Chapter 3

Attempts to Solve the Problem

3.1 A new symmetric basis

Consider this proposal for phase classes. After applying this numerical optimization [describe], The generating parameters are

$$\vec{p} = \begin{bmatrix} 0.3073524627378583 \\ 1/3 \\ -1/3 \\ 1/2 \\ -1 \\ 0 \\ -0.02563931056721057 \\ 2/3 \\ 1/3 \\ 0.8073524630069947 \\ 4/3 \\ -1/3 \\ -0.5256393104295811 \\ 2/3 \\ 1/3 \end{bmatrix} \tag{3.1}$$

for the parametrized set of vectors:

$$\vec{v}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_1 = \begin{pmatrix} 1 \\ e^{2\pi i p_0} \\ e^{2\pi i p_1} \\ e^{2\pi i (p_0+p_1)} \\ e^{2\pi i p_2} \\ e^{2\pi i (p_0+p_2)} \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ e^{2\pi i p_3} \\ e^{2\pi i p_4} \\ e^{2\pi i (p_3+p_4)} \\ e^{2\pi i p_5} \\ e^{2\pi i (p_3+p_5)} \end{pmatrix},$$
$$\vec{v}_3 = \begin{pmatrix} 1 \\ e^{2\pi i p_6} \\ e^{2\pi i p_7} \\ e^{2\pi i (p_6+p_7)} \\ e^{2\pi i p_8} \\ e^{2\pi i (p_6+p_8)} \end{pmatrix}, \vec{v}_4 = \begin{pmatrix} 1 \\ e^{2\pi i p_9} \\ e^{2\pi i p_{10}} \\ e^{2\pi i (p_9+p_{10})} \\ e^{2\pi i p_{11}} \\ e^{2\pi i (p_9+p_{11})} \end{pmatrix}, \vec{v}_5 = \begin{pmatrix} 1 \\ e^{2\pi i p_{12}} \\ e^{2\pi i p_{13}} \\ e^{2\pi i (p_{12}+p_{13})} \\ e^{2\pi i p_{14}} \\ e^{2\pi i (p_{12}+p_{14})} \end{pmatrix}. \tag{3.2}$$

Equation 3.2 are by construction eigenstates of particle exchange. To find a complete analytical solution, we must determine the analytical representations of the 4 floats in equation 3.1: p_0, p_6, p_9, p_{12}. To do this, let us understand the relationships between these parameters by writing out the inner products of the vectors in Equation 3.2. I denote the column vector $\vec{I}_j$ to represent the inner products for $\vec{v}_j$ such that $\vec{I}_{jl} = \vec{v}_j \vec{v}_l^\dagger$. The result is given in Equation 3.3, with p_0, p_6, p_9, p_{12} colored in red, orange, yellow, and green respectively:

$$
\begin{aligned}
\vec{I}_0 &= \begin{bmatrix} 0 \\ e^{-2i\pi(\mathbf{p_0}+p_1)} + e^{-2i\pi(\mathbf{p_0}-p_2)} + 1 + e^{-2i\pi p_2} + e^{-2i\pi p_1} + e^{-2i\pi \mathbf{p_0}} \\ e^{-2i\pi(p_3+p_4)} + e^{-2i\pi(p_3+p_5)} + 1 + e^{-2i\pi p_5} + e^{-2i\pi p_4} + e^{-2i\pi p_3} \\ e^{-2i\pi(\mathbf{p_6}+p_7)} + e^{-2i\pi(\mathbf{p_6}+p_8)} + 1 + e^{-2i\pi p_8} + e^{-2i\pi p_7} + e^{-2i\pi \mathbf{p_6}} \\ e^{-2i\pi(p_{10}+\mathbf{p_9})} + e^{-2i\pi(p_{11}+\mathbf{p_9})} + 1 + e^{-2i\pi \mathbf{p_9}} + e^{-2i\pi p_{11}} + e^{-2i\pi p_{10}} \\ e^{-2i\pi(\mathbf{p_{12}}+p_{13})} + e^{-2i\pi(\mathbf{p_{12}}+p_{14})} + 1 + e^{-2i\pi p_{14}} + e^{-2i\pi p_{13}} + e^{-2i\pi \mathbf{p_{12}}} \end{bmatrix}, \\
\vec{I}_1 &= \begin{bmatrix} e^{2i\pi \mathbf{p_0}} + e^{2i\pi p_1} + e^{2i\pi p_2} + e^{2i\pi(\mathbf{p_0}+p_1)} + e^{2i\pi(\mathbf{p_0}+p_2)} + 1 \\ 0 \\ e^{2i\pi(\mathbf{p_0}-p_3)} + e^{2i\pi(p_1-p_4)} + e^{2i\pi(p_2-p_5)} + e^{2i\pi(\mathbf{p_0}+p_1-p_3-p_4)} + e^{2i\pi(\mathbf{p_0}+p_2-p_3-p_5)} + 1 \\ e^{2i\pi(\mathbf{p_0}-\mathbf{p_6})} + e^{2i\pi(p_1-p_7)} + e^{2i\pi(p_2-p_8)} + e^{2i\pi(\mathbf{p_0}+p_1-\mathbf{p_6}-p_7)} + e^{2i\pi(\mathbf{p_0}+p_2-\mathbf{p_6}-p_8)} + 1 \\ e^{2i\pi(\mathbf{p_0}-\mathbf{p_9})} + e^{2i\pi(p_1-p_{10})} + e^{-2i\pi(p_{11}-p_2)} + e^{2i\pi(\mathbf{p_0}+p_1-p_{10}-\mathbf{p_9})} + e^{2i\pi(\mathbf{p_0}-p_{11}+p_2-\mathbf{p_9})} + 1 \\ e^{2i\pi(\mathbf{p_0}-\mathbf{p_{12}})} + e^{2i\pi(p_1-p_{13})} + e^{-2i\pi(p_{14}-p_2)} + e^{2i\pi(\mathbf{p_0}+p_1-\mathbf{p_{12}}-p_{13})} + e^{2i\pi(\mathbf{p_0}-\mathbf{p_{12}}-p_{14}+p_2)} + 1 \end{bmatrix}, \\
\vec{I}_2 &= \begin{bmatrix} e^{2i\pi p_3} + e^{2i\pi p_4} + e^{2i\pi p_5} + e^{2i\pi(p_3+p_4)} + e^{2i\pi(p_3+p_5)} + 1 \\ e^{-2i\pi(\mathbf{p_0}-p_3)} + e^{-2i\pi(p_1-p_4)} + e^{-2i\pi(p_2-p_5)} + e^{-2i\pi(\mathbf{p_0}+p_1-p_3-p_4)} + e^{-2i\pi(\mathbf{p_0}+p_2-p_3-p_5)} + 1 \\ 0 \\ e^{2i\pi(p_3-\mathbf{p_6})} + e^{2i\pi(p_4-p_7)} + e^{2i\pi(p_5-p_8)} + e^{2i\pi(p_3+p_4-\mathbf{p_6}-p_7)} + e^{2i\pi(p_3+p_5-\mathbf{p_6}-p_8)} + 1 \\ e^{-2i\pi(p_{10}-p_4)} + e^{-2i\pi(p_{11}-p_5)} + e^{2i\pi(p_3-\mathbf{p_9})} + e^{-2i\pi(p_{10}-p_3-p_4+\mathbf{p_9})} + e^{-2i\pi(p_{11}-p_3-p_5+\mathbf{p_9})} + 1 \\ e^{-2i\pi(\mathbf{p_{12}}-p_3)} + e^{-2i\pi(p_{13}-p_4)} + e^{-2i\pi(p_{14}-p_5)} + e^{-2i\pi(\mathbf{p_{12}}+p_{13}-p_3-p_4)} + e^{-2i\pi(\mathbf{p_{12}}+p_{14}-p_3-p_5)} + 1 \end{bmatrix}, \\
\vec{I}_3 &= \begin{bmatrix} e^{2i\pi \mathbf{p_6}} + e^{2i\pi p_7} + e^{2i\pi p_8} + e^{2i\pi(\mathbf{p_6}+p_7)} + e^{2i\pi(\mathbf{p_6}+p_8)} + 1 \\ e^{-2i\pi(\mathbf{p_0}-\mathbf{p_6})} + e^{-2i\pi(p_1-p_7)} + e^{-2i\pi(p_2-p_8)} + e^{-2i\pi(\mathbf{p_0}+p_1-\mathbf{p_6}-p_7)} + e^{-2i\pi(\mathbf{p_0}+p_2-\mathbf{p_6}-p_8)} + 1 \\ e^{-2i\pi(p_3-\mathbf{p_6})} + e^{-2i\pi(p_4-p_7)} + e^{-2i\pi(p_5-p_8)} + e^{-2i\pi(p_3+p_4-\mathbf{p_6}-p_7)} + e^{-2i\pi(p_3+p_5-\mathbf{p_6}-p_8)} + 1 \\ 0 \\ e^{-2i\pi(p_{10}-p_7)} + e^{-2i\pi(p_{11}-p_8)} + e^{2i\pi(\mathbf{p_6}-\mathbf{p_9})} + e^{-2i\pi(p_{10}-\mathbf{p_6}-p_7+\mathbf{p_9})} + e^{-2i\pi(p_{11}-\mathbf{p_6}-p_8+\mathbf{p_9})} + 1 \\ e^{-2i\pi(\mathbf{p_{12}}-\mathbf{p_6})} + e^{-2i\pi(p_{13}-p_7)} + e^{-2i\pi(p_{14}-p_8)} + e^{-2i\pi(\mathbf{p_{12}}+p_{13}-\mathbf{p_6}-p_7)} + e^{-2i\pi(\mathbf{p_{12}}+p_{14}-\mathbf{p_6}-p_8)} + 1 \end{bmatrix}, \\
\vec{I}_4 &= \begin{bmatrix} e^{2i\pi p_{10}} + e^{2i\pi p_{11}} + e^{2i\pi \mathbf{p_9}} + e^{2i\pi(p_{10}+\mathbf{p_9})} + e^{2i\pi(p_{11}+\mathbf{p_9})} + 1 \\ e^{-2i\pi(\mathbf{p_0}-\mathbf{p_9})} + e^{-2i\pi(p_1-p_{10})} + e^{2i\pi(p_{11}-p_2)} + e^{-2i\pi(\mathbf{p_0}+p_1-p_{10}-\mathbf{p_9})} + e^{-2i\pi(\mathbf{p_0}-p_{11}+p_2-\mathbf{p_9})} + 1 \\ e^{2i\pi(p_{10}-p_4)} + e^{2i\pi(p_{11}-p_5)} + e^{-2i\pi(p_3-\mathbf{p_9})} + e^{2i\pi(p_{10}-p_3-p_4+\mathbf{p_9})} + e^{2i\pi(p_{11}-p_3-p_5+\mathbf{p_9})} + 1 \\ e^{2i\pi(p_{10}-p_7)} + e^{2i\pi(p_{11}-p_8)} + e^{-2i\pi(\mathbf{p_6}-\mathbf{p_9})} + e^{2i\pi(p_{10}-\mathbf{p_6}-p_7+\mathbf{p_9})} + e^{2i\pi(p_{11}-\mathbf{p_6}-p_8+\mathbf{p_9})} + 1 \\ 0 \\ e^{2i\pi(p_{10}-p_{13})} + e^{2i\pi(p_{11}-p_{14})} + e^{-2i\pi(\mathbf{p_{12}}-\mathbf{p_9})} + e^{2i\pi(p_{10}-\mathbf{p_{12}}-p_{13}+\mathbf{p_9})} + e^{2i\pi(p_{11}-\mathbf{p_{12}}-p_{14}+\mathbf{p_9})} + 1 \end{bmatrix}, \\
\vec{I}_5 &= \begin{bmatrix} e^{2i\pi \mathbf{p_{12}}} + e^{2i\pi p_{13}} + e^{2i\pi p_{14}} + e^{2i\pi(\mathbf{p_{12}}+p_{13})} + e^{2i\pi(\mathbf{p_{12}}+p_{14})} + 1 \\ e^{-2i\pi(\mathbf{p_0}-\mathbf{p_{12}})} + e^{-2i\pi(p_1-p_{13})} + e^{2i\pi(p_{14}-p_2)} + e^{-2i\pi(\mathbf{p_0}+p_1-\mathbf{p_{12}}-p_{13})} + e^{-2i\pi(\mathbf{p_0}-\mathbf{p_{12}}-p_{14}+p_2)} + 1 \\ e^{2i\pi(\mathbf{p_{12}}-p_3)} + e^{2i\pi(p_{13}-p_4)} + e^{2i\pi(p_{14}-p_5)} + e^{2i\pi(\mathbf{p_{12}}+p_{13}-p_3-p_4)} + e^{2i\pi(\mathbf{p_{12}}+p_{14}-p_3-p_5)} + 1 \\ e^{2i\pi(\mathbf{p_{12}}-\mathbf{p_6})} + e^{2i\pi(p_{13}-p_7)} + e^{2i\pi(p_{14}-p_8)} + e^{2i\pi(\mathbf{p_{12}}+p_{13}-\mathbf{p_6}-p_7)} + e^{2i\pi(\mathbf{p_{12}}+p_{14}-\mathbf{p_6}-p_8)} + 1 \\ e^{-2i\pi(p_{10}-p_{13})} + e^{-2i\pi(p_{11}-p_{14})} + e^{2i\pi(\mathbf{p_{12}}-\mathbf{p_9})} + e^{-2i\pi(p_{10}-\mathbf{p_{12}}-p_{13}+\mathbf{p_9})} + e^{-2i\pi(p_{11}-\mathbf{p_{12}}-p_{14}+\mathbf{p_9})} + 1 \\ 0 \end{bmatrix}.
\end{aligned}
\tag{3.3}
$$

This becomes, after substituting in the known values from Equation 3.1 except p_0, p_6, p_9, p_{12},

$$
\begin{gathered}
\vec{I}_0 = \begin{bmatrix} 0 \\ \left(1+e^{-2/3i\pi}+e^{2/3i\pi}\right)e^{-2i\pi p_0} \\ 0 \\ \left(1+e^{4/3i\pi}+e^{2/3i\pi}\right)e^{-i\pi(2p_6+2)} \\ \left(1+e^{10/3i\pi}+e^{8/3i\pi}\right)e^{-i\pi(2p_9+8/3)} \\ \left(1+e^{4/3i\pi}+e^{2/3i\pi}\right)e^{-i\pi(2p_{12}+2)} \end{bmatrix}, \\
\vec{I}_1 = \begin{bmatrix} e^{2i\pi p_0}+e^{2i\pi(p_0-1/3)}+e^{2i\pi(p_0+1/3)} \\ 0 \\ -e^{2i\pi p_0}-e^{2i\pi(p_0-1/3)}-e^{2i\pi(p_0+1/3)} \\ \left(1+e^{4/3i\pi}+e^{2/3i\pi}\right)e^{i\pi(2p_0-2p_6-2)} \\ 3e^{2i\pi p_0-2i\pi p_9}+3 \\ \left(1+e^{4/3i\pi}+e^{2/3i\pi}\right)e^{i\pi(2p_0-2p_{12}-2)} \end{bmatrix}, \\
\vec{I}_2 = \begin{bmatrix} 0 \\ -\left(1+e^{-2/3i\pi}+e^{2/3i\pi}\right)e^{-2i\pi p_0} \\ 0 \\ \left(-1-e^{2/3i\pi}-e^{4/3i\pi}\right)e^{-i\pi(2p_6+2)} \\ \left(-1-e^{8/3i\pi}-e^{10/3i\pi}\right)e^{-i\pi(2p_9+8/3)} \\ \left(-1-e^{2/3i\pi}-e^{4/3i\pi}\right)e^{-i\pi(2p_{12}+2)} \end{bmatrix}, \\
\vec{I}_3 = \begin{bmatrix} \left(1+e^{4/3i\pi}+e^{2/3i\pi}\right)e^{2i\pi p_6} \\ \left(1+e^{4/3i\pi}+e^{2/3i\pi}\right)e^{2i\pi(-p_0+p_6)} \\ \left(-1-e^{2/3i\pi}-e^{4/3i\pi}\right)e^{2i\pi p_6} \\ 0 \\ \left(1+e^{4/3i\pi}+e^{8/3i\pi}\right)e^{i\pi(2p_6-2p_9-4/3)} \\ 3e^{-2i\pi p_{12}+2i\pi p_6}+3 \end{bmatrix}, \\
\vec{I}_4 = \begin{bmatrix} e^{2i\pi p_9}+e^{2i\pi(p_9-1/3)}+e^{2i\pi(p_9+4/3)} \\ 3e^{-2i\pi p_0+2i\pi p_9}+3 \\ -e^{2i\pi p_9}-e^{2i\pi(p_9-1/3)}-e^{2i\pi(p_9+4/3)} \\ \left(1+e^{4/3i\pi}+e^{8/3i\pi}\right)e^{i\pi(-2p_6+2p_9-4/3)} \\ 0 \\ \left(1+e^{4/3i\pi}+e^{8/3i\pi}\right)e^{i\pi(-2p_{12}+2p_9-4/3)} \end{bmatrix}, \\
\vec{I}_5 = \begin{bmatrix} \left(1+e^{4/3i\pi}+e^{2/3i\pi}\right)e^{2i\pi p_{12}} \\ \left(1+e^{4/3i\pi}+e^{2/3i\pi}\right)e^{2i\pi(-p_0+p_{12})} \\ \left(-1-e^{2/3i\pi}-e^{4/3i\pi}\right)e^{2i\pi p_{12}} \\ 3e^{2i\pi p_{12}-2i\pi p_6}+3 \\ \left(1+e^{4/3i\pi}+e^{8/3i\pi}\right)e^{i\pi(2p_{12}-2p_9-4/3)} \\ 0 \end{bmatrix}.
\end{gathered}
\tag{3.4}
$$

Evaluating the complex exponentials in the above vectors using Wolfram Alpha, we arrive at:

$$\vec{I_0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{I_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3e^{2i\pi(p_0-p_9)}+3 \\ 0 \end{bmatrix}, \vec{I_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$
$$\vec{I_3} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3e^{-2i\pi(p_{12}-p_6)}+3 \end{bmatrix}, \vec{I_4} = \begin{bmatrix} 0 \\ 3e^{-2i\pi(p_0-p_9)}+3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{I_5} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3e^{2i\pi(p_{12}-p_6)} \\ 0 \\ 0 \end{bmatrix}. \tag{3.5}$$

We can see these vectors give us the system of equations:

$$\begin{aligned} 3e^{2\pi i(p_0-p_9)}+3 &= 0 \\ 3e^{2\pi i(p_6-p_{12})}+3 &= 0 \\ 3e^{2\pi i(p_9-p_0)}+3 &= 0 \\ 3e^{2\pi i(p_{12}-p_6)}+3 &= 0 \end{aligned} \tag{3.6}$$

That is,

$$\begin{aligned} p_0 - p_9 &= -\frac{1}{2} \\ p_6 - p_{12} &= -\frac{1}{2} \end{aligned} \tag{3.7}$$

This has infinitely many solutions, but for simplicity sake let us choose:

$$\begin{aligned} p_0 &= 0, \\ p_9 &= \frac{1}{2}, \\ p_6 &= 0, \\ p_{12} &= \frac{1}{2}. \end{aligned} \tag{3.8}$$

Therefore, an exact analytical solution for the parameter vector is

$$\vec{p} = \begin{bmatrix} 0 \\ 1/3 \\ -1/3 \\ 1/2 \\ -1 \\ 0 \\ 0 \\ 2/3 \\ 1/3 \\ 1/2 \\ 4/3 \\ -1/3 \\ 1/2 \\ 2/3 \\ 1/3 \end{bmatrix}. \tag{3.9}$$

The resulting phase classes, combining 3.9 and 3.2, are:

$$\begin{gathered} p = 0 \mapsto \left\{ \begin{matrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{matrix} \right\}, p = 1 \mapsto \left\{ \begin{matrix} 1 \\ 1 \\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\ -\frac{1}{2} - \frac{\sqrt{3}}{2}i \\ -\frac{1}{2} - \frac{\sqrt{3}}{2}i \end{matrix} \right\}, p = 2 \mapsto \left\{ \begin{matrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{matrix} \right\}, \\ p = 3 \mapsto \left\{ \begin{matrix} 1 \\ 1 \\ -\frac{1}{2} - \frac{\sqrt{3}}{2}i \\ -\frac{1}{2} - \frac{\sqrt{3}}{2}i \\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i \end{matrix} \right\}, p = 4 \mapsto \left\{ \begin{matrix} 1 \\ -1 \\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\ \frac{1}{2} - \frac{\sqrt{3}}{2}i \\ -\frac{1}{2} - \frac{\sqrt{3}}{2}i \\ \frac{1}{2} + \frac{\sqrt{3}}{2}i \end{matrix} \right\}, p = 5 \mapsto \left\{ \begin{matrix} 1 \\ -1 \\ -\frac{1}{2} - \frac{\sqrt{3}}{2}i \\ \frac{1}{2} + \frac{\sqrt{3}}{2}i \\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\ \frac{1}{2} - \frac{\sqrt{3}}{2}i \end{matrix} \right\}. \end{gathered} \tag{3.10}$$

This new phase class compliments a new set correlation classes that embed being an eigenstate of particle exchange into their structure.

$$
\begin{aligned}
&c = 0 \mapsto \left\{\begin{array}{c} |00\rangle \\ |11\rangle \\ |22\rangle \\ |33\rangle \\ |44\rangle \\ |55\rangle \end{array}\right\}, \quad c = 1 \mapsto \left\{\begin{array}{c} |01\rangle \\ |10\rangle \\ |23\rangle \\ |32\rangle \\ |45\rangle \\ |54\rangle \end{array}\right\}, \quad c = 2 \mapsto \left\{\begin{array}{c} |02\rangle \\ |14\rangle \\ |20\rangle \\ |35\rangle \\ |41\rangle \\ |53\rangle \end{array}\right\} \\
&c = 3 \mapsto \left\{\begin{array}{c} |03\rangle \\ |15\rangle \\ |24\rangle \\ |30\rangle \\ |42\rangle \\ |51\rangle \end{array}\right\}, \quad c = 4 \mapsto \left\{\begin{array}{c} |04\rangle \\ |13\rangle \\ |25\rangle \\ |31\rangle \\ |40\rangle \\ |52\rangle \end{array}\right\}, \quad c = 5 \mapsto \left\{\begin{array}{c} |05\rangle \\ |12\rangle \\ |21\rangle \\ |34\rangle \\ |43\rangle \\ |50\rangle \end{array}\right\}
\end{aligned}
\tag{3.11}
$$