Entanglement Witness Write-up

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We derive and experimentally verify the efficacy of an adaptive entanglement witnessing procedure on two qubits that utilizes local measurements in fewer bases than would be necessary for full tomography. We begin with six classes of witnesses, $\{W\}$, formulated by Riccardi et al [1], and construct 3 additional triplets sets of optimal entanglement witnesses, $\{W'\}$, that each require local measurements in two further bases. When applied to computationally generated random two-qubit states, our adaptive procedure is able to witness a significantly more complete set of entangled states than the static W witnesses alone. Entangled states which remain undetected have low concurrence. Using polarization-entangled photons generated via spontaneous parametric down conversion, we demonstrate witness behavior discussed in [1] and also create and measure two-qubit entangled states that cannot be witnessed by the static method but that are detected by our adaptive method. Finally, we discuss future experimental implementations and possible directions to improve our procedure.

I. INTRODUCTION

Riccardi *et al.* have proposed a set of extremal entanglement witnesses, $\{W\}$, in the form

$$W_k = |\varphi_k\rangle\langle\varphi_k|^\Gamma \tag{1}$$

for $k = 1,...6$ for two-qubit systems.[1] $|\varphi_k\rangle\langle\varphi_k|$ represents six families of 1-parameter, rank-1 projectors, where

$$\begin{aligned} |\varphi_1\rangle &= a|\Phi^+\rangle + b|\Phi^-\rangle, & |\varphi_2\rangle &= a|\Psi^+\rangle + b|\Psi^-\rangle, \\ |\varphi_3\rangle &= a|\Phi^+\rangle + b|\Psi^+\rangle, & |\varphi_4\rangle &= a|\Phi^-\rangle + b|\Psi^-\rangle, \\ |\varphi_5\rangle &= a|\Phi^+\rangle + ib|\Psi^-\rangle, & |\varphi_6\rangle &= a|\Phi^-\rangle + ib|\Psi^+\rangle, \end{aligned}$$

and $a, b \in \mathbb{R}$ such that $a^2 + b^2 = 1$. $|\Phi^\pm\rangle$ and $|\Psi^\pm\rangle$ are the Bell states. For simplicity, we parameterize the coefficients as $a = \cos\theta$ and $b = \sin\theta$.

The expectation value of W_k is determined by minimizing the expression $\mathrm{Tr}(W_k\rho)$ with respect to θ over $[0, \pi]$:

$$\langle W_k\rangle = \min(\mathrm{Tr}(W_k\rho)). \tag{2}$$

If at least one of the expectation values is negative, the state is identified as entangled. Evaluating $W_k\rho$ requires only three sets of local measurements: $\{\sigma_x\otimes\sigma_x, \mathbb{I}\otimes\sigma_x, \sigma_x\otimes\mathbb{I}\}$, $\{\sigma_y\otimes\sigma_y, \mathbb{I}\otimes\sigma_y, \sigma_y\otimes\mathbb{I}\}$, and $\{\sigma_z\otimes\sigma_z, \mathbb{I}\otimes\sigma_z, \sigma_z\otimes\mathbb{I}\}$. In each set, the detector measures the coincidence count of the incoming two-qubit state in four basis states and calculate their relative probabilities. For instance, $\{\sigma_x\otimes\sigma_x, \mathbb{I}\otimes\sigma_x, \sigma_x\otimes\mathbb{I}\}$ involves a measurement in $\{|DD\rangle, |DA\rangle, |AD\rangle, |AA\rangle\}$ basis. Their expectation values are given by probabilities $P_{DD} - P_{DA} - P_{AD} + P_{AA}$, $P_{DD} - P_{DA} + P_{AD} - P_{AA}$, and $P_{DD} + P_{DA} - P_{AD} - P_{AA}$.

Such tomographic efficiency, however, is achieved at the cost of detection accuracy. While $\{W\}$ requires fewer measurements than full tomography, it produces false negatives, which occur when the incoming state is entangled but the witnesses fail to report any negative value. We measure the significance of false negativity by the false negative rate (r_{FN}), which is defined as

$$r_{FN} = \frac{\text{total number of false negatives}}{\text{total number of entangled states}}. \tag{3}$$

When we have no *a priori* knowledge of the input state, the false negative errors become significant. In this paper, we propose a strategy to reduce r_{FN} by adding two more sets of measurements.

II. NEW SET OF ENTANGLEMENT WITNESSES

We introduce a new set of optimal entanglement witnesses, $\{W'\}$, which takes the same form (1) but involves 2-parameter and 3-parameter projectors:

$$\begin{aligned} |\varphi_1'\rangle &= \cos\theta|\Phi^+\rangle + e^{i\alpha}\sin\theta|\Phi^-\rangle; \\ |\varphi_2'\rangle &= \cos\theta|\Psi^+\rangle + e^{i\alpha}\sin\theta|\Psi^-\rangle; \\ |\varphi_3'\rangle &= \frac{1}{\sqrt{2}}(\cos\theta|HH\rangle + e^{i(\beta-\alpha)}\sin\theta|HV\rangle \\ &\quad + e^{i\alpha}\sin\theta|VH\rangle + e^{i\beta}\cos\theta|VV\rangle); \\ |\varphi_4'\rangle &= \cos\theta|\Phi^+\rangle + e^{i\alpha}\sin\theta|\Psi^+\rangle; \\ |\varphi_5'\rangle &= \cos\theta|\Phi^-\rangle + e^{i\alpha}\sin\theta|\Psi^-\rangle; \\ |\varphi_6'\rangle &= \cos\theta\cos\alpha|HH\rangle + i\cos\theta\sin\alpha|HV\rangle \\ &\quad + i\sin\theta\sin\beta|VH\rangle + \sin\theta\cos\beta|VV\rangle; \\ |\varphi_7'\rangle &= \cos\theta|\Phi^+\rangle + e^{i\alpha}\sin\theta|\Psi^-\rangle; \\ |\varphi_8'\rangle &= \cos\theta|\Phi^-\rangle + e^{i\alpha}\sin\theta|\Psi^+\rangle; \\ |\varphi_9'\rangle &= \cos\theta\cos\alpha|HH\rangle + \cos\theta\sin\alpha|HV\rangle \\ &\quad + \sin\theta\sin\beta|VH\rangle + \sin\theta\cos\beta|VV\rangle; \end{aligned}$$

While $\{W\}$ represents the most general witnesses in form (1) that can be constructed from only $\sigma_i\otimes\sigma_i$-type of measurements, $\{W'\}$ is a set of most general witnesses in the same form that are constructed from those measurements plus an extra pair (TABLE I). For instance the witness $W_1' = |\varphi_1'\rangle\langle\varphi_1'|^\Gamma$ requires an additional pair of measurements $\sigma_x \otimes \sigma_y$ and $\sigma_y \otimes \sigma_x$:

$$\begin{aligned} W_1' = &\frac{1}{4}[\mathbb{I}\otimes\mathbb{I} + \sigma_z\otimes\sigma_z + \cos 2\theta(\sigma_x\otimes\sigma_x + \sigma_y\otimes\sigma_y) \\ &+ \sin 2\theta\cos\alpha(\mathbb{I}\otimes\sigma_z + \sigma_z\otimes\mathbb{I}) \\ &+ \sin 2\theta\sin\alpha(\sigma_x\otimes\sigma_y - \sigma_y\otimes\sigma_x)]. \end{aligned}$$

We evaluate the expectation value of W' using the same way as (2), but now we minimize the expression with respect to θ over $[0,\pi/2]$ and α over $[0,2\pi]$ (and β over $[0,2\pi]$ for W_3', W_6' and W_9').

III. MIN-MAX METHOD

If $\langle W_k\rangle$ is non-negative for all six witnesses, i.e., $\{W\}$ does not report entanglement, we would want to take more measurements to look for evidence for entanglement. To determine which of the three witness triplets in TABLE I we should add, we calculate

$$\max(\mathrm{Tr}(W_k\rho)) \tag{4}$$

with respect to θ for all $W_k \in \{W\}$ and look for the witness that produces the smallest value. In specific, $\{W_1, W_2\}$ corresponds to triplet 1, $\{W_3, W_4\}$ corresponds to triplet 2, and $\{W_5, W_6\}$ corresponds to triplet 3. If (4) is the smallest when $k = 5$, we would add the triplet $\{W_{an}'\} = \{W_7', W_8', W_9'\}$ and measure $\sigma_x\otimes\sigma_z$ and $\sigma_z\otimes\sigma_x$.

The min-max method above comes from an educated guess. We test the effectiveness of this strategy computationally with randomly generated quantum states. We have sampled $10,000$ pure entangled states and $10,000$ mixed entangled states using the standard method.[2,3] We further constrain each of the sampled states to have a concurrence value greater than the threshold ($C > \varepsilon$).

The results for a concurrence threshold of $\varepsilon = 10^{-5}$ are shown in TABLE II. By adding one witness triplet, $\{W_{an}'\}$, to $\{W\}$ using the min-max method, we are able to detect entanglement of almost all pure states and decrease r_{FN} by $\sim 34\%$ for random mixed states. We also compare the results with the ones when we include all nine witnesses in $\{W'\}$, which will be discussed more in Section IV. Notice that most undetected states have small concurrence and low purity (FIG. 1). When the incoming states have concurrence greater than 0.25, we are able to detect 95% of them by performing only five sets of measurements (FIG. 2).

Among prominent analytical methods to detect entanglement, fully entangled fraction (FEF) is able to achieve the lowest r_{FN}, 20.67%, when six sets of non-local measurements are performed on an incoming random entangled state.[3] Our method detects $\sim 14\%$ fewer entangled states than FEF with one less set of measurements, while it allows all measurements to be local.

Witness	Additional measurements
W_1', W_2', W_3' (Triplet 1)	$\sigma_x\otimes\sigma_y$, $\sigma_y\otimes\sigma_x$
W_4', W_5', W_6' (Triplet 2)	$\sigma_y\otimes\sigma_z$, $\sigma_z\otimes\sigma_y$
W_7', W_8', W_9 (Triplet 3)	$\sigma_x\otimes\sigma_z$, $\sigma_z\otimes\sigma_x$

TABLE I. Additional measurements required by the entanglement witnesses in $\{W'\}$.

Random state	$r_{FN}(W)$ (%)	$r_{FN}(W_{an}')$ (%)
Pure	12.04	< 0.01
Mixed	66.09	31.62

TABLE II. A comparison of the average r_{FN} obtained before and after adding the new witness triplet using the min-max method. The threshold for concurrence is set to $\varepsilon = 10^{-5}$. $r_{FN}(W)$: the false negative rate using only $\{W\}$. $r_{FN}(W_{an}')$: the false negative rate after adding one more triplet of witness $\{W_{an}'\}$.

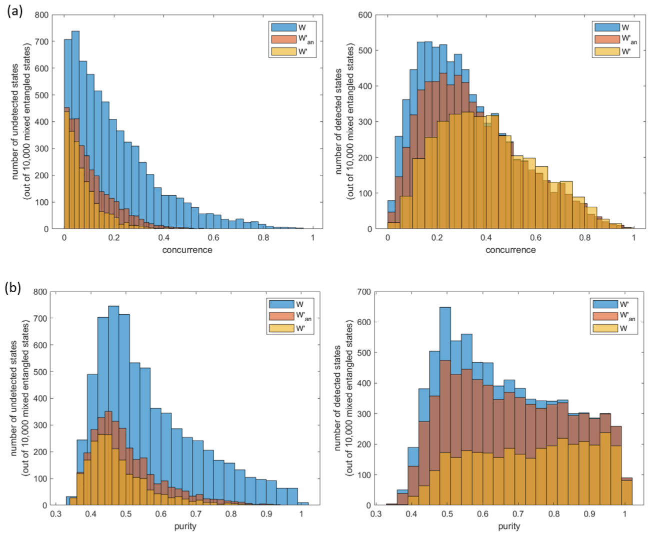

FIG. 1. Concurrence and purity distribution of mixed states undetected (left) and detected (right) by W, W_{an}' and W' when $\varepsilon = 10^{-5}$. (a) The mean concurrence of undetected states is 0.2025 for W, 0.1058 for W_{an}', and 0.0765 for W'. (b) The mean purity of undetected states is 0.5684 for W, 0.5001 for W_{an}', and 0.4788 for W'.

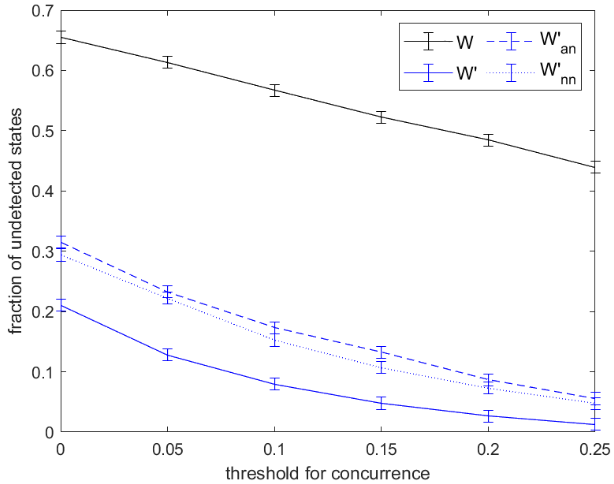

FIG. 2. r_{FN} for different concurrence threshold ε. We computationally sample 10,000 mixed entangled states for each value of ε. W'_{an}: the results from the analytic min-max method. W'_{nn}: the results from neural network training mentioned in Section IV.

IV. SUPERVISED MACHINE LEARNING

The motivation of applying supervised learning is to further decrease r_{FN}. We wish to choose the correct triplet (that give at least one negative witness value) with a higher success rate based on the information of W.

The neural network is constructed with Python Keras and TensorFlow. The model is a regression model that takes in 12 inputs: W_1 minimum value, W_1 maximum value, W_2 minimum, W_2 maximum, W_3 minimum, W_3 maximum, W_4 minimum. The model outputs three pieces of information: predictions for whether or not entanglement will be detected by Triplet 1, Triplet 2, and Triplet 3. The model is trained on 50,000 randomly generated states that are not detected by W. During training, the model has a mean-squared error of 9.52%. The model is then tested on various concurrence thresholds and it slightly outperforms the min-max method (FIG. 2). However, increasing the training set and changing model parameters would likely decrease the model's error. Ideally we would be like to further reduce the r_{FN} by $\sim 10\%$ (from W'_{an} to W' in FIG. 2).

V. EXPERIMENTAL IMPLEMENTATION

We test our method on several sets of entangled states using the setup in FIG. 3.

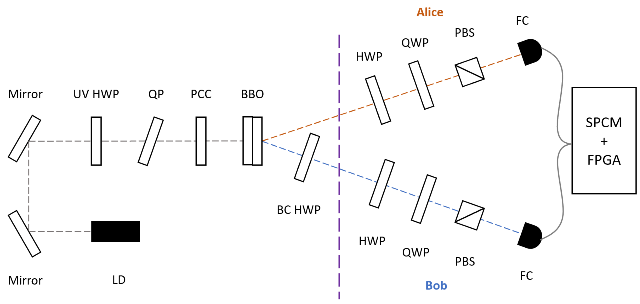

FIG. 3. Schematic of the experimental setup. Entangled photon pairs are prepared on the left of the dotted purple line. The laser diode (LD) generates horizontally polarized photons at 403 nm. After the photons pass through a half-wave plate for near-ultraviolet photons (UV HWP), a quartz plate (QP) and a precompensation crystal (PCC), the β barium borate (BBO) crystal converts $|V\rangle$ to $|H\rangle_A|H\rangle_B$ pairs and $|H\rangle$ to $|V\rangle_A|V\rangle_B$ pairs. Bob's creation half-wave plate (BC HWP) further changes the polarization state of Bob's photon. Photon detection and measurement are completed on the right of the dotted purple line. A HWP, a QWP, and a polarizing beam splitter (PBS) are placed in each path. The photons go through optical fiber couplers (FC) and are detected by the system consisting of a single-photon counting module (SPCM) and FPGA timing circuit.

A. Amplitude-damped states

We first demonstrate why the parameter θ in the construction of $\{W\}$ is important. We test the performance of $\{W\}$ on states from an amplitude-damping channel suggested by Riccardi *et al.*[1] Such states have the form

$$\rho_{ad} = \mathbb{I} \otimes \varepsilon(|\Phi^+\rangle\langle\Phi^+|)$$

with $\varepsilon(\rho) = A_0\rho A_0^\dagger + A_1\rho A_1^\dagger$, where $A_0 = |H\rangle\langle H| + \sqrt{1-\gamma}|V\rangle\langle V|$ and $A_1 = \sqrt{\gamma}|H\rangle\langle V|$. ρ_{ad} is entangled for $\gamma \in [0,1)$, and W_2 is able to detected entanglement for any γ. In FIG. 4, when γ is closer to 1, the parameter θ that minimizes $\mathrm{Tr}(W_2\rho_{ad})$ is further away from the extreme values ($\sin\theta = 0$ or ± 1) that make W_2 only an extremal witness on $|\phi^\pm\rangle$. This suggests that when the projector $|\varphi_k\rangle$ involves a superposition of two Bell states, its corresponding W_k can detect more entangled states than the witnesses that only target Bell states.

B. States undetected by $\{W\}$

We then test our method on a set of pure entangled states that are in theory undetected by $\{W\}$ but detected by $\{W'_{an}\}$:

$$|\varphi\rangle = \cos\phi|H\rangle_A|D\rangle_B - \sin\phi|V\rangle_A|A\rangle_B \quad (5)$$

for $\phi \in (0,\pi/2)$ where $|D\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$ and $|A\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)$. The experimental results in FIG. 5 illustrate how our new set improves the success of entanglement detection.

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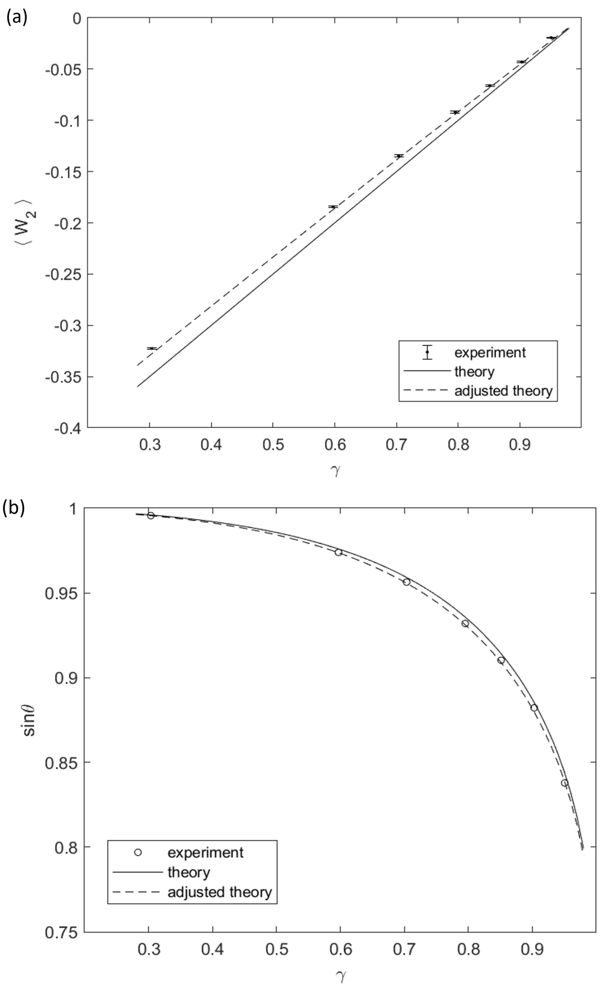

FIG. 4. Entanglement detection pattern of W_2 for amplitude-damped states with different γ. The adjusted theory curve accounts for the entanglement purity in the experiment. (a) The expectation value of W_2 against γ. (b) The value of $\sin\theta$ that minimizes $\mathrm{Tr}(W_2\rho_{ad})$.

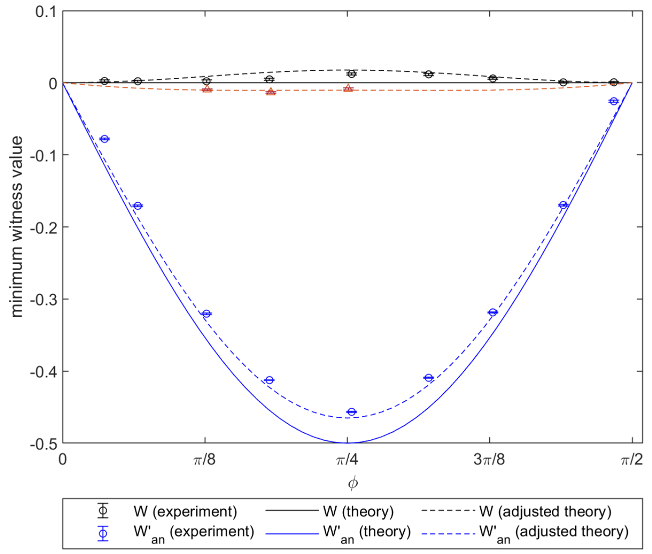

FIG. 5. Entanglement detection pattern of $\{W\}$ and $\{W'_{an}\}$ on the state $\cos\phi|H\rangle_A|D\rangle_B - \sin\phi|V\rangle_A|A\rangle_B$ for different values of ϕ. The curves for adjusted theory account for the entanglement purity in the actual setup. Notice that $\{W\}$ is sensitive to Bob's photon polarization. It is able to detect entanglement when Bob's states $\{|D\rangle, |A\rangle\}$ are instead $\{\cos 46.5°|H\rangle + \sin 46.5°|V\rangle, \sin 46.5°|H\rangle - \cos 46.5°|V\rangle\}$ (red, where the dotted line represents the adjusted theory, and $\triangle$ represents the experimental data).

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APPENDIX A: WITNESS OPERATORS

The nine witnesses in $\{W'\}$ are given by the following operators:

$$\begin{aligned}W'_1 =& \frac{1}{4}[\mathbb{I}\otimes\mathbb{I} + \sigma_z\otimes\sigma_z + \cos 2\theta(\sigma_x\otimes\sigma_x + \sigma_y\otimes\sigma_y)\\ &+ \sin 2\theta\cos\alpha(\sigma_z\otimes\mathbb{I} + \mathbb{I}\otimes\sigma_z)\\ &+ \sin 2\theta\sin\alpha(\sigma_x\otimes\sigma_y - \sigma_y\otimes\sigma_x)],\end{aligned}$$

$$\begin{aligned}W'_2 =& \frac{1}{4}[\mathbb{I}\otimes\mathbb{I} - \sigma_z\otimes\sigma_z + \cos 2\theta(\sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y)\\ &+ \sin 2\theta\cos\alpha(\sigma_z\otimes\mathbb{I} - \mathbb{I}\otimes\sigma_z)\\ &- \sin 2\theta\sin\alpha(\sigma_x\otimes\sigma_y + \sigma_y\otimes\sigma_x)],\end{aligned}$$

$$W_3' = \frac{1}{4}[\cos^2\theta(\mathbb{I}\otimes\mathbb{I}+\sigma_z\otimes\sigma_z) + \sin^2\theta(\mathbb{I}\otimes\mathbb{I}-\sigma_z\otimes\sigma_z) + \cos^2\theta\cos\beta(\sigma_x\otimes\sigma_x+\sigma_y\otimes\sigma_y) + \sin^2\theta\cos(2\alpha-\beta)(\sigma_x\otimes\sigma_x-\sigma_y\otimes\sigma_y) + \sin 2\theta\cos\alpha\,\sigma_x\otimes\mathbb{I} + \sin 2\theta\cos(\alpha-\beta)\mathbb{I}\otimes\sigma_x + \sin 2\theta\sin\alpha\,\sigma_y\otimes\mathbb{I} + \sin 2\theta\sin(\alpha-\beta)\mathbb{I}\otimes\sigma_y + \cos^2\theta\sin\beta(\sigma_y\otimes\sigma_x-\sigma_x\otimes\sigma_y) + \sin^2\theta\sin(2\alpha-\beta)(\sigma_y\otimes\sigma_x+\sigma_x\otimes\sigma_y)],$$

$$W_4' = \frac{1}{4}[\mathbb{I}\otimes\mathbb{I}+\sigma_x\otimes\sigma_x+\cos 2\theta(\sigma_z\otimes\sigma_z+\sigma_y\otimes\sigma_y) + \sin 2\theta\cos\alpha(\mathbb{I}\otimes\sigma_x+\sigma_x\otimes\mathbb{I}) + \sin 2\theta\sin\alpha(\sigma_y\otimes\sigma_z-\sigma_z\otimes\sigma_y)],$$

$$W_5' = \frac{1}{4}[\mathbb{I}\otimes\mathbb{I}-\sigma_x\otimes\sigma_x+\cos 2\theta(\sigma_z\otimes\sigma_z-\sigma_y\otimes\sigma_y) + \sin 2\theta\cos\alpha(\mathbb{I}\otimes\sigma_x-\sigma_x\otimes\mathbb{I}) - \sin 2\theta\sin\alpha(\sigma_y\otimes\sigma_z+\sigma_z\otimes\sigma_y)],$$

$$W_6' = \frac{1}{4}[\cos^2\theta\cos^2\alpha(\mathbb{I}\otimes\mathbb{I}+\sigma_z\otimes\sigma_z+\sigma_z\otimes\mathbb{I}+\mathbb{I}\otimes\sigma_z) + \cos^2\theta\sin^2\alpha(\mathbb{I}\otimes\mathbb{I}-\sigma_z\otimes\sigma_z+\sigma_z\otimes\mathbb{I}-\mathbb{I}\otimes\sigma_z) + \sin^2\theta\cos^2\beta(\mathbb{I}\otimes\mathbb{I}+\sigma_z\otimes\sigma_z-\sigma_z\otimes\mathbb{I}-\mathbb{I}\otimes\sigma_z) + \sin^2\theta\sin^2\beta(\mathbb{I}\otimes\mathbb{I}-\sigma_z\otimes\sigma_z-\sigma_z\otimes\mathbb{I}+\mathbb{I}\otimes\sigma_z) + \sin 2\theta\cos\alpha\cos\beta(\sigma_x\otimes\sigma_x+\sigma_y+\sigma_y) + \sin 2\theta\sin\alpha\sin\beta(\sigma_x\otimes\sigma_x-\sigma_y+\sigma_y) + \sin 2\theta\cos\alpha\sin\beta(\sigma_y\otimes\sigma_z+\sigma_y\otimes\mathbb{I}) + \sin 2\theta\sin\alpha\cos\beta(\sigma_y\otimes\sigma_z-\sigma_y\otimes\mathbb{I}) - \cos^2\theta\sin 2\alpha(\sigma_z\otimes\sigma_y+\mathbb{I}\otimes\sigma_y) - \sin^2\theta\sin 2\beta(\sigma_z\otimes\sigma_y-\mathbb{I}\otimes\sigma_y)],$$

$$W_7' = \frac{1}{4}[\mathbb{I}\otimes\mathbb{I}+\sigma_y\otimes\sigma_y+\cos 2\theta(\sigma_z\otimes\sigma_z+\sigma_x\otimes\sigma_x) + \sin 2\theta\cos\alpha((\sigma_z\otimes\sigma_x-\sigma_x\otimes\sigma_z) - \sin 2\theta\sin\alpha(\sigma_y\otimes\mathbb{I}+\mathbb{I}\otimes\sigma_y)],$$

$$W_8' = \frac{1}{4}[\mathbb{I}\otimes\mathbb{I}-\sigma_y\otimes\sigma_y+\cos 2\theta(\sigma_z\otimes\sigma_z-\sigma_x\otimes\sigma_x) + \sin 2\theta\cos\alpha((\sigma_z\otimes\sigma_x+\sigma_x\otimes\sigma_z) + \sin 2\theta\sin\alpha(\sigma_y\otimes\mathbb{I}-\mathbb{I}\otimes\sigma_y)],$$

$$W_9' = \frac{1}{4}[\cos^2\theta\cos^2\alpha(\mathbb{I}\otimes\mathbb{I}+\sigma_z\otimes\sigma_z+\sigma_z\otimes\mathbb{I}+\mathbb{I}\otimes\sigma_z) + \cos^2\theta\sin^2\alpha(\mathbb{I}\otimes\mathbb{I}-\sigma_z\otimes\sigma_z+\sigma_z\otimes\mathbb{I}-\mathbb{I}\otimes\sigma_z) + \sin^2\theta\cos^2\beta(\mathbb{I}\otimes\mathbb{I}+\sigma_z\otimes\sigma_z-\sigma_z\otimes\mathbb{I}-\mathbb{I}\otimes\sigma_z) + \sin^2\theta\sin^2\beta(\mathbb{I}\otimes\mathbb{I}-\sigma_z\otimes\sigma_z-\sigma_z\otimes\mathbb{I}+\mathbb{I}\otimes\sigma_z) + \sin 2\theta\cos\alpha\cos\beta(\sigma_x\otimes\sigma_x+\sigma_y+\sigma_y) + \sin 2\theta\sin\alpha\sin\beta(\sigma_x\otimes\sigma_x-\sigma_y+\sigma_y) + \cos^2\theta\sin 2\alpha(\mathbb{I}\otimes\sigma_x+\sigma_z\otimes\sigma_x) + \sin^2\theta\sin 2\beta(\mathbb{I}\otimes\sigma_x-\sigma_z\otimes\sigma_x) + \sin 2\theta\cos\alpha\sin\beta(\sigma_x\otimes\mathbb{I}+\sigma_x\otimes\sigma_z) + \sin 2\theta\sin\alpha\cos\beta(\sigma_x\otimes\mathbb{I}-\sigma_x\otimes\sigma_z)].$$

APPENDIX B: ANALYTIC DERIVATION OF STATES GENERATING WITNESSES

Our starting point is a set of equations setting various stokes parameters to be 0.

1. $\sigma_x \otimes \sigma_y = \cos\theta\sin\theta(-\cos\alpha\cos\beta\sin\phi_3 + \sin\alpha\sin\beta\sin(\phi_2-\phi_1)) = 0$
2. $\sigma_x \otimes \sigma_z = \cos\theta\sin\theta(\cos\alpha\sin\beta\cos\phi_2 - \sin\alpha\cos\beta\cos(\phi_3-\phi_1)) = 0$
3. $\sigma_y \otimes \sigma_x = \cos\theta\sin\theta(-\cos\alpha\cos\beta\sin\phi_3 + \sin\alpha\sin\beta\sin(\phi_1-\phi_2)) = 0$
4. $\sigma_y \otimes \sigma_z = \cos\theta\sin\theta(-\cos\alpha\sin\beta\sin\phi_2 + \sin\alpha\cos\beta\sin(\phi_3-\phi_1)) = 0$
5. $\sigma_z \otimes \sigma_x = \cos^2\theta\cos\alpha\sin\alpha\cos\phi_1 - \sin^2\theta\cos\beta\sin\beta\cos(\phi_2-\phi_3) = 0$
6. $\sigma_z \otimes \sigma_y = -\cos^2\theta\cos\alpha\sin\alpha\sin\phi_1 - \sin^2\theta\cos\beta\sin\beta\sin(\phi_2-\phi_3) = 0$

However, we can initially simplify some of the equations by observing that if $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ then the solution state will not be entangled and thus is not a desired solution. This simplifies equations 1-4.

1. $\sigma_x \otimes \sigma_y = -\cos\alpha\cos\beta\sin\phi_3 + \sin\alpha\sin\beta\sin(\phi_2-\phi_1) = 0$
2. $\sigma_x \otimes \sigma_z = \cos\alpha\sin\beta\cos\phi_2 - \sin\alpha\cos\beta\cos(\phi_3-\phi_1) = 0$
3. $\sigma_y \otimes \sigma_x = -\cos\alpha\cos\beta\sin\phi_3 + \sin\alpha\sin\beta\sin(\phi_1-\phi_2) = 0$
4. $\sigma_y \otimes \sigma_z = -\cos\alpha\sin\beta\sin\phi_2 + \sin\alpha\cos\beta\sin(\phi_3-\phi_1) = 0$

5. $\sigma_z \otimes \sigma_x = \cos^2\theta\cos\alpha\sin\alpha\cos\phi_1 - \sin^2\theta\cos\beta\sin\beta\cos(\phi_2-\phi_3) = 0$

6. $\sigma_z \otimes \sigma_y = -\cos^2\theta\cos\alpha\sin\alpha\sin\phi_1 - \sin^2\theta\cos\beta\sin\beta\sin(\phi_2-\phi_3) = 0$

1. Case 1

The first case that we can study begins by removing the assertion that $\sigma_x\otimes\sigma_y = 0$ and that $\sigma_y\otimes\sigma_x = 0$. The 4 equations we have are then:

1. $\sigma_x\otimes\sigma_z = \cos\alpha\sin\beta\cos\phi_2 - \sin\alpha\cos\beta\cos(\phi_3-\phi_1) = 0$
2. $\sigma_y\otimes\sigma_z = -\cos\alpha\sin\beta\sin\phi_2 + \sin\alpha\cos\beta\sin(\phi_3-\phi_1) = 0$
3. $\sigma_z \otimes \sigma_x = \cos^2\theta\cos\alpha\sin\alpha\cos\phi_1 - \sin^2\theta\cos\beta\sin\beta\cos(\phi_2-\phi_3) = 0$
4. $\sigma_z \otimes \sigma_y = -\cos^2\theta\cos\alpha\sin\alpha\sin\phi_1 - \sin^2\theta\cos\beta\sin\beta\sin(\phi_2-\phi_3) = 0$

Rearranging and simplifying:

1. $\cos\alpha\sin\beta\cos\phi_2 = \sin\alpha\cos\beta\cos(\phi_3-\phi_1)$
2. $\cos\alpha\sin\beta\sin\phi_2 = \sin\alpha\cos\beta\sin(\phi_3-\phi_1)$
3. $\cos^2\theta\sin 2\alpha\cos\phi_1 = \sin^2\theta\sin 2\beta\cos(\phi_2-\phi_3)$
4. $\cos^2\theta\sin 2\alpha\sin\phi_1 = -\sin^2\theta\sin 2\beta\sin(\phi_2-\phi_3)$

The three forms for solutions are:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(\cos\alpha\,|00\rangle \pm_0 \sin\alpha e^{i(\phi_3-\phi_2)}\,|01\rangle$$

$$\pm_0 \sin\alpha e^{i\phi_2}\,|10\rangle + \cos\alpha e^{i\phi_3}\,|11\rangle)$$

$$|\psi\rangle = \cos\theta\,|00\rangle + \sin\theta e^{i\phi}\,|11\rangle$$

$$|\psi\rangle = \cos\theta\,|01\rangle + \sin\theta e^{i\phi}\,|10\rangle$$

2. Case 2

The second case that we can study begins by removing the assertion that $\sigma_x\otimes\sigma_z = 0$ and that $\sigma_z\otimes\sigma_x = 0$. The 4 equations we have are then:

1. $\sigma_x\otimes\sigma_y = -\cos\alpha\cos\beta\sin\phi_3 + \sin\alpha\sin\beta\sin(\phi_2-\phi_1) = 0$
2. $\sigma_y\otimes\sigma_x = -\cos\alpha\cos\beta\sin\phi_3 + \sin\alpha\sin\beta\sin(\phi_1-\phi_2) = 0$
3. $\sigma_y\otimes\sigma_z = -\cos\alpha\sin\beta\sin\phi_2 + \sin\alpha\cos\beta\sin(\phi_3-\phi_1) = 0$
4. $\sigma_z \otimes \sigma_y = -\cos^2\theta\cos\alpha\sin\alpha\sin\phi_1 - \sin^2\theta\cos\beta\sin\beta\sin(\phi_2-\phi_3) = 0$

Rearranging and Simplifying:

1. $\cos\alpha\cos\beta\sin\phi_3 = \sin\alpha\sin\beta\sin(\phi_2-\phi_1)$
2. $\cos\alpha\cos\beta\sin\phi_3 = \sin\alpha\sin\beta\sin(\phi_1-\phi_2)$
3. $\cos\alpha\sin\beta\sin\phi_2 = \sin\alpha\cos\beta\sin(\phi_3-\phi_1)$
4. $\cos^2\theta\sin 2\alpha\sin\phi_1 = -\sin^2\theta\sin 2\beta\sin(\phi_2-\phi_3)$

There are 2 forms for solutions that can potentially create entangled states:

$$|\psi\rangle = \cos\theta\cos\alpha\,|00\rangle + \cos\theta\sin\alpha\,|01\rangle$$

$$+\sin\theta\sin\beta\,|10\rangle + \sin\theta\cos\beta\,|11\rangle$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(\cos\alpha\,|00\rangle + \sin\alpha e^{\phi_1}\,|01\rangle \pm_0 \sin\alpha e^{\phi_1}\,|10\rangle \mp_0 \cos\alpha\,|11\rangle)$$

3. Case 3

The third case that we can study begins by removing the assertion that $\sigma_z\otimes\sigma_y = 0$ and that $\sigma_y\otimes\sigma_z = 0$. The 4 equations we have are then:

1. $\sigma_x\otimes\sigma_y = -\cos\alpha\cos\beta\sin\phi_3 + \sin\alpha\sin\beta\sin(\phi_2-\phi_1) = 0$
2. $\sigma_x\otimes\sigma_z = \cos\alpha\sin\beta\cos\phi_2 - \sin\alpha\cos\beta\cos(\phi_3-\phi_1) = 0$
3. $\sigma_y\otimes\sigma_x = -\cos\alpha\cos\beta\sin\phi_3 + \sin\alpha\sin\beta\sin(\phi_1-\phi_2) = 0$
4. $\sigma_z \otimes \sigma_x = \cos^2\theta\cos\alpha\sin\alpha\cos\phi_1 - \sin^2\theta\cos\beta\sin\beta\cos(\phi_2-\phi_3) = 0$

Rearranging and simplifying:

1. $\cos\alpha\cos\beta\sin\phi_3 = \sin\alpha\sin\beta\sin(\phi_2-\phi_1)$
2. $\cos\alpha\sin\beta\cos\phi_2 = \sin\alpha\cos\beta\cos(\phi_3-\phi_1)$
3. $\cos\alpha\cos\beta\sin\phi_3 = \sin\alpha\sin\beta\sin(\phi_1-\phi_2)$
4. $\cos^2\theta\cos\alpha\sin\alpha\cos\phi_1 = \sin^2\theta\cos\beta\sin\beta\cos(\phi_2-\phi_3)$

There are 2 forms for solutions that can potentially create entangled states:

$$|\psi\rangle = \cos\theta\cos\alpha\,|00\rangle + i\cos\theta\sin\alpha\,|01\rangle$$

$$+ i\sin\theta\sin\beta\,|10\rangle + \sin\theta\cos\beta\,|11\rangle$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(\cos\alpha\,|00\rangle + \sin\alpha e^{\phi_1}\,|01\rangle$$

$$\pm_0 \sin\alpha e^{\phi_1}\,|10\rangle \pm_0 \cos\alpha\,|11\rangle)$$

APPENDIX C: THE NEURAL NETWORK MODEL

The model is a linear regression multi-output model. With the current training set, increasing the number of hidden nodes led to the model over fitting and increased the mean-squared error on test sets. Increasing the number of layers, did not have a noticeable increase in performance. Currently the model performs similarly to the analytic min-max method, which shows how the analytic method is incredibly effective. However, increasing the training set and changing the architecture of the model would perhaps improve the model's performance.

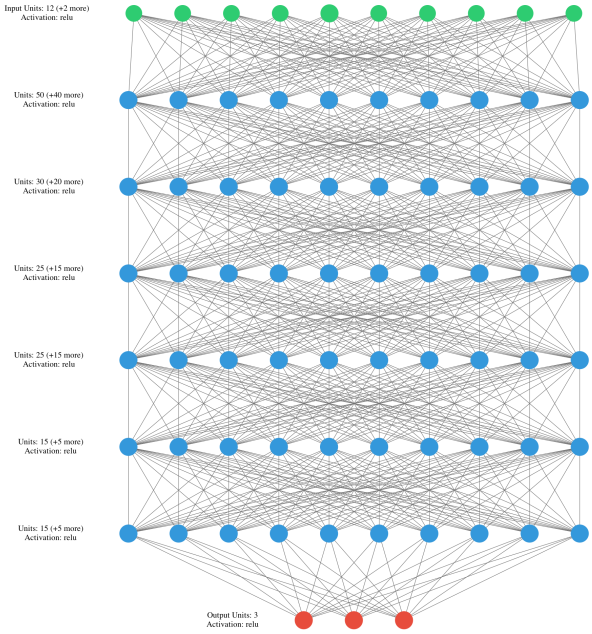

FIG. 6. The Model's Architecture. The structure of the model is constructed of various hidden layers that contain a total of 160 hidden nodes using Python's Keras and Tensorflow.