$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \begin{pmatrix}-\sin\theta e^{-i\varphi}\\ \cos\theta+1\end{pmatrix}$$

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Квантовая механика БФЗ-19-1

Задание 1 Пункт 1

$$\hat{S} = \begin{pmatrix}\cos\theta & \sin\theta e^{-i\varphi}\\ \sin\theta e^{i\varphi} & -\cos\theta\end{pmatrix}$$

$$\hat{S}|\psi\rangle = \lambda|\psi\rangle$$

$$\hat{S}|\psi\rangle - \lambda|\psi\rangle = 0$$

$(\hat{S}-\lambda E)|\psi\rangle = 0$, E – единичн-я матрица

$$\begin{vmatrix}\cos\theta-\lambda & \sin\theta e^{-i\varphi}\\ \sin\theta e^{+i\varphi} & -\cos\theta-\lambda\end{vmatrix} = 0$$

$$-\cos^2\theta - \lambda\cos\theta + \lambda\cos\theta + \lambda^2 - \sin^2\theta = 0$$

$$\lambda^2 = \cos^2\theta + \sin^2\theta$$

$$\lambda = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = \pm 1$$

$\underline{\underline{\lambda=-1}}$ $$\begin{pmatrix}\cos\theta+1 & \sin\theta e^{-i\varphi}\\ \sin\theta e^{i\varphi} & -\cos\theta+1\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{cases}\psi_1(\cos\theta+1) + \psi_2(\sin\theta e^{-i\varphi}) = 0\\ \psi_1(\sin\theta e^{i\varphi}) + \psi_2(-\cos\theta+1) = 0\end{cases}$$

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}\cos\theta-1\\ \sin\theta e^{i\varphi}\end{pmatrix}\frac{1}{\sqrt{(\cos\theta-1)^2+\sin^2\theta}}$$

$\underline{\underline{\lambda=1}}$ $$\begin{pmatrix}\cos\theta-1 & \sin\theta e^{-i\varphi}\\ \sin\theta e^{i\varphi} & -\cos\theta-1\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{cases}(\cos\theta-1)\psi_1 + \sin\theta e^{-i\varphi}\psi_2 = 0\\ \sin\theta e^{i\varphi}\psi_1 + (-\cos\theta-1)\psi_2 = 0\end{cases}$$

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}-\sin\theta e^{-i\varphi}\\ \cos\theta-1\end{pmatrix}\frac{1}{\sqrt{(\cos\theta-1)^2+\sin^2\theta}}$$

Проверка на ортогональность

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}-\sin\theta e^{-i\varphi}\\ \cos\theta-1\end{pmatrix}\frac{1}{\sqrt{(\cos\theta-1)^2+\sin^2\theta}}$$

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}\cos\theta-1\\ \sin\theta e^{i\varphi}\end{pmatrix}\frac{1}{\sqrt{(\cos\theta-1)^2+\sin^2\theta}}$$

$$\begin{pmatrix}-\sin\theta e^{+i\varphi}\\ \cos\theta-1\end{pmatrix}\begin{pmatrix}\cos\theta-1\\ \sin\theta e^{+i\varphi}\end{pmatrix} = \langle\psi_1|\psi_2\rangle$$

$$(\cos\theta-1)(-\sin\theta e^{i\varphi})+(\cos\theta-1)(\sin\theta e^{i\varphi})=0$$

$$\langle\psi_1|\psi_2\rangle=0$$

Пункт 2

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \alpha_1\frac{1}{\sqrt{(\cos\theta-1)^2+\sin^2\theta}}\begin{pmatrix}-\sin\theta e^{-i\varphi}\\ \cos\theta-1\end{pmatrix}+$$

$$+\alpha_2\frac{1}{\sqrt{(\cos\theta-1)^2+\sin^2\theta}}\begin{pmatrix}\cos\theta-1\\ \sin\theta e^{i\varphi}\end{pmatrix}$$

Посчитаю заранее: $\sqrt{(\cos\theta-1)^2+\sin^2\theta} = (\ldots)$

$$(\psi_1) \left(-\sin\theta e^{-i\varphi} \atop \ldots\right)$$

$$(\ldots) = \sqrt{\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta} =$$

$$= \sqrt{\cos^2\theta - 2\cos\theta + 1 + 1 - \cos^2\theta} = \sqrt{2(1-\cos\theta)}$$

Тогда: $|\psi\rangle = \frac{\alpha_1}{\sqrt{2(1-\cos\theta)}}\begin{pmatrix} -\sin\theta e^{-i\varphi} \\ \cos\theta - 1 \end{pmatrix} + \frac{\alpha_2}{\sqrt{2(1-\cos\theta)}}\begin{pmatrix} \cos\theta - 1 \\ \sin\theta e^{i\varphi} \end{pmatrix}$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \frac{\alpha_1}{\sqrt{1-\cos\theta}}\begin{pmatrix} -\sin\theta e^{-i\varphi} \\ \cos\theta - 1 \end{pmatrix} + \frac{\alpha_2}{\sqrt{1-\cos\theta}}\begin{pmatrix} \cos\theta - 1 \\ \sin\theta e^{i\varphi} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\alpha_1\begin{pmatrix} -\sin\theta e^{-i\varphi} \\ \cos\theta - 1 \end{pmatrix} + \alpha_2\begin{pmatrix} \cos\theta - 1 \\ \sin\theta e^{i\varphi} \end{pmatrix} = \begin{matrix} \sqrt{1-\cos\theta} \\ \sqrt{1-\cos\theta} \end{matrix}$$

$$\alpha_1(-\sin\theta e^{-i\varphi}) + \alpha_2(\cos\theta - 1) = \sqrt{1-\cos\theta} \qquad (1)$$

$$\alpha_1(\cos\theta - 1) + \alpha_2(\sin\theta e^{i\varphi}) = \sqrt{1-\cos\theta} \qquad (2)$$

Множу: $\sin\theta e^{i\varphi} \cdot (1)$

$(\cos\theta - 1) \cdot (2)$

$$\begin{cases} \alpha_1(-\sin^2\theta) + \alpha_2(\cos\theta - 1)(\sin\theta e^{i\varphi}) = \sqrt{1-\cos\theta}\,\sin\theta e^{i\varphi} \\ \alpha_1(\cos\theta - 1)^2 + \alpha_2(\cos\theta - 1)(\sin\theta e^{i\varphi}) = \sqrt{1-\cos\theta}\,(1-\cos\theta) \end{cases}$$

Вычту: $(2) - (1)$

$$\alpha_1((\cos\theta - 1)^2 + \sin^2\theta) + \alpha_2 \cdot 0 = \sqrt{(1-\cos\theta)}(1 - \cos\theta - \sin\theta e^{i\varphi})$$

$$\alpha_1(\cancel{\cos^2\theta} - 2\cos\theta + 1 + \cancel{\sin^2\theta}) = \sqrt{(1-\cos\theta)}(1-\cos\theta - \sin\theta e^{i\varphi})$$

$$\alpha_1(2(1-\cos\theta)) = \sqrt{(1-\cos\theta)}(1-\cos\theta - \sin\theta e^{i\varphi})$$

$$\alpha_1 = \frac{1}{2} \cdot \frac{\sqrt{1-\cos\theta}(1-\cos\theta - \sin\theta e^{i\varphi})}{1-\cos\theta}$$

$$\alpha_1 = \frac{1}{2} \frac{\sqrt{1-\cos\theta}\,(1-\cos\theta - \sin\theta e^{i\varphi})}{(1-\cos\theta)} =$$

$$= \frac{1}{2} \frac{(1-\cos\theta - \sin\theta e^{i\varphi})}{\sqrt{1-\cos\theta}}$$

$$\frac{1}{2} \frac{(1-\cos\theta - \sin\theta e^{i\varphi})}{\sqrt{1-\cos\theta}} (\cos\theta - 1) + \alpha_2 \sin\theta e^{i\varphi} = \sqrt{1-\cos\theta}$$

$$\alpha_2 \sin\theta e^{i\varphi} - \frac{1}{2} \frac{(1-\cos\theta - \sin\theta e^{i\varphi})}{\sqrt{(1-\cos\theta)}} (1-\cos\theta) = \sqrt{1-\cos\theta}$$

$$\alpha_2 \sin\theta e^{i\varphi} - \frac{1}{2} \frac{1-\cos\theta - \sin\theta e^{i\varphi}}{1} \sqrt{1-\cos\theta} = \sqrt{1-\cos\theta}$$

$$\alpha_2 \frac{\sin\theta e^{-i\varphi}}{\sqrt{1-\cos\theta}} - \frac{1}{2}(1-\cos\theta - \sin\theta e^{i\varphi}) = 1$$

$$\alpha_2 \frac{\sin\theta e^{i\varphi}}{\sqrt{1-\cos\theta}} = 1 - \frac{1}{2}(1-\cos\theta - \sin\theta e^{i\varphi})$$

$$\alpha_2 = \frac{\sqrt{1-\cos\theta}}{\sin\theta e^{i\varphi}} \left(1 - \frac{1}{2}(1-\cos\theta - \sin\theta e^{i\varphi}\right)$$

$$\alpha_2 = \frac{\sqrt{1-\cos\theta}}{2\cdot\sin\theta e^{i\varphi}} \left(\cos\theta + 1 + \sin\theta e^{i\varphi}\right)$$

$(\psi_1), (-\sin\theta e^{-i\varphi}$

$\lambda = -1$

$\lambda = 1$

Тогда это возможные варианты измерений

$$\langle A_\psi\rangle = \langle\psi|A|\psi\rangle = \sum_i \lambda_i |\langle\psi|A_i\rangle|^2$$

$$\cancel{\tilde{P}_1 = |\langle\psi|A_1\rangle|^2 = \left(\frac{1}{\sqrt{2}}\right)^3\left(\left(\frac{\cos\theta-1}{2\sqrt{\cos\theta+1}} - \frac{\sin\theta e^{-i\varphi}}{2\sqrt{\cos\theta+1}}\right)\right)^2 = \frac{1}{8}\frac{(-\sin\theta e^{-i\varphi}+\cos\theta-1)^2}{\cos\theta+1}}$$

$$\psi = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix},\quad \psi_1 = A_1,\quad \psi_2 = A_2$$

$$\tilde{P}_1 = \left(\frac{1}{\sqrt{2}}\right)^2\left(\left|-\frac{\sin\theta e^{-i\varphi}}{\sqrt{2(1-\cos\theta)}} + \frac{\cos\theta-1}{2(1-\cos\theta)}\right|\right)^2 = (\ldots)$$

$$(\ldots) = \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^2\frac{|(\cos\theta-\sin\theta e^{-i\varphi}-1)|^2}{(1-\cos\theta)}$$

$$\tilde{P}_1 = \alpha_1^2 = \frac{1}{4}\frac{|(\cos\theta-\sin\theta(\cos\varphi-i\sin\varphi)-1)|^2}{(1-\cos\theta)}$$

$$\tilde{P}_1 = \frac{1}{4}\frac{|\cos\theta-1-\sin\theta\cos\varphi+i\sin\varphi\cos\theta|^2}{(1-\cos\theta)}$$

$$\tilde{P}_1 = \frac{1}{4}\frac{(\sin^2\varphi\cos^2\theta) + (\cos\theta-1-\sin\theta\cos\varphi)^2}{(1-\cos\theta)}$$

$\tilde{P}_2 = |\langle \psi | A_2 \rangle| = (\dots) = \alpha_2 = (\dots) = \tilde{P}_1$

$\tilde{P}_1 + \tilde{P}_2 = \sum P$ полная вероятность всегда равна единице

$\tilde{P}_1 + \tilde{P}_2 = 1$, где $\tilde{P}_1 = \tilde{P}_2$

В таком случае мы имеем $\tilde{P}_1 = \tilde{P}_2 = \frac{1}{2}$

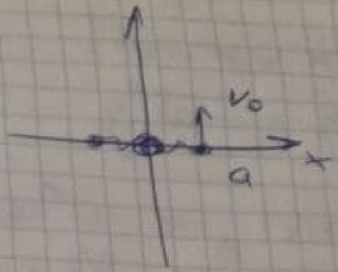
У нас есть верятность 50/50 что результатом будет "1" или "-1"

Ответ: $\tilde{P}_1 = 0,5$

$\tilde{P}_2 = 0,5$

$$\begin{pmatrix} \psi_1 \\ \dots \end{pmatrix}, \begin{pmatrix} -\sin\theta e^{-i\varphi} \\ \cos\theta + 1 \end{pmatrix}$$

Задание 2 Пункт 1



$$U(x,y) = \frac{m\omega^2}{2}(x^2+y^2)$$

$x(0) = a \quad \dot{x}(0) = 0$

$y(0) = 0 \quad \dot{y}(0) = V_0$

$$L = T - U$$

$$L = \frac{m\dot{x}^2}{2} + \frac{m\dot{y}^2}{2} - \frac{m\omega^2}{2}(x^2+y^2)$$

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 - \omega^2(x^2+y^2)\right)$$

П.Н.Д. $\quad \delta S = 0 \Rightarrow \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)\,dt = 0$

$$\int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta\dot{q}, t)\,dt - \int_{t_1}^{t_2} L(q, \dot{q}, t)\,dt = 0$$

$$\int_{t_1}^{t_2} \left(\delta q \frac{\partial L}{\partial q} + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}\right) = 0$$

$$\int_{t_1}^{t_2} \left(\delta q \frac{\partial L}{\partial q} + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\delta q\right) = 0 = \frac{\partial L}{\partial \dot{q}}\delta q\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q}\delta q - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\delta q\right)dt$$

$$\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\delta q\,dt = 0 \ : \forall \delta q, \text{ тогда}$$

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \Rightarrow \begin{cases} -2\omega^2 x - 2\ddot{x} = 0 \\ -2\omega^2 y - 2\ddot{y} = 0 \end{cases} \Rightarrow \begin{cases} \ddot{x} = -\omega^2 x \\ \ddot{y} = -\omega^2 y \end{cases}$$

$$\begin{cases} \ddot{x} = -\omega^2 x \\ \ddot{y} = -\omega^2 y \end{cases} \Rightarrow \begin{cases} x(t) = x_0 \cos\omega t \\ y(t) = y_0 \sin\omega t \end{cases}$$

$$\begin{cases} x(t) = a\cos\omega t \\ y(t) = \dfrac{v_0}{\omega}\sin\omega t \end{cases} \Rightarrow \begin{cases} \dfrac{x(t)}{a} = \cos\omega t \\ y(t)\cdot\dfrac{\omega}{v_0} = \sin\omega t \end{cases}$$

$$\begin{cases} x^2(t) = a^2\cos^2\omega t \\ y^2(t) = \left(v_0/\omega\right)^2\sin^2\omega t \end{cases}$$

$$\cancel{x^2(t) + y^2(t) = a^2\cos^2\omega t + \left(\frac{v_0}{\omega}\right)^2\sin^2\omega t}$$

$$\cancel{a^2x^2(t) + a^2y^2(t) = \cos^2\omega t + \left(a\frac{v_0}{\omega}\right)^2\sin^2\omega t}$$

$$\cancel{\frac{a^2x^2(t)}{a^2\left(\frac{v_0}{\omega}\right)^2} + \frac{a^2y^2(t)}{a^2\left(\frac{v_0}{\omega}\right)^2} =}$$

$$\frac{x^2(t)}{a^2} + \frac{\omega^2 y^2(t)}{v_0^2} = \cos^2\omega t + \sin^2\omega t$$

$$\frac{x^2(t)}{a^2} + \frac{y^2(t)}{\left(v_0/\omega\right)^2} = 1$$ – траектория эллипс

Пункт 2

$$p_x = \frac{\partial L}{\partial \dot{x}} \Rightarrow p_x = m\dot{x}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

$$H = \cancel{\ldots}\ p_x \cdot \dot{x} + p_y \cdot \dot{y} - L = \frac{m}{2}\left((\dot{x}^2+\dot{y}^2) + \omega^2(x^2+y^2)\right) = \frac{m}{2}\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \omega^2(x^2+y^2)\right) =$$

$$= E \Rightarrow \underbrace{E = \frac{m}{2}\cdot\left(\frac{dx^2+dy^2}{dt^2} + \omega^2(x^2+y^2)\right)}$$

$$dt = \frac{\sqrt{dx^2+dy^2}}{\sqrt{\frac{2E}{m} - \omega^2(x^2+y^2)}}$$

$$p_x = m\frac{dx}{dt} = \frac{m\,dx}{\sqrt{dx^2+dy^2}}\sqrt{\frac{2E}{m} - \omega^2(x^2+y^2)}$$

Аналогично $p_y = m\frac{dy}{dt} = \frac{m\,dy}{\sqrt{dx^2+dy^2}}\sqrt{\frac{2E}{m} - \omega^2(x^2+y^2)}$

$$S = \int p\,dq \longrightarrow \max \ : \ S = \int \cancel{\ldots}\ p_x dx + \int p_y dy$$

$$\int p_x dx + \int p_y dy = \int m\sqrt{\frac{2E}{m} - \omega^2(x^2+y^2)}\ \frac{dx^2+dy^2}{\sqrt{dx^2+dy^2}} = (\ldots)$$

$$(\ldots) = \int m\sqrt{\frac{2E}{m} - \omega^2(x^2+y^2)}\ \sqrt{dx^2+dy^2} = (\ldots)$$

$$\Rightarrow \left| \; x' = \sqrt{\frac{\omega^2 (a^2 - x^2)}{\frac{2E}{m} - \omega^2 (y^2 + a^2)}} \; \right|$$

$$\frac{dx}{dy} = \frac{\omega \sqrt{a^2 - x^2}}{\sqrt{\frac{2E}{m} - \omega^2 (y^2 + a^2)}}$$

$$\frac{dx}{\omega \sqrt{a^2 - x^2}} = \frac{dy}{\sqrt{\frac{2E}{m} - \omega^2 (y^2 + a^2)}}$$

$$\frac{dx}{\omega \sqrt{a^2 - x^2}} = \frac{dy}{\omega \sqrt{\left(\frac{2E}{m\omega^2} - a^2\right) - y^2}}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{dy}{\sqrt{\left(\frac{2E}{m\omega^2} - a^2\right) - y^2}}$$

$$\arcsin \frac{x}{a} = \arcsin \frac{y}{\left(\frac{2E}{m\omega^2} - a^2\right)} + C$$

$$E = \frac{m}{2} \left(V_0^2 + 0^2 + \omega^2 (a^2 + 0^2) \right) = \frac{m}{2} (V_0^2 + \omega^2 a^2)$$

$$\frac{2E}{m\omega^2} = \frac{\cancel{2}}{\cancel{m}\omega^2} \frac{\cancel{m}}{\cancel{2}} (V_0^2 + \omega^2 a^2) = \frac{V_0^2}{\omega^2} + a^2$$

$$\arcsin 1 = 0 + C \Rightarrow C = \frac{\pi}{2}$$

$$\begin{pmatrix}\psi_1\\ \psi_1\end{pmatrix} = \begin{pmatrix}-\sin\theta e^{-i\varphi}\\ \cos\theta + 1\end{pmatrix}$$

$$\arcsin\frac{x}{a} = \arcsin\frac{y}{v_0/\omega} + \frac{\pi}{2} = \arcsin\frac{y\omega}{v_0} + \frac{\pi}{2} =$$

$$= \arcsin\left(0 + 1\sqrt{1 - \frac{y^2\omega^2}{v_0^2}}\right); \quad \frac{x}{a} = \sqrt{1 - \frac{y^2\omega^2}{v_0^2}}$$

$$\frac{x^2}{a^2} + \frac{y^2\omega^2}{v_0^2} = 1$$

Задача 3: $|\psi\rangle(0) = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad \hat{S} = \begin{pmatrix}\cos\theta & \sin\theta e^{-i\varphi}\\ \sin\theta e^{i\varphi} & -\cos\theta\end{pmatrix}$

$$\hat{H} = -\mu B\hat{S}$$

$$|\psi\rangle(t) = \exp\left(-\frac{\hat{H}t}{i\hbar}\right)|\psi\rangle(0)$$

$$e^{-\frac{\hat{H}t}{i\hbar}} = \cos\frac{\hat{H}t}{\hbar} + i\sin\frac{\hat{H}t}{\hbar}$$

Если степень чётная выносится $\hat{1}$

Если степень нечётная выносится $\hat{S}$

$$\Rightarrow \hat{1}\begin{pmatrix}1\\0\end{pmatrix}\cos\frac{\mu Bt}{\hbar} - i\begin{pmatrix}\cos\theta\\ \sin\theta e^{i\varphi}\end{pmatrix}\sin\frac{t\mu B}{\hbar}$$

Задача 4

~~$\psi(x) = C x^2 e$~~

$\psi(x) = Ce^{-x/\lambda} \quad x \geq 0$

1.) $\int_0^{+\infty} |\psi(x)|^2 dx = 1 = \langle \psi | \psi \rangle$

интегрирую вероятность по всему пространству...

$$C \int_0^{+\infty} x^4 e^{-2x/\lambda} dx = (\dots)$$

Через калькулятор интегралов

$$C \int_0^{+\infty} x^4 e^{-2x/\lambda} dx = C \frac{3\lambda^5}{4} = 1 \Rightarrow C = \frac{4}{3\lambda^5}$$

Пункт (2) ~~$P(x)$~~ $P([x, x+dx]) = \int_x^{x+dx} |\psi|^2 dx =$

$= |\psi|^2 dx = \left| Ce^{-x/\lambda} \right|^2 dx$

$$|\psi|^2 dx = \frac{4x^4 e^{-2x/\lambda}}{3\lambda^5}$$

Пункт (3)

$$\langle X_\psi \rangle = \int_0^{+\infty} x |\psi(x)|^2 dx = (\dots)$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} -\sin\theta e^{-i\varphi} \\ \cos\theta + 1 \end{pmatrix}$$

$$(\ldots) = \frac{4}{3\lambda^5}\int_0^{+\infty} x^5 e^{-2x/\lambda}dx =$$ | Через калькулятор интегралов => =

$$\Rightarrow \frac{4}{3\lambda^5}\int_0^{+\infty} x^5 e^{-2x/\lambda}dx = \frac{4}{3\lambda^5}\cdot\frac{15\lambda^6}{8} = \frac{5}{2}\lambda$$