

Ralf Schindler: Talks#4 on Logic Summer School of Fudan University, 2020

TODAY:

- Show a characterization of precipitousness;
- V is generically iterable with respect to precipitous ideals;
- Discussion of effective counterexamples to **CH**.
- Illustrations of Admissible Club Guessing(ACG) $\implies \mathfrak{u}_2 = \omega_2$.
- Prove ACG follows from **MM**.

Theorem 1. *The followings are equivalent:*

- \mathbf{NS}_{ω_1} is precipitous;
- Let $x \prec H_\theta$ be countable, and let M_x be its transitive collapse via embedding σ . Define

$$G_x = \{X \in P(\omega_1^{M_x}) \cap M_x : \omega_1^{M_x} \in \sigma(X)\}.$$

Then the collection

$$\mathcal{S} = \{x \prec H_\theta : |x| = \omega \wedge G_x \text{ is } \sigma^{-1}(\mathbf{NS}_{\omega_1}^+)\text{-generic over } M_x\}$$

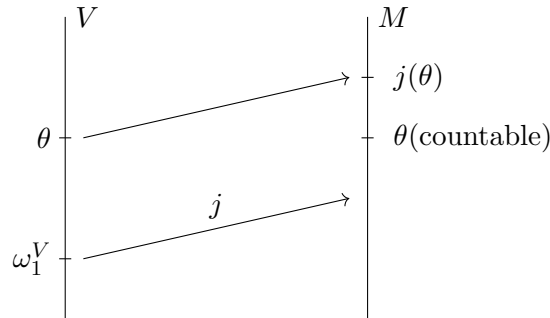
is projective stationary.

Definition. \mathcal{S} is projective stationary iff f.a. $T \subset \omega_1$ stationary, $\{X \in \mathcal{S} : X \cap \omega_1 \in T\}$ is stationary.

Proof. (Sketch) " \implies ": Fix a stationary set $T \subset \omega_1$. Let G be V -generic for $\mathbf{NS}_{\omega_1}^+$ such that $T \in G$. This implies the existence of an elementary embedding:

$$j : V \rightarrow \text{Ult}(V; G) = M.$$

Since $T \in G$, $\omega_1^V \in j(T)$.



Now we consider the structure $j'H_\theta^V$. It is a substructure of $H_{j(\theta)}^N$, $\omega_1^M \cap j''H_\theta^V = \omega_1^V$, and the transitive collapse of $j''H_\theta^V$ is H_θ^V .

" \Leftarrow " (...)

□

Corollary 2. \mathbf{NS}_{ω_1} saturated $\implies \mathbf{NS}_{\omega_1}$ precipitous.

How do you obtain models in which \mathbf{NS}_{ω_1} is saturated/precipitous? We can first pick a measurable cardinal and collapse it to ω_1 . Then there is a precipitous ideal of ω_1 . Then we can shoot a club to the stationary sets of ω_1 and make this precipitous ideal to be \mathbf{NS}_{ω_1} . However, this does not work for \mathbf{NS}_{ω_1} to be saturated. To get it one may need the existence of a Woodin cardinal δ , and some δ -c.c. semi-proper forcing to do that. A proof of this theorem can be found on here¹.

It is also true that $\mathbf{MM} \implies \mathbf{NS}_{\omega_1}$ is saturated. This theorem is obtained by Foreman-Magidor-Shelah in their original \mathbf{MM} paper [2], [3].

1 Generic iterability

Assume $I \subset P(\omega_1)$ is a precipitous normal uniform ideal on ω_1 . Work in $V^{Col(\omega, \theta)}$, where $\theta > \omega_1$ is large enough. We may do the following iteration process:

$$V = M_0 \xrightarrow{j_{01}} M_1 = \text{Ult}(M_0; G_0) \rightarrow \dots \rightarrow M_\omega = \lim \text{dir}_{n \rightarrow \omega} (M_n; G_n) \rightarrow \dots$$

For every $n < \omega$, we let G_n to be the generic filter of $(\mathbf{NS}_{\omega_1}^+)^{M_n}$. By elementarity, $(\mathbf{NS}_{\omega_1}^+)^{M_n}$ is precipitous for every $n < \omega$, so every successor stages are well-founded. We now show that whenever we can construct (M_α, G_α) , it is always well-founded.

Remark. Notice that here, the use of $Col(\omega, \theta)$ is to collapse H_θ to countable size, so we can always pick G_n (clearly not unique) in $V^{Col(\omega, \theta)}$. Because of this, we cannot do this iteration to any ordinal stages, but only as much as we want.

Theorem 3. *Let $I \subset P(\omega_1)$ is a precipitous normal uniform ideal on ω_1 , then V is generically iterable via I^+ and its images.*

Here, being generically iterable means that V can be iterated along the sequence without being ill-founded at limit stages.

Proof. A variant of the argument of the iterability of V via a measure in V and its images. Suppose the statement is false. Then we pick the least triple $(\theta, \lambda, \alpha)$ with respect to lexicographic order, such that

- θ is the least ordinal such that in $V^{Col(\omega, \theta)}$, there is a $\lambda < \theta$ such that the λ -th generic iteration taken inside $V^{Col(\omega, \theta)}$ is ill-founded.
- λ is the least ordinal such that the λ -th generic iteration contains an ill-founded sequence of ordinals;
- α is the least ordinal such that there is a ill-founded sequence of ordinals below $j_{0\lambda}(\alpha)$.

Suppose $\gamma < \lambda$ such that there is an $\bar{\alpha}_1$, $j_{\gamma\lambda}(\bar{\alpha}_1)$ is the first element of the infinite descending sequence in M_λ . By elementarity, M_γ sees that $(j_{0\gamma}\theta, j_{0\gamma}\lambda, j_{0\gamma}\alpha)$ is the lexicographically least triple, however, $(\theta, \lambda - \gamma, \bar{\alpha}_1)$ is lexicographically smaller, and it satisfies our requirements listed above. Contradiction. \square

¹https://ivv5hpp.uni-muenster.de/u/rds/sat_ideal_better_version.pdf

2 Effective counterexamples to CH

In the last lecture, we have proved that $\mathbf{MM} \implies 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$. This implies a surjection $f : \mathbb{R} \rightarrow \omega_2$. We now look at the set

$$R_f = \{(x, y) \in \mathbb{R}^2 : f(x) \leq f(y)\}.$$

What are possible levels of definability of R_f ? And can we have f such that $R_f \in L(\mathbb{R})^2$? Or even: Can(in the presence of large cardinals, or under \mathbf{MM}) such an R_f be projective?

Definition. $R \subset \mathbb{R}^n$ is projective iff R is definable(with parameters) over $(H_{\omega_1}; \in)$.

This is not the usual definition for projectiveness; however, since every element in H_{ω_1} is coded by a real, $H_{\omega_1}^{L(\mathbb{R})} = H_{\omega_1}^V$. So if something is definable over $(H_{\omega_1}; \in)$, then it is certainly inside $L(\mathbb{R})$. Equivalently, R is projective iff we can write $\vec{x} \in R$ iff

$$\exists x_0 \in \mathbb{R} \forall x_1 \in \mathbb{R} \dots Q x_k (\vec{x}, x_0, \dots, x_k) \in C,$$

where C is a Borel set of \mathbb{R}^{n+k+1} .

Let us look at H_{ω_2} . A function ϕ is $\Pi_2^{H_{\omega_2}}$ if it is equivalent(in \mathbf{ZFC}) to a function of the form:

$$\forall A \in H_{\omega_2} \exists B \in H_{\omega_2} \psi(A, B),$$

where ψ is Σ_0 . It turns out that \mathbf{MM} is complete with respect to $\Pi_2^{H_{\omega_2}}$ statements. Important example of $\Pi_2^{H_{\omega_2}}$ statements:

- $\mathfrak{u}_2^3 = \omega_2^V$.
- Admissible Club Guessing(ACG).
- $\varphi_{\mathbf{AC}}$ and $\psi_{\mathbf{AC}}$, etc..

$\mathfrak{u}_2 = \omega_2$ is a $\Pi_2^{H_{\omega_2}}$ statement: Since $\mathfrak{u}_2 = \sup\{(\omega_1^V)^{+L[x]} : x \in \mathbb{R}\}$, and $\mathfrak{u}_2 \leq \omega_2^V$, we have

- $\mathfrak{u}_2 \geq \omega_2^V \iff \forall \alpha < \omega_2 \exists x \in \mathbb{R} [(\omega_1^V)^{+L[x]} \geq \alpha]$;
- $(\omega_1^V)^{+L[x]} \geq \alpha \iff \exists \beta \exists L_\gamma[x] \exists N[N \text{ is a transitive structure of height } \beta \wedge N \models \text{"Everything is atmost countable"} \wedge L_\gamma[x] \models \text{"}\alpha \leq \beta^+\text{"}]$.

Under the hypothesis $\forall x \in \mathbb{R} (\exists x^\#)$ (given by \mathbf{MM}) and $\mathfrak{u}_2 = \omega_2$, we have

$$f : \mathbb{R} \rightarrow \omega_2; \quad \omega \supset x \mapsto \omega_1^{+L[x]}.$$

Since $\mathfrak{u}_2 = \omega_2$, f is cofinal. Now look at R_f we defined above and we want to claim this is projective, actually Δ_3^1 . For any $x, y \in \mathbb{R}$, we have

$$(\omega_1^V)^{+L[x]} \leq (\omega_1^V)^{+L[y]} \iff \exists z \subset \omega \exists (L_\tau[z]; U) \text{ iterable } [(L_\tau[z]; U) \models \kappa^{+L[x]} \leq \kappa^{+L[y]}].$$

² $L(\mathbb{R})$ = the least transitive model of \mathbf{ZF} which contains $\mathbb{R} \cup \text{ORD}$.

³ \mathfrak{u}_2 is the second uniform indiscernible ordinal: Suppose $x^\#$ exists for all $x \in \mathbb{R}$. Since the x -indiscernible ordinal class C_x is a club for every $x \in \mathbb{R}$, $\bigcap_{x \in \mathbb{R}} C_x$ is another class of indiscernibles called the uniform indiscernibles. Clearly countable ordinals can never be uniform indiscernible, and ω_1 is uniform indiscernible, $\mathfrak{u}_1 = \omega_1$.

Proof. Note that here, $\kappa^{+L[x]}$ and $\kappa^{+L[y]}$ is actually the interpretation inside $L_\tau[z]$, that is: $(\kappa^{+L[x]})^{L_\tau[z]}$ and $(\kappa^{+L[y]})^{L_\tau[z]}$. Thus we may need to assume x, y are Turing reducible to z , and we want to prove that $(\kappa^{+L[x]})^{L_\tau[z]} = \kappa^{+L[x]}$. Let $\tau = \kappa^{+L[z]}(\dots)$ \square

So the complexity of this statement is Σ_2 over H_{ω_1} . Thus, this statement is Δ_3^1 .

Definition. ACG is the following statement:

$$\forall C \subset \omega_1 \text{ club } \exists D \subset C \text{ club } \exists x \subset \omega_1 [D \text{ is the set of all countable } x\text{-admissible}].$$

Here, x -admissible are ordinals τ such that $L_\tau[x]$ are model of **KP** set theory. Or just: $D \in L[z]$ for some $z \subset \omega$.

Remark. It can be proved that ACG implies the existence of $x^\#$ for all $x \subset \omega$.

Definition. ψ_{AC} is the following statement:

$$\forall S, T \subset \omega_1 \text{ stationary and co-stationary } \exists \eta < \omega_2 \exists C \subset \omega_1 \text{ club } \forall \xi \in C [\xi \in T \iff f_\eta(\xi) \in S].$$

Here, f_η is the function defined by some surjection $g : \omega_1 \rightarrow \eta$ such that $f_\eta(\xi)$ is the ordertype of $g''\xi$. It is also called the canonical function of η .

All the listed $\Pi_2^{H_{\omega_2}}$ statements are implied by $(*)$, and \mathbf{MM}^{++} implies $(*)$. Next we want to show ACG implies $\mathfrak{u}_2 = \omega_2$ and \mathbf{MM}^{++} implies ACG.

Theorem 4. $ACG \implies \mathfrak{u}_2 = \omega_2$.

Proof. (Sketch) Let $\alpha < \omega_2$ and some bijection $f : \omega_1 \rightarrow \alpha$. Moreover, fix a tower $(X_i : i < \omega_1)$ of countable substructure of H_θ . Let N_i be the transitive collapse of X_i . We then let $f \in X_0$, which gives that there is $\alpha_i > \omega_1^{N_i}$ in N_i such that α_i would be mapped to α in H_θ . We can then modify the tower such that $\alpha_i < \omega_1^{N_{i+1}}$ for every $i < \omega_1$.

Thus ACG gives a club $D \subset \{\omega_1^{N_i} : i = \omega_1^{N_i}, i < \omega_1\}$, and $D \in L[x]$. We may then assume that D is definable only by parameters from ω_1^V . Thus,

$$\xi \in D \iff L[x] \models \phi(\xi, x, \omega_1^V);$$

Assuming that $\eta < \omega_1^V$ is x -indiscernible, we have

$$\xi \in D \cap \eta \iff L[x] \models \phi(\xi, x, \eta).$$

So now $\eta \mapsto \omega_1^V$, and $D \cap \eta \mapsto D$ by the elementary embedding from $L[x]$ to itself. This gives every x -indiscernible in D is a limit point of D . Thus $\xi \in D$ iff $\xi = \omega_1^{N_\xi}$ and the next x -indiscernible $> \xi$ is bigger than $\omega_1^{N_{\xi+1}}$, thus bigger than α_ξ .

Now we pick another tower $(Y_i : i < \omega_1)$ such that $x^\# \in Y_0$ (in particular, $D \in Y_0$), and $f \in Y_0$. So there is a club $E \subset D$ such that $Y_i \cap \alpha = X_i \cap \alpha$ for all $i \in E$. Now if $i \in E$, we denote the transitive collapse of Y_i as M_i , and thus

$$M_i \models \text{"the next } x\text{-indiscernible } > i \text{ is } > \alpha_i\text{"}.$$

By elementarity, this gives

$$H_\theta \models \text{"the next } x\text{-indiscernible } > \omega_1 \text{ is } > \alpha\text{"}.$$

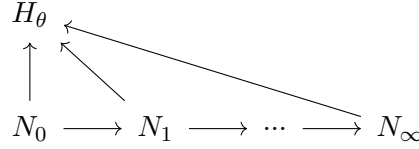
which gives $u_2 = \omega_2$. □

Theorem 5. $\mathbf{MM}^{++} \implies \mathbf{ACG}$.

Proof. (Sketch, Easier) Let $C \subset \omega_1$ be a club. Now we are going to construct a tower $(X_i : i < \omega_1)$ of countable substructures of H_θ . Let N_0 be some countable transitive substructure of H_θ , where

- $C \in N_0$;
- There is some $\alpha_0 \in N_0$ such that the elementary embedding $j_0 : N_0 \rightarrow H_\theta$ maps α_0 to ω_1 .

Let $G_0 = \{s \in P(\alpha_0) \cap N : \alpha_0 \in j_0(x)\}$. Then this filter is saturated and N_0 -generic, with respect to $\mathbf{NS}_{\omega_1}^{+N_0}$. Now by the precitousness, we can do the generic iteration:



Now let $X_i = \text{ran}(j_i)$. Since N_0 is countable, we can find some countable $x \subset \omega$ such that x codes N_0 . Now by the following unproved claim:

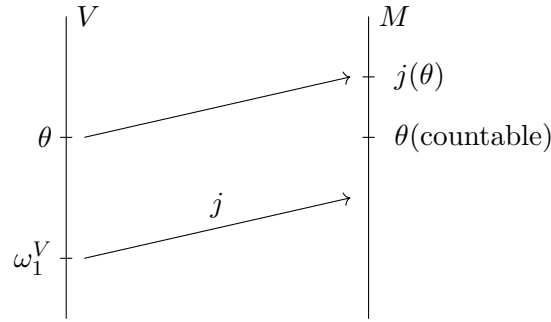
Claim. Suppose $\alpha < \omega_1$ is x -admissible, then α is the limit point of $\{\omega_1^{N_i} : i < \omega_1\}$.

We have that every x -admissible ordinal α is inside C since $\omega_1^{N_i}$ is the limit point of C for every $i < \omega_1$. □

Now we would like to present a harder proof which can be further motified into a way to prove $(*) \implies \mathbf{MM}^{++}$.

Proof. (Sketch, Harder, [1]) We would like to force the existence of some iterable countable structure $(M; I)$, together with its generic iteration $(M_i; I_i : i \leq \omega_1)$ such that $M_{\omega_1} = H_{\omega_2}^V$ ⁴. We do it via a forcing which preserves stationary subsets of ω_1 .

⁴Clearly, the iteration cannot be performed in V .



We aim to find a transitive model N in a stationary set preserving extension such that

$$N \models \text{"}\exists \text{generic iteration } (M_i, G_i : i < \omega_1), |M_i| = \omega \text{ s.t. } M_{\omega_1} = \lim_{i \rightarrow \omega_1} \text{dir}_{i \rightarrow \omega_1} M_i = H_{\omega_2}^V \text{"}.$$

Think of N as a term model. The forcing will consist of finite sets of sentences in a language describing the full theory of such a model + starting to prove that this model is well-founded by ranking the constants:

$$\phi(c_{i_0}, \dots, c_{i_k}), f : c_{i_0} \mapsto \xi \in \text{ORD}$$

such that in some outer model, this finite piece of information can be extended to a maximal consistent theory + a proof that the model which arises is well-founded.

Our forcing notion will actually have size $2^{\omega_2} \geq \omega_3$. We will need to assume $2^{\omega_2} = \omega_3$, which follows from \diamond_{ω_3} ⁵. We will finish this proof in our next lecture. \square

References

- [1] Benjamin Claverie, Ralf Schindler, et al. Increasing \mathfrak{u}_2 by a stationary set preserving forcing. *Journal of Symbolic Logic*, 74(1):187–200, 2009.
- [2] Matthew Foreman, Menachem Magidor, and Saharon Shelah. Martin's maximum, saturated ideals, and non-regular ultrafilters. part i. *Annals of Mathematics*, pages 1–47, 1988.
- [3] Matthew Foreman, Menachem Magidor, and Saharon Shelah. Martin's maximum, saturated ideals and non-regular ultrafilters. part ii. *Annals of Mathematics*, pages 521–545, 1988.

⁵There is a sequence $((Q_\alpha, A_\alpha) : \alpha < \omega_3)$ such that $(Q_\alpha : \alpha < \omega_3)$ is a tower of transitive substructures of H_{ω_3} of size \aleph_2 with $\bigcup_\alpha Q_\alpha = H_{\omega_3}$; Moreover, for all $A \subset H_{\omega_3}$, $\{\alpha : (Q_\alpha, A_\alpha) \prec (H_{\omega_3}, A)\}$ is stationary.