Фаворов Валентин БФЗ-19-1
HOMEWORK - 1
QUANTUM MECHANICS
Задание 1 (1)

$-\infty \xrightarrow{\quad |\psi\rangle \quad} +\infty \quad x$

$\psi(x) = \frac{A}{x^2+a^2}$
$A = ?$

Решение: $\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = \int_{-\infty}^{+\infty} \psi^*(x)\psi(x)dx = 1$

$\int_{-\infty}^{+\infty} \frac{A^2 dx}{(x^2+a^2)^2} \Rightarrow \frac{1}{A^2} = \left[\frac{1}{2a^3} \operatorname{arctg}\frac{x}{a} + \frac{x}{2a^2(x^2+a^2)}\right]\Bigg|_{-\infty}^{+\infty} = (\dots)$

$(\dots) = \frac{1}{2a^2}\left[\frac{1}{a}\operatorname{arctg}\frac{x}{a} + \cancel{\frac{x}{x(x+\frac{a^2}{x})}}\right]\Bigg|_{-\infty}^{+\infty} = \frac{1}{2a^2}\left[\dots\right]\Bigg|_{-\infty}^{+\infty} = (\dots)$

$(\dots) = \frac{1}{2a^2}\left[\frac{1}{a}\left(\frac{\pi}{2} - 0\right) - \frac{1}{a}\left(-\frac{\pi}{2}+0\right)\right] = \frac{1}{2a^2}\cdot\frac{1}{a}\cdot\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{1}{2a^3}\cdot\pi$

$\frac{1}{A^2} = \frac{\pi}{2a^3} \Rightarrow A^2 = \frac{2a^3}{\pi} \Rightarrow A = \sqrt{\frac{2a^3}{\pi}} = \pm a\sqrt{\frac{2a}{\pi}}$

Ответ: $\psi(x) = \pm\sqrt{\frac{2a}{\pi}} \cdot \frac{a}{(x^2+a^2)}$

Задание 2 (1)

$\psi(x) = \frac{B}{x+ib}$; $-\infty \xrightarrow{\quad |\psi\rangle \quad} +\infty \quad x$; $B = ?$

$\int_{-\infty}^{+\infty} \psi^*(x)\psi(x)dx = 1 = \langle\psi|\psi\rangle$

$|\psi(x)\rangle = \frac{B}{(x+ib)}\cdot\frac{(x-ib)}{(x-ib)} = \frac{Bx}{x^2+b^2} - i\frac{bB}{x^2+b^2} \Rightarrow \langle\psi(x)| = \frac{Bx}{x^2+b^2} + i\frac{bB}{x^2+b^2}$

$$\cancel{\varphi^*(x)} = \frac{Bx}{x^2+b^2} + i\frac{bB}{x^2+b^2} = \langle\varphi(x)|$$

$$\langle \cancel{\varphi^*}\varphi(x)|\varphi(x)\rangle = \int_{-\infty}^{+\infty}\varphi^*(x)\varphi(x)dx = \int_{-\infty}^{+\infty}\left[\frac{(Bx)^2}{(x^2+b^2)^2} + \frac{(Bb)^2}{(x^2+b^2)^2}\right]dx$$

$$1 = B^2\int_{-\infty}^{+\infty}\left(\frac{x^2}{(x^2+b^2)^2} + \frac{b^2}{(x^2+b^2)^2}\right)dx = B^2\int_{-\infty}^{+\infty}\frac{\cancel{(x^2+b^2)}}{(x^2+b^2)^{\cancel{2}}}dx = 1$$

$$\frac{1}{B^2} = \int_{-\infty}^{+\infty}\frac{dx}{x^2+a^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a}\Bigg|_{-\infty}^{+\infty} = \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]\frac{1}{b} \Rightarrow$$

$$= \frac{1}{B^2} = \frac{\pi}{b} \Rightarrow B^2 = \frac{b}{\pi} \Rightarrow B = \pm\sqrt{\frac{b}{\pi}}$$

Задание 3 (1)

$\boxed{\langle\varphi|\psi\rangle = ?}$ Решение: $A = \pm a\sqrt{\frac{2a}{\pi}}$

$B = \pm\sqrt{\frac{b}{\pi}}$

$$\int_{-\infty}^{+\infty}\left(\left(\frac{B}{x+ib}\right)^* \frac{A}{x^2+a^2}\right)dx = \ldots$$

$$\frac{B}{x+ib} = \frac{B(x-ib)}{x^2+b^2} \Rightarrow \left(\frac{B}{x+ib}\right)^* = \frac{B(x+ib)}{x^2+b^2}$$

$$\langle\varphi|\psi\rangle = \int_{-\infty}^{+\infty}\frac{AB(x+ib)}{(x^2+a^2)(x^2+b^2)}dx = AB\int_{-\infty}^{+\infty}\frac{(\cancel{x}+ib)dx}{(x^2+a^2)(x^2+b^2)} = ABibI$$

где $I - \int_{-\infty}^{+\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{a^2-b^2}\int_{-\infty}^{+\infty}\frac{(a^2-b^2)dx}{(x^2+a^2)(x^2+b^2)} =$

$$= \frac{1}{a^2-b^2}\int_{-\infty}^{+\infty}\frac{(a^2+x^2-b^2-x^2)dx}{(x^2+b^2)(x^2+a^2)} = \frac{1}{a^2-b^2}\int_{-\infty}^{+\infty}\frac{dx}{x^2+b^2} - \frac{1}{a^2-b^2}\int_{-\infty}^{+\infty}\frac{dx}{x^2+a^2} =$$

$$= \frac{1}{a^2-b^2}\cdot\frac{1}{b}\,\mathrm{arctg}\frac{x}{b}\Big|_{-\infty}^{+\infty} - \frac{1}{a^2-b^2}\frac{1}{a}\,\mathrm{arctg}\frac{x}{a}\Big|_{-\infty}^{+\infty} = (\dots)$$

$$(\dots) = \frac{1}{a^2-b^2}\left(\frac{\pi}{1}\cdot\frac{1}{b} - \frac{\pi}{1}\cdot\frac{1}{a}\right) = \frac{1}{a^2-b^2}\left(\frac{\pi}{b}-\frac{\pi}{a}\right) =$$

$$\Rightarrow I = \left(\frac{\pi}{b}-\frac{\pi}{a}\right)\frac{1}{a^2-b^2} \Rightarrow \langle\varphi|\psi\rangle = ABibI =$$

$$ABib\left(\frac{\pi}{b}\cdot\frac{\pi}{a}\right)\frac{1}{a^2-b^2} = a\frac{\sqrt{2ab}}{\pi}ib\left(\frac{\pi}{b}-\frac{\pi}{a}\right)\cdot\frac{1}{a^2-b^2}$$

Ответ: $\langle\varphi|\psi\rangle = a\frac{\sqrt{2ab}}{\pi}ib\left(\frac{\pi}{b}-\frac{\pi}{a}\right)\cdot\frac{1}{a^2-b^2}$

Задание 4 (1) Д-ть $\delta(f(x)) = \sum_{x_i}\frac{1}{|f'(x_i)|}\delta(x-x_i)$
где x_i – нули первого порядка для функции $f(x)$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \dots$$

допускаю $f(x_0)=0$, откуда $f(x) = f'(x_0)(x-x_0)$

$$\delta(f(x)) = \delta(f'(x_0)(x-x_0)) = \frac{1}{|f'(x_0)|}\delta(x-x_0)$$

Отсюда, зная из x_i, что мы имеем "i" корней...

$$\delta(f(x)) = \sum_{i}\frac{\delta(x-x_0)}{|f'(x_i)|}$$

Задание 5 (3)

$\psi(x) = \sqrt{\frac{2}{a}} \sin \frac{\pi x}{a}$

$f_1(x) = \alpha_1 \exp(i\pi x/a)$

$f_2(x) = \alpha_2 \exp(-i\pi x/a)$

1. Проверить $\langle f_1 | f_2 \rangle = 0$
2. α_1 - ? α_2 - ?
3. разложить $|\psi\rangle$ по базису $|f_{1,2}\rangle$, т.е. найти c_1 - ?, c_2 - ? $|\psi\rangle = c_1|f_1\rangle + c_2|f_2\rangle$

Решение: (1)

$$\langle f_1 | f_2 \rangle = \int_0^a \alpha_1\alpha_2 f_1^* f_2 \, dx = \int_0^a \alpha_1\alpha_2 e^{-\frac{i\pi x}{a}} e^{-\frac{i\pi x}{a}} dx = \int_0^a \alpha_1\alpha_2 e^{\frac{-2i\pi x}{a}}$$

$$= \left| \begin{array}{l} e^{Axi} = \cos Ax + i\sin Ax \\ e^{-Axi} = \cos Ax - i\sin Ax \end{array} \right| = \int_0^a \alpha_1\alpha_2 \left(\cos\frac{2\pi x}{a} - i\sin\frac{2\pi x}{a}\right) dx =$$

$$= \alpha_1\alpha_2 \cdot \frac{a}{2\pi} \cdot \sin\frac{2\pi x}{a}\Big|_0^a + \alpha_1\alpha_2 \frac{ai}{2\pi} \cos\frac{2\pi x}{a}\Big|_0^a = \alpha_1\alpha_2\frac{a}{2\pi}(\sin 2\pi - \sin 0)$$

$$+ \alpha_1\alpha_2 \frac{ai}{2\pi}(\cos 2\pi - \cos 0) = \alpha_1\alpha_2 \cdot \frac{ai}{2\pi}(1-1) = 0$$

Решение. (2)

$$e^{-\frac{\pi x i}{a}} e^{\frac{\pi x i}{a}} = e^0 = 1$$

$$f_1: \langle f_1 | f_1 \rangle = \int_0^a \alpha_1^2 f_1^* f_1 \, dx = 1 = \int_0^a \alpha_1^2 dx = 1 \Rightarrow \alpha^{-2} = x\Big|_0^a$$

$$\alpha^2 = \frac{1}{a} \Rightarrow \alpha_1 = \frac{1}{\sqrt{a}}$$

$$f_2: \langle f_2 | f_2 \rangle = \int_0^a \alpha_2\alpha_2 f_2^* f_2 \, dx \Rightarrow \alpha_2^{-2} = a \Rightarrow \alpha_2 = \alpha_1 = \frac{1}{\sqrt{a}}$$

Решение: (3) $|\psi\rangle = C_1|\psi_1\rangle + C_2|\psi_2\rangle$

$$\frac{\sqrt{2}}{\sqrt{a}}\sin\frac{\pi x}{a} = \alpha_1 C_1 e^{\frac{\pi x i}{a}} + \alpha_2 C_2 e^{-\frac{\pi x i}{a}}$$

$$\frac{\sqrt{2}}{\sqrt{a}}\sin\frac{\pi x}{a} = \frac{C_1}{\sqrt{a}}\left(\cos\frac{\pi x}{a} + i\sin\frac{\pi x}{a}\right) + \frac{C_2}{\sqrt{a}}\left(\cos\frac{\pi x}{a} - i\sin\frac{\pi x}{a}\right)$$

$$\frac{\sqrt{2}}{\sqrt{a}}\sin\frac{\pi x}{a} = (C_1+C_2)\cdot\frac{1}{\sqrt{a}}\cos\frac{\pi x}{a} + (C_1-C_2)\frac{i}{\sqrt{a}}\sin\frac{\pi x}{a}$$

$$\frac{\sqrt{2}}{a}\sin\frac{\pi x}{a} = \frac{(C_1+C_2)}{\sqrt{a}}\cos\frac{\pi x}{a} + \frac{i(C_1-C_2)}{\sqrt{a}}\sin\frac{\pi x}{a}$$

$$\begin{cases}\frac{\sqrt{2}}{\sqrt{a}} = \frac{i(C_1-C_2)}{\sqrt{a}} \\ 0 = \frac{(C_1+C_2)}{\sqrt{a}}\end{cases} \Rightarrow \begin{cases}\sqrt{2} = i(C_1-C_2) \\ 0 = C_1+C_2\end{cases} \Rightarrow \begin{cases}\frac{\sqrt{2}}{i} = C_1 - C_2 \\ -C_2 = C_1\end{cases}$$

$$\begin{cases}C_1 = \frac{\sqrt{2}}{i} + C_2 \\ C_2 = -C_1\end{cases} \Rightarrow \quad C_1 = \frac{\sqrt{2}}{i} - C_1$$

$$C_1 = \frac{\sqrt{2}}{2}i \Rightarrow C_2 = -\frac{\sqrt{2}}{2}i$$

Ответ: $|\psi\rangle = C_1|\psi_1\rangle + C_2|\psi_2\rangle$

$$|\psi\rangle = \frac{\sqrt{2}}{2}i|\psi_1\rangle - \frac{\sqrt{2}}{2}i|\psi_2\rangle$$