The Baltic Seminar Notes #4

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In the last lectures we have showed that:

Theorem 1. *The following statements are equivalent:*

1. *E is a countably complete extender;*
2. *$Ult(V;E)$ is well-founded;*
3. *Whenever $\pi : M \to V$ is a "countable hull"* of V such that $E \in ran(\pi)$, then letting $F = \pi^{-1}(E)$, there exists $k : Ult(M;F) \to V$ such that $\pi = k \circ j_F$;*
4. *There is some $\pi : M \to V$ that is elementary and countable, such that there exists $k : Ult(M;F) \to V$ with $F = \pi^{-1}(E)$ and $\pi = k \circ j_F$.*

Here, j_F is the canonical elementary embedding $j_F(x) = [\{\kappa\}, c_x]$ whenever $\kappa = crit(E)$ and $x \in M$.

Definition. Suppose $\pi : M \to V$ is an elementary embedding, and F is an M-extender. Say F is π-realizable iff there exists $k : Ult(M;F) \to V$ such that $\pi = k \circ j_F$.

Our previous results implies that E is countable complete iff all countable hulls of E are realizable. We shall now work towards a new realizing lemma and a copying lemma, which are useful in building iteration trees. For the realizing lemma, we showed in previous lectures that:

Theorem 2. *Suppose $\pi : M \to V$ is an elementary embedding and $F \in M$ is a countably complete extender. Then there is some elementary embedding $k : Ult(M;F) \to V$ such that the following diagram commutes:*

$$\begin{tikzcd} V & \\ M \arrow[u, "\pi"] \arrow[r, "j_F"'] & Ult(M;F) \arrow[ul, "k"'] \end{tikzcd}$$

For the copying lemma, we mean the following statement:

Theorem 3. *Suppose $\pi : M \to N$ is an elementary embedding and $F \in M$ is a countably complete extender. Then there is some elementary embedding $h : Ult(M;F) \to Ult(N;G)$ such that the following diagram commutes:*

$$\begin{tikzcd} (N;\pi(F)=G) \\ (M;F) \arrow[u, "\pi"] \end{tikzcd} \qquad \begin{tikzcd} N \arrow[r, "j_G"] & Ult(N;G) \\ M \arrow[u, "\pi"] \arrow[r, "j_F"'] & Ult(M;F) \arrow[u, "k"'] \end{tikzcd}$$

*Which means, M is a countable elementary substructure of V and π is the elementary embedding.

and $\pi \restriction lh(F) = k \restriction lh(F)^\dagger$.

Proof. Set $k([a, f]) = [\pi(a), \pi(f)]$ and all details are easy to check. $\square$

Now we prove the new realizing lemma:

Theorem 4. *Suppose that:*

a. $W_i^\omega \subset W_i$ *and* $Ord \subset W_i$*;*

b. $\sigma_i : M_i \to W_i$ *is elementary;*

c. $M_1 \models F$ *is a countably complete* (κ, λ)*-extender;*

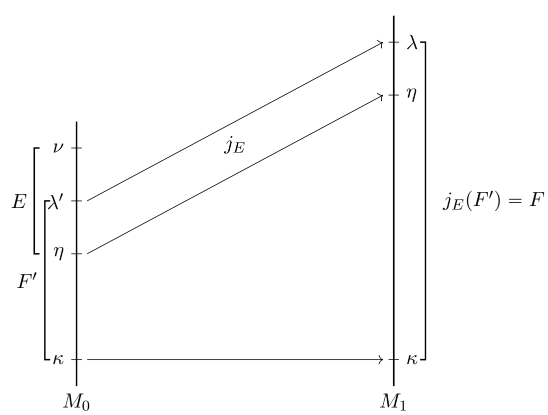
d. $M_0 \sim_{\kappa+1} M_1$ *and* M_i *is countable;*

e. $\sigma_0 \restriction V_{\kappa+1}^{M_0} = \sigma_1 \restriction V_{\kappa+1}^{M_1}$.

$$\begin{tikzcd}
 & W_0 & W_1 \\
Ult(M_0; F) \arrow[ur, "k"] & M_0 \arrow[l, "\pi"'] \arrow[u, "\sigma_0"'] \arrow[r, phantom, "\underset{\kappa+1}{\sim}"] & M_1 \ni F \arrow[u, "\sigma_1"]
\end{tikzcd}$$

Then F *is* σ_0*-realizable, where* $i = 0, 1$ *and* $M \sim_\delta N$ *means* $V_\delta^M = V_\delta^N$.

Remark. Since the relation between M_0 and M_1 is not clearly stated in the theorem, we shall narrate a possible situation of this theorem.



Here, $E \in M_0$ is a (η, ν)-extender and $F \in M_0$ is a (ν, κ)-extender. By doing ultrapower with E, λ' grows much larger, possibly larger than $Ord \cap M_0$. In this case, $F = j_E(F')$ is no longer an extender existing in M_0; however it lives as a proper class outside of M_0 and its extender properties are still preserved. Thus we can still construct $Ult(M_0; F)$ outside of M_0.

†Recall that F is a (κ, λ)-extender, and we denote $crit(F) = \kappa$ and $lh(F) = \lambda$.

Comment. We shall first show that the ultrapower $Ult(M_0; F)$ is well-defined and well-founded.

Well-definedness: Since $M_0 \sim_{\kappa+1} M_1$, it is obvious that each element in F is a κ-complete ultrafilter on $[\kappa]^n$ for some n. Therefore, although F may not be an element of M_0(or even, $\lambda > Ord \cap M$), it is still vaild to construct $Ult(M_0, F)$ by adding the F as a predicate.

Well-foundedness: For each $a \in \lambda^{<\omega}$, we let $A_a = \bigcap \sigma_1\text{''} F_a = \bigcap \sigma_0\text{''} F_a$. By the completeness of each $\sigma_1(F_a)$, $A_a \in \sigma_1(F_a)$. Then by the countably completeness of $\sigma_1(F)$, we pick $g : \lambda \to Ord$ such that $g\text{''}a \in A_a$ for all $a \in \lambda^{<\omega}$. Since $g \in V$, we proved that F is countably complete with respect to M_0, which means $Ult(M_0; F)$ is well-founded by our previous statements.

We shall now define the realizability embedding $k : Ult(M_0; F) \to W_0$. Let $x \in Ult(M_0; F_0)$ and $x = [a, f]_F^{M_0} = j_F^{M_0}(f)(a)$. Set

$$k(x) = k(j_F^{M_0}(f)(a)) = \sigma_0(f)(g\text{''}a),$$

where g is the previously defined fiber through each $A_a = \bigcup \sigma_1\text{''} F_a$. We claim that k is elementary and $\sigma_0(x) = k \circ j_F(x)$ for all $x \in M_0$.

Proof. Elementarity:

$$\begin{aligned}
&Ult(M_0; F) \models \phi[[a, f]_F^{M_0}] \\
\iff &\{s : M_0 \models \phi[f(s)]\} = C \in F_a \\
\iff &\sigma_1(C) \in \sigma_1\text{''} F_a \\
\iff &g\text{''}a \in \sigma_1(C) \\
\iff &g\text{''}a \in \sigma_0(C) \\
\iff &W_0 \models \phi[\sigma_0(f)(g\text{''}a)] \\
\iff &W_0 \models \phi[k(x)].
\end{aligned}$$

Commutativity:

$$k([\{\kappa\}, c_x]) = \sigma_0(c_x)(g\text{''}\{\kappa\}) = c_{\sigma_0(x)}(g\text{''}\{\kappa\}) = \sigma_0(x).$$

□

Remark. Prof. Boban Velickovic mentioned that there is a method of constructing extender ultrapowers by a lifting argument. Readers interested in this topic can refers to [1].

Next we show the new copying lemma:

Theorem 5. *Suppose that:*

a. $W_i^\omega \subset W_i$*;*

b. $\sigma_i : M_i \to W_i$*,* $F \in M_1$ *and* $\sigma_1(F) = G$*;*

c. $W_0 \sim_{\sigma_1(\kappa)+1} W_1$ *and* $W_0 \sim_{\kappa+1} M_1$*,* *Then:*

1. *$Ult(W_0; G)$ is well-founded;*
2. *The embedding $k : Ult(M_0; F) \to Ult(W_0; G)$ defined as $k([a, f]_F) = [\sigma_1(a), \sigma_0(f)]$ is commutative and elementary. Moreover, $k \upharpoonright lh(F) = \sigma_1 \upharpoonright lh(F)$.*

$$\begin{tikzcd}
Ult(W_0, G) & W_0 \arrow[l, "j_G"'] & \underset{\sigma_1(\kappa)+1}{\sim} & W_1 \ni G \\
Ult(M_0, F) \arrow[u, "k"] & M_0 \arrow[l, "j_F"] \arrow[u, "\sigma_0"'] & \underset{\kappa+1}{\sim} & M_1 \ni F \arrow[u, "\sigma_1"']
\end{tikzcd}$$

Proof. Clause 1: We shall need the realizability and collapse the whole picture to a countable size:

$$\begin{tikzcd}
 & W_0' & & W_1' & \\
Ult(W_0; G) \arrow[ur, "h"] & W_0 \arrow[l, "j_G"] \arrow[u, "\sigma_0'"'] & \underset{\sigma_1(\kappa)+1}{\sim} & W_1 \arrow[u, "\sigma_1'"'] & \\
 & M_0 \arrow[u, "\sigma_0"'] & \underset{\kappa+1}{\sim} & M_1 \arrow[u, "\sigma_1"'] & \left.\right\} \text{Countable}
\end{tikzcd}$$

Here h is the realizability embedding. This embedding also shows that the ultrapower is well-founded.

Clause 2: The commutativity is easy to check. Let $x = [a, f]_F^{M_0} \in Ult(M_0; F)$. Then

$$\begin{aligned}
& Ult(W_0; G) \models \phi[k([a, f]_F^{M_0})] \\
\iff & \{s : W_0 \models \phi[\sigma_0(f)(s)]\} \in G_{\sigma_1(a)}; \\
\iff & \sigma_0 (\{s : M_0 \models \phi[f(s)]\}) \in G_{\sigma_1(a)}; \\
\iff & \sigma_1 (\{s : M_0 \models \phi[f(s)]\}) \in G_{\sigma_1(a)}; \\
\iff & \{s : M_0 \models \phi[f(s)]\} \in F_a; \\
\iff & Ult(M_0; F) \models \phi[[a, f]_F^{M_0}].
\end{aligned}$$

$\square$

References

[1] Ronald Jensen. Subcomplete forcing and $\mathcal{L}$-forcing. In *E-recursion, forcing and C*-algebras*, pages 83–182. World Scientific, 2014.