T_\alpha| = \omega^\omega \geq \omega_1$. We shall find a suitable subtree $T^* \subseteq T$.

If $s, t \in {}^\alpha\omega$, we define
$$s \sim t \Leftrightarrow |\{\beta < \alpha \mid s(\beta) \neq t(\beta)\}| < \omega.$$

We will define inductively $s_\alpha \in T_\alpha$ such that if $\alpha < \beta$, then $s_\alpha \sim s_\beta \restriction \alpha$, then we will define

$$T^* = \bigcup_{\alpha<\omega_1} \{t \in T_\alpha \mid t \sim s_\alpha\}.$$

Notice that if $\alpha < \beta$ and $t \in T^*_\beta$, then $t \restriction \alpha \sim s_\beta \restriction \alpha \sim s_\alpha$ and so $t \restriction \alpha \in T^*_\alpha$. Thus T^* is a subtree of T.

Clearly $ht(T^*) = \omega_1$. Also $|T^*_\alpha| \leq \omega$. Thus T^* will be an ω_1-Aronszajn tree.
It just remains to define s_α:

- $s_0 = \emptyset$;
- If s_α has been defined, let $s_{\alpha+1} = s_\alpha \cup \{\langle \alpha, n\rangle\}$ where $n = \min(\omega \setminus ran(s_\alpha))$.
- Finally suppose $\lim \alpha$ and s_β is defined for all $\beta < \alpha$. Choose $\alpha_0 < \alpha_1 < ... < \alpha_n < ...$ such that $\lim_n \alpha_n = \alpha$. We now define $t_n \in T_{\alpha_n}$ inductively by:
  - $t_0 = s_{\alpha_0}$;
  - $t_{n+1} \restriction \alpha_n = t_n$;
  - the first $n+1$ elements of $\omega \setminus ran(t_n)$ are in $\omega \setminus ran(t_{n+1})$;
  - $t_n \sim s_{\alpha_n}$.

  Then we can let $s_\alpha = \bigcup_{n<\omega} t_n$.

$\square$

Question. How to construct an ω_2-Aronszajn tree?

Lecture Notes #3 of Axiomatic Set Theory

Recap We constructed an ω_1-Aronszajn tree

$$T^* \subseteq T = \{t : \alpha \hookleftarrow \omega \mid \alpha < \omega_1\}.$$

Remark The ω_1-Aronszajn tree is not an ω_1-Suslin Tree.

Proof. For each $n < \omega$, let $A_n = \{t \in T \mid \text{For some } \alpha < \omega_1,\ dom(t) = \{\alpha\} \cup \alpha, \text{ and } t(\alpha) = n\}$. Then each A_n is an antichain. For each $\alpha < \omega_1$, choose $t_\alpha \in T^* \cap {}^{(\alpha+1)}\omega$. Then $t_\alpha \in A_n$ for some $n \in \omega$. Hence there exists $n < \omega$ such that $|A_n| = \omega_1$. □

Exercise (Skinny ω_2-Aronszajn tree $\to$ Countable level) Suppose T is a tree with $ht(T) = \omega_2$ and $|T_\alpha| \leq \omega$ for each $\alpha < \omega_2$. Then T has a branch of size ω_2. (Kurepa)

Hint Of course, you can suppose the underlying set of T is ω_2. Now make a suitable application of Fordor's Lemma:

Fordor's Lemma Suppose that κ is a regular cardinal and $S \subseteq \kappa$ be a stationary set. If $fS \to \kappa$ satisfies $f(\alpha) < \alpha$ for all $\alpha \in S$, then there exists $\beta < \kappa$ such that

$$\{\alpha \in S \mid f(\alpha) = \beta\} \text{ is stationary.}$$

Theorem (CH). *There exists an ω_2-Aronszajn tree.*

Proof. Let T be the tree of functions

$$t : \alpha \to \omega_1 \text{ such that } \begin{cases} \alpha < \omega_2 \\ t \text{ is an injection.} \end{cases}$$

ordered by: $s < t \Leftrightarrow s \subset t$. Then clearly T is a tree of height ω_2 with no branches ω_2. Once again, we look for a suitable subtree $T^* \subseteq T$.
If $\alpha < \omega_2$ and $s, t \in {}^\alpha\omega_1$, we define

$$s \approx t \Leftrightarrow |\{\beta < \alpha \mid s(\beta) \neq t(\beta)\}| \leq \omega.$$

Claim. There exist $\{s_\alpha \mid \alpha < \omega_2\} \subseteq T$ such that:

(i) $s_\alpha \in T_\alpha$;

(ii) if $\beta < \alpha$, then $s_\alpha \restriction \beta \approx s_\beta$.

Assuming the Claim, we can complete the proof of Theorem 2.2 as follows. Let

$$T^* = \bigcup_{\alpha < \omega_2} \{t \in T_\alpha \mid t \approx s_\alpha\}.$$

Once again, it is easily seen that T^* is a subtree of T and that $ht(T^*) = \omega_2$. Also if $\alpha > \omega_2$, then

$$\begin{aligned} |T^*_\alpha| &\leq |\alpha|^\omega \cdot \omega_1^\omega \leq \omega_1^\omega \cdot \omega_1^\omega \\ (\text{By CH}) &= (2^\omega)^\omega \cdot (2^\omega)^\omega = \omega_1. \end{aligned}$$

Thus T^* is an ω_2-tree.

Proof of Claim. We will inductively construct $s_\alpha \in T_\alpha$ such that:

(a) if $\beta < \alpha$. then $s_\alpha \upharpoonright \beta \approx s_\beta$;

(b) $ran(s_\alpha)$ is a non-stationary subset of ω_1; ie there exists a club $C_\alpha \subseteq \omega_1 - ran(s_\alpha)$.

Case 1: $\alpha = 0$. We let $s_0 = \emptyset$.

Case 2: $\alpha = \beta + 1$. Let $\gamma = \min(\omega_1 - ran(s_\beta))$. Then we let $s_\alpha = s_\beta \cap \{\langle \beta, \gamma \rangle\}$.

Case 3: $\lim \alpha$ and $cf(\alpha) = \omega$. Choose a sequence of ordinals:

$$\beta_0 < \beta_1 < ... < \beta_n < ... < \alpha$$

such that $\alpha = \sup_n \beta_n$. Let $t_0 = s_{\beta_0}$ and inductively define $t_n \in T_{\beta_n}$ such that:

(i) $t_n \approx s_{\beta_n}$.

(ii) $t_{n+1} \upharpoonright \beta_n = t_n$.

Let $t = \bigcup_n t_n$. Then $t \in T_\alpha$ clearly satisfies (a). To see that t also satisfies (b), note that there exists a countable set $C \subseteq \omega_1$ such that

$$ran(t) \subseteq \bigcup_n ran(s_{\beta_n}) \cup C.$$

Thus $ran(t)$ is non-stationary and we can let $s_\alpha = t$.

Case 4: $\lim \alpha$ and $cf(\alpha) = \omega_1$. Choose a continuous sequence of ordinals

$$\beta_0 < \beta_1 < ... < \beta_\xi < ... < \alpha$$

such that $\alpha = \lim_\xi \beta_x i$. Once again, we will inductively define $t_\xi \in T_{\beta_\xi}$ such that

(i) $t_\xi \approx s_{\beta_\xi}$;

(ii) $t_{\xi+1} \upharpoonright \beta_\xi = t_\xi$.

If $\lim \xi$, then we let $t_\xi = \bigcup_{\gamma<\xi} t_\gamma$. Clearly $t_x i \in T_{\beta_\xi}$ satisfies (i) and (ii). If $\xi = \gamma + 1$, we let $t_\xi = t_\gamma \cup u \in T_{\beta_\xi}$, where u is obtained by changing countably many values of $s_{\beta_\xi} \upharpoonright (\beta_\xi - \beta_\gamma)$ in order to make t_ξ injective. Let $t = \bigcup_{\xi<\omega_1} t_\xi$. Then $t \in T_\alpha$; and if $\beta < \alpha$, then $t \upharpoonright \beta \approx s_\beta$. However, we need to adjust t in order to ensure that $ran(s_\alpha)$ is non-stationary.

First note that since each $t_\xi \approx s_{\beta_\xi}$, it follows that each $ran(t_\xi)$ is non-stationary. For each $\xi < \omega_1$, choose some club C_ξ such that $X_\xi \cap ran(t_\xi) = \emptyset$. Next choose a continuous increasing sequence $\langle \beta_i \mid i \in \omega_1 \rangle$ such that $\beta_i \in \bigcap_{\xi<i} C_\xi$. Then we define

$$\begin{aligned} s_\alpha(\theta) &= \beta_{i+1} \text{ if } t(\theta) = \beta_i; \\ &= t(\theta) \text{ otherwise.} \end{aligned}$$

Then clearly $s_\alpha \in T_\alpha$. Let

$$C = \{\beta_\delta \mid \delta < \omega_1, \lim \delta\}.$$

Then C is a club of ω_1, and it is clear that $C \cup ran(s_\alpha) = \emptyset$.

Finally suppose that $\xi < \omega_1$ and that $\xi < i < \omega)1$. Then $\beta_i \in C_\xi$ and so $\beta_i \notin ran(t_\xi)$. Thus, in modifying t to s_α, we change only countably many values of $t \restriction \beta_\xi$. Thus s_α also has the property that if $\beta < \alpha$, then $s_\alpha \restriction \beta \approx s_\beta$. □

In the remainder of this section, we will attempt to construct an ω_1–Suslin tree in ZFC.

Definition. A tree T is ever-branching if for all $t \in T$, the set $\{s \in T \mid t < s\}$ is NOT linearly ordered.

Exercise Suppose that T is a tree such that $ht(T) = \omega_2$ and $|T_\alpha| \leq \omega$ for all $\alpha < \omega_2$. Then T has a branch of cardinality ω_2.

Hint The usual proof uses Fordor's Lemma. □

Lemma 2.3. *Suppose that the tree $\langle T, < \rangle$ satisfies the following conditions.*

(i) T is ever-branching;

(ii) $ht(T) = \omega_1$;

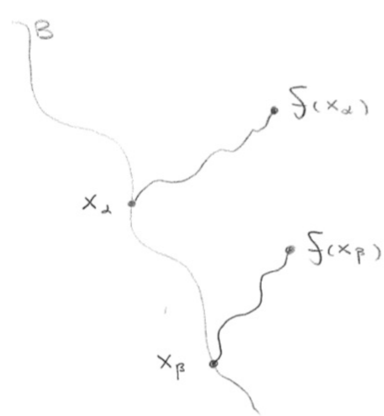
(iii) Every maximal antichain of T is countable.

Then T is a Suslin tree.

Proof. Suppose that B is an uncountable branch. For each $x \in B$, there exists $f(x) \in T - B$ such that $x < f(x)$. Define inductively $x_\alpha \in B$ for $\alpha < \omega_1$ such that

$$ht_T(x_\alpha) > \sup\{ht_T(f(x_\beta)) \mid \beta < \alpha.\}$$

Then $\{f(x_\alpha) \mid \alpha < \omega_1\}$ is an uncountable antichain, which is a contradiction. □



Now imagine that we are trying to define a Suslin tree $T = \bigcup_{\alpha<\omega_1} T_\alpha$. We will define the levels T_α by induction on $\alpha < \omega_1$. We want to ensure that:

$$\left.\begin{array}{l}(i) 1 \leq |T_\alpha| < \omega_1 \text{ for each } \alpha < \omega_1; \\ (ii) T \text{ is ever-branching.}\end{array}\right\} \circledast$$

Then we need only worry about avoiding the creation of uncountable maximal antichains.

$\underline{\alpha = 0}$ We set $T_0 = \{r\}$.

$\underline{\alpha = \beta + 1}$ Suppose that $T_\beta = \{t_n \mid n < \omega\}$. Then we set $T_{\beta+1} = \{u_n, v_n \mid n < \omega\}$ and $t_n < u_n, v_n$. (This ensures that T is ever-branching.)

At limit stages, we try to "kill off" potential uncountable maximal antichains. Suppose that we fail and that $A \subset T$ is an uncountable maximal antichain.

Lemma 2.4(Reflection Lemma). *Suppose that T is an ω_1−tree and that $A \subset T$ is an uncountable maximal antichain. Then for all $\beta < \omega_1$, there exists $\beta < \alpha < \omega_1$ such that*

(a) α is a limit ordinal.

(b) $A \cap \bigcup_{\gamma<\alpha} T_\gamma$ is a maximal antichain of $\bigcup_{\gamma<\alpha} T_\gamma$.

Proof. For each $x \in T$, let $f(x) \in A$ be such that x and $f(x)$ are comparable. Define inductively a strictly increasing sequence of countable ordinals

$$\beta = \beta_0 < \beta_1 < ... < \beta_n < ...$$

such that $f\left[\bigcup_{\gamma<\beta_n} T_\gamma\right] \subseteq \bigcup_{\gamma<\beta_{n+1}} T_\gamma$. Then $\alpha = \sup_{n<\omega} \beta_n$ satisfies our requirements. $\square$

So we get "plenty of warning" that A is growing during the construction. And at each such limit stage, we could have "killed off" A as follows. Let $D = A \cap \bigcup_{\gamma<\alpha} T_\gamma$ be such that D is a maximal antichain.

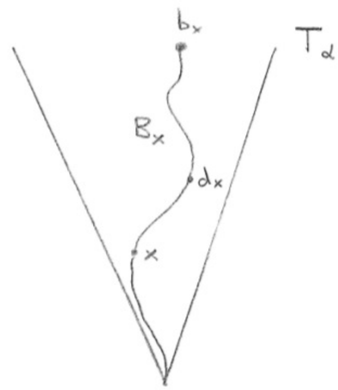
$\underline{\lim \alpha}$ We shall prevent the maximal antichain $D \subseteq \bigcup_{\gamma<\alpha} T_\gamma$ from growing any larger at later stages of the construction.

For each $x \in \bigcup_{\gamma<\alpha} T_\gamma$, let $d_x \in D$ be such that x and d_x are comparable. Then we can inductively construct <u>distinct</u> branches B_x of $\bigcup_{\gamma<\alpha} T_\gamma$ such that:

(1) $x, d_x \in B_x$;

(2) $B_x \cap T_\gamma \neq \emptyset$ for all $\gamma < \alpha$.

Now we let $T_\alpha = \{b_x \mid x \in \bigcup_{\gamma<\alpha} T_\gamma\}$ and set $z < b_x$ iff $z \in B_x$.



Note that each $t \in T_\alpha$ is comparable with some element of D. So D will remain a maximal antichain during the rest of the construction.

Lecture Notes #4 of Axiomatic Set Theory

Definition. The dominance order on ω^ω is the partial order defined by

$$f <^* g \leftrightarrow \exists n_0 \forall n > n_0 f(n) > g(n).$$

Proposition 3.1. $\omega < \mathfrak{b} \leq 2^\omega$.

Question Suppose $\emptyset \neq \mathcal{F} \subseteq \omega^\omega$ is any collection, when does there exists $g \in \omega^\omega$ such that $f <^* g$ for all $f \in F$.

We will translate this problem to an equivalent problem about a suitable poset.

Definition. Let $\mathbb{P}_\mathcal{F}$ consist of the ordered pair $\langle p, F_0 \rangle$ where $p : n \to \omega$ for some $n \in \omega$ and F_0 is a finite subset of $\mathcal{F}$.[1]
We partially order $\mathbb{P}_\mathcal{F}$ by $\langle p', F_0' \rangle \leq \langle p, F_0 \rangle$ iff:

- $p' \supseteq p$;
- $F_0' \supseteq F_0$;
- if $l \in \operatorname{dom} p' - \operatorname{dom} p$ and $f \in F_0$, then $p'(l) > f(l)$.

Definition. If $\mathbb{P}$ is a poset, then $G \subseteq \mathbb{P}$ is a filter iff

1. for all $p, q \in G$, there exists $r \in G$ such that $r \leq p, q$;
2. $\forall p \in G \forall q \in \mathbb{P}(q \leq p \to q \in G)$.

Example Suppose there exists $g \in \omega^\omega$ such that $f <^* g$ for all $f \in \mathcal{F}$. Let $G \subseteq \mathbb{P}_\mathcal{F}$ be the set of elements $\langle p, \mathcal{F}_0 \rangle$ such that

1. $p \subseteq g$;
2. if $f \in F_0$ and $l \in \omega - \operatorname{dom} p$ then $f(l) < g(l)$.

Then G is a filter.

Definition. If $\mathbb{P}$ is a poset, then $D \subseteq \mathbb{P}$ is a dense subset iff, for every $f \in \mathbb{P}$, there exists $g \in D$ such that $g \leq f$.

Example

1. for each $n \in \omega$, $D_n = \{\langle p, F_0 \rangle \in \mathbb{P}_\mathcal{F} \mid n \in \operatorname{dom} p\}$ is dense.
2. for each $f \in \mathcal{F}$, $E_f = \{\langle p, F_0 \rangle \in \mathbb{P}_F \mid f \in F_0\}$ is dense.

The above filter $G \subseteq \mathbb{P}_\mathcal{F}$ satisfies:

[1] Thinks: we imagine that p is a finite approximation to a dominating function, and F_0 is a finite set of promises. i.e., we promise if $l \in \omega - \operatorname{dom} p$ and $f \in F_0$, then $p(l) > f(l)$.

- $G \cap D_n \neq \emptyset$ for all $n < \omega$;
- $G \cap E_f \neq \emptyset$ for all $f \in \mathcal{F}$.

Proposition . *Suppose there exists a filter $G \subseteq \mathbb{P}_{\mathcal{F}}$ such that:*

1. *$G \cap D_n \neq \emptyset$ for all $n < \omega$;*
2. *$G \cap E_f \neq \emptyset$ for all $f \in \mathcal{F}$.*

Then there exists $g \in \omega^\omega$ such that $f <^* g$ for all $f \in \mathcal{F}$.

Proof. Let $g = \bigcup\{p \mid \exists F_0(\langle p, F_0\rangle \in G)\}$.
We first check that $g \in \omega^\omega$. Let $n \in \omega$. Then there exists $\langle p, F_0\rangle \in G$ such that $n \in \operatorname{dom} p$. Suppose that $\langle q, F_1\rangle \in G$ also satisfies $n \in \operatorname{dom} q$. Then there exists $\langle r, F_2\rangle \in G$ such that

$$\langle r, F_2\rangle \leq \langle q, F_2\rangle, \langle p, F_0\rangle.$$

Thus $p(n) = q(n)$, and $g : \omega \to \omega$ is a function. Next, let $f \in \mathcal{F}$. Then there exists $\langle p, F_0\rangle \in G$ such that $f \in F_0$. Suppose that $l \in \omega - \operatorname{dom} p$, there exists $\langle q, F_1\rangle \in G$ with $l \in \operatorname{dom} q$ and $\langle q, F_1\rangle \leq \langle p, F_0\rangle$. Hence: $g(l) = q(l) > f(l)$. Thus $f <^* g$.

Definition. Let $\mathcal{C}$ be a class of posets. Let κ be a infinite cardinal. Then $FA_\kappa(\mathcal{C})$ is the statement:

> Whenever $\mathbb{P} \in \mathcal{C}$ and $\mathcal{D}$ is a collection at most κ dense subsets of $\mathbb{P}$, then there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

Proposition .

1. *If $\mathcal{C}$ is any class of posets, then $FA_\omega(\mathcal{C})$ is true.*
2. *There exists a poset $\mathbb{P}$ such that $FA_{\omega_1}(\{\mathbb{P}\})$ is false.*

Proof.

1. Let $\mathcal{D} = \{D_n \mid n < \omega\}$ be a countable set of dense subsets of some $\mathbb{P} \in \mathcal{C}$. Then we can inductively define that
$$p_0 \geq p_1 \geq ... \geq p_n \geq ...\ (n < \omega).$$
such that, $\forall p_n \in D_n$, clearly:
$$G = \{q \in \mathbb{P} \mid \exists n(p_n \leq q)\}$$
is a filter such that $G \cap D_n \neq \emptyset$ for all $n \in \omega$.
2. Let $\mathbb{P}_{coll}$ be the poset of finite functions:
$$p : n \to \omega_1, \quad n < \omega;$$

Partially ordered by

$$p \leq q \leftrightarrow p \supseteq q.$$

For each $n < \omega$, $D_n = \{p \in \mathbb{P}_{coll} \mid n \in \operatorname{dom} p\}$ is dense, and for each $\alpha < \omega_1$,$E_\alpha = \{p \in \mathbb{P}_{coll} \mid \alpha \in \operatorname{ran} p\}$ is dense.

Suppose $G \subseteq \mathbb{P}_{coll}$ is a filter such that $G \cap D_n \neq \emptyset$ and $G \cap E_\alpha \neq \emptyset$ for all $n \in \omega$ and all $\alpha \in \omega_1$. Then $g = \bigcup G$ is a surjective function from ω to ω_1. Contradiction.

Definition. Let $\mathbb{P}$ be a poset.

1. $p_1, p_2 \in \mathbb{P}$ is comparable iff $p_1 \leq p_2$ or $p_2 \leq p_1$.
2. $p_1, p_2 \in \mathbb{P}$ is compatible iff there exists $r \in \mathbb{P}$ such that $r \leq p_1$ and $r \leq p_2$.
3. $A \subseteq \mathbb{P}$ is an antichain iff every pair $p, q \in A$ not compatible.
4. $\mathbb{P}$ satisfies c.c.c.(countable chain condition) iff every antichain A is countable.

Example $\mathbb{P}_{coll}$ does not satisfy c.c.c.: $\{\langle 0, \alpha\rangle \mid \alpha \in \omega_1\} \subseteq \mathbb{P}_{coll}$ is an uncountable antichain.

Example If $\emptyset \neq \mathcal{F} \subseteq \omega^\omega$ is any nonempty subset, then $\mathbb{P}_\mathcal{F}$ is c.c.c..
Proof. Suppose that $\{\langle p_\alpha, F_\alpha\rangle \mid \alpha \in \omega_1\}$ is an uncountable subset of $\mathbb{P}_\mathcal{F}$, then there exist $\alpha < \beta < \omega_1$ such that $p_\alpha = p_\beta$. Clearly:

$$\langle p_\alpha, F_\alpha \cup F_\beta\rangle \leq \langle p_\alpha, F_\alpha\rangle, \langle p_\alpha, F_\beta\rangle.$$

Definition. MA(κ) is the statement: If poset $\mathbb{P}$ satisfies c.c.c. and $\mathcal{D}$ is a collection of $< \kappa$ dense sets of $\mathbb{P}$, then there exists a filter $G \subseteq \mathbb{P}$ such that $G \cup D \neq \emptyset$ for all $D \in \mathcal{D}$.
MA is the statement: For all $\kappa < 2^\omega$, MA(κ).

Remarks

- By considering $\mathbb{P}_{\omega^\omega}$, we see that MA($2^\omega$) is false.
- MA is only interested in $\neg$CH models.

Theorem (MA(κ)). $\mathfrak{b} \geq \kappa^+$.

Proof. Suppose $\mathcal{F} \subseteq \omega^\omega$ with $|\mathcal{F}| \leq \kappa$. By applying MA($\kappa$) to $\mathbb{P}_\mathcal{F}$, we obtaining $g \in \omega^\omega$ such that $f <^* g$ for all $f \in \mathcal{F}$.

Corollary. (MA) $\mathfrak{b} = 2^\omega$.

Theorem (MA(ω_1)+$\neg$CH). *There are no ω_1-Suslin tree.*

Definition. A $\kappa-$tree is well-pruned iff:

- $|Lev_0(T)| = 1$;
- If $t \in T$ and $ht_T(t) < \alpha < \kappa$, then there exists $s \in Lev_\alpha(T)$ such that $s > t$.

Lecture Notes #4 of Axiomatic Set Theory

Theorem . *There are no ω_1 Suslin trees.*

Proof. Suppose that $\langle T, < \rangle$ is an ω_1 Suslin tree. By Lemma 3.7 we can suppose that T is well-pruned. Consider $\mathbb{P} = \langle T, \prec \rangle$ where $s \prec t \leftrightarrow t < s$. Since T is well-pruned, for each $\alpha < \omega_1$,

$$D_\alpha = \{t \in T \mid ht_T(t) > \alpha\}$$

is dense in $\mathbb{P}$. Applying MA(ω_1), then there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$. But then G is an uncountable branch, contradiction.

Next we begin working towards trying to understand the value of 2^ω under MA.

Recall

Theorem (König's Lemma). *If $\kappa \geq \omega$, then $\mathrm{cf}(2^\kappa) > \kappa$. In particular, $\mathrm{cf}(2^\omega) > \omega$.*

Nothing else can be proved under ZFC. But first, we need to introduce a second cardinal invariant of the continuum.

Definition. A family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ os almost disjoint(a.d.) iff

1. $|a| = \omega$ for all $a \in \mathcal{A}$;
2. $|a \cap b| < \omega$ for all $a \neq b \in \mathcal{A}$.

We write mad family as an abbreviation of a maximal almost disjoint family.

Proposition 3.8. *There exists an a.d. family with $|\mathcal{A}| = 2^\omega$.*

Proof. Consider the branches of the following complete binary tree: For level 0, there is only one element: 0; for level 1, there are 2 elements: 1 and 2, both connected with 0; for level 3, there are 4 elements: 3 to 6, where 3, 4 are connected with 1 and 5, 6 are connected with 2, etc.. Consider the family consists of all the branches of this tree. Clearly every two different branches will have only finite intersection, and every branch is an infinite branch.

Observation For each $1 \leq n < \omega$, there exists a mad family $\mathcal{A}$ with $|\mathcal{A}| = n$,
Proof. Let $\omega = \mathcal{A}_0 \sqcup \mathcal{A}_1 \sqcup ... \sqcup \mathcal{A}_{n-1}$ be a partition of ω.

Definition. $\mathfrak{a} = \min\{|\mathcal{A}| \mid \mathcal{A}$ is an infinite mad family$\}$.

Proposition 3.9. $\omega < \mathfrak{a} \leq 2^\omega$.

Proof. Suppose that $\mathcal{A} = \{A_n \mid n < \omega\}$ is a mad family, then by the diagonal argument, we can find another $B \subseteq \omega$ with $|B| = \omega$ such that B have only finite intersection with every $A \in \mathcal{A}$. Thus $\mathcal{A}$ is not maximal.
As expected, we next work towards:

Theorem . *MA(κ)$\vdash \mathfrak{a} \geq \kappa^+$. In particular, MA$\vdash \mathfrak{a} = 2^\omega$.*

Definition. Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$ be any family. Then the poset $\mathbb{P}_\mathcal{A}$ consists of all $\langle s, F \rangle$ where:

1. $s \subseteq \omega$ with $|s| < \omega$;

2. $F \subseteq \mathcal{A}$ with $|F| < \omega$,

partially ordered by $\langle s', F' \rangle \leq \langle s, F \rangle$ iff:

1. $s \subseteq s'$ and $F \subseteq F'$;

2. if $A \in F$, then $s' \cap A \subseteq s$.

Lemma 3.12. *$\mathbb{P}_{\mathcal{A}}$ satisfies c.c.c..*

Proof. Suppose $\{p_\alpha = \langle s_\alpha, F_\alpha \rangle \mid \alpha < \omega_1\}$ is a subset of $\mathbb{P}_{\mathcal{A}}$, then there exists $\alpha \neq \beta < \omega_1$ such that $s_\alpha = s_\beta = s$. Thus,

$$\langle s, F_\alpha \cup F_\beta \rangle \leq \langle s, F_\alpha \rangle, \langle s, F_\beta \rangle.$$

Lemma 3.13. *For each $A \in \mathcal{A}$:*

$$D_A = \{\langle s, F \rangle \in \mathbb{P}_{\mathcal{A}} \mid A \in F\}$$

is dense in $\mathbb{P}_{\mathcal{A}}$.

Notation If $G \subseteq \mathbb{P}_{\mathcal{A}}$ is a filter, then

$$d_G = \bigcup \{s \mid \exists F (\langle s, F \rangle \in G)\}.$$

Lemma 3.14. *If $G \subseteq \mathbb{P}_{\mathcal{A}}$ is a filter and $G \cap D_A \neq \emptyset$, then $|d_G \cap A| < \omega$.*

Proof. Suppose that $\langle s, F \rangle \in G \cap D_A$, and $l \in (A \cap d_G) - S$. Then there exists $\langle s', F' \rangle \in G$ with $l \in s'$. Also exists $\langle s'', F'' \rangle \leq \langle s, F \rangle, \langle s', F' \rangle$. But $l \in s''$, which contradicts $s'' \cap A \in s$.

Without some assumptions on $\mathcal{A}$, it might be that d_G is finite.

Theorem 3.15(MA(κ)). *Suppose that $\mathcal{A}, \mathcal{C} \subseteq \mathcal{P}(\omega)$ with $|\mathcal{A}|, |\mathcal{C}| \leq \kappa$ satisfies:*

> *(*) For all $C \in \mathcal{C}$ and all finite $F \subseteq \mathcal{A}$, $|C - \bigcup F| = \omega$.*

Then there exists $d \subseteq \omega$ such that

1. *$|d \cap A| < \omega$ for all $A \in \mathcal{A}$;*

2. *$|d \cap c| = \omega$ for all $c \in \mathcal{C}$.*

Proof. For each $C \in \mathcal{C}$ and $n < \omega$, let $E^C_m = \{\langle s, F \rangle \in \mathbb{P}_{\mathcal{A}} \mid s \cap C \not\subseteq n\}$. Then (*) implies that E^C_n is dense in $\mathbb{P}_{\mathcal{A}}$. By MA(κ), there exists a filter $G \subseteq \mathbb{P}_{\mathcal{A}}$ such that $G \cap D_A \neq \emptyset$ and $G \cap E^C_n \neq \emptyset$ for all $A \in \mathcal{A}$, $C \in \mathcal{C}$ and $n < \omega$. Clearly d_G satisfies the requirement.

Theorem 3.10,MA(κ). $\mathfrak{a} \geq \kappa^+$.

Suppose that $\mathcal{A}$ is an infinite a.d. family, with $|\mathcal{A}| \leq \kappa$. Let $\mathcal{C} = \{\omega\}$, then (*) holds. Hence the result follows from **Theorem 3.15**.

We are now ready to prove:

Theorem 3.16(MA(κ)). $2^\kappa = 2^\omega$.

Corollary. 3.17(MA) 2^ω is regular.

Proof. of 3.17 Suppose $\mathrm{cf}(2^\omega) = \kappa < 2^\omega$, then by **Theorem 3.16**,

$$\mathrm{cf}(2^\kappa) = \mathrm{cf}(2^\omega) = \kappa.$$

Contradiction.
Proof of 3.16 Let $\mathcal{B} \subseteq \mathcal{P}(\omega)$ be an a.d. family with $|\mathcal{B}| = \kappa$.[1] Let $\Phi : \mathcal{P}(\omega) \to \mathcal{P}(\mathcal{B})$ be defined by:

$$\Phi(d) = \{B \in \mathcal{B} \mid |d \cap B| < \omega\}.$$

Then it is enough to show that Φ is onto. Let $\mathcal{A} \subseteq \mathcal{B}$ be any subset and let $\mathcal{C} = \mathcal{B} - \mathcal{A}$. Then (*) holds and so by MA(κ), there exists $d \in \mathcal{P}(\omega)$ such that $\Phi(d) = \mathcal{A}$.

Definition. If $A, B \in [\omega]^{\omega}$[2], then $A \subseteq^* B \leftrightarrow |A - B| < \omega$.

Definition. A tower is a sequence $\langle X_\alpha \mid \alpha < \lambda \rangle \subseteq [\omega]^\omega$ for some ordinal $\lambda > 0$ such that:

1. if $\alpha < \beta < \lambda$, then $X_\beta \subseteq^* X_\alpha$;
2. there does not exist $Y \in [\omega]^\omega$ such that $Y \subseteq^* X_\alpha$ for all $\alpha < \lambda$.

And similarly, $\mathfrak{t}$ is the least λ such that there exists a tower $\langle X_\alpha \mid \alpha < \lambda \rangle$.

<u>Exercise</u> $\mathfrak{t}$ is an infinite regular cardinal such that $\omega < \mathfrak{t} \leq 2^\omega$.

<u>Exercise</u>(MA(κ)) $\mathfrak{t} \geq k^+$.

[1] Such a family exists from just ZFC.
[2] The collection of all subsets of ω with size ω.

Lecture Notes #6 of Axiomatic Set Theory

Definition. A tree T is special iff T is the union of countably many antichains.

Remark (1) If T is special, it has no uncountable branches; (2) If T is a special ω_1-Aronszajn tree, then T isn't ω_1-Suslin.

Trivial but important observation: If T is a tree, then the following are equivalent:

1. T is special;
2. there exists $f : T \to \omega$ such that if $f(s) = f(t)$ then s, t are incomparable.

We say that f is a specializing function.

Definition. Let $\langle \mathbb{P}, \prec \rangle$ be a poset, then a tree T is $\mathbb{P}-$embeddable iff there exists $f : T \to \mathbb{P}$ such that if $s < t$ in T then $f(s) \prec f(t)$.

Remark $\mathbb{P}-$embeddings need not be injective.

Exercise If T is a tree, then TFAE(the following are equivalent):

1. T is special;
2. T is $\mathbb{Q}-$embeddable.

What can we say about $\mathbb{R}-$embeddable trees?

Observation If T is $\mathbb{R}-$embeddable, then T has no uncountable branches.(By seperability.)

Observation Suppose that S is an infinite countable set, and T is the tree of all sequences $\langle s_\beta \mid \beta < \gamma \rangle, \gamma < \omega_1$ of elements of S, partially ordered by inclusion. Then T is $\mathbb{R}-$embeddable.
Proof. Let $e : S \to \mathbb{N}$ be a bijection. Then we can define an $\mathbb{R}-$embedding by

$$f(\emptyset) = 0; \quad f(\langle s_\beta \mid \beta < \alpha \rangle) = \sum_{\beta<\alpha} \frac{1}{2^{e(s_\beta)}}, \alpha > 0.$$

Theorem 3.18. *There exists a tree T of size 2^ω which is $\mathbb{R}$-embeddable but not $\mathbb{Q}-$embeddable.*

Proof. Let T consist of strictly increasing sequences $\langle q_\gamma \mid \gamma < \alpha \rangle$ of positive rational numbers $q_\gamma \in \mathbb{Q}^+$ with $\alpha < \omega_1$ such that $\sup_{\gamma<\alpha} q_\gamma < \infty$. From previous observation, T is $\mathbb{R}-$embeddable. We claim that T is not $\mathbb{Q}$-embeddable.

So suppose $f : T \to \mathbb{Q}$ is a $\mathbb{Q}-$embedding. Then we can suppose that $f(T) \subseteq (0, 1) \cap \mathbb{Q}$. We cannow inductively on $\alpha < \omega_1$ define:

1. a strictly increasing sequence $\langle t_\alpha \mid \alpha < \omega_1 \rangle$ of elements of T;
2. a strictly increasing seqence $\langle q_\alpha \mid \alpha < \omega_1 \rangle$ of elements of $\mathbb{Q}$,

which gives 2 contradictions. Let $t_0 = \emptyset$ and $q_0 = f(\emptyset)$. Suppose inductively by that t_β, q_β for $\beta < \alpha$ have been defined such that $t_\beta = \langle q_\gamma \mid \gamma > \beta \rangle$ and $q_\beta = f(t_\beta)$,

1. If $\alpha = \beta + 1$, let $t_\alpha = t_\beta \frown q_\beta$. Let $q_\alpha = f(t_\alpha)$.
2. If $\lim(\alpha)$, then $t_\alpha = \bigcup_{\beta<\alpha} t_\beta$ and $q_\alpha = f(t_\alpha)$.

Theorem 3.19(MA(κ)). *If T is a tree with no uncountable branches, and $|T| \leq \kappa$, then T is special.*

Remark Thus, MA implies that if T is $\mathbb{R}$-embeddable, and $|T| < 2^\omega$, then T is $\mathbb{Q}-$embeddable.

Corollary. 3.20(MA+¬CH). Every ω_1-Aronszajn tree is special. Thus, every such tree is not Suslin.

Proof. Suppose that T with no uncountable branches and $|T| \leq \kappa$. Let $\mathbb{P}$ consist of all finite functions $p : T_0 \to \omega$ with $T_0 \subset T$ is a finite subset and if $s \neq t \in \mathrm{dom}(p)$ with $p(s) = p(t)$, then s, t are incomparable in T.

Claim.3.21 $\mathbb{P}$ is c.c.c..

Assuming the Claim, for each $t \in T$, let $D_t = \{p \in \mathbb{P} \mid t \in \mathrm{dom}(p)\}$. Clearly D_t is dense. By MA(κ), there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_t \neq \emptyset$ for all $t \in T$. Then $g = \bigcup G : T \to \omega$ is a specializing function.

Unlike our earlier MA proofs, Claim 3.21 is highly non-trivial. First we need to develop some infinite combinatorics.

Definition. A family F of sets is a $\Delta-$system if there exists a fixed r such that for all $a \neq b \in F$, $a \cap b = r$.

Lemma (The $\Delta-$system Lemma). *If F is a family of finite sets with $|F| = \omega_1$, then there exists a $\Delta-$system $F_0 \subseteq F$ with $|F_0| = \omega_1$.*

Proof. We can suppose that:

1. There exists $n \geq 1$ such that $|A| = n$ for all $A \in F$; The result is clearly true if $n = 1$.
2. Thus we can suppose that $n > 1$ and that the result holds for all $1 \leq l < n$. Let $S = \bigcap F$.

Case 1 Suppose that there exists $s \in S$ such that $|\{A \in F \mid s \in A\}| = \omega_1$. Then we can suppose that $s \in A$ for all $A \in F$. Let $F^* = \{A - \{s\} \mid A \in F\}$. By 2., there exists a $\Delta-$system $F_0^* \subseteq F^*$ with $|F_0^*| = \omega_1$. Then $F_0 = \{B_0 \cup \{s\} \mid B \in F_0^*\}$ satisfies our requirement.

Case 2 For every $s \in S$, $|\{A \in F \mid s \in A\}| \leq \omega$. Then we can inductively choose pairwise disjoint $\{A_\alpha \mid \alpha < \omega_1\} \subseteq F$ as follows:
Suppose $\{A_\beta \mid \beta < \alpha\}$ has been defined. Let $S_\alpha = \bigcup\{A_\beta \mid \beta < \alpha\}$. Then $|S_\alpha| \leq \omega$. So there are only countably many $A \in F$ such that $A \cup S_\alpha \neq \emptyset$. Hence we find A_α such that $A_\alpha \cap S_\alpha = \emptyset$.

Definition. For any sets I, J we define $F_n(I, J) = \{p \mid p : I \to J \text{ is finite partial function}\}$, ordered by $p \leq q \leftrightarrow p \supseteq q$.

Theorem 3.22. *If I is arbitrary, and J is countable, then $F_n(I, J)$ is c.c.c..*

Proof. Suppose that $\{p_\alpha \mid \alpha < \omega_1\}$ is an antichain. Then we can suppose that if $\alpha \neq \beta$, then $\text{dom}(p_\alpha) \neq \text{dom}(p_\beta)$. Applying the $\Delta-$system lemma, we can suppose there exists a finite R, if $\alpha \neq \beta$, then $\text{dom}(p_\alpha) \cap \text{dom}(p_\beta) = R$. But then there exist $\alpha \neq \beta$ such that $p_\alpha \upharpoonright R = p_\beta \upharpoonright R$ and so $p_\alpha \cup p_\beta \leq p_\alpha, p_\beta$, contradiction.

Definition. The filter F over S is an ultrafilter if for all $x \in \mathcal{P}(S)$ either $x \in F$ or $S - x \in F$.

Example If $s \in S$, then $\{A \in \mathcal{P}(S) \mid s \in A\}$ is an ultrafilter.

Lemma 3.23, Zorn. *A filter F on S is an ultrafilter iff F is a maximal filter.*

Proof. Clearly if F is a ultrafilter, then F is maximal. Conversely, suppose that F is maximal, suppose $x \in \mathcal{P}(S)$ and $x \notin F$, then for all $y \in F$, $Y \cap (S - x) \neq \emptyset$, since otherwise, $y \subseteq x$ and so $x \in F$.

Hence $F^+ = \{z \in \mathcal{P}(S) \mid \exists y(y \cap (s - x) \subseteq z)\}$ is a filter with $S - x \in F^+$, and $F \subseteq F^+$. By maximality, $F = F^+$, and so $S - x \in F$ and so F is an ultrafilter.
Applying Zorn's Lemma, we obtain

Theorem 3.24. *Every filter is contained in an ultrafilter.*

Definition. If κ is an infinite cardinal, then an ultrafilter $\mathcal{U}$ on κ is uniform if $|A| = \kappa$ for all $A \in \mathcal{U}$.

Corollary. 3.25 If κ is an infinite cardinal, then there exists a uniform ultrafilter on κ.

Proof. Let $\mathcal{U}$ be any ultrafilter extending the filter $F = \{A \in \mathcal{P}(\kappa) \mid |\kappa - A| < \kappa\}$.

Lecture Notes #7 of Axiomatic Set Theory

Lemma 3.21. *Let T be a tree with no uncountable branches and let $\mathbb{P}$ consist of all finite function $p : T_0 \to \omega$ such that:*

1. *$T_0 \subset T$ is finite;*
2. *If $s \neq t \in T_0$ with $p(s) = p(t)$, then $s \perp t$(s, t are incompatible).*

Then $\mathbb{P}$ is c.c.c..

Proof. Suppose that $\{p_\alpha \mid \alpha < \omega_1\}$ is an uncountable antichain.

(a) We can suppose that $|\operatorname{dom}(p_\alpha)| = |\operatorname{dom}(p_\beta)|$ for all $\alpha < \beta < \omega_1$;

(b) We can suppose that there is a fixed $R \subseteq T$ such that $\operatorname{dom}(p_\alpha) \cap \operatorname{dom}(p_\beta) = R$ for all $\alpha < \beta < \omega_1$;

(c) We can suppose that $p_\alpha \upharpoonright R = p_\beta \upharpoonright R$ for all $\alpha < \beta < \omega_1$.

For each $\alpha < \omega_1$ let $\operatorname{dom}(p_\alpha) - R = \{x_{\alpha,1}, x_{\alpha,2}, ..., x_{\alpha,n}\}$. If $\alpha \neq \beta$, then since $p_\alpha \perp p_\beta$, there exist $1 \leq k, l \leq n$ such that $p_\alpha(x_{\alpha,k}) = p_\beta(x_{\beta,l})$ where $x_{\alpha,k}$ and $x_{\beta,l}$ are comparable in T.

<u>Remark</u> If $n = 1$, then $\{x_{\alpha,1} \mid \alpha < \omega_1\}$ is contained in an uncountable branch, which is an contradiction. If $n > 1$, then we need to work harder.

For each $\alpha < \omega_1$ and $1 \leq k, l \leq n$, define:

$$Y_{\alpha,k,l} = \{\beta \in \omega_1 - \{\alpha\} \mid p_\alpha(x_{\alpha,k}) = p_\beta(x_{\beta,l}), \text{ and } x_{\alpha,k} \perp x_{\beta,l}\}.$$

Then $\omega_1 - \{\alpha\} = \bigcup_{1 \leq k,l \leq n} Y_{\alpha,k,l}$. Let U be a uniform ultrafilter on ω_1. Then each $\omega_1 - \{\alpha\} \in U$. Hence for each $\alpha < \omega_1$, there exists $k = k(\alpha), l = l(\alpha)$ such that $Y_{\alpha,k,l} \in U$.

There exists uncountable $A \subseteq \omega_1$ and fixed k, l such that $k(\alpha) = k, l(\alpha) = l$ for all $\alpha \in A$. Let $\alpha, \beta \in A$, then $Y_{\alpha,k,l} \cap Y_{\beta,k,l} \in U$, and so $|Y_{\alpha,k,l} \cap Y_{\beta,k,l}| = \omega_1$. Note that if $\gamma \in Y_{\alpha,k,l} \cap Y_{\beta,k,l}$, then $x_{\alpha,k}, x_{\gamma,l}$ are comparable and $x_{\beta,k}, x_{\gamma,l}$ are comparable. Also if $\gamma \neq \gamma' Y_{\alpha,k,l} \cap Y_{\beta,k,l}$, then $x_{\gamma',l} \neq x_{\gamma,l}$.

There only exists countably many $t \in T$ such that $t \leq x_{\alpha,k}$ or $t \leq x_{\beta,k}$, then there exists $\gamma \in Y_{\alpha,k,l} \cap Y_{\beta,k,l}$ such that $x_{\alpha,k} \leq x_{\gamma,l}$ and $x_{\beta,k} \leq x_{\gamma,l}$. Hence $x_{\alpha,k}$ and $x_{\beta,k}$ are comparable, and so $\{x_{\alpha,k} \mid \alpha \in A\}$ is contained in an uncountable branch. Contradiction.

<u>Remark</u> There is an "analogy" between models of forcing axioms and existentially closed structures in model theory. In the latter, universal statements are equivalent to apparently stronger existential statements. The same is true in the former.

<u>Example</u> (MA+¬CH) If T is an ω_1−tree, then TFAE:

($\forall$) T is an ω_1−Aronszajn tree;

($\exists$) T is special.

Definition. An ω_1-tree is an ω_1-Kurepa tree if T has at least ω_2 uncountable branches.

Remark Let $T \subseteq 2^{<\omega_1}$ be the tree of countable binary sequences containing at most one 1. Then T is an ω_1-tree with exactly ω_1 uncuntable branches.

Remark If $V = L$, then there exists an ω_1-Kurepa tree.(See Kunen.)

Definition. An ω_1-tree is weakly special if there exists a function $f : T \to \omega$ such that if $s < t, n \in T$ and $f(s) = f(t) = f(n)$ then t, n are comparable.

Proposition 3.26. *If T is an ω_1-tree with no uncountable branches, then T is special iff T is weakly special.*

Proof. If T is special, then T is clearly weakly special. Conversely, suppose that $f : T \to \omega$ satisfies (*). Fix some $n \in \omega$ and let M_n be the set of $T-$minimal $t \in T$ such taht $f(t) = n$. For each $t \in M_n$ let $C_t = \{s \in f^{-1}(n) \mid t \leq s\}$. Then each C_t is a chain, which is necessarily countable. Also if $t \neq t' \in M_n$, and $c \in C_t, c' \in C_{t'}$, then $c \perp c'$. It follows that M_n is a union of countable many antichains, and hence so is T.

Proposition 3.27. *If T os a weakly special ω_1-tree, then T has at most ω_1 uncountable branches.*

Proof. Suppose $f : T \to \omega$ satisfies (*). Let B be any uncountable branch. Let n_B be the least n such that $|\{b \in B \mid f(b) = n\}| = \omega_1$. Let $t_B \in B$ be the $T-$least element such that $f(t_B) = n_B$. Then clearly:

$$B = \{s \in T \mid s \leq t_B\} \cup \{s \in T \mid (\exists u) t_B < s < u \text{ and } f(u) = n_B\}.$$

Since n_B, t_B determine B, there are only ω_1 possibilities for B.

Theorem 3.28(MA+¬CH). *If T is an ω_1-tree with at most ω_1 uncountable branches, then T is weakly special.*

Proof. After enlarging T if necessary, we can suppose that T has exactly ω_1 uncountable branches, and that every $t \in T$ lies in an uncountable branch. Let $\mathcal{B} = \{B_\alpha \mid \alpha < \omega_1\}$ be the set of uncountable branches. Define $s : \omega_1 \to T$ by

$$s(\alpha) = \text{ the } T\text{−least element of } B_\alpha - \bigcup_{\beta<\alpha} B_\beta.$$

Let $S = \{s(\alpha) \mid \alpha < \omega_1\}$; and regard S as a tree using the restriction of the $T-$order. For each $B \in \mathcal{B}$, we have that $B \cap S$ is countable. Thus S has no uncountable branches. Hence by MA+¬CH there exists a special function $f : T \to \omega$. Now defining $g : T \to \omega$ by $g(t) = f(s(\alpha))$ where α is the least $t \in B_\alpha$. We claim that g satisfies (*). To see this suppose $t < u \in T$ with $g(t) = g(u)$. Let $\alpha, \beta < \omega_1$ be the least such that $t \in B_\alpha, u \in B_\beta$. Then clearly $\alpha \leq \beta$. Suppose that $\alpha < \beta$, then we must have that $t < s(\beta) \leq u$. But then $f(s(\alpha)) = g(t) = g(u) = f(s(\beta))$, which contradicts that f specializing T. Now it is clear that g satisfies (*).

Lecture Notes #8 of Axiomatic Set Theory

We finish this section by considering a few more ordinal invariants of the continuum:

Definition. An ideal $\mathcal{I}$ us a $\sigma-$ideal if whenever $\{X_n \mid n < \omega\} \subseteq \mathcal{I}$, then $\bigcup_n X_n \in \mathcal{I}$.

Example (1) Let $\mathcal{N} \subseteq \mathcal{P}(\mathbb{R})$ be the collection of Lebesgue $0-$measure subsets, then $\mathcal{N}$ is an $\sigma-$ideal.

(2) Let $\mathcal{M} \subseteq \mathcal{P}(\mathbb{R})$ be the collection of meager subsets, then $\mathcal{M}$ is a $\sigma-$ideal.

Definition. Suppose $\mathcal{I}$ is an ideal of S such that $\{s\} \in \mathcal{I}$ for all $s \in S$, then the covering number $\mathrm{cov}(\mathcal{I})$ is the least λ such that there exists $\{X_\alpha \mid \alpha < \lambda\} \subseteq \mathcal{I}$ such that $S = \bigcup_{\alpha<\lambda} X_\alpha$.

Observation If $\mathcal{I}$ is a $\sigma-$ideal in S, then $\omega < \mathrm{cov}(\mathcal{I}) \leq |S|$. Thus:

- $\omega < \mathrm{cov}(\mathcal{N}) \leq 2^\omega$;
- $\omega < \mathrm{cov}(\mathcal{M}) \leq 2^\omega$;

Theorem (MA(κ)). $\mathrm{cov}(\mathcal{N}) \geq \kappa^+$ *and* $\mathrm{cov}(\mathcal{M}) \geq \kappa^+$.

Corollary. (MA) $\mathrm{cov}(\mathcal{N}) = 2^\omega$ and $\mathrm{cov}(\mathcal{M}) = 2^\omega$.

Definition. The Mycielski ideal $\mathcal{B}$ on $\mathcal{P}(\omega)$ consists of the susbsets $X \subseteq \mathcal{P}(\omega)$ for all $A \in [\omega]^\omega$, $\mathcal{P}(A) \neq \{A \cap B \mid B \in X\}$.

Observation $\mathcal{B}$ is a $\sigma-$ideal.

Proof. Suppose that $\{X_n \mid n < \omega\} \subseteq \mathcal{B}$. Let $A \in [\omega]^\omega$. Express $A = \bigsqcup_n A_n$ where each $A_n \in [A]^\omega$. Then for each $n < \omega$, there exists $C_n \in \mathcal{P}(A_n) - \{A_n \cap B \mid B \in \mathcal{B}\}$. Consider $C = \bigsqcup_n C_n$. Clearly $C \in \mathcal{P}(A) - \{A_n \cap B \mid B \in \bigcup_n X_n\}$. Thus $\bigcup_n X_n \in \mathcal{B}$.

Remark Clearly $C \notin \bigcup_n X_n$. Thus $\omega < \mathrm{cov}(\mathcal{B}) \leq 2^\omega$.

Next we attempt to use MA to show that MA+$\neg$CH proves $\mathrm{cov}(B) = 2^\omega$. So suppose for example that $\mathcal{P}(\omega) = \bigcup_{\alpha<\omega_1} X_\alpha$ where each $X_\alpha \in \mathcal{B}$. Let $\mathbb{P}$ be the poset of "suitable" approximations $p : \omega \to 2$ of the characteristic function of some $C \notin \bigcup_{\alpha<\omega_1} X_\alpha$. Thus p is a partial function.

Let $D_\alpha \subseteq \mathbb{P}$ be the set of $p \in \mathbb{P}$ such that $A \in [\omega]^\omega$ such that $p \restriction A = \chi_C \restriction A$ for some $C \in \mathcal{P}(A) - \{A \cap B \mid B \in X_\alpha\}$. Thus we must let $\mathbb{P} = \{p : \omega \to 2 \mid\ p$ is a partial function and $|\omega - \mathrm{dom}(p)| = \omega\}$. This $\mathbb{P}$ is clearly not c.c.c.. It is also not countably closed.

Question Is $\mathrm{FA}_{\omega_1}(\mathbb{P})$ consistent?[1]

[1] Here Prof. Simon Thomas told us that in his first students thesis in 1994, there was a result saying that MA+$\neg$CH does not settle the question whether $\mathrm{cov}(\mathcal{B}) = \omega_1$.

Chapter 4: A Brief Introduction of Forcing

Until further noticed, let $\langle M, \in \rangle$ be a countable[2] transitive model of ZFC, ie, if $x \in M$ and $y \in x$, then $y \in M$.

Convention From now on we only work with posets $\mathbb{P}$ which contain a largest element $\mathbf{1} \in \mathbb{P}$. Namely, the element with least information.

Definition. Suppose that $\mathbb{P} \in M$ is a poset, the filter $G \subseteq \mathbb{P}$ is $M-$generic if $G \cap D \neq \emptyset$ for every dense $D \subseteq \mathbb{P}$ with $D \in M$.

Lemma 4.1. *If $\mathbb{P} \in M$ is a poset and $p \in \mathbb{P}$, then there exists an $M-$generic filter $G \subseteq \mathbb{P}$ such that $G \ni p$.*

Proof. Working in the actual universe V. Let $\{D_n \mid n < \omega\}$ enumerate the dense $D \subseteq \mathbb{P}$ such that $D \subseteq M$. Then we can inductively define $p_n \in \mathbb{P}$ such that

$$p = p_0 \geq p_1 \geq p_2 \geq ... \geq p_n \geq ...$$

such that $p_{n+1} \in D_n$. Then $G = \{q \in \mathbb{P} \mid (\exists n) p_n \leq q\}$ satisfies our requirement.

Lemma 4.2. *Suppose $\mathbb{P} \in M$ satisfies:*

(*) For all $p \in \mathbb{P}$, there exists $r, q \in \mathbb{P}$ such that $r \perp q$.

If $G \subseteq \mathbb{P}$ is $M-$generic then $G \notin M$.

Proof. Suppose that $G \in M$. Then $D = \mathbb{P} - G \in M$. Since $G \cap D = \emptyset$, it is enough to show that D is dense. Let $p \in P$, there exists $q, r \in \mathbb{P}$ such that $q \perp r$. Thus $q \notin G$ or $r \notin G$, ie $r \in D$ or $q \in D$.

Target We next define a countable transitive model $M[G]$ of ZFC such that $M \subseteq M[G]$ and $G \in M[G]$.

Definition. τ is a $\mathbb{P}-$name if

- τ is a set of ordered pairs;
- for all $\langle \sigma, p \rangle \in \tau$, σ is a $\mathbb{P}-$name and $p \in \mathbb{P}$.

Example (1) $\emptyset$ is a $\mathbb{P}-$name; (2) $\{\langle \emptyset, p \rangle\}$ is a $\mathbb{P}-$name for each $p \in \mathbb{P}$.

Definition. $M^{\mathbb{P}} = \{\tau \in M \mid \tau \text{ is a } \mathbb{P}-\text{name}\}$.

Definition. If $G \subseteq \mathbb{P}$ is an $M-$generic filter, and $\tau \in M^{\mathbb{P}}$, then

$$\tau_G = \{\sigma_G \mid (\exists p \in G) \langle \sigma, p \rangle \in \tau\}.$$

Example (1) $\emptyset_G = \emptyset$; (2) $\{\langle \emptyset, p \rangle\}_G = \{\emptyset\}$, if $p \in G$; otherwise, $\{\langle \emptyset, p \rangle\}_G = \emptyset$.

[2]Obtained by Löwenheim-Skolem Theorem.

Definition. $M[G] = \{\tau_G \mid \tau \in M^{\mathbb{P}}\}$.

Theorem (Forcing Theorem A). *With the above assumptions:*

1. *$M[G]$ is a countable transitive model of ZFC;*
2. *$M \subseteq M[G]$ and $G \in M[G]$.*

Sketch of Proof. (2) For each $x \in M$, let $\check{x} \in M^{\mathbb{P}}$ be defined inductively by

$$\check{x} = \{\langle \check{y}, \mathbf{1} \rangle \mid y \in x\}.$$

If $G \subseteq \mathbb{P}$ is $M-$generic, then $\mathbf{1} \in G$ and $\check{x}_G = \{\check{y}_G \mid y \in x\} = \{y \mid y \in x\} = x$. Next define $\Gamma_G = \{\check{p}_G \mid p \in G\} = \{p \mid p \in G\} = G$. Thus $G \in M[G]$.

Lecture Notes #9 of Axiomatic Set Theory

Theorem (Forcing Theorem A).

1. *$M[G]$ is a countable transitive model of ZFC;*
2. *$M \subseteq M[G]$ and $G \in M[G]$;*
3. *M and $M[G]$ contains the same ordinals.*

Notation For each $x \in M$, $\check{x}$ is the canonical $\mathbb{P}-$name of x.

Definition. Let $\phi(x_1, ..., x_n)$ is a formula in the language of set with $x_1, ..., x_n$ free. Let $\tau_1, ..., \tau_n \in M^{\mathbb{P}}$. If $p \in \mathbb{P}$, then

$$p \Vdash \phi(\tau_1, ..., \tau_n)$$

iff whenever $G \subseteq \mathbb{P}$ is an $M-$generic filter with $p \in G$, then

$$M[G] \vDash \phi(\tau_{1G}, ..., \tau_{nG}).$$

Example Let $\tau = \{\langle \emptyset, p \rangle\}$ for some $p \in \mathbb{P}$, then $p \vDash \tau \neq \check{\emptyset}$.

Theorem (Forcing Theorem B).

1. *With the assumptions, $M[G] \vDash \phi(\tau_{1G}, ..., \tau_{nG})$ iff there exists $p \in G$ such that $p \Vdash \phi(\tau_1, ..., \tau_n)$;*
2. *The relation $\Vdash$ is definable in M.*

Corollary. 4.3 With $\phi, \tau_1, ..., \tau_n$ as above, let $D \subseteq \mathbb{P}$ consist of the $p \in \mathbb{P}$ such that either

$$p \Vdash \phi(\tau_1, ...\tau_n) \quad \text{or} \quad p \Vdash \neq \phi(\tau_1, ...\tau_n).$$

Thus $D \in M$, and D is dense in $\mathbb{P}$.

Proof. Since $\Vdash$ is definable in M, we have $D \in M$. Suppose $q \in \mathbb{P}$ is arbitrary. There exists an $M-$generic $G \subseteq \mathbb{P}$ such that $q \in G$. Suppose, for example, that $M[G] \vDash \phi(\tau_{1G}, ..., \tau_{nG})$. Then there exists $r \in G$ such that

$$[*] r \Vdash \phi(\tau_1, ..., \tau_n). \tag{1}$$

Then there exists $p \in G$ such that $p \leq r, q$. Since $p \leq r$ and (*) holds,

$$p \Vdash \phi(\tau_1, ..., \tau_n).$$

Thus $p \in D$.

Conventions (1) From now on, we usually "pretend" force over the actual universe V; (2) We usually wrote $\Vdash \phi$ instead of $\mathbf{1} \Vdash \phi$; (3) We sometimes write $p \Vdash_{\mathbb{P}} \phi$.

Question For which $\mathbb{P}$ is $\mathrm{FA}_{\omega_1}(\mathbb{P})$ obviously false?

Definition. $\mathbb{P}$ collapse ω_1 iff:

$$\Vdash \check{\omega}_1 \text{ is countable.}$$

Proposition 4.4. *If $\mathbb{P}$ collapse ω_1, then $FA_{\omega_1}(\mathbb{P})$ is false.*

Proof. Suppose $\mathbb{P}$ is a counterexample. Then there exists a $\mathbb{P}-$name τ such that $\tau : \check{\omega} \to \check{\omega}_1$ is a surjective function. Note that through we work inside V. For each $n \in \omega$, let $D_n = \{p \in \mathbb{P} \mid \exists\alpha(p \Vdash \tau(\check{n}) = \check{\alpha})\}$ and for each $\alpha < \omega_1$, let $E_\alpha = \{p \in \mathbb{P} \mid \exists n(\tau(\check{n}) = \check{\alpha}))\}$. Then a moment of thought shows that each D_n and E_α is dense in $\mathbb{P}$. By $\mathrm{FA}_{\omega_1}(\mathbb{P})$, there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_n \neq \emptyset$ and $G \cap E_\alpha \neq \emptyset$ for each $n < \omega$ and each $\alpha < \omega_1$. Let

$$\phi = \{\langle n, \alpha\rangle \in \omega \times \omega_1 \mid \exists p \in G(p \Vdash \tau(\check{n}) = \check{\alpha})\}.$$

Then $\phi \in V$. Also for each $n \in \omega$, there exists $\alpha \in \omega_1$, such that $\langle n, \alpha\rangle \in \phi$; and for each $\alpha \in \omega_1$, there exists $n \in \omega$ such that $\langle n, \alpha\rangle \in \phi$.

To reach a contradiction, it is enough to show that ϕ is a function. Suppose not, so there exists $\alpha \neq \beta$ and $n < \omega$ such that $\langle n, \alpha\rangle, \langle n, \beta\rangle \in \phi$. Then there exist $p, q \in G$ such that $p \Vdash \tau(\check{n}) = \check{\alpha}$, and $q \Vdash \tau(\check{n}) = \check{\beta})$. Let $r \in G$ with $r \leq p, q$. Then $r \vDash \tau(\check{n}) = \check{\alpha} \wedge \tau(\check{n}) = \check{\beta}$, which is a contradiction, since $r \Vdash$ "$\tau : \check{\omega} \to \check{\omega}_1$ is a function".

Definition. $\mathbb{P}$ preserves ω_1 iff $\Vdash$ "$\check{\omega}_1$ is uncountable".

Theorem 4.5. *If $\mathbb{P}$ is c.c.c., then $\mathbb{P}$ preserves ω_1.*

Proof. Suppose $\mathbb{P}$ is a counterexample. Then there exists $p \in \mathbb{P}$ and a $\mathbb{P}-$name τ such that $\Vdash$ "$\tau : \check{\omega} \to \check{\omega}_1$ is a surjective function". For each $n < \omega$, let $F(n) = \{\alpha > \omega_1 \mid \exists q < p(q \Vdash \tau(\check{n}) = \check{\alpha})\}$ and let $A = \bigcup_{n\in\omega} F(n)$. Then each $F(n) \in V$ and $A \in V$. Clearly $p \Vdash \mathrm{ran}(\tau) \subseteq \check{A}$.
<u>Claim</u> $|F_n| \leq \omega$ for each $n \in \omega$.
Proof. For each $\alpha \in F(n)$, choose $q_\alpha < p$ such that $q_\alpha \Vdash \tau(\check{n}) = \check{\alpha}$, then $B = \{q_\alpha \mid \alpha \in F(n)\}$. This is an antichain, so $|B| \leq \omega$. Hence $|F(n)| \leq \omega$.

<u>Claim proved.</u>

Since $A \subseteq \omega_1$ is countable, then there exists $\beta < \omega_1$ such that $A \subseteq \beta$. But then $p \Vdash \mathrm{ran}(\tau) \subseteq \check{\beta}$, which is a contradiction.

<u>Remark</u> The above argument easily generalizes to show that if $\mathbb{P}$ is c.c.c., then $\mathbb{P}$ preserves every cardinal and every cardinal cofinally.

Definition. A poset $\mathbb{P}$ is countably closed, iff whenever

$$p_0 \geq p_1 \geq \ldots \geq p_n \geq \ldots, \quad n \in \omega,$$

at elements of $\mathbb{P}$, then there exists $p \in \mathbb{P}$ such that $p \leq p_n$ for all $n \in \omega$.

<u>Example</u> Let $\kappa > \omega_1$ be a cardinal, and let $\mathbb{P}$ consist of all partial functions $p : \alpha \to \kappa, \alpha < \omega_1$, partially ordered by $p \leq q \leftrightarrow p \supseteq q$. Then $\mathbb{P}$ is countably closed, and $\mathbb{P}$ collapses κ.

Theorem 4.6. *If $\mathbb{P}$ is countably closed, then $\mathbb{P}$ preserves ω_1.*

Proof. If $\mathbb{P}$ is a counterexample, then there exists $p \in \mathbb{P}$ and a $\mathbb{P}-$name τ such that $p \Vdash$ "$\tau : \check{\omega} \to \check{\omega}_1$ is a surjective function". Working in V, we can inductively define $p_n \in \mathbb{P}$ and $\alpha_n \in \omega_1$ such that

$$p \geq p_0 \geq ... \geq p_n \geq ...$$

and $p_n \Vdash \tau(\check{n}) = \check{\alpha}_n$. There exists $q \in \mathbb{P}$ such that $q \leq q_n$ for all $n \in \omega$. Let $\phi = \{\langle n, \alpha_n \rangle \mid n \in \omega\} \in V$. Then $q \Vdash \tau = \check{\phi}$, which contradicts $p \Vdash$ "$\tau : \check{\omega} \to \check{\omega}_1$ is a surjective function".

<u>Remark</u> The above argument shows that if $\mathbb{P}$ is c.c.c., and $A \in V$ then $\mathbb{P}$ adds no new countable sequence(functions $\omega \to A$).

<u>Pop Quiz</u> We want to prove the consistency of CH. Unfortunately, $V \vDash 2^\omega = \omega_2$. Let $\mathbb{P} = \{p : \alpha \to \omega_2 \mid \alpha < \omega\}$ and let $G \subseteq \mathbb{P}$ be $V-$generic. Then $\omega_1^{V[G]} = \omega_1^V$. Also there is a surjective function $f : \omega_1^{V[G]} \to \mathcal{P}(\omega)^V$. But $P(\omega)^V = P(\omega)^{V[G]}$, and so $V[G] \vDash$CH.

Question: What is $\mathrm{cf}^{V[G]}(\omega_2^V)$?

Answer: Since there is no new $f : \omega \to \omega_2^V$, $\mathrm{cf}^{V[G]}(\omega_2^V) = \omega_1^{V[G]}$.

Definition. Let $\mathcal{C}_{\omega_1}$ be the class of posets which doesn't collapse ω_1.

Theorem F. *$A_{\omega_1}(\mathcal{C}_{\omega_1})$ is false.*

Proof Delayed.

Definition. Suppose that $S \subseteq \omega_1$ is stationary, then $\mathbb{P}_s$ is the collection of subsets $p \subseteq S$ such that

- p is a closed subset, i.e. if $\alpha_0 < \alpha_1 < ... < \alpha_n < ...$ are elements of p, then $\sup_n \alpha_n \in p$;
- p has a maximum element, partially ordered by $p \leq q$, iff (1) $p \supseteq q$; (2) If $\alpha \in p - q$, then $\alpha > \max(q)$. i.e., p is an end-extension of q.

Lecture Notes #10 of Axiomatic Set Theory

Definition. $\mathcal{C}_{\omega_1}$ is the class of posets which preserves ω_1.

Theorem 4.7. *$FA_{\omega_1}(\mathcal{C}_{\omega_1})$ is false.*

Proof Delayed.

Definition. Suppose $S \subseteq \omega_1$ is stationary, then $\mathbb{P}_s$ is the collection of $p \subseteq S$ such that

- p is a closed subset of ω_1, i.e. if $\alpha_0 < \alpha_1 < ... < \alpha_n < ...$ are elements of p, then $\sup_n \alpha_n \in p$;
- p contains a maximal element. We order $\mathbb{P}_s$ by $p \leq q$, iff p is an end-extension of q, i.e. if $\alpha \in q - p$, then $\alpha > \max(p)$.

Theorem I. *f $S \subseteq \omega_1$ is stationary, then:*

1. *$\mathbb{P}_s$ preserves ω_1;*
2. *If $G \subseteq \mathbb{P}_s$ is $V-$generic, then $C = \bigcup G$ is a club contained in S.*

Proof. Assume for the moment that $\mathbb{P}_s$ preserves ω_1. Let $G \subseteq \mathbb{P}_s$ be $V-$generic and let $C = \bigcup G$. Then clearly $C \supseteq S$. For each $\alpha < \omega_1$,

$$D_\alpha = \{p \in \mathbb{P}_s \mid \max(p) > \alpha\} \in V$$

is dense.[1] Hence $G \cap D_\alpha \neq \emptyset$. It follows that C is unbounded.

Next, suppose that $\alpha_0 < \alpha_1 < ... < \alpha_n < ...$ for all $n \in \omega$ are in C. Let $\alpha = \sup_n \alpha_n$. Choose $p \in G$ such that $\max(p) > \alpha$. Let $n \in \omega$, then there exists $p_n \in G$ such that $\alpha_n \in p_n$. Choose $q \in G$ such that $q \leq p, p_n, ...$, then $\alpha_n \in q$, and so $\alpha_n \in p$. Since $\alpha_n \in p$ for all $n \in \omega$, $\alpha \in p$. Thus p is also closed.

To see that $\mathbb{P}_s$ preserves ω_1, it is enough to show that if $p \in \mathbb{P}_s$, τ is a $\mathbb{P}_s$-name and $p \Vdash \tau : \check{\omega} \to \check{\omega}_1$. Then there exists $q \leq p$ and $\phi : \omega \to \omega_1$ in V such that $q \Vdash \tau = \check{\phi}$. Fix such a $p \in \mathbb{P}_s$ and τ. We will inductively define countable subsets $A_\alpha \subseteq \mathbb{P}_s$ and countable ordinals $h(\alpha) < \omega_1$ such that:

- $A_0 = \{p\}$;
- $h(\alpha) = \sup\{\max(q) \mid q \in A_\alpha\}$;
- $A_{\alpha+1} \supseteq A_\alpha$ is a countable subset of $\mathbb{P}_s$ such that for each $q \in A_\alpha$ and $n \in \omega$, there exists $\gamma < \omega_1$ and $r \in A_{\alpha+1}$ such that $r \leq q$ and $r \Vdash \tau(\check{n}) = \check{\gamma}$ and $\max(r) > h(\alpha)$;
- $A_\delta = \bigcup_{\alpha<\delta}$ if δ is a limit ordinal.

[1] If $\mathbb{P}$ is any poset, $G \subseteq \mathbb{P}$ is $V-$generic, $p \in G$ and E is dense below p, then $E \cap G \neq \emptyset$. This is obvious since $D = E \cup \{q \in \mathbb{P} \mid q \perp p\}$ is dense in $\mathbb{P}$.

Clearly $h : \omega_1 \to \omega_1$ is a strictly increasing continuous function. Hence, $C = \{h(\alpha) \mid \alpha < \omega_1\}$ is a club. It follows that the set $D \subseteq C$ of limit points is also a club. Thus there exists

$$\alpha_0 < \alpha_1 < ... < \alpha_n < ..., \quad n \in \omega$$

such that $\delta = \sup_n h(\alpha_n) \in S$. Hence we can inductively define $q_n \in A_{\alpha_n}$ such that:

- $q_0 = p$;
- $q_{n+1} \leq q_n$ and $q_{n+1} \Vdash \tau(\check{n}) = \check{\gamma}_n$ for some $\gamma_n < \omega_1$.

Note that $\sup(\bigcup_n q_n) = \delta \in S$. Thus $q = \bigcup_n q_n \cup \{s\} \in \mathbb{P}_s$ satisfies our requirements, i.e. $q \Vdash \tau = \check{\phi}$, where $\phi = \{\langle n, \gamma_n\rangle \mid n \in \omega\} \in V$.

Proof of **Theorem 4.7.** Suppose that $\mathrm{FA}_{\omega_1(\mathcal{C}_{\omega_1})}$ is true. By Ulam, there exists a stationary $S \subseteq \omega_1$ such that $\omega_1 - S$ is also stationary. Let $\mathbb{P}_s$ as above and for each $\alpha < \omega_1$, let $D_\alpha = \{p \in \mathbb{P}_s \mid \max(p) > \alpha\}$. By $\mathrm{FA}_{\omega_1}(\mathcal{C}_{\omega_1})$, there exists a filter $G \in V$ such that $G \cap D_\alpha \neq \emptyset$ for each $\alpha < \omega_1$. By the above (?), $C = \bigcup G$ is a club with $C \subseteq S$. But this contradicts the fact that $\omega_1 - S$ is also stationary.

Definition. $\mathbb{P}$ preserves stationary subset of ω_1 if for all stationary $S \subseteq \omega_1$, $\Vdash_{\mathbb{P}}$ "$\check{S}$ is stationary."

Definition. Martin's Maximum(MM) is the following statement:

> If $\mathbb{P}$ preserves stationary subsets of ω_1, and $\mathcal{D}$ is a collection of dense subsets of $\mathbb{P}$, with $|\mathcal{D}| \leq \omega_1$. Then there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

<u>Fact</u> MM is consistent relative to the existence of a supercompact cardinal. However, the notion "preserves stationary subsets of ω_1" is a little complicated. We will focus on a more concrete notion, that works for most problems.

Definition. Poset $\mathbb{P}$ satisfies Axiom A if there exist partial orderings $\{\leq_n \mid n \in \omega\}$ of $\mathbb{P}$ such that

(a) $p \leq_0 q$ iff $p \leq q$;

(b) if $p \leq_{n+1} q$, then $p \leq_n q$;

(c) if $\langle p_n \mid n \in \omega\rangle$ is such that $p_0 \geq_0 p_1 \geq_1 p_2 \geq_2 ... \geq_{n-1} p_n \geq_n p_{n+1} \geq_{n+1} ...$, then there exists $q \in \mathbb{P}$ such that $q \leq_n q_n$ for all $n \in \omega$.

(d) for all $p \in \mathbb{P}$ and $n \in \omega$, if $\mathbb{P}$ forces $\tau \in V$, then there exists a countable set $C \in V$ and $q \leq_n p$ such that $q \Vdash \tau \in \check{C}$.

<u>Remark</u> A typical instance of (d) is when $p \Vdash \tau$ is an ordinal.

Proposition 4.10. *If $\mathbb{P}$ is countably closed, then $\mathbb{P}$ satisfies Axiom A.*

Proof. For all $n \in \omega$, let $\leq_n$ be $\leq$. Then clearly (a), (b) and (c) holds. Next, suppose $p \in \mathbb{P}$, $n \in \omega$ and $o \Vdash \tau \in V$. Then there exists $q \leq p$ such that $q \Vdash \tau = \check{x}$ for some $x \in V$. Thus we have $q \leq_n p$ and $q \Vdash \tau \in \check{C}$, where $C = \{x\}$.

Proposition 4.11. *If $\mathbb{P}$ is c.c.c., then $\mathbb{P}$ satisfies Axiom A.*

Proof. For each $n \geq 1$, let $\geq_n$ be $=$. Clearly (a), (b) and (c) hold. So suppose $p \in \mathbb{P}$, $n \in \omega$ and $p \Vdash \tau \in V$. Let

$$C = \{x \in V \mid \exists q \leq p(q \Vdash \tau = \check{x})\}.$$

For each $x \in C$, choose $q_x \leq p$ such that $q_x \Vdash \tau = \check{x}$. Then $A = \{a_x \mid x \in C\}$ is an antichain. And so $|A| \leq \omega$. This implies that $|C| \leq \omega$and $p \Vdash \tau \in C$.

Definition. The Prikry-Silver poset $\mathbb{P}_{ps}$ consists of the partial function $p : \omega \to 2$ such that $|\omega - \mathrm{dom}(p)| = \omega$, ordered by $p \leq q$ iff $p \supseteq q$.

Definition. For each $n \in \omega$, let $\leq_n$ be the partial order on $\mathbb{P}_{ps}$ defined by: $p \leq_n q$ iff $p \supseteq q$ and the first n elements of $\omega - \mathrm{dom}(p)$ aren't in $\mathrm{dom}(q)$.

Theorem 4.12. *With the above definition, $\mathbb{P}_{ps}$ satisfies Axiom A.*

Lecture Notes #3 of Axiomatic Set Theory

Question If $S \subseteq \omega_1$ is stationary, does $\mathbb{P}_s$ collapse cardinals? (Under ZFC, $|\mathbb{P}_S| = \omega_1^\omega = 2^\omega$.)

Observation $\Vdash_{\mathbb{P}_s}$CH. Thus, $\mathbb{P}_s$ collapse the continuum. (In fact, $\Vdash_{\mathbb{P}_s} \diamond$.)

Proof. ($\mathbb{Q}: \ p : \alpha \to 2, \alpha < \omega_1$.)

Let $S = \{\alpha_i \mid i < \omega_1\}$ be the increasing enumeration. For each $X \subseteq \omega$, let D_X consist of the conditions(posets) p such that there exists a limit $\delta < \omega_1$ such that

1. $\max(p) \geq \alpha_{\delta+\omega}$;
2. $\alpha_{\delta+n} \in p$ is equivalent with $n \in X$.

Clearly D_X is dense. Let $G \subseteq \mathbb{P}_s$ be $V-$generic, and let $C = \bigcup G \in V[G]$. The proof of **Theorem 4.8** shows that $\mathcal{P}^{V[G]}(\omega) = \mathcal{P}^V(\omega)$. Also $G \cap D_X \neq \emptyset$, for each $X \subseteq P^V(\omega)$. So there exists a limit $\delta_X < \omega_1$ such that $\alpha_{\delta_{X+n}} \in C$ is equivalent with $n \in X$.

Also, if $X \neq Y \in \mathcal{P}^V(\omega)$, then clearly $\delta_X \neq \delta_Y$. Hence,

$$|\mathcal{P}^{V[G]}(\omega)| = |\mathcal{P}^V(\omega)| = \omega_1.$$

Theorem 4.12. *With the above definitions, $\mathbb{P}_{ps}$ satisfies Axiom A.*

Proof. Clearly $\mathbb{P}_{ps}$ satisfies (a), (b) and (c). Suppose that $p \in \mathbb{P}_{ps}$, $n \in \omega$ and $p \Vdash \tau \in V$. Clearly we can suppose that $n > 1$. Let I be the first n elements of $\omega - \mathrm{dom}(p)$. Let $2^I = \{f_1, ..., f_{2^n}\}$.

- First let $p_1 = p \cup f_1$. Then there exists $q \leq p_1$ and $x \in V$ such that $q \Vdash \tau = \check{x}_1$.
- Next, let $p_2 = (q_1 - f_1) \cup f_2$. Then there exists $q \leq p_1$ and $x_2 \in V$ such that $q_2 \Vdash \tau = \check{x}$.
- Continue in this fashion, for each $1 \leq i \leq 2^n$. We can define $q_i \in \mathbb{P}_{ps}$ and $x_1 \in V$ such that
$$p \subseteq p_1 - f_1 \subseteq p_2 - f_2 \subseteq ... \subseteq q_{2^n} - f^{2^n},$$
with $f_i \subseteq q_i$ and $q_i \Vdash \tau = \check{x}_i$.

Let $q = q_{2^n} - f_{2^n}$. Then clearly, $q \leq_n p$.

Suppose $r \leq q$ and $x \in V$ are such that $r \Vdash \tau = \check{x}$. After slightly extending r if necessary we can suppose that $I \subseteq \mathrm{dom}(r)$. Then there exists $1 \leq i \leq 2^n$ such that $q_i \subseteq r$. Since $r \leq q_i$, we have $r \Vdash \tau = \check{x}_i$. Thus $q \Vdash \tau \in C$, where $C = \{x_1, ..., x_{2^n}\}$.

Remark It is independent of MA+$\neg$CH whether $\mathbb{P}_{ps}$ preserves 2^ω. This question is equivalent to one about the Silver ideal.

Definition.
- For each $p \in \mathbb{P}_{ps}$, let $[p] = \{x \in 2^\omega \mid p \subseteq x\}$;
- The Silver ideal $\mathcal{I}_s$ consists of the subsets $X \subseteq 2^\omega$ such that for every $p \in \mathbb{P}_{ps}$, there exists $q \leq p$ such that $X \cap [q] = \emptyset$.

Remark Let $\mathcal{B}$ be the Mycielski ideal. Then a moments of thought shows that $\mathcal{B} \subseteq \mathcal{I}_s$.

Recall that $\mathcal{B}$, which is a $\sigma-$ideal, consists of the $X \subseteq \mathcal{P}(\omega)$ such that for all $A \in [\omega^\omega]$, $\mathcal{P}(A) \neq \{B \cap A \mid B \in X\}$.

Exercise Show that $\mathcal{I}_s$ is a $\sigma-$ideal.

Exercise(FAA) $\mathrm{cov}(\mathcal{I}_s) \geq \omega_1$.

Definition. If $\mathcal{I}$ is an ideal, then $\mathrm{add}(\mathcal{I})$ is the least λ such that there exists $\{X_\alpha \mid \alpha < \lambda\} \subseteq \mathcal{I}$ such that $\bigcup_{\alpha<\lambda} X_\alpha \notin \mathcal{I}$.

Remark Clearly, $\mathrm{add}(\mathcal{I}) \leq \mathrm{cov}(\mathcal{I})$.

Remark Less that a moment of thought shows that

$$\mathrm{add}(\mathcal{B}) = \mathrm{cov}(\mathcal{B}).$$

Theorem (MA+$\neg$CH). *The followings are equivalent:*

- $\mathbb{P}_{ps}$ *doesn't collapse* 2^ω*;*
- $\mathrm{add}(\mathcal{I}_s) = 2^\omega$.

Theorem (FAA). $\mathrm{add}(\mathcal{I}_s) = 2^\omega$.

Theorem I. *t is consistent with MA+$\neg$CH that* $\mathrm{add}(\mathcal{I}_s) = \omega_1$.

Forcing Axiom for Axiom A(FAA) If $\mathbb{P}$ satisfies Axiom A, and $\mathcal{D}$ is a collection of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| \leq \omega_1$, there exists a filter G such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

Remark By **Proposition 4.11**, FAA implies MA_{ω_1}. So the consequence of FAA includes: Every ω_1-Aronszajn tree is special; $\mathfrak{b} \geq \omega_2$; etc, etc, etc.

We will eventually show that: FAA implies that $2^\omega = \omega_2$.

Theorem . $\mathrm{cov}(\mathcal{B}) \geq \omega_2$*, which implies that* $\mathrm{cov}(\mathcal{B}) = \omega_2$

Proof. Suppose not, since $\mathcal{B}$ is a $\sigma-$ideal, $\mathrm{cov}(\mathcal{B}) \geq \omega_1$. Hence, $\mathrm{cov}(\mathcal{B}) = \omega_1$ and we will find $\{X_\alpha \mid \alpha < \omega_1\} \subseteq \mathcal{B}$ such that $\bigcup_{\alpha<\omega_1} X_\alpha = \mathcal{P}(\omega)$.

Let $\mathbb{P} = \mathbb{P}_{ps}$. Then $\mathbb{P}$ satisfies Axiom A. For each $S \in \mathcal{P}(\omega)$, let $\chi_S \in 2^\omega$be the characteristic function. For each $\alpha < \omega_1$, let D_α be the set of $p \in \mathbb{P}$ such that

(*) there exists $A \in [\mathrm{dom}(p)]^\omega$ such that $p \restriction A \neq \chi_S \restriction A$ for all $S \in X_\alpha$.

Claim. D_α is dense in $\mathbb{P}$. (Need Prove.)

Also, for each $n \in \omega$, let $E_n = \{p \in \mathbb{P} \mid n \in \mathrm{dom}(p)\}$. Then each E_n is also dense. Applying FAA, there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$ and $G \cap E_n \neq \emptyset$. Let $g = \bigcup G$. Then $g = \chi_B$ for some $B \in \mathcal{P}(\omega)$. Hence, there exists $\alpha < \omega_1$ such that $B \in X_\alpha$. Since $G \cap D_\alpha \neq \emptyset$, there exists $A \in [\omega]^\omega$ such that $B \cap A \in \mathcal{P}(A) - \{A \cap S \mid S \in X_\alpha\}$. Contradiction.

Lecture Notes #12 of Axiomatic Set Theory

- For each $p \in \mathbb{P}_{ps}$, $[p] = \{x \in 2^\omega \mid p \subseteq x\}$.
- The Silver ideal, $\mathcal{I}_s$ consists of the $X \subseteq 2^\omega$ such that for all $p \in \mathbb{P}_{ps}$, there are $q \leq p$ such that $X_n[p] = q$.

Example 1 Let $\mathbb{E} = \{2n \mid n \in \omega\}$ and $\mathbb{O} = \{2n+1 \mid n \in \omega\}$. Let X consist of $s \in \mathcal{P}(\mathbb{E})$. Then $X \in \mathcal{I}_s - \mathcal{B}$.

$$S \sqcup \{2n+1 \mid 2n \in S\}.$$[1]

Example 2 If $A \subseteq \mathbb{P}_{ps}$ is a maximal antichain, then $X = 2^\omega - \bigcup_{r \in A}[r]$ is in $\mathcal{I}_s$.

Proof. Let $p \in \mathbb{P}_{ps}$ be a library. Then there exists $r \in A$ such that p, r are compatible. Let $p, q \geq q$, Since $q \leq r$, we have $q \supseteq r$ and so $[q] \subseteq [r]$. Hence, $[q] \cap X = \emptyset$.

We next prove:

Theorem . *If the additivity* $\operatorname{add}(\mathcal{I}_s)2^\omega$ *and* 2^ω *is regular, then* $\mathbb{P}_{ps}$ *doesn't collapse* 2^ω.

Remark We will later show that FAA implies:

1. $\operatorname{add}(\mathcal{I}_s) \geq \omega_2$;
2. $2^\omega = \omega_2$.

Assuming these results, we obtain:

Corollary. (FAA) $\mathbb{P}_{ps}$ doesn't collapse 2^ω.

To prove the above theorem, we first prove a technical result:

Lemma . *Suppose* $D \subseteq \mathbb{P}_{ps}$ *is dense, then there exists a maximal antichain* A *of* $\mathbb{P}_{ps}$ *such that*

1. $A \subseteq D$;
2. *If* $p \in \mathbb{P}_{ps}$ *satisfies* $[p] \subseteq \bigcup_{q \in A}[q]$. *Then there exists* $A' \subseteq A$ *with* $|A'| < 2^\omega$ *such that* $p \in \bigcup_{q \in A'}[q]$.

Proof. Let $\mathbb{P}_{ps} = \{p_\alpha \mid \alpha < 2^\omega\}$. We will define, by induction, the antichain $A = \{q_\alpha \mid \alpha < 2^\omega\} \subseteq D$ and the elements $\{x_\alpha \mid \alpha < 2^\omega\} \subseteq \mathbb{P}_{ps}$.

- First, if $[p_\alpha] \not\subseteq \bigcup_{\beta<\alpha}[q_\beta]$, then choose $x_\alpha \in [p_\alpha] - \bigcup_{\beta<\alpha}[q_\beta]$. Otherwise, let $x_\alpha = x_0$.
- If p_α compatible with q_β, for some $\beta < \alpha$, let $q_\alpha = q_0$. Otherwise, let $\in [\omega - \operatorname{dom}(p_\alpha)]$ such that $|\omega - (\operatorname{dom}(p_\alpha) \cup s)| = \omega$. For each $f : s \to 2$, let $q_f = q_\alpha \cup f$. Then the set $\{[q_f] \mid f \in 2^s\}$ are pairwise disjoint. Hence, there exists $f \in 2^s$ such that $[q_f] \cap \{x_\beta \mid \beta \leq \alpha\} = \emptyset$.

[1] I don't really know what these lines mean.

We now choose $q_\alpha \in D$, with $q_\alpha \leq q_f$. This completes the construction of the maximum antichain $A \subseteq D$.

Finally, suppose that $p \in \mathbb{P}_{ps}$ satisfies $[p] \subseteq \bigcup_{q\in A}[q]$. Let $p = p_\alpha$. Then $[p] \subseteq \bigcup_{\beta<\alpha}[q_\beta]$. Since otherwise, $x_\alpha \in [p] - \bigcup_{q\in A}[q]$, contradiction.

Exercise 1. Does there exists $C \subseteq \mathbb{P}_{ps}$ such that $|C| < 2^\omega$, such that

$$\mathbb{P}_{ps} = \bigcup_{q\in C}[q].$$

2. Does there exists $C \subseteq \mathbb{P}_{ps}^\infty = \{p \in \mathbb{P}_{ps} \mid |\operatorname{dom}(p)| = \omega\}$ such that $|C| < 2^\omega$ and

$$\mathbb{P}_{ps} = \bigcup_{p\in C}[p].$$

Hint: Suitable algebra filter.

Proof of the **Theorem**. If not, then there exists a $\mathbb{P}_{ps}-$name τ and $\kappa < 2^\omega$ such that

$$\Vdash \tau_G : \check{\kappa} \to \check{2^\omega} \text{ is surjective.} \tag{1}$$

For each $\alpha < \kappa$, let $D_\alpha = \{p \in \mathbb{P}_{ps} \mid p \Vdash \tau(\check{\alpha}) = \check{\beta},$ for some $\beta < 2^\omega\}$. Then D_α is dense in $\mathbb{P}_{ps}$. Let $A_\alpha \subseteq D_\alpha$ be the antichain given by the Lemma, and let $x_\alpha = 2^\omega - \bigcup_{q\in A_\alpha}[q]$. Since A_α is a maximal antichain, $x_\alpha \in \mathcal{I}_s$. Also, since $add(\mathcal{I}_s) = 2^\omega > \kappa$, it follows that $\bigcup_{\alpha<\kappa} x_\alpha \in \mathcal{I}_s$. Hence, there exists a condition such that $[p] \cap \bigcup_{\alpha<\kappa}(2^\omega - \bigcup_{q\in A_\alpha}[q]) = \emptyset$.

In other words, for each $\alpha < \kappa$, $[p] \subseteq \bigcup_{q\in A_\alpha}[q]$. Hence, for each $\alpha > \kappa$, there exists $A'_\alpha \subseteq A_\alpha$, with $|A'_\alpha| < 2^\omega$ such that $[p] \subseteq \bigcup_{q\in A'_\alpha}[q]$. For each $\alpha < \kappa$, let

$$C_\alpha = \{\beta < 2^\omega \mid \exists q \in A'_\alpha(q \Vdash \tau(\check{\alpha}) = \check{\beta})\}.$$

Then $|C_\alpha| < 2^\omega$. Suppose $r \leq p$ and $r \Vdash \tau(\check{\alpha}) = \check{\beta}$. Since $[r \subseteq \bigcup_{q\in A'_\alpha}[q]$, and $|A'_\alpha| < 2^\omega$. There exists $q \in A'_\alpha$ such that $|[r] \cap [q]| = 2^\omega$. It follows that $r \cup q$ is a partial function and $|\omega - \operatorname{dom}(r \cup q)| = \omega$. Thus, $r \cup q \leq r, q$. It follows that $\beta \in C_\alpha$. Let $C = \bigcup\{C_\alpha \mid \alpha < \kappa\}$,then $|C| < 2^\omega$ and $p \Vdash \operatorname{ran}(\tau) \subseteq \check{C}$, which contradicts $\Vdash \operatorname{ran}(\tau) = 2^\omega$.

We now work towards proving:

Theorem (FAA).

$$\operatorname{add}(\mathcal{I}_s) \geq \omega_2.$$

We first prove:

Lemma . *If $X \in \mathcal{I}_s$, $p \in \mathbb{P}_{ps}$ and $n \in \omega$, there exists $q \leq_n p$ such that $[q] \cap X = \emptyset$.*

Proof. Clearly we can see that $n \geq 1$. Let I be the first n elements of $\omega - \operatorname{dom}(p)$ and let $\{f_1, ..., f_{2^n}\} = 2^I$. Then we can inductively define $p \subseteq q_1 - f_1 \subseteq ... \subseteq q_{2^n} - f_{2^n}$ such that

- $f_i \subseteq q_i$;

- $X \cap [q_i] = \emptyset$.

Let $q = q_{2^n} - f_{2^n}$. Then $q \leq_n p$. Suppose $x \in [q]$. Then there exists $1 \leq i \leq 2^n$ such that $f_i \subseteq x$, and so $x \in [q_i]$. Thus $x \notin X$. Thus $X \cup [q] = \emptyset$ as required.

<u>Remark</u> It follows easily that $\mathrm{add}(\mathcal{I}_s) > \omega$.

<u>Notation</u> If $p \in \mathbb{P}_{ps}$ and $n \in \omega_1$, then $p \restriction n$ is the partial function obtained by restricting p to $n \cap \mathrm{dom}(p)$.

Definition. The Silver Amoeba $a(\mathbb{P}_{ps})$ consists of the elements:

$$\langle p \restriction n, p \rangle, \quad p \in \mathbb{P}_{ps},\ n \in \omega,$$

partially ordered by: $\langle q \restriction m, q \rangle \leq \langle p \restriction n, p \rangle$ iff:

- $q \leq p$;
- $m \geq n$ and $q \restriction n = p \restriction n$.

<u>Remark</u> We want to adjoin a "generic" condition $p_{gen} \in \mathbb{P}_{ps}$. The condition $\langle p \restriction n, p \rangle$ tell us that:

- $p \subseteq p_{gen}$;
- $p_{gen} \restriction n = p \restriction n$.

Definition. If $G \subseteq a(\mathbb{P}_{ps})$ is a filter, then

$$p_G = \bigcup \{p \mid \exists n (\langle p \restriction n, p \rangle) \in G\}.$$

Definition. For each $l \in \omega$, let:

$$D_l = \{\langle p \restriction n, p \rangle \mid |n - \mathrm{dom}(p)| \geq l\}.$$

Lemma . *For each $l \in \omega$, D_l is dense in $a(\mathbb{P}_{ps})$.*

Proof. Let $\langle p \restriction n, p \rangle \in a(\mathbb{P}_{ps})$. Since $|\omega - \mathrm{dom}(p)| = \omega$, there exists $m \geq n$ such that $|m - \mathrm{dom}(p)| \geq l$. Then:

$$\langle p \restriction m, p \rangle \leq \langle p \restriction n, p \rangle, \quad \text{and} \quad \langle p \restriction m, p \rangle \in D_l.$$

Lecture Notes #13 of Axiomatic Set Theory

$a(\mathbb{P}_{ps})$ consists of conditions:

$$\langle p \upharpoonright n, p\rangle; \quad p \in \mathbb{P}_{ps},\ n \in \omega,$$

partially ordered by $\langle q \upharpoonright m, q\rangle \leq \langle p \upharpoonright n, p\rangle$ iff: (1) $q \leq p$ and, (2) $m \geq n$ and $q \upharpoonright n = p \upharpoonright n$.

If $G \subseteq a(\mathbb{P}_{ps})$ is a filter, then

$$p_G = \bigcup\{p \mid \exists n(\langle p \upharpoonright n, p\rangle \in G)\}.$$

Lemma . *For each $l \in \omega$, $D_l = \{\langle p \upharpoonright n, p\rangle \mid |n - \mathrm{dom}(p)| \geq l\}$ is dense in $a(\mathbb{P}_{ps})$.*

Lemma . *If $G \subseteq a(\mathbb{P}_{ps})$ is a filter such that $G \cap D_l \neq \emptyset$ for all $l \in \omega$, then $p_G \in \mathbb{P}_{ps}$.*

Proof. Obviously, p_G is the partial function $\omega \to 2$, so it only remains to check that $|\omega - \mathrm{dom}(p_G)| = \omega$. Let $l \in \omega$ be arbitrary, then there exists $\langle p \upharpoonright m, p\rangle \in G \cap D_l$. If $\langle q \upharpoonright m, q\rangle \leq \langle p \upharpoonright n, p\rangle$, then $q \upharpoonright n = p \upharpoonright n$, and it follows that,

$$|\omega - \mathrm{dom}(p_G)| \geq |n - \mathrm{dom}(p_G)| = |n - \mathrm{dom}(p)| \geq l.$$

Definition. For each $X \in \mathcal{I}_s$, let $\mathbb{E}_X = \{\langle p \upharpoonright n, p\rangle \mid X \cap [p] = \emptyset\}$.

Lemma . *If $X \in \mathcal{I}_s$, then $\mathbb{E}_X$ is dense.*

Proof. Let $\langle p \upharpoonright n, p\rangle \in a(\mathbb{P}_{ps})$, and let $l = |n - \mathrm{dom}(p)|$. Then there exists $q \leq_l p$ such that $X \cap [q] = \emptyset$. Note that $\langle q \upharpoonright n, q\rangle \leq \langle p \upharpoonright n, p\rangle$ and $\langle q \upharpoonright n, q\rangle \in \mathbb{E}_X$.

Lemma . *Suppose $G \subseteq a(\mathbb{P}_{ps})$ is a filter such that $G \cap D_l \neq \emptyset$ for each $l \in \omega$. If $X \in \mathcal{I}_s$ and $G \cap \mathbb{E}_X \neq \emptyset$, then $p_G \in \mathbb{P}_{ps}$ and $X \cap [p_G] = \emptyset$.*

Proof. We have already seen that $p_G \in \mathbb{P}_{ps}$. Suppose $\langle p \upharpoonright n, p\rangle \in \mathbb{E}_X$. Then $X \cap [p] = \emptyset$, and since $p \subseteq p_G$, it follows that $[p_G] \subseteq [p]$ and so $X \cap p_G = \emptyset$.

Theorem . *$a(\mathbb{P}_{ps})$ satisfies Axiom A.*

Proof. Delayed.

Corollary. For every $p \in \mathbb{P}_{ps}$, $a(\mathbb{P}_{ps}) = \{\langle q \upharpoonright n, q\rangle \mid \langle p \upharpoonright n, p\rangle \leq \langle \emptyset, p\rangle\}$ also satisfies Axiom A.

Theorem (FAA). $\mathrm{add}(\mathcal{I}_s) \geq \omega_2$.

Proof. Suppose that $\{X_\alpha \mid \alpha < \omega_1\}$ is contained in $\mathcal{I}_s$. Let $\mathbb{P}_{ps} \ni p$ be arbitrary. Applying FAA to $a(\mathbb{P}_{ps})_p$, there exists a filter $G \subseteq a(\mathbb{P}_{ps})_p$ such that $G \cap D_l \neq \emptyset$ and $G \cap \mathbb{E}_{X_\alpha} \neq \emptyset$ for all $l \in \omega$ and $\alpha < \omega_1$. Thus $p_G \in \mathbb{P}_{ps}$ satisfies $p_G \leq p$ and $[p_G] \cap X_\alpha = \emptyset$ for all $\alpha < \omega_1$. Thus $\bigcup_{\alpha > \omega_1} X_\alpha \in \mathcal{I}_s$.

Definition. For each $n \geq 1$,

$$\langle q \upharpoonright l, q\rangle \leq_n \langle p \upharpoonright n, p\rangle \text{ iff:}$$

- $q \leq_n p$,

- $q \restriction l = p \restriction l$.

With these definitions, it is clear that $a(\mathbb{P}_{ps})$ satisfies (1), (2) and (3). We now sketch the proof that (4) holds.
Proof. Suppose that $\langle p \restriction m, p\rangle \in a(\mathbb{P}_{ps})$ with $m \in \omega$ and $\langle p \restriction m, p\rangle \Vdash \tau \in V$.

Let $a = |m - \mathrm{dom}(p)|$. Then we can suppose that $n \geq a$. Now a tedious fusion argument gives $\langle p \restriction m, q\rangle \leq_n \langle p \restriction m, q\rangle$ such that if $\langle r \restriction l, r\rangle \leq \langle p \restriction m, p\rangle$ and $\langle r \restriction l, r\rangle \Vdash \tau = \check{x}$. Then $\langle r \restriction l, q \cup (r \restriction l)\rangle \Vdash \tau = \check{x}$.

Notice that $\langle r \restriction l, q \cup (r \restriction l)\rangle \leq \langle p \restriction m, q\rangle$. Let C consist of $x \in V$ such that there exists a finite extension $q^+ \supseteq q$ with $q^+ \restriction m = q \restriction m = p \restriction m$ and $l \geq m$ such that $\langle q^+ \restriction l, q^+\rangle \Vdash \tau = \check{x}$. Then $C \in V$ is countable and $\langle p \restriction m, q\rangle \Vdash \tau \in C$.

Most interesting applications of FAA make use of $2-$step iterations, which (something) as follows. Suppose $\mathbb{P} \in V$ and $G \subseteq \mathbb{P}$ is a $V-$generic filter. Then we cam form the generic extension $V[G]$. Now suppose $\mathbb{Q} \in V[G]$ and $H \subseteq \mathbb{Q}$ is a $V[G]-$generic filter, then we can form the generic extension:

$$V[G][H] = (V[G])[H].$$

Note that there exists a $\mathbb{P}-$name $\tilde{\mathbb{Q}}$ such that $\mathbb{Q} = \tilde{\mathbb{Q}}_G$. This suggests that there exists a poset $\mathbb{P} * \tilde{\mathbb{Q}} \in V$ such that $V[G][H] = V[K]$ for a suitable $V-$generic filter $K \subseteq \mathbb{P} * \tilde{\mathbb{Q}}$.

Definition. Suppose that $\mathbb{P}$ is a poset and $\Vdash_{\mathbb{P}}$ "$\tilde{\mathbb{Q}}$ is a poset". Then $\mathbb{P} * \tilde{\mathbb{Q}} = \{\langle p, \tilde{q}\rangle \mid p \in \mathbb{P}, \Vdash_{\mathbb{P}} \tilde{q} \in \tilde{\mathbb{Q}}\}$, partially ordered by:

$$\langle p_1, \tilde{q}_1\rangle \leq \langle p_2, \tilde{q}_2\rangle \text{ iff:}$$

- $p_1 \leq p_2$;
- $p_1 \Vdash \tilde{q}_1 \leq \tilde{q}_2$.

<u>Remark</u> We identify $\langle p_1, \tilde{q}_1\rangle$ and $\langle p_2, \tilde{q}_2\rangle$ iff:

$$\langle p_1, \tilde{q}_1\rangle \leq \langle p_2, \tilde{q}_2\rangle \leq \langle p_1, \tilde{q}_1\rangle.$$

<u>Remark</u> In most application, $\tilde{\mathbb{Q}} = \langle \check{\mathbb{Q}}, \tilde{\leq}\rangle$.

Theorem . *Let $G \subseteq \mathbb{P}$ be $V-$generic, let $\mathbb{Q} = \check{\mathbb{Q}}_G$ and let $H \subseteq \mathbb{Q}$ be $V[G]-$generic. Then $G * H = \{\langle p, \tilde{q}\rangle \in \mathbb{P} * \tilde{\mathbb{Q}} \mid p \in G, \tilde{q}_G \in H\}$ is a $V-$generic filter such that $V[G * H] = V[G][H]$.*

Conversely, suppose that $K \subseteq \mathbb{P} * \tilde{\mathbb{Q}}$ is $V-$generic. Then $G = \{p \in \mathbb{P} \mid \exists \tilde{q}(\langle p, \tilde{q}\rangle) \in K\}$ is $V-$generic, and $H = \{\tilde{q}_G \mid \exists p \in \mathbb{P}(\langle p, \tilde{q}\rangle \in K)\}$ is $V[G]-$generic, and $K = G * H$.
Proof. See Kunen.

Theorem I. *f $\mathbb{P}$ satisfies Axiom A, adn $\Vdash_{\mathbb{P}}$ "$\tilde{Q}$ satisfies Axiom A", then $\mathbb{P} * \tilde{\mathbb{Q}}$ satisfies Axiom A.*

Proof. For each $n \in \omega$, define:

$$\langle p_1, \tilde{q}_1\rangle \leq_n \langle p_2, \tilde{q}_2\rangle \text{ iff:}$$

- $p_1 \leq_n p_2$;
- $p_1 \Vdash \tilde{q}_1 \leq_n \tilde{q}_2$.

Clearly properties (1) and (2) holds. To see that (3) holds, suppose $\langle p_n \tilde{q}_n \rangle \in \mathbb{P} * \tilde{\mathbb{Q}}$ satisfying:

$$\langle p_{n+1}, \tilde{q}_{n+1} \rangle \leq_n \langle p_n, \tilde{q}_n \rangle$$

for all $n \in \omega$. Then $p_{n+1} \leq_n p_n$ for all $n \in \omega$ and so there exists $p \in \mathbb{P}$ such that $p \leq_n p_n$ for all $n \in \omega$. It follows that $p \Vdash q_{n+1} \leq_n q_n$ for all $n \in \omega$. Hence, there exists $\tilde{q} \in \tilde{\mathbb{Q}}$ such that:

$$p \Vdash q \leq_n q_n, \quad \text{for all } n \in \omega.$$

Then $\langle p, \tilde{q} \rangle \leq_n \langle p_n \tilde{q}_n \rangle$ for all $n \in \omega$, as required.
Finally, to see that (4) holds, suppose $\langle p, \tilde{q} \rangle \in \mathbb{P} * \tilde{\mathbb{Q}}$, $n \in \omega$ and $\langle p, \tilde{q} \rangle \Vdash \tau \in V$. Then $p \Vdash$ "$\tilde{q} \Vdash_{\tilde{\mathbb{Q}}} \tau \in V$". Hence, there exists $\tilde{s} \in \tilde{\mathbb{Q}}$ and $\mathbb{P}-$names,

$$\tilde{V} = \{\tilde{a}_i \mid i\omega\} \text{ such that } p \Vdash \text{"}\tilde{s} \leq_n \tilde{q}\text{"} \wedge \text{"}\tilde{s} \Vdash_{\tilde{\mathbb{Q}}} \tau \in \tilde{C}\text{"} \wedge \text{"}\tilde{s} \Vdash_{\tilde{\mathbb{Q}}} a_i \in V \text{ for all } i \in \omega\text{"}.$$

Now inductively define $p_i \in \mathbb{P}$ and countable sets $C_i \in V$ such that

- $p_i = p$ for $0 \leq i \leq n$;
- $p_{n+l+1} \leq_{n+l} p_{n+l}$;
- $p_{n+l+1} \Vdash$ "$\tilde{s} \Vdash_{\tilde{\mathbb{Q}}} \tilde{a}_l \in \check{C}_l$".

Then there exists $r \in \mathbb{P}$ such that $r \leq_i p_i$ for all $i \in \omega$ and so $r \leq_n p$. Thus $\langle r, \tilde{s} \rangle \leq_n \langle r, \tilde{s} \rangle \Vdash \tau \in \bigcup_{i \in \omega} C_i$.

Lecture Notes #13 of Axiomatic Set Theory

Chapter 5: Some Applications of FAA

Our first applications involve trees.

Theorem 5.1(FAA). *There are no ω_1-Kurepa trees.*

Theorem 5.2(FAA). *There are no ω_2-Aronszajn trees.*

Remark The statement of 5.1 is equiconsistent with an inaccessible cardinal. The statement of 5.2 is equiconsistent with a weakly compact cardinal.

In both cases, we use a 2−step iteration in which we first collapse a suitable $\kappa > \omega_1$ via a countably closed forcing. The following lemma plays a key role in both proofs.

Lemma 5.3. *Suppose $\mathbb{P}$ is countably closed, and $\lambda \in \{\omega_1, \omega_2\}$. Let $T \in V$ be a $\lambda-$tree and let $G \subseteq \mathbb{P}$ be $V-$generic.*

1. *If $\lambda = \omega_1$ and $B \in V[G]$ is a branch of T which intersects with every level, then $B \in V$.*
2. *($\neg CH$)if $\lambda = \omega_2$ and $B \in V[G]$ is a branch of T which intersects with every level, then $B \in V$.*

Proof. Suppose not, let $\mathcal{B} = \mathcal{P}(T) \cap V$, and let τ be a $\mathbb{P}-$name such that $\tau_G = B$. Then there exists $p \in \mathbb{P}$ such that

$$p \Vdash \tau \text{ is a branch of } \check{T} \text{ which intersects every level and such that } \tau \notin \check{\mathcal{B}}. \tag{1}$$

From now on, we are working in V.

Claim 1. For all $q \leq p, \alpha < \lambda$, there exists $r \leq q$ and $t \in T_\alpha$ such that $t \Vdash \check{t} \in \tau$.

Proof of Claim 1. An immediate consequence of (1).

Claim 2. For all $q \leq p$, there exist $r_0, r_1 \leq q$ and $\alpha < \lambda$ and $a_0 \neq a_1 \in T_\alpha$ such that $r_0 \Vdash \check{a}_0 \in \tau$ and $r_1 \Vdash \check{a}_1 \in \tau$.

Proof of Claim 2. Suppose q is a counterexample, then

$$C = \{t \in T \mid \text{there exists } r \leq q \text{ such that } r \Vdash \check{t} \in \tau\}$$

is a branch which intersects every level of T, and $q \Vdash \tau = \check{C} \in \check{\mathcal{B}}$, which is a contradiction with the fact: $p \Vdash \tau \notin \check{\mathcal{B}}$.

Using Claim 1 and 2, we can inductively define $p_s \in \mathbb{P}$ and $a_s \in T$ for $s \in 2^n$ together with $\alpha_n < \lambda$ such that:

1. $p_\emptyset = p$;
2. $\alpha_0 < \alpha_1 < ... < \alpha_n < ...$;
3. If $s \in 2^n$, then $p_{s\frown 0}, p_{s\frown 1} \leq p_s$, and $a_{s\frown 0} \neq a_{s\frown 1} \in T_{\alpha_n}$; and if $i = 0, 1$, $p_{s\frown i} \Vdash \check{a_{s\frown i}} \in \tau$.

For each $f \in w^\omega$, let p_f satisfy $p_f \leq p_{f\restriction n}$ for all $n < \omega$. Then for all $n \in \omega$, $p_f \Vdash \check{a_{f\restriction n}} \in \tau$. Let $\gamma = \sup_n \alpha_n < \lambda$. Then there exists $r_f \leq p_f$ and $t_f \in T_\gamma$ such that $r_f \Vdash \check{t}_f \in \tau$. Suppose $f \neq g \in 2^\omega$. Let n be minimal such that $f \restriction n \neq g \restriction n$. Then $t_f \Vdash \check{a_{f\restriction n}} \in \tau$ and $r_g \Vdash \check{a_{g\restriction n}} \in \tau$. Since $a_{f\restriction n} \neq a_{g\restriction n} \in T_{\alpha_{n-1}}$. Hence $t_f \neq t_g$. Thus $|T_\gamma| \geq 2^\omega$. In both cases, (1) and (2), this contradicts the fact that T is a $\lambda-$tree.

Proof of **Theorem 5.1.** Suppose that T is an ω_1-Kurepa tree. i.e., T is an ω_1-tree with $\kappa \geq \omega_2$ uncountable branches. Let $\mathcal{B}$ be the set of uncountable branches of T. Let $\mathbb{P}$ be the poset of functions,

$$p : \alpha \to \kappa, \quad \alpha < \omega_1,$$

partially ordered by inverse inclusion. Let $G \subseteq \mathbb{P}$ be $V-$generic, then work inside $V[G]$, we see that $|\mathcal{B}| = |\kappa| = \omega_1$. Also, by **Lemma 5.3(1)**, $\mathcal{B}$ is the set of uncountable branches of T in $V[G]$. By the proofs of the **Theorem 3.28** and **3.19**, since T is a ω_1-tree, with only ω_1 uncountable branches, there exists a c.c.c. $\mathbb{Q} \in V[G]$ such that if $H \subseteq \mathbb{Q}$ is $V[G]-$generic, then

$$V[G][H] \Vdash T \text{ is weakly special.}$$

Let $\tilde{\mathbb{Q}}$ be a $\mathbb{P}-$name such that $\tilde{\mathbb{Q}}_G = \mathbb{Q}$. Then by **Theorem 4.15**, $\mathbb{P} * \tilde{\mathbb{Q}}$ satisfy Axiom A. Let τ be a $\mathbb{P} * \tilde{\mathbb{Q}}-$name such that

$$\Vdash_{\mathbb{P}*\tilde{\mathbb{Q}}} \tau : \check{T} \to \check{\omega} \text{ is a weakly specializing function.}$$

From now on, we work in V. For each $t \in T$, let

$$D_t = \{p \in \mathbb{P} * \tilde{\mathbb{Q}} \mid \text{ there exists } n \in \omega \text{ such that} p \Vdash \tau(\check{t}) = \check{n}\}.$$

Then each D_t is dense in $\mathbb{P} * \tilde{\mathbb{Q}}$. Applying FAA, there exists a filter $G \subseteq \mathbb{P} * \tilde{\mathbb{Q}}$ such that $G \cap D_t \neq \emptyset$ for all $t \in T$. Let

$$g = \{\langle t, b\rangle \in T \times \omega \mid \exists p \in G(p \Vdash \tau(\check{t}) = \check{n})\}.$$

Then $g : T \to \omega$ is a weakly specializing function. But then by **Proposition 3.27**, There are at most ω_1 uncountable functions, which is a contradiction.

Proof of **Theorem 5.2.** Suppose there is a ω_2-Aronszajn tree T. Let $\mathbb{P}$ be the poset of functions:

$$p : \alpha \to \omega_2 \quad \alpha < \omega_1,$$

partially ordered by inverse inclusion. Let $G \subseteq \mathbb{P}$ be $V-$generic. Until further notice, we work in $V[G]$. Since FAA$\vdash$CH, it follows that T has no branches in $V[G]$ which intersect every level. Also, we have $h_\tau(T) = \check{\omega}_2$ is an ordinal of size ω_1. Since $\mathbb{P}$ adds no new countable sets of ordinals, it follows that $\mathrm{cf}^{V[G]}(\omega_2) = \omega_1$. Let $h : \omega_1 \to \check{\omega}_2$ be a strictly increasing cofinal map, and let $S = \bigcup_{\alpha<\omega_1} T_{h(\alpha)}$. Then using the restriction tree order on T, S is a tree of size ω_1 with no uncountable branches. Hence, there exists a c.c.c. $\mathbb{Q} \in V[G]$, which adjoins a function $f : s \to \omega$ such that if $s, t \in S$ satisfies $f(s) = f(t)$, then s, t are incomparable in S and hence in T. Let $\tilde{\mathbb{Q}}$ be a $\mathbb{P}-$name such that $\tilde{\mathbb{Q}}_G = \mathbb{Q}$. Then $\mathbb{P} * \tilde{\mathbb{Q}}$ satisfies Axiom A. Also, for each $\gamma < \omega_2$, let $\phi_\gamma : \omega_1 \to T_\gamma$ be a surjective function. For each $\alpha, i < \omega_1$, let D_{α_i} be the set of conditions $r \in \mathbb{P} * \tilde{\mathbb{Q}}$ such that for some $\gamma < \omega_2$ and $n \in \omega$, $r \Vdash \tilde{h}(\check{\alpha}) = \check{\gamma} \wedge \hat{f}(\check{\phi_\gamma(i)}) = \check{n}$.

Clearly, each D_{α_i} is dense in $\mathbb{P} * \tilde{\mathbb{Q}}$. Hence, by FAA, there exists a filter $G \subseteq \mathbb{P} * \tilde{\mathbb{Q}}$ such that $G \cap D_{\alpha_i} \neq \emptyset$ for each $\alpha, i < \omega_1$. Define: $H : \omega_1 \to \omega_2$ by $H(\alpha) = \gamma$ iff $\exists r \in G(r \Vdash \tilde{h}(\check{\alpha}) = \check{\gamma})$. Then H is a strictly increasing function(which is not a cofinal function). Next, Let $U = \bigcup_{\alpha<\omega_1} T_{H(\alpha)}$. Then we can define $F : U \to \omega$ by $F(u) = n$ iff $\exists r \in G(r \Vdash \tilde{f}(\check{u}) = \check{n})$. Note that if $u, u' \in U$, and $F(u) = F(u')$, then u, u' are incompatible in T. Let $\delta = \sup(\operatorname{ran} H) < \omega_2$. Choose some $t \in T_\delta$, and for each $\alpha < \omega_1$, let $b_\alpha = T_{H(\alpha)}$ satisfy $b_\alpha < t$. Then there exists $\alpha \neq \beta < omega_1$ such that $F(b_\alpha) = F(b_\beta)$. But since $b_\alpha, b_\beta < t$, b_α, b_β are comparable in T, contradiction.

Lecture Notes #15 of Axiomatic Set Theory

We next begin working towards:

Theorem 5.4(FAA). $\mathfrak{b} = \omega_2$

Obviously, FAA$\vdash$MA$(\omega_1)\vdash$ $\mathfrak{b} \geq \omega_2$. First, we need to study the gaps in ω^ω.

Definition. If κ, λ are infinite cardinals, then a (κ, λ) gap is a pair of sequences $\langle f_\alpha \mid \alpha < \kappa\rangle, \langle g_\beta \mid \beta < \lambda\rangle \subseteq \omega^\omega$ such that:

1. If $\alpha < \beta < \kappa$, then $f_\alpha <^*$ [1] f_β;
2. If $\alpha < \beta < \lambda$, then $g_\beta <^* g_\alpha$;
3. If $\alpha < \kappa$ and $\beta < \lambda$, then $f_\alpha <^* g_\beta$.

The gap is filled iff there exists $h \in \omega^\omega$ such that for all $\alpha < \kappa$ and $\beta < \lambda$,

$$f_\alpha <^* h <^* g_\beta.$$

<u>Easy Exercise</u> Every (ω, ω) gap is filled.

<u>Question</u> When does there exist a c.c.c. forcing which fills the given (κ, λ) gap?

Theorem 5.5, Hausdorff. *There exists an unfilled (ω_1, ω_1) gap which remains unfilled in any generic extensions in which ω_1 is preserved.*

<u>Sketch of the basic idea.</u> An involved transfinite induction gives an (ω_1, ω_1) gap satisfies

(*) For all $\alpha < \omega_1, n < \omega$, $|\{\xi < \alpha | f_\alpha(t) < g_\xi(t) \text{ for all } t \geq n\}| < \omega$.

Suppose that $h \in \omega^\omega$ fills this gap. Then there exists $n_0 \in \omega$ and $T \in [\omega_1]^{\omega_1}$ such that for all $\alpha \in I$ and $t \geq n_0$, $f_\alpha(t) < h(t)$. Also, there exists $m_0 \in \omega$ and $J \in [I]^{\omega_1}$ such that for all $\beta \in J$, $t \geq m_0$, $h(t) < g_\beta(t)$. Let $n = \max\{n_0, m_0\}$. Then for all $\xi, \alpha \in J$ and $t \geq n$, $f_\alpha(t) < h(t) < g_\xi(t)$. Let α be the $\omega-$th element of J, we contradicts (*).

Definition. Let $\overline{f} = \langle f_\alpha \mid \alpha < \kappa\rangle$, $\overline{g} = \langle g_\beta \mid \beta < \lambda\rangle$ be a (κ, λ) gap. Then $\mathbb{P}(\overline{f}, \overline{g})$ consists of the conditions: $p = \langle \phi, x, y\rangle$ where:

- $\phi : n \to \omega$ for some $n < \omega$;
- $x \in [\kappa]^{<\omega}, y \in [\lambda]^{<\omega}$;
- For all $t \geq n, \alpha \in x$ and $\beta \in y$, $1 + f_\alpha(t) < g_\beta(t)$,

partially ordered by: $\langle \phi, x, y\rangle \leq \langle \psi, u, v\rangle$ iff:

- $\phi \supseteq \psi, x \supseteq u$ and $y \supseteq v$;
- for all $t \in \mathrm{dom}(\phi) - \mathrm{dom}(\psi)\alpha \in u, \beta \in v$, $f_\alpha(t) < \phi(t) < g_\beta(t)$.

[1] In order to forbid possible misunderstanding, $f <^* g$ means "g surpasses f eventually."

<u>Question</u> When is $\mathbb{P}(\overline{f},\overline{g})$ c.c.c.?

Clearly not always, because of the Hausdorff gap.

Lemma 5.6. *For each $n \in \omega$, $D_n = \{\langle \phi, x, y\rangle \mid n \in \mathrm{dom}(\phi)\}$ is dense.*

Proof. Suppose that $\langle \phi, x, y\rangle \in \mathbb{P}(\overline{f},\overline{g})$ and $n \notin \mathrm{dom}(\phi) = m$. Define:

$$f_x(l) = \max\{f_\alpha(l) \mid \alpha \in x\};$$

$$g_y(l) = \min\{g_\beta(l) \mid \beta \in y\};$$

Then for all $t \geq m$,

$$f_\alpha(t) \leq f_x(t) < f_x(t) + 1 < g_y(t) \leq g_\beta(t).$$

Define: $\psi : n+1 \to \omega$ by:

$$\psi(t) = \begin{cases} \phi(t), & \text{if } t < m; \\ f_x(t) + 1, & \text{if } m \leq t \leq n. \end{cases}$$

Then $\langle \psi, x, y\rangle \leq \langle \phi, x, y\rangle$ for each $\alpha < \kappa$.

Lemma 5.7. *For each $\alpha < \kappa$, $E_\alpha = \{\langle \phi, x, y\rangle \mid \alpha \in x\}$ is dense.*

Proof. Suppose $\langle \phi, x, y\rangle \in \mathbb{P}(\overline{f},\overline{g})$ with $\alpha \notin x$. Let $\max\{x \cup \{\alpha\} < \gamma < \kappa$. Then there exists $n \in \omega$ such that for all $\beta \in x \cup \{\alpha\}$ and $\xi \in y$ for all $t \geq n$,

$$f_\beta(t) < f_\gamma(t) < g_\xi(t).$$

Using **Lemma 5.6**, extend ϕ to ψ such that $n \subseteq \mathrm{dom}(\psi)$ and

$$\langle \psi, x, y\rangle \leq \langle \phi, x, y\rangle,$$

then,

$$\langle \psi, x \cup \{\alpha\}, y\rangle \leq \langle \psi, x, y\rangle.$$

Lemma 5.8. *For all $\beta < \lambda$, $\mathcal{E}_\beta = \{\langle \phi, x, y\rangle \mid \beta \in y\}$ is dense.*

Lemma 5.9. *Suppose that $H \subseteq \mathbb{P}(\overline{f},\overline{g})$ is a filter such that $H \cap D_n \neq \emptyset$, $H \cap E_\alpha \neq \emptyset$ and $H \cap \mathcal{E}_\beta \neq \emptyset$ for all $n \in \omega$, $\alpha < \kappa$ and $\beta < \lambda$. Then $h = \bigcup\{\phi \mid \exists x, y(\langle \phi, x, y\rangle \in H)\}$ fills the gap $(\overline{f},\overline{g})$.*

Theorem 5.10. *Let $(\overline{f},\overline{g})$ be a (κ,λ) gap,*

(a) If the gap is filled, then $\mathbb{P}(\overline{f},\overline{g})$ is c.c.c.;

(b) If $\mathrm{cf}(\kappa) \neq \omega_1$ or $\mathrm{cf}(\lambda) \neq \omega_1$, then $\mathbb{P}(\overline{f},\overline{g})$ is c.c.c.;

(c) If $\kappa = \lambda = \omega_1$, and the gap is not filled, then there exists a c.c.c. poset $\mathbb{Q}$ which adjoins an uncountable antichain to $\mathbb{P}(\overline{f},\overline{g})$.

Proof. (a) Suppose that $h \in \omega^\omega$ fills the gap. Suppose also that $\{p_\alpha = \langle \phi_\alpha, x_\alpha, y_\alpha \rangle \mid \alpha < \omega_1\}$ is an uncountable antichain. For each $\alpha < \omega_1$, there exists $k_\alpha \in \omega$ such that for all $\xi \in x_\alpha$ and $\eta \in y_\alpha$,

$$f_\xi(t) < h(t) < g_\eta(t)$$

for all $t \geq k_\alpha$. Applying **Lemma 5.6**, we can suppose that $k_\alpha \subseteq \mathrm{dom}(\phi_\alpha)$ for all $\alpha < \omega_1$. Choose $\alpha \neq \beta$ such that $\phi_\alpha = \phi_\beta$. Then

$$\langle \psi_\alpha, x_\alpha \cup x_\beta, y_\alpha \cup y_\beta \rangle \leq p_\alpha, p_\beta.$$

(b) Suppose $A = \{\langle \phi_\alpha, x_\alpha, y_\alpha \rangle \mid \alpha < \omega_1\}$ is an antichain. First suppose that $\mathrm{cf}(\kappa) > \omega_1$. Then there exists $\xi < \kappa$ such that $x_\alpha \subseteq \xi$ for all $\alpha < \omega_1$. But the nf_ξ fills the gap

$$\overline{f}_0 = \langle f_\alpha \mid \alpha < \xi \rangle; \quad \overline{g} = \langle g_\beta \mid \beta < \lambda \rangle.$$

And so $\mathbb{P}(\overline{f}_0, \overline{g})$ is c.c.c.. However, $A \subseteq \mathbb{P}(\overline{f}_0, \overline{g})$, contradiction.

Next, suppose that $\mathrm{cf}(\kappa) = \omega$, express $\kappa = \bigcup_{n \in \omega} \xi_n$, where $\xi_0 < \xi_1 < \ldots < \xi_n < \ldots$ where $\kappa = \sup_n \xi_n$. For each $n < \omega$, let $I_n = \{\alpha < \omega_1 \mid x_\alpha \subseteq \xi_n\}$. Then $\omega_1 = \bigcup_n I_n$, and so there exists $n \in \omega$ such that $|I_n| = \omega_1$. Now, by considering $\{\langle \phi_\alpha, x_\alpha, y_\alpha \rangle \mid \alpha \in I_n\}$, we easily get contradiction, as before.

(c) In order to simplify the notation, we will suppose that for all $\alpha < \omega_1$, all $t < \omega$, $g_\alpha(t) > f_\alpha(t) + 1$. It follows that for each $\alpha < \omega_1$, $p_\alpha = \langle \emptyset, \{\alpha\}, \{\alpha\} \rangle \in \mathbb{P}(\overline{f}, \overline{g})$.(Continued after the Remark.)

Remark To obtain (*), we just need to make finitely many changes to each g_α. Let $(\overline{f}, \overline{g}')$ be the resulting slightly modified gap. Clearly in any generic extensions, $(\overline{f}, \overline{g})$ is filled iff $(\overline{f}, \overline{g}')$ is filled. Suppose $\mathbb{Q}$ is a c.c.c. poset which adds an uncountable antichain to $\mathbb{P}(\overline{f}, \overline{g}')$. Let $G \subseteq \mathbb{Q}$ be $V-$generic, and let $A \subseteq \mathbb{P}(\overline{f}, \overline{g}')$ be an uncountable antichain. We claim that there also exists an uncountable antichain $A' \subseteq \mathbb{P}(\overline{f}, \overline{g})$ with $A' \in V[G]$. Suppose not, then $\mathbb{P}(\overline{f}, \overline{g})$ is c.c.c. in $V[G]$. Let $H \subseteq \mathbb{P}(\overline{f}, \overline{g})$ be $V[G]-$generic, then $(\overline{f}, \overline{g})$ is filled in $V[G][H]$. Hence, $(\overline{f}', \overline{g}')$ is filled in $V[G][H]$. However, $A \subseteq \mathbb{P}(\overline{f}, \overline{g}')$ is an uncountable antichain, which contradicts **Theorem 5.10**(a).

(Continue on proving, next time)Let $\mathbb{Q}$ consist of all $z \in [\omega_1]^{<\omega}$ such that $\{p_\alpha \mid \alpha \in z\}$ is an antichain in $\mathbb{P}(\overline{f}, \overline{g})$, ordered by: $z_1 \leq z_2$ iff $z_1 \supseteq z_2$.

Lecture Notes #16 of Axiomatic Set Theory

If $(\overline{f}, \overline{g})$ is a (κ, λ) gap, $\mathbb{P}(\overline{f}, \overline{g})$ consists of $\langle \phi, x, y \rangle$:

- $x \in [\kappa]^{<\omega}$, $y \in [\lambda]^{<\omega}$;
- $\phi : n \to \omega$;
- For all $t \geq n, \alpha \in x, \beta \in y,\ f_\alpha(t) + 1 < g_\beta(t)$.

Theorem 5.10.

1. *If the gap is filled, $\mathbb{P}(\overline{f}, \overline{g})$ is c.c.c.;*
2. *If $\mathrm{cf}(\kappa) \neq \omega_1$, then $\mathbb{P}(\overline{f}, \overline{g})$ is c.c.c.;*
3. *If $\kappa = \lambda = \omega_1$, and $(\overline{f}, \overline{g})$ is not filled, there is a c.c.c. poset $\mathbb{Q}$ which adjoins an uncountable antichain to $\mathbb{P}(\overline{f}, \overline{g})$.*[1]

Target $\mathfrak{b} = \omega_2$.

Claim. $\mathbb{Q}$ is c.c.c..

Proof. Suppose that $\{z_\alpha \mid \alpha < \omega_1\}$ is an uncountable antichain then we can suppose that $\{z_\alpha \mid \alpha < \omega_1\}$ is a $\Delta-$system with root r. If $\alpha \neq \beta$, then there exist $\xi \in z_\alpha - r$ and $\eta \in z_\beta - r$ such that p_ξ, p_η are compatible in $\mathbb{P}(\overline{f}, \overline{g})$. So we can suppose that $r = \emptyset$. Also, by passing to a suitable subsequence, we can suppose that if $\alpha < \beta$, then $\max z_\alpha < \max z_\beta$. Also, we can suppose that there is a fixed $k \in \omega$ such that for all $\alpha < \omega_1$, and $\xi < \eta \in z_\alpha$,

$$f_\xi(t) < f_\eta(t) < g_\eta(t) < g_\xi(t),$$

for all $t \geq k$.

Finally, for each $\alpha < \omega_1$, let $\alpha_0 = \min z_\alpha$.

Case 1 First, suppose there exists $\alpha \neq \beta$, and $t \geq k$ such that $f_{\alpha_0}(t) \geq g_{\beta_0}(t)$. If $\xi \in z_\alpha$ and $\eta \in z_\beta$, then

$$f_\xi(t) \geq f_{\alpha_0}(t) \geq g_{\beta_0}(t) \geq g_\eta(t),$$

and so $p_\xi \perp p_\eta$ in $\mathbb{P}(\overline{f}, \overline{g})$. But this means that $z_\alpha \cup z_\beta \in \mathbb{Q}$ which contradicts the fact that $\{z_\alpha \mid \alpha < \omega_1\}$ is an antichain in $\mathbb{Q}$.

Case 2 Thus for all $\alpha, \beta < \omega_1$, and all $t \geq k$, $f_{\alpha_0}(t) < g_{\beta_0}(t)$. Hence, we can define $h \in \omega^\omega$ such that for all $t \geq k$,

$$h(t) = \max\{f_\alpha(t) \mid \alpha < \omega_1\}.$$

[1] In order to simplify notion, we will suppose that for all $\alpha < \omega_1$ and $t < \omega$, $f_\alpha(t) + 1 < g_\alpha(t)$.(Last time we saw that this assumption is harmless.) It follows that for each $\alpha < \omega_1$, $p_\alpha = \langle \emptyset, \{\alpha\}, \{\alpha\} \rangle \in \mathbb{P}(\overline{f}, \overline{g})$. Let $\mathbb{Q}$ consist of all $z \in [\omega_1]^{<\omega}$ such that $\{p_\alpha \mid \alpha \in z\}$ is an antichain in $\mathbb{P}(\overline{f}, \overline{g})$, partially ordered by $z_1 \leq z_2$ iff $z_1 \supseteq z_2$.

We claim that h fills the gap $(\overline{f}, \overline{g})$, which is a contradiction. To see this, we first note that $h <^* g_{\beta_0}$. For all $\beta < \omega_1$, and this implies that $h <^* g_\gamma$ for all $\gamma < \omega)1$. Next note that if $\alpha < \beta < \omega_1$, then there exists $l \geq k$, such that

$$f_{\alpha_0}(t) < f_{\beta_0}(t) \leq h(t),$$

for all $t \geq l$. It follows that $f_{\alpha_0} <^* h$ for all $\alpha < \omega_1$, and this implies that $f_\gamma <^* h$ for all $\gamma < \omega_1$. Claim Proved.

Finally, let $\tilde{A}$ be the $\mathbb{Q}-$name defined by:

$$\tilde{A} = \{\langle \check{p}_\alpha, z\rangle \mid \alpha \in z\alpha \in \mathbb{Q}\}.$$

Clearly, $\Vdash_{\mathbb{Q}}$ "$\tilde{A}$ is an antichain in $\mathbb{P}(\overline{f}, \overline{g})$". Suppose that $\Vdash_{\mathbb{Q}}$ "$\tilde{A}$ is countable". Since $\mathbb{Q}$ is c.c.c., there exists $\alpha < \omega_1$, such that

$$\Vdash_{\mathbb{Q}} \tilde{A} \subseteq \{\check{p}_\beta \mid \beta < \check{\alpha}\}.$$

However, if $\gamma \geq \alpha$, then $\{\gamma\} \Vdash p_\gamma \in \tilde{A}$, which is a contradiction. Hence,

$$\neg \Vdash_{\mathbb{Q}} |\tilde{A}| = \omega.$$

Hence there exists $z_0 \in \mathbb{Q}$ such that

$$z_0 \Vdash_{\mathbb{Q}} |\tilde{A}| > \omega.$$

Then $\mathbb{Q}_0 = \{z \in \mathbb{Q} \mid z \leq z_0\}$ is a c.c.c. poset which adjoins an uncountable antichain to $\mathbb{P}(\overline{f}, \overline{g})$.

Theorem 5.11(FAA). *If κ, λ are regular, uncountable cardinals, and $\overline{f} = \langle f_\alpha \mid \alpha < \kappa\rangle, \overline{g} = \langle g_\beta \mid \beta < \lambda\rangle$ is an unfilled (κ, λ) gap, then $\kappa = \lambda = \omega_1$.*

Proof. Suppose not. For example, suppose that $\omega_1 \leq \kappa \leq \lambda \leq 2^\omega$ and $\lambda > \omega_1$, let $\mathbb{P}$ consist of all functions $p : \alpha \to \lambda, \alpha < \omega_1$ ordered by $p \leq q$ iff $p \supseteq q$. Let $G \subseteq \mathbb{P}$ be a $V-$generic filter, Until further notice, we work in $V[G]$. Clearly, $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda) = \omega_1$, and so $\overline{f}, \overline{g}$ has a cofinal (ω_1, ω_1) gap $(\overline{f}_0, \overline{g}_0)$. Since $\mathbb{P}$ adds no new $h \in \omega^\omega$, it follows that $\overline{f}_0, \overline{g}_0$ is unfilled. Hence, by **Theorem 5.10**(c), there exists a c.c.c. poset $\mathbb{Q} \in V[G]$ such that $\mathbb{Q}$ adjoins an uncountable antichain to $\mathbb{P}(\overline{f}_0, \overline{g}_0)$ and hence to $\mathbb{P}(\overline{f}, \overline{g})$. Let $\tilde{\mathbb{Q}}$ be a $\mathbb{P}-$name such that $\tilde{\mathbb{Q}}_G = \mathbb{Q}$. For the rest of the proof, we work in V. Since,

$$\Vdash_{\mathbb{P}*\tilde{\mathbb{Q}}} \check{\mathbb{P}}(\overline{f}, \overline{g}) \text{ has an uncountbale antichain,}$$

there exists a $\mathbb{P} * \tilde{\mathbb{Q}}-$name $\tilde{\phi}$ such that

$$\Vdash_{\mathbb{P}*\tilde{\mathbb{Q}}} \phi : \check{\omega}_1 \to \check{\mathbb{P}}(\overline{f}, \overline{g}) \text{ is an injection such that } \mathrm{ran}(\phi) \text{ is an antichain.}$$

For each $\alpha < \omega_1$, let

$$D_\alpha = \{r \in \mathbb{P} * \tilde{\mathbb{Q}} \mid \text{There exists } a \in \mathbb{P}(\overline{f}, \overline{g}) \text{ such that } r \Vdash \tilde{\phi}(\check{\alpha}) = \check{\alpha}\}.$$

Then each D_α is dense. Since $\mathbb{P} * \tilde{\mathbb{Q}}$ satisfies Axiom A, there exists a filter $G \subseteq \mathbb{P} * \tilde{\mathbb{Q}}$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$. Let

$$\theta = \{\langle \alpha, a\rangle \in \omega_1 \times \mathbb{P}(\overline{f}, \overline{g}) \mid \exists r \in G(r \Vdash \tilde{\phi}(\check{\alpha}) = \check{\alpha})\}.$$

Then $\theta : \omega_1 \to \mathbb{P}(\tilde{f}, \tilde{g})$ is an injection such that $\mathrm{ran}(\theta)$ is an antichain, Thus $\mathbb{P}(\tilde{f}, \tilde{g})$ is not c.c.c., which contradicts **Theorem 5.10**(b), since $\lambda = \mathrm{cf}(\lambda) > \omega_1$.

We are finally ready to prove:

Theorem (FAA). $\mathfrak{b} = \omega_2$.

Proof. Since FAA holds, $\mathfrak{b} \geq \omega_2$. Hence we can inductively construct a sequence $\langle f_\alpha \mid \alpha < \omega_1 \rangle \subseteq \omega^\omega$ such that if $\alpha < \beta < \omega_2$, then $f_\alpha <^* f_\beta$.

From now on, assume $\mathfrak{b} > \omega_2$, then there exists $g_0 \in \omega^\omega$ such that $f_\alpha <^* g_0$ for all $\alpha < \omega_1$. Now inductively define g_β for as long as possible such that:

- $f_\alpha <^* g_\beta$ for all $\alpha < \omega_2$;
- If $\gamma < \beta$, then $g_\beta <^* g_\gamma$.

Suppose that we cannot continue past $\langle g_\beta \mid \beta < \theta \rangle$.

<u>Claim 1.</u> θ is a limit ordinal.

Proof. Suppose that $\theta = \beta + 1$. Define $h \in \omega^\omega$ by

$$h(n) = \begin{cases} g_\beta(n) - 1, & \text{if } g_\beta(n) > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\alpha < \omega_2$. Then there exists $k \in \omega$ such that for all $t \geq k$,

$$f_\alpha(t) < f_{\alpha+1}(t) < g_\beta(t),$$

and π follows that $f_\alpha <^* h <^* g_\beta$, which is a contradiction.

<u>Claim 2.</u> $\mathrm{cf}(\theta) = \omega$.

Proof. Suppose that $\mathrm{cf}(\theta) = \lambda > \omega$. Then there exists unfilled (ω_2, λ) gap, which contradicts **5.11** .

By passing to a cofinal subsequence of $\langle g_\beta \mid \beta < \theta \rangle$ we can suppose that $\theta = \omega$. Thus we have constructed an unfilled (ω_2, ω) gap:

$$f_0 <^* f_1 <^* ... <^* f_\alpha <^* ... <^* g_n <^* .. <^* g_0.$$

Lecture Notes #17 of Axiomatic Set Theory

Recap(FAA) Assuming $\mathfrak{b} > \omega_2$, we have constructed an unfilled (ω_1, ω) gap:

$$f_0 <^* f_1 <^* ... <^* f_\alpha <^* ... <^* g_n <^* g_0.$$

So to reach a final contradiction, it is only necessary to show that, $\mathfrak{b} > \omega_2$ implies that this gap is filled.

First Approximation Consider $\phi(n) = \min\{g_l(n) \mid l \leq n\}$. Clearly $\phi <^* g_m$ for all $m \in \omega$. Unfortunately, we might have $\phi(n) = 0$ for all $n \in \omega$ and so $f_\alpha \not<^* \phi$ for all $\alpha > 0$.

Second Approximation Let $\theta \in \omega^\omega$ be a monotonically increasing VERY slow groweing function function. Consider

$$\psi(n) = \min\{g_l(n) \mid l \leq \theta(n)\}.$$

Then clearly $\psi <^* g_m$ for all $m \in \omega$, and with luck $f_\alpha <^* \psi$ for all $\alpha \in \omega_1$. For each $\alpha \in \omega_2$, define $h_\alpha \in \omega^\omega$ by

(*) $h_\alpha(n) =$"the least m such that for all $l \leq n$, $f_\alpha(t) < g_l(t)$ for all $t \geq m$".

Since $\mathfrak{b} > \omega_2$, there exists a strictly increasing $h \in \omega^\omega$ such that $h_\alpha <^* h$ for all $\alpha < \omega_2$. Wlog[1], we can suppose $h(0) = 0$. Define $g \in \omega^\omega$ by:

$$g(n) = \min\{g_l(n) \mid h(l) \leq n\}.$$

We will show that g fills our unfilled (ω_2, ω) gap. Contradiction.

Claim. $g <^* g_m$ for all $m \in \omega$.

Proof. There exists $k \geq h(m+1)$ such that for all $t \geq k$, $g_{m+1}(t) < g_m(t)$, and so

$$g(t) = \min\{g_l(t) \mid h(l) \leq t\} \leq g_{m+1}(t) < g_m(t).$$

Claim. $f_\alpha <^* g$ for all $\alpha < \omega_2$.

Proof. Fix some $\alpha < \omega_2$, and let n_0 be such that for all $n \geq n_0$, $h_\alpha(n) < h)n$.

Let $t \geq h(n_0)$, so there exists $n \geq n_0$ such that $h(n) \leq t < h(n+1)$. Then $t \geq h(n) > h_\alpha(n)$ and so

$$f_\alpha(t) < \min\{g_l(t) \mid l \leq n\} = \min\{g_l(t) \mid h(l) \leq t\} = g(t).$$

[1] Without loss of generality.

The Open Coloring Axiom(OCA)

We next study a Ramsey-theoretic consequence of our forcing axiom. First we review some of the classical Ramsey theory.

Theorem 6.1,Ramsey. *If $[\mathbb{N}]^2 = K_0 \sqcup K_1$ is any partition, then there exists an infinite $H \subseteq \mathbb{N}$ and $i \in 2$ such that $[H]^2 \subseteq K_i$.*

We say that H is $i-$homogeneous. The analogue for 2^ω is false.

Theorem 6.2. *There exists a partition $[\mathbb{R}]^2 = K_0 \sqcup K_1$ with no uncountable homogeneous subsets.*

Proof. Let $\prec$ be a well-ordering of $\mathbb{R}$ and let $<$ be the normal ordering. Define $\{x, y\} \in K_0$ iff $\prec$ and $<$ agrees on $\{x, y\}$. If H is an uncountable homogeneous subset, then there exists either an increasing or decreasing ω_1-sequence of reals, which contradicts the separability of $\mathbb{R}$.[2]

Theorem 6.3,Galvin. *Let X be a non-empty perfect Polish space and let $[X]^2 = K_0 \sqcup K_1$ be a partition such that K_0, K_1 have the Baire property. Then there exists a Cantor set $C \subseteq X$ such that $[C]^2 \subseteq L_i$ for some $i \in 2$.*

Remark here we identify each $K_i \subseteq [X]^2$ with the symmetric subset:

$$\{\langle x, y\rangle \in X^2 \mid x \neq y \wedge \{x, y\} \in K_i\}.$$

Thus it make sense to consider whether K_i is Borel etc as a subset of X^2.

A subset A of a Polish space Z has to have the Baire property, if there exists an open set $U \subseteq Z$ such that $A \Delta U$ is non-meager. Every Borel set has the Baire property.

Corollary. 6.4 If $[\mathbb{R}]^2 = K_0 \sqcup K_1$ is a partition such that K_0, K_1 are Borel, then there exists a Cantor set $C \subseteq \mathbb{R}$ such that $[C]^2 \subseteq K_i$ for some $i \in 2$.

Remark The analogue for $[\mathbb{R}]^3$ is false.

The Open Coloring Axiom(OCA) If X is an uncountable separable space, and $[X]^2 = K_0 \sqcup K_1$ is a partition such that K_0 is open, then either:

(1) There exists an uncountable $0-$homogeneous subset, or

(2) X can be covered by countably many $1-$homogeneous subsets.

Remark The power of OCA comes from that X is any separable metric space, i.e., usually is not complete.

We begin with a couple of consequences of the weak version(wOCA) which only requires an uncountable homogeneous subset.

Theorem 6.5(wOCA). *(1) Every uncountable subset of $\mathcal{P}(\omega)$ contains either an uncountable chain or an uncountable antichain wrt[3] inclusion.*

[2] Of course, $\prec$ is not Borel, measurable, etc.

[3] With respect to

(2) If $A \subseteq \mathbb{R}$ is an uncountable subset, and $f : A \to \mathbb{R}$ is any function, then there exists an uncountable $B \subseteq A$ such that $f \restriction B$ is monotonic.

Proof. (1) As usual, we identify $\mathcal{P}(\omega)$ with the Cantor space 2^ω. Let $X \subseteq \mathcal{P}(\omega)$ be any uncountable subset. Then, with the subset topology, X is a separable metric space. Let

$$K_0 = \{\{a, b\} \in [X]^2 \mid a \perp b \text{wrt} \subseteq\}.$$

Then clearly, K_0 is open. Let $K_1 = [X]^2 - K_0$. Applying wOCA, there exists an uncountable homogeneous $H \subseteq X$. The result follows.

(2) Express $\omega = D \sqcup R$ is the disjoint union of two infinite subsets. Since $\mathcal{P}(D), \mathcal{P}(R)$ are isomorphic to $\mathcal{P}(\mathbb{Q})$ wrt inclusion, by identifying $\mathbb{R}$ with the set of Dedekind cuts, there exists chains $C_0 \subseteq \mathcal{P}(D), C_1 \subseteq \mathbb{P}(R)$ and isomorphisms

$$\phi : \langle \mathbb{R}, \leq \rangle \to \langle C_0, \subseteq \rangle;$$

$$\psi : \langle \mathbb{R}, \leq \rangle \to \langle C_1, \subseteq \rangle.$$

Define

$$\pi : \text{graph}(f) \to \mathcal{P}(\omega), \quad \langle a, f(a) \rangle \mapsto \phi(a) \sqcup \psi(f(a)).$$

Clearly, if $a, a' \in A$, with $a < a'$, then:

$$f(a) \leq f(a') \quad \text{iff} \quad \phi(a) \cup \psi(f(a)) \subseteq \phi(a') \cup \psi(f(a'));$$

$$f(a) > f(a') \quad \text{iff} \quad \phi(a) \cup \psi(f(a)) \perp \phi(a') \cup \psi(f(a')).$$

Thus (2) follows from (1).

Lecture Notes #18 of Axiomatic Set Theory

If X is a separable metric space and $[X]^2 = K_0 \sqcup K_1$ is a partition with K_0 open, then:

wOCA There exists an uncountable homogeneous subset.

OCA Either:

(1) There exists an uncountable $0-$homogeneous subset, or;

(2) X can be covered by countably many $1-$homogeneous subsets.

Last time we proved:

Theorem 6.5,(2)(wOCA). *If $A \subseteq \mathbb{R}$ is uncountable and $f : A \to \mathbb{R}$ is any function, there exists an uncountable $C \subseteq A$ such that $f \restriction C$ is monotonic.*

Application Suppose that $A, B \subseteq \mathbb{R}$ with $|A| = |B| = \omega_1$. Let $f : A \to B$ be a bijection. By **Theorem 6.5**(2), there exists an uncountable $C \subseteq A$ such that $f \restriction C$ is either an isomorphic embedding, or an anti-isomorphic embedding.

Assuming OCA, we obtain a stronger result.

Theorem 6.6(OCA). *If $A, B \subseteq \mathbb{R}$ are uncountable subsets, then there exists partial function $f : A \to B$ such that*

(1) $C = \mathrm{dom}(f)$ is uncountable;

(2) $f : C \to B$ is strictly increasing.

Proof. Let $[\mathbb{R}^2]^2 \supseteq S$ be the collection of those $\{\langle x', y'\rangle, \langle x, y\rangle\}$ such that $x < x'$ and $y < y'$. Clearly, S is open. Let $X = A \times B$. Let $K_0 = S \cap [X]^2$ and $K_1 = [X]^2 - K_0$. Then X is a separable metric space, and $[X]^2 = K_0 \sqcup K_1$ is a partition with K_0 open.

Claim. $A \times B$ is not the union of countably many $1-$homogeneous subsets.

Assuming the Claim., by OCA, there exists an uncountable $0-$homogeneous subset $H \subseteq A \times B$. Clearly H is the graph of some function($H = \mathrm{graph}(f)$) which satisfies (1) and (2).

Proof of Claim. Suppose $A \times B = \bigcup_{n\in\omega} H_n$, where each $[H_n]^2 \subseteq K_1$. For each $a \in A$, let $L_a = \{\langle a, b\rangle \mid b \in B\}$. Then there exists fixed $n \in \omega$ and an uncountable $A_0 \subseteq A$ such that $L_a \cap H_n$ is uncountable for each $a \in A_0$. For each $a \in A_0$, choose $b_1(a) < b_2(a) < b_3(a)$ such that $\{\langle a, b_i(a)\rangle \mid 1 \le i \le 3\} \in H_n$. Then each $(b_1(a), b_3(a)) \cap B$ is a non-empty open interval of a separable space B. Hence there exists $a < a'$ in A_0 usch that

$$(b_1(a), b_3(a)) \cap (b_1(a'), b_3(a')) \neq \emptyset.$$

It follows that $b_1(a) < b_3(a')$. But then $\langle a, b_1(a)\rangle, \langle a', b_3(a')\rangle \in H_n$ satisfy $\{\langle a, b_1(a)\rangle, \langle a', b_3(a')\rangle\} \in K_0$, contradiction.

Remark The following result, together with **Theorem 6.5**(2) implies that

$$\text{wOCA} \Longrightarrow \neg\text{CH}.$$

Theorem 6.7(CH). *There exist $A, B \subseteq \mathbb{R}$ with $|A| = |B| = |\mathbb{R}| = \omega_1$ such that if $C \subseteq A$ is uncountable, then there does not exist either an isomorphic or an anti-isomorphic embedding $f : C \to B$.*

Suppose $X \subseteq \mathbb{R}$, and $f : X \to \mathbb{R}$ is strictly increasing, then the closure $\overline{f}$ of f consists of all $\langle x, y\rangle \in [\mathbb{R}]^2$ such that for $\epsilon > 0$, there exists $x_1, x_2 \in X$ such that $x_1 \leq x \leq x_2$ and $f(x_1) \leq y \leq f(x_2)$ and $0 \leq x_1 - x, x_2 - x, y - f(x_1), f(x_2) - y < \epsilon$. Let $\overline{x} = \{x \in \mathbb{R} \mid \exists y(\langle x, y\rangle \in \overline{f})\}$, then it is easy to check $\overline{f} : \overline{x} \to \mathbb{R}$ strictly increasing.

<u>Observation</u> If $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ is strictly increasing, there exists a countable $f_0 \subseteq f$ such that $f \subseteq \overline{f}_0$.

Similarly for strictly decreasing functions:

Proof of **Theorem 6.7.** Let $\langle f_\alpha \mid \alpha < \omega_1\rangle$ enumerate all countable strictly increasing maps between sets or reals, and let $\langle g_\alpha \mid \alpha < \omega_1\rangle$ enumerate all strictly decreasing maps between set of reals. We define $r_\alpha \in \mathbb{R}$ for all $\alpha < \omega_1$ inductively as follows:

Suppose $\{r_\beta \mid \beta < \alpha\}$ are already been defined. Let R_α be the closure of $\{r_\beta \mid \beta < \alpha\}$ under the functions

$$\{\overline{f}_\beta, \overline{f}_\beta^{-1}; \overline{g}_\beta, \overline{g}_\beta^{-1} \mid \beta < \alpha\}.$$

Then R_α is countable, and so we can choose $r_\alpha \notin R_\alpha$. Express $\omega_1 = X \sqcup Y$ with $|X| = |Y| = \omega_1$. We claim that

$$A = \{r_\alpha \mid \alpha \in X\}, \quad B = \{r_\alpha \mid \alpha \in Y\}$$

satisfies our requirements.

Suppose for example, there exists an uncountable $C \subseteq A$ and a strictly increasing function $f : C \to B$. Then there exists $Z \subseteq X$ with $|Z| = \omega_1$ such that $C = \{r_\beta \mid \beta \in Z\}$. Also, there exists $\alpha \in \omega_1$ such that $f \subseteq \overline{f}_\alpha$. Choose $\alpha < \beta \in Z$, and let $r_\gamma = f(r_\beta) \in B$. Then $r_\gamma = f(r_\beta) = \overline{f}_\alpha(r_\beta) \in R_{\beta+1}$, and so $\gamma < \beta + 1$. Since $\gamma \neq \beta$, it follows that $\gamma < \beta$. But then

$$r_\beta = \overline{f}_\alpha^{-1}(r_\gamma) \in R_\beta.$$

Contradiction.

We next give an example which illustrates the lack of symmetry in OCA. Let T consist of all strictly increasing sequences $\langle q_\gamma \mid \gamma < \alpha\rangle$ of rationals. Clearly, T has no uncountable branches. And **Theorem 3.18** implies that T is not special.(i.e. is not the union of countably many antichains.)

Now consider the isomorphic tree $\tilde{T} \subseteq \mathcal{P}(\mathbb{Q})$ induced by then map $\langle q_\gamma \mid \gamma < \alpha\rangle \mapsto \{q_\gamma \mid \gamma < \alpha\}$. Thus if $a, b \in \tilde{T}$, then $a \leq_{\tilde{T}} b$ iff a is an initial subset of b.

Note that $\tilde{T}$ is a separable metric space via the topology on $\mathcal{P}(\mathbb{Q}) = 2^{\mathbb{Q}}$. Let K_0 be the $2-$sets $\{a, b\} \in [\tilde{T}]^2$ such that a, b are incomparable in $\tilde{T}$. Clearly K_0 is open. Let $K_1 = [\tilde{T}]^2 - K_0$. If $H \subseteq \tilde{T}$ is $1-$homogeneous, then H is contained in a branch of $\tilde{T}$, and if $H \subseteq \tilde{T}$ is $0-$homogeneous, then H is an antichain in $\tilde{T}$.

Thus, there are no uncountable $1-$homogeneous subsets, and $\tilde{T}$ cannot be covered by countably many $0-$homogeneous subsets.

We next begin to work towards:

Theorem 6.8. *FAA implies OCA.*

Suppose that X is a separable metric space, and $[X]^2 = K_0 \sqcup K_1$ with K_0 open. Suppose that X cannot be covered by countably many $1-$homogeneous subsets.

Definition. For each subset $Y \subseteq X$, let $\mathbb{Q}_Y$ be the poset of finite $0-$homogeneous subset of Y, ordered by inverse inclusion.

We would like to find an uncountable $Y \subseteq X$ such that $\mathbb{Q}_Y$ is c.c.c.. If we assume CH, such a set exists.

Lemma 6.9(CH). *With the above hypothesis, there exists an uncountable $Y \subseteq X$ such that $\mathbb{Q}_Y$ is c.c.c..*

Proof Delayed.

Of course FAA implies $\neg$CH. So we need to collapse the continuum using a countably closed poset.

Lemma 6.10. *With the above hypothesis, let $\mathbb{P} \in V$ be countably closed, and let $G \subseteq \mathbb{P}$ be $V-$generic. Then X cannot be covered by countably many $1-$homogeneous subsets in $V[G]$.*

Proof Delayed.

Lecture Notes #19 of Axiomatic Set Theory

Assuming **Lemma 6.9** and **6.10**, we prove:

Theorem 6.8. *FAA implies OCA.*

Proof. Assume FAA, let X be a separable metric space and let

$$[X]^2 = K_1 \sqcup K_0$$

be a partition with K_0 open. Suppose that X cannot be covered by countable many $1-$homogeneous subsets. Let $\mathbb{P}$ be the poset of all functions: $p : \alpha \to 2^\omega$ with α a countable ordinal, and ordered by inverse inclusion. Let $G \subseteq \mathbb{P}$ be $V-$generic. Then $V[G]$ satisfies CH. Until further notice, we work in $V[G]$. By **Lemma 6.10**, X cannot be covered by countably many $1-$homogeneous subsets in $V[G]$. By **Lemma 6.9**, there exists an uncountable $Y \subseteq X$ such that $\mathbb{Q}_Y$ is c.c.c.. Let $Y = \{y_\alpha \mid \alpha < \omega_1\}$. Let $\tilde{H}$ be the $\mathbb{Q}_Y-$name:

$$\tilde{H} = \{\langle \check{y}, z\rangle \mid y \in z \in \mathbb{Q}_Y\}.$$

Then clearly,

$$\Vdash_{\mathbb{Q}_Y} \tilde{H} \text{ is } 0-\text{homogeneous.}$$

Suppose that $\Vdash_{\mathbb{Q}_Y} |\tilde{H}| = \omega$. Since $\mathbb{Q}_Y$ is c.c.c., there exists $\alpha < \omega_1$, such that

$$\Vdash_{\mathbb{Q}_Y} \tilde{H} \subseteq \{y_\beta \mid \beta < \alpha\}.$$

But if $\gamma \geq \alpha$, then $\{y_\gamma\} \Vdash_{\mathbb{Q}_Y} \check{y}_\gamma \in \tilde{H}$, which is contradiction. Hence, there exists $z_0 \in \mathbb{Q}_Y$ such that:

$$z_0 \Vdash_{\mathbb{Q}_Y} |\tilde{H}| > \omega.$$

Let $\tilde{Q}$ be a $\mathbb{P}-$name such that $Q = \tilde{Q}_G$[1]. Then $\mathbb{P} * \tilde{\mathbb{Q}}$ satisfies Axiom A, and

$$\Vdash_{\mathbb{P}*\tilde{\mathbb{Q}}} \text{There exists an uncountable } 0-\text{homogeneous subset of } X.$$

Applying FAA, it follows that there exists an uncountable $0-$homogeneous subset $H \in V$.

Next we prove **Lemma 6.10**. So suppose X is a separable metric space and $[X]^2 = K_0 \sqcup K_1$ with X_0 open. Also suppose X cannot be covered by countably $1-$homogeneous subsets. Let $\mathbb{P}$ be countably closed.

First notice that, since K_1 is closed, if H is $1-$homogeneous and $\mathrm{cl}_X(H)$ is its closure, then $\mathrm{cl}_X(H)$ is also $1-$homogeneous. Thus, we can restrict our attention to coverings by closed $1-$homogeneous.

Next, let $\{x_n \mid n \in \omega\}$ be a dense subset of X. For each $1 \leq t < \omega$, let

$$U_{n,t} = \{x \in X \mid d(x, x_n) < 1/t\}.$$

[1] Let $\mathbb{Q} = \{z \in \mathbb{Q}_Y \mid z \leq z_0\}$.

Then $\{U_{n,t} \mid n \in \omega, t \in \omega - \{0\}\}$ is a basis for the topology on X, and each open set is the union of countably many of these sets. It follows that $\mathbb{P}$ adds no new open sets, and hence adds no new closed sets. Also $\mathbb{P}$ adds no new countable sequences of closed sets. **Lemma 6.10** follows. *Proof of* **Lemma 6.9** For $p \in X^n$ and open $U \subseteq X^n$ with $p \in U$, we define

$$U_p = \{q \in U \mid q_i \neq p_i \& \{p_i, q_i\} \in K_0 \text{ for } i < n.\}$$

If $f : X^n \to X$ is a partial function, and $p \in X^n$, define

$$w_f(p) = \bigcap \{\mathrm{cl}_X(f[U_p \cap \mathrm{dom}(f)]) \mid U \subseteq X^n \text{ is open with } p \in U.\}$$

Let $\{f_\xi \mid \xi < \omega_1\}$ enumerate all countable partial functions $f : X^n \to X$ for some $n \geq 1$ and let $\{T_\xi \mid \xi < \omega_1\}$ enumerate the closed $1-$homogeneous subsets of X. Then we can inductively define:

(a) $x_\alpha \in X - \{x_\beta \mid \beta < \alpha\}$;

(b) $x_\alpha \notin T_\beta$ for $\beta < \alpha$;

(c) if $p \in \{x_\gamma \mid \gamma < \alpha\}^{<\omega}, \beta < \alpha$ and $w_{f_\beta}(p)$ is $1-$homogeneous, then x_α isn't in $w_{f_\beta}(p)$.

To see that $\mathbb{Q}_Y$ is c.c.c., suppose that $F = \{H_\alpha \mid \alpha < \omega_1\}$ is an uncountable family of finite $0-$homogeneous subsets, then

(a) We can suppose that $|H_\alpha| = n$ for all $\alpha < \omega_1$;

(b) We can suppose that there exists a fixed $R \subseteq X$ such that $H_\alpha \cap H_\beta = R$ for all $\alpha \neq \beta$.

Notice that if $\alpha < \beta < \omega_1$, then

$$[H_\alpha \cap H_\beta]^2 \subseteq K_0 \text{ iff } [(H_\alpha - R) \cup (H_\beta - R)]^2 \subseteq K_0.$$

Hence:

(c) We can suppose $\{H_\alpha \mid \alpha < \omega_1\}$ are pairwise disjoint. (ie, $R = \emptyset$.)

Now we agree by induction on $n \geq 1$, there exists $\alpha < \beta < \omega_1$ such that $[H_\alpha \cup H_\beta]^2 \subseteq K_0$. First suppose that $n = 1$, say $H_\alpha = \{y_{\xi_\alpha}\}$. If the result fails, then $H = \{y_{\xi_\alpha} \mid \alpha < \omega_1\}$ is $1-$homogeneous. Let $T = \mathrm{cl}_X(H)$, then T is also $1-$homogeneous and so there exists $\beta < \omega_1$ such that $T = T_\beta$. Choose $\xi_\alpha \geq \beta$, then $y_{\xi_\alpha} \in T_\beta$, contradiction.

Now suppose that $n > 1$, and the result holds for $n - 1$. For each $\alpha < \omega_1$, let $H_\alpha = \{x_{\alpha_0}, ..., x_{\alpha_{n-1}}\}$, where $\alpha_0 < ... < \alpha_{n-1}$. Then we identify $s_\alpha = (x_{\alpha_0}, ..., x_{\alpha_{n-1}}) \in X^n$. For each $\alpha < \omega_1$, we can choose disjoint basic open sets $U_i^\alpha \subseteq X$ such that $x_{\alpha_i} \in U_i^\alpha$. And if $i \neq j$, then $U_i^\alpha \times U_j^\alpha \subseteq K_0$.

(d) We can suppose that there exists a fixed $U_0, ..., U_{n-1}$ such that $U_i^\alpha = U_i$ for all $\alpha < \omega_1$. Of course, this implies:

(d) If $\alpha < \beta < \omega_1$, and $i \neq j$, then $\{x_{\alpha_i}, x_{\beta_j}\} \in K_0$. Thus only the sets $\{x_{\alpha_i}, x_{\beta_j}\}$ are problematic.

Next let $g : X^{n-1} \to X$ be the partial function defined by $g(s_\alpha \restriction n-1) = x_{\alpha_{n-1}}$. From now on, if $U \subseteq X^{n-1}$, we will write $g[U]$ instead of $g[U \cap \mathrm{dom}(g)]$. Define:

$$F_0 = \{s_\alpha \in F \mid x \mid x_{\alpha_{n-1}} \in w_g(s_\alpha \restriction n-1)\}.$$

Claim. $F - F_0$ is countable.

Proof of Claim. Suppose not. For each $s_\alpha \in F - F_0$, there exists a basic open $U^\alpha \subseteq X^{n-1}$ with $s_\alpha \restriction n-1 \in U^\alpha$ such that $x_{\alpha_{n-1}} \notin \mathrm{cl}_X(g[U^\alpha_{s_\alpha \restriction n-1}])$. Hence there exists an basic open $V^\alpha \subseteq X$ with $x_{\alpha_{n-1}} \in V^\alpha$ such that

$$V^\alpha \cap g[U^\alpha_{s_\alpha} \restriction n-1] = \emptyset.$$

There exists an uncountable $F^* \subseteq F - F_0$ and fixed $U \subseteq X^{n-1}$, $V \subseteq X$ such that $U^\alpha = U$ and $V^\alpha = V$ for all $s_\alpha \in F^*$.

By the inductive hypothesis, there exists $s_\alpha \neq s_\beta \in F^*$ such that

$$[(H_\alpha - \{x_{\alpha_{n-1}}\}) \cup (H_\beta - \{x_{\beta_{n-1}}\})]^2 \subseteq K_0.$$

But then $s_\beta \restriction n-1 \in U_{s_\alpha \restriction n-1}$ and $g(s_\beta \restriction n-1) = x_{\beta_{n-1}} \in V$, which is a contradiction.

Lecture Notes #20 of Axiomatic Set Theory

For $p \in X^n$ and open $U \subseteq X^n$ with $p \in U$,

$$U_p = \{q \in U \mid q_i \neq p_i \text{ and } \{p_1, q_i\} \in K_0 \text{ for all } i < n\}.$$

If $f : X^n \to X$ is a partial function and $p \in X^n$,

$$w_f(p) = \bigcap\{\mathrm{cl}_X(f[U_p]) \mid U \subseteq X^n \text{ open with } p \in U\}.$$

Recap For $\alpha < \omega_1$, $s_\alpha = (x_{\alpha_0}, ..., x_{\alpha_{n--1}})$ are pairwise disjoint $0-$homogeneous sequences such that if $\alpha < \beta < \omega_1$, and $i \neq j$, then $\{x_{\alpha_i}, x_{\beta_j}\} \in K_0$.

We have defined $g : X^{n-1} \to X$ by:

$$g(s_\alpha \upharpoonright n-1) = x_{\alpha_{n-1}}; \quad F_0 = \{s_\alpha \in F \mid x_{\alpha_{n-1}} \in w_g(s_\alpha \upharpoonright n-1)\}.$$

Claim. $F - F_0$ is countable.

Remark The proof of this claim used the inductive hypothesis.

Next let $g_0 \subseteq g$ is a countable dense subfunction of g. Then $g_0 = f_\xi$ for some $\xi < \omega_1$. Choose some $s_\alpha \in F_0$ such that α_0 is greater than ξ and greater than all the indices of elements appearing in $g_0 = f_\xi$. Then

$$x_{\alpha_{n-1}} = w_g(s_\alpha \upharpoonright n-1) = w_{f_\xi}(s_\alpha \upharpoonright n-1).$$

Hence by construction(Clause (c)), $w_{f_\xi}(s_\alpha \upharpoonright n-1)$ is not $1-$homogeneous. Thus there exists $u, v \in w_{f_\xi}(s_\alpha \upharpoonright n-1)$ such that $\{u, v\} \in K_0$. Choose disjoint basis open subsets $I, J \subseteq X$ such that $n \in I, v \in J$ and $I \times J \subseteq K_0$.
By definition of $w_{g_0}(s_\alpha \upharpoonright n-1) = w_{f_\xi}(s_\alpha \upharpoonright n-1)$, there exists $p \in \mathrm{dom}(g_0)$ such that $p \cup s_\alpha \upharpoonright n-1$ is $0-$homogeneous and $g_0(p) \in I$. Next pick a basic open $U \subseteq X^{n-1}$ with $s_\alpha \upharpoonright n-1 \in U$ such that $p \cup q$ is $0-$homogeneous for all $q \in U$. Then there exists $q \in U \cap \mathrm{dom}(g_0)$ such that $g_0(q) \in J$. It follows that

$$[p \cup g_0(p)] - [q \cup g_0(q)] \in F$$

satisfy

$$[[p \cup g_0(p)] - [q \cup g_0(q)]]^2 \in K_0.$$

This finally completes the proof: $\mathbb{Q}_Y$ is c.c.c..

Chapter 7: The Continuum

In this section, modulo another Ramsey theorem, we will prove:

Theorem 7.1(FAA). $2^\omega = \omega_2$.

Recall by **Theorem 5.12**, FAA implies $\mathfrak{b} = \omega_2$. Thus it is enough to show that $\mathfrak{b} = 2^\omega$.

Recall that ω^ω is a Polish space with basic open sets:

$$U_\sigma = \{x \in \omega^\omega \mid x \restriction |\sigma| = \sigma\}$$

for $\sigma \in \omega^\omega$. Thus every $X \subseteq \omega^\omega$ is a separable metric space. We will make use of the following easy observation.

Proposition 7.2. *There exists an unbounded family $\{F_\alpha \mid \alpha < \mathfrak{b}\}$ of strictly increasing function such that:*

(1) $f_\alpha(l) > l$ for all $l \in \omega$;

(2) $f_\alpha <^* f_\beta$ for all $\alpha < \beta < \mathfrak{b}$.

Proof. Let $\{g_\alpha \mid \alpha < \mathfrak{b}\} \subseteq \omega^\omega$ be any unbounded family. Then we cam inductively define f_α satisfying (1) such that f_α dominates $\{f_\beta, g_\beta \mid \beta < \alpha\}$. Clearly $\{f_\alpha \mid \alpha < \mathfrak{b}\}$ satisfies our requirements.
From now on, we wrote

$$X = \{f_\alpha \mid \alpha < \mathfrak{b}\}.$$

For each $x \neq y \in X$, define:

$$\Delta(x, y) = \min\{n \mid x(n) \neq y(n)\}.$$

We will make use of the following "anti-Ramsey" theorem:

Theorem 7.3. *There exists a continuous map:*

$$\mathrm{alt} : [X]^2 \to \omega$$

such that whenever $Y \subseteq X$ is unbounded, then there exists $n \in \omega$ such that for every $k \in \omega$, there exists $\{x, y\} \in [Y]^2$ with $\Delta(x, y) = n$ and $\mathrm{alt}(x, y) = k$.

Proof Delayed.

We next convert alt into continuous map:

$$t : [X]^2 \to 2^{<\omega}$$

as follows: For each pair $(k, n) \in \omega^2$, let $\phi(k, n)$ be the unique binary sequence t of length n such that

$$k \equiv \sum_{i=0}^{n-1} t(i) 2^i \mathrm{mod} 2^n.$$

<u>Example</u> Since $13 = 1 \times 2^0 + 0 \times 2^1 + 1 \times 2^2 + 1 \times 2^3$, we see that:

$$\phi(13, 3) = (1, 0, 1);$$

$$\phi(13, 5) = (1, 0, 1, 1, 0).$$

Definition. The continuous map

$$t : [X]^2 \to 2^{<\omega}$$

is defined by $t(x, y) = \phi(\text{alt}(x, y), \Delta(x, y))$.

Applying **Theorem 7.3**, we obtain:

Theorem 7.4. *If $Y \subseteq X$ is unbounded, then for every $r \in 2^\omega$, there exists $\{x, y\} \in [Y]^2$ such that $t(x, y)$ is an initial segment of r.*

Proof. By **Theorem 7.3**, there exists $n \in \omega$ such that $\Delta(x, y) = n$ and $\text{alt}(x, y) = k$. In particular, we can let

$$k = \sum_{i=0}^{n-1} r(i) w^i.$$

Definition. An uncountable subset $H \subseteq X$ is a code iff the elements:

$$\{t(x, y) \mid \{x, y\} \in [H]^2\}$$

are a chain in the complete binary tree 2^ω.

<u>Remark/Define</u> Suppose that $H \subseteq X$ is a code. Then for every $n \in \omega$, there exists $\sigma \in \omega^n$ such that $\{x \in H \mid \sigma \subseteq x\}$ is uncountable. Thus $\Delta(x, y)$ is unbounded on H. Hence there exists a unique $r \in 2^\omega$ such that

(*) $t(x, y) \subseteq r$ for every $\{x, y\} \in [H]^2$.

We say that r is coded by H.

Theorem (. *OCA) For each $r \in 2^\omega$, there exists $H \subseteq X$ such that H codes r.*

Proof. Let $r \in 2^\omega$, and consider the partition

$$[X]^2 = K_0 \sqcup K_1,$$

where $\{x, y\} \in K_0$ iff $t(x, y)$ is an initial segment of r.

<u>Claim.</u> X is not the union of countably 1−homogeneous subsets.

Proof of <u>Claim.</u> Suppose that $X = \bigcup_{n \in \omega} Y_n$, where each Y_n is 1−homogeneous. Then there exists $Y = Y_m$ such that Y is unbounded. But then **Theorem 7.4** gives $\{x, y\} \in [Y]^2$ such that $t(x, y)$ is an initial segment of r, which is a contradiction, wrt Y_m is 1−homogeneous. <u>Claim Proved.</u>

Hence, by OCA, there exists an uncountable 0−homogeneous $H \subseteq X$. Clearly H codes r.

From now on, assume FAA. Thus $\mathfrak{b} = \omega_2$ and OCA holds.

Remark If $H \subseteq X$ codes r, and $H' \subseteq H$ is uncountable, then H' codes r.

By FAA, there exists a family $\{H_r \mid r \in 2^\omega\}$ such that each $H_r \subseteq X$ codes r and $|H_r| = \omega_1$. Hence, there exists $f(r) < \omega_2 = \mathfrak{b}$ such that

$$H_r \subseteq \{f_\alpha \mid \alpha < f(r)\}.$$

Suppose that $2^\omega \geq \omega_3$. Then there exists $\gamma < \omega_2$ such that:

$$|\{r \in 2^\omega \mid H_r \subseteq \{f_\alpha \mid \alpha < \gamma\}\}| \geq \omega_3.$$

Thus to complete the proof that FAA implies $2^\omega = \omega_2$, it is enough to show:

Theorem 7.6(FAA). *If $Z \subseteq X$ with $|Z| = \omega_1$, then*

$$\{r \in 2^\omega \mid \textit{There exists a code } H \subseteq Z \textit{ for } r.\}$$

has size at most ω_1.

The proof of **Theorem 7.6** makes use of another Ramsey theorem:

The Abraham-Rubin-Shelah Axiom(ARS) If X is a separable metric space of size ω_1 and $c : [X]^2 \to n$ is a continuous map, then there exists a decomposition $X = \bigcup_{i \in \omega} X_i$ into countably many $c-$homogeneous subsets.

The proof of the following result will take up the next section of this course.

Theorem 7.7(FAA). *ARS holds.*

Lecture Notes #21 of Axiomatic Set Theory

Assuming FAA:

$$X = \{f_\alpha \mid \alpha\mathfrak{b} = \omega_2\}, \quad t : [X]^2 \to 2^{<\omega} \text{ continuous.}$$

An uncountable $H \subseteq X$ codes $r \in 2^\omega$ if

$$\{t(x, y) \mid \{x, y\} \in [H]^2\} = \{r \restriction n \mid n \in \omega\}.$$

Theorem 7.5(OCA). *Each $r \in 2^\omega$ is coded by some uncountable $H \subseteq X/$*

It is enough to prove:

Theorem 7.6(FAA). *$Z \subseteq X$ with $|Z| = \omega_1$, then $\{r \in 2^\omega \mid$ There exists a code $H \subseteq Z$ for $r\}$ has size at most ω_1.*

The proof of **Theorem 7.6** makes use of:

The Abraham-Rubin-Shelah Axiom(ARS) If X is a separable metric space of size ω_1 and $c : [X]^2 \to n$ is a continuous map, then there exists a decomposition $X = \bigcup_{i\in\omega} X_i$ into countably many $c-$homogeneous subsets.

We also need:

Theorem 7.7(FAA). *ARS holds.*

Theorem 7.8(ARS). *If $Z \subseteq X$, with Z have size ω_1, then there exists a family $\{A_r \mid r \in 2^\omega\}$ of possibly empty of subsets of Z such that:*

(1) If $r \neq s \in 2^\omega$, then $A_r \cap A_s = \emptyset$;

(2) If $r \in 2^\omega$ and $H \subseteq Z$ codes r, then $|H - A_r| \leq \omega$.

Assuming **Theorem 7.7** and **7.8**, we obtain **Theorem 7.6**, by observing that by (2),

$$\{r \in 2^\omega \mid r \text{ is coded by some } H \subseteq Z\} \subseteq \{r \in 2^\omega \mid A_r \text{ is uncountable}\}.$$

and by (1), there are at most ω_1 such $r \in 2^\omega$.

Proof of **Theorem 7.8** Fix some $n \in \omega$ and let $t_n : [Z]^2 \to 2^{\leq n}$ be the continuous map obtained by

$$\begin{aligned} t_n(x, y) &= t(x, y) \restriction n \text{ if } \Delta(x, y) \geq n; \\ &= t(x, y) \qquad\qquad \text{if} \Delta(x, y) < n. \end{aligned}$$

By ARS, there exists a partition $Z = \bigsqcup_{i\in\omega} Y_i$ such that t_n is constant on each $[Y_i]^2$, possibly for the trivial reason that $|Y_i| = 1$. In fact, refining the partition, we can suppose that for each $i \in \omega$, either:

- $|Y_i| = 1$; or
- $|Y_i| = \omega_1$, and $\Delta(x, y) > n$ for all $\{x, y\} \in [Y_i]^2$.

Thus if $|Y_i| > 1$, then there eists a fixed $s \in 2^n$ such that $t_n(x,y) = s$ for all $s \in [Y_]^2$. For each $s \in 2^n$, let

$$A_s = \bigcup \{Y_i \mid |Y_i| > 1 \text{ and } t_n(x,y) = s \text{ for all } \{x,y\} \in [Y_i]^2.\}$$

Of course it is possible that some $A_s = \emptyset$. For each $n \in \omega$, the following holds:

(a) If $s \neq s' \in 2^n$, then $A_s \cap A_{s'} = \emptyset$.

(b) Each non-empty A_s can be decomposed into countably many subsets $A_{s,l}$ such that t_n is constantly s on $[A_{s,l}]^2$.

(c) $Z - \bigcup_{s \in 2^n} A_s$ is countable.

For each $r \in 2^\omega$, define $A_r = \bigcap_{n=0}^{\infty} A_{r \restriction n}$. By (a), if $r \neq r' \in 2^\omega$, then $A_r \cap A_{r'} = \emptyset$. Suppose that $H \subseteq Z$ is code for $r \in 2^\omega$.

<u>Claim.</u> For each $n \in \omega$, $H - A_{r \restriction n}$ is countable.

it Proof of <u>Claim.</u> Suppose not, then applying (b) and (c), there exists an uncountable subset $H_0 \subseteq H$ and $s \in 2^n$ with $s \neq r \restriction n$ such that for all $\{x,y\} \in [H_0]^2$, $t(x,y) \restriction n = t_n(x,y) = s \neq r \restriction n.$, which contradicts the fact that H codes r.
It follows that

$$H - A_r = H - \bigcap_{n \in \omega} A_{r \restriction n} = \bigcup_{n \in \omega} (H - A_{r \restriction n}).$$

is countable. We next define: alt : $[X]^2 \to \omega$ as follows: Suppose that $\{x,y\} \in [X]^2$, then we inductively define $s(x,y) \in \omega^{<\omega}$ as follows:

- Let $s(x,y)_0 = \Delta(x,y)$;
- Suppose that $s(x,y)_n$ has been defined. Let $m(x,y)_n = \max\{x(s(x,y)_n), y(s(x,y)_n)\}$.

If either:

- $x(s(x,y)_n) < y(s(x,y)_n)$, and $x(m(x,y)_n) > y(m(x,y)_n)$; or
- $x(s(x,y)_n) > y(s(x,y)_n)$, and $x(m(x,y)_n) < y(m(x,y)_n)$;

then define $s(x,y)_{n+1} = m(x,y)_m$. Otherwise, stop the procedure and define

$$s(x,y) = (s(x,y)_0, ..., s(x,y)_n),$$

and define $\text{alt}(x,y) = n$.

<u>Remark</u> Since either $x <^* y$ or the other way around, this procedure stops at finitely many steps.

Recall that if $\sigma \in \omega^{<\omega}$, then

$$U_\sigma = \{x \in \omega^\omega \mid \sigma \subseteq x\}.$$

Definition. If $Y \subseteq \omega$, then the neighborhood s of Y is the subsets $U_\sigma \cap Y$ such that $U_\sigma \cap Y \neq \emptyset$.

We will make use of the following easy observations.

Lemma 7.9. *If $Y \subseteq X$ is unbounded and $Y = \bigcup_{n\in\omega} Y_n$, then there exists $n \in \omega$ and an unbounded $Z \subseteq Y_n$ such that every neighborhood of Z is unbounded.*

Proof. Since the union of countably many bounded set is bounded, there exists $n \in \omega$ such that Y_n is unbounded. Let

$$B = \{\sigma \in \omega^{<\omega} \mid U_\sigma \cap Y_n \text{ is bounded}\}.$$

Then $\bigcup\{U_\sigma \cap Y_n \mid \sigma \in B\}$ is bounded, and it follows easily that

$$Z = Y_n - \bigcup U_\sigma \cap Y_n \mid \sigma \in B$$

satisfies our requirement.

Definition. Suppose that $\emptyset \neq Z \subseteq X$ is such that every neighborhood of Z is unbounded, then

$$T_Z = \{\sigma \in \omega^{<\omega} \mid U_\sigma \cap Z \neq \emptyset\}.$$

Remark Clearly T_Z is a subtree of $\omega^{<\omega}$.

Lemma 7.10. *With the above hypothesis on Z, if $\sigma \in T_Z$, then there exists $\sigma \leq \tau \in T_Z$ such that $\tau^\frown l \in T_Z$ for infinitely many $l \in \omega$.*

Proof. Otherwise, $U_\sigma \cap Z$ is a bounded neighborhood, contradiction.

Notation Withe the above hypothesis on Z, each $\sigma \in T_Z$ will be called a node of Z. $\sigma \in T_Z$ will be called a splitting node if there are infinitely many $l \in \omega$ such that $\sigma^\frown l \in T_Z$.

Lemma 7.11. *With the above hypothesis on Z, if $y \in Z$, then for every $l \in \omega$, there exists $y' \in Z$ such that $y \upharpoonright l = U' \upharpoonright l$ and there exists $l' > l$ such that $y' \upharpoonright l'$ is a splitting node for Z.*

Lecture Notes #22 of Axiomatic Set Theory

Theorem 7.3. *If $Y \subseteq Z$ is unbounded, then there exists $n \in \omega$ such that for every $k \in \omega$, there exists $\{x, y\} \in [Y]^2$ with $\Delta(x, y) = n$ and $\mathrm{alt}(x, y) = k$.*

Proof. Suppose that $Y \subseteq Z$ is unbounded, and let $D \subseteq Y$ be a countable dense subset. Then there exists $a \in Y$ such that $d <^* a$ for all $d \in D$. For each $l \in \omega$, let

$$Y_l = \{y \in Y \mid a <^* y \text{ and } a(m) < y(m) \text{ for all } m \geq l\}.$$

Then **Lemma 7.9** implies that there exists $l_0 \in \omega$ and $Z \subseteq Y_{l_0}$ such that every neighborhood of Z is unbounded. Note that if $y \in Z$, then $a(m) > y(m)$ for all $m \geq l_0$.

We will now prove by induction on $k \geq 0$ that there exists $\{x_k, y_k\} \in [Y]^2$ such that the following holds:

(1) $\mathrm{alt}(x_k, y_k) = k$;

(2) $s(x_{k+1}, y_{k+1}) \restriction k + 1 = s(x_k, y_k)$;

In particular, (2) implies that $\Delta(x_k, y_k) = \Delta(x_0, y_0)$ for all $k \in \omega$. Thus it is enough to show that the induction can be carried out.

(3) $x_k(s(x_k, y_k)_k) < y_k(s(x_k, y_k))$;

(4) $y_k \in Z$;

(5) There exists $s(x_k, y_k)_k < i < m(x_k, y_k)_k$ such that $x_k \restriction i$ is a splitting node for Z;

(6) There exists $j > y_k(m(x_k, y_k)_k)$ such that $y_k \restriction j$ is a splitting node for Z.

We first consider the case where $k = 0$. Let $\sigma \in T_Z$(the neighborhood of Z) be a splitting node and let $a_0 \in \omega$ satisfies $\sigma^\frown a_0 \in T_Z$. By **Lemma 7.10**, there exists a splitting node $\tau \in T_Z$ wit $\sigma^\frown a_0 \subseteq \tau$. Let $d \in U_\tau \cap D$(neighborhood of τ). Then there exists $l_1 > l_0$ such that $d(m) < a(m)$ for all $m \geq l_1$. Hence, if $y \in Z$, then $d(m) < y(m)$ for all $m \geq l_1$. Now choose $a_0 < b_0 \in \omega$ such that $\sigma^\frown b_0 \in T_Z$ and such that

$$b_0 > \max\{l_1, |\tau|\}.$$

Choose some $y \in Z \cap U_{\sigma^\frown b_0}$. Applying **Lemma 7.11**, we can suppose that there exists $j > y(b_0)$ such that $y \restriction j$ is a splitting node for Z. We claim that $(x_0, y_0) = (d, y)$ satisfies (1), (3), (4), (5) and (6)[There is no Clause (2) when $k = 0$.]. Clearly (4) holds. Also we have

$$s(x_0, y_0)_0 = \Delta(x_0, y_0) = |\sigma|,$$

and so

$$x_0(s(x_0, y_0)_0) = a_0 < b_0 = y_0(s(x_0, y_0)_0) = m(x_0, y_0)_0.$$

In particular, (3) holds. Also $\tau \subseteq x_0$ is a splitting node for Z such that $s(x_0, y_0)_0 = |\sigma| < |\tau| < b_0 = m(x_0, y_0)_0$, so (5) holds. Also, $y_0 \restriction j$ is a splitting node for Z such that $j > y_0(b_0) =$

$y_0(m(x_0, y_0)_0)$, and so (6) holds. Finally, note that since $b_0 > l_1$, we have $x_0(b_0) = d(b_0) < y(b_0) = y_0(b_0)$, and so $\operatorname{alt}(x_0, y_0) = 0$ and (1) holds.

Suppose inductively that $k \geq 0$ and that $\{x_k, y_k\} \in [Y]^2$ has been defined. Let $s(x_k, y_k)_k < i < m(x_k, y_k)_k$ be such that $x_n \restriction i$ is a splitting node for T_Z and let $j > y_k(m(x_k, y_k)_k)$ be such that $y_k \restriction j$ is a splitting node for Z. Choose $d \in D$ such that $d \restriction j = y_k \restriction j$, and let $l_1 > l_0$ be such that $d(m) < a(m)$ for all $m \geq l_1$. Then there exists $y \in Z \cap U_{x_k \restriction i}$ such that:

- $y(i) > \max\{j, l_1\}$;
- there exists $j' > y(m(x_k, y_k))_k$ such that $y \restriction j'$ is a splitting node of Z.

We claim that $(x_{k+1}, y_{k+1}) = (d, y)$ satisfies (1)-(6). Clearly (4) holds. Also, $(x_{k+1} \restriction i, y_{k+1} \restriction i) = (y_k \restriction i, x_k \restriction i)$ and $i > s(x_k, y_k)_k$. It is clear that (2) holds. Also,

$$x_{k+1}(s(x_{k+1}, y_{k+1})_k) = y_k(s(x_k, y_k)_k); \quad > x_k(s(x_k, y_k), k); = y_{k+1}(s(x_k, y_k)_k).$$

and so,

$$m(x_{k+1}, y_{k+1})_k = x_{k+1}(s(x_{k+1}, y_{k+1})_k) = m(x_k, y_k)_k.$$

Also, since

$$y_{k+1}(m(x_{k+1}, y_{k+1})_k) > y_{k+1}(i) > j > y_k(m(x_k, y_k)_k) = x_{k+1}(m(x_k, y_k)_k) = x_{k+1}(m(x_{k+1}, y_{k+1})_k),$$

we see that $s(x_{k+1}, y_{k+1})_{k+1} = m(x_{k+1}, y_{k+1})_k$ is defined and that (3) holds. Since

$$m(x_{k+1}, y_{k+1}) = y_{k+1}(m(x_k, y_k)_k) > y_{k+1}(i) > l_1 > l_0,$$

It follows that

$$x_{k+1}(m(x_{k+1}, y_{k+1})_{k+1}) = d(m(x_{k+1}, y_{k+1})_{k+1}) < a(m(x_{k+1}, y_{k+1})_{k+1}) < y_{k+1}(m(x_{k+1}, y_{k+1})_{k+1}).$$

So that $\operatorname{alt}(x_{k+1}, y_{k+1}) = k + 1$, and so (1) holds.

Checking the above calculations, we also see that

$$s(x_{k+1}, y_{k+1})_{k+1} = m(x_{k+1}, y_{k+1})_k < j \quad < y_{k+1}(m(x_{k+1}, y_{k+1})_k); = m(x_{k+1}, y_{k+1})_{k+1},$$

and so (5) holds.

Finally, note that $j' > y_{k+1}(y_{k+1}(m(x_k, y_k)_k)) = y_{k+1}(m(x_{k+1}, y_{k+1})_{k+1})$ and so (6) holds.

Chapter 8: The Abraham-Rubin-Shelah Axiom

The Abraham-Rubin-Shelah Axiom(ARS) If X is a separable metric space of size ω_1 and $c : [X]^2 \to n$ is a continuous map, then there exists a decomposition $X = \bigcup_{i\in\omega} X_i$ into countably many $c-$homogeneous subsets.

The main result in this section is

Theorem 8.1. *ARS holds.*

But first we illustrate a typical use of ARS:

Theorem 8.2. *If $A \subseteq \mathbb{R}$ with $|A| = \omega_1$ and $f : A \to \mathbb{R}$ is an injection, then there exists an decomposition $A = \bigcup_{i\in\omega} A_i$ such that each $f \restriction A_i$ is either strictly increasing pr strictly decreasing.*

Lecture Notes #23 of Axiomatic Set Theory

The Abraham-Rubin-Shelah Axiom(ARS) If X is a separable metric space of size ω_1 and $c : [X]^2 \to n$ is a continuous map, then there exists a decomposition $X = \bigcup_{i\in\omega} X_i$ into countably many $c-$homogeneous subsets.

Theorem 8.1. *ARS holds.*

Before proving **Theorem 8.1**, we give an easy consequence, which shows that ARS$\vdash \neg$CH.

Theorem 8.2(ARS). *$A \subseteq \mathbb{R}$ and $|A| = \omega_1$, and $f : A \to \mathbb{R}$ is injective, then there exists a decomposition $A = \bigcup_{i\in\omega} A_i$ such that $f \restriction A_i$ is either strictly increasing or strictly decreasing.*

Proof. Let $X = \text{graph}(f)$ and define $c : [X]^2 \to 2$ by

$$c(\{\langle a.f(a)\rangle, \langle b, f(b)\rangle\}) = 0 \text{ iff } f \restriction \{a, b\} \text{ is order-preserving.}$$

Clearly, c is continuous. Hence, there exists a decomposition $X = \bigcup_{i\in\omega} X_i$ into $c-$homogeneous subsets. Let $A_i = \text{dom}(X_i)$. Then $A = \bigcup_{i\in\omega} A_i$ satisfies our requirements.[1]

Now we begin the proof of **Theorem 8.1**. In order to simplify notation, we first observe:

Observation It is enough to consider the case when $n = 2$.

Proof. The usual induction(via color amalgamation). First we present a short discussion of $H(\omega_1)$, the collection of hereditary countable sets.

Definition. If $A \in V$, then the transitive closure of A, $\text{trcl}(A)$ is the smallest transitive set B such that $A \subseteq B$.

Remark We can also define transitive closure explicitly as follows:

- Let $\text{cl}_0(A) = A$;
- Let $\text{cl}_{n+1}(A) = \bigcup\{x \in x \in \text{cl}_n(A)\}$.

Then $\text{trcl}(A) = \bigcup_{n\in\omega} \text{cl}_n(A)$.

Definition. $H(\omega_1) = \{x \in V \mid |\text{trcl}(x)| < \omega_1\}$. We can compute $|H(\omega_1)|$ as follows: Recall the definition of the cumulative hierarchy:

- $V_0 = \emptyset$;
- $V_{\alpha+1} = \mathcal{P}(V_\alpha)$;
- $V_\delta = \bigcup_{\alpha<\delta} \mathcal{V}_\alpha$ iff $\lim(\delta)$.

[1] In an unpublished work of Dorovich, this theorem is equivalent with ARS, assuming MA(ω_1).

Then $V = \bigcup_{\alpha \in On} V_\alpha$. Let rank$(x)$ = "the least α such that $x \in V_\alpha$". Arguing by induction rank(x) for $x \in H(\omega_1)$, we see $H(\omega_1) \subseteq V_{\omega_1}$. It is also clear that if $\alpha < \omega_1$, then $H(\omega_1) \cap V_{\alpha+1} = [H(\omega_1) \cap V_\alpha]^{\leq \omega}$. It follows that $|H(\omega_1)| = 2^\omega$. It is also easily checked that $H(\omega_1)$ is a model of ZFC - Powerset Axiom.

Remark Since $\omega \in H(\omega_1)$ but $2^\omega \notin H(\omega_1)$, the Powerset Axiom fails in $H(\omega_1)$.

Remark $H(\omega_1) \cap On = \omega_1$.

The proof of **Theorem 8.1** will make use of:

Lemma 8.3(CH). *$H(\omega_1)$ is a transitive model of ZFC - Powerset Axiom of cardinality ω_1.*

Most of our effort will go into proving:

Theorem 8.4(CH). *Suppose X is a separable metric space of size ω_1 and $c = [X]^2 \to 2$ is continuous, then there exists a c.c.c. poset $\mathbb{P}_{X,c}$, which adjoins a map $f : X \to \omega$ such that each $f^{-1}(n)$ is $c-$homogeneous.*

Proof of **Theorem 8.1** Suppose X is a separable metric space of size ω_1 and $c : [X]^2 \to 2$ is continuous. Let $\mathbb{P}$ be the poset of all $p : \alpha \to 2^\omega, \alpha < \omega_1$ ordered by $p \leq q$ iff $p \supseteq q$. Let $G \subseteq \mathbb{P}$ be $V-$generic, then $V[G] \vDash$CH. Hence by **Theorem 8.4**, there exists a c.c.c. poset $\mathbb{P}_{X,c} \in V[G]$ which adjoins a decomposition of X into countably many $c-$homogeneous subsets. Let $\tilde{\mathbb{Q}}$ be a $\mathbb{P}-$name such that $\mathbb{P}_{X,c} = \tilde{\mathbb{Q}}_G$. Then $\mathbb{P} * \tilde{\mathbb{Q}}$ satisfies Axiom A.

Since $|X| = \omega_1$, an application of FAA yields an $f \in V$ with $f : X \to \omega$ such that $f^{-1}(n)$ is $c-$homogeneous for each $n \in \omega$.

Now we begin the proof of **Theorem 8.4**. From now on, we assume CH. Let X be a separable metric space of size ω_1 and let $\mathcal{U} = \{U_n \mid n \in \omega\}$ be a fixed countable base of open sets for the topology. Then we can suppose $X = \omega_1$. Let $T \subseteq \omega \times \omega_1$ be the binary relation $(n, \alpha) \in T$ iff $\alpha \in U_n$.

Thus T codes the topology of X. Finally suppose that $c = [X]^2 \to 2$ is continuous. Define

$$M = \langle H(\omega_1); \in, \omega_1, T, c \rangle,$$

ie, we add a unary predicate of ω_1, a binary predicate T and a binary function c. Express $M = \bigcup_{\alpha \in \omega_1} M_\alpha$ as a strictly increasing continuous chain of countable elementary submodel[2] such that $M_\alpha \cap \omega_1$ is an ordinal for all $\alpha < \omega_1$. Define

$$C = \{\alpha < \omega_1 \mid M_\alpha \cap \omega_1 = \alpha\}.$$

Then clearly C is a club of ω_1.

We are seeking a c.c.c. poset $\mathbb{P}_{X,c}$ which adjoins a function $f : X \to \omega$ such that each $f^{-1}(n)$ is $c-$homogeneous. Our first aim(following an amazing idea of Shelah) is decide in advance for each

[2] N is a substructure of M iff for any $\alpha_1, ..., \alpha_n \in N$ and any sentences $\phi(x_1, ..., x_n)$,

$$N \vDash \phi[\alpha_1, ..., \alpha_n] \text{ iff } M \vDash \phi[\alpha_1, ..., \alpha_n]$$

.

$\alpha \in X$, the color of the $c-$homogeneous subset to which α will belong. Let $C = \{\alpha_i \mid i \in \omega_1\}$ be the increasing enumeration of the club, and for each $i \in \omega_1$, let

$$S_i = \{\beta \mid \alpha_i \leq \beta < \alpha_{i+1}\},$$

then we say that

$$\mathcal{S} = \{S_i \mid i \in \omega_1\}$$

is the set of $C-$slices. For each $i < \omega_1$, fix an enumeration $S_i = \{\alpha_l^i \mid l \in \omega\}$ with $\alpha_0^i = \alpha^i$.

Definition. Fix some $i \in \omega_1$. Then $\bar{t} = \langle t_0, ..., t_{n-1}\rangle \in 2^n$ is a good $n-$tuple iff for any formula $\phi(x_0, ..., x_{n-1})$ with parameters in M_{α_i}, if $M \vDash \phi[\alpha_0^i, ..., \alpha_{n-1}^i]$, then there exists two $n-$tuples of ordinals $\langle \beta_0, ..., \beta_{n-1}\rangle$ and $\langle \beta_0', ..., \beta_{n-1}'\rangle$ and an element $\gamma \in C$ such that

$$\alpha_i \leq \beta_0, ..., \beta_{n-1} < \gamma < \beta_0', ..., \beta_{n-1}' < \omega_1$$

and

$$M \vDash \phi[\beta_0, ..., \beta_{n-1}] \wedge \phi[\beta_0', ..., \beta_{n-1}']$$

and $c(\beta_l, \beta_l') = t_l$ for $0 \leq l \leq n-1$.

Lemma 8.5. *Let $i < \omega_1$. If $\bar{t} = \langle t_0, ..., t_{n-1}\rangle$ is a good $n-$tuple, then there exists $t_n \in 2$ such that $\langle t_0, ..., t_{n-1}, t_n\rangle$ is a good $n+1-$tuple.*

Proof. We argue by induction on n. First suppose that $n = 0$. Suppose there doesn't exists a good $1-$tuple. Then exists formulas $\phi_0(x_0), \phi_1(x_0)$ which witness that $\langle 0\rangle, \langle 1\rangle$ aren't good. Let $\phi(x) = \phi_0(x) \wedge \phi_1(x)$. Then $M \vDash \phi[\alpha_\alpha^i]$.

<u>Claim.</u> There exists uncountably many $\beta \geq \alpha_i$ such that $M \vDash \phi[\beta]$.

Proof of <u>Claim.</u> If not, then

$$\delta = \sup\{\beta < \omega_1 \mid M \vDash \phi[\beta]\}$$

is definable with parameters in M_{α_i}. Since M_{α_i} is a substructure of M, it follows that $\delta \in M_{\alpha_i}$. But this contradicts that $\alpha_0^i = \alpha_i \notin M_{\alpha_i}$ and $M \vDash \phi[\alpha_0^i]$.

Hence, we can find β_0, β_0' such that $\alpha_i \leq \beta_i < \gamma < \beta_0' < \omega_1$ for some $\gamma \in C$ and such that $M \vDash \phi[\beta_0] \wedge \phi[\beta_0']$. Let $\epsilon = c(\beta_0, \beta_0')$, then we have contradicted te choice of $\phi_\epsilon(x_0)$.

Next suppose that $n > 0$ and that $\bar{t} = \langle t_0, ..., t_{n-1}\rangle$ is a good $n-$tuple. Suppose also that $\bar{t}^\frown 0, \bar{t}^\frown 1$ aren't good and let $\phi_0(x_0, ..., x_n), \phi_1(x_0, ..., x_n)$ witness this. Let $\phi = \phi_0 \wedge \phi_1$. Then

$$M \vDash \phi[a_0^i, ..., a_n^i].$$

It follows that

$$(*)\ M \vDash \exists y[\textstyle\bigwedge_{l=0}^{n-1} y \neq \alpha_l^i \wedge y > a_0^i \wedge \phi(a_0^i, ..., a_{n-1}^i, y)].$$

Since $\bar{t}$ is a good $n-$tuple, there exists $\beta_0, ..., \beta_{n-1}, \beta_0', ..., \beta_{n-1}'$ and $\gamma \in C$, such that

$$\alpha_i \leq \beta_0, ..., \beta_{n-1} < \gamma < \beta_0', ..., \beta_{n-1}' < \omega_1 < \omega_1;$$

and $\beta_0, ..., \beta_{n-1}$ satisfies (*) and $\beta_0', ..., \beta_{n-1}'$ satisfies (*), and $c(\beta_l, \beta_l') = t_l$ for $0 \leq l \leq n-1$.

So since $\beta_0, ..., \beta_{n-1} \in M_\gamma$, which is a substructure of M, we can witness the existential statement (*) by some $\beta_n \in M_\gamma$; and it follows that $\beta_n < \gamma = On \cap M_\gamma$. Also, we can witness (*) by some $\beta_n' \in M$. Since $\gamma < \beta_0'$ and $\beta_0' < \beta_n'$, it follows $\gamma < \beta_n'$. Let $c(\beta_n, \beta_n') = \epsilon$, then we have contraficted the choice of ϕ_ϵ.

Lecture Notes #24 of Axiomatic Set Theory

We follow the last lecture. For every $i \in \omega_1$, there exists $\bar{t}_i = \langle t^i_n \mid n < \omega \rangle \in 2^\omega$ such that $E_i \upharpoonright n$ is a good $n-$tuple for every $n \in \omega$. For each $i \in \omega_1$, define

$$D_i = \{a^i_n \mid t^i_n = 0\}; \quad E_i = \{a^i_n \mid t^i_n = 1\}.$$

Let $D = \bigcup_{i<\omega_1} D_i$ and $E = \bigcup_{i<\omega_1} E_i$. Then $\omega_1 - \alpha_0 = D \sqcup E$. And clearly it's enough to find a c.c.c. poset $\mathbb{P}$ which adjoins a function $f : \omega_1 - \alpha_0 \to \omega$ such that each $f^{-1}(n)$ is $c-$homogeneous.

Definition. $\mathbb{P}$ is the poset of finite functions such that $\mathrm{dom}(p) \subseteq \omega$ and for all $n \in \mathrm{dom}(p)$:

(1) $p(n) \neq 0$ is a nonempty finite subset of $\omega_1 - \alpha_0$;

(2) if $n \in \mathrm{dom}(p)$ is even, then $p(n) \subseteq D$ is $0-$homogeneous;

(3) if $n \in \mathrm{dom}(p)$ is odd, then $p(n) \subseteq E$ is $1-$homogeneous;

(4) if $m \neq n \in \mathrm{dom}(p)$, then $p(m) \cap p(n) = \emptyset$.

Clearly, for each $\gamma < \omega_1 - \alpha_0$, $Z_\gamma = \{p \in \mathbb{P} \mid \exists n \in \mathrm{dom}(p)(\gamma \in p(n))\}$ is dense in $\mathbb{P}$. Hence, if $G \subseteq \mathbb{P}$ is $V-$generic, and $X_n = \bigcup\{p(n) \mid \exists p \in G(n \in \mathrm{dom}(p))\}$, then $\omega_1 - \alpha_0 = \bigcup_{n\in\omega} X_n$ is a decomposition into $c-$homogeneous subsets. Hence it is enough to show that $\mathbb{P}$ is c.c.c..
So suppose that $\{p_\alpha \mid \alpha < \omega_1\}$ is an uncountable antichain. Then we can suppose that:

(1) There exists a fixed $F \in [\omega]^{<\omega}$ such that $\mathrm{dom}(p_\alpha) = F$ for all $\alpha < \omega_1$;

(2) For each $n \in F$, there exists a fixed $k_n \in \omega$ such that $|p_\alpha(n)| = k_n$ for all $\alpha < \omega_1$.

If there exists $n \in F$ and uncountable $I \in [\omega_1]^{\omega_1}$ such that $p_\alpha(n) = p_\beta(n)$ for all $\alpha \neq \beta \in I$ then we can suppose that $I = \omega_1$ and replace $\{p_\alpha \mid \alpha < \omega_1\}$ by te antichain $\{p_\alpha \upharpoonright F - \{n\} \mid \alpha < \omega_1\}$. Hence we can also suppose that:

(3) For each $n \in F$, $\{p_\alpha(n) \mid \alpha < \omega_1\}$ is a $\Delta-$system with root r_n.

For each $n \in F$ and $\alpha < \omega_1$, let $p_\alpha(n) = \{\alpha^n_1, ..., \alpha^n_{k_n}\}$(with ordering). Then we can suppose that there exists fixed pairwise disjoint basic open subsets $\{U^n_i \mid 1 \leq i \leq k_n\} \in \mathcal{U}$ such that:

- For all $\alpha < \omega_1$, $\alpha^n_i \in U^n_i$;
- If $1 \leq i < j \leq k_n$, then

$$c[U^n_i \times U^n_j] = 0, \quad \text{iff } n \text{ even};$$
$$c[U^n_i \times U^n_j] = 1, \quad \text{iff } n \text{ odd }.$$

Thus we can also suppose:

(4) The root $r_n \neq \emptyset$ for all $n \in F$.

Finally, we can suppose

(5) The conditions $\{p_\alpha \mid \alpha < \omega_1\}$ all have the same "shape".

Here, the "shape" means the relative order of the elements of $\bigcup\{p_\alpha(n) \mid n \in F\}$ as well as which elements of $\bigcup\{p_\alpha(n) \mid n \in F\}$ are separated by elements of C(ie, whether they lie in the same/different slices).
In order to be intelligible, we will illustrate the general argument by considering a particular not too complicated shape. Suppose that $F = \{0, 1\}$, and each p_α has shape:

- $\alpha_1^1 < \alpha_1^0 < \alpha_2^1 < \alpha_2^0$;
- α_1^1, α_1^0 are separated by an element of C;
- α_1^0, α_2^1 are not separated by C;
- α_1^1, α_2^0 are separated by C.

There exists $\gamma < \omega_1$ such that $\langle p_\beta \mid \beta < \gamma\rangle$ satisfies:

(6) For every $\alpha < \omega_1$ and every sequence $U_1^1, U_1^0, U_2^1, U_2^0 \in \mathcal{U}$ such that $\alpha_l^n \in U_l^n$ for $n = 0, 1; l = 1, 2$ there exists $\beta < \gamma$ such that $\beta_l^n \in U_l^n$ for $n = 0, 1; l = 1, 2$.

<u>Remark</u> We will eventually find $\beta \neq \tilde{\beta} < \gamma$ such that $p_\beta, p_{\tilde{\beta}}$ are compatible. Note that $\langle p_\beta \mid \beta < \gamma\rangle \in H(\omega_1)$ and hence there exists $\alpha < \omega_1$ such that $\langle p_\beta \mid \beta < \gamma\rangle \in M_\alpha$.
<u>Remark</u> This is the only place in the proof where we use CH, since there are 2^ω many possibilities for $\langle p_\beta \mid \beta < \gamma\rangle$ and we need each to appear in some M_α.
Let p_ξ be such that

$$\alpha < \xi_1^1 < \xi_1^0 < \xi_2^1 < \xi_2^0.$$

Then ξ_2^0 satisfies the following formula with parameter $\xi_1^1, \xi_1^0, \xi_2^1$ and $\langle p_\beta \mid \beta < \gamma\rangle$:

> $\phi(x)$: For any sequence $U_1^1, U_1^0, U_2^1, U_2^0 \in \mathcal{U}$ with $\xi_1^1 \in U_1^1, \xi_1^0 \in U_1^0, \xi_2^1 \in U_2^1, x \in U_2^0$, there exists $\beta < \gamma$ such that $\beta_l^n \in U_l^n$ for $n = 0, 1$ and $l = 1, 2$.

By construction, there exists $\delta < \delta'$ such that

$$M \vDash \phi[\delta] \wedge \phi[\delta']$$

and $c(\delta, \delta') = 0$. Let $V_4, V_4' \in \mathcal{U}$ be such that

$$\delta \in V_4, \delta' \in V_4' \text{ and } V_4 \times V_4' \subseteq c^{-1}(0).$$

Thus, the $2-$tuple $\langle \xi_1^0, \xi_2^1\rangle$ satisfies the following formula: $\psi(y, z)$ with parameters ξ_1^1 and $\langle p_\beta \mid \beta < \delta\rangle$:

> $\psi(y, z)$: For any sequence $U_1^1, U_1^0, U_2^1 \in \mathcal{U}$ with $\xi_1^1 \in U_1^1, y \in U_1^0, z \in U_2^1$, there exists $\beta \neq \tilde{\beta} < \gamma$ such that:
>
> - $\beta_1^1 \in U_1^1, \beta_1^0 \in U_1^0, \beta_2^1 \in U_2^1, \beta_2^0 \in V_4$;
> - $\tilde{\beta}_1^1 \in U_1^1, \tilde{\beta}_1^0 \in U_1^0, \tilde{\beta}_2^1 \in U_2^1, \tilde{\beta}_2^0 \in V_4'$.

By construction, there exists

$$\tau_1 < \tau_2 < \tau_1' < \tau_2'$$

such that:

- $M \vDash \psi[\tau_1, \tau_2] \wedge \psi[\tau_1', \tau_2']$;
- $c(\tau_1, \tau_1') = 0$;
- $c(\tau_2, \tau_2') = 1$.

Let $V_3, V_3' \in \mathcal{U}$ be disjoint basic open sets such that

$$\tau_2 \in V_3, \tau_2' \in V_3' \text{ and } V_3 \times V_3' \subseteq c^{-1}(1).$$

Let $V_, V_2' \in \mathcal{U}$ be disjoint basic open sets such that

$$\tau_1 \in V_2, \tau_1' \in V_2' \text{ and } V_2 \times V_2' \subseteq c^{-1}(0).$$

Then ξ_1^1 satisfies the following formulas with parameter $\langle p_\beta \mid \beta < \gamma \rangle$:

$\theta(\alpha)$: For any $U_1^1 \in \mathcal{U}$ with $x \in U_1^1$ there exists $\beta \neq \tilde{\beta} < \gamma$ such that:

- $\beta_1^1 \in U_1^1, \beta_1^0 \in V_2, \beta_2^1 \in V_3, \beta_2^0 \in V_4$;
- $\tilde{\beta}_1^1 \in U_1^1, \tilde{\beta}_1^0 \in V_2', \tilde{\beta}_2^1 \in V_3', \tilde{\beta}_2^0 \in V_4'$.

By construction, there exists $\sigma < \sigma'$ such that $M \vDash \theta(\sigma) \wedge \theta(\sigma')$ and $c(\sigma, \sigma') = 1$. Let $V_1, V_1' \in \mathcal{U}$ be disjoint basic open sets such that

$$\sigma \in V_1, \sigma' \in V_1' \text{ and } V_1 \times V_1' \subseteq c^{-1}(1).$$

Hence there exists $\beta \neq \tilde{\beta} < \gamma$ such that:

- $\beta_1^1 \in V_1, \beta_1^0 \in V_2, \beta_2^1 \in V_3, \beta_2^0 \in V_4$;
- $\tilde{\beta}_1^1 \in V_1', \tilde{\beta}_1^0 \in V_2', \tilde{\beta}_2^1 \in V_3', \tilde{\beta}_2^0 \in V_4'$.

and $p_\beta, p_{\beta'}$ are compatible. This is the end of the theorem:

$$\text{FAA} \vdash \text{ARS}.$$

Lecture Notes #25 of Axiomatic Set Theory

Chapter 9: Grigorieff Forcing and Games

Let $\mathcal{U}$ be a non-principal ultrafilter over ω and let $\mathcal{I} = \{\omega - s \mid s \in \mathcal{U}\}$ be the dual ideal. Then the two-player game $G_{\mathcal{U}}$ is played as follows:

$$\begin{array}{lllll} I: I_0 & & I_1 & & ... \\ II: & F_0 & & F_1 & ... \end{array}$$

where $I_n \in \mathcal{I}, \omega = \bigsqcup_{n\in\omega} I_n$ and $F_n \in [I_n]^{<\omega}$. Player II wins iff $\bigcup_{n<\omega} F_n \in \mathcal{U}$; if Player II fails, then Player I wins.

<u>Pop Quiz</u> Which player (if any) has a wining strategy?

Definition. A non-principal ultrafilter $\mathcal{U}$ over ω is a P-point if whenever $\omega = \bigsqcup_{n\in\omega} I_n$ for $I_n \in \mathcal{I}$, there exists $s \in \mathcal{U}$ such that $|s \cap I_n| < \infty$ for all $n \in \omega$.

<u>Remark</u> Clearly if $\mathcal{U}$ is a Ramsey ultrafilter[1], then $\mathcal{U}$ is a P-point. Hence MA implies the existence of a P-point. MA also implies that there exists a P-point which isn't Ramsey. Shelah has shown that it is consistent that there are no P-points.

<u>Exercise</u> If $\mathcal{U}$ is a non-principal ultrafilter over ω, then TFAE:

1. $\mathcal{U}$ is a P-point;
2. Whenever $\{s_n \mid n \in \omega\} \subseteq \mathcal{U}$, there exists $s \in U$ such that $s \subseteq^* s_n$[2] for all $n \in \omega$.

Theorem 9.1. *If $\mathcal{U}$ is a non-principal ultrafilter over ω, then TFAE:*

(1) Player I has a winning strategy in ω.

(2) $\mathcal{U}$ is not a P-point.

Proof. (2)$\to$(1) Let $\omega = \bigsqcup_{n\in\omega} I_n$ with $I_n \in \mathcal{I}$ be such that there does not exist $s \in \mathcal{U}$ with $s \cap I_n$ is finite for all $n \in \omega$. If I plays I_n on his nth move, then he wins regardless of what Player II plays.

(1)$\to$(2) Suppose $\mathcal{U}$ is a P-point, and suppose also that σ is a winning strategy for I. Then σ is a function $Seq([\omega]^{<\omega}) \to \mathcal{I}$, where $Seq([\omega]^{<\omega})$ denotes the set of finite sequences of elements of $[\omega]^{<\omega}$. Clearly $\operatorname{ran}(\sigma)$ is a countable subset of $\mathcal{I}$ and $\bigcup \operatorname{ran}(\sigma) = \omega$. Let $\{Z_n \mid n < \omega\}$ be an enumeration of $\operatorname{ran}(\sigma)$ with $Z_0 = \sigma(\emptyset)$. Then we can define a partition $\{J_n \mid n \in \omega\}$ of ω(with some pieces possibly empty) with each $J_n \in \emptyset$ by:

- $J_0 = Z_0$;
- $J_{n+1} = Z_{n+1} - \bigcup_{0\leq n\leq l} J_l$.

[1]This concept appears in the take home exam.
[2]$X \subseteq^* Y$ means $|X - Y| < \infty$.

Since $\mathcal{U}$ is a P-point, there exists $s \in \mathcal{U}$ such that $|s \cap J_n| < \infty$ for all $n \in \omega$. It follows that $|s \cap Z_n| < \infty$ for all $n \in \omega$. Consider the play of $G_{\mathcal{U}}$ where Player I use σ and Player II plays

$$F_n = \sigma(\emptyset, F_0, ..., F_{n-1}) \cap s.$$

Then $\bigcup_{n\in\omega} F_n = s \in \mathcal{U}$, and so Player II wins, which contradicts the fact that σ is a winning strategy.

Question Suppose that $\mathcal{U}$ is a P-point. Does Player II has a winning strategy in $G_{\mathcal{U}}$?

This seems hard, so we consider a different question.

Definition. Let $\mathcal{U}$ be a non-principal ultrafilter over ω, then $\mathbb{P}_{\mathcal{U}}$ is the poset of partial functions $p : I \to 2$ where $I \in \mathcal{I}$ ordered by inverse inclusion.

Question When(if ever) does $\mathbb{P}_{\mathcal{U}}$ preserves ω_1?

Theorem 9.2. *If $\mathcal{U}$ is not a P-point, then $\mathbb{P}_{\mathcal{U}}$ collapses 2^ω to a countable set.*

We will make use of the following:

Observation There exists a map: $f : {}^{\omega}2 \to 2^\omega$ such that for every coinfinite[3] $I \subseteq \omega$ and every $\phi \in {}^{I}2$,

$$f[\{\theta \in {}^{\omega}2 \mid \phi \subseteq \theta\}] = 2^\omega.$$

Proof. Let $\{\langle I_\alpha, \phi_\alpha, \gamma_\alpha\rangle \mid \alpha < 2^\omega\}$ enumerates the triples $\langle I, \phi, \gamma\rangle$, where $I \subseteq \omega$ is coinfinite, $\phi \in {}^{I}2$ and $\gamma < 2^\omega$. Then we can inductively find $\phi_\alpha \subseteq \theta_\alpha \in {}^{\omega}2$ such that $\theta_\alpha \notin \{\theta_\beta \mid \beta < \alpha\}$ and set $f(\theta_\alpha) = \gamma_\alpha$.

Proof of **Theorem 9.2**. Let $\omega = \bigsqcup_{n\in\omega} I_n$, where $I_n \in \mathcal{I}$ be such that there does not exists $s \in \mathcal{U}$ such that $|s \cap I_n| < \infty$ for all $n \in \omega$. Clearly infinite many of the I_n are infinite. Hence, after a suitable adjustment if necessary, we can suppose that each I_n is infinite.

For each $n \in \omega$, let $f_n : {}^{I_n}2 \to 2^\omega$ be such that for every co-infinite $J \subseteq I_n$ and every $\phi \in {}^{J}2$,

$$f_n[\{\theta \mid \theta \in {}^{I_n}2, \phi \subseteq \theta\}] = 2^\omega.$$

Consider the $\mathbb{P}_{\mathcal{U}}$ name

$$\tilde{g} = \bigcup_{n\in\omega} \{\langle\langle\check{n}, \check{f_n(\theta)}\rangle, \theta\rangle \mid \theta\, {}^{I_n}2\}.$$

Then clearly,

$$\Vdash_{\mathbb{P}_{\mathcal{U}}} \tilde{g} : \check{\omega} \to \check{2^\omega} \text{ is a funtion.}$$

For each $\gamma < 2^\omega$, let

$$D_\gamma = \{p \in \mathbb{P}_{\mathcal{U}} \mid \exists n(p \Vdash \tilde{g}(\check{n})) = \check{\gamma}\}.$$

Then it is enough to show that each D_γ is dense. So fix some $\gamma < 2^\omega$ and let $p \in \mathbb{P}_{\mathcal{U}}$ be arbitrary. Let $I = \mathrm{dom}(p) \in \mathcal{I}$. Then $s = \omega - I \in \mathcal{U}$ and so there exists $n \in \omega$ such that

$$|s \cap I_n| = |(\omega - I) \cap I_n| = \infty.$$

[3]The word "coinfinite" means that this set is itself infinite and its coset is infinite too.

So choose $\theta \in {}^{I_n}2$ such that $p \restriction I_n \subseteq \theta$ and $f_n(\theta) = \gamma$. Then $p \cup \theta \leq p$ and $p \cup \theta \Vdash \tilde{g}(\check{n}) = \check{\gamma}$.

Question If $\mathcal{U}$ is a P-point, does $\mathbb{P}_{\mathcal{U}}$ preserve ω_1?

Question If $\mathcal{U}$ is a P-point, does $\mathbb{P}_{\mathcal{U}}$ satisfies Axiom A?

Theorem 9.3. *Let $\mathcal{U}$ be a non-principal ultrafilter over ω. If $\mathbb{P}_{\mathcal{U}}$ satisfies Axiom A, then Player II has a winning strategy in $G_{\mathcal{U}}$.*

Proof. Suppose that $\mathbb{P}_{\mathcal{U}}$ satisfies Axiom A.

Claim. If $p \in \mathbb{P}_{\mathcal{U}}, n \in \omega$ and $I \in \mathcal{I}$, there exists $q \leq_n p$ such that $|I - \mathrm{dom}(q)| < \infty$.

Proof of Claim. Clearly we can suppose that $|I| = \infty$. Let ${}^I2 = \{p_\alpha \mid \alpha < 2^\omega\}$, then each $p_\alpha \in \mathbb{P}_{\mathcal{U}}$ and ${}^I2$ is an antichain. Since $\mathbb{P}_{\mathcal{U}}$ satisfies Axiom A, there exists $q \leq_n p$ such that q is compatible with only countably many elements of ${}^I2$, and it follows that $|I - \mathrm{dom}(q)| < \infty$. Claim Proved.

Now we describe a winning strategy for Player II. So suppose Player I plays $I_s \in \mathcal{I}$ as his first move, then Player II plays $\emptyset$, and defines $p_0 \in \mathbb{P}_{\mathcal{U}}$ to be the function with $\mathrm{dom}(p_0) = I_0$ and $p_0(l) = 1$ for all $l \in I_0$.

Suppose that $n \geq 1$, and Player I has played $I_0, I_1, ..., I_{n-1}$ and Player II has played $F_0, ..., F_{n-1}$ and defined $p_0 \geq_0 p_1 \geq_1 ... \geq_{n-1} p_{n-1}$. Suppose Player I next plays $I_n \in \mathcal{I}$. By the CLaim., there exists $p_n \leq p_{n-1}$ such that $|I_n - \mathrm{dom}(p_n)| < \infty$ and Player II plays $F_n = I_n - \mathrm{dom}(p_n)$. At the end of the game, Player II has defined a sequence $\langle p_n \mid n \in \omega\rangle \subseteq \mathbb{P}_{\mathcal{U}}$ such that $p_{n+1} \leq p_n$ for all $n \in \omega$. It follows that there exists $p \in \mathbb{P}_{\mathcal{U}}$ such that $p \leq_n p_n$ for all $n \in \omega$. Hence $p \leq p_n$ and so $\mathrm{dom}(p) \subseteq \mathrm{dom}(p_n)$. Thus $\bigcup_{n\in\omega} \mathrm{dom}(p_n) \subseteq \mathrm{dom}(p) \in \mathcal{I}$. Since $\omega - \bigcup_{n\in\omega} \mathrm{dom}(p_n) \in \mathcal{U}$, and

$$\omega - \bigcup_{n\in\omega} \mathrm{dom}(p_n) = \bigsqcup_{m\in\omega} (I_m - \bigcup_{n\in\omega} \mathrm{dom}(p_n)) \subseteq \bigcup_{m\in\omega} (I_m - \mathrm{dom}(p_m)) = \bigcup_{m\in\omega} F_m.$$

It follows that $\bigcup_{m\in\omega} F_m \in \mathcal{U}$, and so Player II wins.

Theorem 9.4. *If $\mathcal{U}$ is a P-point, then Player II does not have a winning strategy.*

Corollary. 9.5 If $\mathcal{U}$ is a P-point, then $\mathbb{P}_{\mathcal{U}}$ does not satisfy Axiom A.

Corollary. 9.6 If $\mathcal{U}$ is a non-principal ultrafilter over ω, TFAE:

1. $\mathcal{U}$ is a P-point;
2. G_U is non-determined.

Lecture Notes #26 of Axiomatic Set Theory

Theorem 9.4. *If $\mathcal{U}$ is a P-point, then Player II does not have a winning strategy in $G_{\mathcal{U}}$.*

We will make use of a result of Grigorieff:

Definition. (i) If $T \subseteq Seq([\omega]^{<\omega})$ is a tree and $s \in T$, then the ramification of s is:

$$R_T(s) = \{a \in [\omega]^{<\omega} \mid s^\frown a \in T\};$$

(ii) T is a $\mathcal{U}$-tree iff for any $s \in T$, there exists $X \in \mathcal{U}$ such that $[X]^{<\omega} \subseteq R_T(s)$;

(iii) A branch $H = \langle H(n) \mid n \in \omega\rangle$ through T is a $\mathcal{U}-$branch iff $\bigcup_{n\in\omega} H(n) \in \mathcal{U}$.

Theorem 9.7. *If $\mathcal{U}$ is a P-point, then every $\mathcal{U}-$tree has a $\mathcal{U}-$branch.*

Proof Delayed.
We can now prove **Theorem 9.4**. Suppose that σ is a winning strategy for Player II in $G_{\mathcal{U}}$.
Claim. For each sequence $I_0, I_1, ..., I_n$ of pairwise disjoint elements of the ideal $\mathcal{I}$, and any $l \in \omega - \bigcup_{k\le n} I_K$, there exists $Y \in \mathcal{U}$ with $Y \subseteq \omega - \bigcup_{k\le n} I_k$ such that for all $t \in [Y]^{<\omega}$, there exists $I \in \mathcal{I}$ satisfying:

(i) $t \cup \{l\} \subseteq I \subseteq \omega - \bigcup_{k\le n} I_k$;

(ii) $\sigma(I_0, ..., I_n, I) \cap t = \emptyset$.

Proof of Claim. Fix some $I_0, ..., I_n, l$ and suppose the Claim fails. Then we can inductively find pairs $\langle X_m, s_m\rangle$ for $m \in \omega$ such that:

(i) $X_0 = \omega - \bigcup_{k\le n} I_k$;

(ii) $X_{m+1} = X_m - s_m$;

(iii) $s_m \in [X_m]^{<\omega}$ and for all $I \in \mathcal{I}$, such that $s_m \cup \{l\} \subseteq I \subseteq X_0$,

$$\sigma(I_0, ..., I_n, I) \cap s_m \neq \emptyset.$$

Let $J_0 = \bigcup_{i\in\omega} s_{2i}$ and $J_1 = \bigcup_{i\in\omega} s_{2i+1}$. Then $J_0 \cap J_1 = \emptyset$, and so either $J_0 \in \mathcal{I}$, or $J_1 \in \mathcal{I}$. Suppose, for example, that $J_0 \in \mathcal{I}$, then $I = \{l\} \cup J_0 \in \mathcal{I}$, and for all $i \in \omega$,

$$\{l\} \cup s_{2i} \subseteq I \subseteq \omega - \bigcup_{k\le n} I_k,$$

and so, $\sigma(I_0, ..., I_n, I) \cap s_{2i} \neq \emptyset$, which means $|\sigma(I_0, ..., I_n, I)| = \infty$, contradiction.
Next, Player I constructs a tree:

$$T \subseteq Seq(\{\langle I, F, s\rangle \mid I \in \mathcal{I}; F, s \in [I]^{<\omega}\})$$

by induction on the height of the nodes, such that if

$$\langle\langle I_0, F_0, s_0\rangle, ..., \langle I_n, F_n, s_n\rangle\rangle \in T, \text{ then}$$

(i) $I_i \cap I_j = \emptyset$ for $0 \leq i \leq j \leq n$;

(ii) $F_i = \sigma(I_0, ..., I_i)$;

(iii) $s_i \cap F_i = \emptyset$;

(iv) $I_0 = F_0 = s_0 = \emptyset$;

(v) The projective map

$$\langle\langle I_0, F_0, s_0\rangle, ..., \langle I_n, F_n, s_n\rangle\rangle \mapsto \langle s_0, ..., s_n\rangle$$

is injective.

Suppose inductively we have defined which sequences

$$B = \langle\langle I_0, F_0, s_0\rangle, ..., \langle I_n, F_n, s_n\rangle\rangle \in T.$$

Fix some such $B \in T$, and let $l = \min(\omega - \bigcup_{k\leq n} I_k)$. By the Claim, there exists $Y \in \mathcal{U}$ with $Y \subseteq \omega - \bigcup_{k\leq n} I_k$ such that for all $t \in [Y]^{<\omega}$, there exists $I_t \in \mathcal{I}$ such that:

- $t \cup \{l\} \subseteq I_t \subseteq \omega - \bigcup_{k\leq n} I_k$;
- $\sigma(I_0, ..., I_n, I_t) \cap t = \emptyset$.

WE define $R_T(B) = \{\langle I_t, \sigma(I_0, ..., I_n, I_t), t\rangle \mid t \in [Y]^{<\omega}\}$. Let $\tilde{T} \subseteq Seq([\omega]^{<\omega})$ be the tree obtained from T by projecting onto the last entry. Clearly $\tilde{T}$ is a $\mathcal{U}-$tree, and so there exists a $\mathcal{U}-$branch $H = \langle H(n) \mid n \in \omega\rangle$. Let $\langle\langle I_n, F_n, H(n)\rangle \mid n \in \omega\rangle$ be the corresponding branch in T. Then:

$$\begin{array}{lccccc} I: I_0 & & I_1 & & & ... \\ II: & F_0 & & F_1 & & ... \end{array}$$

is a play of $G_{\mathcal{U}}$ in which Player II uses σ and such that $\bigcup_{n\in\omega} H(n) \in \mathcal{U}$, But since $\bigcup_{n\in\omega} H(n) \cap \bigcup_{n\in\omega} F_n = \emptyset$, this means $\bigcup_{n\in\omega} H(n) \notin \mathcal{U}$, so Player II loses, contradiction.

Theorem 9.8. *If $\mathcal{U}$ is a P-point, then $\mathbb{P}_{\mathcal{U}}$ preserves ω_1.*

Proof. Suppose not. Then there exists $p \in \mathbb{P}_{\mathcal{U}}$ and a $\mathbb{P}_{\mathcal{U}}-$name $\tilde{f}$ such that $p \Vdash$ "$\tilde{f} : \check{\omega} \to \check{\omega}_1$ is a surjective map".
We will define a tree $T \subseteq Seq([\omega]^{<\omega})$ together with conditions $q_t, t \in T$, inductively, as follows: First $\emptyset \in T$, and $q_\emptyset = p$. Then for each $a \in [\omega - \mathrm{dom}(p)]^{<\omega}$, we let $\langle a\rangle \in T$.
Now suppose that $t = \langle a_0, ..., a_n\rangle \in T$, and $q_{\langle a_0,...,a_{n-1}\rangle}$ has been defined with $\mathrm{dom}(q_{\langle a_0,...,a_{n-1}\rangle}) \cap \bigcup_{0\leq l\leq n} a_l = \emptyset$. Let $u_0, ..., u_m$ enumerates the functions $u : \bigcup_{0\leq l\leq n} a_l \to 2$, then we can inductively define conditions:

$$q_{\langle a_0,...,a_{n-1}\rangle} \geq r_0 \geq ... \geq r_m$$

with $\mathrm{dom}(r_j) \cap \bigcup_{0\leq l\leq n} a_l = \emptyset$ and such that there exists $\alpha_j \in \omega_1$ such that

$$r_j \cup u_j \Vdash \tilde{f}(\check{n}) = \check{\alpha}_j.$$

Then we define $q_{\langle a_0,...,a_n\rangle} = r_m$. Clearly $q_{\langle a_0,...,a_n\rangle} \Vdash \tilde{f}(\check{n}) \in \{\check{\alpha}_0, ..., \check{\alpha}_m\}$. Finally we let $\langle a_0, ..., a_n, b\rangle \in T$ for all

$$b \in [\omega - (\mathrm{dom}(q_{\langle a_0,...,a_n\rangle}) \cup \bigcup_{0\le i\le n} a_i)]^{<\omega}.$$

Clearly T is a $\mathcal{U}$-tree, and hence there exists a $\mathcal{U}-$branch $H = \{\langle a_0, ..., a_n\rangle \mid n \in \omega\}$. It follows that

$$q = \bigcup_{n\in\omega} q_{\langle a_0,...,a_n\rangle} \in \mathbb{P}_{\mathcal{U}}.$$

By construction, there exists finite sets $F_n \subseteq \omega_1$ such that $q \Vdash \tilde{f}(\check{n}) \in \check{F}_n$ for all $n \in \omega$, which contradicts the choice of p.

Definition. Let $\mathbb{P}$ be a poset, then the corresponding proper game $G_{\mathbb{P}}$ is the two player game, defined as follows:

- First Player I chooses a condition $p \in \mathbb{P}$;
- Then Player I and II play alternatively:

$$\begin{array}{llllll} I: \tilde{\alpha}_0 & & \tilde{\alpha}_1 & & ... & \\ II: & B_0 & & B_1 & & ... \end{array}$$

  where $\tilde{\alpha}_n$ is a $\mathbb{P}-$name for an ordinal and B_n is a countable set of ordinals. Player II wins iff there exists $q \le p$ such that for all $n \in \omega$,

$$q \Vdash \tilde{\alpha}_n \in \bigcup_{k\in\omega} B_k.$$

Definition. $\mathbb{P}$ is proper iff Player II has a winning strategy in $G_{\mathbb{P}}$.

<u>The Proper Forcing Axiom(PFA):</u>

> If $\mathbb{P}$ is proper and $\mathcal{D}$ is a set of dense subsets of size $|\mathcal{D}| \le \omega_1$, then there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for each $D \in \mathcal{D}$.

<u>Observation 9.9.</u> If $\mathbb{P}$ satisfies Axiom A, then $\mathbb{P}$ is proper.
<u>Proof.</u> Suppose Player I first chooses $p \in \mathbb{P}$, then Player II plays the game as follows:

$$\begin{array}{ll} I & : \tilde{\alpha}_0 \quad \tilde{\alpha}_1 \quad \tilde{\alpha}_1 \quad ... \\ II(\text{Thinks}) & : (p_0 \ge_0 p_1 \ge_1 p_1 \ge_2 ...) \\ II & : B_0 \quad B_1 \quad B_1 \quad ... \end{array}$$

where $p \ge p_0$ and $p_n \Vdash \tilde{\alpha}_n \in \check{B}_n$. Then there exists $q \in \mathbb{P}$ such that $q \le_n p_n$ for all $n \in \omega$. So $q \Vdash \tilde{\alpha}_n \in B_n$ for every $n \in \omega$.

Lecture Notes #27 of Axiomatic Set Theory

Definition. Let $\mathbb{P}$ be a poset, then the corresponding proper game $G_{\mathbb{P}}$ is the two player game, defined as follows:

- First Player I chooses a condition $p \in \mathbb{P}$;
- Then Player I and II play alternatively:

$$\begin{array}{lcccc} I: \tilde{\alpha}_0 & & \tilde{\alpha}_1 & & ... \\ II: & B_0 & & B_1 & ... \end{array}$$

  where $\tilde{\alpha}_n$ is a $\mathbb{P}-$name for an ordinal and B_n is a countable set of ordinals. Player II wins iff there exists $q \leq p$ such that for all $n \in \omega$,

$$q \Vdash \tilde{\alpha}_n \in \bigcup_{k \in \omega} B_k.$$

Definition. $\mathbb{P}$ is proper iff Player II has a winning strategy in $G_{\mathbb{P}}$.

<u>The Unnamed Game $G^*_{\mathbb{P}}$</u> As above, except Player II wins iff there exists $q \leq p$ such that for all $n \in \omega$, $q \Vdash \tilde{\alpha}_n \in B_n$.

Theorem 9.9. *If $\mathbb{P}$ satisfies Axiom A, then Player II has a winning strategy in $G^*_{\mathbb{P}}$.*

Proof. Let Player I begin with $p \in \mathbb{P}$. Then Player II defines a sequence

$$p \geq_0 p_1 \geq_1 p_2 \geq_2 p_3 \geq_3 ... \geq_{n-1} p_n \geq_n ...$$

such that $p_n \Vdash \tilde{\alpha}_n \in B_n$, where $B_n \in [On]^{<\omega}$. Then there exists $q \leq_n p_n$ for all $n \in \omega$, and $q \Vdash \tilde{\alpha}_n \in B_n$.

Theorem 9.10. *If $\mathcal{U}$ is a P-point, then Player II has a winning strategy in $G^*_{\mathbb{P}}$.*

Proof. Suppose Player I first chooses $p \in \mathbb{P}_{\mathcal{U}}$, then during the game, Player II constructs a $\mathcal{U}-$tree T and conditions $q_t \in \mathbb{P}_{\mathcal{U}}$ such that $\mathrm{dom}(p_t) \cap \bigcup T = \emptyset$ as follows:

- $T_0 = \{\emptyset\}$ and $q_\emptyset = p$;
- Suppose Player II has defined T_n, together with $\{q_t \mid t \in T_n\}$. Suppose Player I plays $\tilde{\alpha}_n$. Fix some $t \in T_n$; say $t = \langle a_0, ..., a_{n-1} \rangle$. Then as in the proof of **Theorem 9.7**, for each $a = [\omega - (\mathrm{dom}(q_t) \cup \bigcup_{0 \leq l \leq n-1} a_l)]^{<\omega}$, we can find a condition $q_{t^\frown a}$ and a finite subset $F_{t^\frown a}$ such that $q_{t^\frown a} \Vdash \tilde{\alpha}_n \in F_{t^\frown a}$.

We add each $t^\frown a \in T_{n+1}$ and we define $B_t = \bigcup F_{t^\frown a} \mid t^\frown a \in T_{n+1}$ and $B_n = \bigcup_{B_t | t \in T_n}$. Since T is a $\mathcal{U}-$tree, there exists a $\mathcal{U}-$branch, say $\{\langle a_0, ..., a_{n-1} \rangle \mid n \in \omega\}$, and so $q = \bigcup_{n \in \omega} q_{\langle a_0, ..., a_{n-1} \rangle} \in \mathbb{P}_{\mathcal{U}}$ satisfies $q \Vdash \tilde{\alpha}_n \in B_n$ for all $n \in \omega$.

Definition. For each club $C \subseteq \omega_1$, let $e_C : \omega_1 \to C$ be the function which lists C in increasing order.

Definition. Let $\mathbb{P}$ consist of all finite functions $p : \omega_1 \to \omega_1$ such that there exists a club C with $p \subseteq e_C$, ordered by inverse inclusion.

Theorem . *$\mathbb{P}$ is proper, but Player I has a winning strategy in $G^*_{\mathbb{P}}$.*

Definition. An ordinal α is indecomposable if for all $\beta, \gamma < \alpha$, $\beta + \gamma < \alpha$.

Remarks

- $\alpha < \omega_1$ is indecomposable iff $\alpha = \omega^\beta$ for some $\beta < \omega_1$.
- The indecomposable ordinals from a club.
- If $\alpha < \omega_1$ is indecomposable, $\gamma < \alpha$ and $S \subseteq \gamma$, then $S \cup (\alpha - \gamma)$ has order type α.

Lemma . *If $p \in \mathbb{P}$ and $\alpha < \omega_1$ is an indecomposable ordinal such that $\alpha > \max\{\mathrm{ran}(p)\}$, then $p \cup \{\langle \alpha, \alpha \rangle\} \in \mathbb{P}$.*

Proof. Let $C \subseteq \omega_1$ be a club such that $p \subseteq e_C$ and let $\gamma = \max\{\mathrm{ran})p\}$, then

$$p \cup \{\langle \alpha, \alpha \rangle\} \subseteq e_D,$$

where $D = (C \cap \gamma) \cup (\omega_1 - \gamma)$.

Let $A \subseteq \mathbb{P}$ be a maximal antichain. Then we can define

$$g_A : \omega_1 \to \omega_1,$$

by

> $g_A(\alpha) =$"the least $\beta < \omega_1$ such that for all $p \in \mathbb{P} \cap (\alpha \times \alpha)$, there exists $q \in A \cap (\beta \times \beta)$ such that p, q is compatible".

Let $C(A)$ be the set of indecomposable $\beta < \omega_1$ such that $g_A(\alpha) < \beta$ for all $\alpha < \beta$. Then Clearly $C(A)$ is a club.

Theorem . *Player II has a winning strategy in $G_{\mathbb{P}}$.*

Proof. Suppose Player I first plays $p \in \mathbb{P}$. Next, let Player I play α_0. Let

$$D_0 = \{q \in \mathbb{P} \mid \exists \beta (q \Vdash \tilde{\alpha}_0 = \beta)\},$$

and let $A_0 \subseteq D_0$ be a maximal antichain. Let $\gamma_0 \in C(A_0)$ be such that $p \subseteq \gamma_0 \times \gamma_0$. Then Player II plays

$$B_0 = \{\beta \mid q \in A_0 \cap (\gamma_0 \times \gamma_0)(q \Vdash \tilde{\alpha}_0 \in \beta)\}.$$

Next suppose Player I plays $\tilde{\alpha}_1$. Let

$$D_1 = \{q \in \mathbb{P} \mid \exists \beta (q \Vdash \tilde{\alpha}_1 = \beta)\},$$

and let $A_1 \subseteq D_1$ be a maximal antichain. Let $\gamma_0 < \gamma_1 \in C(A_0) \cap C(A_1)$. Then Player II plays

$$B_1 = \{\beta \mid \exists q \in A_1 \cap (\gamma_1 \times \gamma_1)(q \Vdash \tilde{\alpha}_0 = \beta)\}.$$

The game continuous in this fashion. We claim that Player II wins. To see this, let $\gamma = \sup_n \gamma_n$. Then γ is indecomposable and so $q = p \cup \{\langle \gamma, \gamma \rangle\} \in \mathbb{P}$. we claim that for each $n \in \omega$, $q \Vdash \tilde{\alpha}_n \in \bigcup_{k \in \omega} B_k$. To see this, it is enough to show:

<u>Claim.</u> For all $n \in \omega$ and $q_1 \leq q$, there exists $q_2 \leq q_1$ and $\beta \bigcup_{k \in \omega} B_k$ such that $q_2 \Vdash \tilde{\alpha}_n = \beta$.

Proof of <u>Claim</u>. Let $q_1' = q_1 \cap (\gamma \times \gamma)$. Then there exists $n \leq k \in \omega$ such that $q_1 \in \mathbb{P} \cap (\gamma_k \times \gamma_k)$. Since $\gamma_k \in C(A_n)$, there exists $r \in A_n \cap (\gamma_k \times \gamma_k)$ such that $q_1' \cup r \in \mathbb{P}$. It follows that there exists $\beta \in B_k$, such that $q_1' \cup r \Vdash \tilde{\alpha}_n = \beta$.

Finally, suppose C_1, C_2 are clubs such that $q_1 \subseteq e_{C_1}$ and $q_1' \cup r \subseteq e_{C_2}$, and let $\delta = \max \operatorname{ran}(q_1' \cup r)$. Then $q_1 \cup r \subseteq e_C$, where $C = (C_2 \cap \delta) \cup (\gamma - \delta) \cup (\mathcal{C}_1 - \gamma)$. Thus $q_2 = q_1 \cup r \in \mathbb{P}$ satisfies our requirement.

<u>Claim Proved.</u>

Theorem . *Player I has a winning strategy in G_P^*.*

Proof. Player I begins by playing $p = \emptyset$. Then Player I sets $\gamma_0 = 0$ and plays $\tilde{\alpha}_0 = \{\langle \check{\beta}, \langle \gamma_0, \beta \rangle \rangle \mid \beta < \omega_1\}$. Suppose Player II plays B_0. Then Player I chooses an ordinal $beta_0 \geq \gamma_0$ such that $B_0 \subseteq \beta_0$ and an indecomposable $\gamma_1 > \beta_0$, and then plays

$$\tilde{\alpha}_1 = \{\langle \check{\beta}, \langle \gamma_1 + 1, \beta \rangle \rangle \mid \gamma_1 + 1 \leq \beta < \omega_1\}.$$

Suppose Player II plays B_1. Then Player I chooses $\beta_1 > \gamma_1$ such that $B_1 \subseteq \beta_1$ and an indecomposable $\gamma_1 > \beta_1$, and then Player I plays

$$\tilde{\alpha}_2 = \{\langle \check{\beta}, \langle \gamma_2 + 1, \beta \rangle \rangle \mid \gamma_2 + 1 \leq \beta < \omega_1\}.$$

The game continues in this fashion. We claim that Player I wins. To see this, suppose that $q \in \mathbb{P}$ is such that for all $n \in \omega$, $q \Vdash \tilde{\alpha}_n \in B_n$. It follows that if C is any club such that $q \subseteq e_C$, then for all $n \in \omega$, $\gamma_n \leq e_C(\gamma_n) < \beta_n < \gamma_{n+1}$.

Hence $\gamma = \sup_n \gamma_n \in C$. So after extending q if necessary, we suppose $\langle \gamma, \gamma \rangle \in q$. Let $q' = q \cap (\gamma \times \gamma)$. Then there exists $n \in \omega$ such that $q' \subseteq \gamma_n \times \gamma_n$. Let C be a club such that $q \subseteq e_C$ and let $\delta = \max \operatorname{ran}(q')$. Then

$$D = (C \cap \delta) \cup (\gamma_n - \delta) \cup \{\gamma_n\} \cup (\gamma - \gamma_{n=1}) \cup (C - \gamma)$$

is a club such that $q \subseteq e_D$, and $e_D(\gamma_n + 1) = \gamma_{n+1} > \beta_n$, and so

$$r = q \cup \{\langle \gamma_n + 1, \gamma_{n+1} \rangle\} \in \mathbb{P}$$

satisfies $r \leq q$ and $r \Vdash \tilde{\alpha}_n \notin B_n$, contradiction.