MIANO LYNN HIUKO 200664 MFI 8302 ASSIGNMENT 2.

Q2.
$$dS_t = \mu S_t\,dt + \sigma_t S_t\,dZ_t$$
$$dv_t = \kappa(\theta - v(t))dt + \gamma\sqrt{v_t}\,dW_t$$
$$Cov(dZ_t, dW_t) = \rho dt$$
$$v_t = \sigma_t^2$$

Let $U(S,v,t)$ denote the price of a derivative on S_t & v_t

Applying Ito's formula to $U(S_t, v_t, t)$

$$dU = U_t dt + U_S dS + U_v dv + \tfrac{1}{2} U_{SS}(dS)^2 + \tfrac{1}{2} U_{vv}(dv)^2 + U_{Sv} dS dv$$

Substitute the SDEs to compute second order differentials.

$\sigma_t^2 = v_t$ and $(dZ_t)^2 = dt$

$$(dS)^2 = (\sigma_t S_t dZ_t)^2$$
$$= \sigma_t^2 S_t^2 (dZ_t)^2$$
$$= v_t S_t^2 dt$$

$$(dv)^2 = (\gamma\sqrt{v_t}\,dW_t)^2$$
$$= \gamma^2 v_t (dW_t)^2$$
$$= \gamma^2 v_t dt$$

$$(dS dv) = (\sqrt{v_t} S_t dZ_t)\cdot(\gamma\sqrt{v_t}\,dW_t)$$
$$= \gamma v_t S_t dZ_t dW_t$$
$$= \gamma v_t S_t \rho dt$$

Substituting in Ito's formula:

$$dU = U_t dt + U_S(\mu S_t dt + \sqrt{v_t} S_t dZ_t) + U_v(\kappa(\theta - v_t)dt + \gamma\sqrt{v_t}\,dW_t)$$
$$+ \tfrac{1}{2} U_{SS} v_t S_t^2 dt + \tfrac{1}{2} U_{vv}\gamma^2 v_t dt + U_{Sv}\gamma v_t S_t \rho dt$$

Under risk neutral measure:

Replace $\mu \rightarrow r$ and $\kappa(\theta - v) \rightarrow \kappa(\theta - v) - \lambda v$ in the dt terms obtained.

λ is defined as the constant market price of volatility risk

The expected change of the undiscounted option price is then:

$$\mathbb{E}^{Q}[dU] = \{ U_t + U_S r S + U_v(\kappa(\theta - v) - \lambda v) + \tfrac{1}{2} U_{SS} S^2 v + \tfrac{1}{2} U_{vv} \gamma^2 v + U_{Sv} \gamma \rho v S \} \, dt$$

Stochastic terms have zero mean under Q, but they reflect instantaneous randomness.

- If there is no arbitrage, the discounted price process $e^{-rt} U(S_t, v_t, t)$ must be a martingale under Q.
- The drift of U must equal rU, hence must satisfy:

$$U_t + rSU_S + (\kappa(\theta - v) - \lambda v) U_v) + \tfrac{1}{2} vS^2 U_{SS} + \gamma \rho v S U_{Sv} + \tfrac{1}{2} \gamma^2 v U_{vv} - rU = 0$$

Rewriting:

$$\frac{1}{2} v S^2 U_{SS} + \rho \gamma v S U_{Sv} + \frac{1}{2} \gamma^2 v U_{vv} + rSU_S + (\kappa(\theta - v) - \lambda v) U_v - rU + U_t = 0$$

$$\frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \rho \gamma v S \frac{\partial^2 U}{\partial v \partial S} + \frac{1}{2} \gamma^2 v \frac{\partial^2 U}{\partial v^2} + rS \frac{\partial U}{\partial S} + (\kappa(\theta - v) - \lambda v) \frac{\partial U}{\partial v} - rU + \frac{\partial U}{\partial t} = 0$$