Nonlinear Witnesses Writeup

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May 2023

1 Motivation

Using the nonlinear improvement method described in Shen et al (2020), specifically the inequality described in Equation 10, I tested whether the quality of the six entanglement witnesses in Riccardi et al (2020) would improve when converted into nonlinear witnesses. In Example 1 of Shen et al, a nonlinear version of the flip operator, an optimal entanglement witness for a Bell state with white noise, detects entanglement for $p \geq 0.3$. These new nonlinear witnesses were tested on the all_qual_20000.csv data sets, which contained entangled states undetected by Riccardi's witnesses. If the nonlinear versions of Riccardi's six witnesses saw improvements, ideally they could potentially detect these states.

2 Methods

The forms of the six entanglement witnesses in terms of the Pauli matrices are in Appendix A of Riccardi's paper. Each of the witnesses was written in matrix form then decomposed as specified at the start of section 3 of Shen et al to find W_1, W_2, and W_3. Using Riccardi's W_1 as an example, the expectation values of W_1, W_2, and W_3 were then calculated as follows:

$$
\begin{aligned}
W_1 &= \frac{1}{4}[\mathbb{I} \otimes \mathbb{I} + \sigma_z \otimes \sigma_z + (a^2+b^2)\sigma_x \otimes \sigma_x + (a^2-b^2)\sigma_y \otimes \sigma_y + 2ab(\sigma_z \otimes \mathbb{I} + \mathbb{I} \otimes \sigma_z)] \\
&= \frac{1}{4}\begin{bmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{bmatrix} + \begin{bmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{bmatrix} + \begin{bmatrix} 0 & (a^2+b^2)\sigma_x \\ -(a^2+b^2)\sigma_x & 0 \end{bmatrix} \\
&+ \begin{bmatrix} 0 & -i(a^2+b^2)\sigma_y \\ i(a^2+b^2)\sigma_y & 0 \end{bmatrix} + \begin{bmatrix} 2ab\mathbb{I} & 0 \\ 0 & -2ab\mathbb{I} \end{bmatrix} + \begin{bmatrix} 2ab\sigma_z & 0 \\ 0 & 2ab\sigma_z \end{bmatrix}
\end{aligned}
$$

Thus we can decompose this into

$$W_{11} = \frac{1}{4}(\mathbb{I} + \sigma_z + 2ab\mathbb{I} + 2ab\sigma_z)$$
$$W_{22} = \frac{1}{4}(\mathbb{I} - \sigma_z - 2ab\mathbb{I} + 2ab\sigma_z)$$
$$W_{12} = \frac{1}{4}(a^2 - b^2)(\sigma_x - i\sigma_y)$$
$$W_{21} = \frac{1}{4}(a^2 - b^2)(\sigma_x + i\sigma_y)$$

Now using Shen et al, we can construct W_1, W_2, and W_3. We can rewrite the products $\langle 0| |0\rangle$, $\langle 1| |1\rangle$, $\langle 0| |1\rangle$, and $\langle 1| |0\rangle$ in terms of the Pauli matrices as follows.

$$\langle 0| |0\rangle = \frac{\sigma_z + \mathbb{I}}{2}$$
$$\langle 0| |1\rangle = \frac{\sigma_x + i\sigma_y}{2}$$
$$\langle 1| |0\rangle = \frac{\sigma_x - i\sigma_y}{2}$$
$$\langle 1| |1\rangle = \frac{\mathbb{I} - \sigma_z}{2}$$

Now we can rewrite W_1, W_2, and W_3 as

$$W_1 = \frac{\sigma_z + \mathbb{I}}{2} \otimes \frac{1}{4}(\mathbb{I} + \sigma_z + 2ab\mathbb{I} + 2ab\sigma_z)$$
$$W_2 = \frac{\mathbb{I} - \sigma_z}{2} \otimes \frac{1}{4}(\mathbb{I} - \sigma_z - 2ab\mathbb{I} + 2ab\sigma_z)$$
$$W_3 = \frac{\sigma_x + i\sigma_y}{2} \otimes \frac{1}{4}(a^2 - b^2)(\sigma_x - i\sigma_y) + \frac{\sigma_x - i\sigma_y}{2} \otimes \frac{1}{4}(a^2 - b^2)(\sigma_x + i\sigma_y)$$

In order to create a similar detection criterion to the regular entanglement witnesses, the inequality from Equation 10 was changed to

$$\langle W_1\rangle \langle W_2\rangle - \frac{1}{4}|\langle W_3\rangle|^2 \geq 0$$

where if this inequality was true the state would not be entangled. The implementation code is located in nonlinear_test.py.

3 Results

The nonlinear versions of the six entanglement witnesses showed no improvement compared to the original six at detecting the entanglement states in the data. The states were still undetected by the nonlinear witnesses as the minimum value of $\langle W_1\rangle \langle W_2\rangle - \frac{1}{4}|\langle W_3\rangle|^2$ for the six nonlinear witnesses remained above 0.