The Baltic Seminar Notes #3

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Today we talk about extenders.

Definition (Derived Extender)**.** E is a derived (κ, λ)-extender * iff there exists $j : V \to W$ such that

- $crit(j) = \kappa$;
- $j(\kappa) > \lambda > \kappa$;
- E is derived from j, i.e., $E = \{(a, A) : a \in \lambda^{<\omega}, a \in j(A)\}$.

Comment. The above definition refers to the existence of some class-size object. However this is not necessary since V can be replaced by V_Θ provided that Θ is sufficiently large.

A more concrete result is given below:

Definition (Extender)**.** E is a (κ, λ)-extender iff $E = (E_a \mid a \in \lambda^{<\omega})$ and

- E_a is a κ-complete ultrafilter over $[\kappa]^{|a|}$;
- (Coherency) If $a \subset b$ then E_b projects to E_a; i.e., for every $X \in E_b$ and every $u \in X$, suppose $b = \{\beta_1 < \beta_2 < .... < \beta_n\}$ and $a = \{\beta_{j_1} < ... < \beta_{j_m}\} \subset b$, let $u^b_a = \{\xi_{j_1} < .... < \xi_{j_m}\} \subset u$. Then $X^b_a = \{u^b_a \mid u \in X\} \in E_a$.
- (Normality) Suppose $a \in \lambda^{<\omega}$ and $f : [\kappa]^{|a|} \to Ord$ such that $\{s : f(s) \leqslant \max(s)\} \in E_a$, then there is $b \supset a$ such that $\{s : f^{b,a}(s) \in s\} \in E_a$, where $f^{b,a}(t) = t^b_a$ applying the above notion.
- (Completeness) If $(a_i : i \in \omega) \subset \lambda^{<\omega}$ and $A_i \in E_{a_i}$, then there exists some $f : \bigcup_{i<\omega} a_i \to Ord$ such that $f \upharpoonright a_i \in A_i$ for every $i \in \omega$. Such f is often called a fibre through $(A_i : i < \omega)$.

If E satisfies all but the completeness condition, then it is called a pre-extender.

Comment. In the definition of normality, one could further define b precisely. This is done by simply picking $b = a \cup \{\beta\}$ for some $\beta < \max(a)$. Then the result should be $\{s : f^{b,a}(s) = s^b_{\{\beta\}}\} \in E_b$. If the extender is derived from j, then $\beta = j(f)(a)$.

We next narrate the construction of an extender ultrapower. Let $D = \{(a, f) : a \in \lambda^{<\omega} \wedge f : [\kappa]^{<\omega} \to M\}$ and define

$$(a, f) \equiv_E (b, g) \iff \{s : f^{c,a}(s) = g^{c,b}(s)\} \in E_c$$

* κ is often called the critical point of E and λ is called the length of E.

where $c = a \cup b$. Let $\mathbb{D} = \{[a, f]_E : (a, f) \in D\}$ be the collection of all equivalence classes. We next define the membership relation on $\mathbb{D}$:

$$[a, f]_E \in_E [b, g]_E \iff \{s : f^{c,a}(s) \in g^{c,b}(s)\} \in E_c,$$

where $c = a \cup b$. Thus, $Ult(V; E) = (\mathbb{D}, \in_E)$. If it is well-founded then it is recognized as its transitive closure. Just as the ultrapower constructed by an ultrafilter, we define the canonical embedding $j_E : V \to Ult(V; E)$ as $j_E(x) = [\{\kappa\}, c_x]$, where $c_x : [\kappa]^1 \to \{x\}$ is the constant function.

Lemma 1 (Los' Theorem)**.**

$$Ult(V; E) \models \phi([a_1, f_1], ..., [a_n, f_n]) \iff \{s : V \models \phi(f_1^{c,a_1}(s), ..., f_n^{c,a_n}(s))\} \in E_c,$$

where $c = a_1 \cup ... \cup a_n$*.*

Theorem 2. *If E is a (κ, λ)-extender then $Ult(V; E)$ is well-founded.*

Proof. Suppose not. Let $(\alpha_i = [a_i, f_i] : i < \omega)$ be a decreasing ordinal sequence in $Ult(V; E)$. Without loss of generality, we further assume that $a_i \subset a_{i+1}$ for all $i < \omega$. Let

$$A_{i+1} = \{s : f_{i+1}(s) \in f_i^{a_{i+1},a_i}(s)\} \in E_{a_{i+1}}.$$

Let $g : \bigcup_{i<\omega} a_i \to Ord$ be a fiber such that $g \upharpoonright a_i = t_i \in A_i$. So

$$f_{i+1}(t_{i+1}) \in f_i^{a_{i+1},a_i}(t_{i+1}) = f_i(t_i)$$

for every $i < \omega$. Thus $(f_i(t_i) : i < \omega)$ is an ill-founded sequence in V. Contradiction. $\square$

Theorem 3 (Jensen)**.** *Suppose E is a (κ, λ)-pre-extender. Then E is complete iff for all sufficiently large Θ and for all countable $\pi : M \to V_\Theta$ such that $E \in \mathrm{ran}(\pi)$, letting $F = \pi^{-1}(E)$, there is a $\sigma : Ult(M; E) \to V_\Theta$ such that the following diagram commutes:*

$$\begin{tikzcd}
 & V_\Theta & \\
M \arrow[ur, "\pi"] \arrow[rr, "j_F"'] & & Ult(M; F) \arrow[ul, "\sigma"']
\end{tikzcd}$$

Remark. To show $[a, f] = j_E(f)(a)$, one can first by normality, show $a = [a, id]$, where $id : [\kappa]^{|a|} \to V$ is the identity map; then by Los' theorem, prove $[a, f] = j_E(f)(a)$.

Remark. There is another way to construct the extender ultrapower by simply taking $Ult_a = Ult(V; E_a)$. This is a directed system since by coherency one can define a elementary embedding $\sigma_{b,a} : Ult_a \to Ult_b$ for all $a \subset b \in \lambda^{<\omega}$ as $\sigma_{b,a}([f]) = [f^{b,a}]$. Then it is easy to check that the following diagram commutes:

$$\begin{tikzcd}
 & Ult_{a\cup b} & \\
Ult_a \arrow[ur, "\sigma_{a,a\cup b}"] & & Ult_b \arrow[ul, "\sigma_{b,a\cup b}"'] \\
 & V \arrow[ul, "j_a"] \arrow[uu, "j_{a\cup b}"] \arrow[ur, "j_b"'] &
\end{tikzcd}$$

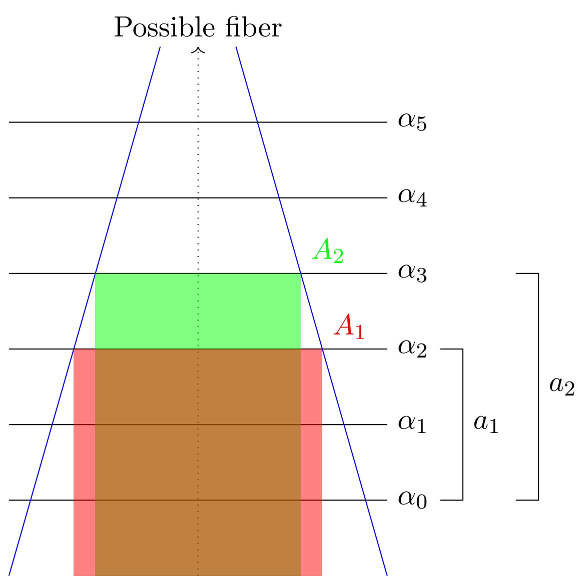
It is then obvious that $Ult(V;E) = \operatorname{dirlim}_{a\in[\lambda]^{<\omega}} Ult_a$.

Proof. The " $\Longleftarrow$ " direction is followed by the following proposition:

Proposition 4. *If E is a pre-extender, then E is complete iff $Ult(V;E)$ is well-founded.*

Remark. One way to prove the " $\Longleftarrow$ " direction of the theorem is that, if $Ult(V_\Theta;E)$ is ill-founded for some sufficiently large Θ, then the ill-founded sequence can be seen for some countable hull M, thus $Ult(M;F)$ is ill-founded. However by our assumption, $Ult(M;F)$ can be embeds into V_Θ, which leads to a contradiction.

Remark. To prove the proposition, let $(A_i : i \in \omega)$ be a sequence with $A_i \in E_{a_i}$ for some $a_i \in (\lambda)^{<\omega}$. Furthermore, $(A_i : i < \omega)$ witnesses the failure of completeness of E. Without loss of generality, we let $a_i \subset a_{i+1}$ and $A_i \supset A_{i+1}^{a_{i+1},a_i}$ for all $i < \omega$.
Consider the following sequence $([a_i, f_i] : i < \omega)$: Let $f_i(s) = n - i$ iff $\exists t \in A_n(t^{a_n}_{a_i} = s)$ for $i \leqslant n < \omega$. Otherwise, let $f_i(s) = 0$. Then, by Los' theorem, $Ult(V;E) \vDash [a_i, f_i] > [a_{i+1}, f_{i+1}$ for all $i < \omega$. Contradiction.



We now prove the " $\Longrightarrow$ " direction of the theorem. Let $\pi : M \to V_\Theta$ be countable such that $E \in \operatorname{ran}(\pi)$. We want to see: there exists $\sigma : Ult(M;F) \to V_\Theta$ where $F = \pi^{-1}(E)$ such that $\pi = \sigma \circ j_F$.

Let $\lambda' = \pi^{-1}(\lambda)$. For $a \in (\lambda')^{<\omega}$, let $X_a = \pi" F_a \subseteq E_{\pi(a)}$. Let $A_a = \bigcap X_a \in E_{\pi(a)}$[†]. Let $(a_i : i \in \omega)$ be an enumeration of $(\lambda)^{<\omega}$. Set $A_i = A_{a_i} \in E_{\pi(a_i)}$. Let $g : \bigcup_{i\in\omega} \pi(a_i) \to Ord$ be a fiber through $(A_i : i \in \omega)$. Given $[a_i, f] \in Ult(M; F)$, set $\sigma([a_i; f]_F) = \pi(f)(g \upharpoonright \pi(a_i))$.

Claim. σ is elementary and $\pi = \sigma \circ j_F$.

Proof. Commutativity:

$$\begin{aligned}\sigma \circ j_F(x) &= \sigma(j_F(x));\\ &= \sigma([\{\pi^{-1}(\kappa)\}, c_x]_F);\\ &= \pi(c_x)(g \upharpoonright \{\kappa\});\\ &= \pi(x).\end{aligned}$$

Elementarity:

$$\begin{aligned}Ult(M; F) \models \phi([a_i, f]_F) &\iff \{s : M \models \phi(f(s))\} \in F_{a_i};\\ &\iff \{s : V_\Theta \models \phi(\pi(f)(s))\} \in \pi(F_{a_i}) = E_{\pi(a_i)};\\ &\implies V_\Theta \models \phi(\pi(f)(g \upharpoonright \pi(a_i)));\\ &\implies V_\Theta \models \phi(\sigma([a_i, f]_F)).\end{aligned}$$

□

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Remark. Following the proof we made in the last lecture, we can prove that:

Proposition 5. *If E is a (κ, λ)-extender, then V is linear iterable via E.*

The proposition can be showed from a same argument of the similar statement we proved in the last lecture. We first make a countable hull M of a sufficiently large V_Θ which witnesses the fact that it is not α-iterable, and then in V we show that M is countably iterable and realizable in V. This leads to a contradiction since M should be ill-founded at some β-th stage with β countable.

Now we can state a useful tool called the copying construction.

Proposition 6. *Suppose $\sigma : M \to N$ is elementary. Let $F \in M$ be an (κ, λ)-extender. Then there is $k : Ult(M; F) \to Ult(N; \sigma(F))$ such that the following diagram commutes:*

$$\begin{tikzcd} N \arrow[r, "j_{\sigma(F)}"] & Ult(N;\sigma(F)) \\ M \arrow[u, "\sigma"] \arrow[r, "j_F"'] & Ult(M;F) \arrow[u, "k"'] \end{tikzcd}$$

[†]The non-emptyness of A_a follows from the countable completeness of E_a

Proof. Let $X \in Ult(M;F)$ and $X = [a,f]_F$. Set $k(X) = [\sigma(a), \sigma(f)]_G$ where $G = \sigma(F)$. k is elementary since:

$$\begin{aligned}
&Ult(M;F) \vDash \phi([a,f]_F) \\
\iff &\{s : M \vDash \phi(f(s))\} \in F_a; \\
\iff &\{s : N \vDash \phi(\sigma(f)(s))\} \in G_{\sigma(a)}; \\
\iff &Ult(N;G) \vDash \phi([\sigma(a), \sigma(f)]_G).
\end{aligned}$$

$\square$