LESSONS IN GEOMETRY

I. Plane Geometry

Jacques Hadamard

AMS
American Mathematical Society

EDC
Education Development Center, Inc.

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Translated from the French by
Mark Saul

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Translator's Preface

In the late 1890s Gaston Darboux was named as the editor of a set of textbooks, resources for the teaching of mathematics (*Cours Complet Pour la Classe de Mathématiques Élémentaires*). Darboux commissioned several mathematicians to write these materials. Jacques Hadamard, having taught on the high school (*lycée*) level,[1] was asked to prepare the materials for geometry. Two volumes resulted: one on plane geometry in 1898 and a volume on solid geometry in 1901.

Hadamard clearly saw this work as important, as he revised it twelve times during his long life, the last edition appearing in 1947. (Hadamard died in 1963 at the age of 97.)

The present book is a translation of the thirteenth edition of the first volume, first printed by Librarie Armand Colin, Paris, in 1947 and reprinted by Éditions Jacques Gabay, Sceaux, in 1988. It includes all the materials that this reprint contains. The volume on solid geometry has not been included here.

A companion volume to this translation, not based on the work of Hadamard, includes solutions to the problems as well as ideas for classroom use.

Hadamard's vision of geometry is remarkably fresh, even after the passage of 100 years. The classical approach is delicately balanced with modern extensions. The various geometric transformations arise simply and naturally from more static considerations of geometric objects.

The book includes a disk for use with the Texas Instruments TI-NSpire™software*. This disk is not meant to exhaust the possibilities of applying technology to these materials. Rather, it is meant to whet the appetite of the user for exploration of this area.

The same can be said about all the materials in the companion volume: Hadamard's book is a rich source of mathematical and pedagogical ideas, too rich to be exhausted in one supplementary volume. The supplementary materials are intended to invite the reader to consider further the ideas brought up by Hadamard.

A word is in order about the process of translation. Hadamard was a master of mathematics, and of mathematical exposition, but not particularly of the language itself. Some of his sentences are stiffly formal, others clumsy, even ambiguous (although the ambiguity is easily resolved by the logic of the discussion). In some cases (the appendix on Malfatti's problem is a good example) footnotes or dependent clauses seem to have been piled on as afterthoughts, to clarify a phrase or logical point. This circumstance presents an awkward dilemma for the translator.

[1]The best account of Hadamard's life, including those episodes alluded to in this preface, can be found in the excellent book by Vladimir Maz'ya and Tatyana Shaposhnikova, *Jacques Hadamard, A Universal Mathematician*, American Mathematical Society, Providence, Rhode Island, 1998.

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Does he make the English elegant and accessible? Or does he convey to the reader the flavor of the original? I have resolved this problem on a case-by-case basis, hoping that the result reads smoothly without distorting the spirit of the original.

In this work, I have received invaluable help from an initial translation prepared by Hari Bercovici, of Indiana University. While most of his work has been altered and fine-tuned, the core of it remains, and Bercovici made significant contributions to the resolution of a number of difficult problems of translation. In addition, the illustrations—faithful copies of Hadamard's own—are almost entirely the work of Bercovici. I am grateful for this opportunity to thank him for generously allowing me access to his work. In return, I take on myself the responsibility for any errors that may have crept in, and that the patient reader will doubtless find.

Others to whom I am grateful for help in this work include Wing Suet Li, Florence Fasanelli, Al Cuoco, Larry Zimmerman, and Sergei Gelfand. My wife, Carol Saul, a great supporter of everything I do, has been immeasurably tolerant of my preoccupation with this work.

This translation was supported by grant number NSF ESI 0242476-03 from the National Science Foundation.

Hadamard's career as a high school teacher does not seem to have ended successfully. He did not stay long in the profession, and there exist notes by his superiors testifying to his difficulties in getting along with his students.[2] However, he seems to have learned from his experiences how to approach them intellectually, thus allowing other teachers, with other skills, the benefit of his own genius.

It is in this spirit of combining the skills of the teacher and the mathematician that I offer these materials to the field.

Mark Saul May 2008

[2]Hadamard's mentoring of Maurice Fréchet, which started in the latter's high school years, is a notable exception to this circumstance.

Author's Prefaces

Preface to the Second Edition

Since the appearance of this work, the teaching of mathematics, and particularly of geometry, has undergone some profound modifications, not just in its details, but in its whole spirit, changes which have been awaited for a long time and are universally desired. In working with beginners, we now tend to rely on practice and intuition, rather than on the Euclidean method, whose utility they are incapable of understanding.

On the other hand, it is clear that we must return to this method when we revisit these early starts, and complete them. It is to this stage of education that our book corresponds, and thus we have not had to change its character.

But even in the area of rigorous logic, the classical exposition was uselessly complicated and scholastic in its first chapter, the one devoted to angles. The convention—unchanged up to the present—which does not permit us to talk of circles in the first book, renders matters obscure which, in themselves, are perfectly clear and natural. Thus this is a place which we have been able to notably simplify things, by introducing arcs of circles into the discussion of angles. We had already departed from the traditional considerations of continuity on which the existence of perpendiculars is often based; the simple artifice which replaced it has itself now become superfluous.

In the same way, the measure of the central angle is naturally integrated into the theory of angles, its correct logical place.

The second book gains no less than the first by this change in order. The fundamental property of the inscribed angle, indeed, is no longer connected to angle measure, a connection which gives one an idea of this property and its significance which is as false as could be.

With this exception, the plan of the work as a whole has been preserved. In fact, the complementary materials introduced by the program of 1902 had been already covered in our first edition. The program of 1905, which has reduced the importance of these materials, has not until now obliged us to do any essential revision. It requires only a single addition: the inverter of Peaucellier. Having made this addition, the only complementary material remaining in this revision, at least in plane geometry[3] is inversion and its applications, which corresponds to Chapters V–VII of our Complements.

[3] I note in this regard, that I have never attempted—despite the advocacy of such a step by M. Méray, whose initiative has proved so fertile and so fortunate in the teaching of geometry—to mix plane and solid geometry together. As this is preferable from a purely logical point of view, I would like very much to do this. But it seems to me that from a pedagogical point of view, we must think, first and foremost, of dividing up the difficulties. That of "spatial visualization" is so serious in and of itself, that I haven't considered adding it to the other difficulties initially.

Another tendency has appeared in the teaching corps in the last few years, which we have had the bad manners not to applaud. We speak here—and I hope we begin to use it a bit—of the method of heuristics. The Note we have added in 1898 to our first edition (Note A) had exactly the goal of describing how, in our view, this method might be understood: how it might be understood, at least, from a theoretical point of view, since both are needed for the application of the heuristic method. I hope that this Note might now be of some use in indicating, at least, how these principles can be put to work.

I have already explained (Preface to *Solid Geometry*) that the method described in Note C for tangent circles belongs in fact to M. Fouché, or to Poncelet himself, and that a solution to the question of areas of plane figures, different, it is true, from that in Note D, is due to M. Gérard. I seize this occasion to add that an objection concerning the theory of dihedral angles has already been noted and refuted by M. Fontené.

J. Hadamard

Preface to the Eighth Edition

The present addition contains no important changes from those that preceded it. We must note, however, that our ideas about the Postulate of Euclid have been modified considerably by recent progress in physics: I have had to recast the end of Note B to take into account this scientific evolution.

A few modification have been made in the present edition, intended to give a bit more importance to properties of the most common articulated systems.

J. Hadamard

Preface to the Twelfth Edition

This edition differs from the preceding only in the addition of several exercises. The elegant Exercise 421b is due to M. Daynac, a teacher in the French School of Cairo; the simple proof of Morley's theorem which is given by Exercise 422, and which brings into play only the first two Books, is due to M. Sasportès;[4] the supplement added to exercise 107 is borrowed from an article of M. Lapierre (*Enseignement Scientifique*, November 1934), with certain modifications intended to reduce the proof to properties of the inscribed angle; Exercise 314b is from Japanese geometers.

J. Hadamard

Preface to the First Edition

In editing these Lessons in Geometry, I have not lost sight of the very special role played by this science in the area of elementary mathematics.

Placed at the entry point to the teaching of mathematics, it is in fact the simplest and most accessible form of reasoning. The importance of its methods, and their fecundity, are here more immediately tangible then in the relatively abstract theories of arithmetic or algebra. Because of this, geometry reveals itself capable of

[4]An essentially equivalent proof was sent me by M. Gauthier, a student at the École Normale Supérieure.

exercising an undeniable influence on the activity of the mind. I have, first of all, sought to develop this influence in awakening and assisting the student's initiative.

Thus it seemed to me necessary to increase the number of exercises as much as the framework of the work would allow. This requirement has been, so to speak, the only rule guiding me in this part of my work. I believed that I must pose questions of very different and gradually increasing difficulty. While the exercises at the end of each chapter, and especially the first few chapters, are very simple, those which I have inserted after each book have solutions which are less immediate. Finally, I have postponed to the end of the volume the statement of problems which are relatively difficult. Certain questions have been borrowed from some important theories—among these we note problems related to the theory of inversion and to systems of circles, many of which come from the note *On the relations between groups of points, of circles, and of spheres in the plane and in space*, of M. Darboux.[5] Others, on the contrary, have no pretensions other than to train the mind in the rules of reason. I have been no less eclectic in the choice of sources I have drawn on: alongside classic exercises which are immediate applications of the theory, and whose absence in this sort of book would be almost astonishing, can be found exercises which are borrowed from various authors and periodicals, both French and foreign, and also a large number which are original.

I have also included, at the end of the work, a note in which I seek to summarize the basic principles of the mathematical method, methods which students must begin to understand starting from the first year of instruction, and which we find poorly understood even by students in our schools of higher education. The dogmatic form which I have had to adopt is not, it must be admitted, the one that fits this topic best: this sort of subject is best taught through a sort of dialogue in which each rule intervenes at exactly the moment when it applies. I believed, despite all, that I had to attempt this exposition, hoping to find readers who are indulgent of the fact that it is presented imperfectly. Let this essay, imperfect as it is, perform a number of services and contribute to the infusion into the classroom of ideas on whose importance we must not tire of insisting.

The other notes, also placed at the end of the volume, are more special in character. Note B concerns Euclid's postulate. The ideas of modern geometers on this subject have assumed a form which is clear and well enough defined that it is possible to give an account of them even in an elementary work.

Note C concerns the problem of tangent circles. As M. Koenigs has noted,[6] the known solution of Gergonne, even when completed by the synthesis neglected by the author, leaves something to be desired. It is this gap that I seek to fill.

Finally, Note D is devoted to the notion of area. The usual theory of area presents, as we know, a serious logical fault. It supposes a priori that this quantity is well-defined and enjoys certain properties. The theory that I give in the note in question, and in which we do without this *postulatum* , must be preferred, especially if one realizes that it applies to space geometry without any significant change.

In the text itself, various classical arguments might be modified to advantage, sometimes to support more rigor, and sometimes in the interests of simplicity.

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[5] *Annales Scientifiques de l'École Normale Superieure*, 2^{nd} series, Vol. I, 1876. Exercise 401 (the construction of tangent circles) was provided to me by M. Gérard, a teacher at the *Lycée Ampère*

[6] *Leçons de l'agrégation classique de Mathématiques*, p. 92, Paris, Hermann, 1892.

Among these, for example, are the proof at the beginning of the first book that a perpendicular can be erected to a line from a point on that line. The considerations of continuity usually raised at this point can be set aside, as long as one assumes, without proof, that a segment or an angle can be divided into two equal parts. The consideration of the sense of rotation of an angle has permitted me to give the statements of theorems in the second book, as well as several following, all their cleanness and all their generality without rendering them less simple or less elementary.

The theories described in the *Complements to the Third Book* are those that, while not included in the elements of geometry as set forth by Euclid, have not taken a lesser place in education in a definitive way. I have limited myself to the elements of these theories and I have systematically eliminated those without real importance. In any case, this work is edited so that these complements, as well as several passages printed in small characters, can be passed over in a first reading without losing the coherence of the rest.

M. Darboux, who has given me the honor of trusting me with the editing of this work, has rendered the task singularly easy by the valuable advice which he has not ceased to give me for its composition. I would not want to end this preface without offering to him the homage of my recognition.

Jacques Hadamard

Introduction

1. A region of space which is bounded in all directions is called a *volume.*

The common part of two contiguous regions of space is called a *surface.* A sheet of paper can give us an approximate idea of a surface. Indeed, it bounds two regions of space, the ones situated on the two sides of the sheet. But more rigorously, such a sheet is not a surface, because these two regions are separated by an intermediate region — the thickness of the paper. We can arrive at the notion of a surface by considering a sheet of paper whose thickness decreases indefinitely.

The common part of two contiguous portions of a surface is called a *line.* This definition is equivalent to the following: *a* line *is the intersection of two surfaces.*

The lines which we draw give us an idea of geometric lines; the idea is approximate because, no matter how thin, they always have some width, which geometric lines do not have.

Finally, the common part of two contiguous portions of a line, or the intersection of two lines which meet, is called a *point.* A point has no dimension.

Any collection of points, lines, surfaces, and volumes is called a *figure.*

1b. Geometric loci. Every line contains infinitely many points. It can be viewed as being generated by a point which moves along it. This is what happens when we trace a line on paper with the point of a pencil or pen (these points are similar to geometric points when they are sufficiently fine).

In the same way, a surface can be generated by a moving line.

Definition. If a point can occupy infinitely many positions (generally, a line or a surface), we call the figure formed by the set of these positions the *geometric locus* of the point.

In the same way, we can view a surface as the geometric locus of a moving line.

2. Geometry is the study of the properties of figures and of the relations between them.

The results of this study are formulated in statements which are called *propositions.*

A proposition consists of two parts: the first, called the *hypothesis*, indicates the conditions which we impose on ourselves; the other, the *conclusion*, expresses a fact which, under these conditions, must necessarily be true.

Thus, in the proposition: *Two quantities A, B equal to a third C are themselves equal*, the hypothesis is *The quantities A, B are both equal to C*; the conclusion is *These two quantities A, B are equal.*

Among propositions, there are some which we consider obvious without proof. These are called *axioms.* One of these is the proposition just considered: "Two quantities equal to a third are themselves equal." All other propositions are called *theorems* and must be proved by a particular line of reasoning. To produce this reasoning, we impose on ourselves the conditions of the hypothesis and, *assuming that these conditions are satisfied*, we must deduce the facts stated in the conclusion.

According to this, we allow that a certain fact is true:

1°. If it is part of the hypothesis;

2°. If it is part of the definition of one of the elements under discussion;[7]

3°. If it follows from an axiom;

4°. If it follows from an earlier proof.

No fact can be considered certain, in a geometric argument, if it does not follow from one of the preceding four reasons.

2b. A proposition whose conclusion is formed, completely or partially, by the hypothesis of a first proposition, and whose hypothesis is formed, completely or partially, by the conclusion of the first proposition, is called a *converse* of the first proposition.

An immediate consequence of a theorem is called a *corollary*.

On the other hand, a *lemma* is a preparatory proposition, intended to facilitate the proof of a later proposition.

3. Congruent figures. An arbitrary geometric figure can be moved through space without deformation in infinitely many ways, as is the case with ordinary solid objects.

Two figures are called congruent *if one can be moved onto the other, in such a way that all their parts coincide*;[8] in other words, *two congruent figures are one and the same figure, in two different places.*

A figure which is subject to displacement without deformation is also called an *invariant* figure.

4. The Straight Line. The simplest of all lines is a straight line, of which a stretched thread gives us an idea. The notion of a straight line is clear by itself; in order to use it in our reasoning, we consider the straight line to be defined by its obvious properties, in particular the following:

1°. *Any figure congruent to a straight line is itself a straight line*; and conversely, *every straight line can be made to coincide with any other straight line, and this may be done so that an arbitrarily chosen point on the first coincides with any point chosen on the second*;

2°. *Through any two points we can draw one, and only one, straight line.*

Thus we can talk about *the* straight line passing through points A and B or, more briefly, of the straight line AB.

It follows immediately from the definition that *two different lines can meet in only one point*, since, if they had two common points, they could not be distinct.

---

[7]It often happens in the course of a proof that one introduces auxiliary elements into the figure. A fact can then be true by virtue of the definition of these new elements. We then say that the fact is true *by construction*.

[8]Hadamard uses the term *equal* throughout to describe figures that can be made to coincide. The term *congruent* is used more often in English. We will use either term for line segments and angles (but most often the term *equal* for these simple figures), and the term *congruent* for other, more complicated figures.–transl.

A line which is composed of portions of straight lines is called a *broken line*. Other lines, which are neither straight nor broken, are called *curves*.

5. The part of a straight line contained between two points A and B is called a *segment* of the line AB, or the distance AB. We can also consider the portion of a line which is not limited in one direction, but limited in the other by a point. This is called a *ray*.

FIGURE 1

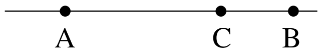


FIGURE 2

It follows from the remarks above that two arbitrary rays are congruent.

The distance AB is said to be *equal* to the distance $A'B'$ if the first segment can be moved onto the second in such a way that A falls onto A' and B onto B'.

After such a move, it follows from the two properties which define the notion of a straight line that the two segments will coincide entirely. Consequently, the definition of equal segments agrees with the definition of congruent figures given earlier.

Two equal segments AB, $A'B'$ can be made to coincide in two different ways; namely, the point A falling on A' and B on B', or A falling on B' and B on A'. In other words, one can turn the segment AB around in such a way that each of the points A, B takes the place of the other.

When two segments AB, BC are on the same line so that one of them is the extension of the other (Fig. 1), the segment AC is called the *sum* of the first two segments. The sum of two, and therefore of many, segments is independent of the order of the summands.[9]

In order to compare segments, we move them onto the same line, so that they start from the same point and are oriented in the same direction: for example AB and AC (Figures 1 and 2). If the points are in the order A, B, C (Fig. 1), the segment AC is the sum of AB and another segment BC; in this case AC is *greater* than AB; if on the contrary, the order is A, C, B, the segment AC is *smaller* than AB. In either case, the third segment BC which, when added to one of the first produces the other, is called the *difference* of the two segments. Finally, the points B and C may coincide, in which case we know that the two segments under consideration are equal.

On each line segment AB, there exists a point M, the *midpoint* of AB, equally distant from A and B; every point on the line between A and M is clearly closer to A than to B, and the opposite is true for points between M and B.

More generally, a line segment can be divided into any number of equal parts.[10]

6. The Plane. An infinite surface such that a line joining any two points on it is contained entirely on the surface is called a *plane*.

We will assume that *there is a plane passing through any three points in space*. A straight line drawn in a plane separates the surface into two regions called *half-planes*, situated on different sides of the line. One cannot travel along a continuous

[9] For two segments this follows immediately from the preceding paragraph.

[10] We mean by this that there *exist* points on AB which divide this segment into equal parts. The question as to whether these points can *actually* be determined with the help of available instruments will be considered later (Book III).

path from one of these regions to the other without crossing the line. These two half-planes can be superimposed by turning one of them about the straight line as a hinge.

We will first study figures drawn in a plane, the study of which is the subject of *plane geometry*.

7. The Circle. A *circle* is the geometric locus of points in a plane situated at a given distance from a given point O (Fig. 3) in this plane.[11]

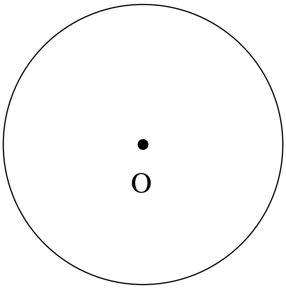


FIGURE 3

A segment joining the center of a circle to a point on the circle is called a *radius* of the circle. All the radii are therefore equal. This is illustrated by the spokes of a wheel, which has the shape of a circle.

According to the preceding definition, in order to express the fact that a point in the plane is on a circle located in that plane, it suffices to say that its distance to the center is equal to the radius.

Any circle divides the plane of which it is part into two regions, the *exterior*, unbounded, formed by the points whose distance to the center is greater than the radius; and the *interior*, bounded in all directions, formed by the points whose distance to the center is less than the radius. This interior region is called a *disk*.[12]

The locus of points in space situated at a fixed distance from a given point is a surface called a *sphere*.

It is clear that a circle is determined when we are given its plane, its center, and its radius.

When no confusion is possible, we denote a circle by the letter denoting its center, or by the two letters denoting one radius, the letter denoting the center being written first. Thus, the circle of figure 4 will be denoted by O, or (if there are several circles with center O) by OA.

Two circles with the same radius are congruent: it is clear that they will coincide if their centers are made to coincide.

Two circles can be superimposed in *infinitely many ways*: we can move one over the other in such a way that a given point M' of the second corresponds with

[11]The locus of the points in a plane situated at a fixed distance from a given point *outside* this plane (if such points exist) is also a circle; we will prove this in space geometry.

The locus of points in space situated at a given distance from a given point is called a *sphere*.

[12]Hadamard uses the standard French terminology, in which the points on a circle form a set called a *circonférence*, while the points inside the circle form a set called a *cercle*. The cognate English terms mean different things. –Transl.

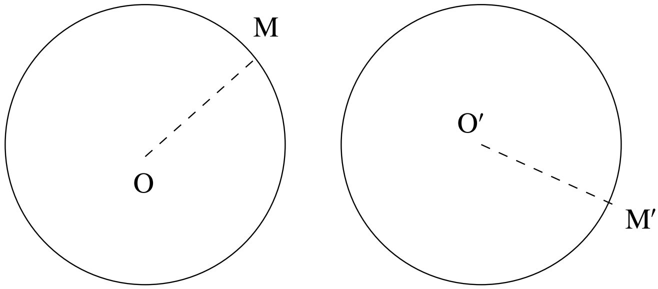


FIGURE 3b

a point M of the first (Fig. 3b). To do this it suffices to move radius OM onto radius $O'M'$ (Fig. 3b), which is possible since the two segments are equal.

8. A portion of a circle is called an *arc* ($\widehat{ApB}$, Fig. 4).

From the fact that two circles can be superimposed in infinitely many ways, it follows that it is possible to compare circular arcs *from the same circle or in two equal circles*, in the same way in which we compare line segments. For this purpose one transports the two arcs to a position where they have the same center, a common endpoint, and they lie on the same side of this common endpoint. Suppose that the two arcs in this positions are $\widehat{AB}$, $\widehat{AC}$; we will say that the first arc is greater than the second if, starting from the point A and moving along arc[13] $\widehat{AB}$ we encounter point C before point B (Fig. 5); the first arc is said to be smaller, if on the other hand the order is ABC (Fig. 4).

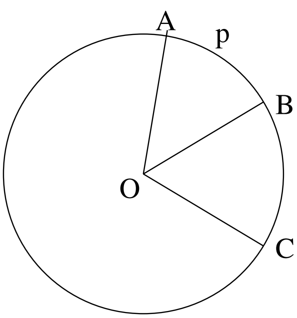


FIGURE 4

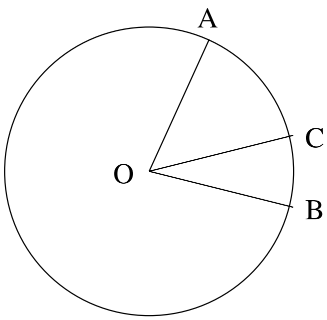


FIGURE 5

8b. We can also define the sum of two arcs $\widehat{AB}$, $\widehat{BC}$ (Fig. 4) on the same circle (or on equal circles) by moving them end-to-end.

A circular arc $\widehat{AB}$ can be divided into two or more equal parts,[14] just as a line segment can.

[13]It is essential here to specify the direction of motion (which was not necessary in the case of a straight line), because the points A, B divide the circle into two arcs, and the order in which we encounter the points B, C depends on the direction of motion.

[14]The remark made about straight line segments in footnote 10 holds here as well.

It is divided by its midpoint into two smaller arcs, of which one consists of those points M on the arc for which the arc $\widehat{AM}$ is greater than $\widehat{MB}$, the other of points for which $\widehat{AM}$ is smaller than $\widehat{MB}$.

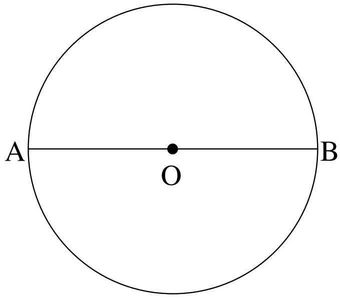


FIGURE 6

9. Two points of a circle are *diametrically opposite* (A, B, Fig. 6) if the line which joins them passes through the center. This segment is called a *diameter*. It is clear that a diameter has a length twice that of a radius.

A circle is clearly determined when we are given one of its diameters. Its center is then the midpoint of the diameter.

A diameter AB divides a circle into two arcs, which are, respectively, the portions of the circle situated in the two half-planes determined by line AB. These two arcs are *equal*: we can make them coincide by superimposing the two half-planes in question (**6**). Thus we have two *semicircles*.

In the same fashion, a disk is divided by a diameter into two equal parts, which can be superimposed in the same manner as the semicircles.

Book I
On the Straight Line

CHAPTER I

On Angles

10. The figure formed by two rays issuing from the same endpoint is called an *angle*. The point is called the *vertex* of the angle, and the two rays are its *sides*.

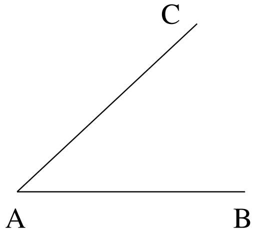


FIGURE 7

We denote an angle by the letter of its vertex, placed between two other letters which indicate the sides, often surmounted by a special symbol. If, however, the figure contains only one angle with the vertex considered, that letter will suffice to designate the angle. Thus, the angle formed by the two rays AB, AC (Fig. 7) will be denoted by $\widehat{BAC}$ or, more simply, by $\widehat{A}$.

Two angles are said to be *congruent*, in agreement with the definition of congruent figures (**3**) if they can be superimposed by a rigid motion.[1]

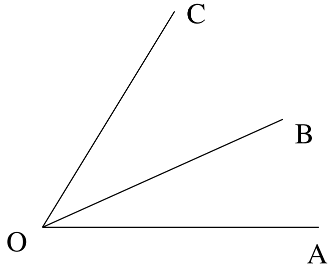


FIGURE 8

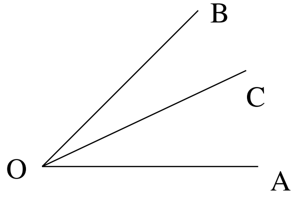


FIGURE 9

Two congruent angles, $\widehat{BAC}$, $\widehat{B'A'C'}$, can be superimposed in two different ways: namely, either AB takes the direction of $A'B'$ and AC the direction of $A'C'$, or the other way around. We pass from one to the other by turning the angle around onto itself, for instance, by moving the angle $\widehat{BAC}$ in such a way that AC occupies the position previously occupied by AB, and vice-versa.

[1]But see the footnote to **3**.–transl.

11. We say that two angles are *adjacent* if they have the same vertex, a common side, and so that they are located on opposite sides of this common side.

When angles $\widehat{AOB}$, $\widehat{BOC}$ are adjacent (Fig. 8), angle $\widehat{AOC}$ is called the *sum* of the two angles.

The sum of two or more angles is independent of the order of the angles added.

To compare angles, we move them so they have a common vertex, and a common side, and lie on the same side of this common side. Assume that $\widehat{AOB}$, $\widehat{AOC}$ are placed in this manner. If, rotating around point O, we encounter the sides in the order OA, OB, OC (Fig. 8), angle $\widehat{AOC}$ is equal to the sum of $\widehat{AOB}$ and another angle $\widehat{BOC}$; in this case angle $\widehat{AOC}$ is said to be *larger* than $\widehat{AOB}$, and the latter is *smaller* than $\widehat{AOC}$; if, on the contrary (Fig. 9) the order is OA, OC, OB, then angle $\widehat{AOC}$ is smaller than $\widehat{AOB}$. The angle $\widehat{BOC}$ which, added to the smaller angle, yields the larger angle, is the *difference* of the two angles.

Finally, in the intermediate case in which OB coincides with OC, the two angles are congruent (see **10**).

In the interior of every angle $\widehat{BAC}$ there is a ray AM which divides this angle into two congruent parts; it is called the *bisector* of the angle. The rays contained in the angle $\widehat{BAM}$ make a smaller angle with AB than with AC; the opposite is true for rays contained in the angle $\widehat{MAC}$.

An angle is said to be the *double, triple,* etc., of another if it is the sum of two, three, etc., angles equal to this other angle. The smaller angle will then be called a *half, third,* etc., of the larger one.

Remark. *The size of an angle does not depend on the size of its sides*, which are rays (**5**), and which we must imagine as extending indefinitely.

12. As we have said, an angle is determined by two rays, such as OA, OB (Fig. 10). If we extend OA past point O to form OA', and in the same way extend OB to form OB', we obtain a new angle $\widehat{A'OB'}$.

Two angles $\widehat{AOB}$, $\widehat{A'OB'}$ *such that the sides of one are extensions of the sides of the other* are called *vertical angles*.

Theorem. *Two vertical angles are congruent.*

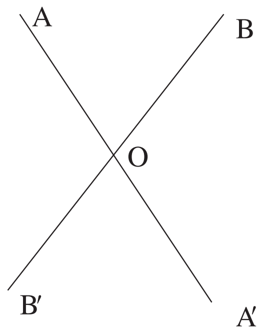


Figure 10

Indeed, let us turn (**10**) the angle $\widehat{BOA'}$ onto itself (Fig. 10). The side OB will take the place of OA', and, since $0A'$ will take the former position of OB, this means that ray OA, which is the extension of OA', will coincide with OB', the extension of OB. Thus angle $\widehat{AOB}$ is superimposed on angle $\widehat{A'OB'}$, and therefore these two angles are congruent.

13. Arcs and angles. Every ray issuing from the center of a circle meets this circle in one and only one point.

Every *angle* $\widehat{AOB}$ (Fig. 4) with its vertex at the center O of a circle determines an *arc* $\widehat{AB}$ having as endpoints the intersections of the sides of the angle with the circle. In general, (see however **20b**), we consider the arc which is less than a semicircle.

Conversely, every arc less than a semicircle can be thought of as determined by an angle with its vertex at the center of the circle, formed by the rays passing through the endpoints of the arc.

Theorem. *On the same circle, or on equal circles*:

1°. *To equal arcs (smaller than a semicircle) there correspond equal central angles*;[2]

2°. *To unequal arcs (smaller than a semicircle) there correspond unequal central angles, and the greater angle corresponds to the greater arc*;

3°. *If an arc (smaller than a semicircle) is the sum of two others, the corresponding central angle is the sum of the angles associated with the smaller arcs.*

1°, 2°. Let $\widehat{AB}$, $\widehat{AC}$ (Fig. 4) be the two arcs, drawn on the same circle, starting from the same point A, in the same direction (**8**). The two angles $\widehat{AOB}$, $\widehat{AOC}$ are therefore placed as in **11**. But then rays OA, OB, OC are placed in the same order as points A, B, C on the circle. Moreover, if rays OB, OC coincide, then so do the points B, C, and conversely.

3°. Let us recall that the sum of two arcs (**8b**) is formed by placing them as arcs $\widehat{AB}, \widehat{BC}$ are placed (Fig. 4). Then the angle $\widehat{AOC}$ corresponding to the sum of these arcs will be the sum of $\widehat{AOB}$ and $\widehat{BOC}$, because these angles are adjacent.

According to this result, in order to compare angles, one can draw circles with the same radius, centered at the vertices of the angles, and compare the arcs intercepted on these circles.

The division of angles into two or more equal parts corresponds to the division into equal parts of the corresponding arc of a circle centered at the vertex of the angle.

14. Perpendiculars. Right angles. We say that two lines are *perpendicular* to each other if, among the four angles they form, two adjacent angles are equal to each other. For example, line AOA' (Fig. 11) is perpendicular to BOB' if the angles numbered 1 and 2 in the figure are equal. In such a case, the four angles at O are equal to each other, since angles $\widehat{3}$ and $\widehat{4}$ (Fig. 11) are equal respectively to $\widehat{1}$ and $\widehat{2}$, because they are vertical angles.

An angle whose sides are perpendicular is called a *right* angle.

[2]This is the standard term in American texts. Hadamard uses the locution 'angles with their vertex at the center' or 'angles at the center'. –transl.

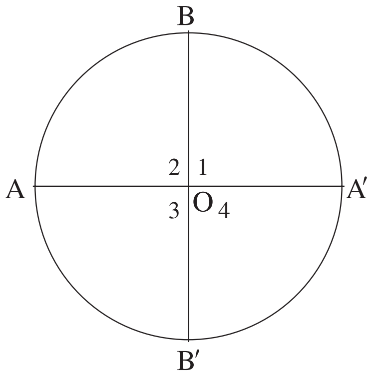


FIGURE 11

THEOREM. *In a given plane, through a given point on a line, one can draw one, and only one, perpendicular to this line.*

Suppose we want to draw a perpendicular to line AA' through point O (Fig. 11). It suffices to draw a circle with center O which cuts the line at A and A', then find point B which divides semicircle AA' into two equal parts. Then OB will be the required perpendicular; and conversely, a perpendicular to AA' drawn through point O must divide arc $\widehat{AA'}$ in half.

COROLLARY. *We see that a right angle intercepts an arc equal to one fourth of a circle centered at the vertex of the angle.*

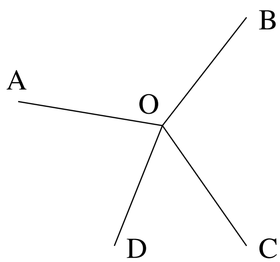


FIGURE 12

All right angles are equal, since two such angles will intercept equal arcs on equal circles centered at their vertices.

15. *If several rays are drawn issuing from the same point, the sum of the successive angles thus formed* ($\widehat{AOB}$, $\widehat{BOC}$, $\widehat{COD}$, $\widehat{DOA}$, Fig. 12) *is equal to four right angles.*

Indeed, the sum of the arcs intercepted by these angles on a circle centered at their vertex is the entire circle.

If several rays are drawn issuing from a point on a line, in such a way that all the rays are on one side of the line (Fig. 13), *the sum of the successive angles thus formed is equal to two right angles.*

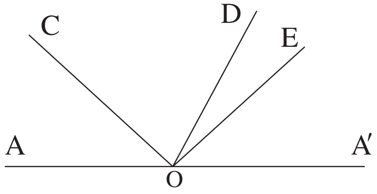


FIGURE 13

Indeed, the sum of the intercepted arcs is a semicircle.

Conversely, *if two or more angles with the same vertex, and each adjacent to the next, ($\widehat{AOC}$, $\widehat{COD}$, $\widehat{DOE}$, $\widehat{EOA'}$, Fig. 13) have a sum equal to two right angles, then their outer sides form a straight line.*

Indeed, these outer sides cut a circle centered at the common vertex of the angles at two diametrically opposite points, since the arc they intercept is a semicircle.

REMARK. The angle AOA' (Fig. 13) whose sides form a straight line is called a *straight* angle.

15b. THEOREM. *The bisectors of the four angles formed by two concurrent lines form two straight lines, perpendicular to each other.*

Let AA', BB' (Fig. 14) be two lines which intersect at O and form the angles $\widehat{AOB}$, $\widehat{BOA'}$, $\widehat{A'OB'}$, $\widehat{B'OA}$, whose bisectors are Om, On, Om', On'. We claim:

1°. That Om, Om' are collinear, as are On, On'.

2°. That the two lines thus formed are perpendicular.

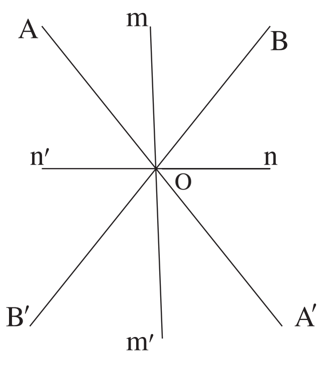


FIGURE 14

First, Om is perpendicular to On because, since the sum of $\widehat{AOB}$ and $\widehat{BOA'}$ is two right angles, half of each, $\widehat{mOB}$ and $\widehat{BOn}$, must add up to a right angle.

Applying the same reasoning to angles $A'OB'$ and $B'OA$, we see that Om' and On are perpendicular. Therefore Om' is the extension of Om, and likewise On' is the extension of On.

16. An angle which is less than a right angle is called an *acute* angle; an angle greater than a right angle is called an *obtuse* angle.

Two angles are said to be *complementary* if their sum is a right angle; *supplementary* if their sum is equal to two right angles.

17. Angle Measure. The *ratio* of two quantities of the same kind is[3] the number which expresses how many times one of the quantities is contained in the other. For instance, if, in dividing the segment AB into five equal parts, one of these parts is contained exactly three times in the segment BC, then the ratio of BC to AB is said to be equal to 3/5. If, on the other hand, a fifth of AB is not contained an exact number of times in BC, for example if it is contained in BC more than three times, but less than four times, then 3/5 would be an *approximate value* of the ratio $\frac{BC}{AB}$: it would be within one fifth less than the value of the ratio (the value 4/5 would be within 1/5 more).

The ratio of two quantities a, b of the same kind is equal to the ratio of two other quantities a', b' also of the same kind (but not necessarily of the same kind as the first two) if, for every n, the approximation to within $1/n$ of the first ratio is equal to the approximation within $1/n$ of the second ratio.

The *measure* of a quantity, relative to a quantity of the same kind chosen as unit, is the ratio of the given quantity to the unit.

In addition we have the following properties (*Leçons d'Arithmétique* by J. Tannery):

1°. Two quantities of the same kind, which have the same measure relative to the same unit, are equal;

2°. The ratio of two quantities of the same kind is equal to the ratio of the numbers which serve as their measures relative to the same unit;

3°. The ratio of two numbers is the same as the quotient of the two numbers; etc.

Theorem. *In the same circle, or in equal circles, the ratio of two central angles is equal to the ratio of their subtended arcs.*

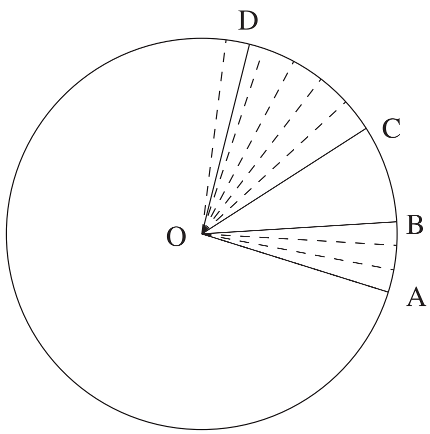


Figure 15

[3]See *Leçons d'Arithmétique* by J. Tannery, chapters X and XIII.

Consider[4] (Fig. 15) two arcs $\widehat{AB}$, $\widehat{CD}$ on circle O. Divide central angle $\widehat{COD}$ into three equal parts (for example), and suppose that one of these thirds can fit four, but not five times, into the angle $\widehat{AOB}$; the value (smaller by less than 1/3) of the ratio $\frac{\widehat{AOB}}{\widehat{COD}}$ is $\frac{4}{3}$.

But dividing the angle $\widehat{COD}$ into three equal parts, we have at the same time divided arc $\widehat{CD}$ into three equal parts (**13**). If a third of $\widehat{COD}$ fits four, but not five times, into angle $\widehat{AOB}$, it follows the same way that a third of arc $\widehat{CD}$ fits four, but not five times, into arc AB. The values to within $1/3$ of the two ratios are therefore equal, and similarly the values to within $1/n$ will be equal for any integer n. The theorem is therefore proved.

Corollary. *If we take as the unit of angle measure the central angle which intercepts a unit arc, then every central angle will have the same measure as the arc contained between its sides.*

This statement is exactly the same as the preceding one, since the measure of a quantity is its ratio to the unit.

Supposing, as we will from now on, that *on each circle we choose as unit arc an arc intercepted by a unit central angle,* the preceding corollary can be stated in the abridged form: *A central angle is measued by the arc contained between its sides.*

18. The definitions reviewed above allow us to establish an important convention.

From now on, we will be able to suppose that all the quantities about which we reason have been measured with an appropriate unit chosen for each kind of quantity. Then, in all the equations we write, the quantities which appear on the two sides of an equality will represent not the quantities themselves, but rather their measures.

This will allow us to write a number of equations which otherwise would have no meaning. For instance, we might equate quantities of different kinds, since we will be dealing only with the *numbers* which measure them, the meaning of which is perfectly clear. We will also be able to consider the product of any two quantities, since we can talk about the product of two numbers, etc.

In fact, whenever we write the equality of two quantities of the same kind, this equality will have the same meaning as before, since the equality of these quantities is the same as the equality of their measures.

Following this convention we can write:

$$\widehat{AOB} = \text{arc } \widehat{\text{AB}},$$

where AB is an arc of a circle and O is the center. It is important however to insist on the fact that the preceding equality assumes that the unit of angle measure and the unit of arc measure have been chosen in the manner specified above.

[4]This theorem becomes obvious if we consider the following proposition from arithmetic (Tannery, *Leçons d'Arithmétique*: *Two quantities are proportional if:* 1° *To any value of the first, there corresponds an equal value of the second, and* 2° *the sum of two values of the first corresponds to the sum of the corresponding values of the second.* These two conditions hold here (**13**).

The argument in the text does no more than reproduce, for this particular case, the proof of this general theorem of arithmetic.

18b. Traditionally, the circle has been divided into 360 equal parts called *degrees*, each of which contains 60 *minutes*, which are themselves divided into 60 *seconds*. One can then measure arcs in degrees and, correspondingly, angles will also be measured in degrees, and the number of degrees, minutes, and seconds of the angle will be the same as the number of degrees, minutes, and seconds of the arc intercepted by this angle on a circle centered at its vertex. A right angle corresponds to one fourth of a circle; that is, to 90 degrees. It follows that the measure of an angle at the center of a circle does not depend on the radius of the circle on which one measure the arcs, since the chosen unit of angle measure (the degree) has a value independent of this radius, namely one ninetieth of a right angle.

In writing angles (or arcs) in degrees, minutes, and seconds, we use an abridged notation: an angle of 87 degrees, 34 minutes, and 25 seconds is written: $87°34'25''$.

The introduction of decimal notation in all other kinds of measurements has led to the use of another mode of division, in which the circle is divided into 400 equal parts called *gradients* (or *grads*). The grad, a little smaller than the degree, is, as we see, one hundredth of a right angle. The grad is divided decimally, so there is actually no need for special names for its parts, which are written using the rules of decimal notation. Thus we can speak of the angle $3^G.5417$ (that is, 3 grads and 5417 ten-thousandths).

Nonetheless, a hundredth of a grad is often called a *centesimal minute*, and is indicated by the symbol ` (to distinguish it from the *hexagesimal* minute, which is a sixtieth of a degree). Likewise, one hundredth of a centesimal minute is called a *centesimal second*, denoted by the symbol ``. The angle just considered could then be written as $3^G.54`17``$.

A grad is equal to $\frac{360}{400}^\circ$, that is $\frac{9}{10}$ of a degree, or $54'$. A degree equals $\frac{400}{360}^G = 1^G.11`11``1\ldots$ (in other words, $\frac{10}{9}$ of a grad).

19. THEOREM. *Given a line, and a point not on the line, there is one, and only one, perpendicular to the line passing through the given point.*

1°. *There is one perpendicular.* Consider point O and line xy (Fig. 16). Using xy as a hinge, let us turn the half-plane containing O until it overlaps the other half-plane. Supposing point O falls on O', we join OO'. This line intersects xy because it joins two points in different half-planes. If I is the point of intersection, the angles $\widehat{O'Ix}$ and $\widehat{OIx}$ are equal because one of them can be superimposed on the other by rotation around xy. Thus the lines xy and OO' are perpendicular.

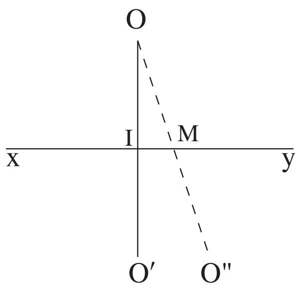


FIGURE 16

2°. *There is only one perpendicular*. Indeed, assume that OM is a perpendicular to xy passing through O. Extend the segment OM by its own length to MO''.

Again turning the half-plane containing O around xy, the segment MO will fall on MO'' because the angles $\widehat{OMx}$ and $\widehat{O''Mx}$ are right angles, and hence are equal. Since $MO'' = MO$, the point O falls on O'', and therefore O' and O'' coincide, as do the lines OO' and OO''. QED

19b. The *reflection* of a point O in a line xy is the endpoint of the perpendicular from O to the line, extended by the length of a segment equal to itself. It follows from the preceding considerations that this reflection is none other than the new position occupied by O after a rotation around xy.

Given an arbitrary figure, we can take the reflection of each of its points. The set of these reflections constitutes a new figure, called the reflection of the first. We see that in order to obtain the reflection of a given figure in line xy, we can turn the plane of the figure around xy, so that each half-plane determined by the line falls on the other, then note the position taken by the original figure. It follows that:

THEOREM. *A plane figure is congruent to its reflection.*

COROLLARY. *The reflection of a line is a line.*

When a figure coincides with its reflection in line xy, we say that it is symmetric in this line, or that this line is an *axis of symmetry* of the figure.

20. To make a figure F coincide with its reflection F', we had to use a motion which took the figure out of its plane. We must note that this superposition is not possible without such a movement; this holds because the *sense of rotation* is reversed in the two figures. We will now explain what this means.

First, let us remark that the plane of the figure divides space into two regions. For brevity, let us call one of these the region situated above the plane; the other, the region below the plane.

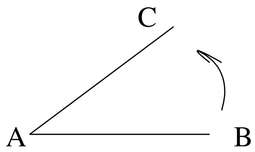


FIGURE 17

Consider now an angle $\widehat{BAC}$ in the figure F, which can be viewed as being described by a ray moving inside this angle from the position AB to the position AC (Fig. 17). *Viewed from above* this angle $\widehat{BAC}$ will be said to have an *inverse* sense of rotation or a *direct* sense, according as the moving ray turns clockwise or counterclockwise.[5] To be definite, let us consider the second situation. In this case, an observer lying along AB, with his feet at A, his head in the direction of B, and looking down, will see the side AC on his left; and therefore, if still lying on AB, he faces AC, the region below the plane will be on his right.

It is clear that to describe the sense of rotation of an angle *viewed from below*, this discussion can be repeated, with the word *above* replaced by the word *below*, and vice-versa.

[5]We note that in order to define the sense of rotation, we must take into account the order of the sides. Thus angle $\widehat{BAC}$ has the opposite sense from $\widehat{CAB}$.

Since an observer lying on AB and facing AC will necessarily have the region above the plane on his left, if the region below is on his right, and vice-versa, we see that *the sense of rotation changes depending on whether we view the angle from one side of the plane or from the other.*[6]

Let us now suppose that an angle is moved in any way at all, without ever leaving the plane. An observer moving along with the angle will not change his position relative to the regions of space above and below the plane, and therefore *the sense of rotation is not altered by a motion which does not leave the plane.*

To prove that such a motion cannot make a figure F coincide with its reflection F', it suffices therefore to see that the senses of rotation of the two figures are opposite. But we have seen that we can make F coincide with F' by turning the plane around xy (**19**). As a result of this rotation, the points above the plane are moved below it, and vice-versa. The sense of rotation of an angle in F, viewed from above, is therefore the same as the sense of F' viewed from below, so that the two angles, viewed from the same side, have opposite senses of rotation. QED

20b. Remarks. I. We say that a plane is *oriented* when a sense of rotation for angles has been chosen as the direct sense. According to the preceding paragraph, orienting a plane amounts to deciding which region of space will be said to be *above* the plane.

II. It is convenient to regard an angle with vertex O as being described by a ray starting from O which at first coincides with the first side, and then turns about O in the plane (in the direct or inverse sense) until it coincides with the second side. If the ray has made a quarter of a complete turn, the angle is a right angle. If it has made one half of a complete turn, the angle is a straight angle, such as $\widehat{AOA'}$ (Figures 11 or 13).

Nothing prevents us, by the way, from considering angles greater than two right angles, since our ray can make more than half a complete turn.

III. Clearly, an arc of a circle, like an angle, can have a direct or an inverse sense, which depends, of course, on the order in which one gives its endpoints.

Two points A, B on a circle divide it into two different arcs of which (unless the two points are diametrically opposite) one is a *minor* arc (less than a semicircle), and the other a *major* arc. It must be noted that, since the endpoints A, B are given in a certain order (for example if A is first and B is second), these two arcs have opposite senses.

Exercises

Exercise 1. Given a segment AB and its midpoint M, show that the distance CM is one half the difference between CA and CB if C is a point on the segment. If C is on line AB, but not between A and B, then CM is one half the sum of CA and CB.

Exercise 2. Given an angle $\widehat{AOB}$, and its bisector OM, show that angle $\widehat{COM}$ is one half the difference of $\widehat{COA}$ and $\widehat{COB}$ if ray OC is inside angle $\widehat{AOB}$; it is the supplement of half the difference if ray OC is inside angle $\widehat{A'OB'}$ which is vertical to $\widehat{AOB}$; it is one half the sum of $\widehat{COA}$ and $\widehat{COB}$ if OM is inside one of the other angles $\widehat{AOA'}$ and $\widehat{BOB'}$ formed by these lines.

[6]For this reason, writing viewed through a transparent sheet of paper appears reversed.

Exercise 3. Four rays OA, OB, OC, OD issue from O (in the order listed) such that $\widehat{AOB} = \widehat{COD}$ and $\widehat{BOC} = \widehat{DOA}$. Show that OA and OC are collinear, as are OB and OD.

Exercise 4. If four consecutive rays OA, OB, OC, OD are such that the bisectors of angles $\widehat{AOB}$, $\widehat{COD}$ are collinear, as are the bisectors of $\widehat{BOC}$, $\widehat{AOD}$, then these rays are collinear in pairs.

CHAPTER II

On Triangles

21. A plane region bounded by line segments is called a *polygon* (Fig 18). The segments are called the *sides* of the polygon, and their endpoints are called its *vertices.*

In fact, we will generally use the term *polygon* only for a portion of the plane bounded by a single contour which can be drawn with a single continuous stroke. The plane region which is shaded in Figure 19 will not be a polygon for us.

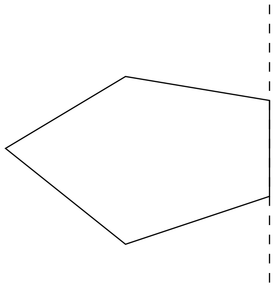


FIGURE 18

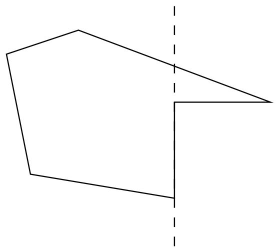


FIGURE 18b

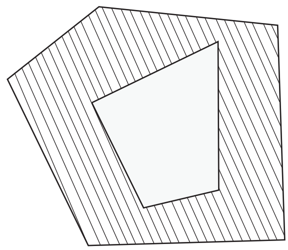


FIGURE 19

A polygon is said to be *convex* (Fig. 18) if, when we extend any side indefinitely, none of the lines thus formed crosses the polygon. In the opposite case (Fig. 18b) the polygon is said to be *concave*.

Polygons are classified according to the number of sides they have. Thus the simplest polygons are: the polygon with three sides or *triangle*, the polygon with 4 sides or *quadrilateral*, the polygon with 5 sides, or *pentagon*, the polygon with six sides or *hexagon*. We will also consider polygons with 8, 10, 12, and 15 sides, called *octagons, decagons, dodecagons, and pentadecagons*.

Any segment joining two non-consecutive vertices of a polygon is called a *diagonal*.

REMARK. More generally, an arbitrary broken line, even one whose sides cross (as in Fig. 19b), will sometimes be called a *polygon*. In this case, when the broken line does not bound a unique region, the polygon is said to be *improper*. However, when we want to specify that the polygon is of the first type (Figures 18 and 18b), and not like the one in Figure 19b, we will speak of a *proper* polygon.

22. Among triangles we distinguish in particular:

The *isosceles* triangle. This name is used for a triangle with two equal sides. The common vertex of the equal sides is called *the vertex* of the isosceles triangle, and the side opposite the vertex is called *the base*;

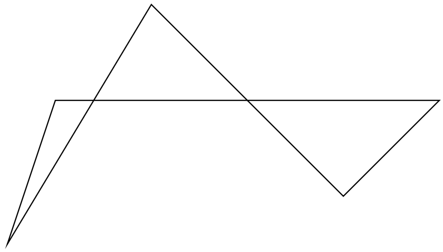


FIGURE 19b

The *equilateral* triangle, which has all three sides equal;

The *right* triangle, which has a right angle. The side opposite the right angle is called the *hypotenuse*.[1]

A perpendicular dropped from a vertex of a triangle onto the opposite side is called an *altitude* of the triangle; a *median* is a line which joins a vertex with the midpoint of the opposite side.

23. THEOREM. *In an isosceles triangle, the angles opposite the equal sides are equal.*

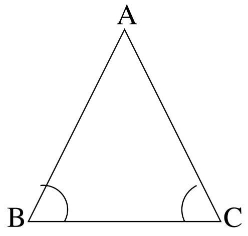


FIGURE 20

Suppose ABC is an isosceles triangle (Fig. 20). Let us turn angle $\widehat{BAC}$ onto itself (**10**) so that AB lies along the line of AC and vice-versa. Since AB and AC are equal, point B will fall on C, and C on B. Angle $\widehat{ABC}$ will therefore fall on $\widehat{ACB}$, so that these angles are equal. QED

CONVERSE. *If two angles of a triangle are equal, the triangle is isosceles.*

In triangle ABC, suppose $\widehat{B} = \widehat{C}$. Let us move this triangle so that side BC is turned around, switching points B and C. Since $\widehat{ABC} = \widehat{ACB}$, side BA will assume the direction of of CA, and conversely. Point A, which is the intersection of BA and CA, will therefore retain its original position, so that segment AB falls on segment AC. QED

COROLLARY. *An equilateral triangle is also equiangular (that is, its three angles are equal), and conversely.*

[1]The original text does not use term "leg" for a side of a right triangle that is not the hypotenuse. However, we will use this standard English term freely in the translation. –transl.

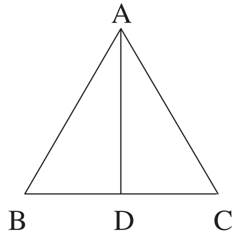


FIGURE 21

THEOREM. *In an isosceles triangle, the bisector of the angle at the vertex is perpendicular to the base, and divides it into two equal parts.*

In isosceles triangle ABC (Fig. 21), let AD be the bisector of $\widehat{A}$. In turning angle $\widehat{BAC}$ around on itself, this bisector does not move, and therefore neither does the point D where this bisector cuts the base. Segment DB falls on DC and angle $\widehat{ADB}$ on $\widehat{ADC}$. Therefore $DB = DC$ and $\widehat{ADB} = \widehat{ADC}$. QED

REMARK. In triangle ABC we can consider:

1°. The bisector of $\widehat{A}$;
2°. The altitude from A;
3°. The median from the same point;
4°. The perpendicular to the midpoint of BC.

In general, these four lines are distinct from each other (see Exercise 17). The preceding theorem shows that, in an isosceles triangle, all of these are the same line, *which is a line of symmetry of the triangle* (**19b**).

This theorem can thus be restated: *the altitude of an isosceles triangle is also an angle bisector and a median*; or *the median of an isosceles triangle is at the same time an angle bisector and an altitude; the perpendicular bisector of the base passes through the vertex and bisects the angle at the vertex.*

COROLLARY. *In an isosceles triangle, the altitudes dropped from the endpoints of the base are equal; the same is true about the medians from the endpoints of the base, about the bisectors of the angles at these points*, etc., because these segments are symmetric to each other.

24. The following propositions, known under the name of **cases of congruence of triangles**, give necessary and sufficient conditions for two triangles to be congruent.

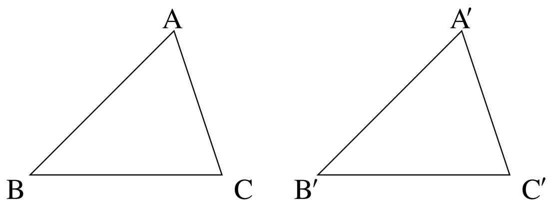


FIGURE 22

1^{st} **case: ASA.** *Two triangles are congruent if they have an equal side contained between corresponding equal angles.*

Suppose the triangles are ABC, $A'B'C'$ (*fig.* 22) in which $BC = B'C'$, $\widehat{B} = \widehat{B'}$, and $\widehat{C} = \widehat{C'}$. Let us move angle $\widehat{B'}$ onto angle $\widehat{B}$ so that side $B'A'$ lies along the line of BA and $B'C'$ along the line of BC. Since $BC = B'C'$, the point C' falls on C. Now, since $\widehat{C'} = \widehat{C}$, side $C'A'$ assumes the direction of CA, and therefore point A' falls on the intersection of BA and CA; that is, on A. This establishes the congruence of the two figures.

2^{nd} **case: SAS.** *Two triangles are congruent if they have an equal angle contained between corresponding equal sides.*

Suppose the two triangles are ABC, $A'B'C'$ (Fig. 22) in which $\widehat{A} = \widehat{A'}$, $AB = AC$, and $AC = A'C'$.

Let us move angle $\widehat{A'}$ over angle $\widehat{A}$ so that $A'B'$ assumes the direction of AB and $A'C'$ assumes the direction of AC. Since $A'B' = AB$, point B' will fall on B, and likewise C' on C. Therefore side $B'C'$ coincides with BC, and the superposition of the two figures is complete.

3^{rd} **case: SSS.** *Two triangles are congruent if they have three equal corresponding sides.*

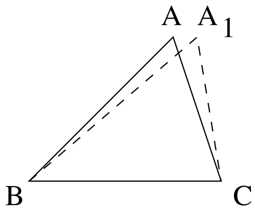


FIGURE 23

Consider triangles ABC and $A'B'C'$ such that $AB = A'B'$, $AC = A'C'$, and $BC = B'C'$, and let us move the second triangle so that side $B'C'$ coincides with BC, and the two triangles are on the same side of line BC. Denote the new position of A' by A_1. We claim that point A_1 coincides with A. This is obvious if BA_1 has the same direction as BA or if CA_1 has the same direction as CA. If this were not the case, we would have formed isosceles triangles BAA_1 and CAA_1 (Fig. 23) and the perpendicular bisector of AA_1 would have to pass through B and C (**23**, Corollary); in other words, this bisector would have to be line BC. This however is not possible, because points A and A_1 are on the same side of BC, so that BC cannot pass through the midpoint of AA_1. Thus the only possibility is that points A and A_1 must be the same point; in other words, the given triangles coincide.

REMARKS. I. In order to establish that $A = A_1$, we have investigated what would happen if these two points were distinct. Arriving, in this case, at a conclusion which is clearly false, we concluded that this possibility does not arise. This is a very useful method of reasoning, called proof *by contradiction*.

II. In a triangle there are six main elements to consider; namely, the three angles and the three sides. We have seen that if we establish the equality of three of these elements (properly chosen) in two triangles, we can conclude that the two triangles are congruent and, in particular, that the remaining elements are equal as well.

III. Two congruent triangles (or, more generally, polygons) may differ in their sense of rotation (**20**). In this case, they can only be superimposed by a motion outside the plane. On the other hand, if the sense of rotation is the same, the two polygons can be superimposed by moving them within the plane, as we will have occasion to see later on.

25. The angle formed by a side of a convex polygon and the extension of the next side is called an *exterior angle* of the polygon.

THEOREM. *An exterior angle of a triangle is greater than either of the non-adjacent interior angles.*

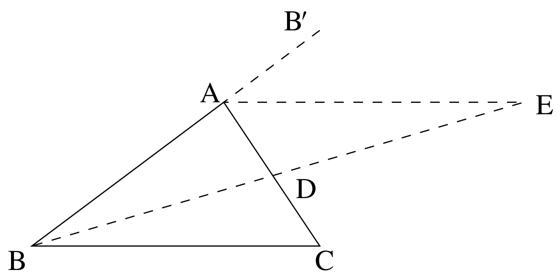


FIGURE 24

Let the triangle be ABC, and construct the exterior angle $\widehat{B'AC}$ (Fig. 24). We claim that this angle is greater, for example, than the interior angle $\widehat{C}$. To show this, we construct median BD, which we extend by its own length to a point E. The point E being inside[2] angle $\widehat{B'AC}$, this last must be greater than angle $\widehat{EAC}$.

But this last angle is precisely equal to $\widehat{C}$; indeed, triangles DAE, DBC are congruent, having an equal angle between equal sides: the angles at D are equal because they are vertical angles, and $AD = DC$, $BD = DE$ by construction. Thus exterior angle $\widehat{B'AC}$ is greater than interior angle C. QED

Exterior angle $\widehat{B'AC}$ is the supplement of interior angle $\widehat{A}$. Since angle C is smaller than the supplement of $\widehat{A}$, the sum $\widehat{A} + \widehat{C}$ is less than two right angles. Our theorem can thus be restated: *The sum of any two angles of a triangle is less than two right angles*. In particular, *a triangle cannot have more than one right or obtuse angle*.

THEOREM. *In any triangle, the greater angle lies opposite the greater side.*

In triangle ABC, suppose $AB > AC$. we will show that $\widehat{C} > \widehat{B}$. To show this, we take, on AB, a length $AD = AC$ (Fig. 25). It follows from the hypothesis that point D is between A and B, and therefore angle $\widehat{ACD}$ is less than $\widehat{C}$. But in

[2]Point E is on the same side of line BAB' as D (otherwise line DE, which has the common point B with BAB', would have to cut it again between D and E, and this is impossible), and thus on the same side as C. But B and E are on different sides of AC, which BE crosses at D.

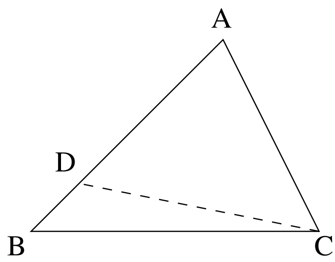


FIGURE 25

isosceles triangle ACD, $\widehat{ACD} = \widehat{ADC}$, which is greater than $\widehat{B}$ by the preceding theorem applied to triangle DCB. The theorem is thus proved.

Conversely, *the greater side corresponds*[3] *to the greater angle.*

This statement is obviously equivalent to the preceding one.

26. THEOREM. *Any side of a triangle is less than the sum of the other two.*

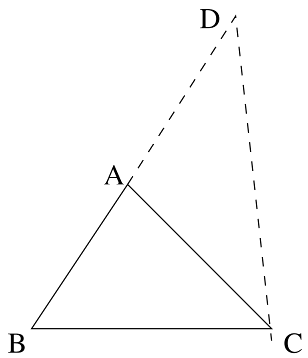


FIGURE 26

In triangle ABC, we extend side AB to a point D such that $AD = AC$ (Fig. 26). We must prove that[4] $BC < BD$. Drawing CD, we see that angle $\widehat{D}$, which is equal to angle $\widehat{ACD}$ (**23**), is therefore less than $\widehat{BCD}$.

The desired inequality thus follows from the preceding theorem applied to the triangle BCD.

COROLLARIES. I. *Any side of a triangle is greater than the difference between the other two.*

Indeed, the inequality $BC < AB + AC$ gives, after subtracting AC from both sides:

$$BC - AC < AB.$$

[3]The side *corresponding* to an angle is the side opposite it.
[4]The theorem is obvious if BC is not the largest side of the triangle.

II. *For any three points A, B, C, each of the distances AB, BC, AB is at most equal to the sum of the other two, and at least equal to the difference of the other two, equality holding if the three points are collinear.*

THEOREM. *A line segment is shorter than any broken line with the same endpoints.*

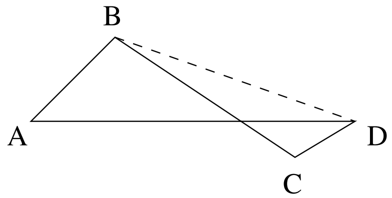


FIGURE 27

If the broken line has only two sides, the theorem reduces to the preceding one. Consider next a broken line with three sides $ABCD$ (Fig. 27). Drawing BD, we will have $AD < AB + BD$ and, since $BD < BC + CD$, we have

$$AD < AB + BD < AB + BC + CD.$$

The theorem is thus proven for a broken line with three sides. The same reasoning can be used successively for broken lines with 4 sides, 5 sides, etc. Therefore the theorem is true no matter how large the number of sides.

27. The sum of the sides of a polygon, or of a broken line, is called its *perimeter*.

THEOREM. *The perimeter of a convex broken line is less than that of any broken line with the same endpoints which surrounds it.*

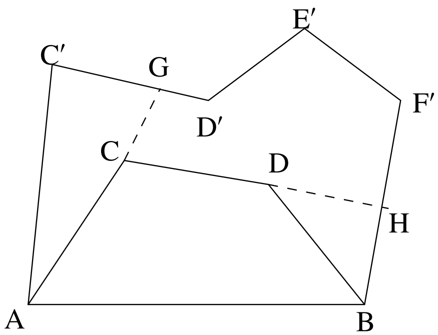


FIGURE 28

Consider the convex line $ACDB$ and the surrounding line $AC'D'E'F'B$ (Fig. 28). We extend sides AC, CD in the sense indicated by $ACDB$; that is, the side AC past C and the side CD past D. These extensions do not intersect the interior of polygon $ACDB$ because of its convexity. Suppose they cut the surrounding line in G and H respectively.

The path $ACDB$ is shorter than $ACHB$ because they have a common portion ACD, and the remainder BD of the first is less than the remainder DHB of the second. In turn, the path $ACHB$ is shorter than $AGD'E'F'B$ because, after removing the common parts AC, HB, the segment CH which is left is smaller than the broken line $CGD'E'F'H$. In the same way, $AGD'E'F'B$ is less than $AC'D'E'F'B$, because AG is less than $AC'G$. Thus

$$ACDB < ACHB < ACGD'E'F'B < AC'D'E'F'B.$$

QED

COROLLARY. *The perimeter of a convex polygon is less than the perimeter of any closed polygonal line surrounding it completely.*

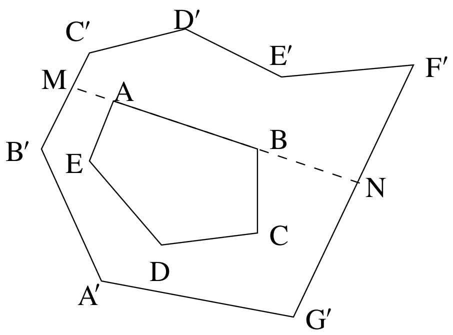


FIGURE 29

Consider (Fig. 29) convex polygon $ABCDE$, and polygonal line $A'B'C'D'E'\ F'G'A'$ which surrounds it completely. We extend side AB in both directions until it intersects the surrounding polygon at M, N. By the preceding result, the length of the path $AEDCB$ is less than $AMB'A'G'NB$, and therefore the perimeter of $AEDCBA$ is less than the perimeter of the polygon $NMB'A'G'N$. This perimeter, in turn, is less than the surrounding line because the part $MB'A'G'N$ is common to both, and $MN < MC'D'E'F'N$.

28. THEOREM. *If two triangles have a pair of unequal angles contained by two sides equal in pairs, and the third sides of the triangles are unequal, then the greater side is opposite the greater angle.*

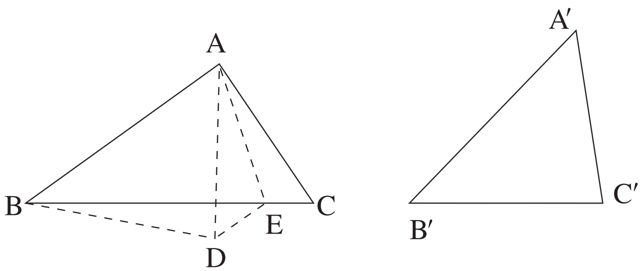


FIGURE 30

Consider triangles ABC, $A'B'C'$ such that $AB = A'B'$, $AC = A'C'$, and $\widehat{A} > \widehat{A'}$ (Fig. 30). We want to prove that $BC > B'C'$. Let us move the second triangle so that $A'B'$ coincides with AB. Since $\widehat{A'}$ is less than A, side $A'C'$ will move to a position AD in the interior of angle $\widehat{BAC}$. We construct the bisector AE of $\widehat{DAC}$. This segment is also inside $\widehat{BAC}$, and, as B and C lie on different sides of this line, it must intersect side BC in some point E located between B and C. If we draw DE, we see that triangles ACE and ADE are congruent, because they have an equal angle (with vertex A) between two corresponding equal sides (AE in common, $AC = A'C' = AD$). Therefore $DE = EC$. The inequality $BD < BE + ED$ provided by the triangle BDE then gives us

$$BD < BE + EC,$$

or

$$BD < BC.$$

QED

Conversely. *If, in two triangles, two pairs of sides are equal, but the third sides are unequal, then the angles opposite the unequal sides are unequal, and the greater angle is opposite the greater side.*

This statement is equivalent to the preceding one.

REMARK. The preceding theorem does not require the third case of congruence for triangles (SSS). It therefore provides another proof of that case.

Indeed, if we have $AB = A'B'$, $AC = A'C'$ and, in addition $BC = B'C'$, the angles $\widehat{A}$ and $\widehat{A'}$ would have to be equal, or else BC and $B'C'$ could not be equal. Knowing now that $\widehat{A} = \widehat{A'}$ the two triangles are congruent by the second case (SAS).

Exercises

Exercise 5. Prove that a triangle is isosceles:
1°. if an angle bisector is also an altitude;
2°. if a median is also an altitude;
3°. if an angle bisector is also a median.

Exercise 6. On side Ox of some angle, we take two lengths OA, OB, and on side Ox' we take two lengths OA', OB', equal respectively to the first two lengths. We draw AB', BA', which cross each other. Show that point I, where these two segments intersect, lies on the bisector of the given angle.

Exercise 7. If two sides of a triangle are unequal, then the median between these two sides makes the greater angle with the smaller side. (Imitate the construction in **25**.)

Exercise 8. If a point in the plane of a triangle is joined to the three vertices of a triangle, then the sum of these segments is greater than the semi-perimeter of the triangle; if the point is inside the triangle, the sum is less than the whole perimeter.

Exercise 8b. If a point in the plane of a polygon is joined to the vertices of the polygon, then the sum of these segments is greater than the semi-perimeter of the polygon.

Exercise 9. The sum of the diagonals of a [convex[5]] quadrilateral is between the semi-perimeter and the whole perimeter.

Exercise 10. The intersection point of the diagonals of a [convex[6]] quadrilateral is the point in the plane such that the sum of its distances to the four vertices is as small as possible.

Exercise 11. A median of a triangle is smaller than half the sum of the sides surrounding it, and greater than the difference between this sum and half of the third side.

Exercise 12. The sum of the medians of a triangle is greater than its semi-perimeter and less than its whole perimeter.

Exercise 13. On a given line, find a point such that the sum of its distances to two given points is as small as possible. Distinguish two cases, according to whether the points are on the same side of the line or not. The second case can be reduced to the first (by reflecting part of the figure in the given line).

Exercise 14. (Billiard problem.) Given a line xy and two points A, B on the same side of the line, find a point M on this line such that $\widehat{AMx} = \widehat{BMy}$.

We obtain the same point as in the preceding problem.

Exercise 15. On a given line, find a point with the property that the difference of its distances to two given points is as large as possible. Distinguish two cases, according to whether the points are on the same side of the line or not.

[5]This condition is not in the original text. See solution. –transl.

[6]See the previous footnote. –transl.

CHAPTER III

Perpendiculars and Oblique Line Segments

29. THEOREM. *If, from a given point outside a line, we draw a perpendicular and several oblique line segments:*

1°. *The perpendicular is shorter than any oblique segment*;

2°. *Two oblique segments whose feet are equally distant from the foot of the perpendicular are equal*;

3°. *Of two oblique segments, the longer is the one whose foot is further from the foot of the perpendicular.*

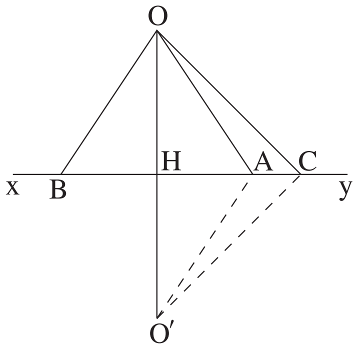


FIGURE 31

1°. Consider perpendicular OH and oblique segment OA from point O to line xy (Fig. 31). If we extend OH by a length HO' equal to itself, then O' is symmetric to O with respect to line xy. Therefore $O'A$ is the symmetric image of OA, and the two are equal. Now in triangle $OO'A$ we have

$$OO' < OA + O'A,$$

and we can replace OO' with $2OH$ and $OA + OA'$ with $2OA$. Thus we find $2OH < 2OA$, or $OH < OA$.

2°. Consider next the oblique segments OA, OB such that $HA = HB$. These two oblique segments will be equal by symmetry with respect to line OH.

3°. Finally, let OA and OC be oblique segments such that $HC > HA$ (Fig. 31). Suppose first that points A and C are on the same side of H. Then point A is inside triangle $OO'C$. By (**27**) we have

$$OA + O'A < OC + O'C$$

and, as we saw before, $OA = O'A$ and $OC = O'C$. Dividing by two, we have, as before,

$$OA < OC.$$

If we had started with an oblique segment OB less distant than OC, but on the other side of H, it would suffice to construct a length $HA = HB$ in the direction of HC. The oblique segment OB then would be equal to OA (2°), and hence smaller than OC as we have seen above.

30. Conversely. *If two oblique segments are equal, their feet are equally distant from the foot of the perpendicular,* otherwise they are unequal; *if they are unequal, the longer is the more distant from the perpendicular.*

COROLLARY. *There are no more than two oblique segments of the same length from the same point O to a line xy.*

This is true because the feet of these oblique segments are equally distant from H, and there are only two points on xy at a given distance from H.

31. The length of the perpendicular dropped from a point to a line is called the *distance from the point to the line*. The preceding theorem shows that this perpendicular is in fact the shortest path from the point to the line.

32. THEOREM.

1°. *Every point on the perpendicular bisector of a segment is equally distant from the endpoints of the segment*;

2°. *A point not on the perpendicular bisector is not equally distant from the endpoints of the segment.*

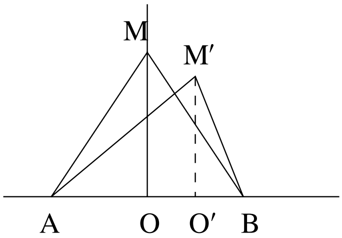


FIGURE 32

1°. If M is on the perpendicular bisector of AB (Fig. 32), the segments MA, MB are equal, because they are oblique lines which are equally distant from the foot of the perpendicular MO.

2°. Let M' be a point not on the perpendicular bisector, and suppose it is on the same side of the bisector as point B. The foot O' of the perpendicular dropped from M onto line AB will be on the same side of the bisector (otherwise this perpendicular would meet the bisector and, from their intersection, there would be two lines perpendicular to AB).

We will then have $O'A > O'B$ and therefore (**29**)

$$M'A > M'B.$$

QED

REMARK I. We could have proved the second part in a different way by establishing the following equivalent proposition: *Any point which is equidistant from A and B is on the perpendicular bisector of AB.* This follows from the converse of **30** (the feet of two equal oblique segments are equidistant from the foot of the perpendicular). One could also use the properties of isosceles triangles (**23**, Remark). However, one must observe that our way of proceeding has the advantage of showing which of the two distances is the greater when they are unequal.

REMARK II. The proposition which we just stated: *Any point which is equidistant from A and B is on the perpendicular bisector of AB,* is the converse of the first part of the preceding theorem. We have here two different ways of proving a converse. The first consists in following the original reasoning in reverse. This is what we did in the preceding remark. The original reasoning (1° in the previous theorem) started from the hypothesis that point M is on the perpendicular bisector; in other words, that the feet of MA, MB were equally distant from the perpendicular from M, and deduced from this that MA and MB are equal. This time, we started from the hypothesis that MA and MB are equal, and we concluded that their feet are equally distant.

The second method of proving the converse consists in proving what we call the *inverse* statement. This name is given to the statement whose hypothesis is the negation of the original hypothesis, and whose conclusion is the negation of the original conclusion. Thus the second part of the preceding theorem is the inverse of the first, and is equivalent to its converse.

We will find later on (see, for example, **41**) a third method of proving a converse.

33. Let us now make use of the definition of **1b**.

Using this definition, the preceding theorem can be restated as follows:

THEOREM. *The locus of points equidistant from two given points is the perpendicular bisector of the segment joining these two points.*

This is true because the figure formed by the points equidistant from A and B is in fact the perpendicular bisector of AB.

We remark that, to establish this fact, one must prove, as we have, that:

1°. every point on the perpendicular bisector satisfies the given condition;

2°. every point satisfying this condition is on the perpendicular bisector; or, equivalently, that no point outside the bisector satisfies the condition. This kind of double argument is necessary in all problems about geometric loci.

Exercises

Exercise 16. If the legs of a first right triangle are respectively smaller than the legs of a second, then the hypotenuse of the first is smaller than the hypotenuse of the second.

Exercise 17. If the angles $\widehat{B}$ and $\widehat{C}$ of a triangle ABC are acute, and the sides AB, AC unequal, then the lines starting from A are, in decreasing order of length, as follows: larger side, median (see Exercise 7 in Chapter II), angle bisector, smaller side, altitude.

Exercise 18. The median of a non-isosceles triangle is greater than the bisector from the same vertex, bounded by the third side.

CHAPTER IV

Cases of Congruence for Right Triangles. A Property of the Bisector of an Angle

34. Cases of congruence for right triangles. The general cases of congruence for triangles apply, of course, to right triangles as well. For instance, two right triangles are congruent if their legs are respectively equal (2^{nd} case of congruence for any triangles).

Aside from these general cases, right triangles present two special cases of congruence.

First case of congruence. *Two right triangles are congruent if they have equal hypotenuses, and an equal acute angle.*

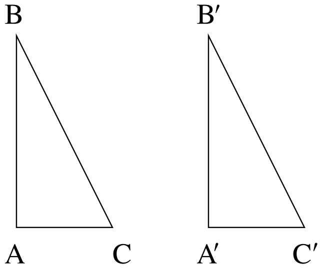


FIGURE 33

Consider two right triangles $ABC, A'B'C'$ (Fig. 33) in which $BC = B'C'$, $\widehat{B} = \widehat{B'}$. We move the second triangle over the first so that the angles B and B' coincide. Then $B'C'$ will assume the direction of BC, and, since these two segments are equal, C' will fall on C. $B'A'$ will assume the direction of BA, and therefore $C'A'$ will coincide with the perpendicular dropped from C on BA; that is, with CA.

Second case of congruence. *Two right triangles are congruent if they have equal hypotenuses and one pair of corresponding legs equal.*

Suppose the two triangles are ABC, $A'B'C'$, and that $BC = B'C'$, $AB = A'B'$. We move the second triangle onto the first so that $A'B'$ coincides with AB. The side AC will assume the direction of $A'C'$. We then have two oblique lines from point B to line AC; namely, BC and the new position of $B'C'$. These oblique lines are equal by hypothesis and therefore (**30**), equidistant from the foot of the perpendicular. This gives $A'C' = AC$, from which the congruence of the triangles follows.

35. THEOREM. *If two right triangles have equal hypotenuses and an unequal acute angle, then the sides opposite the unequal angles are unequal, and the larger side is opposite the larger angle.*

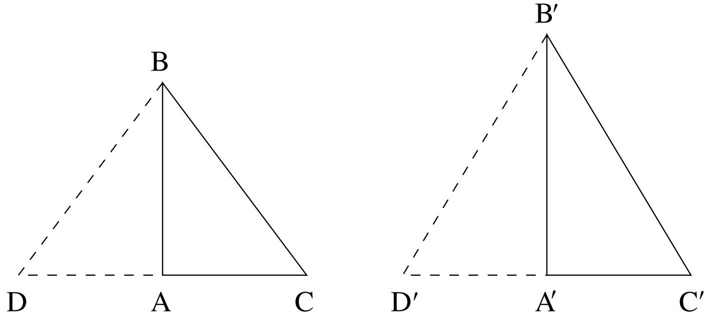


FIGURE 34

Let the two triangles be ABC, $A'B'C'$ (Fig. 34), for which $BC = B'C'$, $\widehat{B} > \widehat{B'}$. We claim that $AC > A'C'$.

To see this, we extend AC by its own length to get AD, and similarly extend $A'C'$ to get $A'D'$. We immediately have (**29**) $BD = BC = B'C' = B'D'$. Moreover, in isosceles triangle BDC median BA is also an angle bisector, so that angle $\widehat{DBC}$ is twice the original angle $\widehat{B}$. Likewise, angle $\widehat{D'B'C'}$ is twice the original angle $\widehat{B'}$, from which we have $\widehat{DBC} > \widehat{D'B'C'}$.

The two triangles DBC, $D'B'C'$ therefore have an unequal angle between equal sides, from which it follows that $DC > D'C'$, and therefore $AC > A'C'$.

36. THEOREM. *The bisector of an angle is the locus of points in the interior of the angle which are equidistant from the two sides.*

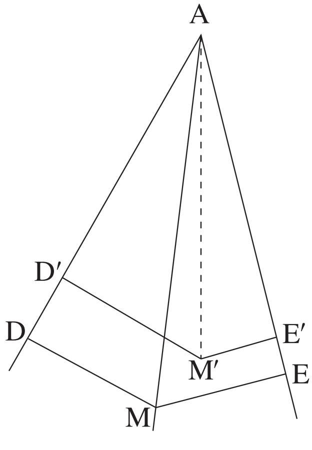


FIGURE 35

As explained earlier (**33**), the proof consists of two parts:

1°. *Every point on the bisector is equidistant from the two sides.*

Suppose the angle is $\widehat{ABC}$, and M is a point on its bisector. If we drop perpendiculars MD, ME to the sides of the angle, then right triangles AMD and AME are congruent, because they have a common hypotenuse, and an equal acute angle (at A) by hypothesis. Therefore the perpendiculars MD, ME are equal.

2°. *The distances from any point inside the angle, but not on its bisector, to the two sides are unequal.*

Suppose point M' lies, for instance, between the bisector and side AC. Then angle $\widehat{BAM'}$ will be greater than $\widehat{M'AC}$. If we drop perpendiculars $M'D'$, $M'E'$ onto sides AB, AC, then right triangles $AM'D'$, $AM'E'$ will have a common hypotenuse and an unequal angle at A. Therefore $M'D'$ will (**35**) be greater than $M'E'$.

As in the case of the theorem of **32**, we could have proved, in place of the inverse proposition in 2° above, the converse proposition: *Any point inside an angle, and equidistant from its two sides, is on its bisector*. To do this, following the original reasoning in reverse, we would have considered a point M equidistant from the two sides, and would have applied the second case of congruence (**34**) to the two right triangles AMD, AME, which have a common hypotenuse and in which $MD = ME$. We would have concluded that the angles at A are equal, so that AM is the angle bisector. However, we would not have established which is the larger distance when they are unequal.

Corollaries. I. *This theorem allows us to give a second* **definition** *of the bisector of an angle, namely: The bisector of an angle is the locus of the points inside the angle which are equidistant from the sides.*

We observe that this second definition is exactly equivalent to the one given in **11**.

II. *The locus of all points equidistant from two intersecting lines is composed of the two bisectors (**16**) of the angles formed by these two lines.*

Exercises

Exercise 19. Show that a triangle is isosceles if it has two equal altitudes.

Exercise 20. More generally, in any triangle, the smaller altitude corresponds to the larger side.

CHAPTER V

Parallel Lines

37. When two lines (Fig. 36) are intersected by a third (called a transversal), this last line forms eight angles with the first two, which are numbered in the figure. The relative positions of these angles are described as follows.

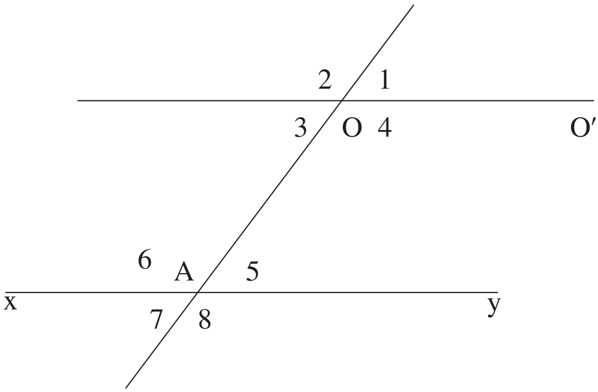


FIGURE 36

Two angles such as $\widehat{3}$ and $\widehat{5}$, situated between the two lines, and on different sides of the transversal, are called *alternate interior* angles.

Two angles such as $\widehat{3}$ and $\widehat{6}$ situated between the two lines, but on the same side of the transversal, are said to be *interior on the same side*.

Two angles such as $\widehat{6}$ and $\widehat{2}$ on the same side of the transversal, one between the two lines, one outside, are said to be *corresponding.*

38. DEFINITION. Two lines in the same plane are said to be *parallel* if they do not intersect, no matter how far extended in either direction.

THEOREM. *Two lines intersected by the same transversal are parallel*:
1°. *If the interior angles on the same side are supplementary*;[1]
2°. *If alternate interior angles are equal*;
3°. *If corresponding angles are equal.*

1°. If the two lines were to intersect, on either side of the transversal, they would form a triangle in which (**25**) the sum of two interior angles on the same side would have to be less than two right angles.

[1]If the angles $\widehat{3}$ and $\widehat{6}$ are supplementary, then so are $\widehat{4}$ and $\widehat{5}$, since the sum of these four angles is four right angles.

The other two cases can be reduced to the first:

2°. If $\widehat{3} = \widehat{5}$, this is equivalent to saying that $\widehat{3}$ is the supplement of $\widehat{6}$, or that the interior angles on the same side are supplementary.

3°. If $\widehat{6} = \widehat{2}$, then again $\widehat{3}$ and $\widehat{6}$ are supplementary, because $\widehat{3}$ is the supplement of $\widehat{2}$.

This theorem can be used to prove that two lines are parallel.

Corollary. *In particular, two lines perpendicular to a third line are parallel.*

39. Theorem. *Through any point not on a given line, a parallel to the given line can be drawn.*

Consider point O and line xy (Fig. 36). If we join point O to any point A of xy, then line OO', which makes an angle with OA such that $\widehat{AOO'} + \widehat{OAy}$ is equal to two right angles, will be parallel to xy.

40. Because the preceding construction can be made in infinitely many ways (since point A can be chosen anywhere on line xy), it would seem that there are infinitely many different parallels.

This is not true, however, if we adopt the following **axiom**:

Axiom. *Through any point not on a given line, only one parallel to the given line can be drawn.*[2]

Corollaries. I. *Two distinct lines parallel to a third line are themselves parallel; since if the two lines had a point in common, two lines would pass through it, each parallel to the third line.*

II. *If two lines are parallel, any third line which intersects one of them must intersect the other,* otherwise two parallels to the second line would intersect each other.[3]

41. The most important proposition in the theory of parallels is the following converse of the theorem in **38**.

Converse. *When two parallel lines are intersected by the same transversal*:

1°. *The interior angles on the same side are supplementary*;

2°. *The alternate interior angles are equal*;

3°. *The corresponding angles are equal.*

The proof is the same in all three cases. Consider the parallel lines AB, CD intersected by the transversal EFx (Fig. 37). We claim, for example, that the corresponding angles $\widehat{xEB}$ and $\widehat{xFD}$ are equal. Indeed, we can construct, at point E, an angle $\widehat{xEB'}$ equal to angle $\widehat{xFD}$. Line EB' will then be parallel to CD, and therefore it will coincide with EB.

Corollaries. I. *If two lines determine, with a common transversal, two interior angles on the same side which are not supplementary, then the two lines are not parallel, and they intersect on the side of the transversal where the sum of the interior angles is less than two right angles.*

II. *When two lines are parallel, any line perpendicular to one of them is perpendicular to the other.*

---

[2]This axiom is known as *Euclid's Postulate*. In fact it should be viewed as part of the definition in fundamental notions. (See Note B at the end of this volume.)

[3]We have here another example of a proof by contradiction (see **24**, Remark I.)

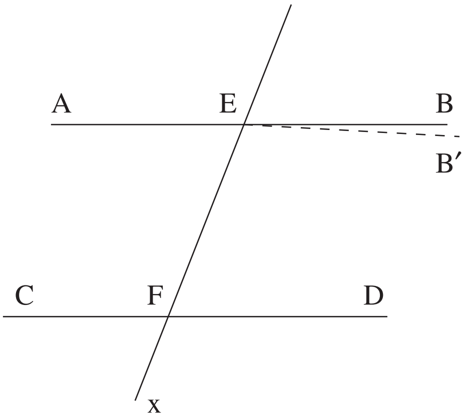


FIGURE 37

This is true because the perpendicular must intersect the other line (**40**, Corollary II) and the angle it forms will be a right angle, because of the theorem we have just proved.

REMARKS. I. The corresponding equal angles $\widehat{xEB}$ and $\widehat{xFD}$ have the same direction of rotation.

The two parallel directions EB, FD, both situated on the same side of a common transversal, are called *parallel and in the same direction.*

II. We have used a new method of proof of a converse, different from the ones used in **32**, and which consists in establishing the converse with the help of the original theorem. It should be noted here that the argument basically depends on the fact that the parallel through E to CD is *unique.*

42. According to the theorem of **38** and its converse, the definition of parallels amounts to the following: *Two lines are parallel if they form, with an arbitrary transversal, equal corresponding angles (or equal alternate interior angles, or supplementary interior angles on the same side of the transversal).*

This definition, equivalent to the first, is usually more convenient to use.

In place of the phrase *parallel lines* we often substitutes *lines with the same direction*, whose meaning is clear from the preceding propositions.

REMARK. Because of what we have just noted, *two lines which coincide must be viewed as a particular case of two parallel lines.*

43. THEOREM. *Two angles with two pairs of parallel sides are either equal or supplementary. They are equal if the sides both lie in the same direction or both in opposite directions; they are supplementary if a pair of sides lie in the same direction, and the other pair opposite.*

First, two angles with a common side, and whose second sides are parallel in the same direction (Fig. 38) are equal because they are corresponding angles. Two angles whose sides are parallel and in the same direction are then equal because one side of the first angle and one side of the second angle will form a third angle equal to the first two. If one of the sides is in the same direction, with the other

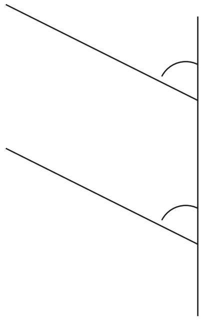


FIGURE 38

opposite, then extending the side which is in the opposite direction we obtain an angle supplementary to the first angle, and equal to the second.

If both sides are in opposite directions, we extend both sides of the first angle. Thus we form an angle equal to the first because they are vertical angles, and also equal to the second.

REMARK. Two corresponding angles and, therefore, two angles with sides parallel and in the same direction, have the same sense of rotation. We can therefore say: *Two angles with parallel sides are equal or supplementary, according as whether they have the same sense of rotation or not.*

THEOREM. *Two angles whose sides are perpendicular in pairs are equal or supplementary, according as whether they have the same sense of rotation or not.*

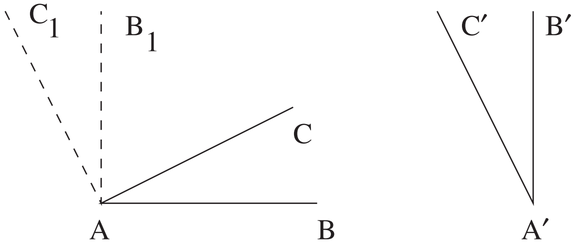


FIGURE 39

Consider angles $\widehat{BAC}$, $\widehat{B'A'C'}$ (Fig. 39) such that $A'B'$ and $A'C'$ are perpendicular respectively to AB and AC. We draw AB_1 perpendicular to AB, and turn angle $\widehat{B_1AC}$ around on itself: side AB_1 falls on AC, and hence line AB, perpendicular to AB_1, will occupy a position AC_1 perpendicular to AC. We now have an angle $\widehat{B_1AC_1}$ which is equal to $\widehat{BAC}$ and has the same sense (because it was constructed by reflection of angle $\widehat{CAB}$, whose sense is opposite to that of $\widehat{BAC}$) and whose sides, perpendicular respectively to those of the first angle, are therefore parallel to those of $\widehat{B'A'C'}$. Since the angles $\widehat{BAC}$, $\widehat{B'A'C'}$ are equal or supplementary, according as their sides have the same or opposite senses of rotation, the same is true of the given angles.

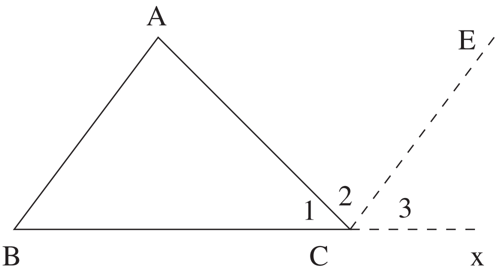


FIGURE 40

44. THEOREM. *The sum of the angles of a triangle is equal to two right angles.*

In triangle ABC (Fig. 40), we extend BC in the direction Cx, and draw CE parallel to AB. We form three angles at C (numbered 1 to 3 in the figure), whose sum is equal to two right angles. These angles are equal to the three angles of the triangle; namely: $\widehat{1}$ is the angle $\widehat{C}$ of the triangle; $\widehat{2} = \widehat{A}$ (these are alternate interior angle formed by transversal AC with the parallel lines AB, CE); $\widehat{3} = \widehat{B}$ (these are corresponding angles formed by transversal BC with the same parallels).

COROLLARIES. I. *An exterior angle of a triangle is equal to the sum of the non-adjacent interior angles.*

II. *The acute angles of a right triangles are complementary.*

III. *If two triangles have two pairs of equal angles, then the third pair of angles is equal as well.*

44b. THEOREM. *The sum of the interior angles of a convex polygon*[4] *is equal to two less than the number of sides times two right angles.*

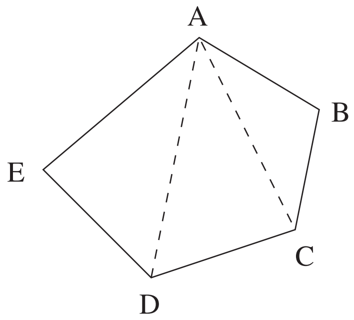


FIGURE 41

Let the polygon be $ABCDE$ (Fig. 41). Joining A to the other vertices, we decompose the polygon into triangles. The number of triangles is equal to the

[4]The theorem remains true if one takes, as the interior angle at any vertex pointing inwards (Fig. 19b), the one which is directed towards the interior of the polygon, an angle which in this case is greater than two right angles. In the case of Fig. 19b the proof would proceed as in the text, drawing the diagonals starting from the vertex pointing inwards. If there are several inward-pointing vertices, the theorem would still be true, but the proof would be more difficult.

number of sides minus two because if A is taken as the common vertex of these triangles, all the other sides of the polygon are the sides opposite A, except the two sides which end at A. The sum of the angles of these triangles gives us the sum of the angles of the polygon. The theorem is proved.

If n is the number of sides of the polygon, the sum of the angles is $2(n-2)$ or $2n-4$ right angles.

Corollary. *The sum of the exterior angles of a convex polygon, formed by extending the sides in the same sense, is equal to four right angles.*

Indeed, an interior angle plus the adjacent exterior angle gives us two right angles. Adding the results for all the n vertices we obtain $2n$ right angles, of which $2n-4$ are given by the sum of the interior angle. The sum of the exterior angles is equal to the four missing right angles.

Exercises

Exercise 21. In a triangle ABC, we draw a parallel to BC through the intersection point of the bisectors of $\widehat{B}$ and $\widehat{C}$. This parallel intersects AB in M and BC in N. Show that $MN = BM + CN$.

What happens to this statement if the parallel is drawn through the intersection point of the bisectors of the exterior angles at B and C? Or through the intersection of the bisector of $\widehat{B}$ with the bisector of the exterior angle at C?

Sum of the angles of a polygon.

Exercise 22. Prove the theorem in **44b** by decomposing the polygon into triangles using segments starting from an interior point of the polygon.

Exercise 23. In triangle ABC we draw lines AD, AE from point A to side BC, such that the first makes an angle equal to $\widehat{C}$ with AB, while the second makes an angle equal to $\widehat{B}$ with AC. Show that triangle ADE is isosceles.

Exercise 24. In any triangle ABC:

1°. The bisector of $\widehat{A}$ and the altitude from A make an angle equal to half the difference between $\widehat{B}$ and $\widehat{C}$.

2°. The bisectors of $\widehat{B}$ and $\widehat{C}$ form an angle equal to $\frac{1}{2}\widehat{A}$ + one right angle.

3°. The bisectors of the exterior angles of $\widehat{B}$ and $\widehat{C}$ form an angle equal to one right angle $-\frac{1}{2}\widehat{A}$.

25. In a convex quadrilateral:

1°. The bisectors of two consecutive angles form an angle equal to one half the sum of the other two angles.

2°. The bisectors of two opposite angles form a supplementary angle to the half the difference of the other two angles.

CHAPTER VI

On Parallelograms. — On Translations

45. Among quadrilaterals, we consider in particular *trapezoids* and *parallelograms.* A quadrilateral is called a *trapezoid* (Fig. 42) if it has two parallel sides. These parallel sides are called the *bases* of the trapezoid. A quadrilateral with two pairs of parallel sides is called a *parallelogram.* (Fig. 43).

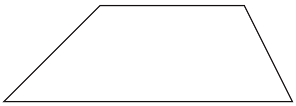


FIGURE 42

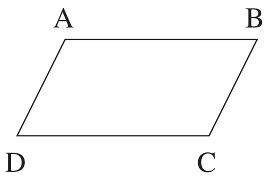


FIGURE 43

THEOREM. *In a parallelogram, opposite angles are equal, and angles adjacent to the same side are supplementary.*

Indeed, in parallelogram $ABCD$ (Fig. 43), angles $\widehat{A}$ and $\widehat{C}$, adjacent to side AB, are interior angles on the same side formed by parallels AD, BC cut by transversal AB; therefore they are supplementary. The opposite angles A, C are equal because they have parallel sides in opposite directions.

REMARK. We see that it suffices to know one angle of a parallelogram in order to know all of them.

Converse. *If, in a quadrilateral, the opposite angles are equal, then the quadrilateral is a parallelogram.*

Indeed, the sum of the four angles of a quadrilateral equals four right angles (**44b**). Since $\widehat{A} = \widehat{C}$, $\widehat{B} = \widehat{D}$, the sum of the four angles $\widehat{A} + \widehat{B} + \widehat{C} + \widehat{D}$ can be writen as $2\widehat{A} + 2\widehat{B}$. We then have $\widehat{A} + \widehat{B} =$ two right angles, so the lines AD, BC are parallel, since they form two supplementary interior angles on the same side of transversal AB. Similarly, we can show that AB is parallel to CD.

46. THEOREM. *In any parallelogram, the opposite sides are equal.*

In parallelogram $ABCD$ (Fig. 44), we draw diagonal AC. This diagonal divides the parallelogram into two triangles ABC, CDA, which are congruent because they have the common side AC between equal angles: $\widehat{A_1} = \widehat{C_1}$ are alternate interior angles formed by parallels AB, CD, and $\widehat{A_2} = \widehat{C_2}$ are alternate interior angles formed by parallels AD, BC.

These congruent triangles yield $AB = CD$, $AD = BC$.

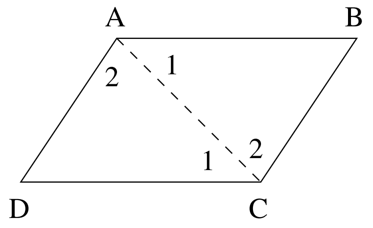


FIGURE 44

In this theorem, the hypothesis consists of two parts:

1°. two sides are parallel;

2°. the other two sides are also parallel.

The conclusion also has two parts:

1°. two opposite sides are equal;

2°. the other two sides are also equal.

Since we can form a converse by using either the whole or a part of the original conclusion, and *vice versa,* this theorem has two converses.

Converses. *A quadrilateral is a parallelogram:*

1°. *If the opposite sides are equal*;

2°. *If two opposite sides are parallel and equal.*

1°. Suppose that $AB = CD$ and $AD = BC$ in quadrilateral $ABCD$ (Fig. 44). We again draw diagonal AC. Triangles ABC, CDA will be congruent, since they have three sides equal in pairs. Thus angles $\widehat{A_1}$ and $\widehat{C_1}$ are equal, and since these are alternate interior angles with respect to transversal AC, lines AB, CD must be parallel. Likewise, the equality of $\widehat{A_2}$, $\widehat{C_2}$ proves that AD and BC are parallel.

2°. Assume now that $AB = CD$ and AB is parallel to CD. Triangles ABC, CDA are again congruent, because they have an equal angle ($\widehat{A_1} = \widehat{C_1}$ are alternate interior angles) between two pairs of equal sides: AC is a common side and $AB = CD$. From this congruence we again find that $\widehat{A_2} = \widehat{C_2}$, and that sides AD, BC are parallel.

46b. REMARK I. A quadrilateral can have two sides AB, CD equal, and the other two BC, AD parallel, without being a parallelogram (it is then called an *isosceles trapezoid*).

Choosing an arbitrary side AB, it suffices to take the line symmetric to AB (D_1C, Fig. 45) with respect to any line xy in the plane, provided that the line xy is not parallel to AB, and that xy intersects the line AB in a point I on the extension of AB (and not on segment AB itself). Then quadrilateral $ABCD_1$ will have two parallel sides (both perpendicular to xy) and the other two equal (because they are symmetric to each other). These last two sides are not parallel because they intersect at point I.

Conversely, every quadrilateral with two sides BC, AD parallel, and the other two equal is either a parallelogram, or (in the case of an isosceles trapezoid) a figure with a line of symmetry. Indeed, let xy be the perpendicular bisector of BC. We have already found two oblique lines from C to the line AD, both equal to AB, namely: CD_1 which is symmetric to AB with respect to xy, and CD_2 which,

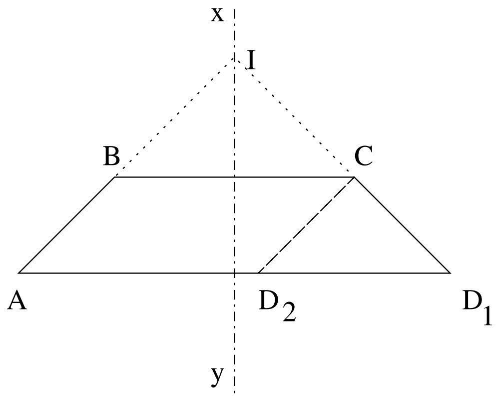


FIGURE 45

together with A, B, and C forms a parallelogram. These oblique lines certainly have endpoints on AD, since that line is parallel to BC. It follows that D must be either D_1 or D_2, since there are only two oblique lines equal to AB from C to AD.

The reasoning above may seem to be incorrect if D_2 coincides with D_1; that is, when CD_1 is parallel to AB. This requires, as we have just seen, that AB also be parallel to xy, and therefore perpendicular to AD, and indeed in this case there is only one oblique line from C with length equal to AB (the perpendicular).

REMARK II. In the first part of the preceding converse it is essential that the quadrilateral be proper (**21**, Remark): it is only if the triangles ABC, ADC of Fig. 44 are on different sides of the common side AC that the angles $\widehat{A_1}$, $\widehat{C_1}$ are alternate interior.

It is easy to construct an improper quadrilateral (called a *anti-parallelogram*) whose opposite sides are equal. It suffices, in parallelogram $ABCD$ (Fig. 45b), to replace the point D by point E symmetric to it with respect to the diagonal AC.

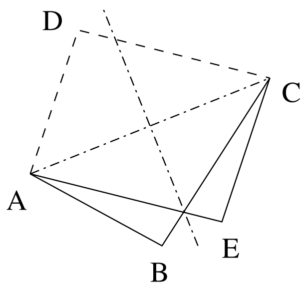


FIGURE 45b

We can also obtain an anti-parallelogram $ABCE_1$ by taking E_1 to be symmetric to B with respect to the perpendicular bisector of AC, so that ABE_1C is an isosceles trapezoid. But this point E_1 is none other than E, because (**24**, case 3°) there is, on any one side of AC, only one point E which is both at a distance $AE = BC$ from A and also at a distance $CE = AB$ from C. Every improper quadrilateral with equal opposite sides is thus formed by the non-parallel sides and the diagonals of an isosceles trapezoid.

REMARK III. Two quadrilaterals with four pairs of equal sides need not be congruent. In other words, we can deform a quadrilateral $ABCD$ (proper or not) without changing the lengths of its sides.

Indeed, let $AB = a$, $BC = b$, $CD = c$, $DA = d$ be the constant lengths of the four sides. With sides a, b and an arbitrary included angle B, we can construct a triangle ABC, whose construction will determine the length of AC (a diagonal of the quadrilateral). To each value of this diagonal (under the conditions for existence given in **86**, Book II) there corresponds a triangle ACD with this base AC and with the other two sides CD, DA having length c, d. The angle $\widehat{B}$ can therefore be arbitrary (at least within certain limits).

A quadrilateral which can be deformed under these conditions is called an *articulated quadrilateral.* This notion is important in practical applications of geometry.

According to the preceding results, a parallelogram remains a parallelogram when it is articulated and, likewise, an anti-parallelogram remains an anti-parallelogram under these conditions.[1]

47. THEOREM. *In a parallelogram, the two diagonals divide each other into equal parts.*

In parallelogram $ABCD$ (Fig. 46), we draw diagonals AC, BD, which intersect at O. Triangles ABO, CDO are congruent because they have equal angles, and an equal side $AB = CD$ (by the preceding theorem). Therefore $AO = CO$, $BO = DO$. QED

Converse. *A quadrilateral is a parallelogram if its diagonals divide each other into equal parts.*

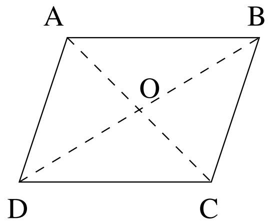


FIGURE 46

Indeed, assume (Fig. 46) that $AO = CO$, $BO = DO$. Triangles ABO, CDO are again congruent because they have an equal angle (the angles at O are vertical) included between pairs of equal sides. Therefore their angles at A and C will be equal, so that AB is parallel to CD. The congruence of triangles ADO, BCO also shows that AD is parallel to BC.

[1]This reasoning fails only if a parallelogram is transformed into an anti-parallelogram, or inversely (since a parallelogram and an anti-parallelogram are the only quadrilaterals with opposite sides equal). If the deformation occurs continuously, this requires that the quadrilateral first flatten into a line, so that one pair of adjacent sides ends up as extensions of each other, as does the other pair.

REMARK. We have proved the converses in **46** and **47** by retracing the original reasoning in the opposite direction, as explained in **32**, Remark II.

48. A quadrilateral whose angles are all equal, and are therefore are all right angles, is called a *rectangle*. A rectangle is a parallelogram, since its opposite angles are equal.

A quadrilateral whose sides are all equal is called a *rhombus*. A rhombus is a parallelogram since it has equal opposite sides.

Thus, in a rectangle, as in a rhombus, the diagonals intersect at their common midpoint.

THEOREM. *The diagonals of a rectangle are equal.*

In rectangle $ABCD$ (Fig. 47), the diagonals are equal because triangles ACD, BCD are congruent: they have side DC in common, $\widehat{ADC} = \widehat{DCB}$ since they are both right angles, and $AD = BC$ since they are opposite sides of a parallelogram.

COROLLARY. *In a right triangle, the median from the vertex of the right angle equals half the hypotenuse.*

This is true because if we draw parallels to the sides of the right angle through the endpoints of the hypotenuse, we form a rectangle, in which the median in question is half the diagonal.

Converse. *A parallelogram with equal diagonals is a rectangle.*

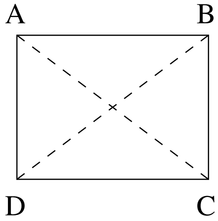


FIGURE 47

Suppose (Fig. 47) that $AD = BC$ in parallelogram $ABCD$. We know that $AD = BC$: consequently, triangles ADC, BCD are congruent, since their three sides are equal in pairs. Angles $\widehat{ADC}$, $\widehat{BCD}$ are therefore equal and, since they are supplementary, they must be right angles, which shows that the parallelogram is a rectangle.

COROLLARY. *A triangle in which a median is one half the corresponding side is a right triangle.*

COROLLARY. *In a rhombus, the diagonals are perpendicular, and bisect the vertex angles.*

If $ABCD$ (Fig. 48) is a rhombus, then triangle ABD is isosceles. Diagonal AC, being a median of this triangle, is also an altitude and an angle bisector.

Converse. *A parallelogram with perpendicular diagonals is a rhombus.*

Indeed, each vertex is equidistant from the adjacent vertices, since it lies on the perpendicular bisector of the diagonal joining these vertices.

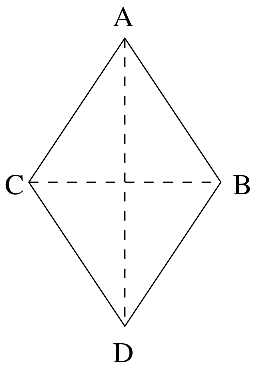


FIGURE 48

49. A *square* is a quadrilateral in which all sides are equal, and all the angles right angles.

Thus a square is both a rhombus and a rectangle, so that the diagonals are equal, perpendicular, and intersect each other at their common midpoint.

Conversely, any quadrilateral whose diagonals are equal, perpendicular, and bisect each other is a square.

Two squares with the same side are congruent.

50. Translations.

LEMMA. *Two figures F, F' are congruent, with the same sense of rotation, if their points correspond in such a way that, if we take points A, B, C from one figure, and the corresponding points A', B', C' from the other, then the triangles thus formed are always congruent, with the same sense of rotation, no matter which point we take for C.*

Indeed, take two points A, B in figure F, and the homologous[2] points A', B'. Segment AB is then clearly equal to $A'B'$. Let us move the second figure onto the first in such a way that these two equal segments coincide. We claim that then the two figures coincide completely. Indeed, let C be a third point of the first figure, and let C' be its homologous point. Since the two triangles ABC, $A'B'C'$ are congruent, angle $\widehat{B'A'C'}$ is equal to $\widehat{BAC}$, and has the same sense of rotation. Therefore, when $A'B'$ is made to coincide with AB, the line $A'C'$ will assume the direction of AC. Since $A'C' = AC$ as well, we conclude that C' coincides with C. This argument applies to all the points of the figure, so the two figures must coincide completely.

REMARKS. I. We have just provided a sufficient condition for two figures to be congruent; this conditions is also clearly necessary.

From the preceding reasoning it also follows that:

II. In order to superimpose two equal figures with the same orientation, it suffices to superimpose two points of one of the figures onto their homologous points.

[2]This is the name given to pairs of corresponding points in the two figures.

51. THEOREM. *If, starting at each point of a figure, we draw equal parallel segments in the same direction, then their endpoints will form a figure congruent to the first.*

First consider two points A, B of the first figure, and the corresponding points A', B' of the second. Since the segments AA' and BB' are parallel and equal, $ABA'B'$ is a parallelogram. Therefore $A'B'$ is equal and parallel to AB, with the same orientation. Thus segments joining homologous pairs of points are equal, parallel, and have the same direction.

It follows that any three points of the first figure correspond to three points forming a congruent triangle, and since the angles of these triangles have their sides parallel and in the same sense, their sense of rotation is the same. The figures are therefore congruent.

The operation by which we pass from the first figure to the second is called a *translation.* We note that a translation is determined if we are given the length, direction, and sense of a segment, such as AA', joining a point to its homologue. We can therefore designate a translation by the letters of such a segment: for example, we speak of the translation AA'.

COROLLARIES. I. *If, through each point on a line, we draw equal parallel segments in the same sense, the locus of their endpoints is a line parallel to the first.*

In particular, *the locus of points on the same side of a line, and a at a given distance from the line, is a parallel line.*

II. *Two parallel lines are everywhere equidistant.*

We can therefore speak of the *distance* between two parallel lines.

III. *The locus of points equidistant from two parallel lines is a third line, parallel to the first two.*

Exercises

Parallelograms.

Exercise 26. The angle bisectors of a parallelogram form a rectangle. The bisectors of the exterior angles also form a rectangle.

Exercise 27. Any line passing through the intersection of the diagonals of a parallelogram is divided by this point, and by two opposite sides, into two equal segments.

For this reason, the point of intersection of the diagonals of a parallelogram is called the *center* of this polygon.

Exercise 28. Two parallelograms, one of which is *inscribed* in the other (that is, the vertices of the second are on the sides of the first) must have the same center.

Exercise 29. An angle of a triangle is acute, right, or obtuse, according as whether its opposite side is less than, equal to, or greater than double the corresponding median.

Exercise 30. If, in a right triangle, one of the acute angles is double the other, then one of the sides of the right angle is half the hypotenuse.

Translations.

Exercise 31. Find the locus of the points such that the sum or difference of its distances to two given lines is equal to a given length.

Exercise 32. Given two parallel lines, and two points A, B outside these two parallels, and on different sides, what is the shortest broken line joining the two points, so that the portion contained between the two parallels has a given direction?

CHAPTER VII

Congruent Lines in a Triangle

52. THEOREM. *In any triangle, the perpendicular bisectors of the three sides are concurrent.*

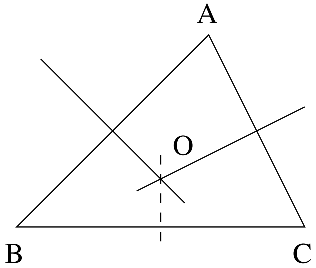


FIGURE 49

Suppose the triangle is ABC (Fig. 49). The perpendicular bisectors of sides AB, AC are not parallel (otherwise lines AB, AC would coincide), so they intersect at some point O. We must show that point O is also on the perpendicular bisector of BC.

Because point O, is on the perpendicular bisector of AB, it is equidistant from A and B; likewise, because it is on the perpendicular bisector of AC, it is equidistant from A and C. It is therefore equidistant from B and C and thus it is on the perpendicular bisector of BC.

53. THEOREM. *In any triangle, the three altitudes are concurrent.*

Suppose the triangle is ABC (Fig. 50). We draw a parallel to BC through A, a parallel to AC through B, and a parallel to AB through C. This forms a new triangle $A'B'C'$. We will show that the altitudes of ABC are the perpendicular bisectors of the sides of the new triangle, from which it follows that they are concurrent.

Parallelogram $ABCB'$ gives us $BC = AB'$, and parallelogram $ACBC'$ gives $BC = AC'$, so that A is indeed the midpoint of $B'C'$. Altitude AD of ABC thus passes through the midpoint of $B'C'$, and is perpendicular to it because it is perpendicular to the parallel line BC.

Since this reasoning can be repeated for the other altitudes, the proof is complete.

54. THEOREM. *In any triangle:*

1°. *the three angle bisectors are concurrent;*

2°. *the bisector of an angle, and the bisectors of the two non-adjacent exterior angles, are concurrent.*

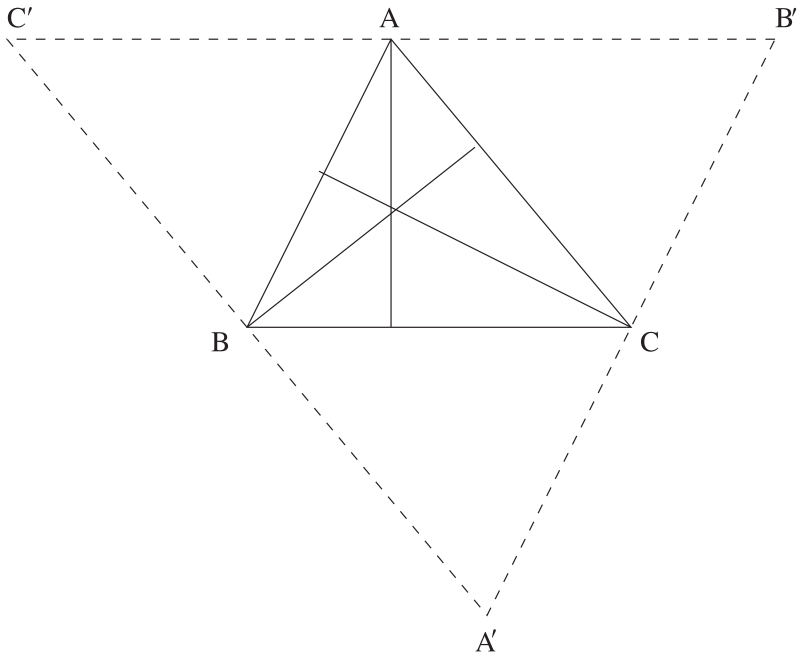


FIGURE 50

1°. In triangle ABC (Fig.51), we draw the bisectors of angles $\widehat{B}$ and $\widehat{C}$; they intersect at a point O inside the triangle. This point, being on the bisector of angle $\widehat{B}$, is equidistant from the sides AB and BC. Likewise, being on the bisector of angle $\widehat{C}$, the point O is equidistant from AC and BC. It is therefore equidistant from AB and AC and, being interior to angle $\widehat{A}$, is on the bisector of this angle.

2°. Since the exterior angles $\widehat{CBx}$, $\widehat{BCy}$ have a sum less than four right angles, halves of each have a sum less than two right angles. The bisectors of these angles will therefore meet (**41**, Corollary) at a point O' inside angle $\widehat{A}$. This point O', like point O, will be equidistant from the three sides of the triangle. Therefore it will belong to the bisector of $\widehat{A}$.

55. THEOREM. *The segment joining the midpoints of two sides of a triangle is parallel to the third side, and equal to half of it.*

In triangle ABC (Fig. 52), let D be the midpoint of AB and let E be the midpoint of AC. We extend line DE past E by its own length, to a point F. Quadrilateral $ADCF$ will be a parallelogram (**47**), and therefore CF will be equal and parallel to DA, or, equivalently, to BD. Thus quadrilateral $BDCF$ is also a parallelogram. Therefore:

1°. DE is parallel to BC;

2°. DE, which is equal to half of DF, is also half of BC.

56. THEOREM. *The three medians of a triangle are concurrent at a point situated on each of them one third of its length from the corresponding side.*

First, let BE, CF be two medians of triangle ABC (Fig. 53). We claim that their intersection G lies at one third the length of each of them. To see this, let M

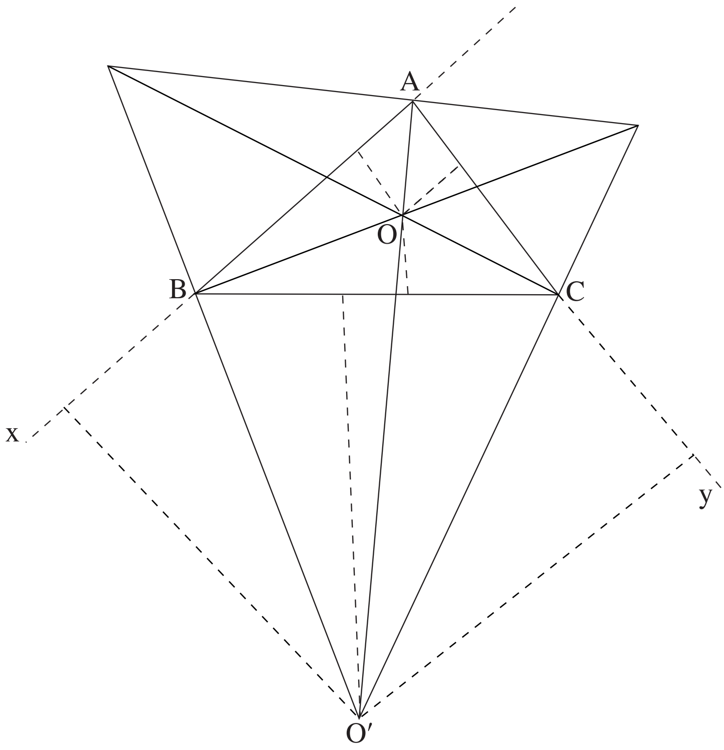


Figure 51

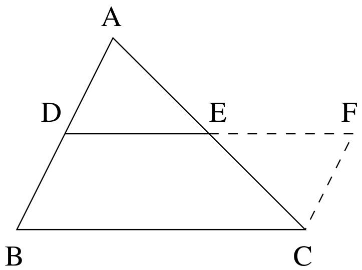


Figure 52

and N be the midpoints of BG and CG. Segment MN, which joins the midpoints of two sides of triangle BCG, is parallel to BC and equal to half of it. But EF is parallel to BC and equal to half of it. This means that $EFMN$ is a parallelogram, whose diagonals divide each other in half. Therefore $EG = GM = MB$ and $FG = GN = NC$.

Thus median BE passes through the point situated at one third the length of CF. But the same reasoning can also be applied to show that the median AD passes through the same point. The thereom is proved.

Remark. The point where the medians meet is also called the *center of mass* of the triangle. The reason for this name is given in the study of mechanics.

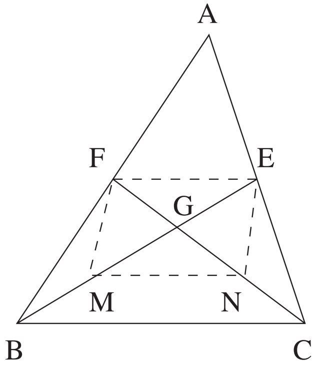


FIGURE 53

Exercises

Exercise 33. Join a given point to the intersection of two lines, which intersect outside the limits of the diagram (**53**).

Exercise 34. In a trapezoid, the midpoints of the non-parallel sides and the midpoints of the two diagonals are on the same line, parallel to the bases. The distance between the midpoints of the non-parallel sides equals half the sum of the bases; the distance between the midpoints of the diagonals is equal to half their difference.

Exercise 35. If, from two points A, B and the midpoint C of AB, we drop perpendiculars onto an arbitrary line, the perpendicular from C is equal to half the sum of the other two perpendiculars, or to half their difference, according as whether these two perpendiculars have the same or opposite sense.

Exercise 36. The midpoints of the sides of any quadrilateral are the vertices of a parallelogram. The sides of this parallelogram are parallel to the diagonals of the given quadrilateral, and equal to halves of these diagonals. The center of the parallelogram is also the midpoint of the segment joining the midpoints of the diagonals of the given quadrilateral.

Exercise 37. Prove that the medians of a triangle ABC are concurrent by extending the median CF (Fig. 53) beyond F by a length equal to FG.

Exercise 38. Given three lines passing through the same point O (all three distinct), and a point A on one of them, show that there exists:

1°. A triangle with a vertex at A and having the three lines as its altitudes (one exception);

2°. A triangle with a vertex at A and having the three lines as its medians;

3°. A triangle with a vertex at A and having the three lines as bisectors of its interior or exterior angles (one exception);

4°. A triangle with a midpoint of one of its sides at point A and having the three lines as perpendicular bisectors of the sides (reduce this to 1°).

Problems for Book 1

Exercise 39. In a triangle, the larger side corresponds to the smaller median[1]. (Consider the angle made by the third median with the third side.) A triangle with two equal median is isosceles.

Exercise 40. Let us assume that a billiard ball which strikes a flat wall will bounce off in such a way that the two lines of the path followed by the ball (before and after the collision) make equal angles with the wall. Consider n lines $D_1, D_2, \ldots, D_n$ in the plane, and points A, B on the same side of all of these lines. In what direction should a billiard ball be shot from A in order that it arrive at B after having bounced off each of the given lines successively? Show that the path followed by the ball in this case is the shortest broken line going from A to B and having successive vertices on the given lines.[2]

Special Case. The given lines are the four sides of rectangle, taken in their natural order; the point B coincides with A and is inside the rectangle. Show that, in this case, the path traveled by the ball is equal to the sum of the diagonals of the rectangle.

Exercise 41. The diagonals of the two rectangles of exercise 26 are situated on the same two lines, parallel to the sides of the given parallelogram (analogous to **54**). One of these diagonals is half the difference, and the other half the sum, of the sides of the parallelogram.

Exercise 42. In an isosceles triangle, the sum of the distances from a point on the base to the other sides is constant.— What happens if the point is taken on the extension of the base?

In an equilateral triangle, the sum of the distances from a point inside the triangle to the three sides is constant. What happens when the point is outside the triangle?

Exercise 43. In triangle ABC, we draw a perpendicular through the midpoint D of BC to the bisector of angle A. This line cuts off segments on the sides AB, AC equal to, respectively, $\frac{1}{2}(AB + AC)$ and $\frac{1}{2}(AB - AC)$.

Exercise 44. Let $ABCD$, $DEFG$ be two squares placed side by side, so that sides DC, DE have the same direction, and sides AD, DG are extensions of each other. On AD and on the extension of DC, we take two segments AH, CK equal to DG. Show that quadrilateral $HBKF$ is also a square.

Exercise 45. On the sides AB, AC of a triangle, and outside the triangle, we construct squares $ABDE$, $ACGF$, with D and F being the vertices opposite A. Show that:

1°. EG is perpendicular to the median from A, and equal to twice this median;

2°. The fourth vertex I of the parallelogram with vertices EAG (with E and G opposite vertices) lies on the altitude from A in the original triangle;

---

[1] For similar statements concerning the altitudes of a triangle, see Exercises 19, 20, and for angle bisectors see exercises 362, 362b at the end of this volume.

[2] In the case where there is only one line, the problem reduces to the subject of Exercises 13–14. Once these exercises are solved, one tries to find a way to use the solution for the case of one line to treat the case for two lines; then to extend it to three lines, and so on.

3°. CD, BF are equal to and perpendicular to BI, CI respectively, and their intersection point is also on the altitude from A.

Exercise 46. We are given a right angle $\widehat{AOB}$, and two perpendicular lines through a point P, the first intersecting the sides of the angle in A, B, and the second intersecting the same sides in C, D. Show that the perpendiculars from the points D, O, C to the line OP intercept on AB segments equal to AP, PB, respectively, but having the opposite sense.

Book II
On the Circle

CHAPTER I

Intersection of a Line with a Circle

57. Theorem. *Through three non-collinear points we can always draw one and only one circle.*

In other words, *a circle is determined by three points not on a straight line.*

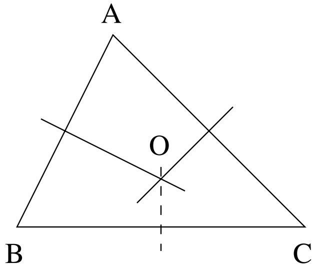


Figure 54

Indeed, assume that points A, B, C (Fig. 54) are not collinear. We proved earlier (**52**) that the perpendicular bisectors of segments BC, CA, AB pass through the same point O, equidistant from A, B, C. The circle with center O and radius OA passes through the three given points. It is the only circle with this property, because the center of a circle passing through A, B, C must necessarily belong to the three perpendicular bisectors just mentioned.

Corollary. We see that a circle *cannot have two different centers* nor, consequently, two different radii.

58. Theorem. *A line cannot intersect a circle in more than two points.*

If the distance from the center to the line is greater than the radius, the line does not intersect the circle.

If this distance is less than the radius, the line intersects the circle in two points.

Finally, if the distance to the line is equal to the radius, the line and the circle have only one common point.

In the last case we say that the line is *tangent* to the circle.

Assume that we are given a circle with center O and a line D. We drop (Fig. 55) perpendicular OH to the line.

1°. The circle cannot have more than two common points with line D. This amounts to saying that there are at most two oblique line segments equal in length to the radius R from O to D (**31**, Corollary).

2°. If OH is greater than the radius, the same is true *a forteriori* of all the oblique segments to D from O (**29**); thus all the points of D are outside the circle.

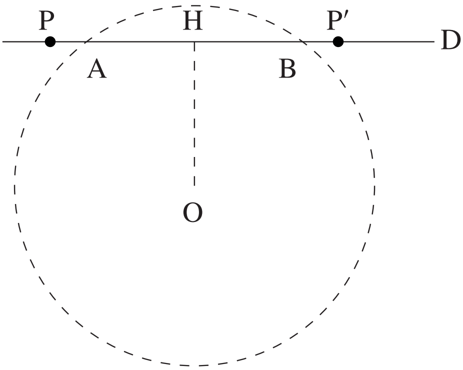


FIGURE 55

3°. If, on the contrary, OH (Fig. 55) is less than the radius, then point H is inside the circle; but on the two rays determined by H, there exist points outside the circle. To obtain such points, it is enough to construct points P, P' on these rays such that HP, HP' are equal to the radius: the distances OP, OP' must then be greater than the radius. Therefore there will be an intersection point between H and P, and another between H and P'; further, these are the only points of intersection (1°).

4°. Finally, if OH is equal to the radius (Fig. 56), the point H is on the circle but, as in 2°, we see that all the other points of the line are outside the circle.

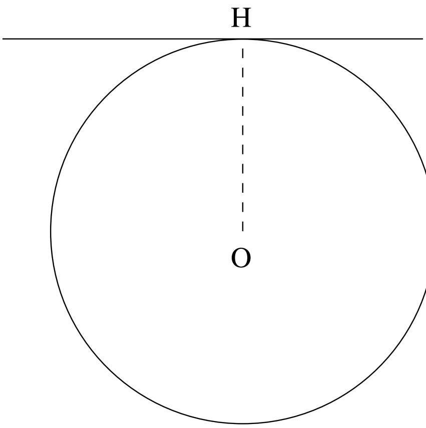


FIGURE 56

COROLLARY. *Through a point on a circle we can draw one, and only one, tangent to the circle; namely, the perpendicular to the radius which ends at that point.*

59. The definition just given for the tangent to a circle is not appropriate for an arbitrary curve.

DEFINITION. A *tangent* to an arbitrary curve (Fig. 57) at a point M of this curve is the limiting position of line MM' as the point M' on the curve approaches the point M.

In other words,[1] line MT will be said to be tangent at M if, for any angle ε, we can find, on the two sides of M, arcs $\widehat{MM_1}$, $\widehat{MM_2}$ such that for any point M' on these arcs, line MM' makes an angle less than ε with line[2] MT.

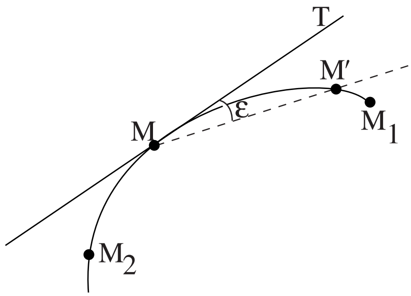


FIGURE 57

In the case of a circle, we will see that this definition is equivalent to the one we gave earlier.

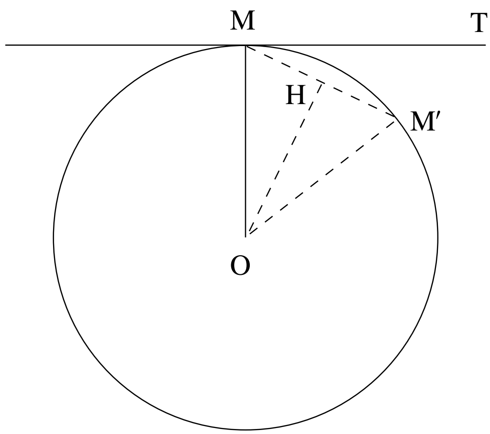


FIGURE 56b

From a point M on the circle we draw perpendicular MT to radius OM (Fig. 56b), and from O we drop a perpendicular OH to chord MM'. This perpendicular is both an altitude of the isosceles triangle OMM', and also the bisector of the angle at O. Angle $\widehat{TMM'}$ is equal to $\widehat{MOH}$ (they have pairs of perpendicular

[1]See the definition of the word *limit* in *Leçons d'Arithmétique* by J. Tannery (Paris, Librarie Armand Colin, 1894), chap. VII, n° 237, and chap. XII.

[2]One can show, as it is shown in arithmetic (*Leçons* de J. Tannery, n°239), that if the line MT exists, then it is unique.

sides), and thus is half of $\widehat{MOM'}$. But this angle can be made arbitrarily small by taking the point M' sufficiently close to M.

60. The line perpendicular to the tangent at its point of contact with a curve is called the *normal* to the curve at that point. The normal to a circle at any point is simply the radius ending at that point.

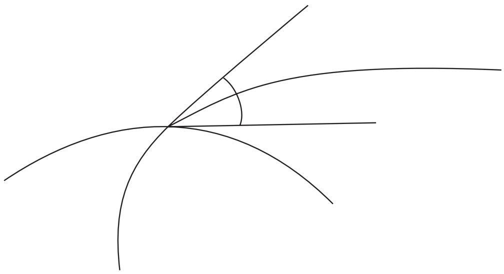


FIGURE 57b

On an arbitrary circle, there exist two points (and only two) such that the normals at these points pass through a given point P on the plane (different from the center): these are the endpoints of the diameter passing through P.

60b. The *angle between two curves* at one of their intersection points is the angle formed by the tangents to the curves at this point (Fig. 57b). The angle at which two circles intersect is therefore equal to the angle between the two radii ending at a common point, or to its supplement.

Exercises

Exercise 47. Through every point on a circle, we draw segments equal and parallel to a given segment. What is the locus of the endpoints of these segments?

Exercise 48. Find the locus of the midpoints of the segments joining a fixed point with each point on a circle.

Exercise 49. Consider a diameter AB of a circle O, a point C on the extension of the diameter, and a secant CDE passing through C which cuts the circle in D, E. If the exterior segment CD of the secant is equal to the radius, show that angle $\widehat{EOB}$ is triple $\widehat{DOA}$.

CHAPTER II

Diameters and Chords

61. Using the definition of **19b**, the remarks in **9** can be stated as follows:

Theorem. *Every diameter is a line of symmetry for the interior of its circle, and also for its circumference.*

We see that a circle has *infinitely many lines of symmetry.*

62. A line segments whose endpoints are the endpoints of an arc of a circle is called a *chord*. The arc is said to be *subtended* by the chord. We note that every chord subtends two different arcs, one larger than, the other smaller than, a semicircle (or both equal to a semicircle if the chord is a diameter).

63. Theorem. *The diameter perpendicular to a chord cuts this chord, and its two subtended arcs, into equal parts.*

Indeed, the diameter of a circle O perpendicular to a chord AB is a line of symmetry for the circle and at the same time a line of symmetry for the isosceles triangle OAB. Therefore this diameter is a line of symmetry for the whole figure formed by the circle and the chord.

Corollary. *The same line is determined by any two of the following conditions*:

1°. *It is perpendicular to a chord*;

2°. *It passes through the center of the circle*;

3°. *It passes through the midpoint of the chord*;

4°, 5°. *It passes through the midpoint of one of the subtended arcs.*

Corollary. *The locus of the midpoints of a series of parallel chords is the diameter perpendicular to these chords.*

Corollary. *A tangent is parallel to any chord bisected by the diameter through the point of contact.*

Theorem. *Two parallel lines intercept equal arcs between them.*

Indeed (Fig. 58), these arcs are symmetric with respect to the diameter which is perpendicular to these lines.

64. Theorem. *On any circle, the closest and furthest point from a given point P of the plane are the feet of the normals* (**60**) *passing through P.*

If A is the one of these points on ray OP, and B the point on the opposite ray (Figures 59 and 60), the distance PA is the difference between OP and the radius, and PB is the sum of the same segments. An arbitrary point M of the circle is

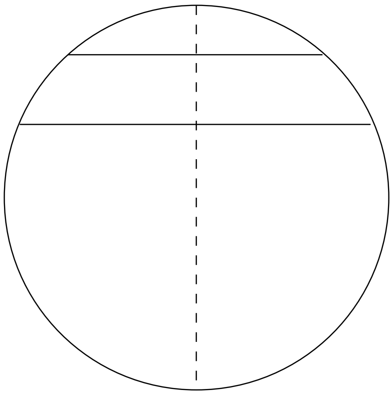


FIGURE 58

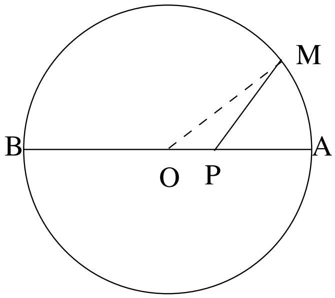


FIGURE 59

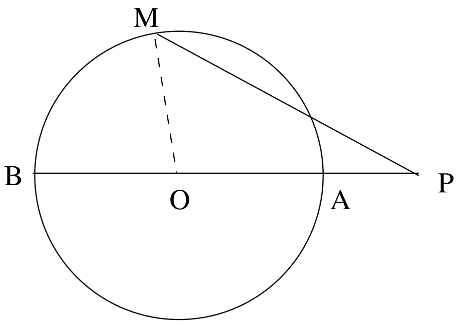


FIGURE 60

therefore at a distance from P greater than PA and smaller than PB, since PM is the third side of triangle OPM.

The distance PM increases as the point M moves along the circle from A to B, because, in triangle OPM, it is opposite to an angle which increases, but is contained between two constant sides.

COROLLARY. *The diameter is the longest chord of a circle.*

Indeed, if we take point P to coincide with A, we see that chord PM is smaller than the diameter PB.

65. THEOREM. *In the same circle, or in equal circles we have*:

1°. *Equal chords correspond to equal arcs, and conversely*;

2°. *For two unequal arcs, both less than a semicircle, the greater arc corresponds to the greater chord.*

1°. If we superimpose the two equal arcs, their endpoints will coincide, and so will the corresponding chords. Conversely, if the chords are equal, the angles at the center corresponding to these chords are also equal, by the third case of congruence for triangles (SSS). Therefore the corresponding arcs are equal as well.

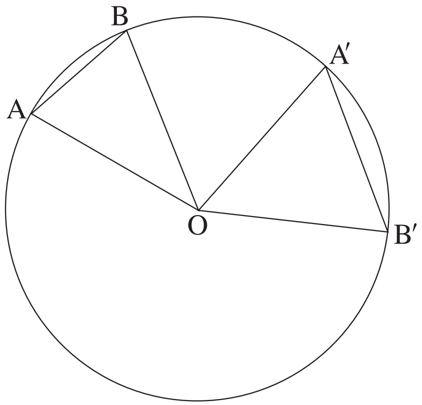


FIGURE 61

2°. If arc $\widehat{AB}$ is less than $\widehat{A'B'}$ (Fig. 61), and therefore the angle $\widehat{AOB}$ is less than $\widehat{A'OB'}$, then chord AB will be less than $A'B'$ by the theorem of **28** applied to triangles OAB, $OA'B'$.

66. THEOREM. *In the same circle, or in equal circles*:
1°. *Two equal chords are equidistant from the center, and conversely*;
2°. *Of two unequal chords, the longer is closer to the center.*

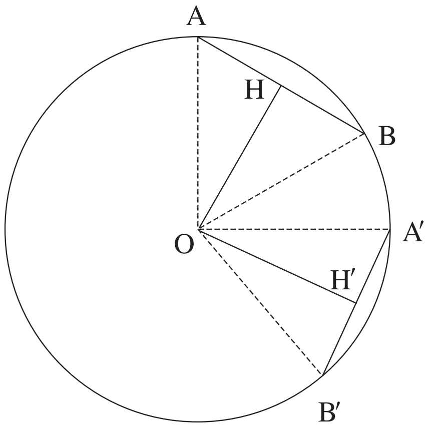


FIGURE 62

1°. Two equal chords of the same circle correspond to equal arcs. It suffices to superimpose the two arcs to see that the midpoints of the two chords are equidistant from the center.

Conversely, if two chords AB, $A'B'$ (Fig. 62) of circle O are equidistant from the center, then right triangles OHA, $OH'A'$ have equal hypotenuses and an equal pair of sides $OH = OH'$; thus $HA = H'A'$ and $AB = A'B'$.

2°. Suppose that chord AB is greater than $A'B'$ (Fig. 63). It follows that angle $\widehat{AOB}$ is greater than $\widehat{A'OB'}$ and, dropping perpendiculars OH, OH', we see that

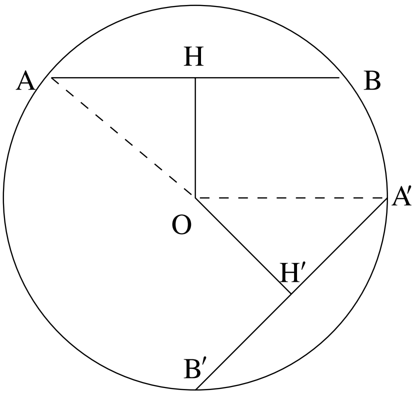


Figure 63

$\widehat{AOH}$ is greater than $\widehat{A'OH'}$. Angle $\widehat{OAH}$, the complement of the first angle, is less than $\widehat{OA'H'}$, the complement of the second. Thus right triangles OHA, $OH'A'$ have equal hypotenuses, and a pair of unequal angles, from which we deduce (**35**) that $OH < OH'$.

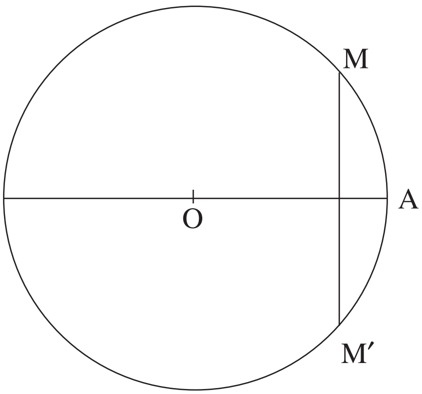


Figure 64

67. Consider a chord MM' (Fig. 64) whose distance to the center, at first less than the radius, increases until it equals the radius. Assume, to be definite, that this chord moves while remaining perpendicular to a fixed diameter OA. According to the preceding theorem, the length MM' decreases as this chord approaches the tangent at A, and we can take point M so close to point A (that is, the chord so close to the tangent), as to make this length as small as we wish, since $MM' < 2MA$. We see thus that the points M, M' approach each other and tend to coincide with the point A; this is expressed informally by saying that *the tangent has a double point in common with the circle*. We will see that this manner of speaking allows us to simplify the statements of certain theorems.

Exercises

Exercise 50. A circle passes through two fixed points A, B. Let C be one of the points where this circle meets a fixed line perpendicular to AB. Find the locus of the point diametrically opposite to C as the circle varies while passing through A and B.

Exercise 51. By dividing a chord into three equal parts and joining the division points to the center, we do not obtain three equal angles. Which is the greatest of the three angles? (Exercise 7.) Generalize to more than three equal parts.

Exercise 52. If two chords of a circle are equal, and we extend them (if necessary) to their point of intersection, the segments joining this intersection point with the endpoints of the two chords form two equal pairs.

Exercise 53. What is the locus of the midpoints of the chords of a given length of a circle?

Exercise 54. What is the shortest chord which passes through a given point inside a circle?

CHAPTER III

The Intersection of Two Circles

68. According to the theorem in **57**, *two circles cannot have more than two points in common.*

THEOREM. *When two circles intersect, the line joining their centers is perpendicular to their common chord and divides it into equal parts. When there is only one common point, this point is on the line joining the centers. Conversely, if there is a common point on the line of their centers, the circles have only one point in common.*

Indeed, the line of the centers is a common line of symmetry of the two circles. If the two circles meet at a point not on this line, they must meet again at the symmetric point. If there is only one intersection point, then it must belong to the line of the centers.

Conversely, if there is a common point on the line of the centers, then this is the only common point. Otherwise, the second common point is either on the line of the centers, in which case the circles would have a common diameter, or outside this line, in which case there would be a third common point. Under either of these circumstances, the two circles must coincide.

69. DEFINITION. We say that two curves are *tangent* if they have the same tangent line at a common point.

According to the preceding theorem, in the case of two circles this definition becomes: *Two circles are tangent when they have exactly one common point*, because the tangents at this common point must be the same (**58**. Corollary).

70. Consider two circles with centers O, O' and radii R, R' and suppose, to be definite, that $R' \leq R$. Five cases are possible:

1°. $OO' > R + R'$ (Fig. 65). Let M be a point inside or on circle O', so that $O'M \leq R'$. We will have (**26**, Corollary)

$$OM \geq OO' - O'M > R + R' - O'M > R.$$

Therefore, every point in the second circle is exterior to the first, and every point on the first is exterior to the second. The two circles are said to be *exterior* to each other.

2°. $OO' = R + R'$. In this case the segment OO' can be viewed as the sum of two segments OA, $O'A$ (Fig. 66) equal to R, R', respectively. The point A is common to the two circles, and for all other points the preceding reasoning applies. The two circles are then *tangent externally.*

3°. $R + R' > OO' > R - R'$. In this case, R' is contained between the sum and the difference of the two lengths OO' and R.

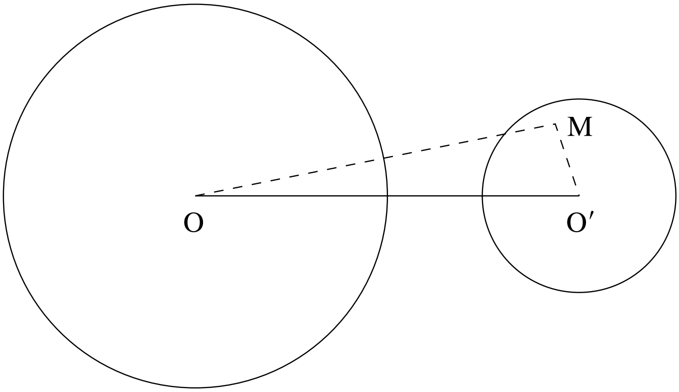


FIGURE 65

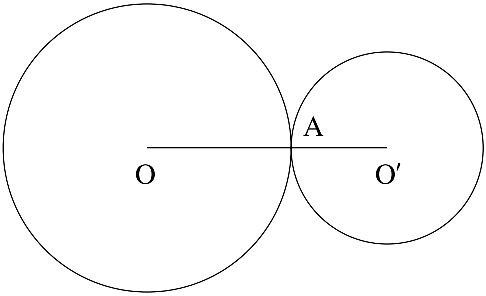


FIGURE 66

Therefore, of the two points A, B where circle O intersects the line of the centers, one is interior and the other exterior to circle O' (Fig. 67). It follows that circle O, which is a curve joining A and B, meets circle O' in a point different from either A or B, and therefore not on the line of the centers.

The circles have two common points, and are said to be *intersecting.*

4°. $OO' = R - R'$. In this case, OO' can be viewed as the difference of two segments OA, $O'A$ (Fig. 68) equal to R, R', respectively. Point A on the line of the centers is common to both circles, so the two circles are tangent.

Let M be any point on or inside the circle O'. We have

$$OM \leq OO' + O'M \leq OO' + R';$$

in other words, $OM \leq R$.

The entire circle O', except for point A, is therefore interior to O. The two circles are said to be *tangent internally.*

5°. $OO' < R - R'$ (Fig. 69). Consider again a point M on or inside circle O'. We obtain

$$OM \leq OO' + O'M \leq OO' + R' < R.$$

Thus circle O' is entirely inside O. The circle O' is said to be *interior* to O.

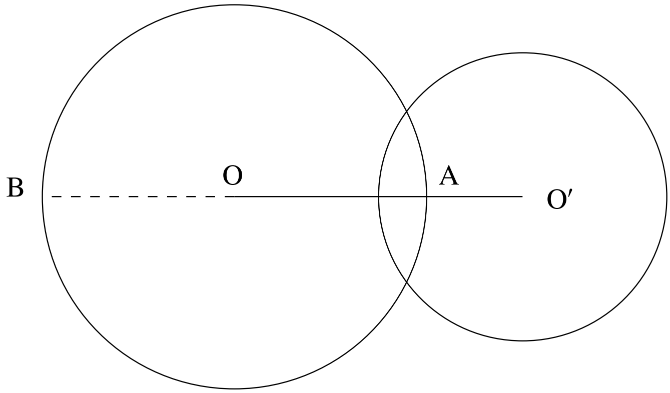
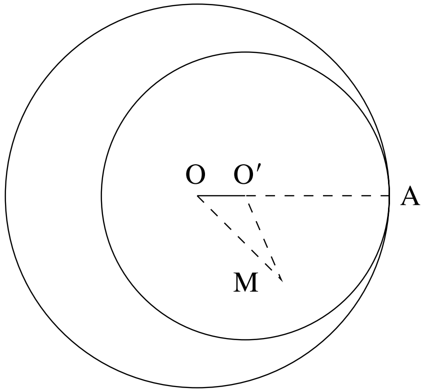


FIGURE 67



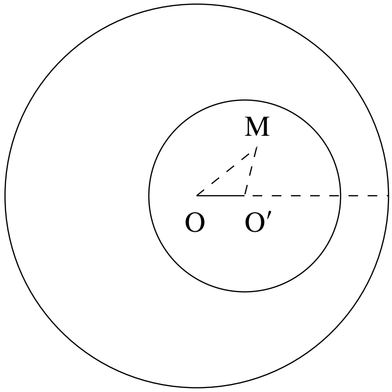


FIGURE 68

FIGURE 69

71. The preceding list exhausts the possibilities. It follows that the converses of the conclusions are also true. For example, if two circles intersect, the distance between the centers is contained between the sum and the difference of their radii: if this distance were greater than the sum of the radii, the circles would be exterior; if this distance were equal to the sum, the circles would be tangent externally, etc.

In any case, all of these converses are easily proved directly. Thus, in the case of intersecting circles, the point of intersection forms a triangle with the two centers, and the theorems of **26** yields the preceding conclusion.

We can therefore state the following theorem.

THEOREM. 1°. *If two circles are exterior, the distance between their centers is greater than the sum of the radii, and conversely.*

2°. *If two circles are tangent externally, the distance between their centers is equal to the sum of the radii, and conversely.*

3°. *If two circles intersect, the distance between their centers is less than the sum of the radii and greater than their difference, and conversely.*

4°. *If the circles are tangent internally, the distance between their centers is equal to the difference of the radii, and conversely.*

5°. *If one circle is interior to another, the distance between their centers is less than the difference of the radii, and conversely.*

One can also say: *Two circles are exterior, interior, intersecting, or tangent according as whether the segments they determine on the line of the centers are exterior to each other, are interior to each other, are extensions of each other, or overlap with a common endpoint.*

72. If two circles, at first intersecting, vary so as to become tangent at A, their common points approach the point A indefinitely(see Exercise 55). One can informally say, as before (**67**), that *two tangent circles have two common points which coincide, or a double common point.*

Exercises

Exercise 55. Consider a circle O of radius R, and a circle O' of radius R' intersecting the first. Let A be one of the points where the line OO' intersects the circle O, and let B be another point on the same circle chosen arbitrarily, with particular interest when it is close to A.

Show that the two circles have one of their common points on the little arc $\widehat{AB}$:

When the point O' is on the extension of OA past A, if the difference $R + R' - OO'$ is less than $OB + OB' - OO'$;

When O' is between O and A, if the difference $OO' - (R - R')$ is less than $OO' - (OB - OB')$;

When O is between O' and A, if the difference $OO' - (R - R')$ is less than $OO' - (O'B - OB)$.

Exercise 56. What are the shortest and longest segments one can draw joining two circles?

Exercise 57. What is the locus of the centers of circles of fixed radius which are tangent to a fixed circle?

Exercise 58. If a line passing through the common point of two tangent circles intersects these circles again in two other points, then the radii ending at these points are parallel.

Exercise 59. Two circles which are tangent internally remain tangent if their radii are increased or decreased by the same quantity, without changing their centers. Two circles which are tangent externally remain tangent if, without changing their centers, one radius is increased by some quantity and the other is decreased by the same quantity.

CHAPTER IV

Property of the Inscribed Angle

73. An angle is said to be *inscribed* in a circle if it is formed by two chords AB, AC with a common endpoint. Thus an inscribed angle has its vertex on the circle.

THEOREM. *The measure of an angle inscribed in a circle equals one half the arc contained between its sides.* (The endpoints B, C determine two arcs on the circle. The arc $\widehat{BC}$ which concerns us is the one which does not contain A.)

An inscribed angle equals *half the central angle which subtends the same arc.*[1]

N.B. The first part of the above statement *assumes the convention of* **17** (corollary).

We distinguish three cases.

First Case. One of the sides of the inscribed angle passes through the center.

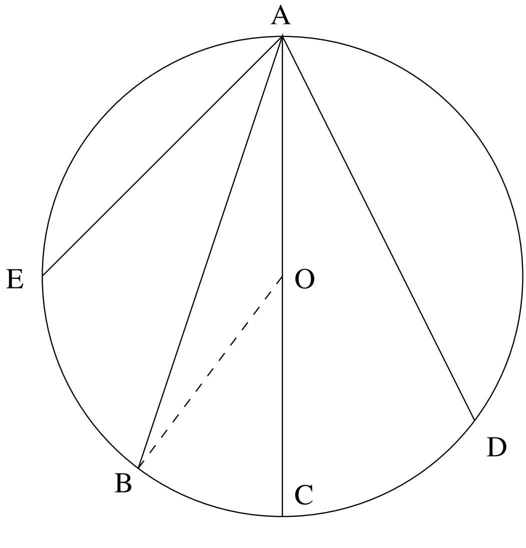


FIGURE 70

Consider inscribed angle $\widehat{BAC}$ (Fig. 70) whose side AC passes through the center O. We join BO. Thus we form an isosceles triangle OAB whose angles $\widehat{A}$

[1]The intercepted arc thus can be either minor arc $\widehat{BC}$ (**20b**) or major arc $\widehat{BC}$. In the last case, the corresponding central angle will be greater than two right angles, and the inscribed angle will be obtuse.

and $\widehat{B}$ are equal. The exterior angle $\widehat{BOC}$, equal to the sum $\widehat{A}+\widehat{B}$, is therefore twice $\widehat{A}$. Since the measure of $\widehat{BOC}$ is precisely that of arc $\widehat{BC}$, the measure of $\widehat{BAC}$ is one half of this arc.

Second Case. The center is inside the inscribed angle.

Consider inscribed angle $\widehat{BAD}$ (Fig. 70). Joining AO, which intersects the circle in C, we decompose the inscribed angle into two parts $\widehat{BAC}$, $\widehat{CAD}$ for which the theorem has already been proved (first case). We have therefore (with the convention of **18**)

$$\widehat{BAC} = \frac{1}{2}\text{arc } \widehat{BC}, \quad \widehat{CAD} = \frac{1}{2}\text{arc } \widehat{CD},$$

and adding,

$$\widehat{BAD} = \frac{1}{2}\text{arc } \widehat{BD}\ .$$

Third Case. The center is outside the inscribed angle.

Consider inscribed angle $\widehat{BAE}$ (Fig. 70). We again construct diameter AC, and we can write

$$\widehat{BAC} = \frac{1}{2}\text{arc } \widehat{BC}, \quad \widehat{EAC} = \frac{1}{2}\text{arc } \widehat{EC},$$

and subtracting,

$$\widehat{BAE} = \frac{1}{2}\text{arc } \widehat{BE}\ .$$

Corollaries. I. *All the angles inscribed in the same arc of a circle are equal,* because they have the same measure.

II. *An angle inscribed in a semicircle is a right angle,* because its measure is a quarter of a circle.

74. Theorem. *The angle formed by a tangent and a chord ending at the point of contact is equal to one half the arc it intercepts on the circle.*

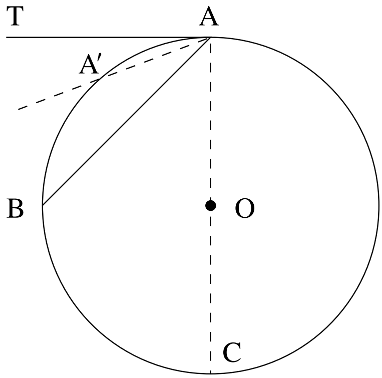


Figure 71

Angle $\widehat{BAT}$, formed by tangent AT and chord AB (Fig. 71) can in effect be considered as the limit of the angle formed by AB with chord AA' when point A' becomes infinitely close to A.

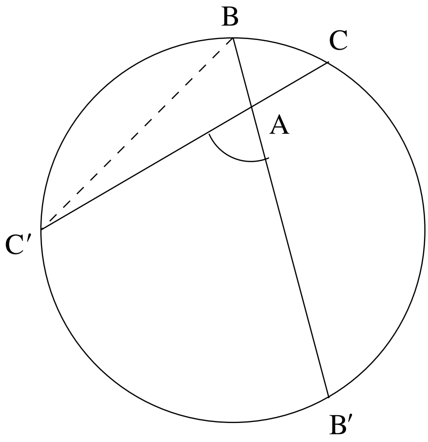


FIGURE 72

Alternatively, the proof given for the third case above can also be applied here; the equality $\widehat{EAC} = \frac{1}{2}\text{arc } \widehat{EC}$ must simply be replaced by

$$\widehat{TAC} = \frac{1}{2}\text{arc } \widehat{AC},$$

which follows from the fact that angle $\widehat{TAC}$ is a right angle and arc $\widehat{AC}$ is a semicircle.

75. THEOREM. *The angle formed by two secants which intersect inside a circle is equal to one half the sum of two intercepted arcs: one which is between its sides and the other which is between the extensions of its sides.*

Consider angle $\widehat{BAC}$ (Fig. 72) whose sides intersect the circle again in B' and C', and join BC'. Angles $\widehat{C'}$ and $\widehat{B}$ are one half the arcs $\widehat{BC}$, $\widehat{B'C'}$. But these two angles have a sum equal to $\widehat{A}$, an exterior angle for triangle ABC.

76. THEOREM. *The angle formed by two secants which meet outside a circle is equal to the half the difference of the two arcs contained between its sides.*

Consider angle $\widehat{BAC}$ formed by secants $AB'B$, $AC'C$. Angle $\widehat{BC'C}$, exterior to triangle ABC, is equal to the sum $\widehat{A} + \widehat{B}$. Thus we can measure angle $\widehat{A}$ as the difference $\widehat{BC'C} - \widehat{B}$. This proves the theorem, since inscribed angle $\widehat{BC'C}$ intercepts arc $\widehat{BC}$, and $\widehat{B}$ intercepts $\widehat{B'C'}$.

REMARK. If one of the secants, say $AB'B$, is replaced by the tangent AT (Fig. 74), the theorem and its proof hold without change, if we let T play the roles of both B and B'; in other words, by using the fact that *the tangent meets the circle in two points coinciding at T*, following the convention of **67**.

77. THEOREM. *The locus of points situated on one side of a line, from which a given segment of the line subtends a given angle, is a circular arc ending at the endpoints of the given segment.*

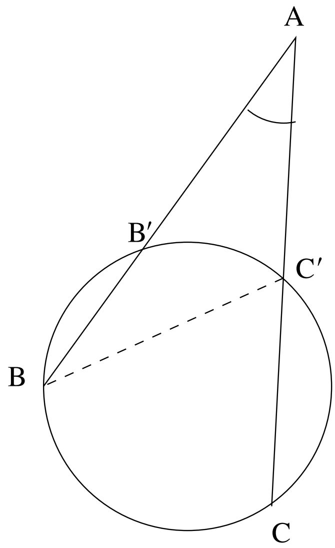


FIGURE 73

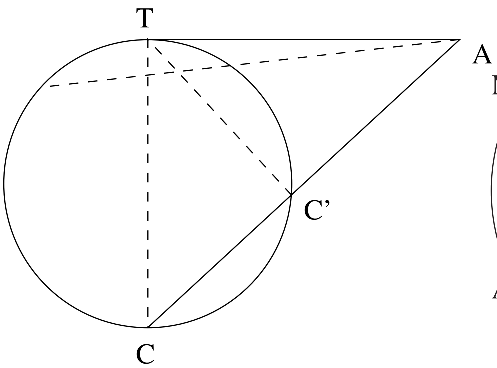


FIGURE 74

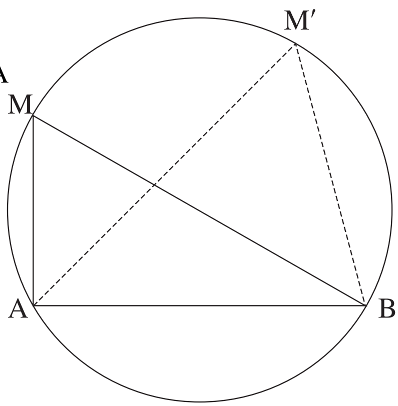


FIGURE 75

Indeed, let the given segment be AB, and let M be a point on the locus;[2] in other words, such that the angle $\widehat{AMB}$ is equal to the given angle (Fig. 75). If we draw an arc of a circle through A, M, and B, with endpoints at A and B, any point on this arc will belong to the locus, because $\widehat{AM'B} = \widehat{AMB}$ (by **73**). On the other hand, any point N of the plane, situated on the same side of AB but not on the arc, is either outside or inside this circle. In the first case, angle $\widehat{ANB}$ will be smaller than the given angle (**76**); in the second it will be larger (**75**), and therefore N will not belong to the locus. QED

[2]Such points certainly exist. One of them is obtained by constructing, through A and B, parallels to any angle equal to the given one and choosing, among the vertices of the parallelogram thus formed, the one which is on the required side of the line AB.

The circular arc described above is called the arc *capable* of the given angle relative to AB.[3]

If we do not require the points on the locus to be on one side of the line AB, then the locus will consist of two circular arcs, symmetric witih respect to AB.

78. When the given angle is a right angle, the preceding result combined with Corollary II in **73** shows that *the locus of points where a given segment subtends a right angle is the circle having the segment as diameter.*

This also follows from the two corollaries in **48**.

79. Four randomly chosen points will not generally belong to the same circle, since any three of them determine a circle which, in general, will not pass through the fourth.

A necessary condition for four points to belong to a circle is given in the following result.

THEOREM. *In any convex quadrilateral inscribed in a circle, the opposite angles are supplementary.*

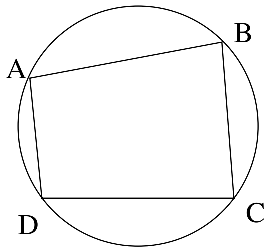


FIGURE 76

In quadrilateral $ABCD$ inscribed in a circle (Fig. 76):

Angle $\widehat{A}$ is equal to half arc $\overparen{DCB}$;

Angle $\widehat{C}$ is equal to half arc $\overparen{DAB}$.

The sum of these two arcs is the whole circle, so that the sum $\widehat{A} + \widehat{C}$ is equal to a central angle corresponding to a semicircle; that is, to two right angles.[4]

80. Converse. *If two opposite angles of a convex quadrilateral are supplementary, then the quadrilateral is cyclic.*

Suppose the quadrilateral is $ABCD$ (Fig. 76) in which angles $\widehat{A}$ and $\widehat{C}$ are supplementary. The arc of the circle joining D, A, B, extended beyond BD, is the locus of the points from which BD subtends a constant angle (**77**), which is (**79**) supplementary to angle $\widehat{A}$. This arc therefore passes through point C.

81. REMARK. In any quadrilateral $ABCD$ one can consider the four angles $\widehat{A}$, $\widehat{B}$, $\widehat{C}$, $\widehat{D}$, and also, by drawing the diagonals AC, BD, the eight angles numbered 1 to 8 in Figure 77. If the quadrilateral is cyclic, then:

[3]The term in French is simply the word *capable*, which has the same ordinary meaning as in English. But it is not used in English in this mathematical sense. We will not use it in this edition.–Transl.

[4]Hadamard uses the term 'inscribable' *(inscriptible)*, but without defining it, for a quadrilateral whose vertices lie on a circle. From this point on, we will use the standard term *cyclic*.–Transl.

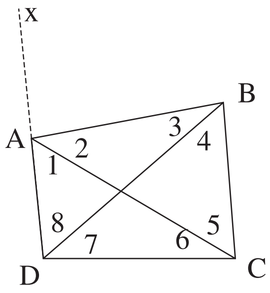


FIGURE 77

1°. $\widehat{A}$ and $\widehat{C}$ are supplementary;

2°. $\widehat{B}$ and $\widehat{D}$ are supplementary;

3°. $\widehat{1}$ and $\widehat{4}$ are equal;

4°. $\widehat{5}$ and $\widehat{8}$ are equal;

5°. $\widehat{2}$ and $\widehat{7}$ are equal;

6°. $\widehat{3}$ and $\widehat{6}$ are equal.

Conversely, if any one of these conditions holds, the quadrilateral is cyclic (**77, 80**).

Thus, *any one of the conditions* 1°, 2°, 3°, 4°, 5°, 6° *implies the others.*

82. Conditions 1° and 2° in the preceding remark can be rephrased in a form closer to 3°–6°.

We note first that since points A, B (Fig. 77) are on the same side of CD, triangles CDA, CDB have the same sense of rotation, so that the equal angles $\widehat{DAC}$, $\widehat{DBC}$ have the same sense. The same is true of angles $\widehat{ACB}$, $\widehat{ADB}$, etc. On the other hand, angles $\widehat{DAB}$, $\widehat{DCB}$ have the opposite sense, since A and C are on different sides of BD. So we extend DA beyond point A along Ax, to form an angle $\widehat{xAB}$ with the same sense as $\widehat{DCB}$: we see that these angles are equal if the angles $\widehat{DAB}$, $\widehat{DCB}$ are supplementary, and we arrive at the following conclusion:

If four points A, B, C, D are on the same circle, the angle of the same sense formed by AC, AD on the one hand, and BC, BD on the other, are equal; and any statement that can be derived from this one by permuting the letters A, B, C, D is also true.

Conversely, if any one of the above six conditions holds, then the quadrilateral is cyclic. Therefore, *each of the conditions* 1°, 2°, 3°, 4°, 5°, 6° *implies all the others.*

82b. The statement of **77** can also be replaced by the following: *The locus of vertices of all angles of a given size, whose sides (extended, if necessary) pass through two given points, is a circle passing through these two points.*

Since every circle can be obtained in this manner, this statement can be considered as a new **definition of a circle**, equivalent to the first, and which can replace it if necessary.

Exercises

Exercise 60. Find the locus of the midpoints of the chords of a circle passing through a fixed point.

Exercise 61. On each radius of a circle we lay off, starting from the center, a length equal to the distance from the endpoint of the radius to a fixed diameter. Find the locus of the endpoints of the segments thus obtained.

Exercise 62. Given a fixed circle, and a fixed chord AB in this circle. Let CD be a variable chord, whose length stays fixed.

1°. Find the locus of the intersection I of the lines AC, BD.

2°. Find the locus of the intersection K of the lines AD, BC.

3°. Find the locus of the centers of the circles circumscribing the two triangles ICD, KCD. Show that these are respectively congruent to the two loci found above.

Exercise 63. Points A and B are fixed on a circle, and M is a variable point on the same circle. We extend segment MA by a length $MN = MB$. Find the locus of point N.

Exercise 64. Points A, B, C are three points on a circle. We join the midpoints of the arcs $\widehat{AB}$, $\widehat{AC}$. Show that this line intercepts equal segments (starting from A) on chords AB, AC.

Exercise 65. If we draw two arbitrary secants through the common points A, B of two circles, the chords joining the new intersections of these lines with the two circles are parallel.

Exercise 66. The bisectors of the angles of an arbitrary quadrilateral form a cyclic quadrilateral. The same is true for the bisectors of the exterior angles.

Exercise 67. Through the midpoint C of arc $\widehat{AB}$ of a circle, we draw two arbitrary lines which meet the circle in D, E, and chord AB in F, G. Show that $DEFG$ is cyclic.

Exercise 68. We are given a circle, a fixed point P on this circle, and also a line, and a fixed point Q on this line. Through points P, Q we draw a variable circle which cuts the given circle again at R, and the given line at S. Show that line RS intersects the given circle in a fixed point.

Exercise 69. Two circles intersect at A and B. Through A we pass a variable line which cuts the circles again in C, C'. Show that the segment CC' subtends a constant angle at B, equal to the angle of the two radii ending in C, C'.

Through point A we draw a second secant, cutting the circles in D and D'. Show that the angle between the chords CD, $C'D'$ is equal to the angle found earlier, or to its supplement.

What becomes of the last statement when the two secants coincide?

Exercise 70. In any triangle, the points which are symmetric to the intersection of the altitudes with respect to the three sides lie on the circumscribed circle.

Exercise 71. Show that the altitudes of a triangle are the bisectors of the triangle formed by their feet.

We can apply the method of **82** to the quadrilaterals formed by two altitudes and two sides, show that these quadrilaterals are cyclic, and use the resulting angle properties.

The same method applies in a number of situations, such as the one in the following problem.[5]

Exercise 72. The feet of the perpendiculars, dropped from a point of the circumscribed circle onto the three sides of a triangle, are on the same line (the *Simson Line*).

Conversely, if the feet of the perpendiculars, dropped from a point in the plane onto the three sides of a triangle, are on the same line, this point must be on the circumscribed circle.

[5]This kind of reasoning is also useful in many of the problems proposed at the end of Book II.

CHAPTER V

Constructions

83. Since ancient times, it has been the practice to reserve the name *geometric constructions* for those constructions done with straightedge and compass. These constructions, theoretically, have absolute precision. In practice, they are effectively very precise, but nonetheless, as with any such process, offer some opportunity for error (such as the width of the marks left by a pencil).

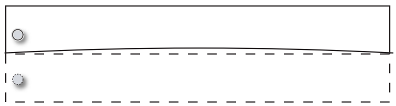


FIGURE 78

The *straightedge* is an instrument designed to draw straight lines. The accuracy of a straightedge, that is, the fact that its edge is really straight, depends on the care taken in its manufacture. To verify that this condition is satisfied, we use the actual definition of a straight line. We trace a first line along the edge of the straightedge (Fig. 78); then we turn the instrument around, placing it on the other side of the drawing. We should be able, in this new position, to trace along the edge of the straightedge a second line coinciding with the first.

The *compass* is composed of two pointed, articulated rods. The *opening* of the compass is the distance which separates the points of the two rods, and this instrument allows one to transfer distances, or to draw a circle with radius equal to this opening, with any center. This instrument is accurate if the opening does not change while the compass is moved.

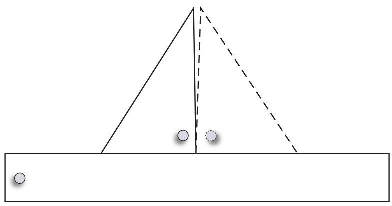


FIGURE 79

84. Besides the instruments just discussed, there are others which are used in practice, such as the set-square and the protractor. The *set-square* is a flat piece of wood (or other material) in the shape of a right triangle. A good set-square must satisfy two conditions:

1°. the sides must be straight;

2°. the angle must be a right angle.

The first of these is verified as in the case of the straightedege; it is usually satisfied in common set-squares. To verify the second one, we place the set-square on the edge of a straightedege (Fig. 79) and we trace a straight line along the side which should be perpendicular. Then, placing the set-square in the opposite sense, we should be able to draw a straight line coinciding with the first. Often, this condition is found to be only approximately satisfied, and one cannot rely on it for exact geometric constructions.

Finally, the *protractor* is an instrument for measuring angles. It is usually in the shape of a semicircle made of horn or copper, and divided into 180 parts. Since this division can only be done approximately, the protractor, while useful in practice, is not a geometric instrument.

85. Construction 1. (First fundamental construction). *Construct the perpendicular bisector of a given segment.*

This perpendicular is the locus of points equidistant from the endpoints A, B of a segment (**33**). So if we draw arcs, with centers A and B, which are sufficiently large that they intersect, the intersection points will belong to the perpendicular bisector; it remains simply to join these points.

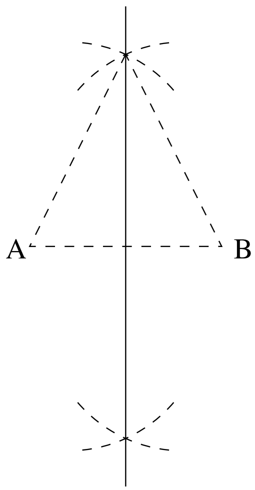


FIGURE 80

The following two constructions reduce to this one.

Construction 2. *Drop a perpendicular from a point to a line.*

With the given point O (Fig. 81 or 81b) as center, we draw a circle which intersects the line in points A, B. The perpendicular bisector of AB passes through O because $OA = OB$; it is therefore the desired perpendicular.

If point O is not on the line, we can take the distance OA as the common radius of the circles from construction 1 (used to find the perpendicular bisector of AB): thus we need not change the opening of the compass. These two circles will intersect at O and again at a point O', symmetric to the first with respect to

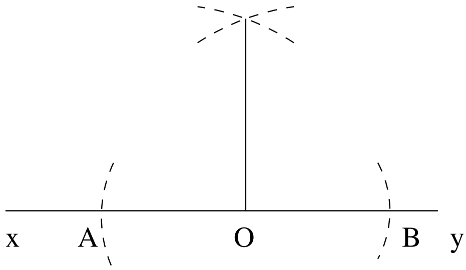


Figure 81

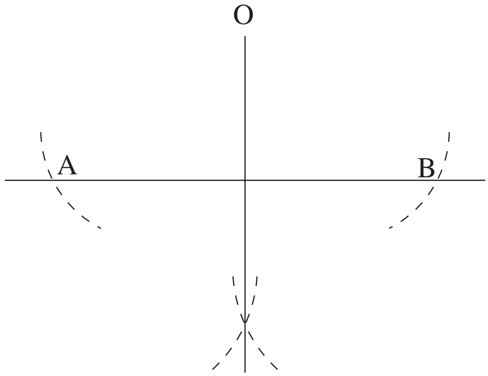


Figure 81b

AB. This simplification is not possible when O is on the line, because in that case $O = O'$. Even when O is very close to the line it is preferable to use a different radius, as the proximity of O and O' will not allow for a good determination of the line between them.

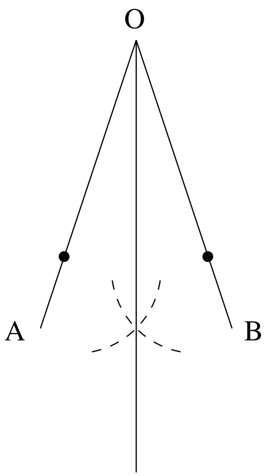


Figure 82

Construction 3. *Construct the bisector of a given angle* $\widehat{AOB}$ (Fig. 82).

We lay off two equal segments OA, OB on the sides of the angle, so as to form an isosceles triangle. The perpendicular bisector of AB is the desired angle bisector (**23**).

The same simplification as before applies to the radii of the circles.

86. Construction 4. (Second fundamental construction.) *Construct a triangle, knowing its three sides.*

Taking segment BC equal to the first side (Fig. 83), we draw circles centered at B, C with radii equal to the other two sides; their intersection A gives the third vertex of the triangle. Thus there are two possible choices for this third vertex, since two circles have two points in common, but the two resulting triangles are symmetric with respect to BC, and so are congruent.

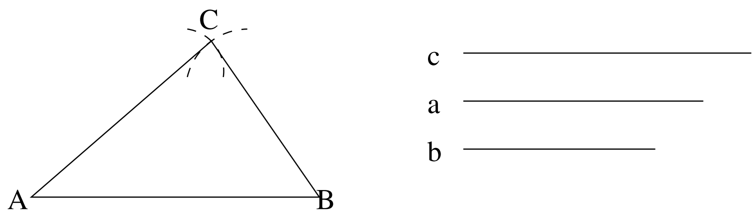


FIGURE 83

Condition for this construction to be possible. In order for the two circles to intersect, it is necessary and sufficient (**70, 71**) that the first side be less than then sum of the others, and greater than their difference or (**26**, Corollary), that each side be less than the sum of the other two.

If the order of the lengths of the three sides is known, it is enough to write that the longest side is smaller than the sum of the other two.

Construction 5. *Through a point on a line, construct another line, forming with the first an angle equal to a given angle.*

On the sides of the given angle O, we take two arbitrary points A, B. We then construct a triangle congruent to OAB, starting with a side equal to OA on the given line, and with an endpoint at the given point.

In practice, the triangle OAB is taken to be isosceles, so that we only need two different settings for the opening of the compass.

Construction 6. *Knowing two angles of a triangle, find the third.*

We construct two adjacent angles equal respectively to the two known angles. By extending one of the exterior sides we forms an angle supplementary to the sum of the first two. This is the required angle.

Condition for this construction to be possible. The sum of the two given angles must be less than two right angles.

Construction 7. *Construct a triangle knowing two sides and the angle they contain.*

On the sides of an angle equal to the given one, we lay off segments, starting from the vertex, equal to the given sides.

Construction 8. *Construct a triangle knowing one side and two angles.*

First, knowing two angles, we can find the third. In particular, we know the two angles adjacent to the given side; we place these angles at the endpoints of this side (Construction 5).

The conditions for this construction to be possible are the same as those for Construction 6.

87. Construction 9. *Construct a triangle knowing two sides, and the angle opposite one of them.*

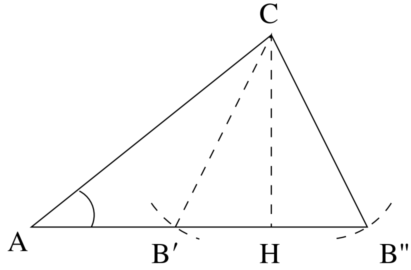


FIGURE 84

Let the triangle be ABC, in which we are given angle $\widehat{A}$ and sides a, b, opposite $\widehat{A}$, $\widehat{B}$, respectively. After constructing an angle equal to $\widehat{A}$ (Fig. 84), we lay off a segment AC equal to b on one of its sides. This determines point C, and we can then find B by drawing a circle with center C and radius a. The intersection of this circle with the second side of angle $\widehat{A}$ gives us vertex B, which determines the triangle.

Discussion. We have a solution whenever the circle constructed above intersects ray Ax (Fig. 84), the second side of angle $\widehat{A}$.

We drop a perpendicular CH onto Ax.

1°. If $CH > a$, the circle does not intersect Ax and *the construction is impossible.*

2°. If $CH < a$, the circle intersects line Ax in two points B', B''. It remains to see whether these points are on ray Ax or on its extension.

We distinguish two cases:

(a) *Angle $\widehat{A}$ is acute.* In this case H, the midpoint of $B'B''$, is on ray Ax. At least one of the points B', B'' will also be on this ray, so that *there always is at least one solution. There is a second solution* if the second point is between A and H, which happens *if and only if $a < b$.*

(b) *Angle $\widehat{A}$ is obtuse.* Then point H will not be on ray Ax; therefore at least one of the points B', B'' will not provide a solution, and thus *there is at most one solution. There is one* if the second point is farther from H than A is, which happens *if and only if $a > b$.*

87b. Finally, if angle $\widehat{A}$ is a right angle, the problem can be restated thus: *Construct a right triangle, knowing the hypotenuse and a side of the right angle.* It has a solution if the hypotenuse is greater than the given side.

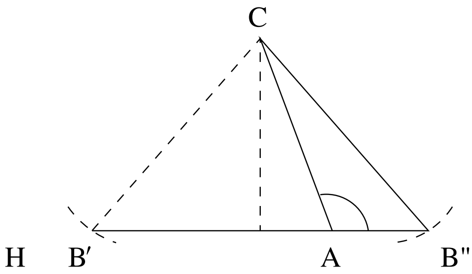


FIGURE 85

REMARK. The following theorem is a result of the previous considerations:

THEOREM. *When two triangles have two sides equal in pairs, as well as the angle opposite one of them, the angles opposite the second side are either equal or supplementary. The first case (which implies the congruence of the triangles) always holds if the given equal angles are right or obtuse.*

88. Construction 10. *Through a point, construct a parallel to a given line.*

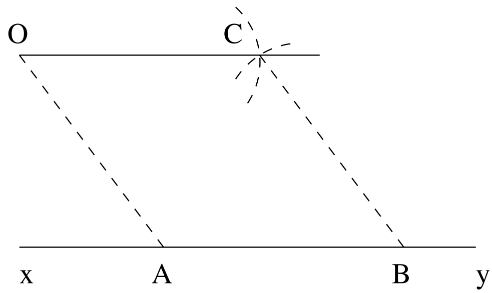


FIGURE 86

On the given line, we take two arbitrary points A, B (Fig. 86), then draw a circle with center O and radius AB, and a circle with center B and radius OA. These two circles will intersect (on the side of OB opposite to A) in a point C which belongs to the desired parallel. Indeed, $OABC$ is a parallelogram because it has equal opposite sides.

It is convenient to take $AB = OA$ so that all the circles drawn have the same radius.

89. Constructions 2 and 10 can also be performed with the set-square. To perform Construction 2 (*To construct a perpendicular to a given line through a given point*), we first apply the straightedge to the given line, and then slide one of the sides of the set-square along the straightedge, until its second side, perpendicular to the first, passes through the given point (Fig. 87) and thus gives the desired perpendicular.

This construction assumes the condition, usually not fully satisfied, that the angle of the set-square is a right angle, so it is not used in very precise sketches.

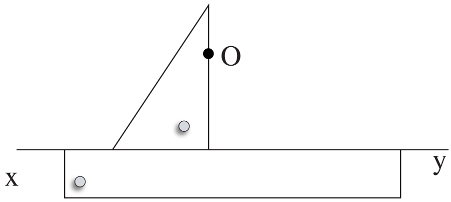


Figure 87

To perform Construction 10: *Through a point, construct a parallel to a given line*, we first place one side of the set-square along the given line (Fig. 88). Next, applying the straightedge to another side of the set-square and holding it in this position, we slide the set-square along the straightedge until the edge originally placed on the given line passes through the given point. The line thus obtained is parallel to the original one because it makes equal corresponding angles ($\widehat{A}$, $\widehat{A'}$ in the figure) with the same transversal (the edge of the straightedge).

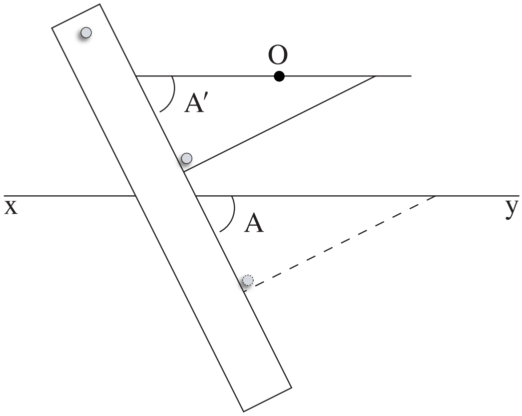


Figure 88

This construction supposes only that the edges of the straightedge and of the set-square are straight. Since it is as accurate as, and simpler than, the compass construction, it is the one that is used in practice.

90. Construction 11. *Divide a circular arc into two equal parts.*

We draw the perpendicular bisector of the corresponding chord (Construction 1).

Construction 12. *Construct the circle passing through three given points A, B, C, not on the same line.*

We construct the perpendicular bisectors of two of the segments joining the points: their intersection O is the center of the circle, and OA is its radius.

The circle thus obtained is said to be *circumscribed* about the triangle formed by the three points.

In general a polygon (or a broken line) is said to be *inscribed* in a curve, which is then said to be *circumscribed* about the polygon, when all the vertices of the polygon are on the curve.

REMARK. If the three points A, B, C were collinear, there would not be any circle passing through them; we can consider line ABC as replacing it. *A straight line can be viewed as a limiting case of a circle,* whose center becomes infinitely distant. In fact, line AB is the locus of points at which segment AB subtends a constant angle; namely, an angle of zero degrees, or two right angles. One can also show directly that a circle passing through two fixed points AB, and whose center recedes indefinitely, has the line AB as its limit (one must, of course, give a precise meaning to this statement.)

Construction 13. *Construct a circle knowing two of its points A, B, and the tangent AT at one of them.*

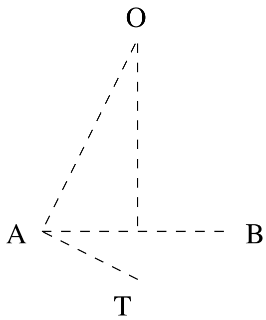


FIGURE 89

The center O will be the intersection of the perpendicular bisector of AB with the perpendicular to AT at the point A. The circle with center O and radius OA will indeed be tangent to AT, and will pass through B (Fig. 89).

REMARK. This construction is a limiting case of the preceding one, following the general definition of tangents (**59**).

Construction 14. *On a given segment, construct the arc of a circle from which the segment subtends a given angle.*

Let AB be the given segment. Construct one point M of the locus (see footnote 2 to **77**) and construct the circle AMB; or, note that the tangent at A to this circle forms with AB an angle equal to the given one (**73, 74**), which allows us to construct it (Construction 5), and we have reduced the problem to the preceding construction.

91. Construction 15. *Construct the tangent to a given circle at a given point.*

We draw a perpendicular to the radius at the point of contact.

Construction 16. *Construct a tangent to a circle, parallel to a given line.*

The diameter passing through the point of contact is precisely the diameter perpendicular to the given line; this diameter has two endpoints, and therefore there are two solutions.

Construction 17. *Construct a tangent to a circle through a point in the plane.*

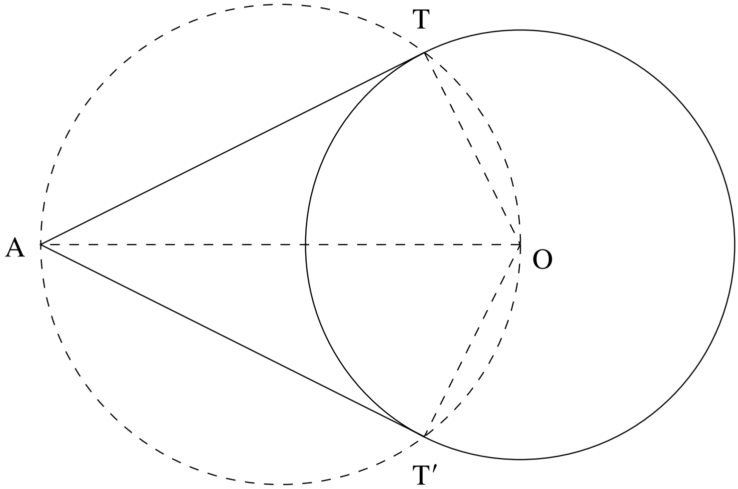


FIGURE 90

Let the given point be A, and the given circle have center O (Fig. 90). Assume that the problem is solved, and let T be the point of contact of a tangent passing through A. Since $\widehat{ATO}$ is a right angle, point T belongs to the circle with diameter OA. Conversely, if a point T is on the given circle, and on the circle with diameter OA, then the tangent through T passes through A, because $\widehat{ATO}$ is a right angle.

Therefore the points T which satisfy the conditions of the problem are the common points of the given circle and the circle with diameter OA.

Another method. Let the desired point of contact be T. Extend the segment OT beyond T by a length $TO' = OT$ (Fig. 91). Since TA must be perpendicular to OT, point O' is symmetric to O with respect to AT, and therefore $AO = AO'$. Thus point O' belongs to the circle with center A and radius AO. As it also belongs to a circle centered at O and with a radius twice the radius of the given circle, it is determined by the intersection of these two curves. Conversely, if point O' is such that $AO' = AO$, the midpoint T of OO' is the point of contact of a tangent from A to the circle with center O and radius OT, because of the properties of an isosceles triangle.

Discussion. Let us start, for example, with the second construction. In order for a solution to exist, it is necessary that the two circles which determine O' intersect; that is, (**66**) that the distance between their centers be less than the sum of the radii, and more than the difference. If R is the radius of the given circle,

$$AO \leq 2R + AO \tag{1}$$

and

$$AO \geq AO - 2R \tag{2}$$

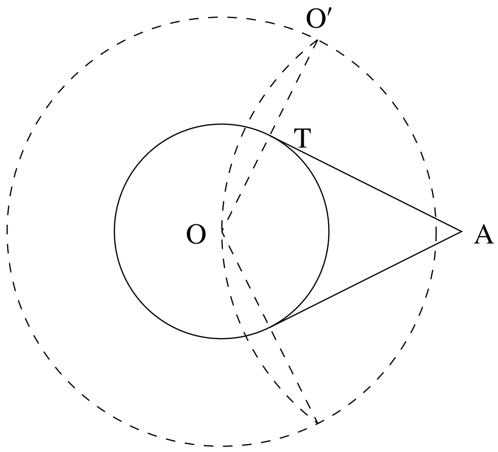


FIGURE 91

or

$$AO \geq 2R - AO, \tag{3}$$

depending on whether AO is greater than or less than $2R$. The inequalities (1) and (2) are obviously satisfied. Inequality (3) can be written $2AO \geq 2R$, or $AO \geq R$. Thus *the problem is impossible if A is inside the circle. On the other hand, if A is outside, there are two solutions*, since the circles intersect in two points. Finally, *if point A is on the circle,* the two circles which determine point O' are tangent, and *there is only one solution,* the tangent at A. Or rather, *there are two identical solutions* because, as we remarked (**72**), if the point A approaches infinitely close to the circle, the two points of contact, and therefore the two tangents, tend to coincide.

92. THEOREM. *The two tangents to a circle passing through a given point are equal, and they form equal angles with the diameter passing through the point. This diameter is perpendicular to the chord joining the two points of contact.*

Indeed, the two points of contact are symmetric with respect to OA (Fig. 90) because they are the intersection points of two circles with centers on this line (see **68**).

93. Construction 18. *Draw the common tangents to two circles.*

Consider two circles with centers O, O' and radii R, R'. The common tangents can be of two kinds: they can be *external* (Fig. 92), which means that both circles are on one side of the line, or *internal* (Fig. 93) when the two circles are on different sides of the tangent.

1°. *Common external tangents.* Suppose that a solution exists, and let AA' be the required tangent (Fig. 92), with A, A' being the points of contact, so that OA, $O'A'$ are perpendicular to AA' . We daw a parallel through O' to AA', extended until it intersects OA in C. We will determine this point C.

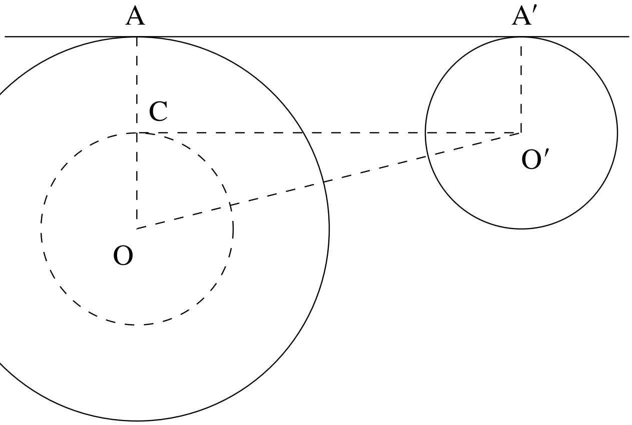


FIGURE 92

To this end, observe that $CAA'O'$ is a rectangle, so that $AC = R'$ and therefore[1] $OC = R - R'$.

The point C therefore lies on a circle we can draw, since we have its center, O, and its radius, which is the difference between the given radii. Also, line CO' is tangent to this circle, since angle $\widehat{OCO'}$ is a right angle. Thus we are led to Construction 17.

Once point C is determined, extending radius OC gives us point A. The tangent at A to the circle O is then the common tangent, because if we drop a perpendicular $O'A'$ onto AA', the equality $OC = R - R'$ gives $O'A' = AC = OA - OC = R'$.

The construction is possible (**91**) *only if* $OO' \geq R - R'$, in other words (**70**), *if the two circles are external to each other, tangent externally, intersecting, or tangent internally.* In the first three cases there are two solutions, since we can draw two tangents from O' to circle OC; in the fourth case, there is only one solution, which can be considered two coinciding solutions, since the two tangents from O' to circle OC coincide.

2°. *Common internal tangents.* Let BB' be a common internal tangent, with points of contact at B and B' (Fig. 93). As in the first case, we draw a parallel $O'D$ to BB', where D is the intersection of the parallel with line OB. Quadrilateral $DBB'O'$ is a rectangle, and therefore $BD = R'$, and D belongs to the circle centered at O with radius $R + R'$. We can draw this circle, and draw a tangent to it from O'. Once point D is determined, the intersection of OD with the first circle provides the point B. Following this reason backwards, we see that the tangent at B is a common tangent.

The construction is possible only if $OO' \geq R + R'$, that is, *if the two circles are exterior to each other, or externally tangent.* In the first case there are two solutions, in the second only one (or two identical solutions).

In summary:

• Two circles external to each other have two common external tangents and two common internal tangents;

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[1]To be definite, we assume that the circle O has the greater radius. This assumption, which we have the right to make, is not essential here. In the opposite case, we will have $OC = R' - R$.

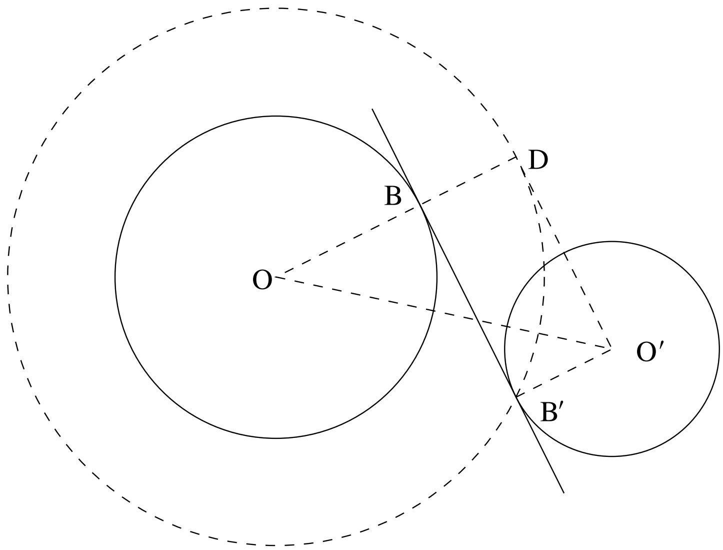

FIGURE 93

• Two circles which are tangent externally have two common external tangents, and one common internal tangent (at their point of contact);

• Two intersecting circles have two common external tangents;

• Two circles which are tangent internally have one common external tangent (at their point of contact);

• Two circles, one of which is interior to the other, have no common tangents.

94. Construction 19. *Construct the circles tangent to three lines.*

The problem amounts to this: *find the points equidistant from three lines.* A circle with its center at such a point, and the common distance as radius, will answer the question.

Now, the locus of points equidistant from two intersecting lines consists of the two bisectors of the angles formed by these lines.

Let us assume first that each pair of the three lines intersect, but not all three pass through the same point, so that they form a triangle (Fig. 94). In this triangle we construct the bisectors of the interior and exterior angles. We already know that: 1° the bisectors of the interior angles are concurrent; 2° the bisector of any interior angle has a point in common with the bisectors of the non-adjacent exterior angles. The four points thus defined (the intersection of the interior bisectors, and the three vertices of the triangle formed by the exterior bisectors) are therefore the desired points, and there are no others.

The first of the circles with centers at these points is inside the triangle. It is called the *inscribed circle*: in general, a curve is *inscribed* in a polygon when it is tangent to all of its sides. The three other circles, also tangent to the three sides, but outside the triangle, are said to be *escribed*: each one of them is situated inside one of the angles of the triangle, but separated from its vertex by the opposite side, as indicated in the figure.

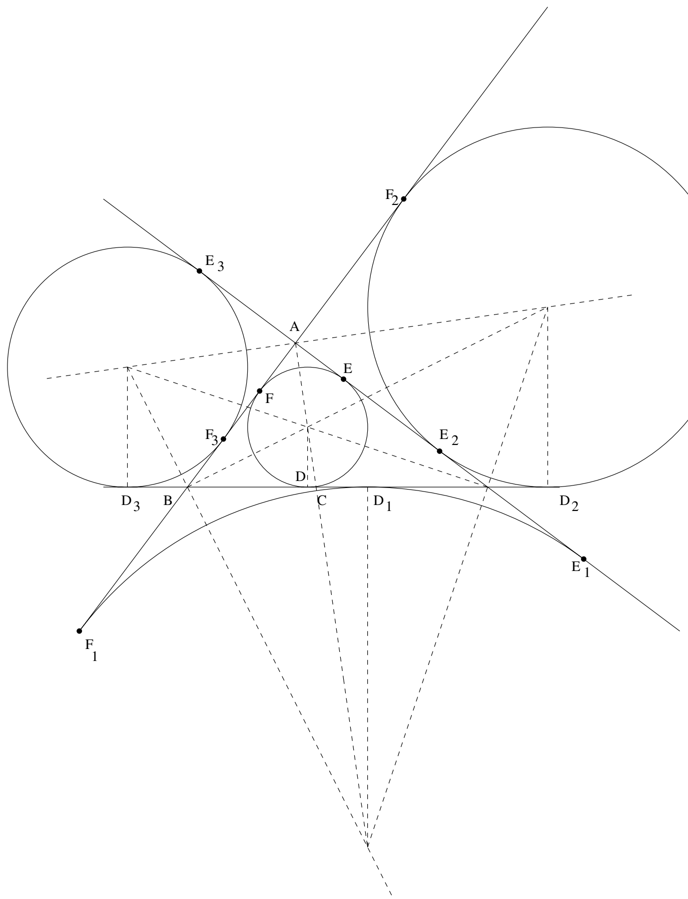

FIGURE 94

If the three given lines pass through the same point, there is no solution properly speaking, because one can draw only two tangents from a point to a circle. The preceding construction gives a circle reduced to one point; namely the, common point of the three lines.

If two of the lines are parallel, and the third is a transversal (Fig. 95), the bisectors of the angles formed by the transversal with the other two lines form two parallel pairs. Therefore we obtain only two circles, on either side of the transversal.

Finally, if the three lines are parallel, there is no solution, because a circle has only two tangents with a given direction.

Exercises

Exercise 73. Construct a circle of given radius:

(i) which passes through two given points;

(ii) which passes through a given point and is tangent to a given line;

(iii) which is tangent to two given lines;

(iv) which is tangent to a given line and a given circle.

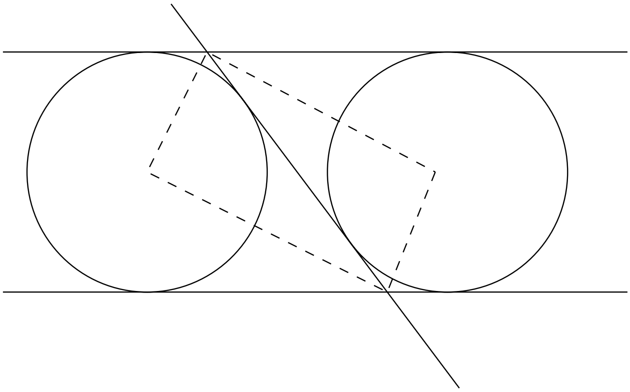

Figure 95

Exercise 74. Construct a circle tangent to a given line at a given point and also tangent to a given circle.

Exercise 75. Between two given lines (or circles), construct a segment of given length and direction.

Exercise 76. Construct a tangent to a given circle, on which a given line determines a segment of a given length.

Exercise 77. Construct a triangle knowing a side, the opposite angle, and an altitude (distinguish two cases, according as the given altitude corresponds to the given side or not).

Exercise 78. Divide a segment into three equal parts using the theorem of no. 56 on the medians of a triangle.

Exercise 79. Construct a triangle knowing:

(i) two sides and a median (distinguish two cases, according as the given median falls on one of the given sides or is contained between them);

(ii) one side and two medians (two cases);

(iii) the three medians.

Exercise 80. Construct a triangle knowing a side, an altitude, and a median (five cases).

Exercise 81. Construct a triangle knowing an angle, an altitude, and a median (five cases).

Exercise 82. Construct a triangle knowing one side, the opposite angle, and the sum or the difference of the other sides.

Exercise 83. Same problem, but the angle is adjacent to the given side.

Exercise 84. Construct a triangle knowing an angle, an altitude, and the perimeter (two cases).

Exercise 85. Construct a triangle knowing one side, the difference of the adjacent angles, and the sum or difference of the other sides.

Exercise 86. Construct the bisector of the angle formed by two lines which cannot be extended to their intersection.

Theorems on Tangents (see 92).

Exercise 87. In a quadrilateral circumscribed about a circle (and containing the circle in its interior), the sum of two opposite sides is equal to the sum of the other two opposite sides. Conversely, a convex quadrilateral in which the sums of the opposite sides are equal can be circumscribed about a circle.

While we cannot be sure that a quadrilateral circumscribed about a circle contains the circle in its interior, we can nonetheless always say that the sum of two side of the quadrilateral, *appropriately chosen*, is equal to the sum of the two others, and conversely.

Exercise 88. A circle tangent to three sides of a convex quadrilateral is on the same side as the interior of the quadrilateral, relative to these three sides. Show that this circle intersects or does not intersect the fourth side according as the sum of this side and the opposite side is greater than or less than the sum of the other sides.

Exercise 88b. If two points are outside a circle, the line joining them is exterior to or secant to this circle according as this segment between the two points is or is not contained between the sum and the difference of the tangents from these points to the circle.

Exercise 89. The segment intercepted by two fixed tangents of a circle on a variable tangent subtends a constant angle at the center. Special case: the fixed tangents are parallel.

Exercise 90. If a variable tangent to a circle intersects two fixed tangents, and the point of contact of the variable tangent is on the smaller arc determined by the point of contact of the fixed tangents, the triangle formed by the three tangents has a constant perimeter.

What can we say about this situation if the point of contact of the variable tangent lies on the larger arc determined by the fixed tangents?

Exercise 90b. Denote by a, b, c the three sides of a triangle ABC, and let p be the semiperimeter of the triangle (so that $2p = a + b + c$). Let D, E, F be the points of contact of the inscribed circle, and D_1, E_1, F_1, D_2, E_2, F_2, D_3, E_3, F_3 be the points of contact of the escribed circles in angles A, B, C, respectively. Then the segments intercepted by these points have the following values:

$$AE = AF = CD_2 = CE_2 = BD_3 = BF_3 = p - a;$$
$$BF = BD = AE_3 = AF_3 = CE_1 = CD_1 = p - b;$$
$$CD = CE = BF_1 = BD_1 = AF_2 = AE_2 = p - c;$$
$$AE_1 = AF_1 = BF_2 = BD_2 = CD_3 = CE_3 = p.$$

Exercise 91. Construct three pairwise tangent circles with centers at the three vertices of a given triangle.

CHAPTER VI

On the Motion of Figures

95. We have defined (**20**) the notion of congruent figures with the same orientation. We know (**50**) that if two points of a figure coincide with the corresponding points in a congrent figure, then the two figures coincide completely. In other words, *there is only one way to place a figure F' congruent to a given figure F with the same orientation*[1] *in such a way that two points of this figure (the points corresponding to two given points A, B of F) have a given position.*

In fact, with this information (knowledge of the homologues A', B' of A, B), *we can determine the point C' corresponding to another point C of F.*

Indeed, it suffices to construct the triangle $A'B'C'$ congruent to A, B, C (Construction 4) then choose, from the two possibilities for the third vertex, the one which corresponds to the same orientation.

Since this can be done for every point of F', we see that we can solve (at least for figures composed of finitely many points and lines) the following:

Problem. Construct a plane figure F' congruent to a given figure F (and with same orientation), knowing the points corresponding to two given points.

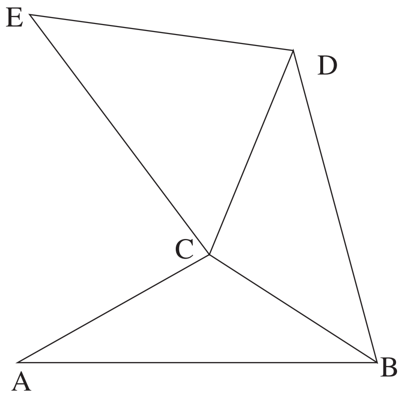

FIGURE 96

We should note that in constructing each point F', we are not obliged to start directly from the given points A', B'.

[1]It is clear that without this restriction, nothing prevents us from replacing the figure F' with the figure symmetric to it, with respect to the line joining the two points.

On the contrary, we can obtain the point corresponding to any point using the triangle formed, not with the original points, but with two other points already constructed.[2]

In this way we can construct a figure $ABCDE$ (Fig. 96) by constructing successively three triangles congruent to ABC, BCD, CDE.

96. The operation called *translation* transforms every figure into a congruent figure with the same orientation. This motion can be thought of as a *continuous* sliding of the figure in its plane. It suffices to make the translation vary continuously from the value zero (for which the transformed figure coincides with the original one) to the final value of the translation.

In this parallel motion, *the points move along parallel lines, in the same direction.*

97. A *rotation* is the name given to a motion in which each point of the figure turns around a fixed point, called the *center* of the rotation, in a determined sense, through an angle which defines the *magnitude* of the rotation. In other words, a point A of the original figure F being given, we find the corresponding point A' of F' by joining A to the center O of rotation, forming an angle $\widehat{AOA'}$ equal to the angle of rotation (and with the same orientation) then taking, on the second side of this angle, a length OA' equal to OA (Fig. 97).

We see that a rotation is determined by its center, its magnitude, and its sense.

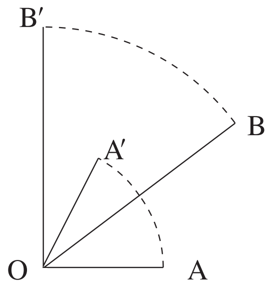

FIGURE 97

The figure F' obtained from F by an arbitrary rotation is congruent to F. To see this, take any two points A, B of F (Fig. 97) and consider the homologues A', B'. Angle $\widehat{A'OB'}$ is equal to $\widehat{AOB}$ (and has the same orientation). Suppose, to be definite, that $\widehat{AOB}$ has the same orientation as the rotation.[3] Replacing OB by OB', we increase this angle by a quantity equal to the angle of rotation, but then replacing OA by OA' diminishes the angle by the same quantity.

We conclude that the triangles AOB, $A'OB'$ are congruent, with same orientation, because they have an equal angle contained between equal sides. ($OA = OA'$, $OB = OB'$). Since this holds for all points B, the congruence of the figures is established.

[2]We will see how we can proceed in just the same manner in drawing maps (see Solid Geometry, Book X, **565**).

[3]Recall (**20**) that angle $\widehat{AOB}$ is taken to be drawn in the sense going from AO towards OB. We can make the assumption of the text by taking, as side AO, the first side we encounter, as we rotate in the given sense.

A rotation can also be obtained as a sliding of the figure in the plane, making the angle of rotation vary continously from zero to its final value. In this motion, each point moves along a circular arc with its center at the center of rotation. These arcs correspond to equal central angles.

98. Theorem. *In two equal figures, derived from each other by a rotation around a point O* (Fig. 98):

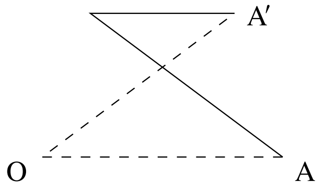

Figure 98

1°. *Two corresponding lines, taken in the corresponding sense, form an angle equal to the angle of rotation, and in the same sense*;

2°. *The center of rotation, two arbitrary corresponding points A, A', and the intersection I of two corresponding lines passing through A and A', respectively, belong to the same circle.*

1°. The first part of the theorem is true, by definition, for two corresponding lines passing through the center of rotation. It is therefore true for arbitrary corresponding lines, because each one can be replaced by a parallel line passing through the center of rotation.

2°. The four points A, A', O, I are on the same circle; namely, the locus of the vertices of angles, equal to the angle of rotation, whose sides pass through points A, A'.

99. Rotations through an angle equal to two right angles, or 180°, are of particular interest. The point A' corresponding to a point A can be constructed by joining OA, and extending this segment by a length $OA' = OA$. The point A' thus obtained is called the point *symmetric to A with respect to the point O.* Thus, in *plane geometry,* symmetry with respect to a point, and rotation about the same point by 180°, are two identical operations.

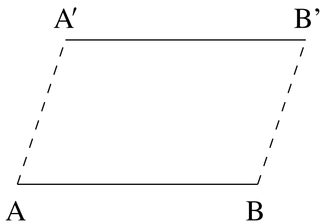

Figure 99

100. *If, in two equal figures with the same orientation, two corresponding segments AB, $A'B'$* (Fig. 99) *are parallel and have the same sense, the two figures* *can be superimposed by a translation.* Indeed, the quadrilateral $ABA'B'$ is a parallelogram, and therefore AA' is also parallel to BB', equal to it, and in the same sense: the translation AA' moves the points A, B to their respective homologues, which implies that this translation will make the two figures coincide completely.

If, in two equal figures with the same orientation, a point O coincides with its homologue, the two figures can be superimposed by a rotation around this point.

Indeed, if A, A' are two other corresponding points, the rotation with angle AOA' around O brings the point A to A', while the point O remains its own homologue.

101. We have seen (**20**) that two congruent figures with different orientations cannot be superimposed without one of them leaving the plane. On the other hand, two congruent figures with the same orientation can always be superimposed by a motion in the plane; namely, by *a translation followed by a rotation.* Indeed, if A, A' are two corresponding points, the translation AA' applied to the first figure will make these two points coincide, after which a rotation about the point A' will suffice to make the figures coincide (by the preceding section).

Observing that a translation does not change the directions of lines, we deduce that *two arbitrary corresponding lines of two congruent figures with same orientation form a constant angle:* thus we can talk of *the angles between the two figures.*

102. However, of the two preceding operation (translation and rotation), we can show that one only is necessary. Indeed, we have the following theorem:

Theorem. *Two congruent figures with the same orientation can be superimposed either by a translation, or by a rotation about a properly chosen point.*

This point is called the *center of rotation.*

Proof. Consider two points A, B of the first figure, and their homologues A', B' in the second figure. We will have superimposed the two figures if we make the segements AB and $A'B'$ coincide.

1°. If the segments AB, $A'B'$ are parallel in the same sense (Fig. 99), the figures can be superimposed by a translation;

2°. Assume that this is not the case. Let us see if these figures can be superimposed by a rotation. We know the angle of this rotation, which is equal to the angle between these figures, for instance the angle between AB and $A'B'$ (Fig. 100). We must therefore find a point O such that the rotation around O with this angle takes A to A'. This amounts to finding the center of a circular arc ending at A, A' and subtended by an angle equal to half the angle of rotation, and in the same sense (since the angle at the center equals twice the inscribed angle intercepting the same arc).

The point O will therefore be on the perpendicular bisector of the segment AA'; on the other hand, it will also be on the perpendicular to the tangent to the circle, constructed as indicated in **90** (Construction **14**).

Conversely, point O being thus determined, the rotation with this center and with an angle equal to the angle of the two figures, will take A onto A'; it will also take the segment AB to a new position, parallel to $A'B'$ and with same direction. Since the two segments have the endpoint A' in common, this new position must be $A'B'$ itself. Thus the figures are superimposed. QED

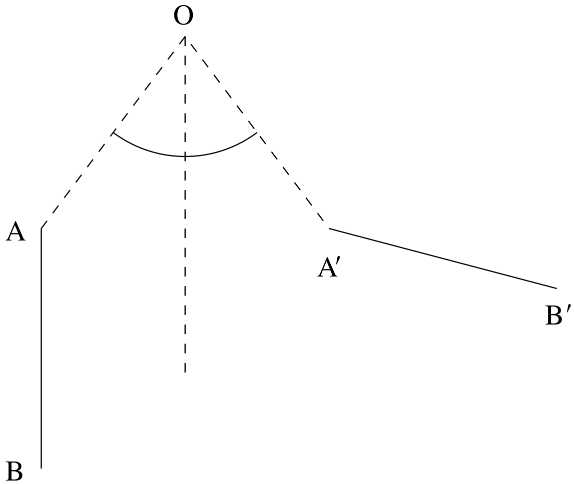

FIGURE 100

REMARK. We can also prove this theorem by finding point O as the intersection of the perpendicular bisectors of AA' and BB' or (**98**) as the intersection of the circumscribed circles of triangles IAA', IBB', where point I is the intersection of AB and $A'B'$. With either of these constructions, one would establish that the triangles OAB and $OA'B'$ are congruent.

102b. The preceding theorem can also be proved by decomposing rotations and translations into two successive symmetries.

LEMMA. *Successive reflection in two lines D_1, D_2 is equivalent to:*

1°. *a translation perpendicular to the common direction of the lines, and twice the translation which would bring the first line onto the second, when the lines are parallel*;

2°. *a rotation around their intersection point and double the one which would bring the first line onto the second, when the lines intersect.*

PROOF. Let F be a figure, let F' be its reflection in D_1, and let F'' be the reflection of F' in D_2.

1°. Suppose that D_1 and D_2 are parallel (Fig. 101). If M is a point on F, we can find its reflection M' in D_1, by extending the perpendicular MH_1 from M to D_1 by its own length. We can then find the reflection M'' of M' in D_2 by extending the perpendicular MH_2 from M' to D_2 by its own length. We see immediately that the three points M, M', M'' are on the same common perpendicular to D_1 and D_2. To be definite, we assume that M' is between D_1 and D_2. In this case the segment MM'', equal to the sum of MM' and $M'M''$ (Fig. 101), is twice the segment H_1H_2 which is the sum of $M'H_1$ and $M'H_2$.

This conclusion remains true if the point M' is exterior to the two parallel lines: in that case segment MM'' is the difference of MM' and $M'M''$, and H_1H_2 is the difference of $M'H_1$ and $M'H_2$. Thus the translation represented by a segment perpendicular to the two lines and double in length to H_1H_2 in fact does take any point of F onto the corresponding point of F''.

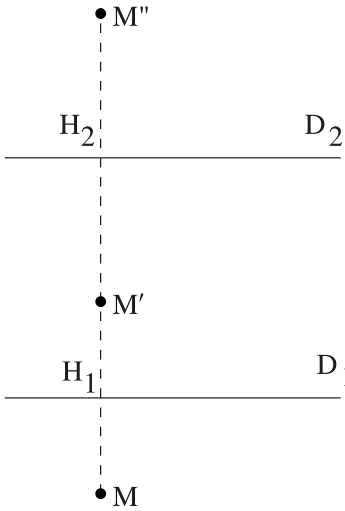

Figure 101

2°. Assume that the two lines of symmetry meet at O (Fig. 102), and let M, M', M'' be corresponding points of F, F', F'', so that M, M' are symmetric with respect to D_1 and M', M'' are symmetric with respect to D_2. Points M, M', M'' are clearly on the same circle with center O. Moreover, angle $\widehat{MOM''}$ is equal to the sum or difference of $\widehat{MOM'}$ and $\widehat{M'OM''}$, and therefore is twice the angle between D_1 and D_2, and can be thought of as the sum or the difference of the angles made by OM' with D_1 and D_2. Therefore the rotation defined in the statement of the lemma moves each point M onto its corresponding point M''. QED

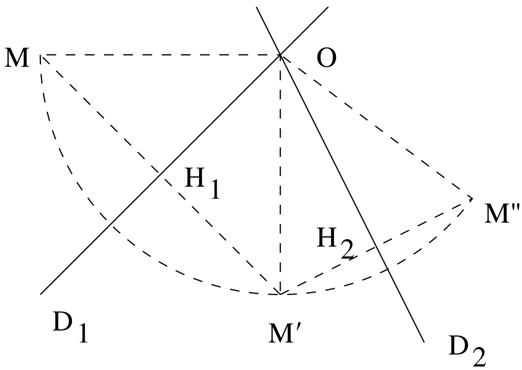

Figure 102

Remark. Two successive reflections in the same line cancel each other, since each point returns to its original position.

Corollary. *Conversely, every translation can be replaced by two successive line reflections, in lines perpendicular to the direction of the translation, one of which can be chosen arbitrarily*;

Every rotation can be replaced by two successive reflections in lines passing through the center of the rotation, and one of these can be chosen arbitrarily.

Indeed, in the case of a translation, it is enough to take the line that is not arbitrarily chosen to be at a distance from the arbitrary line equal to half that of the translation; and, in the case of a rotation, to take it so as to make, with the arbitrarily chosen line, an angle half that of the rotation.

103. Theorem. *Any number of successive translations or rotations can be replaced with a single translation or a single rotation.*

This process called *composition* of displacements: the final motion is called the *result* of the given displacements.

It suffices to know how to compose two displacements; if we have three, we begin by composing the first two, and then compose the result with the third; one would continue in an analogous fashion if there were more than three displacements.

Two successive translations replace a line with a parallel one, in the same sense, and therefore the result is a translation (**100**).

We need therefore consider only a translation followed or preceded by a rotation, or two rotations.

In the first case, we assume, to be definite, that the translation is performed first. This translation can be decomposed into two reflections in the lines D_1, D_2, and the rotation can also be decomposed into reflections in the lines D_1', D_2', so that we must perform four reflections altogether, in lines D_1, D_2, D_1', D_2'. Now the line D_2 can be chosen arbitrarily from among those perpendicular to the direction of the translation, and the line D_1' can be chosen arbitrarily from among those passing through the center of the rotation. So we can choose both to coincide with the line through the center of rotation and perpendicular to the direction of translation.

With this choice, the reflections in these two lines will cancel, and we are left with two reflections, in lines D_1, D_2' which will yield (by the preceding numbered paragraph) a translation or a rotation.

If we had to compose two rotations, we would have taken $D_2 = D_1'$ to be the line joining the two centers of rotation. This proves the theorem in all cases.

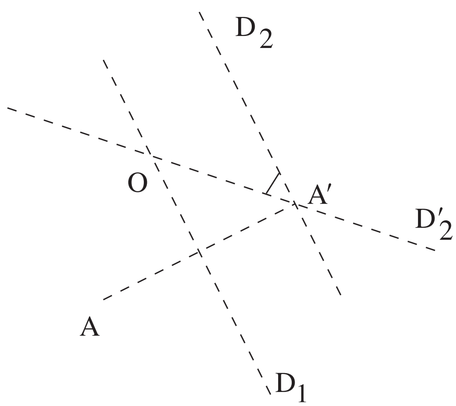

Figure 103

The theorem in **102** can now be regarded as a corollary of the one we just proved. An arbitrary motion of a figure in the plane is a translation followed by a rotation, namely (if A, A' are two corresponding points of the original figure and of the displaced figure) the translation AA' followed by a rotation with center A' through an angle equal to the angle of the two figures (Fig. 103). Lines D_2, D_1' will coincide with the perpendicular from A' to AA'; line D_1 will then be the perpendicular bisector of AA', and D_2' will be obtained by rotating D_2 about A' through half the angle of rotation. The intersection point of D_1, D_2' gives the center of rotation. We see that the construction is entirely in agreement with the one given earlier, because in constructing (Construction 14) the tangent to the circular arc on AA' which subtends half the angle of rotation, we are actually constructing a perpendicular to D_2'.

104. Theorem. *When a figure moves in the plane, without change shape, at each instant the normals* (**60**) *to the trajectories of the various points (if they are not all parallel) will be concurrent at a point* which we call the *instantaneous center of rotation.*

Leaving aside the case where all normals to the moving points $M, N, P, \ldots$ are parallel, we assume that two of these normals, the normals at M, N for instance, intersect in a point O (Fig. 104).

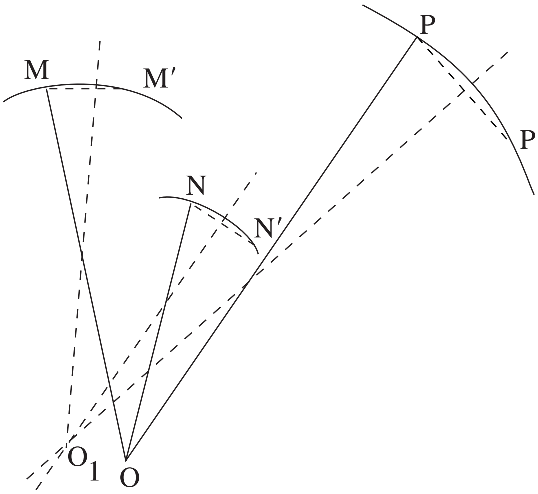

Figure 104

Consider a second position of the moving figure, close to the first, where the new positions of M, N, $P, \ldots$ are M', N', $P' \ldots$ The perpendicular bisectors of MM', NN', $PP', \ldots$ pass through the same point O_1, the center of the rotation which brings the first position of the figure onto the second. If we now imagine the second position coming arbitrarily close to the first, the midpoint of MM' will approach M, the line MM' will approach the tangent at M to the trajectory of this point, and the perpendicular bisector of MM' will approach the normal to this trajectory. In the same way, the perpendicular bisector to NN' approaches the normal to the trajectory of N. The point O_1 will then approach the point O, so that the normals to the trajectories of $P\ldots$, which are the limiting positions of the perpendicular bisectors of $PP', \ldots$, will also pass through O.[4]

Corollary. *If we know the tangents to the trajectories of two points of a moving figure which does not change shape, we can find the tangent to the trajectory of any other point of the figure.*

Indeed, the perpendicular to the known tangents determine the instantaneous center of rotation and, therefore, allow us to draw the normal to the trajectory at any point.

Exercises

Exercise 92. If a figure moves, without changing shape, in such a way that two of its lines each pass through a fixed point, then there exist infinitely many lines in the figure which, during the motion, also pass through a fixed point. Every other line remains tangent to a fixed circle. The instantaneous center of rotation for any two positions of the figure remains on a certain fixed circle.

Exercise 93. Given three figures F_1, F_2, F_3, all congruent and with the same orientation, we determine the three centers of the rotations which would superimpose one of these figures on another. Show that the angles of the triangle formed by these centers of rotation are equal or supplementary to half the angles between pairs of these figures.

Exercise 94. Compose two rotations with the same magnitude but opposite sense, and with different centers of rotation.

Exercise 95. Two congruent figures with opposite orientation can be superimposed:

1°. In infinitely many ways, by three successive line reflections;

2°. In infinitely many ways, by a translation followed by a line reflection;

3°. In only one way, by a translation preceded or followed by a reflection in a line parallel to the translation.

Exercise 96. Construct an equilateral triangle with vertices on three given parallel lines, or on three given concentric circles.

Exercise 97. On a given circle, find an arc, equal to a given arc, such that the lines joining the endpoints to two given points are parallel.

[4]We assume here, as axioms, propositions such as these: *If a variable point M_1 has a limit position M, and a variable direction D_1 has a limit position D, the parallel through M_1 to D_1 has as limiting position the parallel through M to D. If two lines D_1, D_1' have limiting positions D, D', then the intersection point of D_1, D_1' has limiting position the intersection of D, D'.* In reality, these propositions are theorems which should be proved; we omit the proofs because they could not be performed with the desired simplicity, without assuming a knowledge of limits relying on notions of differential calculus.

Problems for Book II

Exercise 98. We join a point M on a circle with the points of contact A, B of the tangents passing through a point P in the plane. We then draw a parallel through P to the tangent at M. Show that lines MA, MB intercept on this parallel a segment whose length is independent of M, and which is divided by P into two equal parts.

Exercise 99. Let ABC be an equilateral triangle inscribed in a circle, and let M be a point on arc $\widehat{BC}$. Show that $MA = MB + MC$.

Exercise 100. Through the three vertices of an equilateral triangle, draw lines such that the bisectors of the angles they form with the altitudes from the same vertex are parallel. Show that these three lines intersect on the circumscribed circle.

Exercise 101. The midpoints of the sides of a triangle, the feet of the altitudes, and the midpoints of the segments, on each altitude, between the vertex and their intersection, are nine points on the same circle (the *nine point circle*). The center of this circle is the midpoint of the segment which joins the intersection of the altitudes to the center of the circumscribed circle; its radius is half that of the circumscribed circle. Deduce from this the result of Problem 70.

Exercise 101b. In any triangle, the circle passing through the centers of the three escribed circles has its center at the intersection of the radii of these three circles to the points of contact with the corresponding sides; also, its center is symmetric to the center of the inscribed circle with respect to the center of the circumscribed circle. Its radius is twice that of the circumscribed circle.

Exercise 102. In any triangle, the six feet of the perpendiculars dropped from the foot of each altitude onto the other sides lie on the same circle.

Exercise 103. In triangle ABC, the perpendicular bisector of BC and the angle bisector of $\widehat{A}$ intersect on the circumscribed circle. Their point of intersection is equidistant from B, C, from the center of the inscribed circle, and from the center of the escribed circle contained in the angle $\widehat{A}$.

Exercise 103b. Construct a triangle knowing the lengths of the altitude, the median, and the bisector from the same vertex.

Exercise 104. Two tangents to a circle and the chord joining their points of contact divide into equal parts the perpendicular dropped from an arbitrary point on this chord to the line which joins it with the center.

Exercise 105. On the three sides of the triangle ABC, and outside the triangle, we construct equilateral triangles BCA', CAB', ABC', and join AA', BB', and CC'. Show that:

1°. These three line segments are equal;

2°. They pass through the same point, at which the three sides of the original triangle subtend equal angles;

3°. If this point is inside the triangle, the sum of its distances to the three vertices is equal to the common length of AA', BB', CC' (Problem 99).

Exercise 106. Lines ABE, CDE,BCF, ADF intersect in pairs, and form four triangles. Show that the circles circumscribing these triangles all pass through the same point I.

The centers of these four circles lie on another circle G, which also passes through point I (use Problem 69).[5]

A necessary and sufficient condition for quadrilateral $ABCD$ to be cyclic is that line EF, which connects the intersections of opposite sides of the quadrilateral, contain point I. This line is then the bisector of the angles $\widehat{AIC}$, $\widehat{BID}$ subtended by the diagonals of the quadrilateral at point I, or of their supplements (the angle formed by IA and the extension of IC, or by IB and the extension of ID).[6]

The circumcenter O of quadrilateral $ABCD$ lies on circle G. This circumcenter also lies on the circles AIC and BID. Conclude from this that I is the foot of the perpendicular from O to EF.

Exercise 107. The bisectors of the angles formed by the (extensions of) the opposite sides of a cyclic quadrilateral are perpendicular. They are parallel to the bisectors of the angles formed by the diagonals of the quadrilateral, and conversely.

Exercise 107b. If any two lines intersect two given circles, the chords joining any pairs of points of intersection of the lines with the first circle intersect the chords joining any pairs of points of the intersection of these lines with the second circle in four concyclic points.

Exercise 108. Given triangle ABC, find the locus of points M such that the three perpendiculars erected from A, B, C to AM, BM, CM respectively are concurrent.

Exercise 109. Find the locus of the vertices of the rectangles considered in Exercise 41, if the given parallelogram is *articulated* (**46b**) while one of its sides is held fixed.

Exercise 110. On two consecutive segments AB, BC of the same line, we draw two variable circles with the same radius. These circles intersect at B, and at one additional point, whose locus is required.

Exercise 111. Let OA, OB be two variable perpendicular radii of a circle O. Through A, B construct two lines, each parallel to one of two fixed perpendicular directions. Find the locus of the intersection of these lines, as the right angle $\widehat{AOB}$ turns around the center.

Exercise 112. Two circles, each of which touches a fixed line at a fixed point, vary while remaining tangent to each other. Find the locus of their point of contact.

Exercise 113. Find the locus of the vertex of the right angle of a right triangle which does not change shape, as the two other vertices slide along two perpendicular lines.

Exercise 114. Draw a line along which two given circles intercept given lengths.

[5]The reference to Problem 69 is probably an error in the original text. We give a proof using Problem 72. –transl.

[6]The second case is the one that always holds. [This footnote, in the original, is the author's correction of an error in an earlier edition. –transl.]

Exercise 115. In a given circle, inscribe a triangle whose sides are parallel to three given lines; or two of whose sides are parallel to given lines, and whose third side passes through a given point.

Exercise 116. Given a line xy and two points A, B, find a point M on the line such that the angle $\widehat{AMx}$ is twice $\widehat{BMy}$.

Exercise 117. Construct a pentagon (or, more generally, a polygon with an odd number of sides) knowing the midpoints of the sides. What happens when the polygon has an even number of sides?

Exercise 118. Construct a trapezoid knowing its sides. More generally, contruct a quadrilateral knowing its four sides and the angle between (the extensions of) two opposite sides.[7]

Exercise 119. Cut the sides AB, AC of a triangle with a transversal having a given direction, and intercepting equal segments on these sides, starting at B and C.

Exercise 120. Construct a segment perpendicular to two given parallel lines, which subtends a given angle at a given point.

Exercise 121. We are given a circle, points P, Q on the circle, and a line. Find a point M on the circle such that the lines MP, MQ intercept on the line a segment IK of given length.

Exercise 121b. Same problem as above, except that we are given not the length, but the midpoint, of IK.

Exercise 122. Construct a square whose sides pass through four given points. Can the problem have infinitely many solutions? In that case, what will be the locus of the centers of the squares described in the problem?

Exercise 123. In an articulated quadrilateral $ABCD$ (**46b**, Remark III), what conditions must the lengths of the sides satisfy if a vertex, say B, is to *pivot through a complete revolution*; in other words, such that the size of the angle at B can be arbitrary? Show that the vertex D is not, in general, the center of a complete revolution. What is the exceptional case?

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[7]With relation to this problem and the following ones, see **284**, Note A.

Book III
On Similarity

CHAPTER I

Proportional Lines

105. A proportion[1] is the equality of two ratios, and two variable quantities are *proportional* when their values correspond to each other in such a way that the ratio of any two values of the first is equal to ratio of the corresponding values of the second. Thus, two segments AB, CD will be said to be proportional to $A'B'$, $C'D'$ if $\frac{AB}{CD} = \frac{A'B'}{C'D'}$.

106. In the preceding equality, $\frac{AB}{CD}$ denotes the ratio of the two *segments* AB and CD, but we will apply, now and in all that follows, the convention formulated in **18**: choosing arbitrarily, but once and for all, a unit of length, we will replace the segments themseves by the *numbers* which are their measures. Thus, $\frac{AB}{CD}$ will always mean $\frac{\text{measure of } AB}{\text{measure of } CD}$. We can always operate this way, since the value of the ratio is not changed by this substitution, because of the theorem already referred to (**17**).

From this point on we will be able to apply to ratios of segments the usual properties of proportions proven in arithmetic,[2] such as:

The value of a ratio does not change if both the numerator and the denominator are multiplied or divided by the same quantity;

To multiply two ratios, we multiply the numerators and the denominators;

In a sequence of equal ratios, the sum of the numerators is to the sum of the denominators as any numerator is to its denominator, etc.

And the properties of proportions of lines are the same as those of proportions of numbers, such as the following:

In any proportion:

The product of the extremes is equal to the product of the means;

We can change the order of the extremes or of the means;

The sum (or the difference) of the terms of the first ratio is to one of them as the sum (or the difference) of the terms of the second ratio is to one of them.

Conversely, each equation obtained this way implies the original one, etc.

Thus, the equation $\frac{a}{b} = \frac{c}{d}$ is equivalent to the following:

$$ad = bc, \quad \frac{a}{c} = \frac{b}{d}, \quad \frac{a+b}{b} = \frac{c+d}{d}, \quad \frac{a-b}{b} = \frac{c-d}{d}, \ldots$$

that is, each of these equations implies the others.

In particular, if we know that three collinear points A, B, C are in the same order as three other collinear points A', B', C', the proportions

$$\frac{AB}{BC} = \frac{A'B'}{B'C'}, \quad \frac{AB}{AC} = \frac{A'B'}{A'C'}, \quad \frac{AC}{BC} = \frac{A'C'}{B'C'}$$

[1]See *Leçons d'Arithmétique* of J. Tannery, chap. VI and XIII.

[2]*Leçons d'Arithmétique*, chap.VI

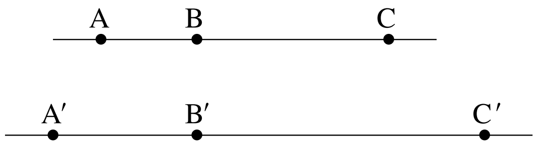


FIGURE 105

are equivalent to each other.

We also recall that *knowing three terms of a proportion, one can deduce the unknown term*: if we assume that a, b, and c are given, the proportion $\frac{a}{b} = \frac{c}{d}$ gives us:

$$d = \frac{b \times c}{a}.$$

The term d is called *the fourth proportional* of a, b, and c.

We see that *if two proportions have three corresponding terms in common, then the fourth terms must also be equal.*

107. We say that a number b is the *geometric mean* of two others a and c, if we can write a proportion where a, c are the extremes, and the middle terms are both b:

$$\frac{a}{b} = \frac{b}{c}.$$

The preceding equality is equivalent to

$$b^2 = ac,$$

so that *to find the geometric mean of two numbers a, c, we must extract the square root of the product of these numbers.* The number c is called the *third proportional* of a and b.

Of course, according to the preceding conventions, we will say that a segment b is the *geometric mean* of two others a, c, if the measure of b is the geometric mean of the measures of a and c; in other words, if we can write the proportion $\frac{a}{b} = \frac{b}{c}$ (using the measures of the segments, or using the segments themselves).

108. *There exists one point, and only one, which divides a given segment in a given ratio.*

Indeed, let the given segment be AB (Fig. 106), and let r be the given ratio. We are looking for a point M such that $\frac{AM}{BM} = r$. This can also be written as

$$\frac{AM}{BM} = \frac{r}{1}$$

and is therefore equivalent to

$$\frac{AM}{AB} = \frac{r}{1+r},$$

a proportion which is true for one, and only one, value of AM:

$$AM = AB \cdot \frac{r}{1+r}.$$

The segment thus determined is less than AB. Laying off a segment with this length starting from A, we obtain the required point M. QED

109. Consider a point M moving along the segment AB, from A towards B. The ratio $\frac{AM}{BM}$ will keep on increasing, because the numerator increases, while the denominator decreases. Also, it can take any given value, by the preceding proposition, and in particular values as small as one likes (when M is sufficiently close to A) and values as large as one likes (when M is sufficiently close to B). In short, this ratio, starting from 0, increases without limit as M travels from A to B.

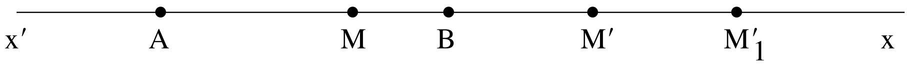


FIGURE 106

110. Let us now study the ratio $\frac{AM'}{BM'}$ when the point M' (Fig. 106) is on the extension of AB. The ratio $\frac{AM'}{BM'}$ is then called the ratio into which M' divides the segment AB *externally*.

There exists one point, and only one, which divides a given line segment in a given ratio externally, provided that this ratio is different from 1.

Let us look for a point M' on the extensions of AB such that the ratio $\frac{AM'}{BM'}$ is equal to a given number r. Suppose, to be definite, that $r > 1$, in which case we will look for points M' on the extension of AB past B. The proportion $\frac{AM'}{BM'} = \frac{r}{1}$ is equivalent to

$$\frac{AM'}{AB} = \frac{r}{r-1},$$

which is true for one, and only one, value of AM':

$$AM' = AB \cdot \frac{r}{r-1}.$$

This segment is greater than AB (because $\frac{r}{r-1}$ is greater than 1).Constructing a segment with this length, starting from A in the direction of AB, we will find the required point M'.

If the ratio r were less than 1, similar reasoning would yield

$$AM' = AB \cdot \frac{r}{1-r},$$

a length which would be measured from A in the sense opposite to that of AB in order to obtain M'. The proposition is thus proved.

Consider a point M' which starts at B and moves in the direction of Bx (Fig. 106). The ratio $\frac{AM'}{BM'}$ is greater than 1. As the point passes from a position M' to a more remote position M'_1, this ratio approaches 1, since we pass from the ratio $\frac{AM'}{BM'}$ to $\frac{AM'_1}{BM'_1}$ by adding the same quantity $M'M'_1$ to both terms.[3] The ratio can take values as large as one likes (when M' is sufficiently close to B) and values as close as one likes to 1 (when M' is sufficiently remote). In short, *the ratio* $\frac{AM'}{BM'}$ *starts from infinity, keeps on decreasing, and approaches 1 as the point M' starts from B and moves in the direction of Bx.*

In the same way we see that *if the point M' starts from A and travels in the direction of Ax'* (Fig. 106), *the ratio* $\frac{AM'}{BM'}$ *starts from* 0, *keeps increasing, and approaches* 1.

111. DEFINITION. Two points C, D (Fig. 107) which divide the same segment AB in the same ratio, one internally, the other externally, are said to be *harmonic*

[3]See Tannery, *Leçons d'Arithmétique*, chap. VI, n°212.

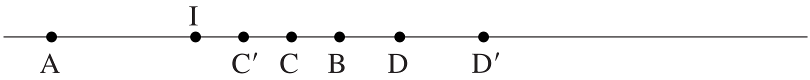


FIGURE 107

conjugates with respect to this segment (we also say that they divide this segment *harmonically*).

If segment CD divides AB harmonically then, conversely, AB divides CD harmonically, because the proportion $\frac{CA}{CB} = \frac{DA}{DB}$ gives $\frac{CA}{DA} = \frac{CB}{DB}$ after interchanging the means.

Every point on a segment AB or on one of its extensions has a harmonic conjugate with respect to this segment, except the midpoint I of AB (the ratio $\frac{IA}{IB}$ being equal to 1).[4]

112. On the other hand, of *two segments CD, $C'D'$ which divide a third AB harmonically, either they are both entirely exterior to one another, or one of them is entirely interior to the other.*

Indeed, of the ratios $\frac{CA}{CB}$ and $\frac{C'A}{C'B}$, if one is greater, and the other less, than 1, the segments are exterior to each other: they are on different sides of the midpoint I of AB. If that is not the case (Fig. 107), and the two ratios just mentioned are both greater than 1, the two segments CD, $C'D'$ will be on the same side of I, and will both contain the point B. But the discussion of **109-110** shows that the ratio $\frac{AM}{MB}$ approaches 1 as M moves away from point B (without passing I). Therefore, the segment which corresponds to the ratio closer to 1 will contain the other in its interior.

113. FUNDAMENTAL THEOREM. *Two arbitrary transversals are cut into proportional segments by parallel lines.*

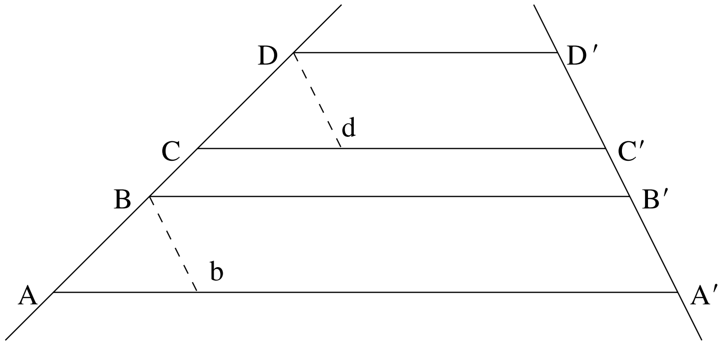


FIGURE 108

Consider two transverals $ABCD$, $A'B'C'D'$, cut by parallel lines AA', BB', CC', DD'. The points $A'B'C'D'$ will be in the same order as A, B, C, D (otherwise

[4]The conjugate of I can be considered to be a point at infinity. What this means is that when the point M is very remote from AB, the ratio $\frac{AM}{BM}$ gets close to 1, which is the value of $\frac{AI}{IB}$.

two of the lines AA', BB', CC', DD' would intersect). We claim that we have $\frac{CD}{AB} = \frac{C'D'}{A'B'}$. We distinguish two cases:

1°. *If the segments AB, CD are equal* (Fig. 108), *the same is true of $A'B'$, $C'D'$.* Indeed, we draw a parallel to $A'B'$ through point B, and extend it to intersect AA' at b, and we draw a parallel through D, and extend it to intersect CC' at d. If $AB = CD$ the two triangles ABb, CDd are congruent, having a pair of equal sides $AB = CD$ adjacent to pairs of equal angles (because they are corresponding angles). We therefore have $Bb = Dd$. But Bb and Dd are equal to $A'B'$ and $C'D'$, respectively, as can be seen from parallelograms $BbA'B'$, $DdC'D'$.

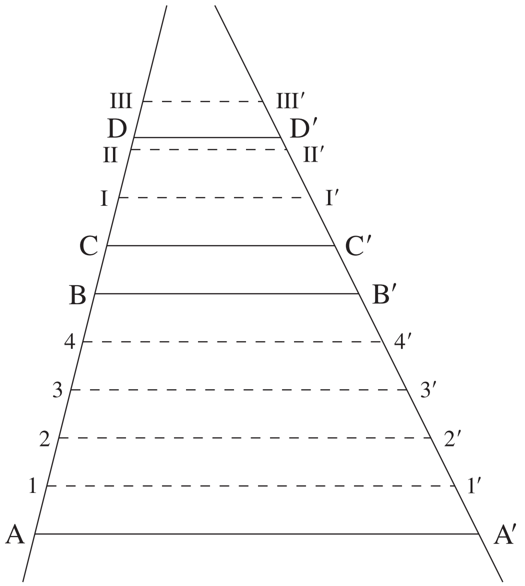


FIGURE 109

2°. *General case.*[5] Suppose the points A, B, C, D are arbitrary. We will show that for every n, the ratios $\frac{CD}{AB}$ and $\frac{C'D'}{A'B'}$ have the same approximation to within $\frac{1}{n}$. Assume for example that $n = 5$: divide AB into five equal parts at the points 1, 2, 3, 4 (Fig. 109) and suppose that one fifth part of AB is contained twice, but not three times, in CD. Let I, II, III be the endpoints of three segments, each equal to one fifth of AB and laid off successively along CD, starting from C. Thus the points I, II are between C and D, while III is beyond D. Through each of these points 1, 2, 3, 4, I, II, III we draw parallels to the common direction of AA', BB', CC', DD', extending them until they meet the line $A'B'C'D'$ at $1'$, $2'$,

[5]This proof is complete if we start with a theorem of arithmetic already referred to (see *Leçons d'Arithmétique* by J. Tannery, Chap. XIII, n°493): *Two quantities are in proportion if: 1°. A given value of the first always corresponds to the same value of the second; and 2° To the sum of two values of the first corresponds the sum of the two corresponding values of the second.* That the first condition is satisfied it is proved in the text (1°). The second condition is made clear by the remark that the corresponding points occur in the same order on the two lines $ABCD$, $A'B'C'D'$. Within the text, we have reproduced the proof of the theorem of arithmetic, adapting it to the particular case which concerns us.

$3'$, $4'$, I', II', III'. We have thus divided $A'B'$ into five equal parts, and laid off one of these lengths starting at point C' in the direction of $C'D'$ (1°). Points I', II' are between C' and D', while III' is beyond D', so the theorem is proved.

114. COROLLARY. *A parallel DE to side AB of triangle ABC divides the two other sides AB, AC proportionately* (Fig. 110).

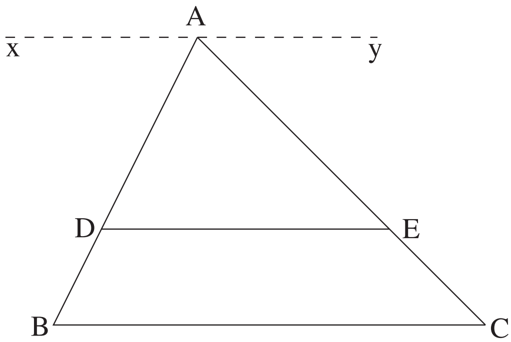


FIGURE 110

Indeed, if, through A, we draw a parallel xy to BC, the preceding theorem shows that transversals AB, AC are divided by parallels BC, DE, xy into proportional parts.

REMARK. The division referred to can be internal or external, but it must be of the same nature on both sides.

Converse. *If a line divides two sides of a triangle into proportional parts, it must be parallel to the third side.*

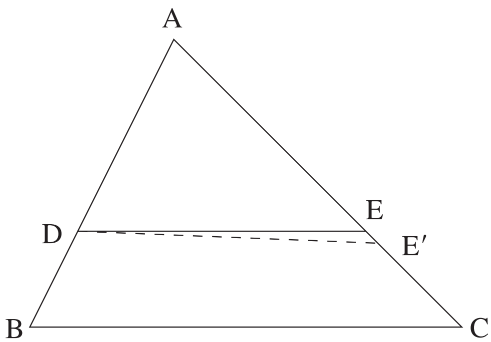


FIGURE 111

Assume that the line DE divides AB, AC proportionately. We draw a parallel to BC through the point D, and let E' be the point where this parallel meets AC. The ratio $\frac{AE'}{E'C}$ is equal to $\frac{AD}{DB}$ (by the previous theorem), and therefore equal to $\frac{AE}{EC}$ by hypothesis. The points E and E' must therefore coincide (**108**) and DE is parallel to BC.

115. THEOREM. *In any triangle:*

1°. *The bisector of any angle divides the opposite side into parts proportional to the adjacent sides;*

2°. *The bisector of an exterior angle also determines on the opposite side segments proportional to the adjacent sides.*

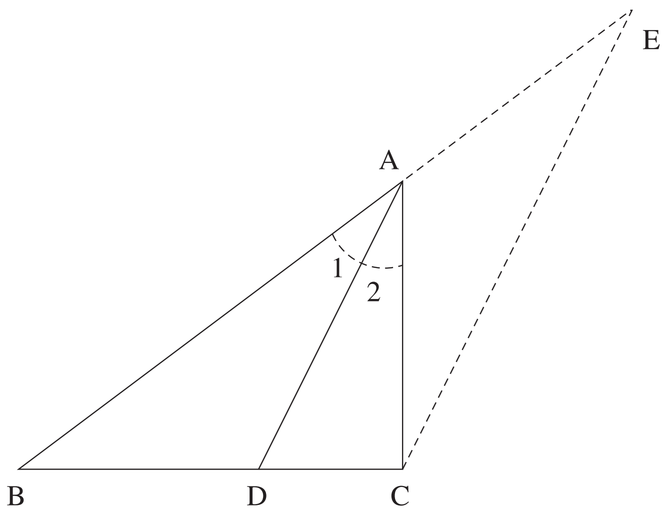


Figure 112

1°. Consider bisector AD of angle $\widehat{A}$ in triangle ABC (Fig. 112). We want to prove that $\frac{BD}{DC} = \frac{AB}{AC}$.

To this end, we draw CE parallel to AD and extend it to its intersection E with AB. In triangle BAD cut by parallel CE to AD, we have:

$$\frac{BD}{DC} = \frac{BA}{AE}.$$

But triangle ACE has equal angles at C and E; they are equal respectively to the two halves A_1, A_2 of $\widehat{A}$ ($\widehat{E} = \widehat{A_1}$ because they are corresponding angles, $\widehat{C} = \widehat{A_2}$ because they are alternate interior angles). This triangle is thus isosceles, so in the preceding proportion we can replace AE by AC, which implies the desired result.

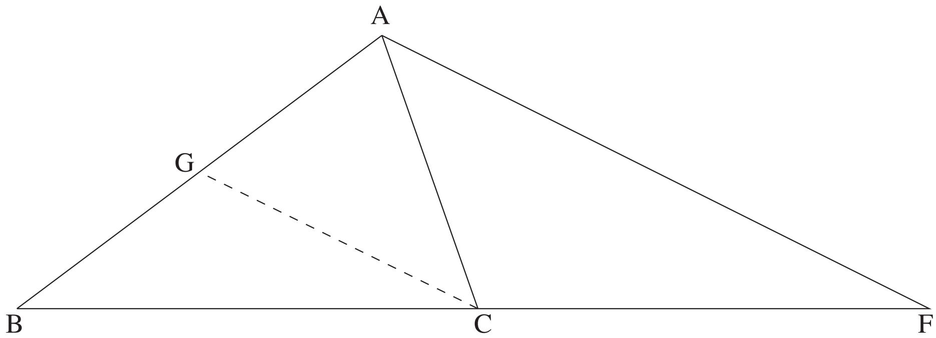


Figure 113

2°. The bisector AF of the exterior angle at A (Fig. 113) in triangle ABC meets BC in F (unless the triangle is isosceles). We want to show that $\frac{BF}{CF} = \frac{AB}{AC}$. The proof is very similar to the preceding one. We draw a parallel CG to AF and we note: 1° that the parallels CG, AF yield $\frac{BF}{CF} = \frac{BA}{AG}$; 2° that triangle ACG is isosceles (having angles at C and G equal respectively to the two halves of the exterior angle at A, and therefore equal to each other), which allows us to replace of AG with AC.

Remark. We see that *the bisector of an angle of a triangle, and the bisector of the adjacent exterior angle, divide the opposite side harmonically.*

Converse. 1°. *If a line from a vertex of a triangle divides the opposite side internally into parts proportional to the adjacent sides, this line must be the bisector of the angle at that vertex.* 2°. *If a line from a vertex of a triangle divides the opposite side externally into parts proportional to the adjacent sides, this line must be the bisector of the exterior angle at that vertex.*

This is true because (**108**) there is only one point on BC (Fig. 112) which divides BC internally into parts proportional to the two adjacent sides, and that is the endpoint of the bisector of $\widehat{A}$. In the same way, there is (**110**) only one point which divides side BC externally into parts proportional to the adjacent sides, and this point is the endpoint of the bisector of the exterior angle at A.

116. Theorem. *The locus of points whose distances to two fixed points are in a fixed ratio* (*different from* 1) *is a circle.*

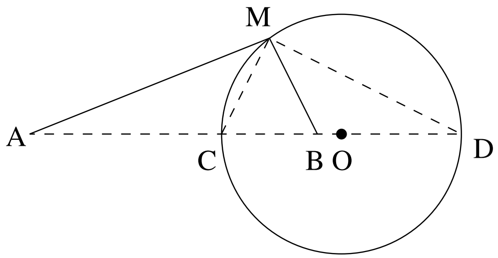


Figure 114

Let A, B (Fig. 114) be the given points: let us find the locus of points M such that $\frac{MA}{MB}$ is equal to a given number m. There exist two such points, one C between A and B, the other D on the extension of this segment; these are the points whose existence we noted in **108–110**. Let M be an arbitrary point on the locus. Line MC, which divides side AB into parts CA, CB proportional to MA, MB, is the bisector of the angle at M in triangle AMB. Similarly, line MD divides side AB of this triangle externally into segments proportional to the adjacent sides, and must therefore be the bisector of the exterior angle at M. These two lines are thus perpendicular, so that M must belong (Book I, **17**) to circle **C** with diameter CD (Book II, **78**).

Conversely, suppose that M is a point of this circle **C**. We draw line MA', symmetric to MB with respect to MC, with point A' on line AB. Lines MC, MD are then the bisectors of angle $\widehat{A'MB}$ and of its supplement, so that point A' is the harmonic conjugate of B with respect to segment CD (**115**, Remark). It must therefore coincide with A. Since lines MC, MD are the bisectors of angle $\widehat{AMB}$ and of its supplement, we have $\frac{MA}{MB} = \frac{CA}{CB} = m$. QED

COROLLARY. *If a diameter of a circle is divided harmonically by two points A and B, the ratio of the distances of a point of the circle to the points A, B is constant.*

Exercises

Exercise 124. We are given two fixed lines, and two points A, B on these lines. The points M, N vary on the two lines in such a way that the segments AM, BN are proportional. We draw parallels through M and N to two given directions. Find the locus of the intersection of these lines.

Exercise 125. Two chords of a circle with a common endpoint divide harmonically the diameter perpendicular to the chord joining their other endpoints.

Exercise 126. In what region of the plane are the points with the property that the ratio of their distances to two given points A, B is greater than a given number (see **112, 71**)?

Exercise 127. Find a point whose distances to the vertices of a triangle are proportional to three given numbers.

The problem, when possible, generally has two solutions. Show that the two points satifying the conditions of the problem belong to the same diameter of the circle circumscribing the given triangle, and they divide this diameter harmonically.

Exercise 128. Through a common point A of two circles, draw a variable secant which cuts these circles again in M, M'. Find the locus of the point which divides MM' in a given ratio (see Exercise 65).

CHAPTER II

Similarity of Triangles

117. Definition. Two triangles are *similar* if they have equal angles and proportional corresponding sides.

Remarks. Two congruent triangles are similar. Two triangles similar to a third are also similar to each other. In particular, if two triangles are similar, any triangle congruent to the first is similar to the second.

Theorem. *Every parallel to one side of a triangle forms a triangle with the other two sides which is similar to the first.*

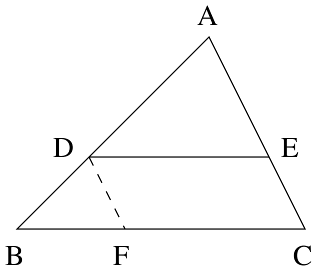


Figure 115

Consider parallel DE to side BC of triangle ABC (Fig. 115). We claim that the new triangle ADE is similar to ABC.

First, the angles of the two triangles are equal, $\widehat{A}$ being common to both, and angles $\widehat{D}$, $\widehat{E}$ being corresponding angles with $\widehat{B}$, $\widehat{C}$, respectively.

Second, we have (**114**) $\frac{AD}{AB} = \frac{AE}{AC}$, and it remains only to prove that the common value of these ratios is also equal to $\frac{DE}{BC}$.

To show this, we draw a parallel DF to AC, forming parallelogram $DEFC$. Line DF cuts sides BA, BC into proportional parts, so that $\frac{DE}{BC} = \frac{FC}{BC} = \frac{AD}{AB}$.
QED

Remark. Line DE can be inside (Fig. 115) or outside (Figs. 116, 117) triangle ABC without affecting the validity of the proof. However, in the case represented in figure 117, the corresponding angles of the two triangles are equal because they are alternate interior angles, rather than corresponding angles.

118. The following theorems (*cases of similarity for triangles*) give necessary and sufficient conditions for two triangles to be similar. They correspond to the three cases of congruence, to which they are reduced by use of the preceding theorem.

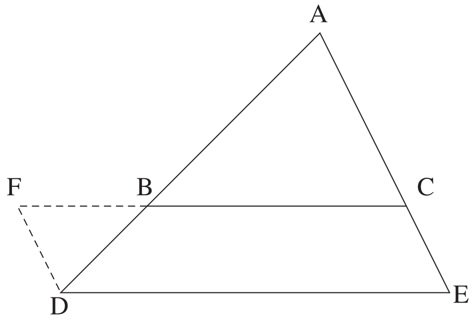


Figure 116

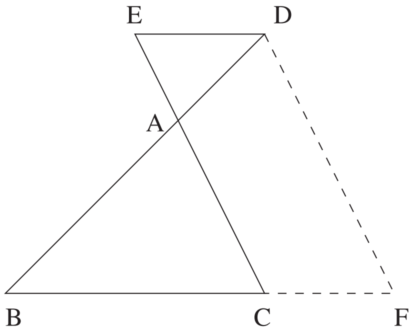


Figure 117

First case of similarity. *Two triangles are similar if they have two equal corresponding angles.*

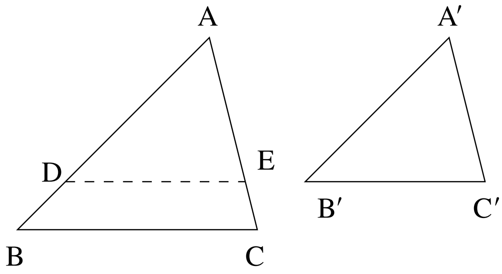


Figure 118

Consider triangles ABC, $A'B'C'$ in which we have $\widehat{A} = \widehat{A'}$, $\widehat{B} = \widehat{B'}$ (Fig. 118). On AB we construct a length $AD = A'B'$ and through point D we draw a parallel DE to BC. Triangle ADE is similar to ABC (by the preceding theorem) and congruent to $A'B'C'$, having an equal side between two equal angles.

Second case. *Two triangles are similar if they have an equal angle contained between proportional sides.*

Consider triangles ABC, $A'B'C'$ in which we have $\widehat{A} = \widehat{A'}$ and $\frac{AB}{AC} = \frac{A'B'}{A'C'}$. On side AB we take a length $AD = A'B'$ and we draw parallel DE to BC. Triangle ADE is similar to ABC and congruent to $A'B'C'$. Indeed, they have an equal angle ($\widehat{A} = \widehat{A'}$) contained between equal sides: $AD = A'B'$ by construction, and the two proportions $\frac{AB}{AC} = \frac{A'B'}{A'C'}$ (given) and $\frac{AB}{AC} = \frac{AD}{AE}$, which have three common terms, give us $A'C' = AE$.

Third case. *Two triangles are similar if they have three proportional sides.*

Suppose the two triangles are ABC, $A'B'C'$ in which $\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$. On side AB we take a length $AD = A'B'$ and draw parallel DE to BC. Triangle ADE is similar to ABC and congruent to $A'B'C'$, having three pairs of corresponding sides equal. Indeed, $AD = A'B'$ by construction, and the proportions $\frac{AB}{AD} = \frac{AC}{AE}$ and $\frac{AB}{AD} = \frac{BC}{DE}$ (by the previous result) have three terms in common with the proportions $\frac{AB}{A'B'} = \frac{AC}{A'C'}$ and $\frac{AB}{A'B'} = \frac{BC}{B'C'}$ (given by hypothesis), respectively. We conclude that in fact $AE = A'C'$, $DE = B'C'$.

REMARK. The definition of similar triangles consists of five conditions: three given by the equality of the angles, two by the proportionality of the sides. Corollary III of **44** (Book I) shows that one of the first three conditions can be omitted, which reduces the total to four. The cases of similarity show that, to establish similarity, we only need to prove *two* of these conditions, if chosen properly.

119. THEOREM. *Two triangles whose corresponding sides are parallel or perpendicular are similar.*

Indeed, in this situation, the corresponding angles are either equal in pairs or supplementary (Book I, **43**), so that if the triangles are ABC and $A'B'C'$, we have

$$\widehat{A} = \widehat{A'} \quad \text{or} \quad \widehat{A} + \widehat{A'} = 2 \text{ right angles,}$$
$$\widehat{B} = \widehat{B'} \quad \text{or} \quad \widehat{B} + \widehat{B'} = 2 \text{ right angles,}$$
$$\widehat{C} = \widehat{C'} \quad \text{or} \quad \widehat{C} + \widehat{C'} = 2 \text{ right angles.}$$

The three equations in the second column cannot all be satisfied simultaneously, because they would imply that the total sum of the angles in the two triangles is equal to six right angles, rather than four. Nor can even two of these equalities hold simultaneously; for instance, $\widehat{A} + \widehat{A'} = 2$ right angles and $\widehat{B} + \widehat{B'} = 2$ right angles would give a total sum for the angles of the two triangles equal to 4 right angles $+\widehat{C} + \widehat{C'}$. Therefore at least two equalities in the first column must hold, and then the triangles have angles which are equal in pairs (first case of similarity).

120. THEOREM. *Two right triangles are similar if the ratio of one of the sides of the right angle to the hypotenuse is the same in the two triangles.*

Indeed, we can proceed as in **118** by forming a third triangle similar to the first, and having the same hypotenuse as the second. The second and third triangles will then be congruent, by the second case of congruence for right triangles.

REMARK. Aside from this case of similarity, we can apply to right triangles the cases for similarity for arbitrary triangles. For instance, two right triangles will be similar if they have an equal acute angle (first case of similarity for arbitrary triangles); or if the sides of the right angle are proportional (second case of similarity).

Because of this, all right triangles ABC which have the same acute angle $\widehat{C}$ have proportional sides. If we are given the angle $\widehat{C}$, we can then determine:

1°. The ratio of the opposite side AB to the hypotenuse BC (this ratio is called the *sine* of $\widehat{C}$ and is denoted $\sin C$);

2°. The ratio of AC to the hypotenuse (called the *cosine* of $\widehat{C}$ and denoted $\cos C$;

3°. The ratio of AB to AC (called the *tangent* of $\widehat{C}$ and denoted $\tan C$).

These facts form the basis of trigonometry.

121. THEOREM. *A family of concurrent lines cuts off proportional segments on two parallel lines.*

Consider, for instance, the three concurrent lines SAA', SBB', SCC' (Fig. 119) which meet two parallels ABC, $A'B'C'$. Similar triangles (**117**) SAB, $SA'B'$

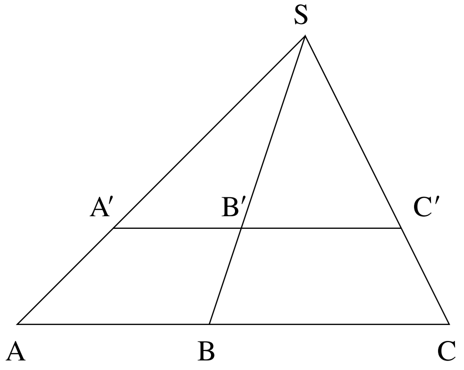


FIGURE 119

yield

$$\frac{A'B'}{AB} = \frac{SB'}{SB},$$

and similar triangles SBC, $SB'C'$ give

$$\frac{B'C'}{BC} = \frac{SB'}{SB},$$

from which it follows that $A'B'$, $B'C'$ are proportional to AB, BC.

REMARK. The corresponding segments all have the same sense, or each pair has the opposite sense, according as point S is exterior or interior to the two parallels.

Converse. *If three lines intercept proportional segments on two parallel lines (all in the same sense, or each pair in the opposite sense), these three lines are either concurrent or parallel.*

If the corresponding segments are equal and in the same sense, the three lines are parallel (**46**, converse). Assume then that lines AA', BB', CC' intercept proportional segments on parallel lines ABC, $A'B'C'$, so that

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC};$$

the value of this ratio being different from 1 if the segments are in the same sense. This last condition makes it impossible (**46**) for lines AA', BB' to be parallel; they therefore intersect at some point S.

Joining SC, we see that this line cuts the parallel $A'B'C'$ in a point C_1' which must coincide with the point C' because $\frac{A'C_1'}{B'C_1'} = \frac{AC}{BC}$ (by the preceding theorem), which equals $\frac{A'C'}{B'C'}$(by hypothesis).

Exercises

Exercise 129. Through the intersection of the diagonals of a trapezoid we draw a parallel to the bases. Show that this line is divided into equal segments by the non-parallel sides.

Exercise 130. Let $a = AB$ and $b = CD$ be the two bases of a trapezoid. We divide one of the non-parallel sides into the ratio $\frac{EA}{EC} = \frac{m}{n}$, and through point E we draw a parallel to the bases. Show that the segment of this parallel contained inside the trapezoid is equal to $\frac{m \cdot CD + n \cdot AB}{m+n}$. Special case: E is the midpoint of AC.

Exercise 131. From the vertices of a triangle, and from the intersection of its medians, we drop perpendicular segments onto a line exterior to the triangle. Show that the last of these segments is equal to the arithmetic mean of the first three. (Preceding exercise.)

Exercise 132. Through vertex A of parallelogram $ABCD$ we draw a line which intersects diagonal BD in E, and sides BC, CD in F, G. Show that AE is the geometric mean of EF and EG.

Exercise 133. Find the locus of the points which divide the segment intercepted by the sides of an angle, on the lines parallel to a given direction, in a given ratio.

Exercise 134. Find the locus of points at which two given circles subtend the same angle.[1]

[1]The angle subtended by a curve at a given point is the angle formed by the tangents to the curve through that point.

CHAPTER III

Metric Relations in a Triangle

122. Definition. The foot of the perpendicular dropped from a point onto a line is called the *orthogonal projection* (or, briefly, *projection*) of the point onto that line. The segment whose endpoints are the projections of the endpoints of a given segment is called the *projection of the segment* onto the line.

123. Let ABC (Fig. 120) be a triangle with a right angle at A. From this vertex A, drop altitude AD onto the hypotenuse. We will discover certain relations among the elements of this figure.

Theorem. *In a right triangle, each side of the right angle is the geometric mean between the whole hypotenuse and its projection onto the hypotenuse.*

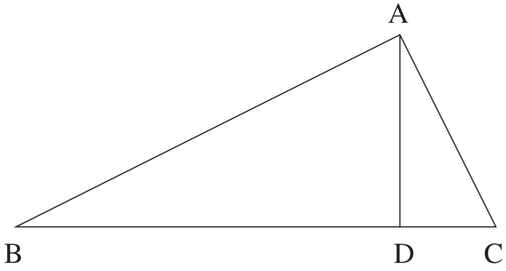


Figure 120

For instance, AB is the geometric mean between BD and BC. Indeed, right triangles ABD, ABC, having angle $\widehat{B}$ in common, are similar (**120**, Remark). The sides corresponding to AB, BD in the first are sides BC, AB in the second, so that

$$\frac{BD}{AB} = \frac{AB}{BC}, \quad \text{or} \quad AB^2 = BD \cdot BC.$$

QED

Corollary. *Any chord in a circle is the geometric mean between a diameter and its projection on the diameter passing through one of its endpoints* (Fig. 121).

This is true because the diameter and the chord are the hypotenuse and a side of a right triangle (**73**, Corollary II).

Remark. Similar triangles ABD, ABC (Figures 120, 121) also give

$$\frac{AB}{BC} = \frac{AD}{AC} \quad \text{or} \quad AB \cdot AC = BC \cdot AD.$$

Thus, *the product of the sides of the right angle in a right triangle, is equal to the product of the hypotenuse and the corresponding altitude.*

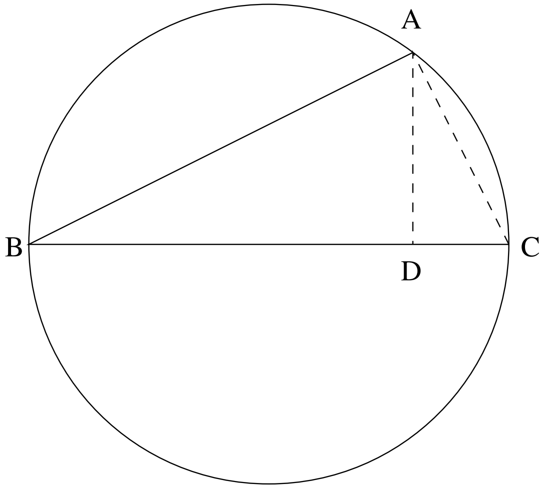


FIGURE 121

124. THEOREM OF PYTHAGORAS. *In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.*

We have just seen that

$$AB^2 = BC \cdot BD.$$

The same theorem applied to the side AC gives

$$AC^2 = BC \cdot CD.$$

Adding these equalities we obtain

$$AB^2 + AC^2 = BC(BD + CD) = BC^2.$$

QED

This theorem can be used to calculate any side of a right triangle when we know the other two sides. Thus, let us calculate the hypotenuse of a right triangle in which the legs are 3 and 4 meters. If the unit is the meter, the square of the number representing the measure of the hypotenuse will be the sum of the squares of the numbers 3 and 4. This sum is $3^2 + 4^2 = 25$, whose square root is 5; thus the hypotenuse will equal 5 meters.

Let us now calculate one leg of a right triangle, knowing that the hypotenuse is equal to 10 meters and the other leg is 7 meters. The square of the required side, added to $49 (= 7^2)$ must give $100\ (= 10^2)$; this square is therefore equal to $100 - 49 = 51$. The square root of 51, (which is 7.14, correct to the nearest centimeter) will give the measure, in meters, of the desired side.

125. THEOREM. *In a right triangle, the altitude from the right angle is the geometric mean between the two segments which it determines on the hypotenuse.*

Indeed, triangles ABD, ACD (Fig. 120) are similar because they have perpendicular sides. Segments BD, AD are therefore proportional to AD, DC.

COROLLARY. *The perpendicular from any point on a circle to a diameter is the geometric mean between the two segments it determines on the diameter.*

126. THEOREM. *The difference of the squares of two sides of a triangle is equal to the difference of the squares of their projections on the third side.*

In triangle ABC (Fig. 123), we project vertex B to point H on AB. The equations

$$AB^2 = AH^2 + BH^2,$$
$$BC^2 = CH^2 + AH^2$$

yield, by subtraction, $BC^2 - AB^2 = CH^2 - AH^2$.

THEOREM. *In any triangle*:

1°. *The square of the side opposite an acute angle is equal to the sum of the squares of the other two sides, minus twice the product of one of these sides and the projection of the other onto it*;

2°. *The square of the side opposite an obtuse angle is equal to the sum of the squares of the other two sides, plus twice the product of one of these sides and the projection of the other onto it.*

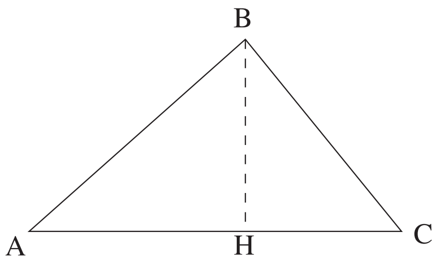


FIGURE 122

1°. In triangle ABC, consider side BC opposite acute angle $\widehat{A}$. Drop altitude BH from B to AC. By the preceding theorem we have

$$BC^2 = AB^2 + CH^2 - AH^2.$$

Now CH is the difference between AC and AH, so that CH^2 can be replaced[1] by

$$AC^2 - 2AC \cdot AH + AH^2,$$

yielding the desired result

$$BC^2 = AB^2 + AC^2 - 2AC \cdot AH.$$

2°. In triangle ABC, consider side BC opposite obtuse angle $\widehat{A}$. Drop altitude BH from B to AC. Again we have

$$BC^2 = AB^2 + CH^2 - AH^2,$$

[1] We assume that the formulas for the squares of the sum and the difference of two numbers are known:

$$(a+b)^2 = a^2 + 2ab + b^2,$$
$$(a-b)^2 = a^2 - 2ab + b^2.$$

See Tannery, *Leçons d'Arithmétique*, chap. II, n^{os} 83, 85.

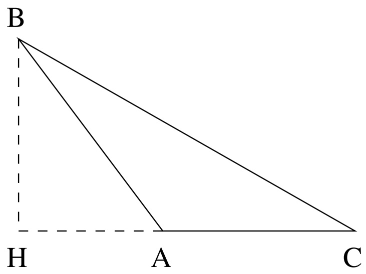


FIGURE 123

but CH is the sum of AC and AH, so that CH^2 can be replaced by

$$AC^2 + 2AC \cdot AH + AH^2,$$

yielding

$$BC^2 = AB^2 + AC^2 + 2AC \cdot AH.$$

COROLLARY. *An angle of a triangle is acute, right, or obtuse, according as the square of the opposite side is less than, equal to, or greater than the sum of the squares of the two other sides.*

127. STEWART'S THEOREM. *Given triangle ABC and point D on the base between B and C, we have*

$$AB^2 \cdot DC + AC^2 \cdot BD - AD^2 \cdot BC = BC \cdot DC \cdot BD.$$

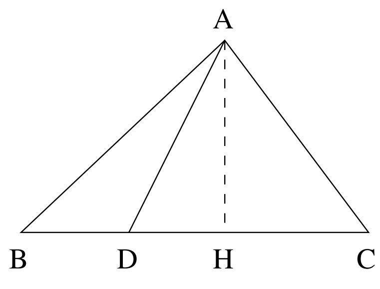


FIGURE 124

We drop a perpendicular AH from A to BC (Fig. 124) and suppose, to be definite, that H and C are on the same side of D. The two theorems of the preceding section apply, one to triangle ACD and the other to triangle ABD, to yield

$$AC^2 = AD^2 + DC^2 - 2CD \cdot DH,$$
$$AB^2 = AD^2 + BD^2 + 2BD \cdot DH.$$

We multiply the first equation by BD, and the second by CD, then add the resulting equations. The term $2BD \cdot CD \cdot DH$, which appears once with a + sign and once with a $-$ sign, disappears and we have

$$\begin{aligned} AC^2 \cdot BD + AB^2 \cdot CD &= AD^2(BD + CD) + DC^2 \cdot BD + BD^2 \cdot CD \\ &= AD^2 \cdot BC + BC \cdot DC \cdot BD. \end{aligned}$$

128. Application to the calculation of various notable lines in a triangle. Consider an arbitrary triangle ABC whose sides BC, CA, AB have known measures a, b, c, respectively. Let us calculate the lengths of the medians, bisectors, and altitudes of this triangle.

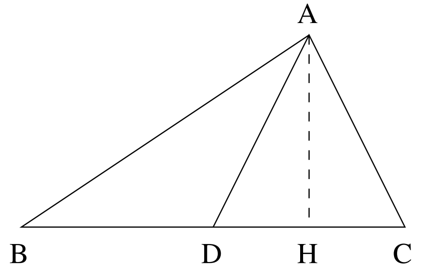


FIGURE 125

1°. **Medians.** Let AD be the median from vertex A (Fig. 125). In the equation given by the previous theorem, we replace BC, CA, AB by a, b, c, respectively; DC and BD by $a/2$. Dividing by a we obtain

$$\frac{b^2 + c^2}{2} = AD^2 + \left(\frac{a}{2}\right)^2,$$

and therefore

$$AD^2 = \frac{b^2 + c^2}{2} - \left(\frac{a}{2}\right)^2.$$

Thus, *the sum of the squares of two sides of a triangle equals twice the square of half the third side, plus twice the square of the corresponding median.*

128b. On the other hand, if, in the equalities

$$AC^2 = AD^2 + DC^2 - 2CD \cdot DH,$$
$$AB^2 = AD^2 + BD^2 + 2BD \cdot DH$$

we replace BC, CA, AB by a, b, c and DC and BD by $a/2$, then substracting them term by term, we get

$$c^2 - b^2 = 2a \cdot DH.$$

Thus *the difference between two sides of a triangle is equal to twice the product of the third side and the projection of the median onto this side.*

COROLLARY. *The locus of points C, such that the difference of the distances from C to two fixed points A and B is constant, is a line perpendicular to AB.*

The reason is that the difference $AB^2 - AC^2$ is constant, so the projection H of C onto AB is a fixed point.

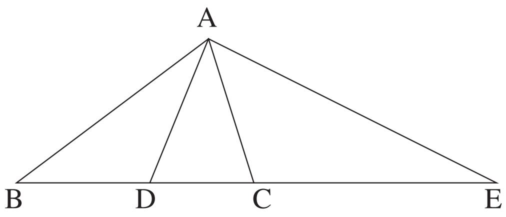


FIGURE 126

129. 2°. **Bisectors.** We now denote by AD the bisector of $\widehat{A}$ (Fig. 126). Point D divides side BC into segments proportional to AB and AC, so that

$$\frac{BD}{c} = \frac{CD}{b} = \frac{BC}{b+c} = \frac{a}{b+c},$$

or

$$BD = \frac{ac}{b+c}, \quad CD = \frac{ab}{c+b}.$$

In the result of the theorem in **127**, we replace BD and CD with these values, and BC, CA, AB with their values. Dividing by a, we obtain

$$\frac{c^2b}{b+c} + \frac{b^2c}{b+c} - AD^2 = \frac{ab}{b+c} \cdot \frac{ac}{b+c},$$

so that

$$AD^2 = \frac{c^2b + b^2c}{b+c} - \frac{a^2bc}{(b+c)^2} = bc\frac{(b+c)^2 - a^2}{(b+c)^2}.$$

Now we consider bisector AE of the exterior angle at A (Fig. 126) and, to be definite, we assume that AB is greater than AC, so that point E, which divides segment BC externally into parts proportional to AB and AC, is on the extension of BC past C. We then have

$$\frac{BE}{c} = \frac{CE}{b} = \frac{BC}{c-b} = \frac{a}{c-b},$$

and therefore

$$BE = \frac{ac}{c-b}, \quad CE = \frac{ab}{c-b}.$$

We apply the theorem of **127** to triangle ABE with point C on base BE. Replacing BC, CA, AB, BE, CE by their values, and dividing by a, we have

$$\frac{c^2b}{c-b} + EA^2 - \frac{b^2c}{c-b} = \frac{ac}{c-b} \cdot \frac{ab}{c-b},$$

so that

$$AE^2 = \frac{a^2bc}{(c-b)^2} - \frac{c^2b - b^2c}{c-b} = bc\frac{a^2 - (c-b)^2}{(c-b)^2}.$$

130. 3°. **Altitudes.** We drop altitude AH from A (Fig. 127). Of the two angles $\widehat{B}$, $\widehat{C}$, at least one is acute. We assume, to be definite, that $\widehat{B}$ is acute, and apply the theorem of **126** to the side $AC = b$ opposite this angle to obtain

$$b^2 = a^2 + c^2 - 2a \cdot BH,$$

or

$$BH = \frac{a^2 + c^2 - b^2}{2a}.$$

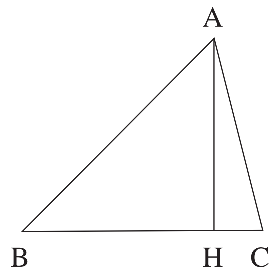


FIGURE 127

Now in right triangle AHB we have

$$AH^2 = c^2 - BH^2 = c^2 - \frac{(a^2+c^2-b^2)^2}{4a^2}.$$

This equation, whose right-hand member is a difference of two squares, can be written as[2]

$$\begin{aligned} AH^2 &= \left(c - \frac{a^2+c^2-b^2}{2a}\right)\left(c + \frac{a^2+c^2-b^2}{2a}\right) \\ &= \frac{(2ac-a^2-c^2+b^2)(2ac+a^2+c^2-b^2)}{4a^2} = \frac{[b^2-(a-c)^2][(a+c)^2-b^2]}{4a^2}. \end{aligned}$$

But each of the factors appearing in the numerator of the right-hand member of this equation is also a difference of two squares. Thus this last formula can be written as

$$AH^2 = \frac{(b-a+c)(b+a-c)(a+c-b)(a+b+c)}{4a^2}.$$

If we denote by p the semi-perimeter of the triangle, so that $a+b+c=2p$, the quantities $b+c-a$, $c+a-b$, $a+b-c$ are equal to $2p-2a$, $2p-2b$, $2p-2c$, respectively, so that

$$AH^2 = \frac{4p(p-a)(p-b)(p-c)}{a^2}.$$

On the other hand, if we want to use the product $[(a+c)^2-b^2][b^2-(a-c)^2]$, we can write the the following formula:

$$AH^2 = \frac{1}{4a^2}[4b^2c^2 - (b^2+c^2-a^2)^2] = \frac{1}{4a^2}(2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4).$$

130b. 4°. **Radius of the circumscribed circle.** *The product of two sides of a triangle is equal to the altitude drawn to the third side, multiplied by the diameter of the circumscribed circle.*

In triangle ABC (Fig. 128), let AH be the altitude from A, and let AA' be the diameter of the circumscribed circle. Triangles AHC, ABA' have right angles at H and B, respectively. They have an equal acute angle ($\widehat{A'} = \widehat{C}$ since they intercept the same arc $\widehat{AB}$). They are therefore similar, so that $\frac{AH}{AC} = \frac{AB}{AA'}$, or $AB \cdot AC = AH \cdot AA'$. QED

[2]See *Leçons d'Arithmétique*, by J. Tannery, chap. II, n°87.

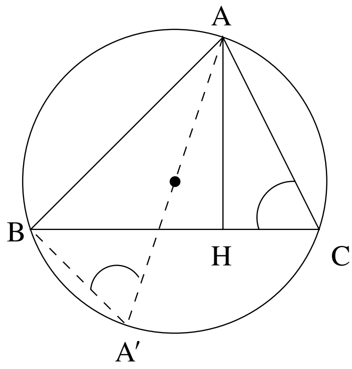


FIGURE 128

Using the value just found for AH, this theorem gives the following formula for the radius R of the circumscribed triangle:

$$R = \frac{AA'}{2} = \frac{bc}{2AH} = \frac{abc}{4\sqrt{p(p-a)(p-b)(p-c)}}.$$

Exercises

Exercise 135. The product of the segments intercepted by a variable tangent of a circle on two parallel tangents is constant.

Exercise 136. The inverse of the square of the altitude of a right triangle is equal to the sum of the inverses of the squares of the sides of the right angle.

Exercise 137. What is the ratio of the sum of the squares of the medians to the sum of the squares of the sides of a triangle?

Exercise 138. The sum of the squares of the distances from an arbitrary point in the plane to two opposite vertices of a parallelogram differs by a constant quantity from the sum of the squares of the distances from the same point to the other two vertices. Special case: the rectangle.

Exercise 139. The sum of the squares of the four sides of a quadrilateral is equal to the sum of the squares of its diagonals plus four times the square of the segment joining their midpoints.

Exercise 140. If ABC is a triangle, G the intersection of its medians, and M an arbitrary point in the plane, we have $MA^2 + MB^2 + MC^2 = GA^2 + GB^2 + GC^2 + 3MG^2$.

Exercise 141. Find the locus of points such that the squares of their distances to two fixed points, multiplied by two given numbers, have a given sum or difference. Use this to find a new proof of the theorem in **116**.

Exercise 142. Find the locus of points such that the squares of their distances to three points in the plane, multiplied by three given numbers, has a given constant sum. Same problem with an arbitrary number of points.

Exercise 143. The square of the bisector of an angle of a triangle is equal to the product of the sides which contain it, diminished by the product of the segments it intercepts on the opposite side. State and prove an analogous statement for the bisector of the exterior angle.

Exercise 144. Deduce the inequalities indicated in Exercises 11 and 18 from the formulas found for the medians and bisectors.

Exercise 145. If the median of a triangle is the geometric mean between the sides which contain it, then the square with side $b - c$ has a diagonal equal to the third side of the triangle.

Exercise 146. The altitudes of a triangle are inversely proportional to the sides to which they are drawn.

Exercise 147. The product of the distances from a point on a circle to two opposite sides of an inscribed quadrilateral is equal to the product of the distances of the same point to the other sides, or to the diagonals. What happens to this statement when the two opposite sides become tangents?

CHAPTER IV

Proportional Segments in a Circle. Radical Axis

131. Theorem. *If a variable secant to a circle is drawn from a point A in the plane of the circle, the product of the distances from A to the two intersection points with the circle is constant.*

We distinguish two cases.

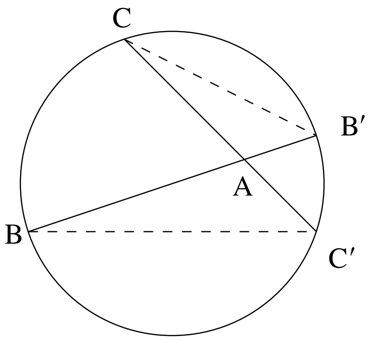


Figure 129

1°. *Point A is inside the circle* (Fig. 129). Consider two secants ABB', ACC', and join BC', CB'. Triangles ABC', ACB' are similar, since they have equal angles. Indeed, the angles at A are equal because they are vertical angles, and $\widehat{B} = \widehat{C}$ because both have measure equal to one half arc $\widehat{B'C'}$. The proportionality of the sides then gives $\frac{AB}{AC} = \frac{AC'}{AB'}$, and the equality of the products of the means and the extremes gives $AB \cdot AB' = AC \cdot AC'$. QED

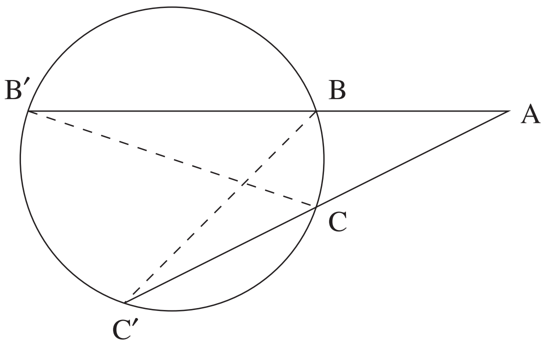


Figure 129b

2°. *Point A is outside the circle* (Fig. 129b). Consider two secants ABB', ACC', and join BC', CB'. Triangles ABC', ACB' are again similar, having equal angles: the angle at A in common, and $\widehat{B} = \widehat{C}$ intercept the same arc. The conclusion follows as before.

131b. Converse. *If, on two lines ABB', ACC' through a point A* (Fig. 130) *we take four points A, A', B, B' (the point A being either exterior to both segments BB', CC', or interior to both) such that $AB \cdot AB' = AC \cdot AC'$, then these four points are on the same circle.*

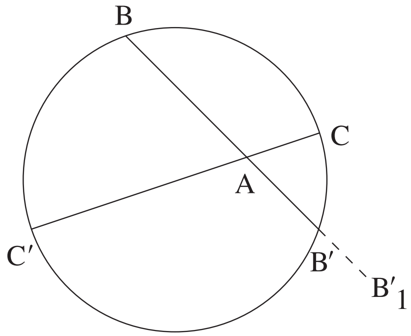


FIGURE 130

Indeed, the three points B, C, C' determine a circle, and this circle intersects ABB' in a point B_1' such that $AB \cdot AB_1' = AC \cdot AC'$. This equality, combined with the one in the hypothesis, shows that $AB_1' = AB'$, and therefore the point B_1' coincides with B'.

132. THEOREM. *If, through a point exterior to a circle, we draw a tangent and a secant to the circle, the tangent is geometric mean between the entire secant and its external segment.*

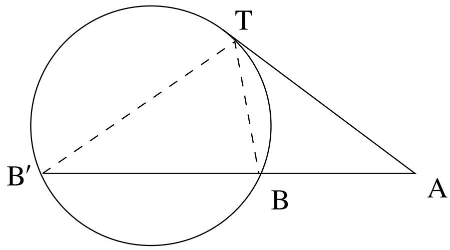


FIGURE 131

Indeed, consider secant ABB' and tangent AT (Fig. 130). We repeat the proof of **131** (2°) with no modification except replacing both letters C, C' by the letter T.

The preceding argument is another example of the fact (Book II, **67**) that the tangent should be thought of as having a double intersection with the circle at the point of contact.

Conversely, *If we take points B, B' on line ABB', both on the same side of A, and on another line AT we take point T such that AT is geometric mean between AB and AB', then the three points B, B', T are on a circle tangent to AT at T.*

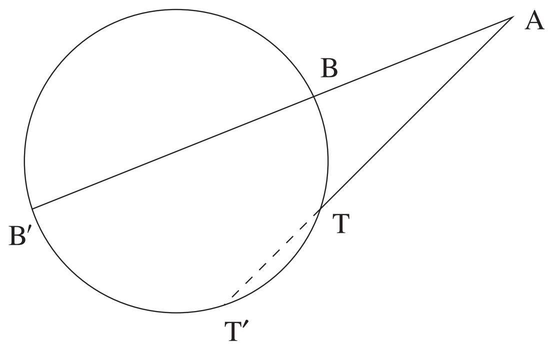


FIGURE 132

Indeed, circle $BB'T$ has the common point T with line AT (Fig. 132); if it had another common point T', then (**131**) $AB \cdot AB' = AT \cdot AT'$. This equation, compared with the hypothesis $AB \cdot AB' = AT^2$, shows that point T' must coincide with T.

133. DEFINITION. For any point A, the product of the segments from A to the points of intersection of a secant with a given circle passing through the point is called the *power* of point A with respect to the given circle. According to **131**, this product is independent of the secant. The product is preceded by a $+$ sign if the point A is outside the circle, and by a $-$ sign if A is inside. When the point is exterior to the circle, the power is equal to the square of the tangent from this point.

134. *The power of the point A with respect to a circle of center O is equal to the difference of the square of OA and the square of the radius.*

Indeed, let us take line OA as our secant (in place of ABB'). The segments AB, AB' represent the sum and the difference between OA and the radius of the circle; their product is therefore the difference of the squares of these quantities.

If one takes into account the sign of the power, this is always equal to $d^2 - R^2$ (where d is the distance OA and R is the radius).

135. *When two circles meet at a right angle, the square of the radius of each one is equal to the power of its center with respect to the other, and conversely.*

If circles O, O' (Fig. 133) intersect in A at a right angle, the tangent to circle O' at this point is precisely OA, and the power of point O with respect to circle O' is indeed OA^2.

Conversely, if the power of point O with respect to circle O' is OA^2, then OA is tangent to this circle, and therefore the two circles are orthogonal.

136. THEOREM. *The locus of points which have the same power with respect to two given circles is a line perpendicular to the line of the centers.*

This line is called the *radical axis* of the two circles.

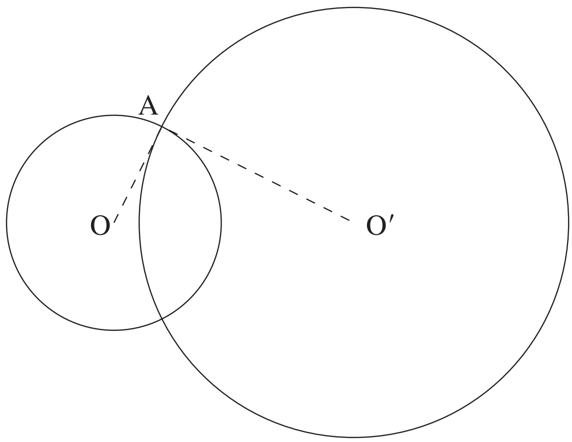


FIGURE 133

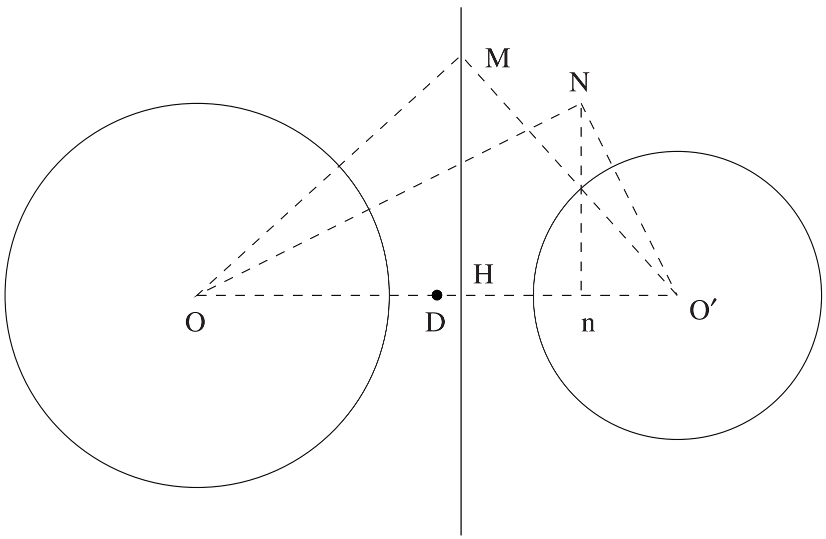


FIGURE 134

Consider two circles with centers O, O' (Fig. 134) and radii r, r'. If the point M has the same power with respect to the two circles, we have $OM^2 - r^2 = O'M^2 - r'^2$, which can be rewritten as $OM^2 - O'M^2 = r^2 - r'^2$ and gives us the desired conclusion (**128b**).

According to the results of **128b**, the distance between the intersection H of the radical axis with the line of the centers, and the midpoint D of this segment, is given by the formula $2DH \cdot OO' = r^2 - r'^2$.

REMARKS. I. The preceding proof remains valid if one of the circles has a radius equal to zero, and so reduces to its center. Thus, *the locus of points such that their power with respect to a given circle is equal to the square of their distance to a given* *point is a line perpendicular to the line joining the given point to the center of the circle.* We can call this line the *radical axis of the circle and the point.*

II. *Two concentric circles do not have a radical axis.*

III. *The difference between the powers of an arbitrary point with respect to two circles is equal to twice the product of the distance of this point to the radical axis and the distance between the centers* (Fig. 134).

Indeed, let N be the point in question, and suppose n is its projection onto OO'. The difference of the powers of this point with respect to the two circles will be

$$ON^2 - r^2 - (O'N^2 - r'^2) = ON^2 - O'N^2 - (r^2 - r'^2).$$

Since $ON^2 - O'N^2 = 2Dn \cdot OO'$ and $r^2 - r'^2 = 2DH \cdot OO'$, we obtain

$$ON^2 - O'N^2 - (r^2 - r'^2) = 2Hn \cdot OO'.$$

137. *If two circles intersect, their radical axis is their common chord.*

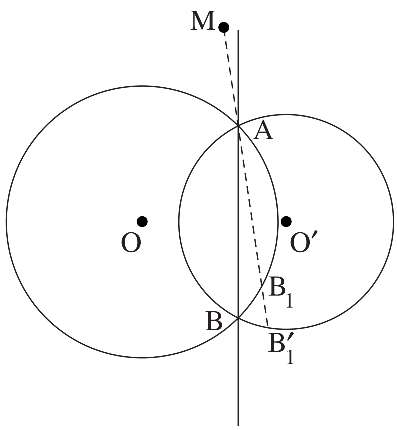


FIGURE 135

Indeed, it is clear that every point of the common chord AB (Fig. 135) belongs to the locus. It follows that, conversely, every point of the radical axis belongs to this chord. This can be seen directly by drawing MA: if this line intersects the two circles in two distinct new points B_1, B_1', the two powers $MA \cdot MB_1$ and $MA \cdot MB_1'$ will be different.

In the same way, *if two circles are tangent, their radical axis is their common tangent.*

138. *The radical axis of two circles* (or at least the part of the axis which is exterior to the two circles) *is the locus of the centers of circles cutting the first two at right angles.*

This is true because the center of such a circle has the same power with respect to the given circles; namely, the square of the radius.

The radical axis divides a common tangent into two equal parts.

139. THEOREM. *Given three circles, the three radical axes determined by pairs of them are either parallel or concurrent.*

This is true because the point of intersection of two of the radical axes will have same power with respect to the three axes, and will therefore belong to the third radical axis as well. This point is called the *radical center* of the three circles; if it is exterior, it is the center of a circle cutting all three of them at right angles.

REMARK. If two of the radical axes coincide, the preceding argument shows that the third must coincide with the first two. The three circles then have *the same radical axis*. Every circle orthogonal to the first two will be orthogonal to the third as well.

Exercises

Exercise 148. We are given a circle and two points A, B in its plane. Through A we draw a variable secant AMN which cuts the circle in M, N. Show that the circle passing through M, N, B also passes through another fixed point.

Exercise 149. The locus of points such that the ratio of their powers with respect to two given circles is equal to a given number is a circle which has the same radical axis with the first two.

Exercise 150. The preceding exercise is equivalent to exercise 128 if the two given circles intersect in two points.

Exercise 151. Consider a triangle ABC, points D, D' on BC, E, E' on CA, and F, F' on AB. If it is known that there is a circle passing through D, D', E, E', a circle passing through E, E', F, F', and a circle passing through F, F', D, D', it follows that all six points D, D', E, E', F, F' are on the same circle.

Exercise 152. When do all the circles orthogonal to two given circles O, O' intersect line OO'? Show that if they all do, then the intersection points are always the same (the *limit points* of Poncelet); namely, the points that have the same radical axis with the given circles.

Exercise 153. Four points A, B, C, D are given on the same line. A variable circle passes through A, B, and another variable circle passes through C, D. Show that the common chords of these circles pass through a fixed point.

Exercise 154. If the radical center of three circles is interior to these circles, then it is the center of a circle divided into two equal parts by each of the first three circles.

CHAPTER V

Homothecy and Similarity

140. Definition. If we choose a point S as the *center of homothecy*, and a number k called the *ratio of homothecy* or *ratio of similarity*, the *homothetic image* of an arbitrary point M is the point M' obtained by joining SM and taking a segment SM' on this line or on its extension, starting from S (Fig. 136), such that $\frac{SM'}{SM} = k$.

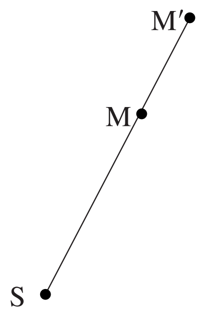


Figure 136

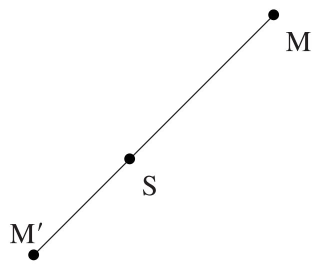


Figure 137

The homothecy is said to be *direct* if SM' is taken in the sense of SM (*fig.* 136.) and *inverse* if these segments are in opposite senses (Fig. 137). The *homothetic image* of a figure F is the figure formed by the set of all points M', homothetic to the points which constitute the figure F.

Remarks. I. *The center of homothecy is its own homothetic image; it is the only point with this property* except, of course, in the case of the direct homothecy with a ratio of similarity equal to 1, in which case every point coincides with its homothetic image.

II. Symmetry with respect to a point (**99**) is a special case of inverse homothecy.

141. Theorem. *In two homothetic figures, the line segment which joins two arbitrary points of one figure, and the segment which joins their homothetic images in the other, are always parallel, and their ratio is the ratio of similarity; they are in the same or opposite sense according as the homothecy is direct or inverse.*

Indeed, assume that A, B are two points of the first figure, A', B' their homothetic images (Fig. 138), S the center, and k the ratio of homothecy. The proportion $\frac{SA'}{SA} = \frac{SB'}{SB} = k$ shows (**114**, Converse) that the lines AB, $A'B'$ are parallel, and the similarity of the triangles SAB, $SA'B'$ shows that the ratio of these segmants is the same as the ratio of SA, SA'.

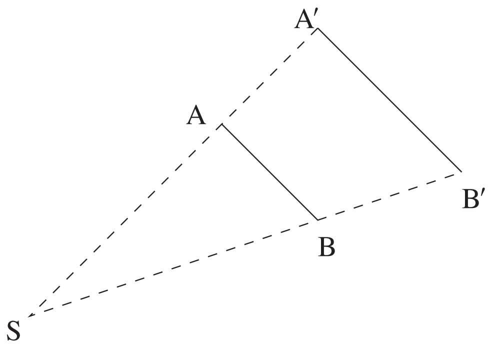


FIGURE 138

COROLLARY. I. The homothetic image of a line is a line.

This is true because, as point B moves along a line AB, and this line remains fixed, the point B' will move along the parallel through A' to the given line.

II. The homothetic image of a circle is a circle, and the centers are corresponding points.

This is true because, if the point B moves such that its distance to a fixed point A remains constant, the point B' will move along a circle with center A' and radius $A'B' = k \cdot AB$.

III. The homothetic image of a triangle is a triangle similar to the original one.

142. THEOREM. *Conversely, if there exist two points O, O' in the plane of two figures with the property that the segment which joins O to an arbitrary point M in the first figure, and the one which joins O' to its corresponding point M', are always parallel and have a constant ratio k (always in the same sense or in always in the opposite sense), the two figures are homothetic.*

For this theorem to be entirely correct, *one must consider two figures* in which OM, OM' *are equal and in the same sense* (that is, figures *which are congruent, and obtained from each other by translation)* as *directly homothetic*.

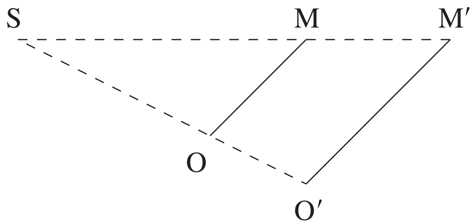


FIGURE 139

PROOF. Consider two arbitrary corresponding points, and draw MM' (Fig. 139). If this line is parallel to OO', then quadrilateral $OMM'O'$ is a parallelogram and $OM = O'M'$: the two figures are obtained from each other by a translation. Otherwise, line MM' will intersect OO' in a point S, which will not depend on the choice of points M, M'. In fact, this point S divides segment OO' in the given ratio k (externally, if OM, $O'M'$ have the same sense, or internally, if they have the opposite sense); this follows from the similarity of triangles SOM, $SO'M'$. This similarity also yields $\frac{SM'}{SM} = \frac{O'M'}{OM} = k$. QED

143. Corollary. *Two circles can always be considered homothetic in two different ways.*

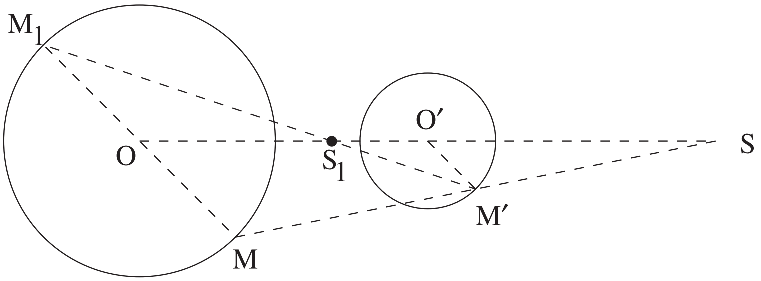


Figure 140

Indeed, if O, O' are the centers of the two circles, the endpoints M, M' of two parallel radii in the same sense satisfy the conditions of the preceding theorem, and will move along two directly homothetic figures (Fig. 140). In the same way, the endpoints M, M' of two parallel radii with opposite senses will be corresponding points in an inverse homothecy. In both cases, the ratio of similarity is the ratio of the two radii.

Thus two circles have two centers of homothecy[1] (or *centers of similarity*) S, S_1, one direct and the other inverse. These points divide the segment between the circles' centers harmonically, the ratio of division being the ratio of the radii.

The points of contact of a common external tangent are homothetic because the radii with these endpoints are parallel and in the same sense; the points of contact of an interior common tangent are likewise inversely homothetic.

Thus, *the exterior common tangents* (*when they exist*) *meet at the center of direct similarity; the interior common tangents* (*if they exist*) *meet at the center of inverse similarity.*

If the two circles are tangent, their point of contact is a center of similarity.

Remark. Two circles cannot be homothetic in more than two different ways.

This is true because, in case of homothecy, the centers correspond to each other (**141**, Corollary II) and corresponding rays are parallel. We again find one of the two homothecies described above, according as they are in the same or the opposite sense.

144. Theorem. *Two figures homothetic to a third are themselves homothetic, and the three centers of homothecy are on the same line.*

Consider figures F_2, F_3 (Fig. 141) homothetic to the same figure F_1. Let points O_2, O_3 be the points in F_2, F_3 corresponding to a fixed point O_1 in F_1, and let M_2,

[1]However, two congruent circles do not have a center of direct homothecy; they can only be considered as directly homothetic if we extend the meaning of the word as in the previous paragraph.

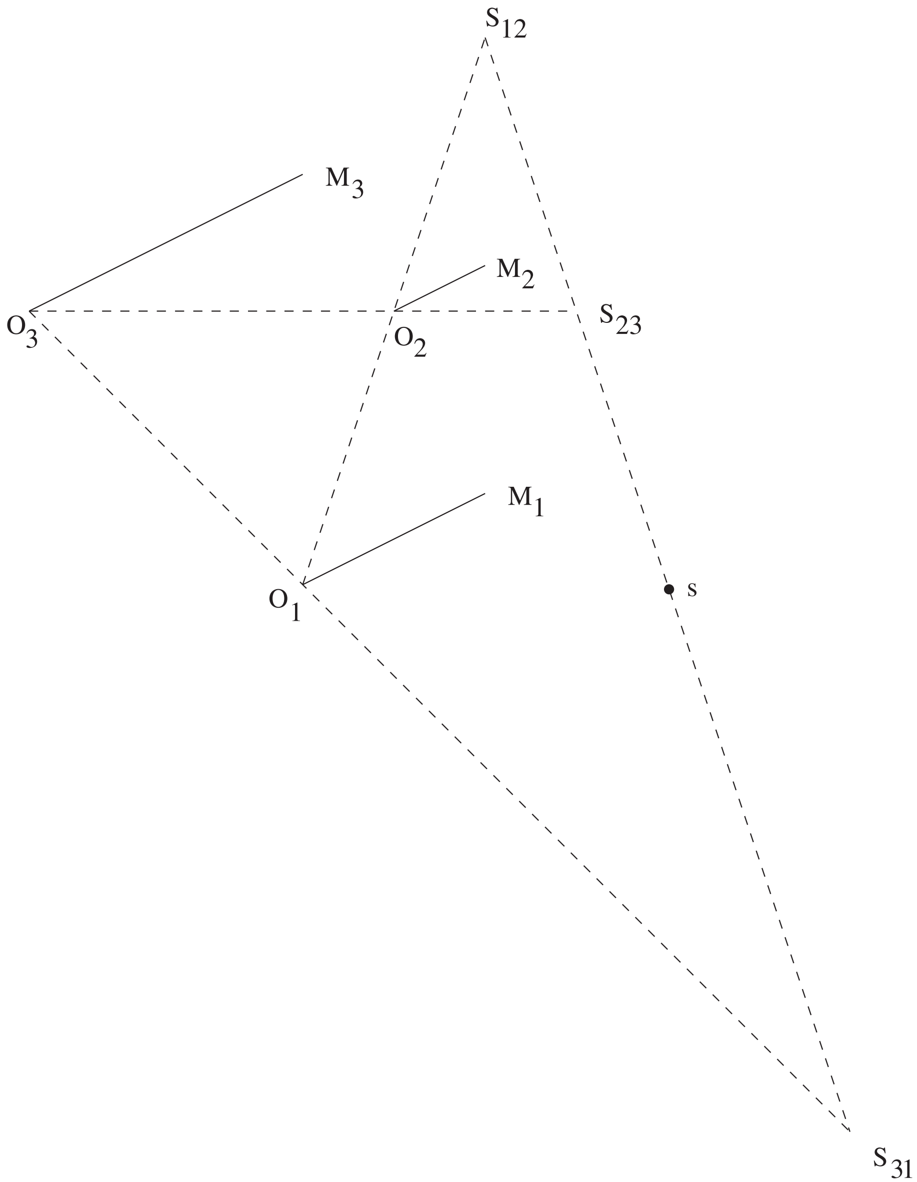


Figure 141

M_3 correspond to an arbitrary point M_1 in F_1. Then segments O_2M_2, O_3M_3 are parallel, since both are parallel to O_1M_1; they are always in the same sense (if they are both in the same sense as O_1M_1, or both opposite to O_1M_1; that is, if both given homothecies are direct, or both inverse) or always in opposite sense (if one of the given homothecies is direct and the other inverse); finally, their ratio is constant, since the equalities $\frac{O_2M_2}{O_1M_1} = k$ and $\frac{O_3M_3}{O_1M_1} = k'$ imply by division $\frac{O_3M_3}{O_2M_2} = \frac{k'}{k}$. Thus figures F_2, F_3 are homothetic, and their homothecy is direct if the first two are both direct or both inverse, and inverse if only one of the first two is direct.

Now consider the centers of homothecy S_{23} of F_2, F_3, S_{31} of F_3, F_1, and S_{12} of F_1, F_2. We claim that these three points lie on a straight line. Indeed, point S_{23}, considered as a point in figure F_2, corresponds to itself in F_3; it has a certain corresponding point[2] s in F_1. Line sS_{23} will pass through S_{31} (as it joins two corresponding points in F_1, F_3) and through S_{12} (as it joins correpsonding points in F_1, F_2) QED

Remark. We see that if three figures are homothetic in pairs, either one or three of the homothecies are direct.

145. Three given circles can be considered to be pairwise homothetic in four different ways, because we can (**143**) choose the sense of the first two homothecies arbitrarily (which gives four possible homothecies). The preceding theorem shows that *the three centers of direct similarity are collinear, and each center of direct similarity is collinear with the centers of inverse similarity which are not its conjugates.*[3] The four lines defined by this are called *axes of similarity*; one of them is *direct* and three *inverse*; they intersect in pairs at the six centers of similarity.

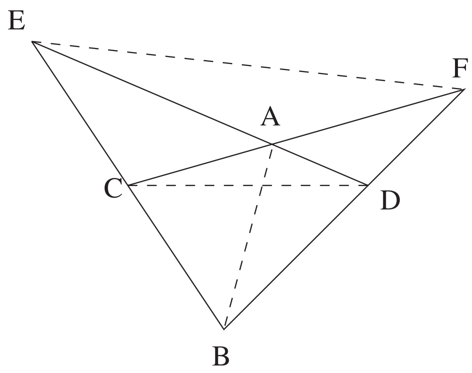


Figure 142

Definition. The figure obtained by extending the opposite sides of a quadrilateral to their intersections is called a *complete quadrilateral* (Fig. 142). A complete quadrilateral has six *vertices* A, B, C, D, E, F (Fig. 142) forming three pairs of opposite vertices, so that a complete quadrilateral has three diagonals AB, CD, EF.

146. Definition. Two figures are *similar* if they can be placed in such a way as to be homothetic. In other words, two figures are similar if one of them is congruent to a homothetic image of the other.

Theorem. *Two similar polygons have equal angles and proportional corresponding sides.*

[2]The point s may coincide with S_{23} but, in this case, it would correspond to itself in all three figures, and therefore would be the common center of homothecy.

[3]In this context, the author uses the term ‘conjugate [centers]’ to mean the centers of similarity for the same pair of circles. –transl.

Place the two polygons in their homothetic position; corresponding angles will be equal, having parallel sides in the same, or opposite, sense (according to the type of homothecy) and the ratio of corresponding sides is equal to the ratio of similarity.

Corollary. *The ratio of the perimeters of two similar polygons is equal to their ratio of similarity.*

This follows from the proportionality of the sides, because in a series of equal ratios, the sum of the numerators and the sum of the denominators are in the same ratio as any numerator and denominator.

147. Theorem. *Conversely, if two polygons have equal angles in the same order, all in the same sense or all in the opposite sense, and they have proportional corresponding sides, then they are similar.*

Consider two polygons P $(ABCDE)$ and P' $(A'B'C'D'E')$ satisfying the conditions of the theorem, so that $\widehat{A} = \widehat{A'}$, $\widehat{B} = \widehat{B'}$, $\widehat{C} = \widehat{C'}$, $\widehat{D} = \widehat{D'}$, $\widehat{E} = \widehat{E'}$,

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{E'A'}{EA}.$$

Take the homothetic image P_1 of P, with respect to an arbitrary point, and with a ratio of similarity equal to $\frac{A'B'}{AB} = \frac{B'C'}{BC}$. The polygon $P_1(A_1B_1C_1D_1E_1)$ will have angles equal to those of P' (all in the same sense, or all in the opposite sense) and sides equal to those of P'; we claim it is congruent to P'.

First, we may assume that the angles are in the same sense; otherwise we replace P' by its symmetric image with respect to a line.

Under these conditions, move the polygon P', without changing the sense of rotation, onto P_1 in such a way that $A'B'$ coincides with A_1B_1. The equality of the angles $\widehat{B}$, $\widehat{B'}$ in the same sense shows that $B'C'$ assumes the direction of B_1C_1, and since these two segments are equal, they must coincide. Continuing in this way, we conclude that $C'D'$ coincides with C_1D_1, and so on. Therefore the two polygons coincide. QED

Remark. Two quadrilaterals can have equal angles without being similar. Example: a square and a rectangle. It does not suffice for the sides to be proportional for two quadrilaterals to be similar, since equality of the sides (**46b**, Remark III) does not ensure congruence of the quadrilaterals. (See Note A later in this volume (**281**), and Volume II, Note E.)

148. *Every polygon can be decomposed into triangles, and this can be done in infinitely many ways.*

Indeed:

1°. If the polygon is convex, join any vertex to all the others (Fig. 143) or a point inside the polygon to all the vertices (Fig. 143b);

2°. If the polygon is not convex (Fig. 144), it can be decomposed into convex polygons (which can then be decomposed into triangles, as we have already seen).

To this end, we extend each side indefinitely. Then the plane is decomposed into a number of regions (for example, in Figure 145, the regions numbered 1 to 16) such that we cannot cross any of the lines without changing region, and conversely.[4]

[4]The author means here that if we do cross one of the lines, then we are changing regions.–transl.

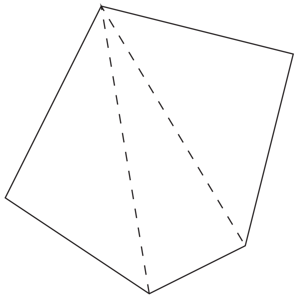


FIGURE 143

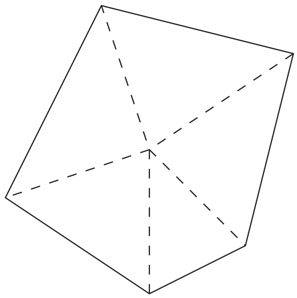


FIGURE 143b

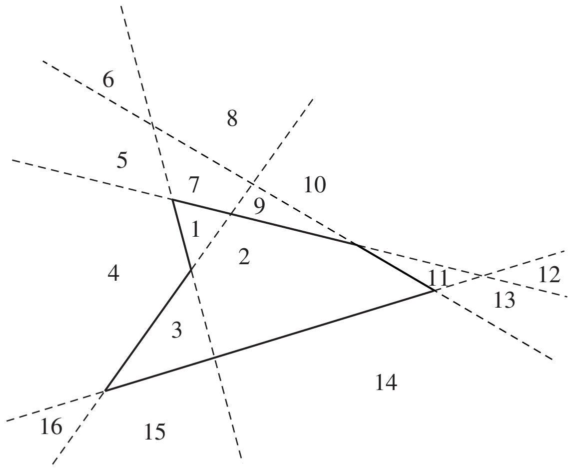


FIGURE 144

Each one of these regions is either entirely inside, or entirely outside, the polygon (since we can pass from any of its points to any other without crossing a side or the extension of a side). The polygon will consist of the regions contained in its interior (in Figure 144, regions 1, 2, and 3) which are obviously convex.

149. THEOREM. *Two similar polygons can be decomposed into similar triangles, arranged in the same way.*

After having placed the polygons in homothetic positions, it suffices to divide one of them into triangles, and the other into triangles homothetic to the first.

Conversely, *two polygons which can be decomposed into similar triangles and arranged in the same way are similar.*

First, two polygons decomposed into congruent triangles arranged the same way are congruent. Indeed, consider the triangles ABC, BCD, CDE, etc. (Fig. 145) forming the first polygon P, and $A'B'C'$, $B'C'D'$, $C'D'E'$, etc., the congruent triangles forming the second polygon P'. If we were to construct, on $A'B'$ as base, a polygon congruent to P (**95**) using the triangles ABC, BCD, CDE, we would recover the polygon P'.

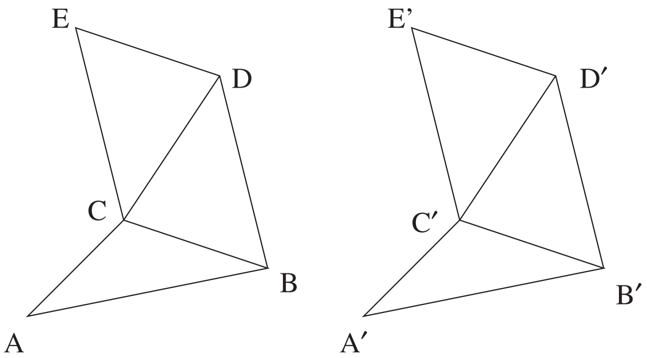


FIGURE 145

Now, if two polygons are formed by similar triangles arranged similarly, with a fixed ratio of similarity[5] k, then taking a polygon homothetic to the first with this same ratio of similarity k, we form a polygon congruent to the second by the preceding argument.

150. THEOREM. *On the segments joining a fixed point O with each point M of a figure F, we construct similar triangles OMM' with the same orientation. The points M' obtained in this way form a figure F' similar to F.*

This is true because if we turn the figure F around O through an angle equal to $\widehat{MOM'}$, we obtain a figure congruent to F and homothetic to F' with respect to O.

Converse. *Given two similar figures with the same orientation, there exists* (except if these figures are translates of each other) *a point such that the triangles with vertices at this point and two arbitrary corresponding points are all similar.*

Let F and F' be two figures which are similar in the same sense, so that F' is homothetic to a figure F_1, which is congruent to F and has the same orientation. First, it follows that two corresponding lines AB, $A'B'$ (Fig. 146) form a constant angle (which we can call the *angle of the two figures*) because this is true for the congruent figures F, F_1, and figure F' has sides parallel to their corresponding sides in F_1. Since these lines are also proportional, if we construct segment AB_0 through point A, so that it is equal, parallel and in the same sense as $A'B'$, then, for any points A, B, triangle ABB_0 will be similar to a fixed triangle T.

Having noted this, let us look for a point which corresponds to itself when viewed as a point in F and in F'. Let O be such a point: if we repeat the preceding construction, taking point O in place of A, then point B will coincide with B', so that triangle OBB' must be similar to T, and have the same orientation. There exists one and only one point which satisfies this condition.[6]

Conversely, having chosen point O in this way, if we view it as a point in F, the corresponding point in F' will be on the line corresponding to BO, which is the line $B'O$ (since it must form an angle with BO equal to the angle of the two figures). Therefore point O coincides with its corresponding point.

[5]This ratio must be the same for contiguous triangles because they have a common side; it is therefore the same for all the triangles.

[6]The construction of this point will be given later (**152**, Construction 3).

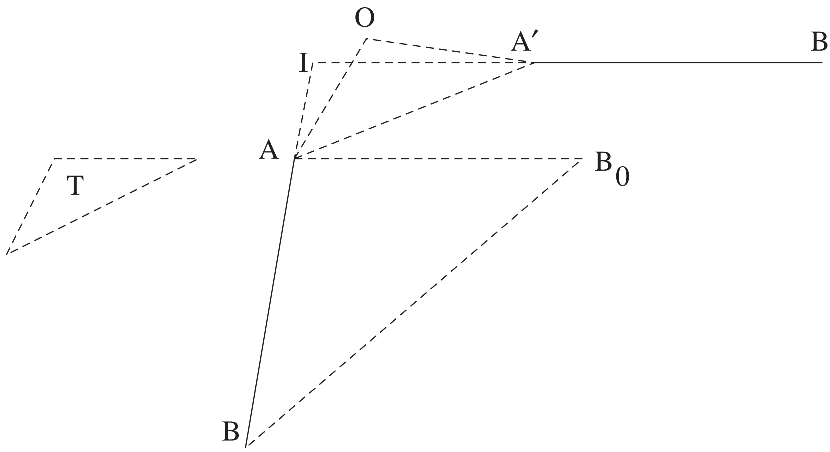


FIGURE 146

Now all the triangles OBB' are similar to triangle T; moreover, if we rotate figure F about O by an angle equal to the angle of the two figures, it will be homothetic to F', with O as the center of homothecy. QED

REMARK. The two angles $\widehat{AOA'}$, $\widehat{BOB'}$ must be equal to the angle made by lines AA', BB'. If we extend these lines to their intersection point I, we can determine point O as the intersection of the circles circumscribed about the triangles AIA', BIB'.

150b. Pantograph.

THEOREM. *Let $PRMQ$ be a parallelogram. Let O, N be two points taken on the adjacent sides PQ, PR, respectively, such that O, N and M are collinear* (Fig. 147).

If parallelogram $PRMQ$ is deformed in such a way that the lengths of its sides remain constant (an articulated parallelogram) as well as the lengths PO, PN, and so that one of the points O, M, N remains fixed, the two others describe two figures homothetic to each other.

First, $PRMQ$ will remain a parallelogram, since we will always have $PQ = RM$, $PR = QM$.

Moreover, the points O, M, N will remain collinear.

Indeed, since this is true in the original position, we have

$$\frac{RM}{OP} = \frac{NR}{NP}.$$

Since the lengths of the various segments remain fixed, this equality remains true, so that triangles MRN, OPN will always be similar (having equal angles at R, P between proportional sides). Therefore the angles $\widehat{PNO}$, $\widehat{RNM}$ are always equal. Finally, the ratio $\frac{OM}{MN}$ is constant: it is equal to $\frac{PR}{RN}$. QED

This theorem is the principle behind the operation of the *pantograph*, an instrument design to reproduce figures, with or without enlargement. The lines OPQ, PRN, MQ, MR are rigid rods, attached at points P, M, Q, R, around which they

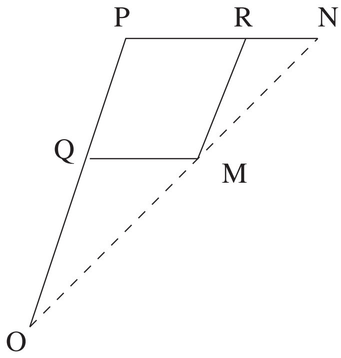


FIGURE 147

can turn. One of the points O, M, N is fixed, and a pencil is placed at another of the points, while the third follows the contours of the figure to be reproduced.

Exercises

Exercise 155. Inscribe a square in a triangle.

Exercise 156. A fixed point A is joined to a variable point B on circle O. Find the locus of the intersection of line AB with the bisector of angle $\widehat{AOB}$.

Exercise 157. On side Ox of a given angle xOy, we take a variable point M, and draw a circle with diameter OM. Next we draw a circle tangent to the first, and to the sides Ox, Oy. Find the locus of the point of contact of the two circles.

Exercise 158. The intersection G of the medians of a triangle lies on the segment joining the center of the circumscribed circle to the intersection of the altitudes, and divides this segment internally in the ratio 1 to 2. (Prove that G is the center of homothecy of the two triangles ABC, $A'B'C'$ which appear in the proof in **53**, Book I.)

Exercise 159. Consider three figures which are pairwise homothetic, with known ratios of similitude. Find the ratio into which one of the centers of homothecy divides the segment formed by the other two.

Exercise 160. We are given two parallel lines, and a point O in their plane. A variable transversal through this point cuts these lines at A, A'. Find the locus of the endpoint of a segment perpendicular to this transversal at A' and of length OA.

Exercise 161. Consider two similar (but not congruent) figures F, F' with opposite orientations. Show that one can find, in two different ways, a figure F'' which is homothetic to F', and symmetric to F with respect to a line. The center of homothecy is the same in both cases, but one of the homothecies is direct, the other inverse.

Exercise 162. On the segments joining pairs of corresponding points of two similar figures with the same orientation, we construct a triangle similar to, and with the same orientation as, a fixed triangle T (or, alternatively, we divide these line segments in a constant ratio). The third vertices of the triangles constructed in this way form a figure similar to the first two.

CHAPTER VI

Constructions

151. Construction 1. *Divide a segment into equal parts, or into parts proportional to given segments.*

First method (Fig. 148). Suppose segment AB must be divided, say, into three equal parts. On an arbitrary line passing through A, and starting from this point, we construct three equal segments AC, CD, DE.

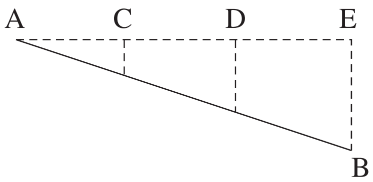


Figure 148

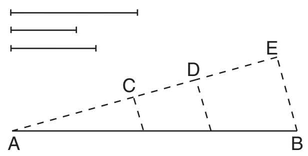


Figure 149

It suffices to draw BE, then draw parallels to this line through C and D.

These parallels will divide AB into three equal parts (**113**). If we need to divide AB into parts proportional to three given segments (Fig. 149), we take AC, CD, DE equal to the three segments in question.

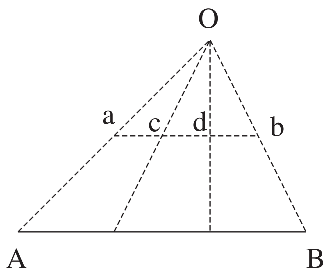


Figure 150

Second method (Fig. 150). Again, suppose segment AB must be divided into three equal parts. We draw an arbitrary parallel ab to AB, and on it take equal segments ac, cd, db. We then draw Aa, Bb, which will intersect in O. Lines Oc, Od then divide AB into three equal parts (**121**).

If we need to divide AB into parts proportional to three given segments (Fig. 149), we take ac, cd, db equal to the three segments in question.

Construction 2. *Find the fourth proportional to three given segments.*

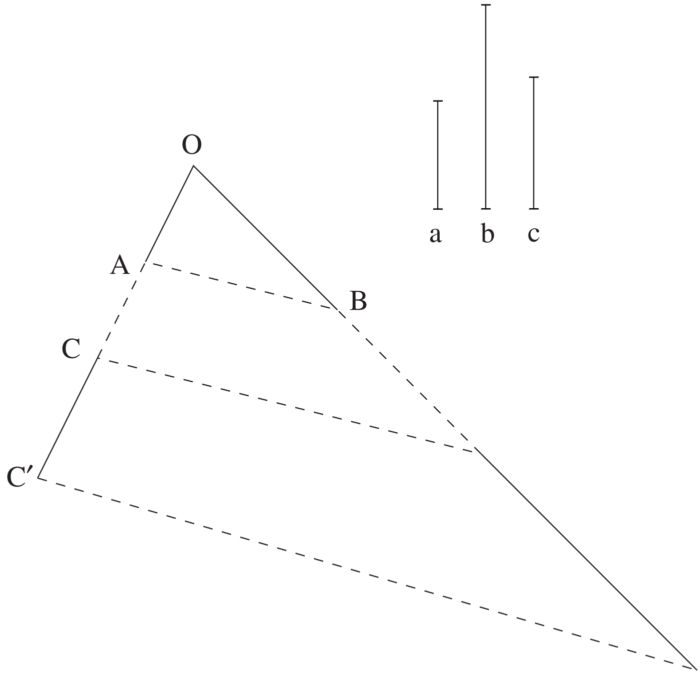


FIGURE 151

The constructions are similar to the preceding.

First method (Fig. 151). Consider the given segments a, b, c. We must find x such that $\frac{a}{b} = \frac{c}{x}$. On the sides of an arbitrary angle $\widehat{AOB}$, we construct segments $OA = a$ and $OB = b$. Then segment c can be placed anywhere on the side OA (in the figure, it is CC'). The lines parallel to AB through the endpoints of this segment determine on OB a segment equal to the required fourth proportional.

If we take one of the endpoints of segment c to coincide with point A or point O, we need, in general, to draw only a single parallel.

We leave it to the reader to apply the second method used in the preceding construction to this problem.

152. Construction 3. *On a given segment, construct a triangle similar to a given triangle.*

Suppose the given the triangle is ABC, and the given segment is A_1B_1. We find a point D on AB such that $AD = A_1B_1$, and draw a parallel DE to BC (Fig. 152), extending it to intersect AC at E. Then it remains only to construct a triangle on A_1B_1 congruent to ADE (Book II, **86**, Construction 4).

This construction obviously gives a solution to the following problem:

Problem. *Given two points A, B of a figure F and their corresponding points A', B' in a similar figure F', find the point corresponding to an arbitrary third point of F.*

In particular, repeated application of this construction will allow us to perform

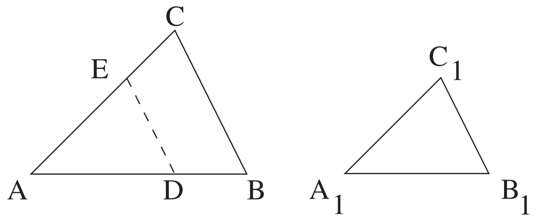


FIGURE 152

Construction 3b. *On a given segment, construct a polygon similar to one given.*

153. Construction 4. *Construct the geometric mean of two segments.*

We have stated three theorems leading to geometric means (**123, 125, 132**). Each of these theorems provides a method for solving the proposed problem.

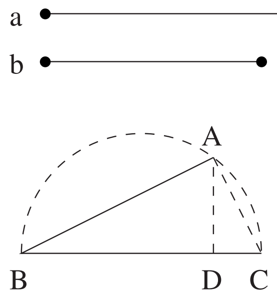


FIGURE 153

First method (Fig. 153). Starting from a point B on a line, we mark off, in the same sense, segments BD, BC equal to the given segments a, b (labeling the smaller of the two BD). We will construct a right triangle with hypotenuse BC, in which BD will be the projection of another side onto the hypotenuse. To do this, it suffices to place the vertex A of the right angle at the intersection of the perpendicular erected from D to BC with the circle on diameter BC. Then AB will be the required geometric mean (**123**).

Second method (Fig. 154). From an arbitrary point D on an arbitrary line, and in opposite senses, we construct segments DB, DC equal to the given segments a, b. We will consider them to be the projections of the legs of a right triangle on its hypotenuse. To do this it will suffice to place the vertex A of the right angle at the intersection of the perpendicular from D to BC with the circle with diameter BC. Then AD will be the geometric mean (**125**).

Third method (Fig. 155). Starting from a point O on a line, and in the same sense, we construct segments OA, OB equal to the given segments a, b. We draw an arbitrary circle through A and B, and draw a tangent OT to this circle. This tangent is the geometric mean (**132**).

154. Construction 5. *Construct a segment whose square is the sum of the squares of two given segments.*

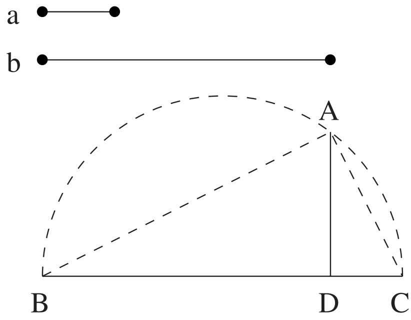


FIGURE 154

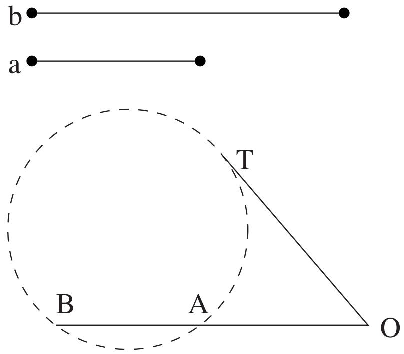


FIGURE 155

We construct segments equal to the given ones on the two sides of a right angle, starting from the vertex. The required segment will be the hypotenuse of the right triangle thus formed.

Construction 6. *Construct a segment whose square is the difference of the squares of two given segments.*

We construct (Book II, **87b**, Construction 9) a right triangle having the larger of the given segments as hypotenuse, and the other as a side of the right angle. The other side of the right angle gives the required segment.

155. Construction 7. *Construct two segments, knowing their sum and their product.*

Supppose we must find two segments whose sum is equal to a given segment $a = BC$ and whose product is equal to the product of two given segments b, c. We may assume that these last two lengths are laid off (as BB', CC' along the perpendiculars to the endpoints of BC) in the same sense (Fig. 156). Since the sum of the required segments is BC, they can be represented as BM and CM, for some point M on BC. Now, the product $BM \cdot CM$ must equal $BB' \cdot CC'$, and therefore

$$\frac{BB'}{BM} = \frac{CM}{CC'},$$

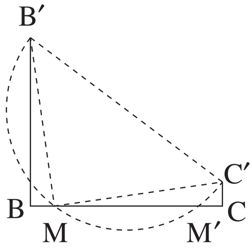


FIGURE 156

which shows that the right triangles $BB'M, CMC'$ are similar. Thus angle $\widehat{BMB'}$ is equal to the corresponding angle $\widehat{MC'C}$, and therefore complementary to $\widehat{CMC'}$, so that angle $\widehat{B'MC'}$ is a right angle. Point M will be on the circle with diameter $B'C'$.

Conversely, if the circle with diameter $B'C'$ intersects BC in M, triangles $BB'M$, $CC'M$ will be similar, since they have perpendicular sides, and the equality $BM \cdot CM = BB' \cdot CC'$ follows.

Conditions for existence. We have taken lines b and c arbitrarily, but we may assume that they are equal: we may replace them, without changing their product, by their geometric mean. Assume thus that $BB' = CC'$, so that $BB'CC'$ is a rectangle. The midpoint of $B'C'$ is then at a distance from BC equal to BB', and the radius of the circle is one half $B'C'$, which is the same as one half BC. Therefore the circle will intersect BC if BB' is at least equal to half of BC; in other words, if *the given product is at most equal to the square of half of* BC.

If M is a point satisfying the conditions of the problem, then, turning the segment BC around on itself or taking the symmetric figure with respect to the midpoint of BC (which is the same thing), we obtain a point M' which will have the same property as the first, since $BM' = CM$, $BM = CM'$. We see thus that the intersections M, M' of BC with the circle with diameter $B'C'$ are symmetric with respect to the midpoint of BC, which can also be verified directly with the help of **62**.

If the given product is equal to the square of half of BC (which, as we have just shown, is the greatest value it can take), the circle with diameter $B'C'$ is tangent to BC; the points M, M' coincide with each other and with the midpoint of BC, so that $BM = CM$. We thus see that *the product of two segments whose sum is constant is maximal when these two segments are equal.*

REMARK. Let x denote segment BM. We have $CM = a - x$, and the equality $BM \cdot CM = BB' \cdot CC'$ can be written $x(a - x) = bc$, or

$$x^2 - ax + bc = 0.$$

The preceding construction allows us to find a segment x satisfying

$$x^2 - ax + q = 0,$$

where a is a given segment, and q is the product of two given segments.

Construction 8. *Construct two segments, knowing their difference and their product.*

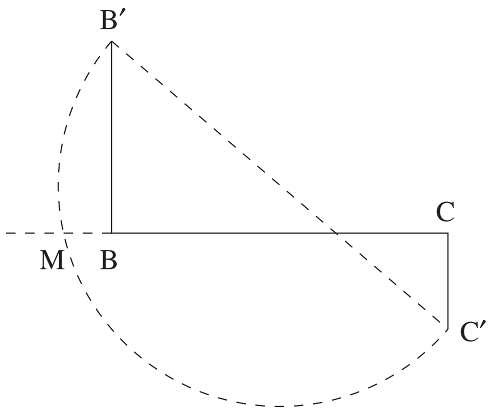


FIGURE 157

Suppose we wish to find two segments whose difference is equal to a given segment $a = BC$, and whose product is equal to the product of two given segments b, c. We may assume that these last two lengths are laid off as BB', CC', perpendicular to the endpoints of BC in opposite senses (Fig. 157). Since the difference of the required segments is BC, they can can be represented as BM, CM, where M is a point on an extension of BC (in one of two directions) satisfying $BM \cdot CM = BB' \cdot CC'$. As in the preceding construction, we conclude that point M must belong to the circle with diameter $B'C'$ and, conversely, a point of intersection of this circle with an extension of BC satisfies the conditions of the problem.

Since the points B', C' are on different sides of BC, the circle always intersects the line, and therefore a solution is always possible.

REMARK. Let x be the smaller of the segments found above; the other can be represented as $y = a + x$, and the equality $xy = bc$ becomes either

$$x(a + x) = bc, \quad x^2 + ax - bc = 0,$$

or

$$(y - a)y = bc, \quad y^2 - ay - bc = 0.$$

We can therefore find segments satisfying either of the following two conditions

$$x^2 + ax - q = 0,$$

$$y^2 - ay - q = 0,$$

where a is a given segment, and q is the product of two given segments.

156. We say that a line is divided into *extreme and mean ratio* if the greater segment is geometric mean between the whole segment and the smaller part.

Construction 9. *Divide a segment into extreme and mean ratio.*

Let the given line segment be BC (Fig. 158), on which we need to find a point D such that $\frac{BC}{BD} = \frac{BD}{CD}$. In this proportion, let us add the numerators and denominators. We obtain

$$\frac{BC}{BD} = \frac{BC + BD}{BC}.$$

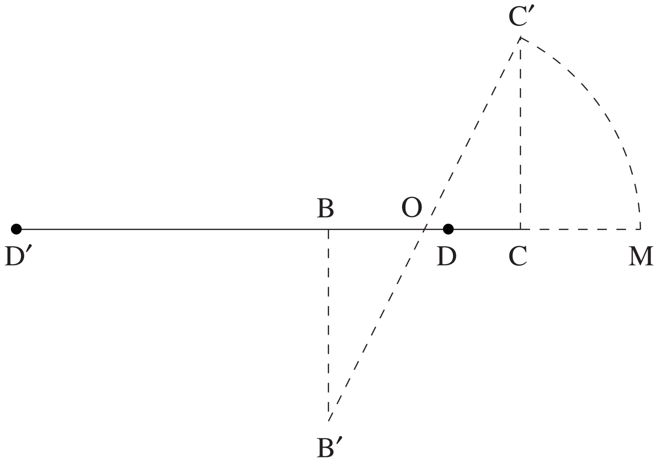


FIGURE 158

Thus the two segments BD and $BC+BD$, whose difference is equal to BC, have a product equal to BC^2. This observation leads us to the preceding construction. We must construct perpendicular segments BB', CC' on the two endpoints of BC, and in opposite senses, both equal to BC. The circle with diameter $B'C'$ will intersect the extension past C of BC in a point M. Then if we construct $BD = CM$, point D is the required point of division.

Starting from B, and in the sense opposite to BC, we construct a segment $BD' = BM$. The point D' thus constructed has a property analogous to that of D: *its distance to B is the geometric mean between its distance to C and the segment BC.* Indeed, we have $BD' = BC + BD$, so that the proportion $\frac{BC}{BD} = \frac{BC+BD}{BC}$ can be rewritten as $\frac{BC}{BD} = \frac{BD'}{BC} = \frac{CD'}{BD'}$.

The point D' is said to divide the segment BC into mean and extreme ratio *externally*.[1]

Note that in order to have this property, the point D' must be chosen as described, because the proportion $\frac{BD'}{BC} = \frac{CD'}{BD'}$ gives, conversely, $\frac{BD'}{BC} = \frac{BC}{BD'-BC}$ (by taking the difference of the numerators and denominators), so that BD' is one of the segments whose difference is BC and whose product is BC^2.

Let $BD = a$: we propose to calculate BD and BD'. We remark first that since BB' and CC' are equal, the circle with diameter $B'C'$ has as its center the midpoint O of BC, so that $OC = \frac{a}{2}$. The theorem about the hypotenuse of right triangle, applied to triangle OCC', tells us that:

$$OC' = OM = \sqrt{\left(\frac{a}{2}\right)^2 + a^2} = \sqrt{\frac{5}{4}a^2} = \frac{a\sqrt{5}}{2}.$$

We therefore have

$$CM = BD = a\frac{\sqrt{5}-1}{2}, \quad BM = BD' = a\frac{\sqrt{5}+1}{2}.$$

[1]The points D and D' are not conjugate with respect to AB (see Exercise 173).

157. Problem. *Find the locus of points with the property that the ratio of their distances to two given lines is equal to a given ratio.*

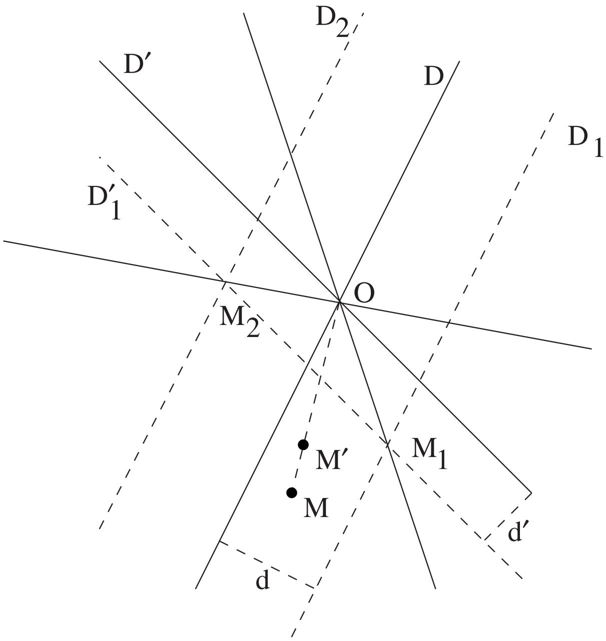


FIGURE 159

Let D, D' be the given lines, and assume that they meet at point O (Fig. 159). We are looking for points whose distances to these lines are in the ratio $d : d'$ of two given segments. If M is a point on this locus, every point M' on line OM will also be on the locus, because the perpendicular segments from M to D, D', and the corresponding perpendicular segments from M', obviously form two homothetic figures with center of homothecy O. It follows that the locus consists of lines passing through O. We can obtain some points on this locus by finding points whose distances to D, D' are equal to d, d', respectively. We know that the locus of points at a distance d from D consists of two lines D_1, D_2 parallel to D. Likewise, the locus of points at a distance d' from D' consists of two lines D'_1, D'_2 parallel to D'. Let M_1, M_2 be the intersections of D_1, D_2 with D'_1. The lines OM_1, OM_2 are part of our locus. They are in fact the entire locus. Indeed, if M is a point on the locus, then line OM intersects D'_1 in a point M' which, being at distance d' from D', must be at distance d from D, and must therefore coincide with M_1 or with M_2.

If the two lines D, D' are parallel, there would be two points of the locus on any arbitrary common perpendicular:[2] these would be the two conjugate points which divide this perpendicular in the given ratio. The locus consists of parallels to the original lines through these two points.

[2]Except when the given ratio is one. In this case there is only one point, the midpoint of the common perpendicular, the other point being thrown to infinity.

Construction 10. *Find all points whose distances to three given lines have given ratios.*

We first construct the two lines which are the loci of points whose distances to the first two lines are in the given ratio; we then do the same for the second and third given line. In this way we obtain two new lines which will intersect the first two parallels in four points which satisfy the conditions of the problem.

158. Construction 11. *Construct the common tangents to two circles.*

We have solved this problem in Book II, **93**. The preceding theorems allow us to give an entirely different solution. It suffices to draw two parallel radii, so as to determine the centers of similarity (**143**), and to draw from either of these centers a tangent to one of the circles: this line will be tangent to the second as well because of the homothecy of the two figures.

Construction 12. *Find the radical axis of two circles.*

If the two circles are secant or tangent, we need only draw the common chord or the common tangent.

If not, we draw an arbitrary third circle which intersects the two circles. The intersection of its two common chords with the given circles will belong (**139**) to the required radical axis. It suffices then to drop a perpendicular from this point to the line of the centers. Alternatively, we can repeat the construction with another circle in order to determine a second point.

Remark. This construction can be applied to a circle and a line, which leads us to state that *the radical axis of a circle and a line is the line itself.*

The construction can also be applied in the limiting case when one of the circles is reduced to one point. The auxilliary circle must then pass through this point.

The preceding construction also allows us to find the radical center of three given circles, and therefore (**139**) to solve the following problem:

Construction 13. *Construct the circle orthogonal to three given circles.* In particular, *construct*:

- *a circle passing thorough a given point and orthogonal to two given circles*;

or

- *a circle passing through two points and orthogonal to a given circle.*

159. Construction 14. *Construct a circle passing through two given points and tangent to a given line.*

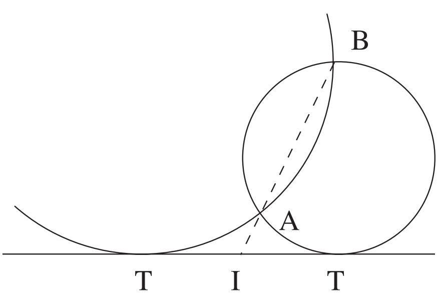


Figure 160

Consider points A, B (Fig. 160) and line xy; let T be the point of contact of the required circle with xy. Extend AB to its intersection I with xy. The segment IT is known: it is the geometric mean of IA and IB.

Conversely, if point T verifies this condition, there exists (**132**) a circle passing through A, B and tangent to xy at T: it is determined by construction 13 of Book II, **90**.

Since the geometric mean of IA, IB can be placed on either side of point I on xy, there are two solutions to the problem.

The solution is clearly possible only when A, B are on the same side of line xy.

REMARKS. I. The preceding method does not apply when AB is parallel to xy. In this case the problem has only one solution, the point of contact being the intersection of xy with the perpendicular bisector of AB.

II. On the other hand, this method does apply when A and B *coincide on the line* IA intersecting xy at I; that is, when we are required *to find a circle tangent*

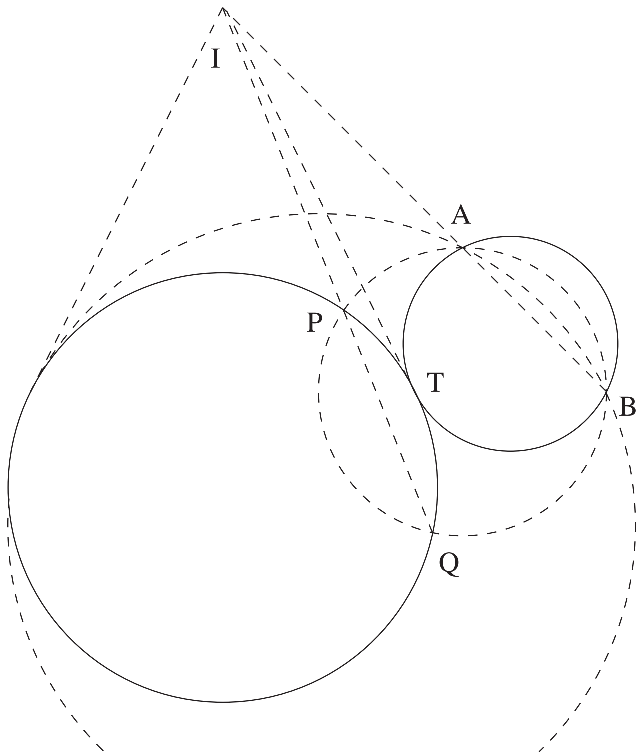


FIGURE 161

to IA *at* A *and also tangent to* xy. The point of contact T can be found by laying off, in either sense, the segment $IT = IA$.

Construction 15. *Construct a circle passing through two given points and tangent to a given circle.*

Consider two points A, B (Fig. 161) and circle C. We are looking for the point of contact T with the required circle.

To find this point, let I be the point of intersection of AB with the tangent at T. Take an arbitrary circle passing through A, B, and intersecting C in P, Q. Point I will be on line PQ (**139**).

Since the intersection of AB with PQ gives us point I, it suffices to construct tangents to C through this point. The problem thus has two solutions.

Condition for existence. The point I must be exterior to circle C. For this to happen, it is necessary and sufficient that this point lie on the extention in one direction or the other of segment PQ; that is, outside the auxiliary circle. This will happen when points A, B are both on the same arc determined on the auxilliary circle by P, Q; that is, *when A and B are both inside or both outside the given circle.*

The remarks of the preceding construction apply here, with respect to the case when A and B are two identical points on a given line.

Exercises

Exercise 163. Find a segment whose ratio to a given segment is equal to the ratio of the squares of two given segments.

Exercise 164. Find a segment such that the ratio of its square to the square of a given segment is equal to the ratio of two given segments.

Exercise 165. Draw a segment through a given point which is divided into a given ratio by two given lines.[3]

Exercise 166. Through a point exterior to a circle, draw a secant which is divided into mean and extreme ratio by the circle.

Exercise 167. Given three consecutive segments AB, BC, CD on a line, find a point at which they subtend the same angle.

Exercise 168. Through two points on the same diameter of a circle, draw two equal chords which intersect on the circle.

Exercise 169. Construct a triangle knowing two sides and the bisector of the angle they include.

Exercise 170. Construct a triangle knowing a side, the corresponding altitude, and the product of the other two sides.

Exercise 171. Construct a triangle knowing the angles and the perimeter; or the angles and the sum of the medians; or the angles and the sum of the altitudes, etc.

Exercise 172. Construct a triangle knowing its three altitudes (Exercise 146).

Exercise 173. The harmonic conjugate of point B with respect to segment DD' (Fig. 158, **156**) is obtained by extending segment BC by its own length. The harmonic conjugate of D' with respect to BC is the point symmetric to D with respect to the midpoint of BC. Also show that the circle with diameter DD' passes through the vertices (other than B, C) of the square with diagonal BC.

[3]See solution for a note on this problem statement.–transl.

Exercise 174. Given a line and two points A, B, find a point on the line at which AB subtends the greatest possible angle.

This problem can be treated *indirectly*, by first seeking points on the line at which AB subtends a given angle, then finding the greatest angle for which the construction is possible.

The next two problems can be approached in the same way.

Exercise 175. Draw a common perpendicular to two given parallel lines which subtends the greatest possible angle at a given point.

Exercise 176. Find a circle passing through two given points, and intercepting a chord of given length on a given line. Find the minimum possible length when the two points are on different sides of the line.

Exercise 177. Construct a circle passing through a given point and having the same radical axis as two given circles.

CHAPTER VII

Regular Polygons

160. A *regular polygon* is a convex polygon all of whose sides are equal, and all of whose angles are equal. A *regular broken line* is a broken line all of whose sides are equal, and all of whose angles are equal and have the same orientation.

161. THEOREM. *If a circle is divided into an arbitrary number n of equal parts, then*:

1°. *The division points are the vertices of a regular polygon*;

2°. *The tangents at these points are the sides of a second regular polygon.*

1°. Two consecutive sides of the polygon with vertices at the points of division are obviously symmetric with respect to the radius ending at their common endpoint; two consecutive angles of this polygon are symmetric with respect to the radius perpendicular to their common side.

2°. Two consecutive angles of the polygon formed by the tangents at the points of division are obviously symmetric with respect to the radius perpendicular to their common side; two consecutive sides of this polygon are symmetric with respect to the radius perpendicular to the chord joining their points of contact.

162. THEOREM. *Conversely, every regular polygon and, more generally, every regular broken line, can be inscribed in a circle and circumscribed about another circle.*

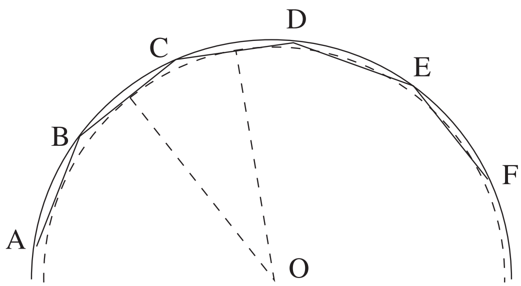


FIGURE 162

Consider, for instance, the regular broken line $ABCDEF$ (Fig. 162). We draw the circle cirumscribing triangle ABC, whose center O is on the perpendicular bisector of BC. We claim that this circle passes through the point D. To see this, it suffices to observe that the sides BA, CD are symmetric with respect to the perpendicular bisector of BC; indeed, the segment symmetric to BA coincides with CD in direction (because of the equality of the angles at B and C) and length (because $BA = CD$). Thus we have $OA = OD$. Likewise, we see that this same circle, circumscribed about BCD passes through E, and so on.

Now, chords AB, BC, etc., are equal chords of the circumscribed circle, and they are therefore equally distant from the center O. Therefore the sides of the broken line are tangent to a second circle with center O.

The radius of this second circle, or the distance from an arbitrary side to the center, is called the *apothem* of the regular broken line (or of the regular polygon).

Remark. Every regular polygon with n sides can be superimposed on itself by certain rotations (namely the rotations whose angle is a number of times the n^{th} part of the circle) *and by certain line reflections* (reflections in the perpendicular bisectors of the sides or the bisectors of the angles).

The number of axes of symmetry is always n. If $n = 2p$ is even, the vertices form diametrically opposite pairs, and this gives p axes of symmetry, each of which is the common bisector of a pair of opposite angles. Likewise, the sides give p axes of symmetry, each perpendicular to two opposite sides.

If $n = 2p+1$ is odd, each axis of symmetry bisects an angle, and is at the same time the perpendicular bisector of the opposite side.

163. Thus there are infinitely many regular polygons with the same number n of sides, but all of them are similar.

Theorem. *Two regular polygons with the same number of sides are similar.*

Indeed, two regular polygons with the same number of sides inscribed in congruent circles are obviously congruent: they will coincide if we superimpose their circumscribed circles and one vertex.

If we now are given two regular polygons P, P' inscribed in circles C, C', we can draw a circle concentric to C and congruent to C'. If we look at the intersections of this circle with the radii of C drawn to the vertices of P, we will find a regular polygon which is homothetic to P with respect to the center of C, and congruent to P'.

Remark. In particular, all regular polygons with the same number n of sides have the same angle. It is easy to calculate this angle. Since the sum of the angles of one of these polygons is $2n-4$ right angles, each one will be the nth part of this measure; that is, $2 - \frac{4}{n}$ right angles.

164. Having divided a circle into n equal parts, let us join the points of division separated by p equal arcs, starting from one division point, until we visit some vertex for the second time. If P is the first vertex which is visited twice, this vertex can only be the one where we started, because if it were the endpoint of a side NP, already traversed in the sense NP, it is clear that we would have visited N first before returning to P.

The figure thus obtained is (**21**, Remark) an improper polygon, and is called a *regular star polygon.* For instance, Fig. 163 represents the star pentagon obtained by dividing the circle into five equal parts at the points A, B, C, D, E and joining them in the order $ACEBDA$ ($n = 5$, $p = 2$).

We may assume that n and p are relatively prime. Indeed, if they had a common divisor $d > 1$, we would obtain the same construction replacing n, p by $\frac{n}{d}$, $\frac{p}{d}$.

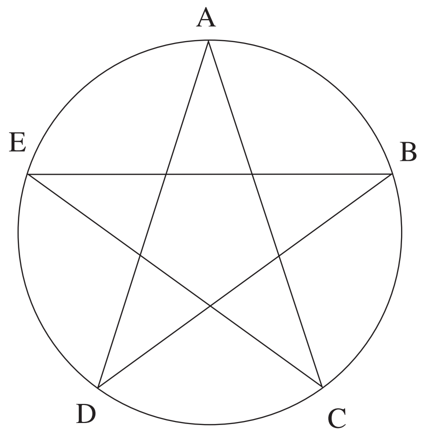


FIGURE 163

If n and p are relatively prime, the star polygon will have precisely n sides. Indeed, since each side spans p parts of the circle, tracing k sides we will have spanned kp parts. We will have returned to the starting point after traveling a whole number of circles, that is if kp is a multiple of n. It is shown in arithmetic[1] that this happens for some value $k < n$ if p and n are not relatively prime; but if, on the other hand, p is relatively prime to n, this happens for the first time when $k = n$.

We can thus form star polygons with n sides by dividing the circle into n equal parts, and joining every p^{th} division point, where p is any number less than, and relatively prime to, n. Every star polygon is obtained in two ways by this construction, because if its side is the chord corresponding to an arc equal to p divisions, it is also the chord corresponding to $n - p$ divisions. For instance, the star polygon in Figure 163 can be obtained either by joining every second point, or by joining every third point. Thus one half of the values of p must be discarded if we want to find each polygon once. The value $p = 1$ (and therefore $p = n - 1$) corresponds to the usual regular polygon.

EXAMPLE. Consider $n = 15$. Besides 1 and 14, the numbers less than 15 and relatively prime to it are 2, 4, 7, 8, 11, 13. We need not consider the values $p = 8$, 11, 13, which give the same polygons as 7, 4, 2. Thus there is one proper regular pentadecagon, and three star pentadecagons.

165. Construction of inscribed regular polygons. *If we know how to inscribe a regular polygon, we also know how to circumscribe the regular polygon with the same number of sides.*

This last is formed by the tangents to the circle through the vertices of the first.

If we know how to inscribe a regular polygon, we also know how to inscribe a regular polygon with double the number of sides.

It suffices to divide the arc intercepted by the first polygon into two equal parts.

[1]See Tannery, *Leçons d'Arithmétique*, chap. IV.

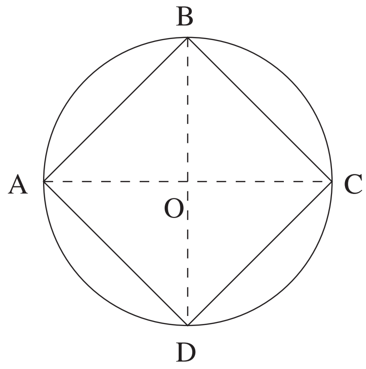


FIGURE 164

166. The Square. To inscribe a square in a circle O (Fig. 164), it suffices to draw two perpendicular diameters AC, BD: they will divide the circle into four equal parts.

The preceding remark allows us to construct the inscribed regular octagon, then the regular polygons with 16, 32, ... and, generally, 2^n sides.

The side of the square inscribed in a circle of radius R is $c_4 = R\sqrt{2}$.

This is true because in right triangle AOB (Fig. 164), we have

$$AB^2 = AO^2 + BO^2 = 2R^2.$$

The apothem of an inscribed regular polygon can be calculated by the following rule: *the square of the apothem equals the square of the radius of the circumscribed circle minus the square of half a side.* This follows from the fact that the apothem, radius, and half a side form a right triangle.

Thus, the apothem a of a polygon with side c inscribed in a circle of radius R is $a = \sqrt{R^2 - \frac{c^2}{4}}$.

For the square, this apothem is

$$a_4 = \sqrt{R^2 - \frac{c_4^2}{4}} = \sqrt{R^2 - \frac{2R^2}{4}} = \frac{R\sqrt{2}}{2}.$$

It is equal to half the side, which was evident *a priori.*

167. The Hexagon.

THEOREM. *The side of a regular hexagon is equal to the radius of its circumscribed circle.*

Let AB (Fig. 165) be the side of a regular hexagon inscribed in a circle with center O. We claim that triangle OAB is equilateral. We see right away that it is isosceles. Also, the angle at O intercepts a sixth of the circle, so it is equal to $\frac{4}{6} = \frac{2}{3}$ of a right angle. There remain $2 - \frac{2}{3} = \frac{4}{3}$ of a right angle for the sum of the angles at A and B. Since these two angles are equal, both are $\frac{2}{3}$ of a right angle. The triangle is equiangular, and so equilateral.

The apothem of the regular hexagon is

$$a_6 = \sqrt{R^2 - \frac{R^2}{4}} = \frac{R\sqrt{3}}{2}.$$

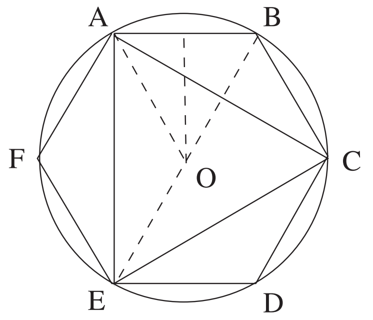


FIGURE 165

The Equilateral triangle. Joining every other vertex of our hexagon, we form inscribed equilateral triangle ACE (Fig. 165).

Side AC of this triangle, being perpendicular to radius OB, is equal to twice the altitude of triangle AOB; that is (since the three altitudes of an equilateral triangle are equal), twice the apothem of the hexagon: $c_3 = R\sqrt{3}$.

This can also be seen by observing that B and E are diametrically opposite (since the arc between them is $\frac{3}{6}$ of the circle). Triangle ABE thus has a right angle at A, and side AE is twice the length of the perpendicular from the midpoint O of BE to AB.

The apothem of the inscribed equilateral triangle is $a_3 = \frac{R}{2}$.

Knowing the construction for the inscribed hexagon, we can derive the construction of inscribed polygons with $12, 24, \ldots, 3 \times 2^n$ sides.

168. The Decagon.

THEOREM. *The side of the inscribed regular decagon is equal to the greater segment of the two that divide the radius into mean and extreme ratio.*

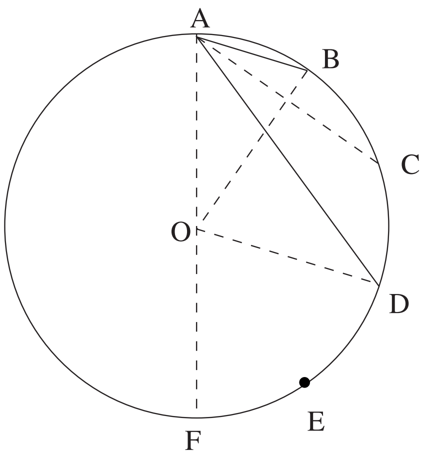


FIGURE 166

Let AB (fig. 166) be the side of the regular decagon inscribed in a circle with center O. The angle at O is $\frac{4}{10} = \frac{2}{5}$ of a right angle. The sum of the angles $\widehat{OAB}$, $\widehat{OBA}$ is therefore $2 - \frac{2}{5} = \frac{8}{5}$ of a right angle, so each of them is $\frac{4}{5}$ of a right angle. We construct the bisector AI of $\widehat{OAB}$ (with I on OB), which forms, with OA and AB, angles equal to $\frac{2}{5}$ of a right angle. Angle $\widehat{AIB}$, exterior to triangle AIO is equal to the sum of the two remote interior angles, which is here $\frac{4}{5}$ of a right angle. It follows that triangles AIB, AIO are isosceles and that $AB = AI = IO$. The property of the angle bisector (**115**), $\frac{OI}{IB} = \frac{OA}{AB}$, can thus be rewritten as

$$\frac{OI}{IB} = \frac{OB}{OI}.$$

QED

The side of the regular inscribed decagon can then be determined by Construction 9 in **156**. This side is:

$$c_{10} = R\frac{\sqrt{5}-1}{2},$$

and the apothem is:

$$a_{10} = \sqrt{R^2 - R^2\left(\frac{\sqrt{5}-1}{4}\right)^2} = \frac{R}{4}\sqrt{10+2\sqrt{5}}.$$

169. Aside from the ordinary regular decagon, there is also a star regular decagon, obtained by joining every third vertex of the vertices of the ordinary one.

We can find the side of the ordinary decagon, and the side of the star decagon, from the following result:

THEOREM. *The difference of the sides of the two regular inscribed decagons is equal to the radius, and their product is the square of the radius.*

Once again, we let AB (Fig. 166) be the side of the ordinary regular decagon, so that semicircle $\widehat{AF}$ is divided into five equal parts by the points B, C, D, E. Then AD is the side of the star decagon, and line AD coincides with the bisector AI of angle $\widehat{OAB}$, since the inscribed angles $\widehat{FAD}$, $\widehat{DAB}$ intercept equal arcs on the circle. Radius OD is parallel to AB because alternate interior angles $\widehat{ODA}$, $\widehat{DAB}$ are both equal to $\widehat{OAD}$. Thus triangles AIB, DOI are similar; they are also isosceles, as we have seen in the preceding section. We then have $AD - AB = AD - AI = ID = OD$.

As for the relation $AB \cdot AD = AI \cdot AD = OA^2$, this follows from the similarity of triangles AIO, AOD which have equal angles.

We observe that the sides of the two regular decagons correspond to the two divisions (interior and exterior) of the radius into mean and extreme ratio, obtained in Construction 9.

The side of the star decagon is $c'_{10} = R\frac{\sqrt{5}+1}{2}$, and its apothem is:

$$a'_{10} = \sqrt{R^2 - R^2\left(\frac{\sqrt{5}+1}{4}\right)^2} = \frac{R}{4}\sqrt{10-2\sqrt{5}}.$$

170. The Pentagon. Joining every other vertex of a regular decagon we obtain the ordinary regular pentagon. Joining every other vertex of the pentagon, which amounts to joining every fourth vertex of the decagon, we obtain the regular star pentagon.

Side AC of the ordinary regular pentagon is twice the altitude from A in triangle AOB (Fig. 166). Thus we can compute it, knowing the side of the decagon, since we can compute the altitudes of a triangle, given its sides. But, more simply, *the side of the ordinary regular pentagon is twice the apothem of the star decagon.* This can be seen either from triangle AIO whose altitudes (namely, half the side of the pentagon, and the apothem of the star decagon) are equal, or in right triangle ADF, where side DF is equal to the side of the pentagon, and the parallel through O to this side is equal to the apothem of the star decagon. The side and apothem of the ordinary regular pentagon are therefore

$$c_5 = 2a'_{10} = \frac{R}{2}\sqrt{10-2\sqrt{5}}, \quad a_5 = \frac{R}{4}(\sqrt{5}+1).$$

In the same way, by considering right triangle ABF, we see that the side of the star pentagon is twice the apothem of the ordinary decagon, and therefore

$$c'_5 = \frac{R}{2}\sqrt{10+2\sqrt{5}}, \quad a'_5 = \frac{R}{4}(\sqrt{5}-1).$$

Remark. In general, the same reasoning shows that the calculation of the side and apothem of a polygon (ordinary or star) obtained by dividing the circle into $2n+1$ equal parts, and joining every p^{th} point of division, amounts to the calculation of the polygon obtained by dividing the circle into $4n+2$ equal parts and joining every q^{th} point of division, where q is derived from p by the relation[2]

$$\frac{p}{2n+1} + \frac{q}{4n+2} = \frac{1}{2}, \quad \text{or} \quad 2p+q = 2n+1.$$

171. The Pentadecagon. Next we consider inscribing a proper regular pentadecagon in a circle. We will know how to do this, once we have constructed the hexagon and decagon, from the following result.

Theorem. *The arc intercepted by the side of the regular inscribed pentadecagon is the difference of the arcs intercepted by the side of the hexagon and the side of the decagon.*

Indeed, this arc is a 15^{th} of the circle, and the ratio $\frac{1}{15}$ is equal to the difference $\frac{1}{6} - \frac{1}{10}$.

Thus we can start from point A (Fig. 167), and lay off two openings of the compass AB, AC, on the circle, one equal to the side of the regular hexagon, the other to the side of the decagon. Segment BC will be the required side.

Aside from the ordinary regular pentadecagon, there exist three star pentadecagons (**164**) whose sides intercept $\frac{2}{15}$, $\frac{4}{15}$, and $\frac{7}{15}$ of the circle. These arcs can be easily deduced from the first, and can also be obtained by constructions analogous to the one used for the ordinary pentedecagon.

[2]Conversely, this remark allows to reduce the calculations for a polygon with $4n+2$ sides to calculations for a polygon with $2n+1$ sides. This is true because a polygon with $4n+2$ sides corresponds to an odd number q (see **164**); $2n+1-q$ will then be even and the relation $2p = 2n+1-q$ will yield an integer p, relatively prime to $2n+1$ and corresponding to a polygon with $2n+1$ sides, for which it will suffice to calculate the side and the apothem. In this way, the calculations done for the pentadecagon (**174**,**175**) can be used to find the corresponding results for polygons with 30 sides.

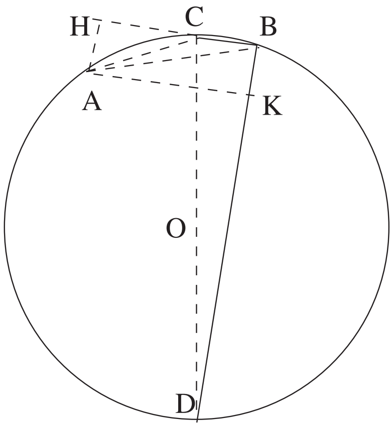


FIGURE 167

The arc intercepted by the side of the first star pentadecagon is the difference of the arcs intercepted by the sides of the star decagon and that of the hexagon.

The arc intercepted by the side of the second star pentadecagon is the sum of the arcs intercepted by the sides of the proper decagon and that of the hexagon.

The arc intercepted by the side of the third star pentadecagon is the sum of the arcs intercepted by the sides of the star decagon and that of the hexagon.

Indeed, we have:

$$\frac{2}{15} = \frac{3}{10} - \frac{1}{6},$$
$$\frac{4}{15} = \frac{1}{10} + \frac{1}{6},$$
$$\frac{7}{15} = \frac{3}{10} + \frac{1}{6}.$$

172. More generally, *if we know how to inscribe regular polygons with m sides and n sides, with m, n relatively prime, we also know how to inscribe a regular polygon with mn sides.*

Indeed, the sides of the two known polygons intercept an m^{th} and an n^{th} of the circle. By adding p times the first arc, and subtracting q times the second, we obtain the arc intercepted by the new polygon if

$$\frac{p}{m} - \frac{q}{n} = \frac{1}{mn}, \quad \text{or} \quad np - mq = 1.$$

It is shown in arithmetic[3] that this relation is not possible if m and n have a common factor (different from one), but that it is possible if m and n are relatively prime.

For instance, the regular dodecagons can be inscribed using the constructions for the square and the triangle, because $\frac{1}{12} = \frac{1}{3} - \frac{1}{4}$, $\frac{5}{12} = \frac{2}{3} - \frac{1}{4}$.

The remark of **170** is a special case of this one, corresponding to $m = 2$ and n odd.

[3]Tannery, *Leçons d'Arithmétique*, chap. XIV, n°511

173. The geometer Gauss proved that we can inscribe, with straightedge and compass, any regular polygon whose number of sides is of the form 2^n+1. Thus we were able to inscribe the regular triangle $(2+1=3)$ and pentagon $(2^2+1=5)$. Next come the polygons with $17(=2^4+1)$, $257(=2^8+1)$, etc., sides.[4]

Combining this proposition with those already obtained, we see that we can inscribe, with straightedge and compass, a regular polygon with N sides if the prime factorization of N contains only: 1°. prime factors of the form 2^n+1, *all distinct*; 2° the factor 2 to an arbitrary power.

Conversely, we can prove that the construction cannot be done by straightedge and compass if the number N is not of the type just described.

Thus, the regular polygon with $170(=2\times 5\times 17)$ can be inscribed, but not the polygon with 9 sides, because the number 9 is a power of 2 plus one, but is not prime; alternatively, its factors $(9=3\times 3)$ are certainly of the form 2^n+1, but are equal.

174. To calculate the side of the ordinary regular pentadecagon inscribed in a circle of radius R, we again consider sides AB, AC (Fig. 167) of the inscribed decagon and hexagon, so that BC is the required side. If we drop a perpendicular AH onto BC, we will have $BC=BH-CH$.

Now angle $\widehat{ABH}$, as an inscribed angle, intercepts arc $\widehat{AC}$, and is therefore equal to half the central angle corresponding to a side of the decagon, so that BH is the apothem of the regular decagon inscribed in a circle with radius AB. Thus

$$BH=\frac{AB}{4}\sqrt{10+2\sqrt{5}}=\frac{R}{4}\sqrt{10+2\sqrt{5}},$$

since $AB=R$.

Similarly, angle $\widehat{ACH}$ is equal to the inscribed angle intercepting arc $\widehat{ACB}$ (since both these angles are supplementary to $\widehat{ACB}$), so it is equal to half the central angle corresponding to the side of the regular hexagon. Thus CH is the apothem of the regular hexagon inscribed in the circle with radius AC, or:

$$CH=\frac{AC\sqrt{3}}{2}=R\frac{\sqrt{5}-1}{2}\cdot\frac{\sqrt{3}}{2},$$

and consequently

$$BC=\frac{R}{4}\left[\sqrt{10+2\sqrt{5}}-\sqrt{3}(\sqrt{5}-1)\right].$$

It is clear that the preceding method applies whenever we must solve problems of the following type: *knowing the chords of two arcs in a circle of given radius, find the chord of the difference of the two arcs.*

In particular, to calculate the side of the first star pentadecagon, we use the same reasoning, replacing the side and apothem of the ordinary decagon with the side and apothem of the star decagon. The required side will be

$$R\frac{\sqrt{5}+1}{2}\frac{\sqrt{3}}{2}-\frac{R}{4}\sqrt{10-2\sqrt{5}}=\frac{R}{4}\left[(\sqrt{5}+1)\sqrt{3}-\sqrt{10-2\sqrt{5}}\right].$$

[4]It is easily proved (it is enough to apply n°48 and n°49, application III, of Bourbet, *Leçons d'Algèbre*) that for 2^n+1 to be prime it is necessary that n itself be a power of 2; this condition is not sufficient.

Next let us calculate the side of the second star pentadecagon. We again consider the sides AB, AC' of the hexagon and ordinary decagon, but we draw them in opposite senses along the circle, so that BC' is the required segment (Fig. 167b).

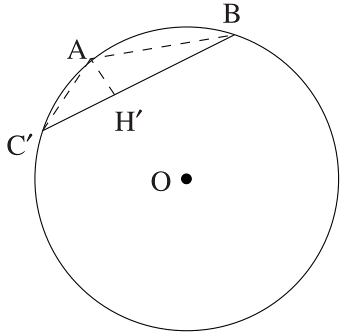


FIGURE 167b

We drop perpendicular AH' onto BC'. The two segments BH', $C'H'$ will again be equal, the first to the apothem of the ordinary decagon inscribed in the circle with radius AB, the second to the apothem of the hexagon inscribed in the circle with radius AC'. Their sum BC' will then be

$$BC' = \frac{R}{4}\left[\sqrt{10+2\sqrt{5}} + \sqrt{3}(\sqrt{5}-1)\right].$$

Clearly we can operate the same way whenever we find ourselves in the position of *knowing the chords of two arcs inscribed in a circle of given radius, and seeking the chord of the sum of the two arcs.*

In particular, for the side of the third star pentadecagon we find the value

$$\frac{R}{4}\left[(\sqrt{5}+1)\sqrt{3} + \sqrt{10-2\sqrt{5}}\right].$$

175. We could deduce the apothem of the pentadecagon from the value found for its side, but this would require the extraction of an additional square root. We can instead compute the apothem (without requiring an additional square root), using a method similar to the one used for the side.

In Fig. 167, we construct diameter CD from point C. Triangle BCD shows that BD is again twice the required apothem. We now drop perpendicular AK from A to BD. Angle $\widehat{ADB}$ (as an inscribed angle) is equal to half the central angle corresponding to a side of the hexagon; the segment DK is equal to the apothem of the hexagon inscribed in the circle of radius AD; and thus

$$DK = AD\frac{\sqrt{3}}{2} = \frac{R}{2}\sqrt{10+2\sqrt{5}}\frac{\sqrt{3}}{2},$$

because AD is twice the apothem of the regular decagon inscribed in the given circle.

On the other hand, angle $\widehat{ABD}$ is equal to $\widehat{ACD}$; that is, to the complement of half the central angle corresponding to the regular decagon; angle $\widehat{KAB}$ is equal to half this central angle; and BK is half the side of the decagon inscribed in the circle with radius AB, so:

$$BK = AB\frac{\sqrt{5}-1}{4} = R\frac{\sqrt{5}-1}{4}.$$

The required apothem is thus

$$\frac{BD}{2} = \frac{BK+DK}{2} = \frac{R}{8}\left[\sqrt{3}\sqrt{10+2\sqrt{5}}+\sqrt{5}-1\right].$$

Analogous reasoning gives the following value for the apothems of the first star pentadecagon:

$$\frac{R}{8}\left[\sqrt{3}\sqrt{10-2\sqrt{5}}+\sqrt{5}+1\right].$$

For the second star pentadecagon, we get:

$$\frac{R}{8}\left[\sqrt{3}\sqrt{10+2\sqrt{5}}-(\sqrt{5}-1)\right].$$

For the third star pentadecagon, we get:

$$\frac{R}{8}\left[\sqrt{3}\sqrt{10-2\sqrt{5}}-(\sqrt{5}+1)\right].$$

176. Measurement of the circumference of a circle.

THEOREM. *Let P be the perimeter of a regular circumscribed polygon, and let p be the perimeter of the inscribed polygon with the same number of sides. If we keep doubling the number of sides indefinitely, then P and p tend to a common limit L.*

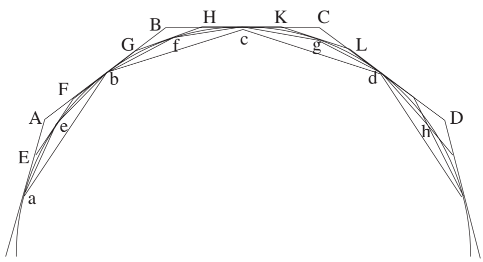


FIGURE 168

Consider inscribed polygon $abcd\cdots$ (see Fig. 168) and circumscribed polygon $ABCD\cdots$ whose sides have, as points of contact, the vertices of the first. Let us double the number of sides of these polygons, dividing each of the arcs $\overset{\frown}{ab}$, $\overset{\frown}{bc}$, $\overset{\frown}{cd}, \ldots$ into two equal parts, so as to form the new regular inscribed polygon $aebfcg\cdots$ and the corresponding new circumscribed polygon $EFGHKL\cdots$ (Fig. 168). We operate in the same way on the new polygons, and so on, indefinitely. We claim that the perimeters p of the inscribed polygons and the perimeters P of the circumscribed polygons tend to a common value.

We will prove this with the help of the following remarks.

1°. The perimeters p increase as we double the number of sides; for instance, polygon $aebfcg\cdots$ has a greater perimeter than polygon $abcd\ldots$, since this last polygon is inside it;

2°. The perimeters P decrease as we double the number of sides; for example polygon $EFGHKL\cdots$ has a smaller perimeter than polygon $ABC\cdots$, which contains it in its interior;

3°. Any perimeter p is smaller than any perimeter P, because each inscribed polygon is inside each circumscribed polygon.

The quantity p, which increases while always remaining less than a fixed quantity (namely, any value of P) tends to a limit (*Leçons d'Arithmétique* de M. Tannery, chap. XII, n° 470).

Likewise, the quantity P, which decreases while always remaining greater than a fixed quantity (namely, any value of p), also tends to a limit.

These limits are equal. Indeed, the two polygons are always similar, and their perimeters are in the same ratio as their apothems. Now the apothem of the circumscribed polygon is equal to the radius R of the given circle, so that

$$\frac{P}{p} = \frac{R}{a},$$

where a denotes the apothem of the inscribed polygon. But as we keep doubling the number of sides indefinitely, the apothem a tends to R. Thus the limit of the ratio $\frac{P}{p}$, which is the ratio of the two limits referred to above (*Leçons d'Arithmétique* chap. XII, n°466) is equal to one. QED

177. We now claim that *the perimeter of any convex polygon, inscribed or circumscribed,*[5] *all of whose sides decrease indefinitely, tends to the limit of the preceding theorem.*

In particular, this limit is independent of the initial regular polygon.

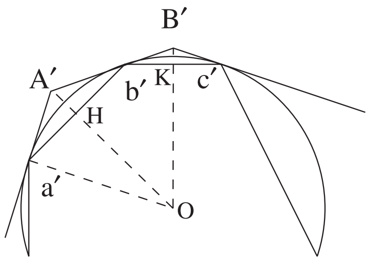


FIGURE 169

Indeed, consider an arbitrary inscribed polygon $a'b'c'\cdots$ (Fig. 169), and also the convex circumscribed polygon $A'B'\cdots$ formed by the tangents at $a', b', c'\ldots$. Let p' and P' be the perimeters of the two polygons. The limit L of the preceding theorem is contained between p' and P' because p', for instance, is smaller than any of the perimeters P considered above, which have limit L, so that $p' \leq L$. In the same way, we find that $P' \geq L$.

[5]It is understood that a circumscribed polygon includes the circle in its interior.

On the other hand, line OA' intersects $a'b'$ at its midpoint H, and

$$\frac{a'A' + A'b'}{a'b'} = \frac{a'A'}{a'H} = \frac{R}{OH},$$

since right triangles $Aa'H, oa'A'$ are similar, having a common acute angle. Likewise,

$$\frac{b'B' + B'c'}{b'c'} = \frac{R}{OK},$$

where K denotes the midpoint of $b'c'$, and so on.

Let us add the numerators and denominators of the left hand sides of these equalities. We obtain a ratio contained between the smallest and the greatest[6] of the original ratios; thus

$$\frac{P'}{p'} = f,$$

where f is between the smallest and largest of the quantities $\frac{R}{OH}, \frac{R}{OK}, \ldots$. Now if the polygon varies in such a way that all the sides decrease indefinitely, all the distances OH tend to R, and all the ratios $\frac{OH}{R}, \frac{OK}{R}$ tend to one. Therefore the same happens for $\frac{P'}{p'}$, and therefore for $\frac{L}{p'}$ and $\frac{P'}{L}$ which are both between 1 and $\frac{P'}{p'}$. Thus P' and p' tend to L. QED

Definition. The length L, the common limit of the inscribed and circumscribed convex polygons as the sides decrease indefinitely (a limit whose existence has just been proved), is called the *circumference* or *length of the circle.*

178. Theorem. *The ratios of the lengths of two circles is the ratio of their radii.*

Consider circles C, C_1 with radii R, R_1. In these circles, we inscribe regular polygons with the same number of sides. These polygons are similar, and therefore (**146**, Corollary) the ratio of their perimeters P, P_1 is the ratio of their radii. As the number of sides increases indefinitely, the ratio $\frac{P}{P_1}$ tends to the ratio $\frac{C}{C_1}$ of the circumferences of the circles. This proves the theorem.

Corollary. *The ratio of the length of a circle to the diameter is constant.*

Indeed, the proportion $\frac{C}{C_1} = \frac{R}{R_1}$ can be rewritten $\frac{C}{R} = \frac{C_1}{R_1}$ or $\frac{C}{2R} = \frac{C_1}{2R_1}$. Thus the ratio of the circumference to the diameter is the same for any two circles.

This constant number, the ratio of the circumference to the diameter, is denoted by the Greek letter π.

Corollary. *The circumference of a circle of radius R is $2\pi R$.*

179. Length of an arc of a circle.

Definition. *The length of an arc of a circle* is the limit of the perimeters of convex broken lines inscribed in the arc, ending at the endpoints of the arc, as all the sides decrease indefinitely.

[6] *Leçons d'Arithmétique*, chap. VI, n° 212.

The existence of this limit is proved by the same reasoning used for the whole circle. We first consider a regular inscribed or circumscribed broken line, and we double the number of sides indefinitely. Then we pass to an arbitrary broken line.[7]

Every arc of a circle is longer than its chord, since it is the limit of broken lines longer than this chord, whose perimeters increase.

For analogous reasons, *it is shorter than any broken line surrounding it, with the same endpoints.*[8]

The ratio of an arc of a circle to its chord tends to one as the arc tends to zero (if the circle remains fixed). This is true because the arc $\widehat{a'b'}$ (Fig. 169) is contained between its chord and the broken line $a'A'b'$ formed by the tangents at its endpoints: as we have seen (**177**) the ratio of these quantities goes to one when the arc $\widehat{a'b'}$ decreases indefinitely.

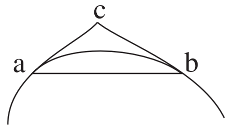
179b. Two congruent arcs have equal lengths, since the broken lines used in the definition of their lengths can be taken to be equal in pairs. The sum of two arcs has a length equal to the sum of the lengths of the two arcs, since two broken lines inscribed in the two arcs, taken together, constitute a broken line inscribed in the total arc.

Thus, according to a principle invoked several times before, the ratio of the lengths of two arcs of the same circle is equal to the ratio of the arcs (**17**). In particular, the ratio of these lengths is equal to the ratio of the measures of the arcs in degrees or gradients.

Since the circumference of a circle of radius R is $2\pi R$, and this circle is divided into 360 degrees, it follows that the length of each degree will be $\frac{2\pi R}{360} = \frac{\pi R}{180}$. The length of a minute will be equal to one sixtieth of this quantity, or $\frac{\pi R}{180\times 60}$, and a second $\frac{\pi R}{180\times 60\times 60}$. Thus, *on a circle with radius R, an arc of m degrees, n minutes,*

---

[7]This method of proof can be extended to any *convex* curve A; that is, to a curve such that none of its arcs intersects its chord. It suffices to show that to every number $m < 1$ (but as close to 1 as we want) we can associate a length ε such that for any arc $\widehat{ab}$ contained in A whose endpoints are closer to each other than ε, the ratio of the chord ab to the sum $ac + bc$ (see figure) of the tangents at a, b (bounded by their points of intersection) is contained between m and 1. This condition is satisfied (as is proved in differential calculus) under certain assumptions which are always satisfied in the case of the curves we consider. We are then assured (as in **177**), that the ratio of the perimeter of an inscribed broken line to the perimeter of the corresponding circumscribed line tends to one, as the number of sides increases and each of them tends to zero.



This can be used to prove the existence of a limit so long as the vertices of each broken line are included among the vertices of the next one (by the argument in **176**). Then we pass to the general case as in **177**.

Finally, a non-convex arc can often be broken into convex pieces, allowing us to define its length.

[8]These conclusions extend to any arc of a curve defined in accordance with the considerations of the preceding note. In particular, *the line segment is the shortest path from one point to another.* A convex arc *(see previous note)* is shorter than any line surrounding it (a broken line or a curve) with the same endpoints.

p seconds, has length

$$\frac{\pi R}{180}\left(m+\frac{n}{60}+\frac{p}{60\times 60}\right);$$

an arc of n *gradients will have length* $\frac{\pi R n}{200}$.

180. Thus we see that we can calculate the length of any arc of a given circle, if we know the number π.

Calculation of π. Method of perimeters. To calculate this number, or equivalently, to calculate the length of a circle of a given radius R, we have to calculate the perimeters of regular inscribed polygons whose number of sides doubles indefinitely. Each of these perimeters will furnish an estimate from below of the required length, and this approximation gets close as the number of sides increases. If, at the same time that we calculate the perimeter of the inscribed polygon, we can calculate the perimeter of the corresponding circumscribed polygon, we will obtain an estimate from above, and the difference of these two estimates will give an upper bound to the error committed by adopting one of them.

We have learned to calculate the sides of a number of regular inscribed polygons, for instance, that of a square. To deduce from this the sides of the polygons with 4×2, 4×2^2, …, 4×2^n sides, it will suffice to solve the following problem:

Problem. Knowing the side c of a regular polygon inscribed in a circle with given radius R, calculate the side of the polygon inscribed in the same circle, with twice the number of sides.

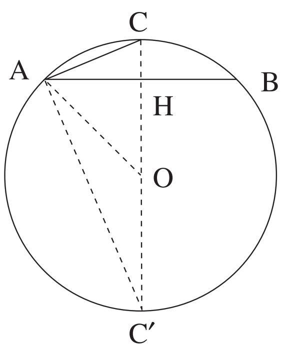


FIGURE 170

Let $AB = c$ (Fig. 170) be the side of the first polygon inscribed in a circle with radius $OA = R$. Dividing arc $\widehat{AB}$ into two equal parts with radius OC, perpendicular to the midpoint H of AB, we obtain side AC of the inscribed polygon with twice the number of sides. Now in triangle OAC we have:

$$AC^2 = OA^2 + OC^2 - 2OC\cdot OH = 2R^2 - 2R\cdot OH = 2R(R-OH),$$

where OH equals $\sqrt{R^2-\frac{c^2}{4}}$. Thus the required side c_1 will be

$$c_1=\sqrt{2R\left(R-\sqrt{R^2-\frac{c^2}{4}}\right)}.$$

Thus, since the side of the square is $R\sqrt{2}$ and its perimeter $R \times 4\sqrt{2}$, the side of the octagon and its perimeter will be

$$R\sqrt{2-\sqrt{2}}, \quad R \times 8\sqrt{2-\sqrt{2}}.$$

The side and perimeter of the regular inscribed polygon with sixteen sides will be

$$R\sqrt{2-\sqrt{2+\sqrt{2}}}, \quad R \times 16\sqrt{2-\sqrt{2+\sqrt{2}}},$$

and so on.

180b. In order to get an upper estimate for the length of the circle, we need to calculate the perimeter of a circumscribed regular polygon, and we must therefore solve the following problem:

Problem. Knowing the side c of a regular polygon inscribed in a circle of radius R, calculate the side of the regular polygon with the same number of sides circumscribed about the same circle.

The required side c' is found immediately by noting that the two polygons are similar, and the ratio of their sides is the ratio of their apothems. Since the apothem of the circumscribed polygon is R, and that of the inscribed polygon $\sqrt{R^2 - \frac{c^2}{4}}$, the required side is given by the proportion

$$\frac{c'}{c} = \frac{R}{\sqrt{R^2 - \frac{c^2}{4}}}.$$

We have just learned how to calculate the sides of the inscribed polygons with large numbers of sides: the preceding proportion gives us the sides of the corresponding circumscribed polygons.

181. Knowing the side c of a regular polygon inscribed in a circle of radius R, we can also calculate the side c_1 and the apothem a_1 of the regular polygon with twice the number of sides inscribed in the same circle (which gives us the side of the circumscribed polygon without an additional square root) by the following method.

In Fig. 170, let C' be the point where the extension of radius OC intersects the circle. Triangle ACC' is a right triangle and gives us (**123**, Remark)

$$AC \cdot AC' = CC' \cdot AH = 2R \times \frac{c}{2} = Rc.$$

On the other hand,

$$AC^2 + AC'^2 = CC'^2 = 4R^2.$$

Multiplying the first equation by 2 and adding it to the second, we get:

$$AC^2 + AC'^2 + 2AC \cdot AC' = (AC + AC')^2 = 4R^2 + 2Rc.$$

Thus

$$AC + AC' = \sqrt{4R^2 + 2Rc} = 2\sqrt{R\left(R + \frac{c}{2}\right)}.$$

If we multiply the first equation by 2, then subtract it from the second, we have:

$$AC^2 + AC'^2 - 2AC \cdot AC' = (AC' - AC)^2 = 4R^2 - 2Rc.$$

Thus

$$AC' - AC = \sqrt{4R^2 - 2Rc} = 2\sqrt{R\left(R - \frac{c}{2}\right)}.$$

Adding and subtracting these equalities member by member, these two equations give us

$$AC = \sqrt{R\left(R + \frac{c}{2}\right)} - \sqrt{R\left(R - \frac{c}{2}\right)}; \quad AC' = \sqrt{R\left(R + \frac{c}{2}\right)} - \sqrt{R\left(R - \frac{c}{2}\right)},$$

which solves the problem, since AC is equal to the required chord c_1 and AC' is twice the apothem a_1.

We have thus learned how to *calculate the chords of half of the two arcs intercepted by a given chord on a circle of given radius.*

182. The isoperimetric method. We can present the preceding method in a slightly different form.

In effect, the problem consist in calculating the ratio of the perimeter p of an inscribed regular polygon to the radius r of the circumscribed circle, or to the apothem a, which is the radius of the inscribed circle, and to do this for polygons whose number of sides increases indefinitely. But it makes no difference whether or not these polygons are inscribed in the same circle, since the ratios $\frac{p}{r}$ and $\frac{p}{a}$ depend only on the number of sides of the polygon (**163**).

In the *isoperimetric method* we consider regular polygons whose number of sides doubles indefinitely, and which have the same perimeter. We must therefore solve the following problem:

Problem. Knowing the radius r and apothem a of a regular polygon, calculate the radius r' and apothem a' of a regular polygon with twice as many sides, isoperimetric to the first.

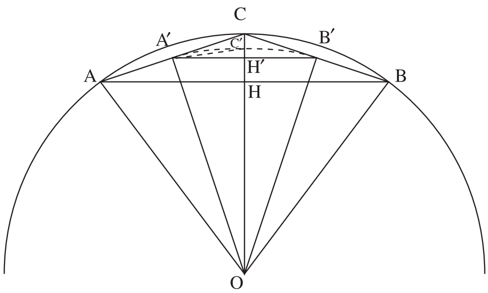


FIGURE 171

Let AB (Fig. 171) be a side of the regular polygon with n sides inscribed in a circle with radius $OA = r$. We drop perpendicular $OH = a$ to AB, and extend it to its intersection C with the circumscribed circle, which is also the midpoint of arc $\widehat{AB}$. We draw AC and BC, and let A', B' be the midpoints of these segments. Let $A'B'$ intersects OC in a point H'.

Segment $A'B'$ is the side of a regular polygon with $2n$ sides inscribed in the circle with radius OA'.

Indeed, since angle $\widehat{AOB}$ intercepts an arc equal to the n^{th} part of the circle with radius OA, angle $\widehat{A'OB'}$ which is half of it (because $\widehat{A'OC}$ is half of $\widehat{AOC}$ and

$\widehat{B'OC}$ half of $\widehat{BOC}$) will intercept an arc equal to the $2n^{th}$ part of the circle with radius OA'.

This polygon is isoperimetric to the first.

This is true because side $A'B'$ joins the midpoints of AC, BC, and is therefore half of the original side AB, and because the new polygon has twice as many sides.

The required lengths will thus be $r' = OA'$, $a' = OH'$.

Now H' is the midpoint of CH, so we have

$$OH' - OH = OC - OH'.$$

This can be written as

$$2OH' = OC + OH,$$

or

$$OH' = \frac{OC + OH}{2}.$$

In other words, OH' is the arithmetic mean of OC and OH.

On the other hand, right triangle $OA'C$ gives

$$OA' = \sqrt{OC \times OH'}.$$

That is, OA' is the geometric mean of OC and OH'.

Thus we can calculate a' by the formula

$$a' = \frac{r + a}{2},$$

and r' by the formula

$$r' = \sqrt{ra'}.$$

Repeating this operation of doubling, we obtain values of r and a for polygons with larger and larger numbers of sides. The ratios of these quantities to the common semi-perimeter of these polygons will give approximate values for $\frac{1}{\pi}$, one less than the limit, the other greater, and both growing closer.

Suppose, to simplify the situation, that the common perimeter of the polygons is equal to two units of length, and take a square as the initial polygon. The apothem is half the side, or $\frac{1}{4}$, and the radius is equal to the side divided by $\sqrt{2}$, or $\frac{1}{2\sqrt{2}}$. Taking $a = \frac{1}{4}$, $r = \frac{1}{2\sqrt{2}}$ in the preceding formulas will give the values of a', r' for the octagon, and so on.

But if we take[9] $a = 0$, $r = \frac{1}{2}$, our formulas yield precisely $a' = \frac{1}{4}, r' = \frac{1}{2\sqrt{2}}$. We then obtain the following proposition (a theorem of *Schwab*).

Theorem. *If we form a sequence of numbers starting with 0 and $\frac{1}{2}$, and each successive term is alternately the arithmetic mean or the geometric mean of the preceding two terms, then the numbers in this sequence tend to $\frac{1}{\pi}$.*

183. Since the numbers a and r, consecutive terms of the preceding sequence, approach the limit $\frac{1}{\pi}$ from below and from above respectively, the error committed in taking either for an approximate value of $\frac{1}{\pi}$ is less than $r - a$.

[9]These numbers correspond to the division of the circle into two equal parts. If the division points are A and B, diameter AB can be thought of as the side of a regular polygon with two sides, whose perimeter is $2AB$. If this perimeter is 2, the radius must be $\frac{1}{2}$ and the apothem (the distance from the side to the center) is zero.

Also we have

$$r' - a' < \frac{r-a}{4}.$$

To prove this, we revisit Figure 171 and lay off on OC a length $OC' = OA' = r'$. Segment $C'H'$ can represent $r'-a'$. The two angles $\widehat{C'A'H'}$, $\widehat{C'A'C}$ are equal, since on the circle OA' they are measured by one half the equal arcs $\widehat{C'B'}$, $\widehat{A'C'}$ (one is an inscribed angle, the other is formed by a tangent and a secant). The angle bisector theorem (see **115**) then shows that the ratio of $H'C'$ to $C'C$ is equal to the ratio of $H'A'$ to $A'C$. Thus $H'C'$ is less than half of $H'C$, which is equal to a quarter of CH; that is, to $\frac{r-a}{4}$.

The first two terms in the Schwab sequence give $r - a = \frac{1}{2}$. Thus, replacing $\frac{1}{\pi}$ by the $2n$th term in this sequence, we commit an error less than

$$\frac{1}{2 \times 4^n} = \frac{1}{2^{2n+1}}.$$

184. The Greek geometer *Archimedes*, who was the first to define and calculate the length of the circle, used the method of perimeters to find the value for π

$$\frac{22}{7}$$

which is greater than π by less than one hundredth.

Note that one can obtain only approximate values for π, and never the exact value, because it can be proved that this number is irrational; that is, it is not an integer or a ratio of two integers.

The first decimals of π and $\frac{1}{\pi}$ are[10]

$$\pi = 3.141592653\ldots, \qquad \frac{1}{\pi} = 0.3183098\ldots.$$

We often use the value 3.1416, which is slightly too large, for π.

The famous problem of the *quadrature of the circle* amounts (as we will see later) to constructing a segment equal to the length of a circle with given radius. It is impossible to solve this problem using only straightedge and compass. This impossibility, which does not necessarily follow[11] from the irrationality of π, has also been proved.[12]

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[10]These values have not been obtained by the methods just explained, but by incomparably faster ones borrowed from the differential calculus. One can easily remember these decimals, using the following verse: *"Que j'aime à faire apprendre un nombre utile aux sages"* in which the number of letters of the words give the consecutive digits of π. (This French mnemonic given above has various analogues in English; for example, *"May I have a large container of coffee?"* –transl.)

The value $\frac{355}{113}$, greater than π by less than one millionth, was given by Adrien Métius.

[11]For instance, the ratio of the diagonal of a square to its side is irrational, but one can easily construct the diagonal knowing the side.

[12]The irrationality of π was proved by Lambert (1770). The impossibility of the quadrature of the circle is due to Lindemann, who proved it in 1882, by generalizing results of the great French mathematician Hermite.

Exercises

Regular Polygons.

Exercise 178. Tile the plane with congruent regular polygons. Show that this can be done only with three kinds of regular polygons.

Exercise 179. Construct a regular pentagon knowing its side.

Exercise 180. In a regular pentagon, two diagonals with no common endpoint divide each other into mean and extreme ratio.

Exercise 181. The right triangle whose legs are the sides of the two regular decagons inscribed in a circle has a hypotenuse equal to the side of the equilateral triangle inscribed in the same circle.

Exercise 182. Same proposition for the convex decagon, the hexagon, and the convex pentagon. Also for the star decagon, the hexagon, and the star pentagon.

Exercise 183. On the sides of a regular hexagon, and outside the hexagon, we construct six squares. The exterior vertices of these squares are those of a regular dodecagon.

Exercise 184. Verify that the two expression for $AC = c_1$ obtained in **180** and **181** are indeed equal.

Length of the Circle.

Exercise 185. Find the radius of the circle on which an arc of $18°15'$ has a length of 2 meters.

Exercise 186. On radius OA of circle O as diameter, we draw a second circle. A radius OB of the first circle intersects the second in C. Show that the arcs $\overset{\frown}{AB}$, $\overset{\frown}{AC}$ have the same length.

Exercise 187. If two circles are tangent internally to a third, and the sum of the radii of the two circles is equal to the radius of the third, show that the arc of the third circle between the points of contact is equal to the sum of the two arcs on the smaller circles contained between the same points of contact, and their intersection which is closer to the greater circle.

Exercise 188. The sum of the sides of the square and equilateral triangle inscribed in the same circle give an approximate value for the length of the semicircle; the error is less than one hundredth of the radius.

Exercise 189. The perimeter of the right triangle whose right angle has sides equal to $\frac{3}{5}$ and $\frac{6}{5}$ of the diameter gives an approximate value for the circumference; the error is less than one ten-millionth of the radius.

Problems for Book III

Exercise 190. Given two concentric circles, find a line on which these circles intercept chords of which one is twice the other.

Exercise 191. On sides AB, AC of a triangle we take, in inverse senses,[13] two equal lengths $BD = CE$. The segment DE is divided by BC in a ratio inverse to that of the sides AB, AC.

Exercise 192. Let A, A' be the points of contact of a common tangent to two given circles, and let M, M' be two points of intersection with the circles of a line parallel to AA'. Find the locus of the intersection of AM, $A'M'$.

Exercise 193. The sides of a variable polygon remain parallel to fixed directions, while all its vertices, except one, slide along given lines. What is the locus of the remaining vertex? (Exercise 124.)

Exercise 194. In a given polygon, inscribe a polygon whose sides are parallel to given directions. Is the solution uniquely determined?

Exercise 195. Divide the altitudes of a triangle into given ratios and, through each division point, draw a parallel to the corresponding side. Find the ratio of similarity of the triangle formed to the original triangle.

Exercise 196. Let a, b, c be the points symmetric to an arbitrary point O in the plane, with respect to the midpoints of the sides BC, CA, AB of a triangle. The lines Aa, Bb, Cc are concurrent at some point P. If O varies along an arbitrary figure, show that point P describes a similar figure.

Exercise 197. Through the three vertices of a triangle, we draw three lines which are concurrent at O; we then consider the lines symmetric to each of these in the angle bisector from the vertex through which it passes. Show that these three lines are also concurrent at a point O'.

Show that this conclusion holds if the three original lines are parallel, rather than concurrent. Deduce from this statement that the altitudes of a triangle are concurrent.

Exercise 198. Rhombus $ABCD$ is circumscribed about a circle. Show that a variable tangent MN to the circle intercepts, on the adjacent sides AB, CB, segments AM, CN whose product is constant.

Exercise 199. Through a point A in the plane of a circle we draw a variable secant AMM'. The lines joining M and M' with an endpoint of the diameter passing through A intercept, on the perpendicular at A to this diameter, two segments whose product is constant.

Exercise 200. A common internal tangent to two circles divides a common external tangent internally (and the latter divides the former externally) into two segments whose product is equal to the product of the radii. The segment intercepted by the two common internal tangents on a common external tangent has the same midpoint as this common external tangent, and same length as one of the common internal tangents.

Exercise 201. A variable right triangle has a right angle at a fixed point A, while the other two vertices B, C move along a fixed circle with center O. Find: 1° the locus of the midpoint of BC; 2° the locus of the projection of A on BC.

[13]The author means here, for example, that point D is inside segment AB, while point E is on the extension of segment AC. See solution. – transl.

Exercise 202. Construct a triangle knowing a side, the corresponding altitude, and the product of the other two sides.

Exercise 203. Construct a triangle knowing two medians and an altitude (two cases).

Exercise 204. Inscribe an isosceles triangle in a given circle, knowing the sum or the difference of the base and altitude.

Exercise 205. Calculate the diagonals of a trapezoid, knowing the four sides.

Exercise 206. Given a circle and two points A, B, construct a chord $A'B'$ parallel to AB such that the lines AA', BB' intersect on the circle.

Exercise 207. Through the endpoints A, B of a diameter of a circle, we draw two chords AC, BD intersecting at point P inside the circle. Show that $AB^2 = AC \cdot AP + BD \cdot BP$.

Exercise 208. Through two fixed points A, B we draw a variable circle, and through a fixed point C on line AB we draw tangents to this circle. Find the locus of the midpoints of the chords joining their points of contact.

Exercise 209. Given three points as centers, draw three pairwise orthogonal circles.

Exercise 210. Given a circle, two points A, B on the circle, and an arbitrary line, find a point M on the circle such that the lines MA, MB intercept, on the given line, a segment divided into a given ratio by a given point on the line.

Exercise 211. Using sides AB, AC, BC of triangle ABC as bases, construct similar isosceles triangles ABP, ACQ, BCR: the first two outside the triangle, and the third on the same side of BC as point A (or inversely). Show that $APQR$ is a parallelogram.

Exercise 212. A figure varies while remaining similar to a fixed figure, in such a way that three non-concurrent lines in the figure each pass through a fixed point. Prove: 1° that every other line of the figure also passes through a fixed point; 2° that every point of the figure moves along a circle.

Exercise 213. Construct a quadrilateral similar to a given quadrilateral, and with sides passing through four given points. Can the problem be undetermined? If so, find the locus of the intersections of the diagonals of the quadrilaterals which satisfy the conditions of the problem.

Exercise 214. A figure varies while remaining similar to a fixed figure, in such a way that three points of the figure each move along a straight line. Show that one point of the figure stays in a fixed position. Deduce that all its other points move along straight lines as well.

Exercise 215. Two figures are similar and have the same orientation. Find the locus of the points in the first figure with the property that the line joining them to their corresponding points in the second figure passes through a given point.

Exercise 215b. We are given two lines, two points A, B on these lines and a point O. Through point O, draw a line which intersects the given line in points M, N such that the ratio of AM to BN is equal to a given number.

Exercise 216. Through a point inside a circle, we draw two perpendicular chords, which determine on the circle the four vertices of a quadrilateral. Show that the sum of the squares of two opposite sides of this quadrilateral is equal to the square of the diameter.

Complements to Book III

CHAPTER I

Directed Segments

185. Up to this point, we have been using only positive numbers to measure line segments. It is useful, when considering segments on the same line, to associate numbers with them which are sometimes positive and sometimes negative, using a convention which we now introduce.[1]

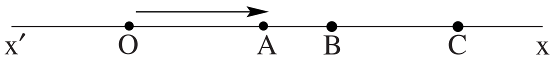

FIGURE 172

On line $X'X$ (Fig. 172) we choose, once and for all, a certain direction, called the *positive direction*; for instance, the direction $X'X$ indicated in Figure 172 by an arrow. *To any segment AB we associate a number which is equal to its length preceded by a + sign if this segment is in the positive direction, and by a − sign if it is in the opposite direction.*

It should be noted that the sign of a segment depends essentially on the order in which we write its endpoints; with this convention we have

$$AB = -BA.$$

186. The preceding convention allows us to write certain relations in a form which is independent of the position of the points under consideration.

For example, if A, B, C are three points on the given line, BC is equal to the sum or difference of AB and AC, depending on the order of the three points. However, when we take signs into account, these various relations are replaced by the single equality

$$AB + BC + CA = 0, \tag{1}$$

which holds regardless of the order of the points.

Indeed if, following the line in the positive direction, we meet the points in the order A, B, C (as in Figure 172), the segments AB, AC, BC are positive and we have[2] $AC = AB + BC$, which is equivalent to (1).

[1]Here in **185–187** we review and summarize the convention for signed line segments. This convention is studied and described in a more complete fashion in *Leçons d'Algèbre Élémentaire* by Bourlet (*Librarie Armand Colin*; introduction and chapters I and II). The reader will find there the demonstrations of certain propositions which we have here assumed, such as *The algebraic sum of several segments is independent of their order; this sum does not change if we replace one or several of the segments by the result of their algebraic sum,* etc.; as well as the conventions which allow us to treat signed numbers, in computation, like ordinary numbers.

[2]The segment AC is called the *resultant* of segments AB, BC (cf. Bourlet, *Leçons d'Algèbre*, introduction).

Now this relation does not change if we permute two of the points; for instance, exchanging B and C, we have the equality

$$AC + CB + BA = 0,$$

which is equivalent to the preceding one, so that (1) is true when the points are in the order A, C, B.

Now we know[3] that by successive permutation of two terms, we can change the order in any way we want. Therefore relation (1) is always true.

This relation extends immediately to any number of points on a line. For instance, for five points A, B, C, D, E on the same line $X'X$ we have

$$AB + BC + CD + DE + EA = 0.$$

To prove this, we observe that the relation is already proved for three points. It suffices to show that, if the relation is established for a certain number of points, it will also be true for the next higher number of points. Thus we can assume that the proposition has been established for four points, and write

$$AB + BC + CD + DA = 0.$$

On the other hand, the three points A, D, E yield

$$AD + DE + EA = 0.$$

Adding, the two terms $DA+AD$ cancel each other out, and we get the equality as claimed.[4]

187. Let us take a point O on line $X'X$ (Fig. 172). A segment OA, given with both magnitude and sign, then determines a point A on this line. This segment is called the *abscissa* of the point A with respect to the origin O. Thus, knowledge of the magnitude and sign of the abscissa of a point determines without ambiguity the position of that point on the line.[5]

Given two points A, B with known abscissas with respect to the same origin, the distance AB is given by

$$AB = OB - OA,$$

which is equivalent to equation (1) of the preceding section.

188. Given a point C on line AB, the ratio $\frac{CA}{CB}$ is negative if C is between A and B, and positive if C is outside the segment AB.[6]

Thus, there is only one point which divides a given segment in a given ratio, both in magnitude and sign. If points C and D are harmonic conjugates with respect to the segment AB, the ratios $\frac{CA}{CB}$ and $\frac{DA}{DB}$ have the same magnitude and opposite signs.

189. Let O be the midpoint of segment CD (Fig. 173), whose endpoints are harmonic conjugates with respect to the segment AB. We have

$$OC^2 = OA \cdot OB.$$

[3]Tannery, *Leçons d'Arithmètique*, ch. II, n°77.

[4]Cf. Bourlet, *Leçons d'Algèbre*, ch. II, n°23.

[5]Cf. Bourlet, *Leçons d'Algèbre*, ch. I, n°23.

[6]Cf. Bourlet, *Leçons d'Algèbre*, ch. I, n°16.

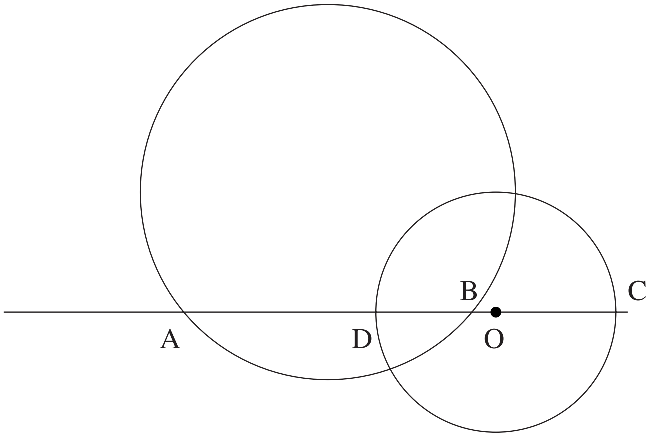

FIGURE 173

Indeed, we can take points A, B, C, D as determined by their abscissas with respect to the point O. The equality $\frac{CA}{CB} = -\frac{DA}{DB}$ becomes

$$-\frac{CA}{CB} = \frac{DA}{DB} = \frac{OC - OA}{OB - OC} = \frac{OA - OD}{OB - OD}.$$

If we add the two last numerators and also the two last denominators, and also take the difference of these same terms, we will preserve these ratios. Observing that $OD = -OC$, we obtain

$$-\frac{CA}{CB} = \frac{DA}{DB} = \frac{2OC}{2OB} = \frac{2OA}{2OC}.$$

Thus we have the following theorem:

THEOREM. *Half of a line segment is the geometric mean of the distances from the midpoint of the segment to any two points which divide it harmonically.*

In addition, we see that the common value of the ratios $\frac{OC}{OB}$ and $\frac{OA}{OC}$ is also equal to the common value of the ratios $-\frac{CA}{CB}$ and $\frac{DA}{DB}$.

Converse. *If we take lengths OA, OB, on the same side of the midpoint O of a segment CD, whose geometric mean is half the segment, then the points A and B divide the given segment CD harmonically.*

This is true because the equation $\frac{OC}{OB} = \frac{OA}{OC}$ yields, by addition and subtraction of numerators and denominators, the equal ratios

$$\frac{OA - OC}{OC - OB} = \frac{OA + OC}{OB + OC},$$

and, since $OD = -OC$,

$$-\frac{CA}{CB} = \frac{DA}{DB}.$$

COROLLARY. *If two segments are harmonic conjugates, the circle whose diameter is the first segment cuts any circle passing through the endpoints of the other at right angles* (Fig. 173).

This is true because the square of the radius of the first circle is equal to the power of its center with respect to the second circle.

Conversely, *when two circles intersect at right angles, any diameter of one of them is divided harmonically by the other.*

190. When two lines are parallel, we will generally adopt the same positive direction on both of them.

If we do this, we can say that *the ratio of the segments BC, DE intercepted by the same angle on two parallel lines is equal, in magnitude and sign, to the ratio of the segments AB, AD which these parallel lines intercept on one of the sides of the angle* (Figures 115, 116, 117 in **117**).

This equality has been proved in **117** for the absolute values of the ratios. It remains to observe that the two ratios have the same sign: the segments BC, BD have the same direction (Figures 115, 116) or opposite directions (Fig. 117) as the segments AB, AD.

In particular, if we consider two homothetic figures, it is clearly appropriate to consider the ratio of similarity as positive or negative, according as whether the homothecy is direct or inverse. From now on, *the ratio of similarity is equal, in magnitude and sign, to the ratio of any two homologous lines.*

191. The convention made in **133**, pertaining to the power of a point with respect to a circle, is just a particular case of the convention made in this chapter.

Indeed, we can choose one direction to be positive on secant ABB' (Figures 128 and 129 in **131**) from point A to circle O. The segments AB, AB' have the same direction or opposite directions according as whether A is exterior or interior to the circle; their product is thus positive in the first case, and negative in the second.[7]

Exercises

Exercise 217. When four points A, B, C, D divide each other harmonically, we have (in magnitude and sign) $\frac{2}{AB} = \frac{1}{AC} + \frac{1}{AD}$.

Exercise 218. How can we modify the statement of Stewart's theorem (**127**) so that it is independent of the order of the points B, C, D on the line which contains them? Prove the modified statement directly when the point A belongs to the line BCD. Deduce that the statement always holds.

Exercise 219. If, in the theorem of **116**, we vary the given ratio but keep the points A, B fixed, the different circles thus found have the same radical axis. Their limit points[8] are the points A and B.

Deduce from this a solution to the following problem: find, on a given line or circle, the position that a point M must occupy so that the ratio of its distances to two given points is as large or as small as possible. (Constructions analogous to those in **159**.)

Exercise 219b. Find a segment which divides two given segments harmonically (the solution of Exercise 153 allows us to construct the midpoint of the required segment).[9] Is a solution always possible?

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[7] Cf. Bourlet, *Leçons d'Arithmètique*, ch. 1, n°14.

[8] See Exercise 152. –transl.

[9] See the comment about the interpretation of this problem in the solution. –transl.

Exercise 220. The circle with its diameter at the endpoints of the two centers of similarity of two given circles has the same radical axis as these circles (**138**).

Exercise 221. What happens to Exercise 130 when E is taken on an extension of AC? Same question for Exercise 131, when the line crosses the triangle.

Exercise 222. Through the vertices A, B, C, D of a square (labeled in their natural order) we drop perpendiculars Aa, Bb, Cc, Dd onto a line in the same plane, outside the square. Show that under these conditions, the quantity $Aa^2+Cc^2-2Bb\cdot Dd$ is independent of the line. (We transform this quantity by introducing the sums $Aa+Cc$, $Bb+Dd$ and the differences $Aa-Cc$, $Bb-Dd$, the first two of which are equal by Exercise 130.) When the line intersects the square, the statement remains true if we take into account the signs.

CHAPTER II

Transversal

192. THEOREM. *If sides BC, CA, AB of a triangle are intersected by a line at a, b, c* (Fig. 174) *we have the following relation among the segments determined on the sides*:

$$\frac{aB}{aC} \cdot \frac{bC}{bA} \cdot \frac{cA}{cB} = 1. \tag{2}$$

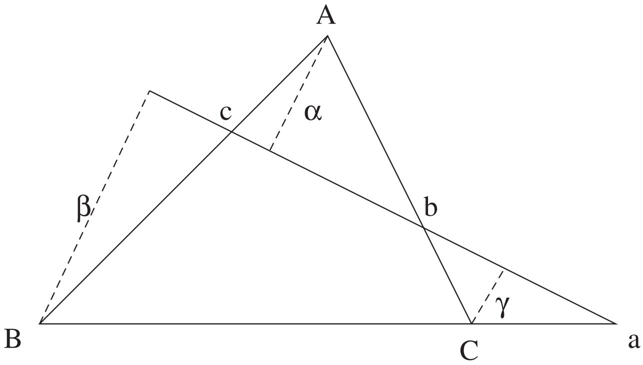

FIGURE 174

To prove this, we draw three segments parallel to an arbitrary direction through the vertices of the triangle , on which we choose the same positive sense, and extend them to intersect the transversal. Let α, β, γ be the distances from the vertices to the transversal, measured along these parallel lines. We have, in magnitude and sign,

$$\frac{aB}{aC} = \frac{\beta}{\gamma},$$
$$\frac{bC}{bA} = \frac{\gamma}{\alpha},$$
$$\frac{cA}{cB} = \frac{\alpha}{\beta}.$$

Multiplying, we obtain $\frac{aB}{aC} \cdot \frac{bC}{bA} \cdot \frac{cA}{cB} = \frac{\beta\gamma\alpha}{\gamma\alpha\beta} = 1.$[1] QED

[1] If the transversal is parallel to BC, the point a should be considered as thrown to infinity, and the ratio $\frac{aB}{aC}$ taken equal to one. The relation in the statement then reduces to $\frac{bA}{bC} = \frac{cA}{cB}$; that is, to the theorem of **114**. If sides AB, AC of the triangle are parallel, and point A is at

193. Converse. Theorem of Menelaus.[2] *If we choose three points a, b, c on sides BA, CA, AB of triangle ABC which satisfy the the relation*

$$\frac{aB}{aC} \cdot \frac{bC}{bA} \cdot \frac{cA}{cB} = 1, \tag{2}$$

then these three points lie on a straight line.

This is true because line ab intersects side AB in a point c' such that

$$\frac{aB}{aC} \cdot \frac{bC}{bA} \cdot \frac{c'A}{c'B} = 1.$$

Comparing this relation with relation (2), we see that

$$\frac{cA}{cB} = \frac{c'A}{c'B},$$

and therefore the points c and c' coincide. QED

Remark. This theorem is actually equivalent to the one proved in **144**. Indeed, we can find three lengths α, β, γ such that we have (in magnitude and sign):

$$\frac{aB}{aC} = \frac{\beta}{\gamma}, \quad \frac{bC}{bA} = \frac{\gamma}{\alpha},$$

which, together with (2), imply

$$\frac{cA}{cB} = \frac{\alpha}{\beta}.$$

Therefore, if we draw three homothetic figures in which A, B, C are three corresponding points, and α, β, γ three corresponding segments, then their centers of similarity are a, b, c.

194. This theorem is useful in showing that three points are collinear.

Example I. *The midpoints of the three diagonals of a complete quadrilateral* (see **145**) *lie on a straight line.*

Consider the complete quadrilateral $ABCDEF$ (Fig. 175), whose diagonals AB, CD, EF have midpoints L, M, N. Consider the triangle ACE formed by three sides of the quadrilateral, and let a, c, e be the midpoints of CE, EA, AC. Line ce is parallel to CE and passes through point L; line ea is parallel to EA and passes through M; line ac is parallel to AC and passes through N. To show that L, M, N are on a straight line it suffices to prove the relation

$$\frac{Lc}{Le} \cdot \frac{Me}{Ma} \cdot \frac{Na}{Nc} = 1.$$

Parallel lines Lec, BCE give

$$\frac{Lc}{Le} = \frac{BE}{BC},$$

and likewise we obtain

$$\frac{Me}{Ma} = \frac{DA}{DE}, \quad \frac{Na}{Nc} = \frac{FC}{FA}.$$

---

infinity, the expression $\frac{aB}{aC} \cdot \frac{bC}{bA} \cdot \frac{cA}{cB}$ can be rewritten in the form

$$\frac{aB}{aC} \cdot \frac{bC}{cB} \cdot \frac{cA}{bA},$$

and we can replace $\frac{cA}{bA}$ by 1. We then obtain the statement in **117**.

[2]The name 'Theorem of Menelaus' in more recent sources refers variously to this theorem, to its converse in **192**, or to both statements taken together.–transl.

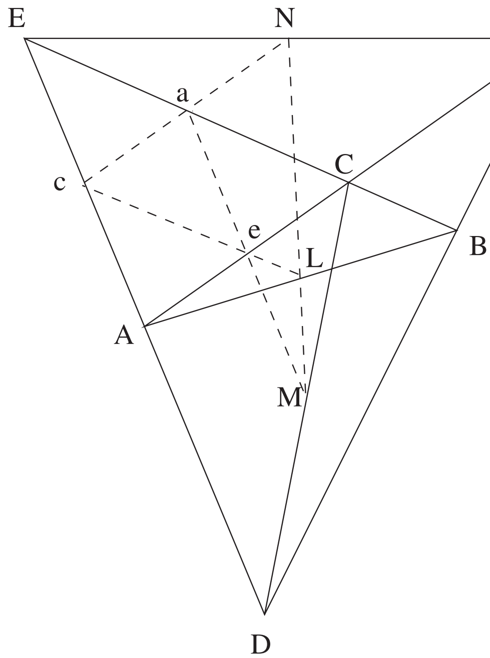

FIGURE 175

The product of the right hand sides

$$\frac{BE}{BC} \cdot \frac{DA}{DE} \cdot \frac{FC}{FA}$$

is indeed 1 since points B, D, F, taken on the sides of triangle ACE, lie on a straight line.

195. EXAMPLE II. *If the vertices of triangles abc, $a'b'c'$ are situated so that the lines aa', bb', cc', joining corresponding vertices, intersect in a point o, then the intersection points of the corresponding sides are on a straight line.*

Consider the intersections ℓ of bc, $b'c'$, m of ca, $c'a'$, and n of ab, $a'b'$ (Fig. 176). To show that these points are on a straight line, we must verify that

$$\frac{\ell b}{\ell c} \cdot \frac{mc}{ma} \cdot \frac{na}{nb} = 1.$$

Now triangle abc is cut by transversal $\ell b'c'$, giving us

$$\frac{\ell b}{\ell c} \cdot \frac{c'c}{c'o} \cdot \frac{b'o}{b'b} = 1.$$

Likewise, triangles oca, oab cut, respectively, by $mc'a'$, $na'b'$ give

$$\frac{ma}{mc} \cdot \frac{a'a}{a'o} \cdot \frac{c'o}{c'c} = 1, \qquad \frac{na}{nb} \cdot \frac{b'b}{b'o} \cdot \frac{a'o}{a'a} = 1.$$

Multiplying these equalities, the factors $a'a$, $b'b$, $c'c$, $a'o$, $b'o$, $c'o$ cancel out, and we obtain the required relation.

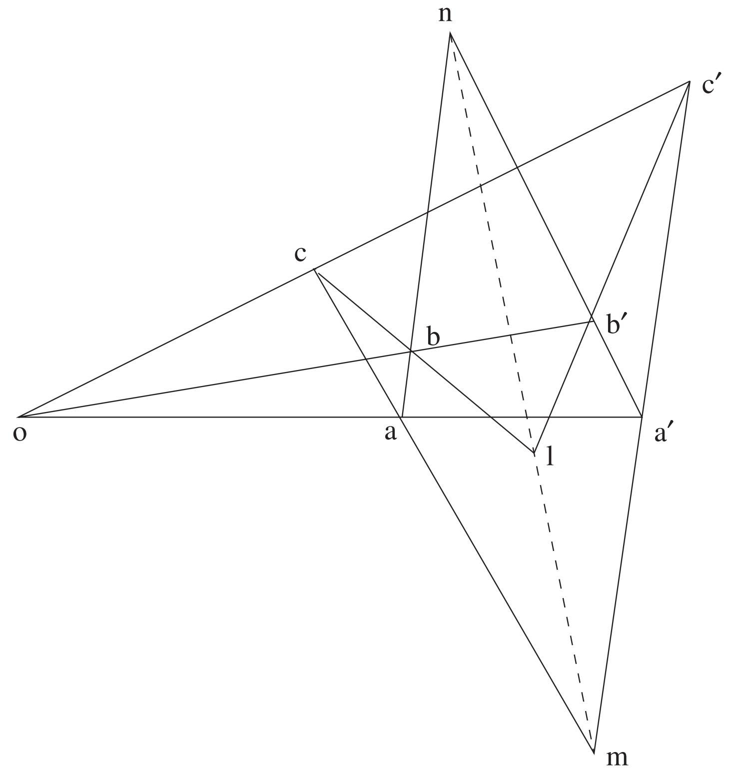

Figure 176

Two triangles, situated so that the lines joining the corresponding vertices are concurrent, are said to be *homologic*.[3]

196. Example III (Pascal's Theorem). *In a hexagon inscribed in a circle, the intersection points of opposite sides are on a straight line.*

Suppose the inscribed hexagon is $ABCDEF$ (Fig. 177), and that opposite sides AB, DE intersect in L, BC, EF meet in M, and CD, FA meet in N. Points L, M, N are on sides JK, KI, IJ of the triangle IJK formed by the lines AB, CD, EF. They will be on a straight line if

$$\frac{LJ}{LK} \cdot \frac{MK}{MI} \cdot \frac{NI}{NJ} = 1. \tag{3}$$

Cutting the triangle IJK by the remaining sides DE, BC, FA of the hexagon, we obtain

$$\frac{LJ}{LK} \cdot \frac{EK}{EI} \cdot \frac{DI}{DJ} = 1, \quad \frac{MK}{MI} \cdot \frac{CI}{CJ} \cdot \frac{BJ}{BK} = 1, \quad \frac{NI}{NJ} \cdot \frac{AJ}{AK} \cdot \frac{FK}{FI} = 1.$$

We now multiply these relations, grouping the factors in the numerator and denominator appropriately, to obtain

$$\frac{LJ}{LK} \cdot \frac{MK}{MI} \cdot \frac{NI}{NJ} \cdot \frac{CI \cdot DI}{EI \cdot FI} \cdot \frac{AJ \cdot BJ}{CJ \cdot DJ} \cdot \frac{EK \cdot FK}{AK \cdot BK} = 1.$$

[3]The more modern terminology for such triangles is *perspective*. – transl.

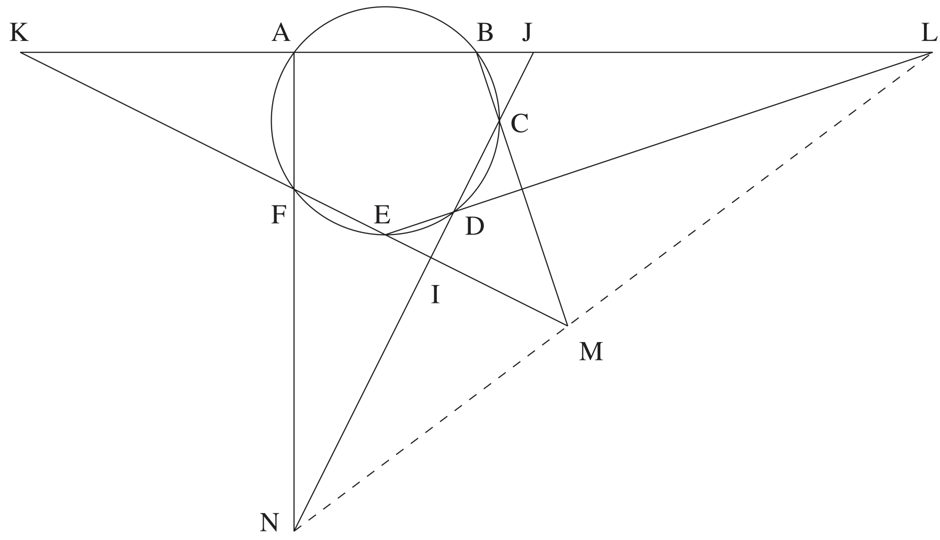

FIGURE 177

Each of the last three fractions is equal to one; for instance, the products $CI \cdot DI$ and $EI \cdot FI$ are both equal to the power of I with respect to the circle.

We conclude that (3) is true, and the theorem is proved.

REMARK. The preceding proof remains valid if the pairs of points A, B; C, D; E, F are made to coincide, so that the sides of the triangle IJK become tangents to the circle. The theorem becomes the following statement: *The tangents to the circumscribed circle at the vertices of a triangle intersect the corresponding sides in three collinear points.*

197. THEOREM. *Through the vertices A, B, C of a triangle, we draw segments Aa, Bb, Cc concurrent at a point O, where a, b, c are on the corresponding sides BC, CA, AB. Then*:

$$\frac{aB}{aC} \cdot \frac{bC}{bA} \cdot \frac{cA}{cB} = -1. \tag{4}$$

Indeed, triangle AaC (Fig. 178), cut by transversal BOb gives us[4]

$$\frac{Ba}{BC} \cdot \frac{bC}{bA} \cdot \frac{OA}{Oa} = 1;$$

triangle AaB, cut by transversal Cc gives

$$\frac{Ca}{CB} \cdot \frac{cB}{cA} \cdot \frac{OA}{Oa} = 1.$$

We divide these two equalities; the ratio $\frac{OA}{Oa}$ cancels out, and we obtain

$$\frac{aB}{aC} \cdot \frac{bC}{bA} \cdot \frac{cA}{cB} \cdot \frac{CB}{BC} = 1,$$

[4]This reasoning remains correct when point O is thrown to infinity; that is, when lines Aa, Bb, Cc are parallel. Indeed, we have seen that the theorem in **192** is true when a transversal is parallel to one of the sides of the triangle.

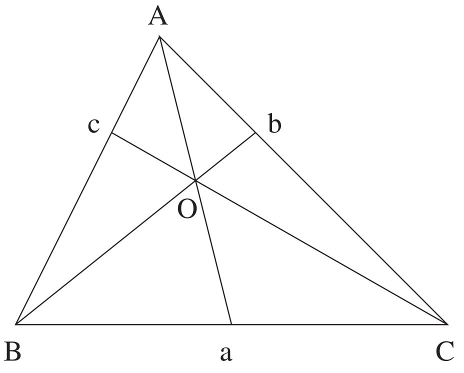

FIGURE 178

which is equivalent to relation (4), because $CB = -BC$.

198. Converse (Theorem of Ceva[5]). *If we take points a, b, c on the sides BC, CA, AB of a triangle, such that*

$$\frac{aB}{aC} \cdot \frac{bC}{bA} \cdot \frac{cA}{cB} = -1,$$

then the lines Aa, Bb, Cc are concurrent.

Indeed, consider the intersection O of Aa and Bb. The line CO intersects AB in a point c' such that

$$\frac{aB}{aC} \cdot \frac{bC}{bA} \cdot \frac{c'A}{c'B} = -1.$$

This, combined with the hypothesis, gives $\frac{cA}{cB} = \frac{c'A}{c'B}$, so that the points c, c' coincide. QED

If the lines Aa, Bb were parallel, the line Cc would be parallel to them, and the three lines must be considered as concurrent at infinity.

This result is useful in proving that three lines are concurrent.

EXAMPLE. The medians of a triangle are concurrent.

This is true because, if a, b, c denote the midpoints of the three sides of the triangle, each of the ratios $\frac{aB}{aC}, \frac{bC}{bA}, \frac{cA}{cB}$ is equal to -1.

One can prove analogously that the angle bisectors of a triangle are concurrent, etc.

Exercises

Exercise 223. Deduce the theorem of Menelaus, in the case for which the division points are external, from the fact that the three circles which determine the required point in Exercise 127 have the same intersection (see **189**).

[5]The term "Ceva's Theorem" in recent sources, can refer to this theorem, to its converse, or to both statements taken together. See the footnote to **193**. – transl.

Exercise 224. A line intersects the sides of a triangle ABC in three points a, b, c. Take the point symmetric to each of these points with respect to the midpoint of the side which contains it. Show that the points a', b', c' obtained in this way are also collinear.

If the points a, b, c are the projections of a point on the circumscribed circle onto the three sides (Exercise 72), the same is true for a', b', c'.

Exercise 225. One side OAB of an angle is fixed, as well as the two points A, B on it. The other side, $OA'B'$, turns around O, carrying with it two points A', B' at fixed distances from O. What is the locus of the intersection of AA' and BB'?

Exercise 226. What does Ceva's theorem (**198**) give us when applied to the altitudes of a triangle?

Exercise 227. Three concurrent lines through the vertices of triangle ABC intersect the opposite sides in a, b, c. We take points a', b', c' symmetric to these points with respect to the midpoints of the sides which contain them. Show that Aa', Bb', Cc' are also concurrent.

Exercise 228. The lines joining the vertices of a triangle to the points of contact of the opposite sides with the inscribed circle are concurrent.

Exercise 229. Consider triangle $A'B'C'$ obtained (as in **53**) from triangle ABC by drawing, through each vertex, a parallel to the opposite side. Let a, b, c be points on BC, CA, AB, respectively.

If Aa, Bb, Cc are concurrent, so are $A'a$, $B'b$, $C'c$.

Exercise 230. Suppose lines Aa, Bb, Cc are concurrent. The harmonic conjugates of a, b, c with respect to the sides BC, CA, AB, on which they are situated, are on a straight line.

Application: Aa, Bb, Cc are the angle bisectors.

Exercise 231. If lines Aa, Bb, Cc, bounded by the sides BC, CA, AB of triangle ABC are concurrent, and if circle abc intersects the sides again at a', b', c', then lines Aa', Bb', Cc' are also concurrent.

CHAPTER III

Cross Ratio. Harmonic Concurrent Lines

199. Let A, B, C, D be four collinear points. The cross ratio, or anharmonic ratio, of these four points is the quotient of the ratio of the distances of C to A and to B and the ratio of the distances of D to the same points, or $\frac{CA}{CB} : \frac{DA}{DB}$, where all these segments are considered both in magnitude and in sign. We denote the cross ratio by the symbol $(ABCD)$.

Remark. This expression can also be written $\frac{CA \cdot DB}{CB \cdot DA}$ and is therefore the ratio of two of the three products we can form with segments joining the four points, and without a common endpoint. The cross ratio depends on the order in which the four points are listed, but it is easy to see that *it does not change when we interchange two of the points, provided that we also interchange the other two.*

On the other hand, if only the points A and B are interchanged, or only the points C, D, then the cross ratio $r = (ABCD)$ is replaced by its inverse. The cross ratio remains the same only when $r = \frac{1}{r}$; that is, when $r^2 = 1$ or $r = \pm 1$.

The case $r = 1$ gives $\frac{CA}{CB} = \frac{DA}{DB}$, which cannot happen (**188**) if the four points are distinct.

The case $r = -1$ occurs if and only if $ABCD$ form a harmonic division (this condition is both necessary and sufficient); the ratios $\frac{CA}{DB}$ and $\frac{DA}{DB}$ have the same magnitude but opposite signs. In this case D is the harmonic conjugate of C with respect to AB, and vice-versa.

If the point D is thrown to infinity, the ratio $\frac{DA}{DB}$ tends to 1. Thus the cross ratio of the points[1] A, B, C, ∞ is equal to $\frac{CA}{CB}$.

200. Theorem. *Four lines passing through the same point determine on any transversal four points whose cross ratio is independent of the transversal.*

Suppose the four lines are OAA', OBB', OCC', ODD' (Fig. 179). If they are intersected by two transversals $ABCD$, $A'B'C'D'$, we claim that

$$\frac{CA}{CB} : \frac{DA}{DB} = \frac{C'A'}{C'B'} : \frac{D'A'}{D'B'}.$$

To prove this, we draw lines Bcd, $B'c'd'$, through the points B, B', both parallel to OA, the first intersecting OC in c and OD in d, the second intersecting OC in c' and OD in d'. We have

$$\frac{CA}{CB} = \frac{OA}{cB}, \quad \frac{DA}{DB} = \frac{OA}{dB},$$

and dividing these relation

$$\frac{CA}{CB} : \frac{DA}{DB} = \frac{dB}{cB}.$$

[1] As in algebra, the symbol ∞ represents infinity.

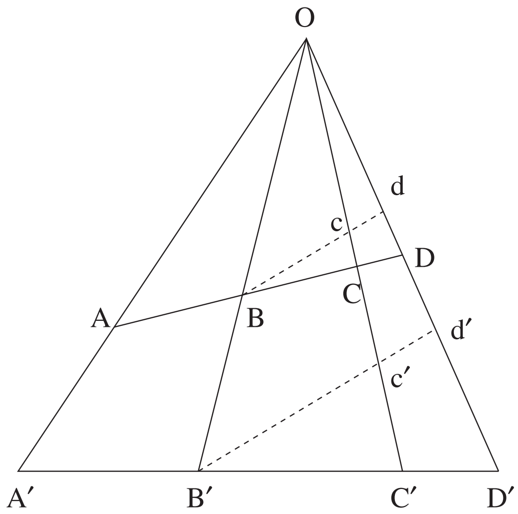

FIGURE 179

Analogously, we have

$$\frac{C'A'}{C'B'} : \frac{D'A'}{D'B'} = \frac{d'B'}{c'B'}.$$

The theorem then follows because the ratios $\frac{dB}{cB}$ and $\frac{d'B'}{c'B'}$ are equal, as the transversals $Bcd, B'c'd'$ are parallel.

The constant cross ratio determined in the preceding theorem is called *the cross ratio of the four concurrent lines.*

The operation by which we pass from the points A, B, C, D to the points A', B', C', D' is among the operations called *perspectives* or *central projections.*[2]

The length of a segment, AB for instance, is changed by such an operation; even the ratios of such segments are changed, but we see that the cross ratio of four points is not changed: *the cross ratio is a projective property.*

201. In particular, if the points C, D divide the segment AB harmonically, and we join the four points A, B, C, D to an arbitrary point O, we will obtain four lines which will divide any transversal harmonically. Such a collection of four lines is said to be a *harmonic pencil*; the rays OC, OD are *harmonic conjugates* with respect to OA, OB, and vice-versa.

THEOREM. *In order for four concurrent lines to form a harmonic pencil, it is necessary and sufficient that a parallel to one of them be divided into equal parts by the other three.*

This is true because, in the figure of the preceding section, we know that the cross ratio is equal to $\frac{dB}{cB}$, which is equal to -1 if and only if B is the midpoint of cd.

COROLLARY 1. *Lines OM_1, OM_2,* (see **157** and Fig. 159) *which make up the locus of points whose distances to two given lines D, D' have a given ratio, are harmonic conjugates with respect to the given lines.*

[2]These operations are defined in space geometry.

Indeed, with the notation in Figure 159, the segment M_1M_2 is clearly divided into equal parts by the line D.

Conversely, if OC, OD are conjugate with respect to OA, OB, the ratio of the distances from an arbitrary point on OC to OA, OB is equal to the ratio of the distances of an arbitrary point on OD to these same lines.[3]

COROLLARY 2. *The bisectors of the angles formed by two lines are harmonic conjugates with respect to the two given lines. Conversely, if two lines are conjugate with respect to two other lines, and they form a right angle, then they are the bisectors of the angles formed by the other lines.*

Indeed, if OA, OB, OC, OD form a harmonic pencil, then the parallel Bcd to OA is divided into equal parts by B. If OA, OB are perpendicular, then OB is the perpendicular bisector of cd. Hence it bisects angle $\widehat{cOd}$.

202. THEOREM. *Each diagonal of a complete quadrilateral is divided harmonically by the other two.*

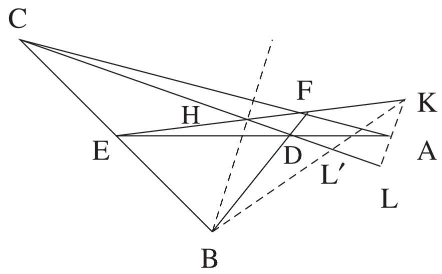

FIGURE 180

Consider the complete quadrilateral $ABCDEF$ (Fig. 180) formed by lines ADE, AFC, BDF, BEC, no three of which are concurrent. The diagonals CD, EF intersect at a point H. Let K be a point on line EF such that the cross ratio $(EFHK)$ is some number r.

Using A as the center, we project these four points onto diagonal CD, and denote by L the intersection of AK with CD, so that $r = (DCHL)$.

Let us project again onto CD, using B as the center. If BK intersects CD in L' we will have

$$r = (CDHL') \quad \text{or (see } \mathbf{199}\text{)} \quad (DCHL') = \frac{1}{r}.$$

In particular, if K is taken to be the intersection of CD with the third diagonal AB, the point L will coincide with L', so that we have $r = \pm 1$. We cannot have $r = 1$ because no three of the lines are concurrent, so we must have $r = -1$.

203. Given an angle, and a point in the plane of the angle, we can draw a variable transversal passing through the point. Consider the harmonic conjugate of the given point with respect to the segment intercepted by the angle; its locus will

[3] This conclusion must be modified if one takes signs into account (having chosen a positive direction on the perpendicular to OA and a positive direction on the perpendicular to OB): in this case the two ratios have the same magnitude, but opposite signs.

obviously be a line; namely, the harmonic conjugate of the line joining the vertex of the angle to the given point.

This line is called the *polar* of the point with respect to the angle.

THEOREM. *Through a point A in the plane of an angle $\widehat{CBD}$* (Fig. 180)*, we draw two transversals ADE, AFC. We then connect pairs of the points of intersection C, D, E, F of these transversals with the sides of the angle. The locus of the intersection H of the lines CD, EF is the polar of A with respect to the given angle.*

Indeed, the lines BC, BD, BA, BH form a harmonic pencil (Fig. 180) since they divide the line EF harmonically.

This theorem gives a simple method for the construction of the polar of a point with respect to an angle.

Exercises

Exercise 232. The property of the complete quadrilateral (**202**) can be deduced from the theorems of **192** and **197**.

Exercise 233. What happens to the cross ratio $r = (ABCD)$ if we permute the four points in all 24 possible ways? Express the six possible values in terms of r. (As in **200**, one of the points may be thrown to infinity.)

Exercise 233b. If two harmonic pencils of lines, one centered at O and the other at O', have the same cross ratio, and one pair of corresponding lines coincide (the line OO'), then the intersections of the other pairs of corresponding lines are collinear.

Exercise 234. On two lines $OABC$, $OA'B'C'$, both passing through O, we take points A, B, C and A', B', C' such that $(OABC) = (OA'B'C')$. Show that the lines AA', BB', CC' are concurrent.

Exercise 235. What is a necessary and sufficient condition for it to be possible to construct a parallelogram whose sides and diagonals are parallel to four given lines?

Exercise 236. We are given a line xy and two points A, B not on the line. We join a point M in the plane to A, B, and let P, Q be the intersections of MA, MB with xy. Find the locus of the intersection M' of PB and QA as M moves along a given line.

Examine the case when the lines AP, BQ, instead of intersecting on a given line, are parallel.

CHAPTER IV

Poles and Polars with respect to a Circle

204. Theorem. *Through a point a in the plane of a circle we draw a variable secant, which intersects the circle at M, N. The harmonic conjugate of the given point with respect to the segment MN is a line* (Fig. 181).

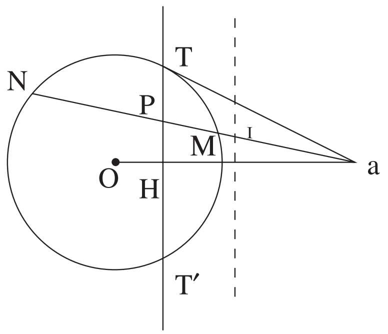

Figure 181

Indeed, the midpoint I of aP satisfies (**189**) the relation $Ia^2 = IM \cdot IN$. Therefore it belongs to a fixed line, the radical axis of the circle and of point a (**136**, Remark).

The point P describes a line homothetic to the radical axis, with respect to a, with ratio of similarity equal to 2.

Conversely, an arbitrary point P of the line thus determined belongs to the locus if line aP intersects the circle (which always happens if a is inside the circle.)

This line, the locus of the point P, is called the *polar* of point a with respect to the given circle, and point a is called the *pole* of the line.

The polar of the point a is the perpendicular at H to the line joining a with the center O of the circle, where H is a point on the same side of O as a which satisfies the relation

$$Oa \cdot OH = R^2, \tag{5}$$

where R is the radius of the circle.

This relation expresses, essentially, the fact that the points a, H divide the diameter situated along line Oa harmonically.

The radius R is between the lengths of Oa and OH because it is their geometric mean, thus *the polar intersects or does not intersect the circle according as the pole is exterior or interior to the circle. If point a is exterior, the polar is simply the chord joining the points of contact of the tangents from this point.*

Indeed, the reasoning which established the existence of the polar remains valid when the secant aMN in Figure 181 becomes a tangent; that is, when both M and N coincide with the point of contact T of this tangent. The point P also coincides with T, and the polar must pass through this point.

When the point a is on the circle, the definition of the polar as a locus no longer makes sense;[1] but we can define this line by the second construction: the point H coincides with the point a, and *the polar is the tangent at this point.*

The segment OH, derived from (5), is infinite in only one case, when the point a coincides with the center O, and the second construction fails only in this case. It is clear that the polar in this case is thrown to infinity, because the point O is the midpoint of all the chords passing through this point.

The same relation allows us to find the pole when the polar is known. We drop a perpendicular OH from the center onto the given line, and we lay off on it a distance Oa given by relation (5). This is impossible only when OH is zero; in other words, *when the line is a diameter, the pole is thrown to infinity in the direction perpendicular to this diameter.*

205. The most important property of the polar is as follows:

THEOREM. *If a point a is on the polar of a point b then b is on the polar of a.*

(Points a and b are said to be *conjugate* with respect to the circle. Their polars, that is, two lines, each of which contains the pole of the other, will also be said to be conjugate.)

Since point a (Fig. 182) belongs to the polar of b, its projection on Ob is a point K such that $OK \cdot Ob = R^2$. Let H be the projection of b onto Oa: the quadrilateral $aHbK$ is then cyclic (the circumscribed circle has diameter ab) and therefore

$$OH \cdot Oa = OK \cdot Ob = R^2.$$

Thus Hb is indeed the polar of a. QED

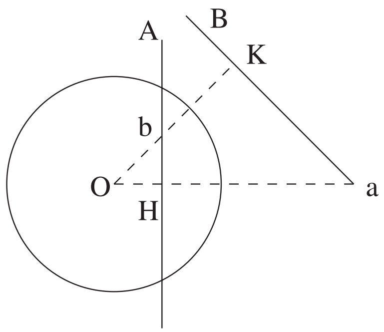

FIGURE 182

REMARK. The theorem is obvious if line ab intersects the circle. In this case, both the hypothesis and the conclusion say the same thing; namely, that the circle divides ab harmonically.

[1] One of the points M, N always coincides with a, and is therefore the same as P. When the secant becomes a tangent, both M and N coincide with a, and P is not determined on this tangent.

206. The preceding theorem allows us to pass from the properties of a figure F to those of a figure F', which we will define, called the *polar reciprocal* or *correlative*, or simply the *polar* of the first.

Let F be a figure composed of an arbitrary number of points and lines.[2] To each point a in this figure we will associate a line A; namely the polar of a with respect to a certain circle, fixed once and for all, called the *directing circle.* To each line B of F we will associate a point b; namely, the pole of B with respect to the directing circle. The lines A and the points b will constitute the polar reciproal F' of F.

According to the preceding theorem: If a line B of figure F passes through point a, the corresponding point b in F' is on the line A which corresponds to a.

Consequently, *if a line in the figure F turns about a fixed point, the corresponding point of F' moves along a line, and conversely.*

In other words, *if three lines in F are concurrent, the corresponding points in F' are collinear, and conversely.*

207. Consider, for instance, the figure F of **195** (Fig. 176), formed by two triangles abc, $a'b'c'$ such that the lines aa', bb', cc' pass through the same point O. The polars A, B, C, A', B', C' of the points a, b, c, a', b', c' will form two new triangles, and the points corresponding to the lines aa', bb', cc' will correspond to the intersection points of the sides A, A'; B, B'; C, C'. Since the original lines are concurrent, the corresponding points are collinear.

Conversely, every pair of triangles whose sides are paired in such a way that their points of intersection are collinear can be considered as the polar of a pair of triangles analogous to abc, $a'b'c'$.

Now we have proved that the intersection points of the corresponding sides of abc, $a'b'c'$ are collinear; therefore the lines corresponding to these points—which is to say the lines joining the corresponding vertices of the triangles formed by A, B, C; A', B', C'—are concurrent. In other words, *the converse of the theorem proved in* **195** *is true.*

208. Next suppose the figure F is a hexagon inscribed in a circle, and let A, B, C, D, E, F denote its sides (Fig. 183). Let us take the circumscribed circle as the directing circle: the polars of the vertices of the hexagon are the tangents at these points, which will intersect in pairs at the points a, b, c, d, e, f, the poles of the sides of the original hexagon. These points are the vertices of a hexagon circumscribed about the circle. Conversely, every hexagon circumscribed about the circle can be considered as the polar of an inscribed hexagon.

We have already proved that, in an inscribed hexagon, the intersection points of A, D; B, E; C, F are collinear (**196**). We thus obtain the following

THEOREM OF BRIANCHON. *In every hexagon $abcdef$ circumscribed around a circle, the three diagonals which join opposite vertices, are concurrent.*

In the same way, the limiting case with respect to an inscribed triangle (**196**, Remark) gives us: *In any triangle circumscribed about a circle, the lines joining each vertex to the point of contact of the opposite side are concurrent.*

[2]The definition of polar figures can be extended, with the help of notions we will study in space geometry, to figures which contain curves.

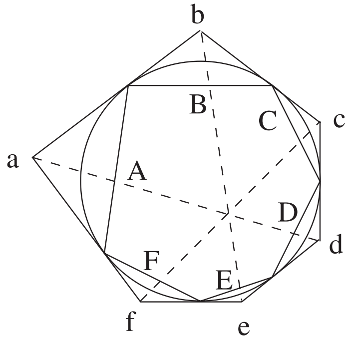

FIGURE 183

209. We have just investigated the elements that correspond, in figure F', to three collinear points or three concurrent lines in figure F. These properties are said to be *descriptive*, as opposed to properties involving the magnitudes of angles or segments, which are called *metric.* We will now investigate what happens to some of these properties when we transform them.

First of all, *if two lines are parallel, the lines joining their poles passes through the center O of the directing circle* (it is the diameter perpendicular to the common direction of these lines), *and conversely.*

Using this observation, we could have given an easy proof of the theorem of **195**, by taking the polar of the figure with respect to a directing circle with center O (Fig. 176). Points a, a'; b, b'; c, c' would then correspond to pairs of parallel lines, which would therefore form a pair of homothetic triangles. The lines joining their corresponding vertices are concurrent (at their center of homothecy). Returning to the original figure, we obtain the desired conclusion.

More generally, *the angle formed by two lines is either equal to the angle subtended at point O by the segment joining their polars, or to the supplement of this angle.* This follows because the two angles in question have perpendicular sides (Fig. 182).

Using this observation, let us transform the theorem about the concurrence of the altitudes of a triangle (Book I, **53**). The vertices a, b, c of an arbitrary triangle correspond to the sides A, B, C of a new triangle. The altitude from a will give, in the new figure, a point which belongs to A, and also to the radius of the directing circle which is perpendicular to the radius joining O to the corresponding vertex of the new triangle. The three altitudes are concurrent, so that we obtain[3] the following proposition:

Through an arbitrary point O in the plane of a triangle we draw lines perpendicular to those joining O to the vertices of the triangle. These lines intersect the corresponding sides in three collinear points.

210. *The cross ratio of four lines D_1, D_2, D_3, D_4, concurrent at a point a, is equal to that of their polars d_1, d_2, d_3, d_4.* Indeed, the two pencils of lines Od_1, OD_2, Od_3, Od_4 and d_1, d_2, d_3, d_4 are congruent figures: they can be superimposed

[3]As in the preceding examples, the proof is not complete unless we remark that we can choose points a, b, c such that point O and lines A, B, C can assume any position at all.

by translating O to a, and then rotating through a right angle about a. They therefore have the same cross ratio.

In particular, this theorem allows us to transform the ratio of the distances from d_3 to d_1 and d_2. To do this, it suffices to send point d_4 to infinity. Then line D_4 passes through O. Thus the ratio $\frac{d_3d_1}{d_3d_2}$ is equal to the cross ratio of the lines D_1, D_2, D_3, aO.

211. THEOREM. *Through a point a in the plane of a given circle, we draw two secants aMN, $aM'N'$* (Fig. 184). *We then connect pairs of the intersections M, N, M', N' of the secants with the circle. The lines joining these points intersect in two new points H, K whose locus, as the secants turn about a, is the polar of this point.*

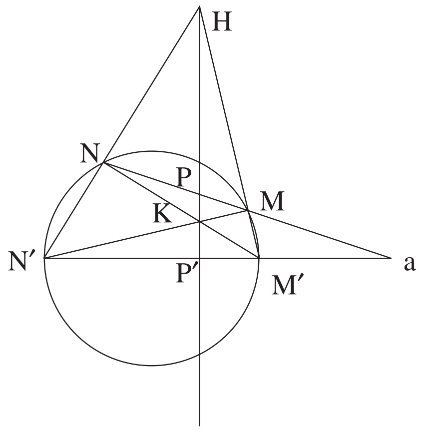

FIGURE 184

Indeed, let H be the intersection of MM' with NN', and let K be the intersection of MN' with $M'N$. Line HK intersects chords MN, $M'N'$ at two points P, P' which are the harmonic conjugates of a with respect to these chords (**202**). Line HK is therefore the polar of a.

This theorem gives us a method for constructing the polar of a point with respect to a circle. Therefore it allows us to draw the tangents from a point to a line with the aid of a straightedge alone.

If the two secants coincide, the preceding locus becomes the locus of the intersection points of tangents at the endpoints of chord MN. But this locus is already known (**205**), since the point in question is none other than the pole of MN.

REMARK. We could have reasoned by starting with one of the points H, K, just as we started from a. The triangle aHK is formed by the three diagonals of the complete quadrilateral $HMKN$. In this triangle, each vertex is the pole of the opposite side. Such a triangle is said to be *conjugate* with respect to the circle.

212. THEOREM. *The pencil of lines formed by joining four given points A, B, C, D on a circle to an arbitrary point M on the circle has a cross ratio independent of M.*

Indeed, let us take points M, M' on the circle. Then the pencil of lines (MA, MB, MC, MD) and the pencil $(M'A, M'B, M'C, M'D)$ are congruent figures, because they form angles (see **82**) which are equal and have the same sense of rotation.

REMARK. Nothing prevents us from taking M as one of the given points, say A. The line MA will then be replaced by the tangent at A, and the above argument still holds.

The constant cross ratio just noted is called the *cross ratio of the four points* A, B, C, D *on the circle.* When this cross ratio is equal to -1, these four points are said to form a *harmonic division* of the circle.

COROLLARY. *Four fixed tangents to a circle determine a constant cross ratio on a variable tangent.*

This is true because the figure formed by the four intersection points of the fixed tangents with the variable tangent is the polar reciprocal of the figure formed by the lines MA, MB, MC, MD of the preceding theorem (where M is the point of contact of the variable tangent).

The cross ratio determined by four tangents to a circle on a variable tangent is called the *cross ratio of the four tangents. The cross ratio of four tangents is equal to the cross ratio of their contact points.*

213. THEOREM. *Two conjugate chords of a circle divide the circle harmonically.*

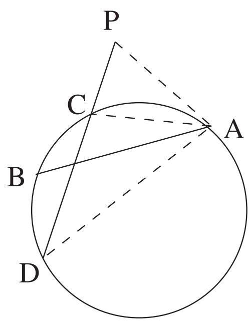

FIGURE 185

Suppose the two chords are AB, CD (Fig. 185), where CD passes through the pole P of AB. The lines AP, AB, AC, AD form a harmonic pencil, since they divide the secant PCD harmonically. This cross ratio is equal to the cross ratio of the points A, B, C, D, since AP is tangent to the circle.

COROLLARY. *The four tangents to a circle passing through two conjugate points divide an arbitrary tangent harmonically.*

The converses of these propositions are true. We leave their proofs to the reader.

Exercises

Exercise 237. If two points A, B are conjugate with respect to a circle O:

1°. What relation must there be between the sides of triangle OAB and the radius R of the circle?

2°. The circles with centers A, B (which we assume are outside circle O) and orthogonal to O are also orthogonal to each other.

3°. The circle with diameter AB is orthogonal to circle O.

Exercise 238. From the preceding exercise (part 2°), deduce the locus of points such that their polars with respect to three circles are concurrent, and the locus of the point of concurrence.

Exercise 239. If, through the vertices of a quadrilateral inscribed in a circle, we draw tangents to the circle to form a circumscribed quadrilateral, then:

1°. The diagonals of the two quadrilaterals are concurrent, and form a harmonic pencil;

2°. The third diagonals of the (complete) quadrilaterals lie along the same line, and divide each other harmonically.

Exercise 240. Through two points of a line D we draw tangents to a circle, forming a quadrilateral, one of whose diagonals is the line D itself. Show that the other two diagonals pass through the pole of D.

Exercise 241. Consider two circles O, O' and their limit points P, Q (Exercise 152).

1°. The polar of a limit point with respect to the first circle is also its polar with respect to the second one. It passes through the other limit point.

2°. There is no other point (at finite distance) which has the same polar with respect to the two circles.

3°. The perpendicular to their common centerline, dropped from the intersection of a common interior tangent with a common exterior tangent, passes through one of the points P, Q (show that this line has the same pole with respect to the two circles).

4°. The line which passes through the points of contact of one of the circles with a common interior tangent and a common exterior tangent also passes through one of the points P, Q.

CHAPTER V

Inverse Figures

214. The *inverse* of a point M with respect to some point O (called the *pole of inversion,* Figures 186, 187) is a point M' on line OM such that:

$$OM \cdot OM' = k,$$

for some constant k, called the *power of inversion*. Point M' is also called the *transform of M with respect to reciprical radius vectors*.

As usual, the segment[1] OM' is taken in the direction of OM (Fig. 186) or in the opposite direction (Fig. 187) according as k is positive or negative.

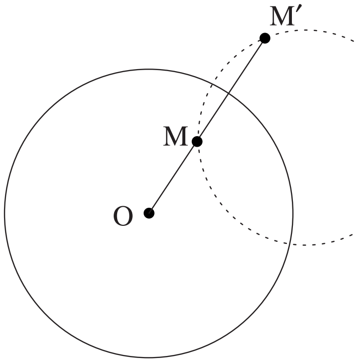

FIGURE 186

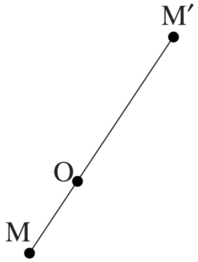

FIGURE 187

We see immediately that:

1°. Every point has an inverse except for the pole, whose inverse is thrown to infinity.

2°. If M' is the inverse of M then, reciprocally, M is the inverse of M'.

The figure F' formed by the inverses of all the points in a figure F is called the *inverse figure* of F.

215. THEOREM. *Two figures, both inverse to a third figure F with respect to the same pole O, are homothetic to each other.*

Indeed, let M be a point of F; let M', M_1' be the points which correspond to it under the two inversions with common pole O and with powers k, k_1. Division of the equation

$$OM \cdot OM' = k$$

[1]The segments OM, OM' are called *radius vectors* of the points M, M'.

by the equation

$$OM \cdot OM_1' = k'$$

gives the equation $\frac{OM_1'}{OM} = \frac{k_1}{k}$. QED

Thus the power of the inversion does not influence the shape of the figure obtained. This shape is changed only by changing the pole of inversion.

216. When the power of inversion k is positive, we can construct a circle centered at the pole of inversion and with radius $\sqrt{k}$ (Fig. 186). This circle, which clearly suffices to define the inversion, is called the *circle of inversion.* It is the locus of points which coincide with their inverses.

Two inverse points are conjugate with respect to the circle of inversion, and belong to the same diameter.

Every circle passing through two inverse points intersects the circle of inversion at a right angle (**135**). *Conversely, if two points M, M' have the property that every circle passing through them is orthogonal to a given circle, the two points are inverses with respect to this circle* (**135–137**).

When two points are symmetric with respect to a line, any circle passing through these points cuts the line at right angles, since the line is a diameter of the circle. *We are thus led to consider symmetry with respect to a line as a limiting case of inversion,* the circle of inversion being replaced by the line and, consequently, the pole thrown to infinity.

217. *Any two points A, B and their inverses A', B' are on the same circle* (**131b**).

It follows (**82**) that *the angle made by AB with the line OAA' is equal to the angle made by $A'B'$ with OBB', but in the opposite sense of rotation.* Thus if we know the direction of AB and of the radius vectors, we also know the direction of $A'B'$.

This direction $A'B'$ is said to be *antiparallel* to AB with respect to angle $\widehat{AOB}$. It is parallel to the reflection of AB in the bisector of this angle since, in this reflection, the line OA becomes OB and angle $\widehat{BAO}$ becomes an angle which is equal, but in the opposite sense.

218. Problem. Knowing the distance between two points A, B, and their radius vectors, find the distance between their inverses A', B'.

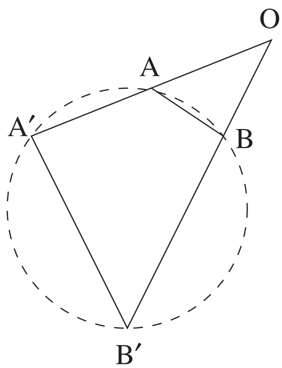

FIGURE 188

Similar triangles OAB, $OB'A'$ (Fig. 188) give us $\frac{A'B'}{OA'} = \frac{BA}{OB}$.

Solving this equation for $A'B'$, then replacing OA' by $\frac{k}{OA}$, we obtain $A'B' = BA \cdot \frac{k}{OA \cdot OB}$.

REMARK. This argument does not work when A, B, O are collinear, but the result is still true, even in magnitude and sign (if a positive direction has been chosen on the common radius vector). This is easily seen if we replace $A'B'$ and AB by $OB' - OA' = \frac{k}{OB} - \frac{k}{OA}$ and $OA - OB$.

219. THEOREM. *The tangents at corresponding points of two inverse curves make equal angles, but in opposite senses, with the common radius vector of their points of contact.*

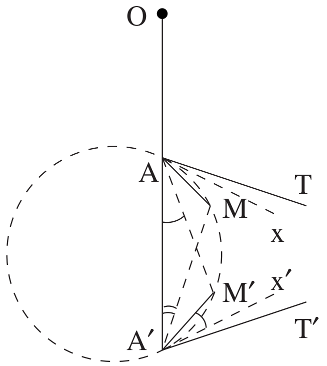

FIGURE 189

Let A (Fig. 189) be a point on curve C, and A' the corresponding point on the inverse C'. Consider a point M on C, close to A, and let M' be its inverse. We draw tangents Ax, $A'x'$ to the circle passing through A, A', M, M' (**217**). When M approaches A as close as we like (and, consequently, M' approaches A') angles $\widehat{AA'M}$, $\widehat{A'AM'}$ will tend to zero, and the same will be true for angles $\widehat{MAx}$, $\widehat{M'A'x'}$, which are equal to the first two. Therefore, if the line AM tends to a limiting position AT, then Ax will tend to the same limiting position. Therefore line $A'x'$ will have a limiting position $A'T'$ which makes the same angle as AT with line OAA', but in the opposite sense. Since angle $\widehat{M'A'x'}$ tends to zero, $A'M'$ will also tend to $A'T'$. QED

COROLLARY. *The tangents at corresponding points of two inverse curves are symmetric with respect to the perpendicular bisector of their points of contact.*

REMARK. The preceding results are also obviously true if the inversion becomes a line reflection.

THEOREM. *Two curves intersect at the same angle as their inverses* (*or their symmetric curves*), *except for the sense of rotation.*

Indeed, the angle formed by the tangents to the curves at their common point A, and the angle formed by the tangents to the inverse curves at the corresponding point A', are symmetric with respect to the perpendicular bisector of AA'.

Corollary. *If two curves are tangent, the same is true for their inverses.*

220. Inverse of a line.

Theorem. *The inverse figure of a line is a circle passing through the pole of inversion.*

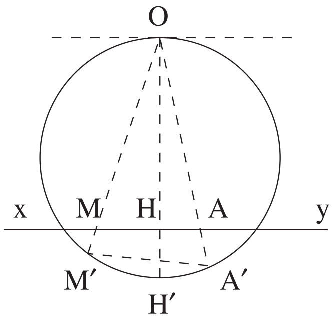

Figure 190

Let xy (Fig. 190) be a line whose inverse we seek, and let O be the pole of the inversion. Take any point A,[2] and let A' be its inverse. Next take a variable point M on xy and its inverse M'. The angle between $M'A'$ and OM' is equal to the angle between AM and OA, and in the opposite sense. Therefore this angle is constant as point M varies along the line, and the locus of M' is a circle passing through O and A. QED

In particular, we can choose, for point A, the projection H of O onto xy. Let H' be its inverse. Angle $\widehat{OM'H'}$ is therefore a right angle, and we see that *the tangent at O of the inverse circle is parallel to the given line, and its diameter is* $\frac{k}{\delta}$, where k is the power of the inversion, and δ is the distance OH from the pole to the line.

Remark. The preceding statement is valid for lines which do not pass through the pole. A line passing through the pole is its own inverse.

Corollary. *The inverse figure of a circle passing through the pole is a line;* namely, the line perpendicular to the diameter of the pole at the inverse point of the point diametrically opposite to the pole.

221. The inverse of an arbitrary circle.

Theorem. *The inverse of a circle not passing through the pole is a circle.*

1°. First, if the power of inversion k is equal to the power p of the pole with respect to the given circle, this circle is its own inverse, as the intersection points of any secant from the pole are inverses of each other.

2°. If the power k is arbitrary (Fig. 191), the inverse figure will be a circle homothetic to the one considered above (**215**) with the pole as center of similarity and ratio of similarity $\frac{k}{p}$.

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[2] i.e. on line xy.–transl.

222. Conversely, two arbitrary circles C, C' (Fig. 191) can be considered as inverses of each other in two different ways, since they are homothetic in two different ways. The pole of inversion is then a center of similarity, and the power of inversion is equal to the power of this pole with respect to the first circle, multiplied by the ratio of similarity.

These similarities are the only two which transform one of these circles into the other. Indeed, if they are inverse with respect to some pole S, the two circles must be homothetic with respect to S, since one of them is its own inverse with respect to this pole.

223. By the preceding argument, if a secant from a center of similarity S intersects C in M, N, and C' in M', N', so that M' corresponds to M and N' corresponds to N under a homothecy beween the two circles, then M and N' on the one hand, and M' and N on the other, are pairs of inverse points.

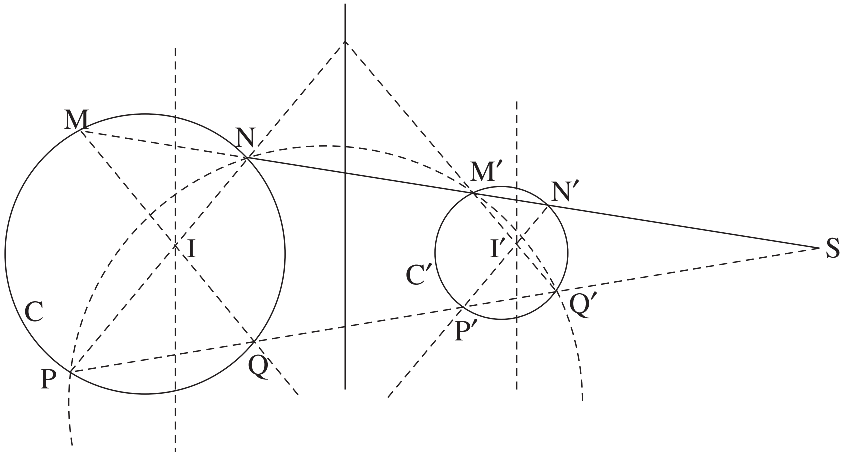

FIGURE 191

The points M and N' are sometimes called *antihomologous* points.

The only pairs of points which are both homologous and antihomologous are the points of contact of the common tangents from point S.

The common points of two circles, if any exist, are their own anti-homologues.

The *antihomologous* chord of a chord given by two points in the first circle, is the chord in the second circle determined by the anti-homologues of the first two points.

224. *Two pairs of antihomologous points lie on the same circle* (**217**) and consequently *two antihomologous chords intersect on the radical axis* (**139**).

Moreover, *two antihomologous chords intersect the polars of the center of similarity, with respect to their respective circles, in two corresponding points.* This is true because the chord $N'P'$, antihomologous to MQ (Fig. 191), corresponds to NP, which intersects MQ in a point I situated on the polar of S with respect to circle C (**211**) and whose corresponding point I' is the intersection of $N'P'$ with the polar of S with respect to C'.

This double property allows us to determine the anti-homologue of a given chord without involving the endpoints of the chord.

This applies, of course, to the tangents at antihomologous points, which are the limiting cases of the preceding chords.

REMARK. If the two circles are identical (case 1° of **221**), the two chords MQ, NP, joining two points to their inverses, intersect on the polar of the pole S of the inversion. From this point of view, *this polar plays the role of the radical axis of the two identical circles,* if they are considered as inverses of each other with respect to the point S.

225. If the two circles are congruent, the exterior center of similarity is thrown to infinity, and the corresponding homothecy reduces to a translation, while the corresponding inversion reduces to a line reflection; the properties of antihomologous points are the same as in the general case.

226. *A line and a circle can also be regarded as inverse figures in two different ways.* The poles of inversion will be (**220**) the endpoints of the diameter perpendicular to the line. The theory of antihomologous points leads us to consider these as centers of similarity of the line and the circle.

Of two antihomologous chords, one is on the given line, and therefore we can still say that these two chords intersect on the radical axis, in the sense of **158** (Construction 12, remark).

Finally, *two lines are symmetric to each other in two different ways,* the axes of symmetry being the bisectors of the angles formed by these lines. The antihomologous chords are the lines themselves, and therefore intersect in a fixed point.

227. *When two circles are inverse to each other, every circle Σ which passes through two antihomologous points is its own inverse,* since the power of the pole with respect to Σ is equal to the power of inversion. *Such a circle must therefore intersect the two circles at the same angle and, if it is tangent to one, it must be tangent to the other.*

Conversely, *any circle Σ which intersects two circles C, C' at the same angle* (Fig. 192) *intersects them in two pairs of antihomologous points.*

Indeed, let A, B; A', B' be the four points of intersection, labeled so that the angles formed by circles C, Σ at A, and of the circles C' and Σ at A' (which are equal by hypothesis) are in opposite senses, and so that the same is true for B, B'. If S is the intersection of AA' and BB', and k is the power of S with respect to circle Σ, then the inversion with pole S and power k does not change Σ: it takes A onto A' and B onto B'. Thus this inversion transforms C into a circle which passes through A', B', and is tangent to C' at A' (by hypothesis, and **219**); that is, this circle coincides with C' (**90**).

This conclusion remains true if Σ is tangent to C and C'. To see this, it suffices to repeat the preceding argument while taking S to be the intersection of AA' and BB', with A and A' being the points of contact of Σ, and B, B' the second intersections of C, C' with any circle passing through A, A' (Fig. 192).

In this case, the proposition also follows from the theorem of **144**, because the points of contact are centers of similarity, one for Σ and C, the other for Σ and C'. Therefore they are collinear with a center S of similarity of C and C'. We also see that *the center of similarity is external if the points of contact are of the same kind*[3]*, and internal otherwise.*

[3]That is, both external or both internal.

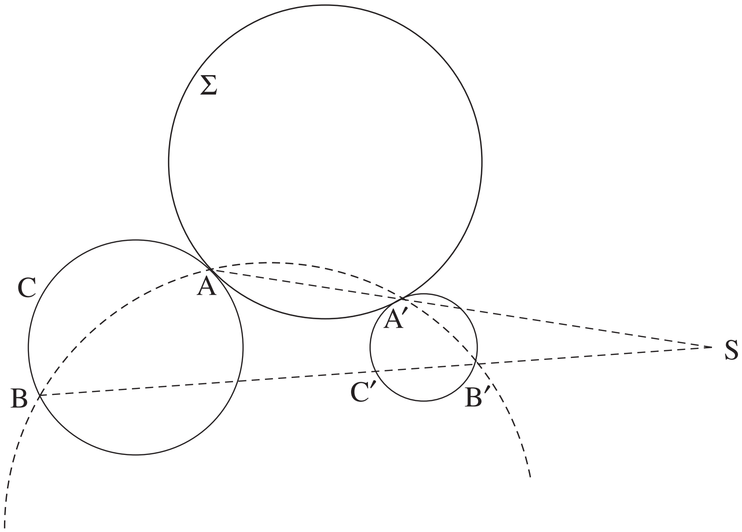

FIGURE 192

Of course, the arguments just presented remain valid when one or both of the circles C, C', or circle Σ, are replaced by lines.

227b. If circle Σ cuts circles C, C' at right angles, we can consider any pair of the points of intersection, chosen arbitrarily, as antihomologous. Indeed, two right angles can always be considered equal and of opposite senses (by replacing, if necessary, one side by its extension). Thus, a circle which cuts C, C' at right angles corresponds to itself in either of the two inversions which transform C into C'.

228. If the inversion with pole S interchanges circles C and C' and it has a circle of inversion Γ, this circle cuts Σ at right angles, since the power of point S with respect to Σ is equal to the power of the inversion.

This property corresponds to the property of the bisectors of the angles formed by two lines. Indeed, these bisectors are the locus of the centers of circles tangent to the two lines, which amounts to saying that any one of these circles intersects one of these bisectors at right angles. Here we see that any circle Σ, tangent to C and C', or, more generally, any circle Σ which cuts C and C' at equal angles, is orthogonal to circle Γ, or to the analogous circle centered at the other center of similarity (if these circles exist).

The circle Γ always exists if the circles C, C' intersect: in this case it passes through their common points.

In general, when it exists, it has the same radical axis as C and C' since these three circles have the same set of orthogonal circles (by the preceding section).

Exercises

Exercise 242. When two points are inverses with respect to a circle, the ratio of their distances to a variable point on the circle is constant.

Exercise 243. Show that Exercise 68, in the case where the given circle intersects the line in a point I, can be reduced to Exercise 65 by means of an inversion with pole I.

Exercise 244. Through a common point A of two circles C, C' we draw secants AMM', ANN', which intersect C in M, N and C' in M', N'. Circles AMN', ANM' intersect at A and at another point. Find the locus of this point as the two secants turn, independent of one another, about point A.

Exercise 245. The inverses of circles with a common radical axis are also circles with a common radical axis.

Exercise 246. Transform the definition of a circle by inversion (**7**) to obtain a new proof of the theorem of **116**.

Exercise 247. When we take the inverse of a circle, what is the point whose image is the center of the transformed circle?

Exercise 248. Two given circles can always be transformed, by the same inversion, either into two parallel lines, or into two concentric circles.

Exercise 249. Given three points on a line, find a fourth point on the same line such that if we invert with respect to this point, the three given points divide the line into two equal parts.

Exercise 250. Two figures are inverse to each other with respect to an inversion S. We apply some inversion T to both figures. Show that the new figures obtained are again inverse to each other, and find the new pole of inversion. In particular, examine the case when the power of the inversion S is positive. (The new circle of inversion is obtained by applying the inversion T to the circle of inversion of S.)

Exercise 251. An inversion S is applied to a figure A to obtained a figure B, which is then transformed into a figure A' by an inversion S'. Assume that the powers of inversion are positive.

1°. Show that by applying an appropriate inversion T to A, A', we can transform these figures into congruent or homothetic figures, these two possibilities being mutually exclusive;[4]

2°. Show that we can find infinitely many pairs of inversions S_1, S_1', equivalent to S, S'; that is, such that consecutive application of S_1 and S_1' to the figure A yields figure A'. In particular, we can always replace the two given inversions by an inversion preceded or followed by a line symmetry, except in one case (the case where figures A, A' are similar);

3°. When are the figures obtained by applying T (of 1°) to A, A' images of each other under translation?

4°. What happens if we apply the operations S, S' several times in succession (that is, we apply S to transform A' into B', then apply S' to transform B' into A'', then apply S to transform A'' into B'', then apply S' to transform B'' into A''', etc.)? Can it happen that we eventually obtain the original figure A?

[4]Except in the case mentioned in 3°, which can be considered a common limiting case, since a translation can be viewed as a limiting case of a homothecy.

Exercise 252. We successively apply inversions S, S', S'', etc. (as in the preceding exercise). Show (limiting the discussion to the case of positive powers of inversion) that this sequence of operations can be replaced by a single inversion preceded or followed by one, two, or three line symmetries, unless the final figure and the original figure are similar (preceding exercise, 2°, and Exercise 94).

Exercise 253. In the preceding exercise, assume that an odd number of inversions S, S', S'', ... has been applied. Find a point which returns to its original position.

Exercise 253b. In a given circle, inscribe a polygon whose sides pass through given points, or are parallel to given directions (preceding exercise).

Exercise 254. On a fixed tangent to a circle, with point of contact T, we draw variable segments TM, TN whose product is constant. Let T' be the point on the give circle which is diametrically opposite T.

1°. Show that the line joining the second intersections of $T'M$, $T'N$ with the circle passes through a fixed point;

2°. Same problem for the line joining the points of contact of the second tangents from M, N (reduce this to the preceding case).

3°. Find the locus of the intersection of these two tangents.

Exercise 255. Given a point O on the plane of a circle, we draw a variable secant which intersects the circle in M, N. The circles with diameters OM, ON intersect the given circle again in M', N'. Find the locus of the intersection of MN, $M'N'$ as the secant turns about point O.

Exercise 256. When a variable circle intersects two fixed circles at a constant angle, it also cuts any fixed circle having the same radical axis as the first two at a constant angle.

Exercise 257. On two segments AB, CD on the same line, we draw arcs of circles with the same central angle $\widehat{V}$. As the angle $\widehat{V}$ varies, find:

1°. the locus of the midpoint of the common chord of the two circles of which these arcs are a part;

2°. the locus of the intersection of these circles; in other words, the locus of points at which the given segments subtend equal or supplementary angles. (Use Exercise 219b.)

CHAPTER VI

The Problem of Tangent Circles

229. Problem. Draw a circle passing through two given points, and tangent to a given line or a given circle.

We have already solved this problem (**159**). Inversion allows us to give another solution. Let us transform the figure by an inversion with its pole at one of the given points. The required circle becomes a line passing through a known point and tangent to a known circle (the inverses of the second point and of the given circle).

230. Problem. Draw a circle passing through a given point and tangent to two given lines or circles.

First method. The required circle corresponds to itself in one of two inversions (or line symmetries) which transform one of the lines into the other. Thus, in addition to the given point, we know a second point; namely the point which is inverse or symmetric to the first. We are thus led to the preceding problem. There can be four solutions, since the preceding problem has two.

Second method. Invert the given figure, with the given point as pole. We are led to the construction of the common tangents to two circles.

231. Problem. Draw a circle tangent to three given circles.

This can be reduced to the preceding problem.

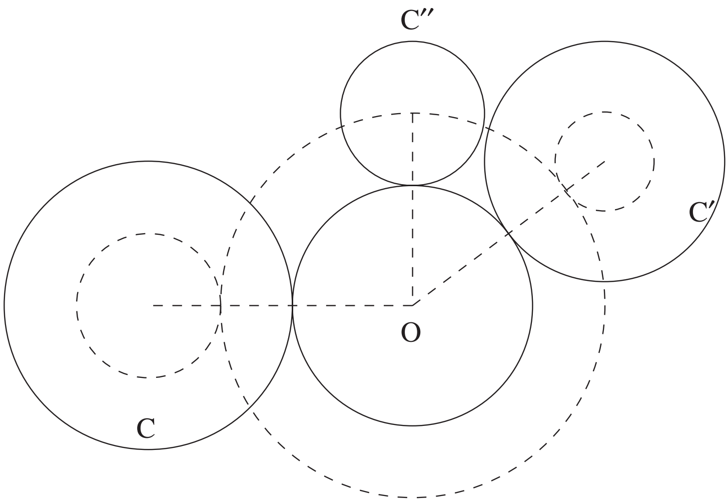

Figure 193

Indeed, consider, the given circles C, C', C'' (Fig. 193) with radii r, r', r'', and a circle with center O and radius R, which, let us say, is externally tangent to the three circles. Then a circle with center O and radius $R+r''$ passes through the center of C'', and is tangent to the circles concentric to C, C' with radii equal to the difference between r'' and the radii r, r', respectively.

The cases where the required circle is tangent internally to one or more of the given circles are treated analogously, except that certain differences of radii are replaced by sums.

232. We can also solve the same problem using an entirely different method (due to *Gergonne*).

Let A, B, C be the given circles. We first look for a circle Σ (Fig. 194) which is tangent *in the same way* to the three given circles, at points a, b, c.

If such a circle exists, there must necessarily be a second[1] circle Σ'. Indeed, the radical center I of the three circles has same power with respect to these circles. The inversion with I as a pole and this common power as the power of inversion will not change the given circles, and will transform Σ into a circle Σ' tangent to A, B, C at points a', b', c' which are the inverses of a, b, c. The contacts will be of the same type; that is, all of the same type as for Σ or the opposite type, according as whether I is an external or internal center of similarity for the circles Σ and Σ'.

We will prove that the radical axis xy of the circles Σ, Σ' is precisely the axis of direct similarity (**145**) of the given circles. To see this, consider the intersection S of the lines ab, $a'b'$. This point is on the radical axis of Σ, Σ', because the points a, a' and b, b' are inverse pairs with respect to point I. This point S is the center of direct similarity for the circles A, B (**227**). In the same way, we can show that the radical axis xy passes through the other two centers of direct similarity of the given circles.

Having established this, we draw the common tangents of A and Σ, Σ' which pass through a, a'. The point α where these tangents intersect will be on xy, because points a, a' are inverse with respect to I. This point α will be the pole of aa' with respect to circle A.

Thus *the chord aa' passes through the pole p of the direct axis of similarity with respect to the circle A.*

We are thus led to the following construction:

Determine the radical center I and the axis of direct similarity xy of the given circles. Join I to the poles p, q, r of xy with respect to these circles. The lines obtained in this way intersect the corresponding circles in the required points a, a'; b, b'; c, c'.

233. It remains to prove that this construction will indeed provide circles satisfying the conditions of the problem.

To do this, we will show that the points a, b, c, a', b', c' obtained above are antihomologous in pairs in the given circles.

Let us look for the chord in circle B which is antihomolgous to chord aa' in circle A. This chord is determined (**224**) by the following two conditions:

1°. the two chords intersect on the radical axis of the two circles;

[1] Circle Σ' is different from Σ. Indeed, unless the line Ia is tangent to A, point a' is different from a. Likewise, Ib would be tangent to B, and Ic to C. But if a, b, c were chosen this way, the circle abc would be orthogonal, rather than tangent, to A, B, C.

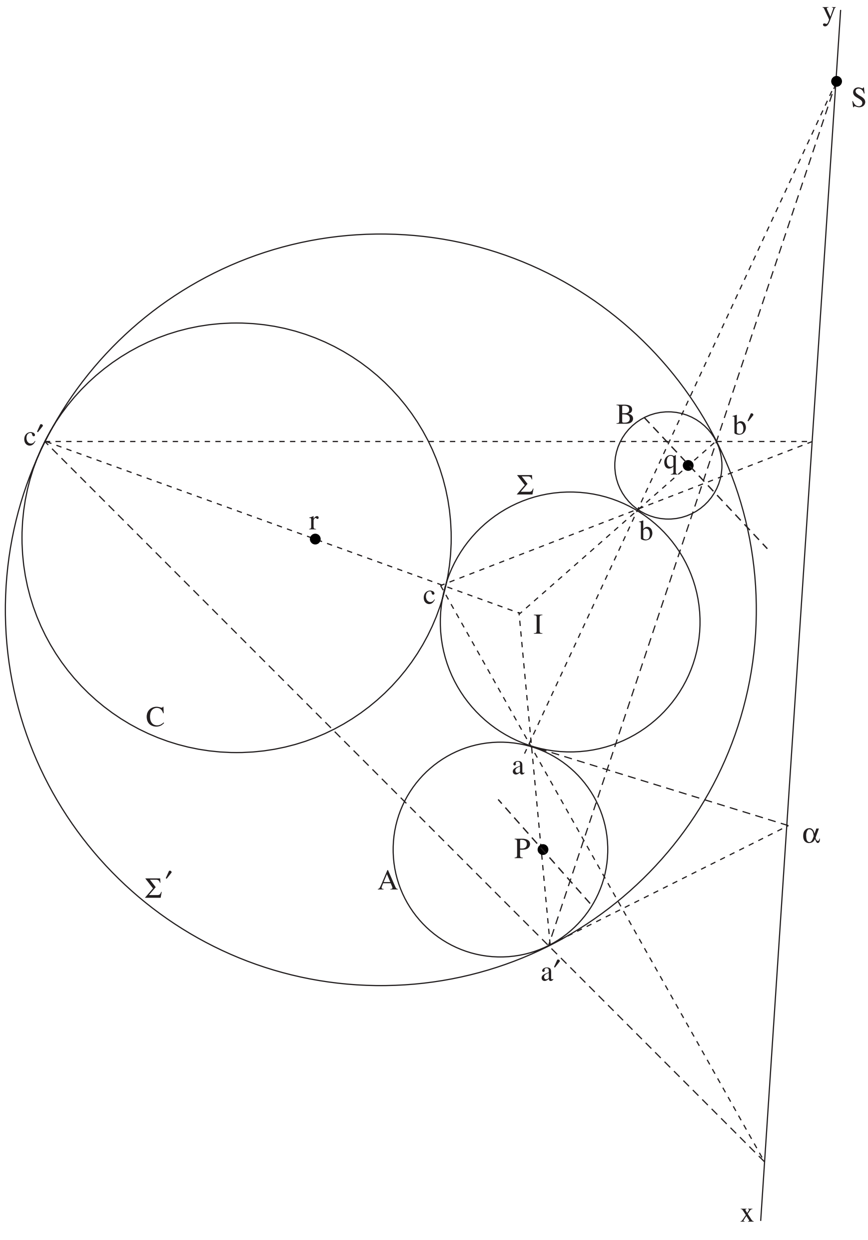

FIGURE 194

2°. the points where they meet the polars of S with respect to the two circles correspond to each other in the homothecy with center S taking one circle onto the other.

Now condition 1° is satisfied by aa' and bb' since their intersection I in fact belongs to the radical axis of A and B. Condition 2° is also satisfied. Indeed, points p, q are the poles of xy with respect to circles A, B, so they lie on the polars of S with respect to these circles. And these points correspond to each other in the homothecy with center S which brings circle A onto circle B, since line xy corresponds to itself, and so its poles in the two circles correspond to each other. The chord antihomologous to aa' is therefore bb'. In the same way, we can show that chords cc', aa' are antihomolgous in circles C, A, and chords cc', bb' are antihomoogous in circles C, B.

We will denote by b, c the two antihomologous points for a (without having yet proved that these two points are antihomologous); then b', c' will be the antihomologues points for a'. Let us draw a circle Σ through a, b, c and a circle Σ' through a', b', c'.

These two circles are inverse to each other with respect to point I, since the the pairs a, a'; b, b'; c, c' are inverse with respect to this point. The circle inverse to that of abc is therefore $a'b'c'$.

The radical axis of these circles is xy because this radical axis must pass through the intersection of ab, $a'b'$, and through the intersection of ac, $a'c'$.

Now let α_1 be the intersection of the tangents to circle A at a, a'. This point is on xy, since aa' passes through point p; it is also on the perpendicular bisector of aa'.

But the point α, where the tangents at a, a' to Σ, Σ' intersect, is also on xy, because this line is the radical axis of Σ, Σ'; it is also on the perpendicular bisector of aa' because the tangents αa, $\alpha a'$, to circles Σ and Σ' respectively, are equal.[2]

Therefore *points α and α_1 coincide*, and the circles Σ, Σ' are tangent to A at a, a'. They are likewise tangent to B and C at b, b'; c, c' (**227**).

234. We have sought a circle Σ tangent to A, B, C with contacts of the same kind. We would find circles which have contacts of different kinds with the circles A, B, C by replacing the direct axis of similarity with an inverse axis of similarity in the preceding reasoning.

As there are four axes of similarity, the problem of tangent circles can have eight solutions. However, some or all of these solutions may be missing: this will happen, for instance, if line Ip does not intersect circle A. We can see that if the power of the point I with respect to the three circles is negative, and therefore the point I is interior to the circles, all eight solutions exist. These eight solutions also exist if the three circles are exterior to each other, as can easily be seen solving the problem by the method of **231**.

235. Gergonne's solution can also be applied when two of the given circles are replaced by points or lines. This can be seen by repeating the reasoning of **232** using these new conditions. However, the construction only yields the points of contact with the circles. In particular, it does not yield any result if *all* circles are replaced by points or lines.

236. Gergonne's solution fails to apply in certain special cases. Indeed, if the three centers are collinear, the radical center and the three poles of each axis of similarity are thrown to infinity in the direction perpendicular to this axis. This position can be avoided by first applying an inversion to the figure.

Conversely, three circles can, in general, be transformed by an inversion into three circles with collinear centers. For this to be possible, it is sufficent that the

[2]The original text here refers to "the tangents $\alpha_1 a$, $\alpha_1 a'$". Since the points α and α_1 in fact coincide, this statement is correct, but the passage seems clearer if we refer to the point here as α.–transl.

common power of the radical center be positive and consequently that there exist a circle cutting the three given circles at right angles. If we take any point at all on this circle as the pole of inversion, the given circles are transformed into three circles which are intersected at right angles by the same line, and therefore their centers will be on that line.

The inconvenience which we just noted would therefore disappear if we could formulate a solution in terms of properties which do not change under an inversion, such as the tangency of two circles, the angle between them, etc. One can in fact modify Gergonne's solution in this way.[3]

Exercises

Exercise 258. Construct a circle passing through two points and cutting a given circle at a given angle.

Exercise 259. Construct a circle orthogonal to two given circles and tangent to a third given circle. More generally, construct a circle orthogonal to two given circles and cutting a third given circle at a given angle. (See Exercise 248.)

Exercise 260. Through two points A, B we draw two circles tangent to the same circle C, and a third one which is orthogonal to C. Show that this last circle divides the angle made by the other two into equal parts, and that its center is the intersection of their common tangents (see **229**).

Exercise 261. Through a point A we draw two tangent circles (with contacts of the same kind) to two given circles, and a third one which is orthogonal to them. Show that this last circle passes through the second common point B of the first two, and possesses the properties indicated in the preceding exercise.

Exercise 262. Two tangent circles are drawn through a point A as in the preceding exercise. Let P, Q be their points of contact with the first given circle, and P', Q' the points of contact with the second.

1°. The circles APQ, $AP'Q'$ are tangent. They are orthogonal to the third circle of the preceding problem;

2°. The circles APQ, BPQ (where B is the second common point of the circles APP', AQQ', as in the preceding exercise) are inverse to each other with respect to the first of the given circles; the circles $AP'Q'$, $BP'Q'$ are inverse with respect to the second.

3°. What happens to these statements when the given circles are replaced by lines? Show that in this case the four circles APQ, $AP'Q'$, BPQ, $BP'Q'$ are equal.

Exercise 263. Draw a circle passing through a given point and tangent to two given lines by first drawing an arbitrary circle C tangent to the two lines, then noting that the required circle must be homothetic to C with respect to the intersection point O of the lines.

Exercise 264. Use an analogous method to construct a circle tangent to two given lines and to a given circle. (The line joining O to the point of contact cuts circle C and the given circle at equal angles.)

[3]See Note C at the end of this volume.

Exercise 265. The same problem, but the two lines are replaced by two concentric circles. (Use a circle concentric to the given circles, and passing the point of contact of the required circle with the third given circle.)

Exercise 266. Two variable tangent circles are each tangent to two fixed circles. Find the locus of their point of contact. (The first method of **230** shows when two circles tangent to two given circles can have two identical intersection points.)

Exercise 267. Pairs of the centers of the eight circles tangent to two given circles are on the perpendiculars dropped from the radical center to the four axes of similarity.

Exercise 268. Through a point inside an angle, draw a transversal forming a triangle of minimum perimeter with the sides of the angle (not extended past the vertex). (Use the indirect method (Exercise 174) and Exercise 90.)

CHAPTER VII

Properties of Cyclic Quadilaterals. Peaucellier's Inverter

237. Ptolemy's Theorem. *The product of the diagonals of a cyclic quadrilateral is equal to the sum of the products of opposite sides.*

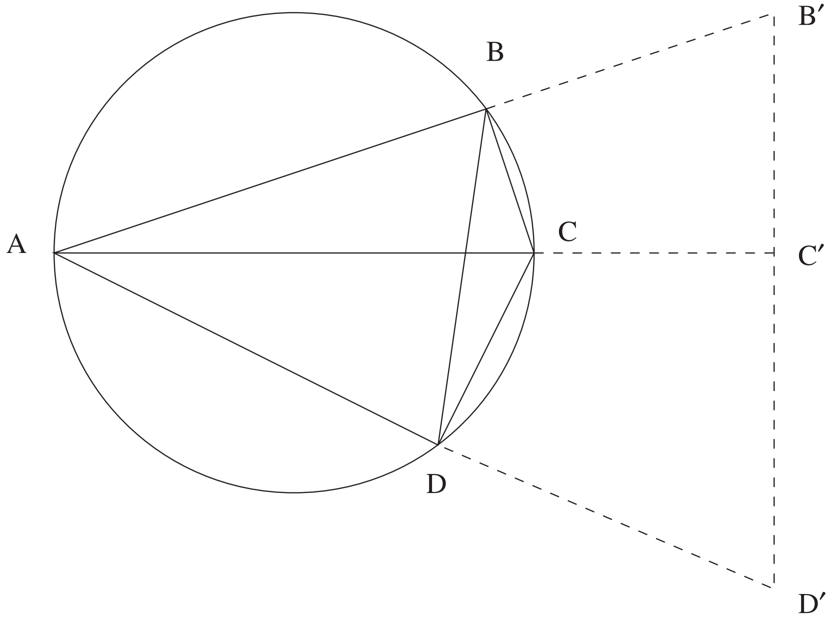

FIGURE 195

Suppose quadrilateral $ABCD$ (Fig. 195) is inscribed in a circle. We transform the figure by an inversion with its center at point A. Circle $ABCD$ is transformed into a line which passes through the inverses B', C', D' of B, C, D. If C is the vertex of the given quadrilateral opposite to A, the points B, D will be on different sides of the line AC, and therefore so will points B', D'. Thus point C' is between B' and D', and we have (in absolute value)

$$B'D' = B'C' + C'D'.$$

But if k is the power of inversion, we have (**218**):

$$B'D' = DB \cdot \frac{k}{AB \cdot AD}, \quad B'C' = CB \cdot \frac{k}{AB \cdot AC}, \quad C'D' = DC \cdot \frac{k}{AC \cdot AD}.$$

Substituting these values into the preceding equality, then multiplying by $AB \cdot AC \cdot AD$, and dividing by k, we obtain the stated relation:

$$AC \cdot BD = AD \cdot BC + AB \cdot CD.$$

237b. The preceding property actually characterizes cyclic quadrilaterals. This is the result of the following theorem:

THEOREM. *If a quadrilateral is not cyclic, then the product of the diagonals is smaller than the sum of the products of opposite sides (and greater than their difference).*

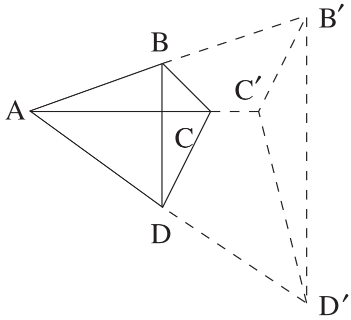

FIGURE 195b

Suppose $ABCD$ (Fig. 195b) is a quadrilateral which is not cyclic. If we repeat the preceding construction, points B', C', D' are no longer collinear, and they will form a triangle.

The values found in the previous section for $C'D'$, $D'B'$, $B'C'$ are proportional to the products $DC \cdot AB$, $BD \cdot AC$, $CB \cdot AD$. Therefore any of these products is less than the sum of the other two.[1]

238. The relation in Ptolemy's theorem is still true, and is proved in the same way, when the four points A, B, C, D are situated in this order on a straight line rather than a circle (Fig. 196). Thus, *if four points are collinear, the product of the overlapping segments they form is equal to the sum of the products of the non-overlapping segments.*

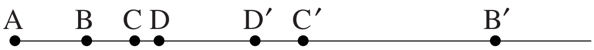

FIGURE 196

Taking signs into account, this relation can be written as

$$AB \cdot CD + AC \cdot DB + AD \cdot BC = 0.$$

[1]Hadamard's text does not mention this directly, but this observation also proves the assertion made about the difference between the products of opposite sides.–transl.

This is the result of repeating the argument of **237**: the relationships we used are, in fact, true in magnitude and in sign (**218**, Remark).

239. Problem. Knowing the chords AB, AC (Fig. 197) of two arcs in a circle of radius R, find the chord of the sum or difference of the two arcs.

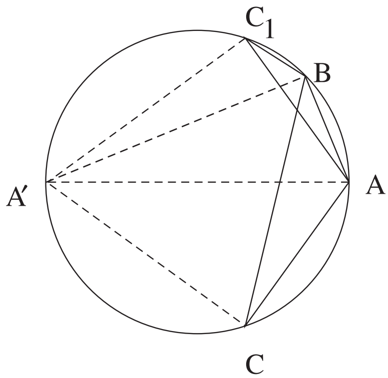

FIGURE 197

This is the problem we had to solve in **174** in order to calculate the sides of the various regular pentadecagons.

1°. Assume that arcs $\widehat{AB}$, $\widehat{AC}$ are described in contrary senses, so that arc $\widehat{BC}$ is their sum. If we draw diameter AA', we have:

$$AA' = 2R, \quad A'B = \sqrt{4R^2 - AB^2}, \quad A'C = \sqrt{4R^2 - AC^2}.$$

The cyclic quadrilateral $ABA'C$ gives

$$BC \cdot AA' = AB \cdot A'C + AC \cdot A'B,$$

a relation in which every quantity is known, except BC, and which gives us:

$$BC = \frac{AB\sqrt{4R^2 - AC^2} + AC\sqrt{4R^2 - AB^2}}{2R}.$$

2°. Assume that the arcs $\widehat{AB}$, $\widehat{AC_1}$ are described in the same sense, so that arc $\widehat{BC_1}$ is their difference. The cyclic quadrilateral $AA'BC_1$ now gives us:

$$A'B \cdot AC_1 = AB \cdot A'C_1 + AA' \cdot BC_1$$

so that

$$BC_1 = \frac{A'B \cdot AC_1 - AB \cdot A'C_1}{AA'} = \frac{AC_1\sqrt{4R^2 - AB^2} - AB\sqrt{4R^2 - AC_1^2}}{2R}.$$

240. THEOREM. *The ratio of the diagonals of a cyclic quadrilateral is equal to the ratio of the sums of the products of the sides which meet at their endpoints.*

Suppose the diagonals of cyclic quadrilateral $ABCD$ (Fig. 128) intersect at O. Triangles OAD, OBC are similar (**131**), and give us:

$$\frac{OA}{OB} = \frac{AD}{BC} = \frac{OD}{OC},$$

which can be rewritten as

$$\frac{OA}{AB \cdot AD} = \frac{OB}{AB \cdot BC}\,, \qquad \frac{OC}{BC \cdot CD} = \frac{OD}{AD \cdot CD}.$$

Likewise, the similar triangles OAB, OCD yield

$$\frac{OA}{OD} = \frac{AB}{CD}, \quad \text{or} \quad \frac{OA}{AB \cdot AD} = \frac{OD}{AD \cdot CD}.$$

Thus the four ratios

$$\frac{OA}{AB \cdot AD}, \quad \frac{OB}{AB \cdot BC}, \quad \frac{OC}{BC \cdot CD}, \quad \frac{OD}{CD \cdot DA}$$

are equal. Adding the numerators and denominators of the first and third, and of the second and fourth, we obtain the desired relation

$$\frac{AC}{AB \cdot AD + BC \cdot CD} = \frac{BD}{AB \cdot BC + CD \cdot AD}.$$

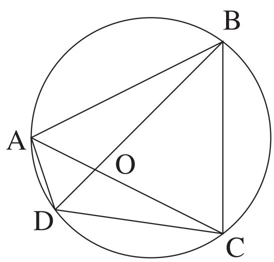

FIGURE 198

Another proof. Let $a = AB$, $b = BC$, $c = CD$, $d = DA$ be the four sides of the quadrilateral. Permuting these sides in all possible ways, we obtain other quadrilaterals inscribed in the same circle, because the sum of the arcs corresponding to these four chords will always be the whole circle. Clearly, we can always take a to be the first side, so we have the six possible permutations

$$\begin{array}{ll} abcd, & adcb, \\ acdb, & abdc, \\ adbc, & acbd. \end{array}$$

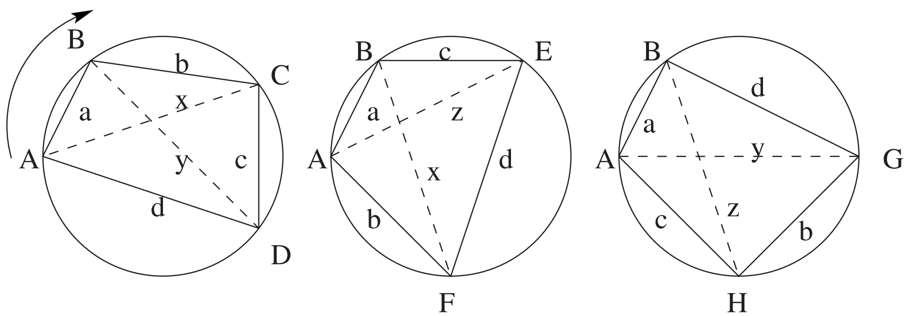

FIGURE 198b

However, the permutations in the same row give the same quadrilateral: for example, in quadrilateral $ABCD$ (Fig. 198b) the order of the sides is a, b, c, d if we read them in the order indicated by the arrow, and a, d, c, b if we read in the opposite direction. The remaining three permutations are *abcd*, *acdb*, *adbc* which correspond to quadrilaterals $ABCD$, $ABEF$, $ABGH$ (Fig. 198b). Arc $\overset{\frown}{BF}$ is equal to $\overset{\frown}{AC}$, since it is the sum of equal arcs. Likewise, arc $\overset{\frown}{BH}$ is equal to $\overset{\frown}{AE}$ and $\overset{\frown}{BD}$ to $\overset{\frown}{AG}$. The three quadrilaterals have only three distinct diagonals: $AC = BF = x$, $BD = AG = y$, $AE = BH = z$, and applying Ptolemy's theorem to quadrilaterals $ABEF$, $ABGH$ gives

$$xz = ad + bc, \quad yz = ab + cd,$$

which gives us, by division, the required ratios.

240b. Problem. Calculate the diagonals x, y of a cyclic quadrilateral with sides a, b, c, d.

We know the product and the ratio of the diagonals:

$$xy = ac + bd, \quad \frac{x}{y} = \frac{ad + bc}{ab + cd}.$$

Multiplication yields

$$x^2 = \frac{(ac + bd)(ad + bc)}{ab + cd},$$

and we can find y^2 similarly.[2]

Problem. Calculate the radius of the circumscribed circle for a cyclic quadrilateral with sides a, b, c, d.

In triangle ABD (Figures 195 or 198), the radius R of the circumscribed circle is given by the formula (**130, 130b**):

$$R^2 = \frac{a^2 d^2 \cdot BD^2}{[(a + d)^2 - BD^2][BD^2 - (a - d)^2]}.$$

Replacing $BD = y$ by the value found above we obtain

$$\begin{aligned}(a + d)^2 - BD^2 &= \frac{(a + d)^2(ad + bc) - (ac + bd)(ab + cd)}{ad + bc} = \frac{ad[(a + d)^2 - (b - c)^2]}{ad + bc} \\ &= \frac{ad(a + d + b - c)(a + d + c - b)}{ad + bc}; \\ BD^2 - (a - d)^2 &= \frac{ad(b + c + a - d)(b + c + d - a)}{ad + bc}.\end{aligned}$$

Thus we find

$$R^2 = \frac{(ac + bd)(ab + cd)(ad + bc)}{(b + c + d - a)(c + d + a - b)(d + b + a - c)(b + a + c - d)},$$

an expression independent of the order of the sides, which is consistent with our earlier remarks (**240**). Denoting by p the semi-perimeter (so that $2p = a+b+c+d$), we can rewrite the formula as follows:

$$R^2 = \frac{(ac + bd)(ab + cd)(ad + bc)}{16(p - a)(p - b)(p - c)(p - d)}.$$

[2]That is, by dividing.–transl.

241. Peaucellier's inversion cell.

THEOREM. *Consider a rhombus $MPM'Q$ and a point O such that $OP = OQ$* (Fig. 199). *Assume that the rhombus is articulated* (**46b**), *the* (*equal*) *lengths OP, OQ held constant, and the point O remains fixed. Then the points M and M' move along two figures inverse to each other.*

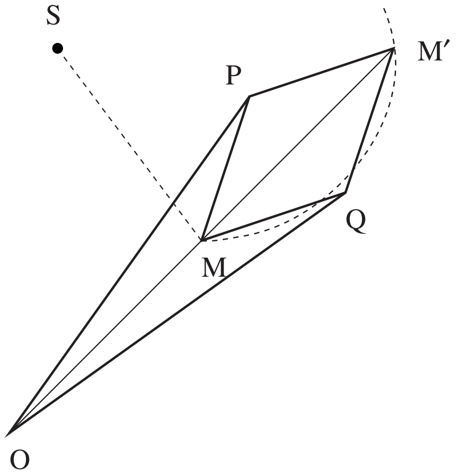

FIGURE 199

Indeed, points M, M', O are equidistant from P, Q, and they are therefore on the same line; namely, the perpendicular bisector of PQ. Let us draw a circle with center P and passing through M and M'. The power $OM \cdot OM'$ of O with respect to this circle is then equal to $OP^2 - PM^2$, and it therefore remains constant. QED

This theorem allows one to solve a problem of theoretical, if not practical, importance.

The apparatus used to draw circles, the compass, is accurate by definition: the two points will remain at a constant distance provided there is sufficient friction at the hinge of the compass.

On the other hand, the correctness of a ruler depends on its edge having the form of a straight line. This can be realized only by approximation, which is more or less exact depending on the care with which the instrument is manufactured.

Can one draw a straight line without having such a line in advance, using only properties of figures we can be sure of?

This is precisely what the apparatus described in the preceding result, due to *Peaucellier,* allows us to do. Indeed, assume that OP, OQ, PM, PM', QM, QM' are rigid rods[3] attached to each other, and the point O is fixed. Then M, M' must be inverses of each other with respect to point O. Let us make point M describe a circle, by attaching it to a fixed point S using a seventh rod (Fig. 199). If this circle passes through point O (that is, if $SM = SO$), point M' will describe a straight line.

[3]In Figure 199, we have represented OP, OQ, ... by straight lines, but it is not necessary that they should have this form. It is only necessary that the points O and P, for instance, should be kept at a constant distance, no matter how this is done.

241b. Hart's inversion cell. *Let $ABCD$ be a anti-parallelogram* (**46b**, Remark II; Fig. 199b) *formed by the diagonals and the non-parallel sides of an isosceles trapezoid. Let O, M, M' be the three points where sides AB, AD, BC intersect a line parallel to the bases of the trapezoid. If the anti-parallelogram is articulated while the common length a of sides AB, CD remains constant, as well as the common length b of BC and AD, and while points O, M, M' remain attached to the corresponding sides* (that is, the lengths AO, AM, BM' remain constant in magnitude and sense) *and finally, if the point O remains fixed, then the points M and M' move along two figures inverse to each other with respect to the pole O.*

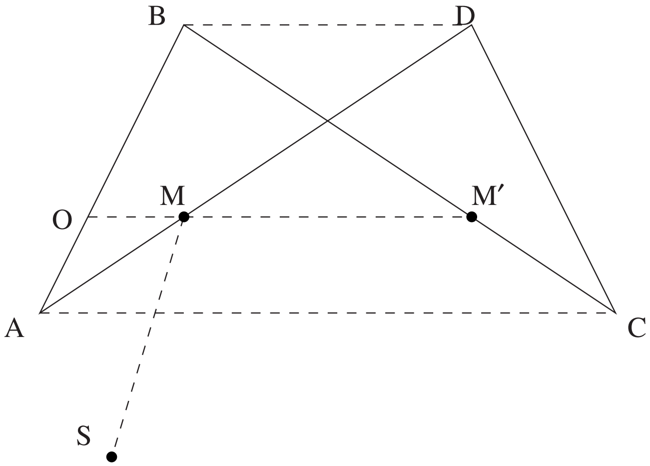

FIGURE 199b

PROOF. We know (**46b**) that the quadrilateral will remain an anti-parallelogram.

On the other hand, the lines OM, OM', which we assumed were parallel to the bases of the trapezoid in the initial position of the figure, will remain so, because this parallelism is characterized by the proportions

$$\frac{AM}{AD} = \frac{AO}{AB}, \quad \frac{BM'}{BC} = \frac{BO}{BA},$$

which, by hypothesis, do not change.

In particular, we see that points O, M, M' remain collinear.

Having seen this, assume for the moment that not only O, but the entire side AB remains fixed. Then points M, M' will move along circles with centers A, B, and radii AM, $BM' = MD$, respectively.

Since the ratio of these radii is equal to the ratio of the distances OA, OB, point O is one of the centers of similarity for the two circles. Since the radii with endpoints M, M' are not parallel, these points will be (**223**) antihomologous, and therefore inverse. The product $OM \cdot OM'$ therefore remains constant as the anti-parallelogram is deformed, and while points A, B remain fixed.

Since this product is not changed by a rigid rotation around O, the theorem is proved. QED

Corollary. *If point M is attached to a fixed point S by a rigid rod of length equal to SO, point M' will move along a straight line.*

Remark. *The product of the bases AC, BD remains constant as the figure is deformed.* Indeed, the ratio of BD to OM is constant and equal to $\frac{1}{h} = \frac{AB}{AO}$, and the ratio of AC to OM' is equal to $\frac{1}{h'} = \frac{BA}{BO}$.

Exercises

Exercise 269. The converse of Exercise 99: If a point M in the plane of an equilateral triangle ABC is such that $MA = MB + MC$, this point is on the circumscribed circle. Otherwise, $MA < MB + MC$.

Exercise 270. If we had started the proof in **237** and **238** with a vertex other than A, we would have obtained a triangle analogous to $B'C'D'$.

1°. Show that all of these triangles are similar;

2°. Calculate the angles of any one of these triangles, knowing the angles made by the sides and diagonals of the given quadrilateral;

3°. Show that we would obtain triangles similar to the preceding ones by dropping perpendiculars from one of the four vertices A, B, C, D to the sides of the triangle formed by the other three; or by joining B and D to the third vertex of a triangle similar to ABC, with the same sense of rotation, in which the side corresponding to AC is AD;

4°. Show that the shape of the similar triangles considered above is preserved, except for the sense of rotation, if points A, B, C, D are all subject to the same inversion; in other words, show that if we operate on these transformed points as we have operated in 1°, 2° or 3° on the points A, B, C, D, we will obtain triangles similar to the first ones;

5°. Conversely, if two quadrilaterals $ABCD$, $A_1B_1C_1D_1$ are such that the triangles resulting from the preceding constructions applied to each of them are similar, show that there exists[4] an inversion which transforms A, B, C, D into four points forming a quadrilateral congruent to $A_1B_1C_1D_1$. Determine this inversion.

Exercise 270b. Find an inversion which transforms three given points into three other points forming a triangle congruent to a given triangle.

Exercise 271. Peaucellier's inversion cell would still function if the articulated quadrilateral $MPM'Q$ were not a rhombus, but if $MP = M'P$, $MQ = M'Q$, and $OP^2 - OQ^2 = MP^2 - MQ^2$.

Exercise 271b. In Hart's inversion cell, calculate the power of the inversion knowing the sides a, b of the anti-parallelogram, as well as the ratios h and $h' = 1-h$. Calculate the product of the bases of the trapezoid knowing a and b.

Problems for the Complements to Book III

Exercise 272. The radical axis and the center of similarity divide the segment joining two antihomologous points in a constant cross ratio.

Exercise 273. The cross ratio of four points on a circle is equal to the cross ratio of their inverses on a circle (or line) inverse to the first.

[4]The case where $ABCD$, $A'B'C'D'$ are similar is an exception – transl.

Exercise 274. The cross ratio of four points on a circle is obtained by dividing the products of the opposite sides of the quadrilateral they form, or dividing one of these products by the product of the diagonals.

Exercise 275. Which inversions transform two given circles into equal circles? Find an inversion which transforms three given circles into three equal circles.

Exercise 276. Given three circles, we draw the three circles Γ (**228**) with respect to each pair of these circles, and with centers at the three centers of similarity situated on the same axis. Show that these three circles have the same radical axis.

Exercise 277. What is the locus of the centers of the inversions which transform two given circles into two others such that the first divides the second into equal parts; or, more generally, such that the first intercepts on the second an arc corresponding to a given angle at the center of the second?

Exercise 278. The segments intercepted by two circles on a line subtend, at the limit points (Exercise 152), angles having the same bisector, or perpendicular bisectors. Special case: the line is tangent to one of the circles.

Exercise 279. Prove Pascal's theorem (**196**) by forming three circles, each pair of which has a center of similarity at the intersections of two opposite sides of the hexagon. (Each of these circles passes through two opposite vertices.)

Exercise 280. Use Exercise 253 to construct the circles tangent to three given circles.

Exercise 281. A right angle pivots around a fixed point in the plane of a circle. We draw tangents to the circle at the intersection points with the sides of the angle. Find the locus of the vertices of the quadrilateral obtained in this way. (This locus is the inverse of that obtained in Exercise 201.)

Exercise 282. A quadrilateral $ABCD$ is inscribed in a circle O and circumscribed around a circle o. Let a, b, c, d be its points of contact with o.

1°. The point P where the diagonals of $ABCD$ and $abcd$ intersect (exercise 239) is a limit point of the circles O, o (Exercise 241, part 2).

2°. The diagonals of $abcd$ bisect the angles formed by the diagonals of $ABCD$ (Exercise 278).

3°. If we apply the preceding exercise to circle o and point P, we obtain circle O.

4°. Conclude that, given two circles, we cannot generally find a quadrilateral inscribed in one and circumscribed about the other. For this to be possible, a certain relation must be satisfied by the radii and the distance between the centers. If this relation is satisfied, there are infinitely many quadrilaterals satisfying the conditions.

Exercise 283. If a triangle has an obtuse angle, there exists a circle (and only one) with respect to which this triangle is conjugate (**211**, Remark). The center of this circle is the intersection of the altitudes of the triangle.

Exercise 283b. For the existence of a triangle inscribed in a given circle, and conjugate to another given circle, it is necessary that the square of the radius of the second circle be equal to one half the power of its center with respect to the other circle. If this condition is satisfied, there are infinitely many triangles with both required properties. (Use Exercise 70.)

Exercise 284. Given three circles, each pair of which intersect, we construct a curvilinear triangle whose three vertices are intersection points of the circles, and whose sides are arcs of these circles which do not contain any other intersection points. Show that the sum of the interior angles of the three circles is less than, or greater than, two right angles, according as whether there exists or not a circle orthogonal to the three given circles.[5]

Exercise 285. Since the necessary and sufficient condition for a quadrilateral $ABCD$ to be cyclic is that the points B', C', D', inverse to B, C, D with respect to A, are collinear, all properties of cyclic quadrilaterals can be deduced from this condition.

In particular, deduce the theorem on the ratio of the diagonals. (Application of Stewart's theorem, **127**.)

Exercise 286. Through two points which divide a diameter of a circle harmonically, we draw perpendiculars to this diameter. These lines are intersected by an arbitrary tangent at two points such that the ratio of their distances to the center is constant.

[5]See Note A, **290**.

Book IV
On Areas

CHAPTER I

The Measure of Areas

242. Two polygons are *adjacent* (Figs. 200, 200b) if they have one or more common sides or portions of sides (Fig. 200b), without having any interior point in common.

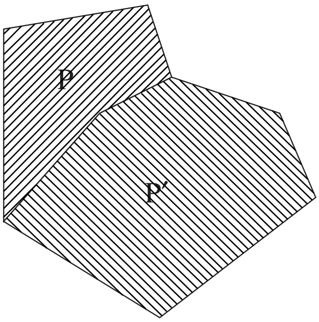

FIGURE 200

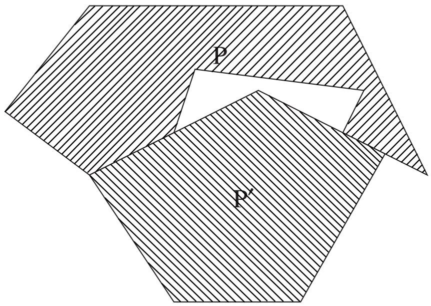

FIGURE 200b

If we remove the common sides of two adjacent polygons P, P', we form[1] a third polygon P'' called the *sum* of the first two. The interior of this polygon consists of the interior points of the original polygons, and only these points.

243. We define the areas of plane polygons by associating with each polygon a quantity (called the *area* or *surface area* of the polygon) satisfying the following properties:

I. Two congruent polygons have the same area, regardless of their position in space;

II. The sum P'' of two adjacent polygons P, P' has an area equal to the sum of the areas of P and P'.

We assume that such a correspondence can be made.[2]

244. Areas can be defined in infinitely many ways, because if we associate a quantity with each planar polygon so as to satisfy conditions I and II, a quantity proportional to the one we have used will also satisfy these conditions. In order to *measure* areas, as we are about to do, we must start by designating a polygon whose area is taken as the *unit* area. The *measure* of an arbitrary area will be the ratio of this area to the unit.

[1]In Book IV we suppress the restriction on the meaning of the word *polygon* formulated in **20** of Book I. The shaded portion of the plane in Figure 19 will therefore be treated in this book as a polygon. This polygon can be formed by taking the sum of two ordinary adjacent polygons, as shown in Figure 200. On the other hand, we consider in this book only *proper* polygons.

[2]This assumption is not really necessary, as it can be proved. See note D.

We agree from now on to take, as the unit of area, the area of a square whose side is the unit of length. The theorems we will state assume this convention, and it will not be necessary to repeat it in each individual case.

245. Two polygons with the same area are said to be *equivalent.* Two congruent polygons are therefore equivalent. The converse, naturally, is not true: two equivalent polygons need not be congruent. For instance, we will show that one can construct a square equivalent to any given polygon.

246. A *base* of a rectangle is one of its sides. The length of the side perpendicular to the base is then called the *altitude* of the rectangle. More generally, we can call one of the sides of a parallelogram its *base*; the *altitude* is then the distance from the base to the opposite side (measured, of course, along a common perpendicular).

Finally, a *base* of a triangle is just an arbitrary side, and an *altitude* is the perpendicular dropped on this side from the opposite vertex.

247. Theorem. *The ratio of the areas of two rectangles with the same base is equal to the ratio of their altitudes.*

Indeed:

1. Two rectangles with the same base and same altitude are congruent, and therefore equivalent by property I.

2. If three rectangles $ABCD$, $A'B'C'D'$, $A''B''C''D''$ (Fig. 201) have the same base, and the altitude of the third is the sum of the altitudes of the first two, the area of the third will be the sum of the areas of the first two (property II). Indeed, the third can be considered as the sum of two rectangles $A''B''EF$, $C''D''EF$ which are congruent, respectively, to the first two.

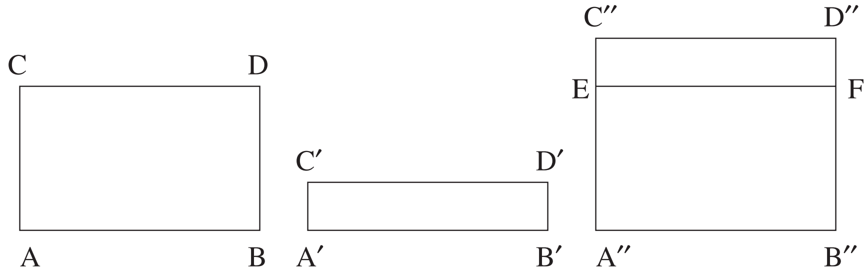

Figure 201

From these two facts we can prove, as in **113** (Book III) or **17** (Book I), and as is explained in the study of arithmetic[3] that an approximation to within $\frac{1}{n}$ of the ratio of the two areas is equal to an approximation within $\frac{1}{n}$ of the two altitudes, which proves the theorem.

Corollary. *Since either side of a rectangle can be taken as a base, or as an altitude, any result concerning altitudes is also true for bases. In other words, the ratio of the areas of two rectangles with one equal side is equal to the ratio of the unequal sides.*

[3] *Leçons d'Arithmétique*, chap. XIII, n°493.

THEOREM. *The ratio of the areas of two rectangles is equal to the product of the ratios of the corresponding dimensions.*

Indeed, we have just seen that the area of a rectangle is proportional to its base, and it is also proportional to its altitude, when these dimensions vary separately. The area is therefore proportional to the product of the base and the altitude.

We could also repeat the reasoning usually followed in arithmetic in such cases. Let A be the area of a rectangle with sides a, b, and A' the area of a rectangle with sides a', b'. Consider a rectangle with sides a' and b. The area A'' of this rectangle satisfies

$$\frac{A}{A''} = \frac{a}{a'}, \quad \frac{A''}{A'} = \frac{b}{b'}.$$

In this equation we can think of A, A', A'' as representing the numbers measuring the corresponding areas (relative to a given unit), rather than the areas themselves. For example, we might choose (as we have here), the unit area to be the area of a square constructed on a line segment of unit length. Under these conditions, we can multiply the two equations, and write the product $\frac{A}{A''} \cdot \frac{A''}{A'}$ as $\frac{AA''}{A''A'} = \frac{A}{A'}$. Thus the ratio $\frac{A}{A'}$ is indeed equal to the product of the ratios $\frac{a}{a'}$ and $\frac{b}{b'}$.

THEOREM. *The area of a rectangle is equal to the product of its two dimensions.*

This theorem is the same as the preceding one, applied to the given rectangle and to the square with unit side. The area A' of the square is the unit area, and its dimensions a', b' are both equal to one, so that the ratio $\frac{A}{A'}$ is just the measure of the area A, and the ratios $\frac{a}{a'}$, $\frac{b}{b'}$ are the measures of the lengths a, b.

REMARK. 1°. The statement above only makes sense if read according to the conventions of **18** (which we also recalled in **106**, Book III). The meaning of the statement is: *the* **number** *which measures the area of a rectangle is equal to the product of the* **numbers** *which measure the base and the altitude.*

2°. This statement is precise only if read according to the convention of **244**, which is therefore assumed here implicitly. It is clear *a priori* that the stated equality does not hold if the unit of length and the unit of area are chosen arbitrarily, and independently of each other.

We see from this that we can choose the unit of length arbitrarily, but once this choice is made, the choice for the unit of area is determined. We sometimes say that the unit of area is a *derived* unit.

248. THEOREM. *The area of a parallelogram is the product of its base and its altitude.*

Suppose the parallelogram is $ABCD$ (Fig. 202). We draw perpendiculars from A, B to side AB, and extend them to their intersections c, d with line CD, to form rectangle $ABcd$. This rectangle is equivalent to the parallelogram, because if we add to these quadrilaterals the right triangles BdD, AcC (which are congruent, having a pair of equal angles and a pair of equal sides) we obtain the same sum; namely, trapezoid $AcBD$. The rectangle has dimensions equal to the base and altitude of the parallelogram, so the theorem is proved.

249. THEOREM. *The area of a triangle is equal to half the product of the base and the altitude.*

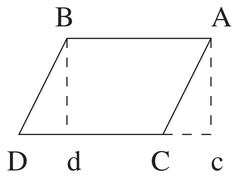

FIGURE 202

Suppose the triangle is ABC (Fig. 203). We draw parallel AD to BC and parallel CD to AB. The parallelogram $ABCD$ thus formed has the same base and altitude as the triangle. This parallelogram is divided into two congruent triangles ABC, ADC by its diagonal AC (**46**). Thus triangle ABC is half the parallelogram.

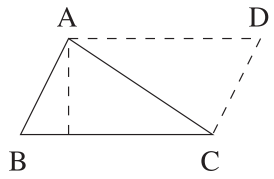

FIGURE 203

250. COROLLARY. *The locus of the vertices of a triangle with base AB and constant area is composed of two lines parallel to AB.*

This is because the vertices in question (Fig. 204) are just those points which are at a fixed distance from AB.

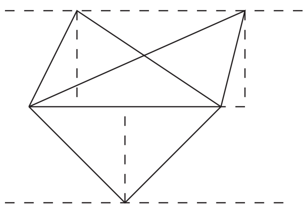

FIGURE 204

251. Problem. Calculate the area of a triangle knowing the three sides.

We denote the sides of the triangle by a, b, c, and the semi-perimeter by p (so that $a + b + c = 2p$). We have seen (**130**) that the altitude AH corresponding to side a is

$$AH = \sqrt{\frac{4p(p-a)(p-b)(p-c)}{a^2}} = \frac{2}{a}\sqrt{p(p-a)(p-b)(p-c)}.$$

The area is then given by

$$\begin{aligned} S = \frac{a \cdot AH}{2} &= \sqrt{p(p-a)(p-b)(p-c)} \\ &= \frac{1}{4}\sqrt{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}. \end{aligned}$$

Remark. The product of the three sides of a triangle is equal to four times the product of its area and the radius of the circumscribed circle.

In triangle ABC, where AH is the altitude from A, and R the radius of the circumscribed circle, we have (**130b**) $AB \cdot AC = 2R \cdot AH$. Multiplying both sides by BC, we obtain

$$AB \cdot AC \cdot BC = 2R \cdot AH \cdot BC = 4RS.$$

252. The area of a polygon. The area of an arbitrary polygon can be obtain by decomposing it into triangles, then adding their areas.

The following two results are just examples of this method.

Theorem. *The area of a trapezoid is equal to half the sum of the bases times the altitude.*

Suppose the trapezoid is $ABCD$ (Fig. 205). Drawing diagonal AD, we divide the base into triangles ABD, ACD. We take their bases to be AB, CD. Then the corresponding altitudes DD', AA' are equal to each other and to altitude h of the trapezoid. The area of $ABCD$ is therefore equal to

$$h \cdot \frac{AB}{2} + h \cdot \frac{CD}{2} = \frac{h(AB + CD)}{2}.$$

252b. Corollary. *The area of a trapezoid is the product of its altitude and the segment joining the midpoints of the non-parallel sides.*

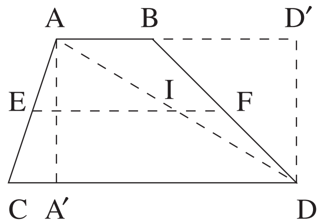

Figure 205

Indeed, the midpoints E, F of sides AC, BD (Fig. 205) and midpoint I of AD are on the same parallel to the bases, and segment EF is equal to half the sum of the bases, since it consists of the pieces EI, IF which are equal to $\frac{CD}{2}$, $\frac{AB}{2}$, respectively.

253. Theorem. *The area of a regular polygon is equal to the perimeter times half the apothem.*

Indeed, we decompose regular polygon $ABCDEF$ (Fig. 206) into triangles OAB, $OBC, \ldots$, which are all congruent and therefore have the same altitude, equal to the apothem OH. Thus

$$\operatorname{area} OAB = OH \cdot \frac{AB}{2}$$
$$\operatorname{area} OBC = OH \cdot \frac{BC}{2}$$
$$\cdots\cdots\cdots$$
$$\operatorname{area} OFA = OH \cdot \frac{FA}{2},$$

and addition gives us:

$$\operatorname{area} ABCDEF = \frac{OH(AB + BC + \cdots + FA)}{2}.$$

QED

REMARK. If the polygon has an even number of sides, its area is equal to half the radius times the perimeter of the polygon obtained by joining every other vertex.

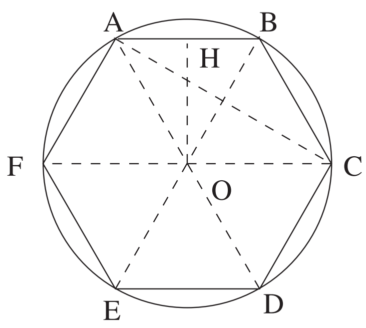

FIGURE 206

This is true because the area of triangle OAB (Fig. 206) is also equal to half the product of OB and the altitude AI from A, which is equal to half of AC.

253b. A polygon bounded by a broken line inscribed in a circle (Fig. 207), and by the two radii of the circumscribed circle corresponding to the endpoints of the line, is called a *polygonal sector.*

The polygonal sector is said to be *regular* if the broken line which serves as its base is regular.

THEOREM. *The area of a regular polygonal sector is the product of the perimeter of the base and half the apothem.*

The proof of this result is identical to the preceding one.

254. THEOREM. *The area of a convex polygon circumscribed about a circle (and containing the circle in its interior) is equal to the product of its perimeter and half the radius of the inscribed circle.*

The polygon (Fig. 208) can be decomposed into triangles with a common vertex at the center of the circle. These triangles, which have as bases the various sides, also have a common altitude equal to the radius.

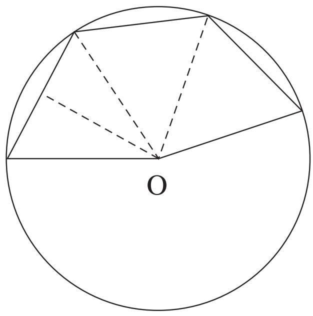

FIGURE 207

255. PROBLEM. *Find the area of a cyclic quadrilateral knowing its four sides.*

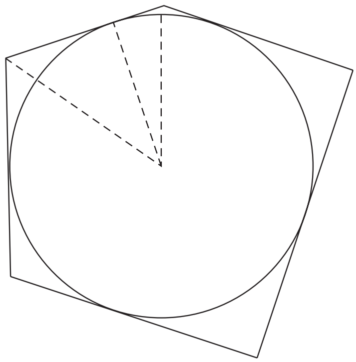

FIGURE 208

Consider cyclic quadrilateral $ABCD$ in which $AB = a$, $BC = b$, $CD = c$, and $DA = d$. If R is the radius of the circumscribed circle, we have (**251**, Remark)

$$a \cdot b \cdot AC = 4R \times \text{area}\, ABC, \quad c \cdot d \cdot AC = 4R \times \text{area}\, ACD.$$

The required area S is the sum of the areas of ABC and ACD, so that

$$4RS = AC(ab + cd) = \sqrt{(ac + bd)(ad + bc)(ab + cd)}.$$

Replacing R by the value found in **240b**,we have:

$$S = \sqrt{(p - a)(p - b)(p - c)(p - d)}.$$

REMARK. The result is independent of the order of the sides, which is obvious *a priori* if we note that quadrilaterals $ABCD$, $ABEF$, $ABGH$ (see **240**) are equivalent, being composed of pairs of congruent triangles.

Exercises

Exercise 287. Find the area of an equilateral triangle with side a.

Exercise 288. What is the side of an equilateral triangle with an area of 1 square meter?

Exercise 289. We travel around the perimeter of a square in a determined direction, and join each vertex to the midpoint of the side which precedes the opposite vertex. Show that the lines obtained in this way form a new square, whose area is one-fifth the area of the original square.

Exercise 290. Through a point on diagonal AC of parallelogram $ABCD$ we draw lines parallel to the sides. The parallelogram is divided into four smaller parallelograms, two of which have a diagonal on AC. Show that the other two are equivalent.

Exercise 291. Among all triangles with the same base and same angle at the opposite vertex, which one has the greatest area?

Exercise 292. In a trapezoid, the triangles formed by each diagonal with one of the non-parallel sides are equivalent, and conversely.

Exercise 293. Through the midpoint of each diagonal of a quadrilateral, we draw a parallel to the other diagonal, then join the intersection point of these parallels to the midpoints of the four sides. Show that the quadrilateral is divided by these segments into four parts with equal areas.

Exercise 294. Through each vertex of a quadrilateral, we draw a parallel to the diagonal which does not pass through that vertex. Show that the parallelogram formed this way has twice the area of the quadrilateral.

Two quadrilaterals in which corresponding diagonals are equal and intersect at the same angle are equal in area.

Exercise 295. Find a point inside a triangle such that when this point is joined to the three vertices, three equivalent triangles are formed. More generally, find a point such that the three triangles formed are proportional to three given lengths or numbers.

Exercise 295b. Using area considerations (see the preceding exercise), show that if we join a point to the three vertices of a triangle, the lines formed divide the opposite sides into ratios whose product is one. (This theorem was proved in **197**.)

Exercise 296. We join a point O in the plane to the vertices of a parallelogram $ABCD$ (in which AC, BD are diagonals).

1°. If the point O is inside the parallelogram, the sum of the triangles OAB, OCD is equivalent to the sum of OBC, ODA.

2°. For any point O, the triangle OAC is equivalent to the sum or difference of OAB, OAD.

Exercise 297. The area of a trapezoid is equal to the product of a non-parallel side and the perpendicular dropped to this side from the midpoint of the opposite side. Prove this in two different ways:

1° by rewriting the expression given in **252b**;

2° by showing directly that the trapezoid is equivalent to a parallelogram with the base and altitude indicated above.

If we join the midpoints of the non-parallel sides of a trapezoid to the endpoints of the opposite side, the triangle obtained has half the area of the trapezoid.

Exercise 298. The sum of the distances of a point inside a regular polygon to its sides is constant.

Exercise 299. The area of a triangle is equal to the product of the semi-perimeter and the radius of the inscribed circle.

The area of a triangle is also equal to the product of the radius of an escribed circle and the difference between the semi-perimeter and the corresponding side (or the radius of an escribed circle and half the difference between this side and the sum of the other two).

Exercise 300. The inverse of the radius of the inscribed circle of a triangle is equal to the sum of the inverses of the radii of the three escribed circles.

Exercise 301. If x, y, z denote the distances from a point inside a triangle to the three sides, and h, k, ℓ are the corresponding altitudes, then

$$\frac{x}{h} + \frac{y}{k} + \frac{z}{\ell} = 1.$$

What happens if the point is outside the triangle?

CHAPTER II

Comparison of Areas

256. THEOREM. *The ratio of the areas of two triangles with a pair of equal (or supplementary) angles is equal to the ratio of the products of the sides containing these angles.*

We can superimpose the angles if they are equal, or make them adjacent if they are supplementary. The two triangles will then be ABC, $AB'C'$, with sides AC, AC' in the same direction, while the sides AB, AB' are either in the same direction (Fig. 209), or are extensions of each other (Fig. 210).

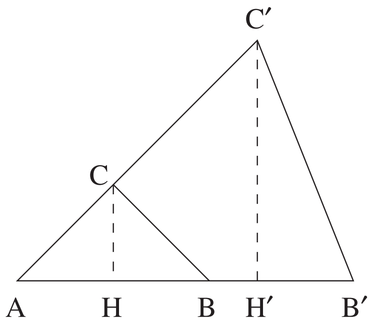

FIGURE 209

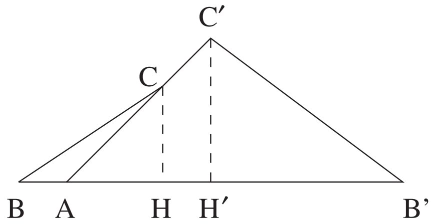

FIGURE 210

The ratio of the two triangles will be equal to the ratio of their bases AB, AB', multiplied by the ratio of their altitudes CH, CH', and this last ratio is obviously equal to $\frac{AC}{AC'}$. QED

257. THEOREM. *The ratio of the areas of two similar polygons is equal to the square of their ratio of similarity.*

We will distinguish two cases.

1°. If we are dealing with two similar triangles ABC, $A'B'C'$, it suffices to observe that these triangles will have angle $\widehat{A} = \widehat{A'}$. Therefore the ratio of the areas is equal to the product of the ratios $\frac{AB}{A'B'}$ and $\frac{AC}{A'C'}$, that is, to the square of one of them, since these two ratios are equal.

2° Now consider two arbitrary similar polygons, $ABCDE$, $A'B'C'D'E'$ (Fig. 211), with a ratio of similarity equal to k.

These polygons can be decomposed into similar triangles, arranged similarly: ABC, ACD, ADE; $A'B'C'$, $A'C'D'$, $A'D'E'$, and as seen in case 1,

$$\frac{ABC}{A'B'C'} = \frac{ACD}{A'C'D'} = \frac{ADE}{A'D'E'} = k^2.$$

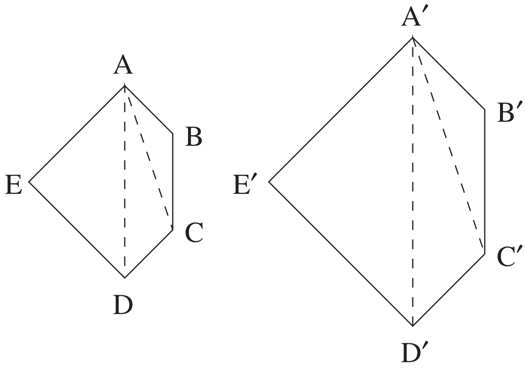

FIGURE 211

Taking the sum of the numerators and of the denominators we obtain

$$\frac{ABCDE}{A'B'C'D'E'} = k^2.$$

QED

258. THEOREM. *The square constructed on the hypotenuse of a right triangle is equivalent to the sum of the squares constructed on the two legs.*

Let the right triangle be ABC, and construct squares $ABEF$, $ACGH$, and $BCIJ$, outside this triangle (Fig. 212), on legs AB, AC and on hypotenuse BC.

We drop perpendicular AD from A to BC, and extend it to its intersection K with IJ. We claim that rectangle $BDIK$ is equivalent to square $ABEF$.

To prove this, we join AI and CE. Then triangle ABI has the same base BI and the same altitude $AL = BD$ as rectangle $BDIK$; it is therefore equivalent to half this rectangle. Also, triangle EBC has the same base BE and the same altitude $CM = AB$ as square $ABEF$, so it is equivalent to half this square. Now triangles ABI, EBC are congruent, since they have an equal angle at A (angle $\widehat{B}$ of the triangle plus a right angle) contained between equal corresponding sides ($AB = BE$ and $BI = BC$). Thus we see that rectangle $BDIK$ is indeed equivalent to square $ABEF$.

In the same way, we can show that rectangle $CDKJ$ is equivalent to square $ACGH$. Thus square $BCIJ$, the sum of the two rectangles, is indeed equivalent to the sum of the two squares.

REMARK. This theorem is the same as the one we proved in **124**, since the three squares $ABEF$, $ACBH$, $BCIJ$ have areas equal to the squares of the numbers which measure the sides. The proof also follows the same line, because the equivalence of rectangle $BDIK$ to square $ABEF$ expresses the same idea as the equation $AB^2 = BC \cdot BD$, which we used at that point.

Exercises

Exercise 302. The three sides of a triangle are divided into given ratios, and a new triangle is formed whose vertices are the points of division. Find the ratio of the area of the new triangle to that of the original triangle. Examine the case

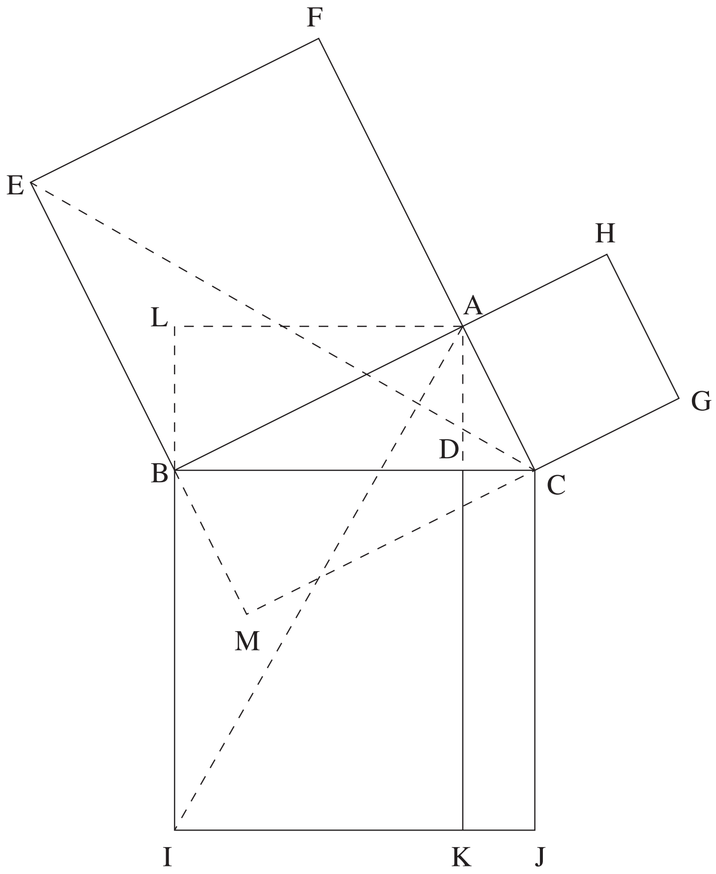

FIGURE 212

in which one or several of the division points is outside the triangle. Deduce from this proofs for the theorems in **192–193**.

Exercise 303. Divide a triangle into two equivalent parts by a line parallel to the base.

Exercise 304. Divide a trapezoid into two equivalent parts by a line parallel to the bases.

Exercise 305. An equilateral triangle is inscribed in a given circle. The radii passing through its vertices are extended to intersect a new circle, the one circumscribing a square which is circumscribed about the original circle. Prove that these points form a triangle equal in area to the regular hexagon inscribed in the original circle.

Exercise 306. Through each vertex of a quadrilateral, we draw a line parallel to a fixed direction, and find the intersection of each of these parallels with the diagonal which does not pass through that vertex. Prove:

1°. that the quadrilateral formed by these intersections is equivalent to the original quadrilateral;

2°. that this new quadrilateral is a trapezoid or a parallelogram, if the original is. More generally, the ratios in which the opposite sides (extended) divide each other are the same in the two quadrilaterals.

Exercise 307. Given some arbitrary polygons in the plane, we draw parallels through each of their vertices in a fixed direction, then measure off on each parallel a distance proportional to the distance from the vertex considered to a fixed line (in the sense corresponding to the sense of the segment from the vertex to the fixed line).

Show that the endpoints of these segments form polygons whose areas are proportional to those of the original polygons (Exercise 297).

When are the new polygon equal in area to the original polygons?

Exercise 308. Show that the preceding exercise contains Exercise 306 as a special case (use Exercise 129).

Alternative Proofs of the Theorem on the Square of the Hypotenuse

Exercise 309. In Exercise 43 (Book I), show that the third square $HBKF$ is equivalent to the sum of the two given squares. Deduce from this the theorem of **258**.

Exercise 310. Prove the theorem on the square of the hypotenuse using the theorem of **257**, and observing that the right triangle is the sum of the triangles into which it is divided by its altitude.

Exercise 311. Given an arbitrary triangle ABC, we construct any parallelograms $ABEF$, $ACGH$, on bases AB, AC, and outside the triangle. Then we extend EF, GH to their intersection point M. Show that the parallelogram having BC for one side, and the other parallel and equal to AM, is equivalent to the sum of the first two parallelograms.

Show that the previous result is a special case of the theorem on the square of the hypotenuse.

CHAPTER III

Area of the Circle

259. We define the *area of a circle* as the limit approached by the area of an inscribed or circumscribed polygon, all of whose sides tend to zero.

To show that this limit exists and is independent of the way in which the sides tend to zero, we follow reasoning similar to that used to study the length of the circle. We start by considering regular inscribed polygons with a number of sides which is doubled indefinitely, and the corresponding circumscribed polygons (Fig. 168). Under these conditions:

The areas of the inscribed polygons increase, because each one contains the preceding one. Any of these areas is smaller than the area of an arbitrary circumscribed polygon. Therefore *these areas tend to a limit.*

In the same way, *the areas of the circumscribed polygons decrease*, because each one is contained in the preceding one. Each of these areas is greater than the area of any inscribed polygon. Therefore *the areas of the circumscribed polygons also tend to a limit.*

These two limits are equal, since the ratio of the area of an inscribed polygon and the corresponding circumscribed polygon is equal to the square of the ratio of similarity, which tends to one (**176**).

260. Let S be the common value of these limits, obtained by starting from the square, for instance, and considering the polygons with $4, 8, 16 \ldots, 2^n \ldots$ sides. Now consider arbitrary inscribed polygons $a'b'c' \ldots$ (Fig. 169) and the corresponding circumscribed polygons $A'B' \ldots$, imposing only the condition that the number of sides of these polygons increases indefinitely, in such a way that all the sides tend to zero.

The area of the inscribed polygon $a'b'c' \ldots$ is smaller than S because S is the limit of areas of circumscribed polygons, which are all greater than $a'b'c' \ldots$.

Since S is contained between the inscribed and circumscribed polygons, it differs from each of them by less than they differ from each other.

Now *this difference tends to zero.* Indeed, it is composed of triangles $a'b'A'$, $b'c'B'$,..., and is thus equal in area to

$$\frac{1}{2}(a'b' \cdot A'H + b'c' \cdot B'K + \cdots) < \frac{1}{2}(a'b' + b'c' + \cdot)\ell,$$

where $A'H, B'K, \ldots$ (Fig. 169) are the altitudes of these triangles, and ℓ denotes the greatest of these altitudes.

The factor $a'b' + b'c' + \cdots$ approaches the length of the circle, while the altitudes $A'H, B'K, \ldots$ all tend to zero. Indeed, the lengths OH, OA', for instance, both tend to the radius of the circle, and therefore their difference $A'H$ tends to zero.

Thus, *the areas of inscribed and circumscribed polygons tend to a unique limit* S which is the area of the circle.[1]

261. THEOREM. *The area of a circle is numerically equal to the length of the circumference, multiplied by half the radius.*

Indeed, the area of a regular convex polygon is the product of its perimeter and half the apothem. As the number of sides increases indefinitely, the perimeter approaches the length of the circle, and the apothem approaches the radius.

REMARK. We obtain the same result using the theorem of **254** about the circumscribed polygon. Indeed, the perimeter of such a polygon will approach the length of the circle as the number of sides increases indefinitely.

COROLLARY. *The area of the circle with radius* R *is* πR^2.

Indeed, $2\pi R \times \frac{R}{2} = \pi R^2$.

262. A *circular sector* is a portion of the plane bounded by a circular arc and the radii corresponding to its endpoints.

The *area* of such a sector is the limit of the areas of inscribed polygonal sectors as all the sides decrease indefinitely. The existence of this limit is established as in the case of the circle.

THEOREM. *The area of a circular sector is equal to the arc which bounds it, multiplied by half the radius.*

Indeed, the perimeter of the regular inscribed broken line approaches the length of the arc, and the apothem approaches the radius.

The length of an arc of n gradients in a circle of radius R is $\frac{\pi n R}{200}$, and therefore the area of the corresponding sector is $\frac{\pi R^2 n}{400}$.

Analogously, the area of a sector of a circle with radius R, corresponding to an angle equal to $m^\circ n' p''$ will be

$$\frac{\pi R^2}{360}\left(m + \frac{n}{60} + \frac{p}{3600}\right).$$

263. Areas bounded by circular arcs. A *segment* of a circle (Figures 213 and 214) is the area contained between an arc and its chord.

Clearly, the area of a segment is obtained from the area of the corresponding sector by subtracting the triangle with the chord as base and the center as its vertex, provided that the arc is less than a semicircle (Fig. 213), and by adding the same triangle in the opposite case (Fig. 214).

More generally, the calculation of the area of a region bounded by arcs of circles can be reduced to that of a polygonal area by adding or subtracting sectors or segments of circles.

---

[1]The distance ab approaches zero, independent of the position of arc $\overset{\frown}{ab}$ on the given curve. This condition is satisfied if the analogous condition in the note to **179** is satisfied. We are then assured that the difference between the areas of the inscribed and circumscribed polygons approaches zero as the sides of these polygons approach zero.

As has been already indicated in the note mentioned above, the argument can be divided into two parts, corresponding to sections **259** and **260** in the text.

As for non-convex areas, they may be evaluated as sums or differences of convex ones.

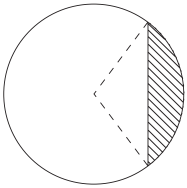

FIGURE 213

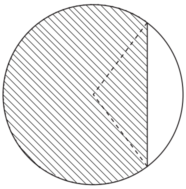

FIGURE 214

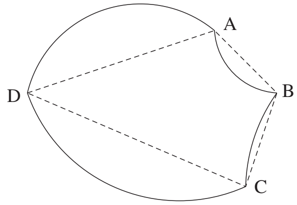

FIGURE 215

For instance, the curvilinear area $ABCD$ (Fig. 215) is equal to the area of the ordinary quadrilateral $ABCD$, diminished by the segments AB, BC, and increased by the segments CD, DA.

In conclusion, we know how to *find the area of any portion of the plane bounded by lines and circular arcs.*

Exercises

Exercise 312. What is the radius of a circle with an area of one square meter?

Exercise 313. What is the radius of the circle in which a sector of $15^G 25'$ has an area of one square meter?

Exercise 314. What is the radius of a circle in which the segment contained between an arc of $60°$ and its chord has an area of one square meter?

Exercise 315. Prove that the area of the annulus bounded by two concentric circles is equal to the area of a circle whose diameter is the chord of the larger circle which is tangent to the smaller one.

Exercise 316. On a quarter circle AC we take points B, D at equal distances from the endpoints. We then drop perpendiculars BE, DF to radius OC. Show that the curvilinear trapezoid $BEFD$ is equivalent to sector OBD.

Exercise 317. On the legs of a right triangle as diameters, we draw semicircles outside the triangle. We draw a third semicircle on the hypotenuse as diameter, but this time on the same side of the hypotenuse as the triangle. Show that the sum of the two *lunes,* or crescents, contained between the smaller semicircles and the large one is equal to the area of the triangle.

Exercise 318. On side AB of a square inscribed in circle O, we draw a new semicircle, outside the square. Radius OMN intersects this semicircle at N and the original circle at M. Show that the area bounded by segments MN and by arcs $\widehat{MA}$, $\widehat{NA}$ can be *squared*; that is, we can construct, with straightedge and compass, a square equivalent to it.

CHAPTER IV

Constructions

264. Problem. Construct a triangle with a given base, equivalent to a given triangle.

The altitude of the required triangle is obviously the fourth proportional to the given base, the base of the given triangle, and the altitude of the given triangle. Once this altitude is found, we place it on an arbitrary perpendicular to the given base to find the third vertex of one of the (infinitely many) triangles which solve the problem.

It is also clear that we can find a triangle with a given altitude and equivalent to a given triangle.

265. Problem. Construct a triangle equivalent to a given polygon.

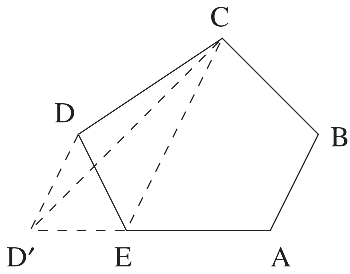

Figure 216

The polygon can be decomposed into several triangles, all of which can be given the same base by the preceding construction. A triangle with this common base and an altitude equal to the sum of the altitudes of these triangles will be equivalent to the sum of the triangles, thus to the given polygon.

This construction can often be simplified. We do this now for the case of a convex polygon.

Consider polygon $ABCDE$ (Fig. 216). We draw diagonal CE to join the two vertices adjacent to D, and draw parallel DD' to CE, intersecting side AE (extended) at D'. Triangle CED' is equivalent to CED (**250**), and therefore polygon $ABCD'$ is equivalent to $ABCDE$. We have now replaced the original polygon with an equivalent polygon with one fewer side. We continue this process until we arrive at a triangle.

Problem. Construct a square equivalent to a given polygon.

The side of the required square will be the geometric mean between the base and half the altitude of the triangle provided by the preceding construction.

266. Problem. Construct a polygon equivalent to a given polygon, and similar to another given polygon.

Suppose we want to construct a polygon P similar to the given polygon P', and equivalent to the given polygon P_1.

Let a' be the side of the square equivalent to P', and a the side of the square equivalent to P_1, obtained using the construction just mentioned above. The ratio of the areas of P' and P_1, which is the ratio of the area of polygon P' to that of the required polygon, is

$$\frac{a'^2}{a^2} = \left(\frac{a'}{a}\right)^2.$$

Consequently, if $A'B'$ is an arbitrary side of P', the corresponding side AB of the required polygon is given by the proportion

$$\frac{a'}{a} = \frac{A'B'}{AB}.$$

This side will be found by constructing the fourth proportional (**151**), after which we use the construction of **152**.

267. The celebrated problem of the *quadrature of the circle* consists of finding a square equivalent to a given circle.

The side of this square is the geometric mean between half the radius and the length of the circle, and the problem would be solved if we could construct this length.

Conversely, if we knew the side of the square equivalent to the circle, the length of the circle would be the third proportional of the radius and this side.

The problem of the quadrature is therefore the same as the one we discussed in **184**: *Construct the length of a circle with a given radius.* As we said already, this problem, and therefore the quadrature of the circle, cannot be solved with straightedge and compass alone.

Exercises

Exercise 319. Construct a rectangle knowing its area and its perimeter. Which is the largest rectangle with a given perimeter?

Exercise 320. Inscribe a rectangle with a given area in a given circle. What is the largest rectangle inscribed in the given circle?[1]

Exercise 321. Divide a triangle into equivalent parts by lines having a given direction.

The same problem for an arbitrary polygon.

Exercise 322. Divide a quadrilateral into equivalent parts by lines passing through a vertex. Show that it suffices to divide the diagonal not passing through this vertex into equal parts, then draw parallels to the other diagonal through the points of division, extending them to their intersections with the perimeter of the polygon.

Exercise 323. Divide an arbitrary polygon into equivalent parts by lines passing through a vertex.

[1]For an algebraic solution of this problem, see Bourlet, *Leçons d'Algèbre Élémentaire*, Book IV, Chap. V.

Problems Proposed for Book IV

Exercise 324. Show that the result of Exercise 296 is correct for any point O in the plane, provided that the areas of the triangles are preceded by a $+$ or a $-$ sign, according to their orientation.

Exercise 325. Two homothetic polygons are positioned so that the larger one contains the small one. Show that the area of any polygon inscribed in one of them and circumscribed about the other is equal to the geometric mean of the areas of the two original polygons.

Exercise 326. Find the ratio of the area of a triangle to the area of the triangle with sides equal to the medians of the first.

Exercise 327. Two lines, one through each of two vertices of a triangle, divide the opposite sides into given ratios. Find the ratios among the various areas into which the triangle is divided.

Exercise 328. Three lines, one through each of three vertices of a triangle, divide the opposite sides into given ratios. Find the ratio of the area of the triangle formed by these three segments to the area of the original triangle. Deduce the theorem of **197–198**, which gives a condition that the three lines pass through the same point.

Exercise 329. Through a given point inside an angle, draw a secant forming a triangle of given area with the sides of the angle. (Start by constructing a parallelogram with two sides on the sides of the given angle, a third one passing through the given point, and having the given area. The required secant must cut a triangle from this parallelogram equivalent to the sum of those it forms with its sides, and outside the parallelogram.)

Exercise 330. Among all the lines passing through a point inside an angle, and cutting the sides of the angle, but not their extensions, which one forms the smallest triangle?

Exercise 331. Show that among all the polygons with the same number of sides, inscribed in a circle, the largest is the regular polygon.[2]

Exercise 332. Construct a triangle knowing a side, the corresponding altitude, and the radius of the inscribed circle.

Exercise 333. Inscribe a trapezoid in a given circle, knowing one of its angles and its area.

Exercise 334. A triangle and a parallelogram have the same base, one pair of equal angles adjacent to this base, and equal areas. Decompose one of these polygons into two pieces which, assembled differently, form the other.

[2]An irregular polygon inscribed in a circle has at least one side smaller and one side greater than the side of a regular inscribed polygon with the same number of sides. One can move the sides around, without changing the area or the radius of the circumscribed circle, so that these two sides are consecutive.

Suppose these sides are AB and BC. Move the point B to a new position so that AB equals the side of the regular polygon, and show that this operation increases the area. Repeating this operation sufficiently many times, we reach the desired conclusion.

Exercise 335. Two triangles have the same base and the same altitude. Decompose one of them into pieces which, assembled differently, form the other. (Use the preceding problem to reduce this problem to the corresponding question for parallelograms.)

Exercise 336. Same problem for two arbitrary equivalent triangles.

Exercise 337. Same problem for two arbitrary equivalent polygons.

Exercise 338. Given four points A, B, C, D, the product of the area of triangle BCD times the power of A relative to the circle circumscribed about this triangle is equal to the analogous product formed with point B and triangle CAD, or with point C and triangle ABD, or with point D and triangle ACB.

Show that these can be viewed as equalities in magnitude and sign, using the convention of Exercise 324.

Exercise 339. Each of the products of the preceding exercise is equal to the area of a triangle whose sides are measured by the products $AB \cdot CD$, $AC \cdot DB$, $AD \cdot BC$ (**239**, Exercise 270b).

Exercise 340. Decompose a triangle into isosceles triangles. Same problem for an arbitrary polygon.

Exercise 341. Evaluate the area of the triangle formed by three circular arcs of radius R, intersecting at right angles.

Exercise 342. If two triangles are symmetric with respect to the center of their common inscribed circle, the areas of the eight triangles formed by their sides have a product equal to the sixth power of the radius of this circle.

Note A: On the Methods of Geometry

268. We would like to collect here some advice which we believe to be very useful for the understanding of mathematics in general and, in particular, for solving problems.

We must convince students that they will not be able to benefit from their study of mathematics, will not be able to pursue this study without excessive effort, and will not be able to form a proper idea of what geometry consists in, unless they not only understand the proofs they read, but also construct for themselves, to a greater or lesser extent, proofs of theorems or solutions to problems.

Contrary to common prejudice, this understanding can be achieved by anyone, or at least by anyone willing to think and to direct their thoughts methodically. The precepts which we will indicate derive from the most ordinary common sense. There is not one among them which may not seem a pure banality to the reader. However, experience shows that forgetting one or another of these obvious rules is the cause of almost every difficulty encountered in the solution of elementary problems; this is also true, more often than one might think, in research in more or less advanced parts of the mathematical sciences.

(a) Theorems to Prove

269. To prove a theorem means to pass, by way of reasoning, from the hypothesis to the conclusion.

Consider the theorem (Book I, **36**): *Every point on the bisector of an angle is equidistant from the two sides of this angle.* The hypothesis and conclusion are: Hypothesis: If the point M is on the bisector of the angle $\widehat{BAC}$ (Fig. 35) Conclusion: Then it will be equidistant from AB and AC.

We have to deduce the second from the first, to *transform* the properties stated in the hypothesis, so as to derive the truth of those which constitute the conclusion.

Obviously, it is necessary *to know very precisely what the hypothesis is and what the conclusion is,* of the theorem we want to prove. The student must therefore practice stating them without any hesitation.

270. Before we continue, we can already make an important observation. In every proof, one proposes to show that the conclusion holds *when the hypothesis is supposed true.* If the hypothesis is not considered certain, nothing can assure us of the truth of the conclusion. Thus, in the preceding example, if the point M were not on the bisector, we in fact know (**36**) that it would not be equidistant from the two sides.

It is clear that it would not serve any purpose to suppose that something is true, if the reasoning carried no trace of this assumption, if this assumption were not used at some point. We see that *the hypothesis must be used in the proof* and even, generally, *the entire hypothesis.*

271. We will return a little later to the preceding rule. But first, we have an immediate need to formulate another, analogous rule. We must call special attention to it, since, while indispensible, it is often misunderstood. It refers to the *definitions* of the terms used.

On the one hand, it is obvious that we could not know how to reason about notions which have not been defined; on the other, as before, it is clear that it is the same thing to ignore a definition, or not to use it in our reasoning.

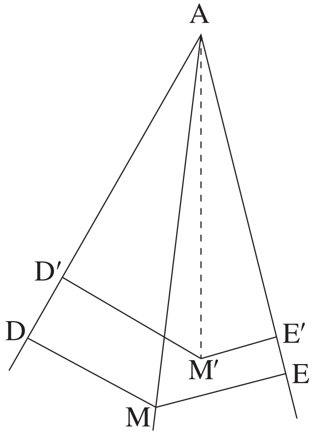

FIGURE 35

Therefore it is of paramount importance that we *refer to the definitions* of all the notions involved in the situation at hand.

EXAMPLE. Consider the same theorem:

Every point on the bisector of an angle is equidistant from the two sides of this angle.

We must start by asking the following questions:

What does it mean that the line MA is the angle bisector?

ANSWER: That it divides the angle $\widehat{A}$ into two equal parts.

What is the distance from M to line AB?

ANSWER: It is the length of the perpendicular dropped from this point to this line.

So our statement becomes:

Hypothesis: $\widehat{MAD} = \widehat{MAE}$; MD is perpendicular to AD; ME is perpendicular to AE.

Conclusion: $MD = ME$

We hope that this example will clarify sufficently the meaning of this rule, which Pascal places at the basis of all logic: we must **substitute the definition in the place of the term which it defines**.[1]

[1]It is even necessary, in general, to use all the parts of the definition when there are several. We may note in this respect the remarks made, a little later, about the hypothesis (**275**).

272. The definition of the same term can sometimes be given in several forms, among which we must choose the one which is the most convenient for the purpose we have in mind. Thus, we can define the bisector AM of the angle as the line which forms an angle equal to one half of the original angle with one of its sides (in the appropriate sense of rotation).

This formulation of the definition would not have been advantageous for the preceding theorem. On the other hand, it is the form which served best, for instance, in the proof of the theorem in **16** (Book I).

Certain theorems allow us, in the same way, to replace a definition with an equivalent one. Thus, the original definition of parallels (**38**) is no longer used in the original form of **39**, where we learned to replace it by the following form, which is exactly equivalent to the first: *Two lines are parallel if they form equal alternate interior angles with the same transversal (or equal corresponding angles, or supplementary interior angles on the same side).*

273. The rule just stated is perhaps the most important one that we are going to discuss. Its importance is apparent when we think of the great number of auxiliary constructions, which sometimes seem arbitrary, but which are a direct consequence of this rule.

To give just one example, at the beginning of Book II, when we reason about an arbitrary point on a circle, we always start by joining it to the center. The reader who has reflected on the preceding remarks will understand that there is nothing artificial about this construction: its necessity is immediately apparent. In fact, it comes from the definition of a circle, according to which, if we say that a point M belongs to a circle with center O, we are saying that the distance OM is equal to the radius of the circle.

Starting with Chapter IV (**73** and later) we see things change: it is not always the case that a point on the circle must be joined to the center in order to reason about it. We have now learned to replace the original definition of the circle with another, given in **82**. Using this definition to express the fact that a point M belongs to a circle, we can join M to three points A, B, C on the curve, and show that quadrilateral $ABCM$ is cyclic, using one of the conditions listed in **81**. From this point on, in every proof involving a circle, we can choose between these two definitions. The one which we use depends on the circumstances.[2]

274. Having been cautioned by what we have just mentioned, we must, as we have said, transform the given hypothesis so as to find in it evidence for the conclusion.

In the simplest cases, we can find a theorem which allows us to make such a transformation.

Example. The hypothesis we discuss in **271** immediately furnishes the corresponding conclusion, through one of the cases of congruence for right triangles.

In other cases, however, we must pass through one or several intermediary transformations. For example, we can seek to state the hypothesis in a form as close a possible to the conclusion.

Example. Suppose we want to prove the theorem (**25**): *In any triangle, the greater angle is opposite the greater side.*

[2]Later, we encounter other definitions, equivalent to the original; for instance, in Book III, **131**.

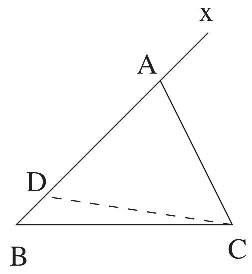

FIGURE 217

We need to express the fact that AB is greater than AC, so we take point D on AB, between A and B, such that $AD = AC$. The hypothesis and conclusion are therefore

(I) Hypothesis: DA is the extension of DB, $DA = AC$.

Conclusion: $\widehat{ACB} > \widehat{ABC}$.

Using isosceles triangle ADC, in which the base angles must be equal, allows us to reformulate the hypothesis in a new way.

(II) Hypothesis: DA is the extension of DB, $\widehat{ADC} = \widehat{ACD}$.

Conclusion: $\widehat{ACB} = \widehat{ADC} + \widehat{DCB} > \widehat{ABC}$.

Or, more simply,

(III) Hypothesis: Dx is the extension of DB (Fig. 217).

Conclusion: $\widehat{xDC} + \widehat{DCB} > \widehat{xBC}$,

which is obvious from the theorem on the exterior angle of a triangle (same section).

We have arrived at the result by a series of *successive transformations.*

275. In performing these transformations, it is especially important that we not neglect our first rule (**270**) and that we examine whether any part of the hypothesis remains unused or has been abandoned. In this way we can make sure that the new hypothesis is the same as the old, or that the two are entirely *equivalent*.

EXAMPLE. In the preceding example, form (II) of the hypothesis is completely equivalent to form (I). In other words, if hypothesis (I) is true, then so is hypothesis (II), **and conversely**. Indeed, the only difference is that the equality $AC = AD$ has been replaced by $\widehat{ADC} = \widehat{ACD}$. But we know that *either* of these conditions implies the other. Thus hypothesis (II) can be entirely substituted for hypothesis (I): to assume one or the other amounts to the same thing.

While it may happen that an element of the hypothesis can be abandoned without consequence,[3] this is not generally the case.[4] When we find ourselves

[3]This is what happens in the preceding example with hypothesis (III) and hypothesis (II). Hypothesis (II) is exactly equivalent to (II′): Dx is the extension of DB, and *there is a point on Dx such that the triangle with this vertex, and with base DC, has equal angles at the base.* Indeed, such a point could obviously be labeled A. But this point might not exist, even if hypothesis (III) (which states that Dx is the extension of DB), is satisfied. For such a point to exist, the angle $\widehat{xDC}$ must be acute, according to the exterior angle theorem.

Thus hypothesis (III) can hold, without hypothesis (II) being true.

[4]One must take care to state results in a form in which the hypothesis does not contain any useless element. In the most advanced research, and particularly in applications of mathematics,

blocked in the course of a proof, we must ask ourselves whether the difficulty in reaching the goal is not due to the fact that we have lost part of the given hypothesis on the way.

EXAMPLE. Suppose we must prove the following theorem:

Given point M in the plane of triangle ABC, we construct angle $\widehat{BAP} = \widehat{MAC}$ and length $AP = AM$. Likewise, we construct $\widehat{CBQ} = \widehat{MBA}$, $BQ = BM$ and $\widehat{ACR} = \widehat{MCB}$, $CR = CM$. Then points M, P, Q, R lie on the same circle (Fig. 218).

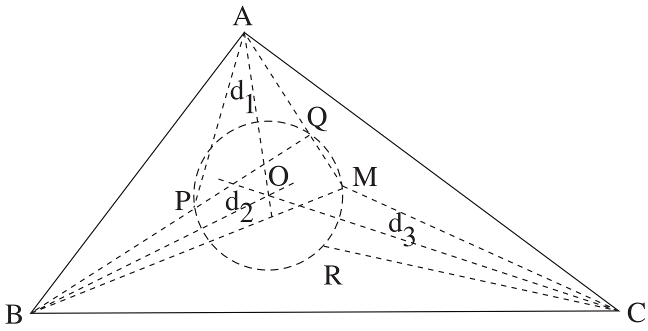

FIGURE 218

In other words,

(I) Hypothesis:

$$\widehat{BAP} = \widehat{MAC}, \quad AP = AM$$
$$\widehat{CBQ} = \widehat{MBA}, \quad BQ = BM$$
$$\widehat{ACR} = \widehat{MCB}, \quad CR = CM.$$

Conclusion: M, P, Q, R lie on the same circle.

Let d_1 be the bisector of angle $\widehat{A}$. Then sides AB, AC are symmetric with respect to this line, and because $\widehat{BAP} = \widehat{MAC}$, we see that AM and AP are also symmetric with respect to d_1. Thus point P is symmetric to M with respect to d_1. Likewise, Q is symmetric to M with respect to the bisector d_2 of $\widehat{B}$, and R is symmetric to M with respect to the bisector d_3 of $\widehat{C}$. Thus we are tempted to transform the hypothesis as follows:

(II) Hypothesis:

P is symmetric to M with respect to d_1,
Q is symmetric to M with respect to d_2,
R is symmetric to M with respect to d_3.

Conclusion: M, P, Q, R lie on the same circle.

However, if we started from this form of the statement, it would be impossible to find a proof; indeed, the proposition thus formulated is false. It is not true that a point M and its symmetric points P, Q, R with respect to three *arbitrary* lines

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the greatest difficulty often consists in recognizing, among the given elements of the problem, those that will be useful.

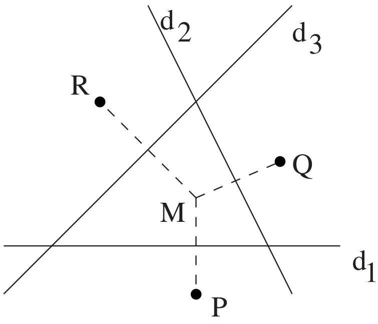

FIGURE 219

are on the same circle. We can see this simply by inspecting figure 219, or by noting that any three points P, Q, R can be viewed as symmetric to a point M with respect to three lines d_1, d_2, d_3; namely, the perpendicular bisectors of MP, MQ, MR.

Thus we must have made an error in substituting form (II) of the hypothesis in place of (I). The error consists in the fact that lines d_1, d_2, d_3 are not arbitrary: they are the angle bisectors of the given triangle and, as such, are concurrent. The correct form of (II) is then[5]

(II′) Hypothesis:

P is the point symmetric to M with respect to d_1,
Q is the point symmetric to M with respect to d_2,
R is the point symmetric to M with respect to d_3;
lines d_1, d_2, d_3 are concurrent at point O,

and this leads immediately to a proof, because we see that the four points M, P, Q, R are on a circle with center O.

276. Instead of transforming the hypothesis in order to make it more similar to the conclusion, it is often advantageous to operate on the conclusion, and to replace it by another which implies the original, and which can be deduced more easily from the hypothesis.

EXAMPLE. Suppose we want to demonstrate the following theorem (Exercise 72):

From a point M on the circumscribed circle of triangle ABC, we drop perpendiculars MP, MQ, MR to the sides of the triangle. Then the feet of these perpendiculars are on a straight line.

We will have proved that P, Q, R are collinear (Fig. 220) if (joining PQ, PR) we show that the angles $\widehat{BPR}$, $\widehat{CPQ}$, which are opposite and have the same vertex, are equal. Thus, if the statement is:

Hypothesis: Points A, B, C, M are on the same circle;
MP, MQ, MR are perpendicular to BC, CA, AB, respectively,

we can give the conclusion the form

Conclusion: $\widehat{BPR} = \widehat{CPQ}$.

[5]This hypothesis can indeed be substituted for the original one: three concurrent lines can generally be viewed as the bisectors of the angles of a triangle (Exercise 38).

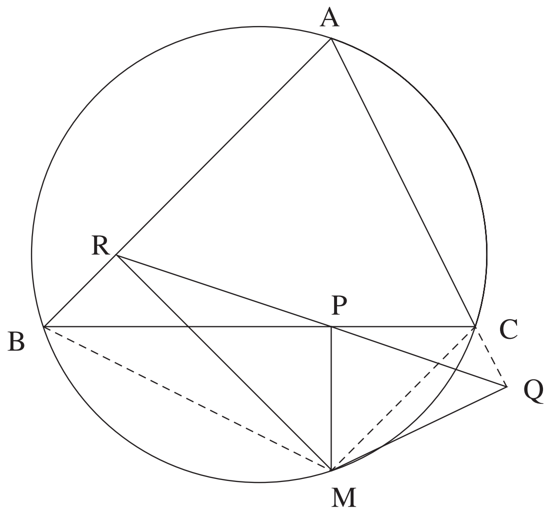

FIGURE 220

But, because of the right angles $\widehat{BRM}$, $\widehat{BPM}$, quadrilateral $BRPM$ is cyclic (**81**), so that $\widehat{BPR} = \widehat{BMR}$; likewise, quadrilateral $CQMP$ is cyclic, so that $\widehat{CPQ} = \widehat{CMQ}$. It will suffice to establish (with the same hypothesis):

Conclusion: $\widehat{BMR} = \widehat{CMQ}$.

The original conclusion is therefore replaced by another, whose proof is simpler and can easily be found by the reader.[6]

277. We now return to an important remark which we were constrained to neglect, but which must be applied to the reading of any statement of a problem.

This is the observation that many theorems can be stated in several different forms. We have already noted some examples in the text.

EXAMPLE I. We have observed in **32** that the proposition:

Every point equidistant from two points A, B is on the perpendicular bisector of AB.

can also be stated as follows: *A point which is not on the perpendicular bisector of AB is not equidistant from A and B.*

We have seen that this is an instance of a general principle: the contrapositive of an arbitrary proposition is equivalent to the original proposition.

This idea is related to the mode of proof *by contradiction* which consists in showing that if we assume that the hypothesis true, and at the same time the conclusion false, we are led to a contradiction.

EXAMPLE II. Consider the statement (**23**):

The bisector of the angle at the vertex of an isosceles triangle is perpendicular to the base, and divides it into equal segments.

[6]We have given an argument based on the diagram as drawn in fig. 220. We can modify this proof to make it independent of this particular position. This is easily done using the remarks in **83**. However, in solving the problem itself, we can always arrange the argument to use the case we have described; it suffices to change, if necessary, the order of the letters A, B, C.

This statement amounts, as we have seen,[7] to either of the following statements:

The altitude from the vertex falls on the midpoint of the base, and divides the angle at the vertex into two equal parts.

The perpendicular bisector of the base passes through the vertex, and is the bisector of the angle at the vertex.

Clearly this example, like the preceding one, is an instance of a general principle. We have encountered this principle, for instance, in **63**, Book II. Indeed, it is encountered at the very beginning of geometry. The proof of **41** (the converse of the theorem of **38**) is obviously just a remark of this kind.

These are two general categories of situations in which a statement can be replaced by an equivalent statement, but they are not the only such situations. Some reflection should, in any particular case, help find the different forms which a proposition can take. It is essential to review these forms, in order to choose the one which lends itself the most to proof; in other words, *to ask the question in the form in which the solution is easiest to find.*[8]

278. This last observation concludes the exposition of the fundamental rules which we wanted to indicate. It will be useful to study the proofs given in the book from the point of view of the application of these principles, and to ask questions such as the following:

• To prove the theorem on the medians of a triangle (**56**), we have taken the midpoints of BG and CG (Fig. 53). Was it logically necessary to think of this construction? Could it be replaced by others?[9]

• In Exercise 8, do both parts of the conclusion assume that the given point is inside the triangle? Does the answer to this question indicate which theorem must be used for the proofs of the two parts?

• In the proof of **27**, at which point do we use the fact that the enveloping polygon is convex?

Etc., etc.

(b) Geometric Loci – Construction Problems

279. What we have just said about the proof of theorems allows us to shorten our remarks on other possible types of questions, whose solutions must be sought according to the same principles. We are about to see this in problems concerning geometric loci and construction.

Geometric loci. In the text, we have learned how to find a number of geometric loci: the locus of points equidistant from two given points, or two given lines, the locus of points at a given distance from a given line, etc.

Other geometric loci are obvious on their own. For instance, the locus of points with the property that the lines joining them to a fixed point A are parallel to a fixed line xy, is simply the parallel to xy passing through A.

Having seen this, if we are given a property of a point M or, more generally, properties of a variable figure containing this point, the locus of M can be found

[7] We recall (see **41**) that this equivalence depends on the fact that there is only *one* bisector of the angle at the vertex, only *one* altitude, only *one* perpendicular bisector of the base, etc.

[8] "One should give the problem a form in which it can always be solved" (Abel).

[9] One of these is given in Exercise 37.

by transforming the given properties into others which correspond, for this point, to a known locus.

We are thus faced with a question analogous to the one encountered in proving a theorem. In that case, we had to deduce, from a collection of given properties (the hypothesis), another collection, also provided (the conclusion). Here the same sort of transformation must be performed. The only difference is that now, while the starting point is given, the end point, or conclusion, is not. We only know that it must be of the sort that will furnish the required locus. It is clear that we must look for common properties of the various positions of the variable figure and, particularly, the elements of this figure which do not change as the figure varies.

The path we must follow is the same as in the preceding case, and we would only need to repeat here the precepts just formulated.

One of these applies even more strictly here than in the case of the proof of a theorem. For proofs, we have seen that that part of the hypothesis may be abandoned in our successive transformations. This can never be allowed in searching for a geometric locus, because the locus must contain the points possessing the given properties, *and only these.* We must therefore make sure that the conclusion is exactly equivalent to the hypothesis.

This is in fact what we have done for most of the loci obtained in the text (**33, 36, 77,** etc.); we have omitted this second part of the reasoning only in those cases where it was so easy to supply that it was not necessary to do so explicitly.

280. Construction problems. Suppose now that we must construct a figure with given properties. The nature of the answer to such a question can vary significantly, depending on whether or not the given conditions are enough to determine exactly the unknown figure.

Examples. (1) Construct a line tangent to a given circle.

The tangent at an arbitrary point on the circle will solve this problem. There are infinitely many solutions: the given conditions *do not suffice* to determine the figure. The problem is said to be *underdetermined.*

(2) Construct a line tangent to two given circles.

Here we have an additional condition. This time the problem is *determined*: there are (at most) four solutions (**93**).

(3) *Construct a line tangent to three given circles.*

In general there is no line solving this problem. The first two circles have (at most) four common tangents, so the required line can only be one of these four; but an arbitrary third circle, will in general be tangent to none of these lines. Thus the problem is, except in special cases, *impossible. Too many* conditions are required of the figure described.

The problems proposed for students will, generally, be determined.

281. The problem often reduces to the construction of a single point.

Example I. Construct a circle passing through three given points (**90**).

It is enough to determine the center.

Example II. Construct a triangle knowing a side, the opposite angle, and the corresponding altitude.

If we place the given side arbitrarily, we need only find the opposite vertex.

In this situation, the method generally used is the construction by *intersecting geometric loci,* which consists in deducing from the given conditions two curves on which the unknown point must be found: the intersection of the curves will locate this point.

EXAMPLE I. *In order to find the center of the circle passing through three points A, B, C,* it was enough to observe that the condition that this center be equidistant from A and B provides a first geometric locus, and the condition that it be equidistant from B and C a second locus.

EXAMPLE II. *In order to construct a triangle ABC knowing the side BC, the angle $\widehat{A}$, and the corresponding altitude,* having chosen a location for BC, we have two loci for A: 1° The circular arc along which segment BC subtends the given angle; 2° a line parallel to BC, at a distance equal to the given altitude.

Let us say that a condition on a point is a *simple condition* if the set of points satisfying the condition is a curve. A point is then determined by *two* simple conditions, and if the two loci corresponding to the simple conditions are known[10], the required point will be their intersection.

More generally, a figure is determined by a certain number of simple conditions (we will return to this point in the second volume, Chapter X, and Note E). A polygon with n sides is determined in size and position by $2n$ simple conditions, since there are n points to be determined. To determine the polygon in size and shape only, $2n - 3$ simple conditions will suffice, since we can choose one of the vertices, and the direction of one of the sides ending in that vertex, both arbitrarily. This number $2n - 3$ is equal to n for a triangle, but larger when n is greater than 3: thus the remarks of **46b** (Remark III) and **147**.

282. According to the observation above, when we are given a construction problem, we try to change it so that it is reduced to questions whose solution we know; for example, by deducing two geometric loci for one of the points of the required figure.

Thus, one can consider a figure which is assumed to satisfy the given conditions,[11] and these conditions will serve as our hypothesis. We will then have to find the conclusion and, for this purpose, we will have to ask what the known elements in the figure are.

In this instance again, the general rules are the same as in the proof of a theorem. As in the case of a geometric locus, the conclusions we obtain must be fully equivalent to the hypothesis. This is true because, on the one hand, the conditions of the problem must imply the construction we find, and, on the other hand, every figure formed by this construction must satisfy all the conditions in question.

We will not dwell any longer on the particular features presented by construction problems, but simply refer the reader to the excellent work *Methods and Theories for the Solution of Construction Problems in Geometry* by Petersen,[12] which we have found very useful in the preparation of this volume.

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[10] In elementary geometry, we limit ourselves to the solution of construction problems with *straightedge and compass*, so that the loci in question must always be straight lines or circles.

[11] This is what is meant, for instance in **93**, by saying that we *assume the problem solved.*

[12] Translated by O. Chemin, Gauthier-Villars, 1880

(c) The Method of Transformations

283. The student who is familiar with the advice in the preceding sections, who can substitute, more or less mechanically, the defined terms for the definitions, who knows how to find quickly the various forms in which a problem he is trying to solve can be presented, will soon be able to solve a great number of problems posed in elementary geometry. Other problems, however, will seem to him inaccessible or very difficult, although they admit in reality of very simple solutions. It is just that these solutions do not depend simply on the direct form of reasoning which we have discussed, but require the aid of simplifications which we have yet to discuss: the method of transformations.

Properly speaking, after the preceding discussion, all geometric methods can be legitimately called "transformation methods". However, we reserve this name for methods which consist in passing from the properties of a figure to the properties of *another* figure.

To define a *transformation* is to associate to each figure another figure, following a certain rule, such that the second figure is completely determined by the first, and conversely. From each property of one of the figures one can derive a property of the other, which is, in some sense, a restatement of that property.

Examples. We have defined a figure F' *homothetic* to a figure F, with respect to a given center of homothecy and ratio of similitude. Given figure F, we can construct figure F'. We have seen that to each line of F there corresponds a parallel line in F', to each triangle in F there corresponds a similar triangle in F', to each circle in F there corresponds a circle in F', and so on.

In the same way, given a figure F, we can construct a figure F' derived from the first by a rotation, or a translation, or a given line reflection; or more generally, a figure F' similar to F, and such that the points corresponding to two given points A, B are two other given points A', B'. The properties of F immediately imply corresponding properties of F'.

284. In some cases, a transformation need not be applied to the *entire* figure. It is often useful to transform only part of a figure.

This happens, in particular, with the simple transformations mentioned above: translation, line reflection, homothecy, or a general similarity. There is usually no advantage in applying such a transformation to the entire figure, since the properties of the transformed figure are neither simpler nor more complicated than those of the original figure: they are exactly the same.[13] On the other hand, it is often necessary to apply one of these simple transformations to a certain part of the figure.

Example I. Consider the problem (Exercise 32): *Given two parallel lines, and two points A and B outside these parallels, and situated on different sides, what is the shortest broken line joining these two points in such a way that the portion contained between the parallel lines has a given direction?*

Suppose the required line is $AMNB$ (Fig. 221). Point N is obtained from M by a translation which is obviously known, since the length of the segment intercepted by the given parallels on any line with a given direction is the same. We can apply this translation to segment AM; point A is changed into a known point C, and

[13] Geometry studies exactly those properties which are not changed by rigid motions.

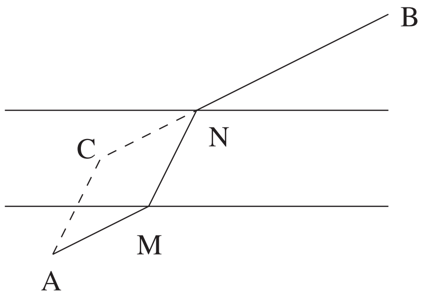

FIGURE 221

segment AM into an equal segment CN. We deduce easily that points B, N, C are on a straight line.

Analogous considerations apply to Exercises 118–122 (Book II).

EXAMPLE II. In Exercise 14, segment AM, after reflection in the given line xy, will be collinear with the segment BM.

285. In solving certain construction problems, however, it is useful to transform an entire figure, by similarity or rigid motion. The advantage obtained by such a transformation is that it can transform unknown parts of the figure into known parts of the transformed figure.

EXAMPLE. In quadrilateral $ABCD$, inscribe another quadrilateral similar to a given quadrilateral $mnpq$.

There exists a figure similar to the one required, in which the required quadrilateral $MNPQ$ corresponds to quadrilateral $mnpq$. Quadrilateral $ABCD$ will correspond, in this similar figure, to a quadrilateral $abcd$, circumscribed about $mnpq$, and which can be constructed[14] by the solution of Exercise 213.

286. Other transformations that we have encountered change the figures to which they are applied in a more or less remarkable way.

An example is transformation by *inversion*, which changes a line of the original figure into a circle of the transformed figure. Another example is transformation by *reciprocal polars.* These transformations have been used in the complements to Book III to simplify the proofs of a number of theorems.

Another transformation, which we have only alluded to, but must be mentioned, is *perspective.* As we have defined it in plane geometry, it is only applied to figures composed of points on a straight line. We obtain the transformed figure by joining every point on the original figure to a fixed point, outside the line, then cutting the rays obtained with another line, then subjecting the figure obtained on this transversal to an arbitrary displacement.

287. It is appropriate to note an essential difference which distinguishes the transformation by polars from others considered above, such as similarity, inversion,

[14]The author has in mind here the following construction: using the method of Exercise 213, construct a quadrilateral similar to $ABCD$ with sides passing through points m, n, p, q. Then the required points M, N, P, Q are those that correspond to m, n, p, q under the similarity in question.–transl.

or perspective. These last are *point transformations,* in the sense that each point in the original figure corresponds to a definite point in the transformed figure. This is not true of a transformation by polars, in which a line in one figure corresponds to a point in the other.

Another transformation used in the text, and which shares this character of not being a point transformation, is the one which may be called a *dilation*, and which we have applied to figures formed by circles and lines. In this operation one increases or decreases (as the case requires) the radius of each circle by a certain quantity a. It may happen, of course, that the radius of a circle becomes zero, and the circle becomes a point. Conversely, under this transformation a point can be considered as a circle of zero radius, and may be transformed into a circle of radius a. Each line in the figure will be transported in a perpendicular direction, in one sense or the other, by the same quantity.[15]

The most important property of a dilation is that two tangent curves remain tangent, after the dilation has been applied appropriately (Exercise 59).[16] It is this property that we have used in **93** and **231**.

288. Reduced forms. Since the purpose of a transformation is to simplify the figure we are working with, we should try to simplify the situation as much as possible.

To this end, we remark that the various kinds of transformations just mentioned involve some arbitrary elements. For example, for a homothecy we must specify the center and the ratio of similarity; for an inversion, we must specify the pole and the power;[17] for a dilatation, we must specify the quantity a discussed in the previous paragraph; etc.

Each type of transformation therefore includes infinitely many particular transformations.

Among these we must choose a particular transformation, such that the transformed figure satisfies one or several required conditions, and in general we want it to satisfy as many of these conditions as we can, taking into account what we noted in **280**. The transformed figure will then be called a *reduced form* of the original figure.

Example I. We can always subject a circle to a dilation which reduces it to a point; it suffice to dilate by a quantity equal to the radius of the circle.

A *reduced form*, reduced by dilation, of a figure containing a circle, is one in which this circle is reduced to a point. We have used such a reduced form in **93** and **231**.

Example II. Two circles with a common point can be transformed by a single inversion into two lines (by taking the common point as the pole). Two circles with no common point can be transformed into concentric circles (Exercise 248). In other words, the reduced form of a figure consisting of two circles is made up of

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[15] A dilation of a circle or a line can be performed in two different ways, between which one can choose according to each particular case

[16] Dilation can be extended (using methods which go beyond the framework of this book) to figures which contain arbitrary curves, and it can be proved that this extended transformation still preserves the property of two curves being tangent. This property also holds for the transformation by polars, generalized in an analogous fashion. (See Volume II (Solid Geometry), **771**.)

[17] We have seen (**215**) that the choice of the pole is the more important.

two lines or two concentric circles, according as whether the original circles had or did not have a common point.

Example III. In the example of **285**, we have used a reduced form of the required figure. The figure was reduced using a similarity, and the reduced form was determined by the requirement that quadrilateral $MNPQ$ be transformed into the given quadrilateral $mnpq$.

289. Invariants. The advantage of a transformation can prove illusory, if, by simplifying certain properties of the figure, we have made others more complicated. We should apply a transformation only in those cases where the various properties in the statement are sufficiently simplified.

As it happens, in almost all the categories of transformations reviewed above, there are certain properties which are not changed at all, which remain, as we say, *invariant.*

Example I. We have seen that in a homothecy, the angles of the transformed figure are the same as those in the original figure. Likewise, the ratios of lengths do not change, and so on.

Example II. We have seen that a dilation changes tangent circles into tangent circles. Thus tangency is an invariant property for dilation.

Example III. The angle of two curves is an invariant for inversion (**219**).

Etc.

290. There can be no inconvenience in applying a transformation if only invariant properties are mentioned in the statement of the problem. In these cases, we can always assume the figure is in its reduced form.

Examples. Since dilations preserve tangency, we can assume, in constructing the common tangents to two circles, that one of the circles is reduced to a point. We followed this path in **93**.

The same observation applies to the construction of a circle tangent to three given circles (**231**).

Tangency is also invariant for inversion. Thus, when searching for circles tangent to three given circles, we may assume that two of them are reduced to two lines or two concentric circles (Exercises 264–265).

Similar observations apply to the proof of theorems such as the following: *All the circles which intersect two given circles at constant angles, are tangent to two fixed circles;* or to the result of Exercise 285, etc.

291. Groups.[18] The fact that the various transformations we discussed possess invariants is a consequence of another property about which we will say a few words.

The transformation equivalent to the successive[19] application of two transformations is called the *product* of these two transformations. In other words, if the transformation S changes figure F into figure F', and the transformation T, applied

[18] The final part of this note is addressed especially to those readers who have studied the complements to Book III.

[19] In general, the product of two transformations depends on the order of the factors. Thus, the product of two reflections in two different lines is a rotation or a translation which is double the rotation or translation which would make the *first* line coincide with the second.

to figure F', changes it into figure F'', the product of these transformations will be the transformation that changes F to F''.

Example. The result of **102** can be stated as follows: *The product of two line reflections is a rotation or a translation.*

Having said this, a set of transformations is said to form a *group* if the product of any two of them is also a member of this set.[20]

For example, *the set of all homothecies is a group:* this amounts to saying that two figure homothetic to a third are homothetic to each other. The set of all homothecies whose centers are on a given line also form a group because, as we have seen, the center of the product of two homothecies is on the line passing through their centers.

The set of all rigid motions form a group, because two plane figures, congruent to a third and with the same orientation, are also equal.

292. The collection of inversions does not form a group. The product of two inversions is not an inversion, but we may consider the group of transformations obtained by successive application of several inversions.

For the sake of brevity, let us call the product of any number of inversions, or line reflections, an S-transformation. This category includes, in particular, all rigid motions, because each rigid motion is the product of two line reflections; all homothecies, because a homothecy results from the application of two inversions with the same pole and different powers. Therefore all similarities are also S-transformations since a similarity is a homothecy, possibly followed by a rigid motion and a line reflection.

It would seem, from their definition, that to obtain all S-transformations we should consider all products of n inversions, n denoting an arbitrary positive integer, and that stopping at some fixed value of n would not suffice.

This however is not the case: every S-transformation amounts to an inversion followed or preceded by one, two, or three line reflections, except for the case when the transformation is a similarity (Exercise 252). Therefore, four simple operations (inversions or line reflections) suffice to produce any S-transformation.

Since the product of two S-transformations is again an S-transformations, the S-transformations form a group which one may call, for the sake of brevity, *the group of inversions.*

293. Two figures F, F' which are changed into one another by a transformation in a group are said to be *homologous* with respect to this group. For instance, two congruent figures with the same orientation are homologous with respect to the group of rigid motions.

According to the definition of a *group*, two figures F, F'', homologous to a third F', are also homologous to each other, since the transformation that passes from F to F'' is the product of the transformations which pass from F to F' and from F' to F''.

The reduced form of a figure F with respect to a given group is a figure homologous to F with respect to this group, which satisfies certain additional conditions. If these conditions are sufficient in number, and well chosen, they will suffice to determine completely the reduced form F_0 of F.

[20]The original text makes no mention of the existence of an inverse transformation for each element of a group of transformations. –transl.

Suppose this has been done. If we then apply one of the transformations of the group to F, the new figure F' obtained will also have the reduced form F_0, since the figures homologous to F' are exactly those which are homologous to F.

The reduced form of F is thus also the reduced form of any figure homologous to F, and therefore *every property of the reduced form is an invariant property of F.*

294. We will clarify this remark by an example.

Consider the figure formed by four arbitrary points A, B, C, D. We propose to find the properties of this figure which are invariant for the group of inversions. For this purpose, take the inverse of the figure, using an inversion I with point A as its pole. If the inverse points for B, C, D are b, c, d respectively, they form a figure homologous to the original (a reduced form) with respect to the group, in which one of the four points (the one that corresponds to A) is thrown to infinity. We will see, by retracing our general argument as applied to this special case, that the angles of triangle bcd are the invariants of the figure $ABCD$. (A direct proof of this fact is the object of Exercise 270.)

Let A', B', C', D' be the points corresponding to A, B, C, D under an arbitrary transformation T of the group. We operate on these points just as we operated on the original points; that is, we apply an inversion I' with pole A' to obtain the homologous points b', c', d'. We can pass from the figure formed by triangle bcd and a point at infinity to the figure formed by triangle $b'c'd'$ and a point at infinity by a transformation in our group; namely, the product of the three inversions I, T, I'. As we know, this transformation is either a similarity, or an inversion followed by line reflections. The second alternative occurs only if T itself is an inversion, because then it would send a point at infinity into a finite point (namely, the pole). We conclude that the triangles bcd, $b'c'd'$ are in fact similar. QED

Thus, the angles and the ratios of the sides of the triangle bcd are indeed the invariants of the figure $ABCD$ relative to the group of inversions. There are only two *independent* invariants, that is, knowledge of two of them is sufficient to calculate the others: two angles of triangle bcd are indeed sufficient to construct a triangle similar to it.

295. We have just seen that the existence of invariants follows from the existence of of a group; conversely, the set of all transformations which have the same invariants forms a group. Indeed, if two transformations do not change a certain property, the same is true for their product.

Example I. A homothecy changes every segment into a parallel, proportional segment. Conversely, every point transformation with this property is a homothecy (**142**). Thus, the collection of homothecies forms a group. The reasoning of **144** is simply a different form of the argument just made.

Example II. A perspective preserves the cross-ratio of four points and, conversely, every transformation applied to a system of points on a straight line, and which conserves the cross-ratio, is a perspective.[21]

Thus *the set of perspectives forms a group.*

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[21] This can be proved by moving the transformed figure so that one point coincides with its corresponding point, while the two lines remain distinct. We are then led to Exercise 233.

We will limit ourselves to these remarks on the general properties of transformations. As for their use, we refer the reader to the work of Petersen already mentioned.

Note B:
On Euclid's Postulate

I

296. We have assumed, as an axiom (**40**), that *through a point there cannot pass more than one parallel to a given line.*

This statement appears in the *Elements* of the Greek geometer Euclid,[22] where the principles of geometry are described completely for the first time, and with remarkable perfection, as well as in all subsequent expositions (which are all based on this first exposition). More precisely, Euclid gives the statement of **41** (Corollary I), which is equivalent to it:

If two lines make, with the same transversal, two interior angles on the same side whose sum is different from two right angles, then the lines are not parallel, and they intersect on the side of the transversal where the sum of the interior angles is less than two right angles.

This statement is taken to be among those considered obviously true.

The followers of Euclid, from antiquity to the middle ages and into modern times, have not stopped marveling at the place assigned to this proposition, which indeed does not exhibit the same obvious character shared by other propositions accepted without proof. Indeed, it does not seem any more evident than many of the propositions in whose proof it is used. In particular (and we will soon return to this point), they found it strange that Euclid accepted as obvious for straight lines a fact which is not at all true for curves in general.[23]

297. Many attempts were therefore made to prove this postulate: all of them failed. In particular, trying to argue by contradiction, and deducing all the possible consequences if the postulate is assumed false, one obtains a series of conclusions which are very different from the conclusions of the usual theory of parallel lines. But however far we push these conclusions, they never yield (at least when they are deduced correctly) a contradiction among themselves, or with earlier propositions, a contradiction which would prove the impossibility of the initial hypothesis.

298. The great mathematician Gauss[24] asked himself whether such a contradiction did in fact exist, if perhaps the opposite of Euclid's postulate is in fact compatible with the other axioms of geometry and with the consequences which would follow. In other words, perhaps it is *impossible* to prove the proposition in question.

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[22] Around 300 B.C.

[23] *Wallis, 1663.* Cf. *Stäckel et Engel, Die Theorie des Parallelinien, von Euklid bis Gauss,* Leipzig, Teubner, 1895.

[24] 1777–1855.

Around the same time Lobachevsky[25] and Bolyai[26] looked at this same hypothesis independently, and constructed a geometry having in common with ordinary geometry all the propositions preceding, or independent of, Euclid's postulate, but in which all the others statements are modified. The results arrived at in such a geometry (called *non-Euclidean* geometry) often take an intriguing, paradoxal form, and contradict our usual way of visualizing things. But no matter how amazing these results may seem at first, there is none whose absurdity can be proved. Here are some of the simplest examples.

In non-Euclidean geometry:

• Through a point A outside a line D, but in the plane which contains them, there are *infinitely many* lines which do not intersect[27] D. All of these *non-secant lines* are situated within a certain angle with vertex A (and in the corresponding vertical angle). This angle, called *angle of parallelism,* increases with the distance from the point to the line;

• Every line D' *interior* to the angle of parallelism has a unique common perpendicular with line D, which gives the shortest distance between points on the two lines. If a point M moves along D' and away from the foot of the common perpendicular, its distance to D increases constantly and indefinitely. The same is not true of lines D'_1, D'_2 which are the sides of the angle of parallelism. These lines do not intersect D, but they approach D more and more closely as we move along them away from the point A, in the direction which makes an acute angle with the perpendicular from A to D. If a point M moves along one of these lines in this direction, its distance to D tends to zero. – The locus of the points in the plane which are equidistant from a fixed line is a curve.

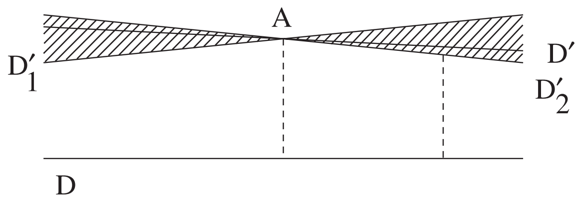


FIGURE 222

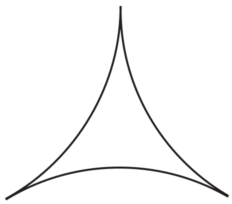
• The sum of the angles of a triangle is *less* than two right angles, and the difference is proportional to the area of the triangle. It follows that this area is always less than a certain quantity, no matter how long the sides of the triangle may be (when the sides are very long, the angles become very small, and the triangle becomes very thin, as indicated in the figure). – In the same way, the sum of the angles of a polygon with n sides is less than $2n - 4$ right angles, and the difference is proportional to the area of the polygon. Consequently, *there are no rectangles:* if a quadrilateral has three right angles, the fourth must be acute.

[25]1793–1856.

[26]1802–1860. Non-Euclidean geometry was also discovered, more or less completely, in a variety of places (Cf. the reference above to *Stäckel* and *Engel*).

[27]We can show that the Euclidean postulate is always true, or always false: we cannot show the existence of a single point outside a single line, for which the parallel through the point to the line is unique, without also being able to prove this uniqueness for all possible points and lines.

• *There are no similar figures*[28] except, of course, congruent figures. Two triangles are congruent if they have equal corresponding angles. – The assumption that there exist two non-congruent similar triangles is equivalent to the assumption of Euclid's postulate.
Etc.



There are in fact infinitely many non-Euclidean geometries. The relations between the various parts of a figure, under Lobachevsky's hypothesis, include in fact a certain number k^2, determined once and for all, but which can have an arbitrary value: for instance the ratio (which is constant, as we just mentioned) between the area of a triangle and the supplement of the sum of its angles. *Euclidian geometry can be viewed as a limiting case of the non-Euclidean geometries:* it corresponds to $k = \infty$.

299. Is non-Euclidean geometry, which seems paradoxical, but also seems to be *logically possible,* really possible, as believed by its advocates? It may be that this is only an impression, due to the fact that the consequences have not been pushed far enough. Would it be possible, through deeper or better investigation, to encounter a contradiction which presented itself neither to Gauss, nor to Bolyai, nor to Lobachvsky?

We can state with confidence that this is not the case. Without developing in detail the reasoning which validates this statement,[29] we will borrow a simple and striking manner of presenting them from H. Poincaré.[30]

"Imagine a sphere S inside of which there is a medium whose temperature and index of refraction are variable. Suppose there are objects moving in this medium, and that their their movements are sufficiently slow and their specific heat sufficiently small that they immediately achieve temperature equilibrium with the medium; moreover, all of these objects will have the same coefficient of dilation, so that we can define the surrounding temperature by the length of any one of them. Let R be the radius of the sphere, and ρ the distance from a point in this medium to

[28] Contrary to our usage in the rest of this book, here we are speaking of similar figures as figures with equal corresponding angles, and proportional corresponding sides.

[29] We may note an analogy between the results mentioned above and certain results of spherical geometry, to which they are opposed in a certain sense. Thus, the sum of the angles in a spherical triangle is *greater* than two right angles, and the difference is proportional to the area of the triangle; two spherical triangles with equal corresponding angles are either congruent or symmetric, etc. In fact, the geometry of certain surfaces (called *pseudospherical surfaces*) is identical to Lobachevski's geometry. It is in this way that the logical possibility of non-Euclidean geometries was established, but this proof is not complete: 1° because pseudospherical surfaces cannot be considered to be limitless in all directions, as we would expect from a plane; 2° because they only pertain to *plane* non-Euclidean geometry, and would not prevent in principle a proof of the postulate by considerations of geometry in space.

[30] *Revue générale des sciences pures et appliqués*, Volume III, 1892, p. 75.

the center of the sphere: we will assume that the absolute temperature[31] is $R^2 - \rho^2$ and that the index of refraction is $\frac{1}{R^2-\rho^2}$.

"What would an intelligent being think, who had never left such a world?

(1) Since the dimensions of two small objects transported from one point to another vary in the same ratio, because they have the same coefficient of dilation, these beings would believe that these dimensions have not changed. They would have no idea of what we call differences in temperature. No thermometer could reveal it, since the dilation of the container would be the same as that of the thermometric liquid.

(2) They would believe that this sphere S is infinite: they could never reach its surface because, as they approach it, they enter colder and colder regions; they would become smaller and smaller and, without realizing it, would make smaller and smaller steps.[32]

(3) What they would call straight lines would be circles orthogonal to the sphere S, for three reasons:

1°: They would be the trajectories of light rays;

2°: Measuring various curves with a ruler, our imaginary beings would find that these circles are the shortest paths from a point to another: indeed, their ruler would contract or dilate when passing from one region to another, and they would not doubt this phenomenon;

3°: If a solid body rotated so that one of its lines remained fixed, this line could only be one of these circles: this is because, if a cylinder turns slowly and is heated on one side only, the locus of the points which do not move will be a convex curve on the heated side, and not a straight line.

"It follows that these beings would adopt Lobachevski's geometry.[33]"

We see now that it is in fact impossible to prove Euclid's postulate with the help of earlier propositions: if such a proof existed, it would have to be admitted by the fictitious beings just described (since all earlier propositions are still true, from their point of view); but it would lead to an incorrect result since, for these beings, the postulate is false.

II

300. What role then must we assign to this proposition, which is not as obvious as the axioms, and which cannot be proved as a theorem?

This role is that of a definition. To explain what we mean by this, we refer to the preceding note (Note A, **271**).

As we have seen there, we must have a definition for each term figuring in our statements, a definition whose role in the reasoning can be made clear by substituting it for the defined term each time it occurs.

[31] In physics, this is the temperature measured from a zero point, so that it is proportional to the volume of a thermometric body. (In physics, it is specifically assumed that this body is a gas sufficiently far from its liquefying point; here it can be any object, since all of them have the same coefficient of dilation, and the absolute temperature is proportional not to the volume, but to an arbitrary linear dimension.)

[32] We could add that, by the properties we have assumed for the index of refraction, they could not see what happens outside the sphere.

[33] The arbitrary constant introduced in non-Euclidean geometry is represented here by the radius R of the sphere.

Of course, there are terms which have not been, and cannot be, defined, because a notion can only be defined by use of earlier notions,[34] which is impossible for the *first* notions to be introduced.

But, since these notions are clear in themselves and have a number of obvious properties, the role of the definition (which is still needed in each case, as we have just recalled) is replaced by these properties, which we assume without proof. We proceeded this way in the case of the straight line, which we did not actually define in the proper sense, but rather defined *indirectly*, by assuming its fundamental properties.

301. It is important that we assume enough of these propreties to be able to *characterize* the notion they define, that is, to distinguish it from every different notion. For instance, we would have given a bad definition of the straight line if, instead of assuming, as we have done:

(1) that every figure congruent to a straight line is a straight line; and conversely, two lines can be superimposed in infinitely many ways;
(2) that one and only one line passes through two points,

we would have assumed the first property, but not the second. This first property is not, in fact, peculiar to straight lines: it is also satisfied, for instances, by circles with a radius of 1 meter. It would have been impossible to prove, starting from this definition, any property of straight lines which is not also shared by these circles. For instance, we could not have proved that the sum of two angles of a triangle is less than two right angles because this is not always true for curvilinear triangles whose sides are arcs of congruent circles.

302. All geometry rests on a fundamental notion, that of a *rigid motion,* which we have introduced in **2**. At that point, we considered as self-evident the idea of a figure which can be moved without changing its shape or size; in other words, what we have called an *invariant figure.* Let us try to study how this notion is defined by its properties.

Given a figure which is subject to a rigid motion, and an arbitrary point M, we can imagine that this point is permanently attached to this figure and follows it in its motion, so that it comes to occupy some position M'. We can thus say that the rigid motion associates to each point M in space another point M', which is the position to which the point M is transported. A rigid motion is therefore a *point transformation*[35] of space.

It is also an obvious property of the notion we consider that two rigid motions, applied successively, are equivalent to another rigid motion. In other words, *these transformations form a group.*[36]

We can thus say:

An invariant figure is a figure to which we apply only transformations from a certain group (*called the group of rigid motions*)*, having the following properties*:

(I)

- *There are infinitely many transformations in the group which transport a point A onto an arbitrary position A'.*

[34]For instance, we cannot define a circle as "*the locus of points in a plane situated at a given distance from a given point of this plane*", without having defined earlier the notions of *distance, plane,* and *geometric locus.*

[35]See Note A, **287, 291**, and the passages that follow.

[36]Ibid.

- *In general, there are no transformations of the group which, at the same time, bring two given points A, B to two given positions A', B'. In order for such a transformation to exist, a certain quantity, depending on A and B, must be equal to the corresponding quantity for A', B'.*
- *If there is a transformation in the group which takes A, B onto A', B', then there are infinitely many. In particular, there exist infinitely many transformations which leave two points A, B fixed but, for all of these transformations, there are infinitely many other points which are fixed. These points form a unique line, which extends indefinitely, called a straight line. One and only one straight line passes through two points.*
- *There exist surfaces* (*called planes*) *such that every line which has two points in a plane is entirely contained in it; one such surface passes through three arbitrary points in space,*

etc.[37]

303. It is nonetheless essential to observe that the preceding definition, like all other indirect definitions, implies the following axiom:

AXIOM (A): *There is a group possessing property* (I).

We will also remark that the notions of *straight line* and *plane* derive from the notion of rigid motion, without which they cannot be defined.

304. Euclideans and non-Euclideans agree that the group of rigid motions satisfies property (I), and they therefore assume axiom (A). They are divided by the question as to whether the following statement is true or not:

(II)

- *Through a point outside a line there is only one line parallel to a given line.*

This statement must then express a property of the group of rigid motions, because straight lines and planes are defined by this group. The question then takes the following form:

Does the group of rigid motions defined by property (I), *also satisfy property* (II)?

Now, if it is not possible to answer this question, the reason can only be that it is *ill posed*: that the question does not have a precise meaning.

Indeed, the group of rigid motions *is not* well defined by property (I). If (in agreement with axiom (A), which we assume) there exists a group with these properties, then *there are infinitely many* such groups. The inhabitants of Poincaré's world would have an entirely different notion from ours of an 'invariant figure', since the objects which they move would dilate or contract in the process. However, their group of rigid motions would satisfy property (I) in the interior of the sphere (that is, for all of space about which they could reason, which is infinite from their point of view).

The resolution of the question is now clear. Statement (II) will be precise if, among all the groups satisfying property (I), we reserve the name *group of rigid motions* for a group satisfying the condition of this statement; in other words, if this group is defined *not just by property* (I), *but by properties* (I) and (II).

[37] We do not pretend to list here all the properties needed to define the group of rigid motions. By "Property (I)" we mean those properties which we have assumed in Book I, prior to Euclid's postulate.

305. But an objection can still be raised. Just as the original definition implied axiom (A), the one we now propose is not possible unless we resolve the following question:

Does there exist a group which satisfies both conditions (I) *and* (II)?

The answer is in the affirmative. One proves (assuming, of course, axiom (A)) that among the infinitely many groups satisfying condition (I) there are some which are *non-Euclidean* (for which Euclid's postulate is false) and some which are *Euclidean* (for which it is true).

Now every difficulty is removed. *Euclid's postulate is part of the definition of the fundamental notions of geometry.*

306. Are we then saying that one should not ask whether this postulate is true or false, that such a question is totally devoid of meaning?

We would have the right to do this, if we were free to define the notions of geometry in a completely arbitrary manner. But this is not the case: these notions are provided us by experience. The idea of an invariant figure is suggested by those invariant figures (namely, the solid bodies) which nature offers. We must define invariant figures and rigid motions with these solids and their displacements in mind if we want geometry to be applicable to real objects.

307. We see therefore that there is indeed a question about Euclid's postulate: the question is whether the definition given above accords with experience, whether or not the properties of *natural* motions which we observe are analogous to the properties of the Euclidean group.

But this is no longer a mathematical question. The solution of such a problem does not depend on reasoning, but on observation.

If we have been led to develop Euclid's conception rather that Lobachevsky's, it is because our senses, within their limitations, show us that the postulate is reasonably exact. We see that two lines parallel to a third are parallel, that there exist similar figures with arbitrary ratios of similarity, that there exist rectangles, etc.

308. This is, of course, a crude verification. To submit it to deeper examination, we must measure, with the greatest possible precision, the angles of a triangle, to discover whether their sum is equal to two right angles. We should use as large a triangle as possible, since it is for large triangles (**298**) that the discrepancy between the two hypotheses is greatest. Experiments have shown (to the extent allowed by experimental difficulties) that this equality is verified (or at least that the discrepancy is smaller than observational errors).

We would then be justified in asserting that the geometry which represents reality most accurately is Euclidean or, at least, it differs very little from Euclidean (in other words, the constant k is very large, or, if the geometry is comparable to Poincaré's imaginary world, the radius R is enormous relative to all usual lengths). In one word, Euclidean geometry is *physically true.*

308b. Things have changed following the recent[38] developments in our ideas about physics (*the Theory of Relativity*).

This new theory has profoundly modified the science of motion or *kinematics* (which is a direct application of geometry). The models used for velocities in the old

[38]That is, at the time the author was writing. –transl.

kinematics were based on principles of Euclidean geometry (mainly on properties of parallelograms). This is still assumed to be practically correct for 'very small' speeds — that is, measured by very small numbers in comparison to the speed V of light (which is approximately 300000 kilometers per second) — and so for all ordinary speeds,[39] but not for speeds comparable to V. For such speeds, *it is Lobachevski's geometry which corresponds to physical reality.* The role of the radius R is played by V.

Non-Euclidean geometry (and even other, more general, geometries) enter into still other parts of the theories of physics (*General Relativity*). However, one can see from the preceding discussion that none of these changes has any influence on everyday life (for instance, on the work of the engineer). Euclid's geometry retains its validity for all figures which we can view at a glance, or measure directly.

[39]Since V is so large, the speed of a military projectile, for instance, is 'very small'.

Note C: On the Problem of Tangent Circles

309. As we discussed in **236**, Gergonne's method for finding the circles tangent to three given circles does not apply in all possible cases: it gives no result if the three centers are collinear. We added that this inconvenience disappears if the solution is presented in a form in which the properties utilized do not change when an arbitrary inversion is applied. This is what we propose to do in this note.

Let the given circles be A, B, C, and take point b on B, antihomologuous[40] to a point a of A, with respect to a center of similarity S_{12} of the circles A and B. Let c be the point antihomologuous in C to b, with respect to a center of similarity S_{23} of B, C. The circle σ passing through a, b, c cuts the three circles again in a', b', c'; it cuts them at equal angles (**227**), so that a', b' are antihomologous with respect to S_{12}, b', c' with respect to S_{23} and, moreover, the pairs a, c'; c, a' are antihomologous[41] with respect to a center of similarity S_{13} of circles A, C.

Replacing a by another point a_1 of circle A, we obtain a new circle σ_1, analogous to σ. Point S_{12} has the same power with respect to σ and σ_1 (equal to the power of the inversion which transforms A into B); the same is true for point S_{23}. Thus the radical axis xy of circles σ, σ_1 is an axis of similarity for the given circles, and it is clear therefore that the center of similarity S_{13} which we are discussing, belongs, along with the first two, to this axis of similarity.[42]

310. We see now that there are four series of circles σ (corresponding to the four axes of similarity), and that the circles in each series have the same radical axis. Conversely, every circle which has a common radical axis with two circles σ of one of these series must belong to that series (since it corresponds to itself in the two inversions with poles S_{12} and S_{13}). Therefore each of these series is determined by two of its circles, or by one circle and the radical axis xy. We can, generally, take this circle to be the circle σ_0, belonging to the series, which is orthogonal to the three given circles (**227b**).

The locus of the centers of the circles in one of the series is the perpendicular from the radical center of A, B, C to one of their axes of similarity.

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[40] See **227b.** – transl.

[41] The fact that a is antihomologous to c' rather than c is established by noting (cf. **227**) the angles made by σ with A, C at the points a, c have the same orientation (rather than opposite orientation, as would be the case for antihomologous points).

[42] The reasoning does assume that the point S_{13} is the same for σ and for σ_1. But if this were not true, at least one of the two points S_{13} corresponding to σ and σ_1, for instance, the one that corresponds to σ_1, would be collinear with S_{12} and S_{23}. This point would then have the same power with respect to σ_1 as to σ; namely, the power of the inversion which transforms A into C. Therefore the circle σ_1 would intersect A and C in points which correspond to each other in this inversion.

311. The circles which are tangent to the given circles obviously belong to the series just mentioned and, conversely, every circle in one of these series which is tangent to one of the given circles must be tangent to the others as well.

The problem of tangent circles is thus reduced to the following:

Find a circle with the same radical axis as two given circles σ, σ_1, and tangent to a third given circle A (Fig. 223).

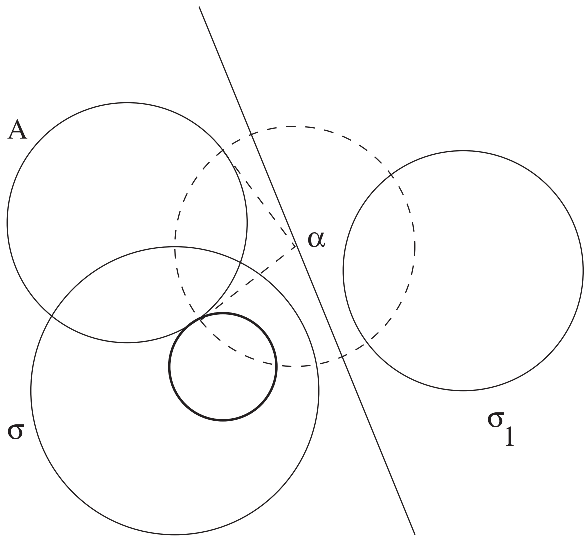

FIGURE 223

This last question is easily solved: if a is the point of contact of the required circle with the circle A, the circle passing through a and orthogonal to σ and σ_1 will also be orthogonal to the required circle, and therefore to A. These conditions allow us to find a (**158**, Construction 13). In other words, we have to draw a tangent to A passing through the radical center of σ, σ_1 and A, to find the required point of contact. Conversely, it is easy to see that the points obtained in this manner correspond to solutions of the problem.

As we remarked above, one of the circles σ, σ_1 can be replaced by their radical axis xy, so that the solution to the problem of tangent circles can also be reformulated as follows:

Through a point on circle A and its two antihomologous points, draw a circle σ. The common chord of A and σ intersects the axis of similarity xy at a point α. The tangents from α to circle A give the points of contact of the required circles.

311b. When circle σ is taken to be the circle σ_0 whose center is the radical center I of the three given circles, and which intersects them at right angles, we recover Gergonne's original solution. The common points of A and σ_0 are indeed the points of contact of the tangents from I to A, so that the common chord of the two circles is the polar of the point I with respect to A. The polar of the intersection point of this chord and xy is thus in fact the line joining I to the pole of xy.

We see now why this solution does not work when the centers of the given circles are collinear: both the circle σ_0 and the line xy coincide with the line of the centers. In order to remove the difficulty, it suffices, as we have said, to use a circle σ different from σ_0.

The method we offer also has the advantage (over Gergonne's method) of being applicable when one or several of the given circles are replaced by points or lines, yielding directly the points of contact with one of the lines: these are situated on a circle whose center is the intersection of the given line with the axis of similarity xy, and which intersects the circle σ_1 at right angles. This construction obviously generalizes Construction 14 (**159**). This construction even applies when there is no circle left, which is not the case for Gergonne's solution. It fails only when all three circles are reduced to points.

312. It follows from **309** that the common chord of circles A and σ is precisely the line we have called aa', where a' is the point antihomologus to c.

Thus the construction indicated above consists in:

- determining the point b antihomologous to a (which is taken arbitrarily on circle A) with respect to the center of similarity S_{12}, the point c antihomologous to b with respect to S_{23}, and the point a' antihomologus to a with respect to S_{31};
- joining aa';
- repeating this construction starting with another point a_1 on A.

The intersection of the two chords aa', a_1a_1' thus obtained is the point α, from which we must draw tangents to circle A in order to obtain the points of contact of the required circles.

Discussion. Unlike Gergonne's form of the solution, ours allows a simple discussion of the *number* of actual circles which solve the problem.

We observe first that the above construction is or is not possible according as whether the point α is exterior or interior to A.

Take an arbitrary point a_1 close to a. Clearly, the point α will be inside A if the small arcs aa_1, $a'a_1'$ have the same sense, and outside if they have the opposite sense.

We must therefore determine whether, as point a moves along the circle A, point a' moves in the same sense, or in the opposite sense.

To answer this question, we start by studying the sense in which the point b will move.

Since the point homothetic to a with respect to S_{12} clearly varies on B in the same sense as a on A, we see immediately (cf. **223**) that b turns in the same sense as a if S_{12} is interior to B, and in the opposite sense if S_{12} is exterior.

Let us agree to say that a center of similarity S_{12} of two circles A, B is *positive* if it is outside these two circles, and *negative* if it is inside the circles. This amounts to saying that a center of similarity is considered positive or negative depending on whether, through this point, there are or are not real common tangents to the two circles.

If two circles are exterior, their centers of similarity are positive. If they intersect at two points, the exterior center of similarity is positive, and the other negative. If one circle is inside the other, both centers of similarity are negative.

According to this convention, whether the sense of the motion of b is the same as that of a depends on whether the center of similarity S_{12} is negative or positive.[43]

Similarly, the point c moves on C in the same sense as b on B, or in the opposite sense, according as S_{23} is negative or positive; and a' moves in the same sense as c, or in the opposite sense, according as S_{31} is negative or positive.

Combining these remarks with the earlier ones, we see that α is outside A — in other words, *there are two real points of contact* — if, among the three centers of similarity on the axis xy, *one or three are positive.*

To calculate the total number of real circles which answer the problem, it will suffice to apply this remark to each of the four groups of collinear centers of similarity.

In what follows, we will use the digit 0 to denote the group of three direct centers of homothecy; the digit 1 for the group for which B, C are directly homothetic, and for which both are inversely homothetic to A; and the digits 2 and 3 for the analogous groups obtained by replacing A by B, and then by C.

312b. Having said this, it is easy to see that (excluding the cases when two of the circles are tangent[44]) there are eleven possible positions for the three circles.[45]

In each case, the preceding considerations would immediately yield the result. For instance, in case I, all the centers of similarity are positive, so that there are eight solutions, while in case V, there are only four, because the centers of similarity of A with B or C are all negative. We would have to use the *positive* center of B and C, (that is, the exterior center of similarity) in order to obtain real solutions.

We thus arrive at the following table which indicates, in each of the eleven cases, the positive and negative centers (the letters S indicate the exterior centers of similarity, and the letters S' the interior ones), the number of solutions, and the homothecy groups (numbered 0, 1, 2, 3, as above) which produce them.

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[43]That is, the sense of motion is the same if the center of similarity is negative, and opposite if the center of similarity is positive.–transl.

[44]These must be considered as limiting cases of those where the circles have 0 or 2 intersections, between which they are intermediary. If two of the given circles are tangent, then two of the solutions will coincide.

[45]In each of the positions shown in Figure 223b, the letters A, B, C can be permuted. The different figures which then correspond to these permutations will allow the same number of solutions, which is why we have kept only one of them in our enumeration.

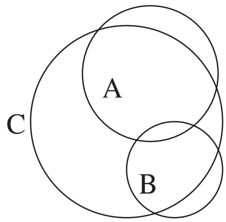

FIGURE XI

Figure XI is meant to include other possible positions in which the intersections of A and B, for instance, are both interior to C (instead of one inside and one outside, as in Fig. 223b). This would not change the result of the current discussion. The same remark could be applied to other problems, such as finding the circle orthogonal to A, B, C. These are numbered I to XI in Figure 223b.

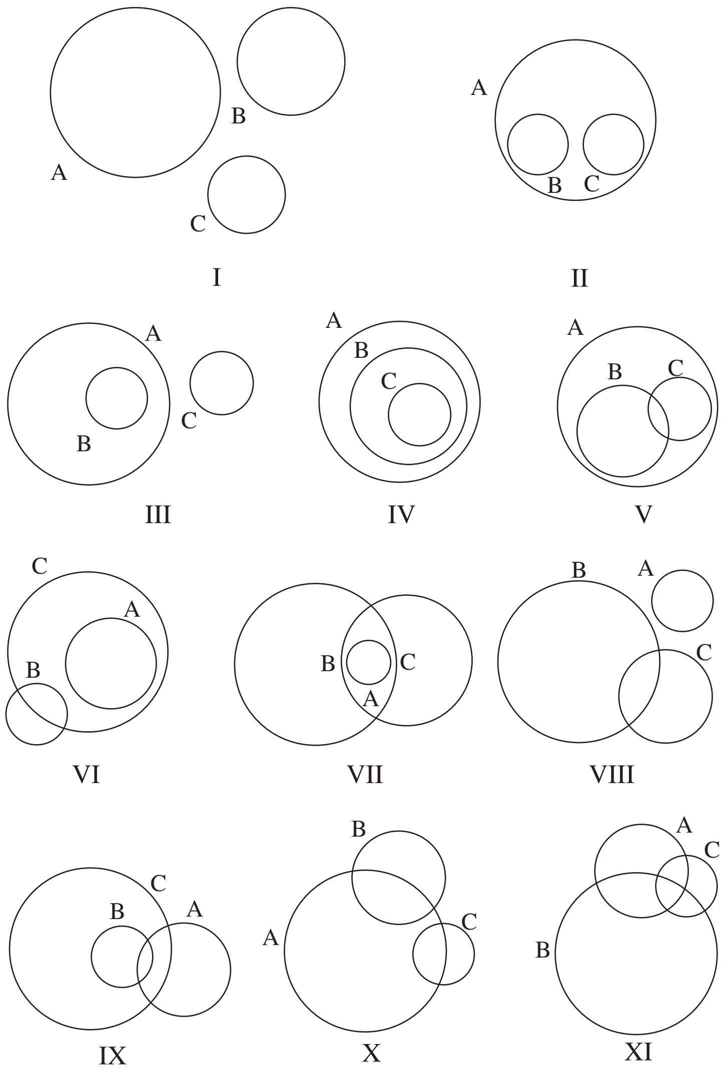

Figure 223b

	positive centers of similitude	negative centers of similitude	number of solutions	corresponding groups
I	all	none	8	
II	S_{23}, S'_{23}	S_{12}, S'_{12}, S_{13}, S'_{13}	8	
III	S_{13}, S'_{13}, S_{23}, S'_{23}	S_{12}, S'_{12}	none	
IV	none	all	none	
V	S_{23}	S'_{23}, S_{12}, S'_{12}, S_{13}, S'_{13}	4	(0,1)
VI	S_{12}, S'_{12}, S_{23}	S'_{23}, S_{13}, S'_{13}	4	(2,3)
VII	S_{23}	S'_{23}, S_{12}, S'_{12}, S_{13}, S'_{13}	4	(0,1)
VIII	S_{12}, S'_{12}, S_{13}, S'_{13}, S_{23}	S'_{23}	4	(0,1)
IX	S_{12}, S_{13}	S'_{12}, S'_{13}, S_{23}, S'_{23}	4	(2,3)
X	S_{12}, S_{13}, S_{23}, S'_{23}	S'_{12}, S'_{13}	4	(0,1)
XI	S_{12}, S_{13}, S_{23}	S'_{12}, S'_{13}, S'_{23}	8	

We may observe that the number of cases to examine could have been reduced considerably by taking into account that cases I and II can be reduced to each other by an inversion; the same is true of cases III, IV, cases V, VI, VII, VIII, and cases IX, X.

Note D:
On the Notion of Area

313. In Book IV of this work, we have taken the usual path of assuming *a priori* (**243**) that it is possible to define the area of a polygon; that is, that we can associate a number to each polygon (called its area) possessing the following properties:

I. *Two congruent polygons have the same area, regardless of their position in space;*

II. *The polygon P'', the sum of two adjacent polygons P, P', has area equal to the sum of the areas of P and P'.*

The possibility of such a correspondence amounts, in this theory, to a *postulate.* But this postulate is useless. The fact in question need not be assumed, but can be proved by the following method, which must be regarded, for this reason, as superior to the previous one.

314. THEOREM. *In any triangle, the product of a side and the corresponding altitude is the same, regardless of the chosen side.*

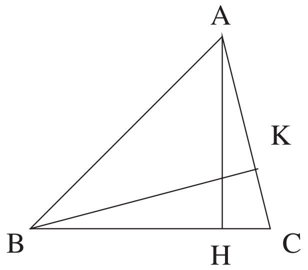

FIGURE 224

Let the given triangle be ABC (Fig. 224), in which altitudes AH, BK correspond to the sides BC, CA respectively. Right triangles ACH, BCK have angle C in common; therefore they are similar and give us

$$\frac{AH}{AC} = \frac{BK}{BC},$$

so that $BC \cdot AH = AC \cdot BK$. QED

The *area* of the triangle is what we call the preceding product, multiplied by a constant k, chosen once and for all, and whose choice will be discussed soon. This area is zero when the triangle ceases to exist properly speaking (when the three vertices are collinear) and only in that case.

The areas of triangles with the same altitude are proportional to their bases. It is clear that two congruent triangles have the same area.

315. Consider an arbitrary point O in the plane of a triangle. By joining it to the three vertices, we form three new triangles with the three sides of the original triangle as bases, and vertex O. One of these triangles will be said to be *additive* (for instance, triangle OBC in figure 228) if it is on the same side of the original triangle, relative to the common base; it will be called *subtractive* (for instance, triangle OBC in Figure 229) otherwise.[46]

THEOREM. *If an arbitrary point O is given in the plane of a triangle, and joined to its vertices, the sum of the areas of the additive triangles, minus the sum of the areas of the subtractive triangles, equals the area of the original triangle.*

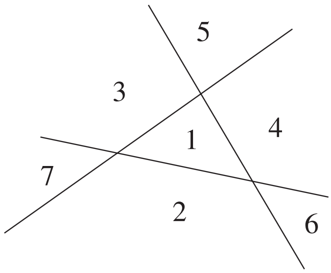

FIGURE 225

The sides of the given triangle divide the plane into seven regions (figure 225): one interior region; three regions (2–4 in the figure) separated from the first by one of the three sides; and three last regions (5–7) in the vertical angles corresponding to the angles of the triangle.

We now distinguish five cases.

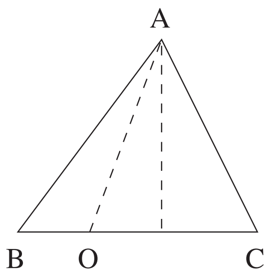

FIGURE 226

1°. *Point O is on one of the sides.* If point O is taken on side BC of triangle ABC (Fig. 226), there is no triangle OBC. Then the sum of the areas of the two

[46]If the vertices of the triangle are given in a fixed (but arbitrary) order, and those of one of the triangles with vertex O are given in such a way that the common vertices are in the same order, we can also say that *this last triangle is additive or subtractive, depending on whether it has the same sense of rotation as the original triangle or not.*

triangles OAB, OAC is equal to the area of ABC because these triangles have the same altitude (the perpendicular from A to BC) and the third base BC is the sum of the first two bases OB, OC.

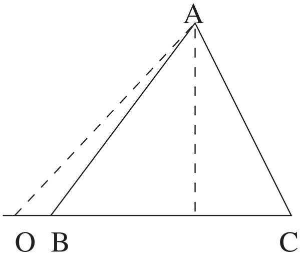

FIGURE 227

2°. *Point O is on the extension of a side.* If point O is on the extension of BC (Fig. 227), there is no triangle OBC. As for the triangles OAC, OAB, their difference is indeed ABC: these triangles have the same altitude, and the third base BC is the difference of the first two OC, OB.

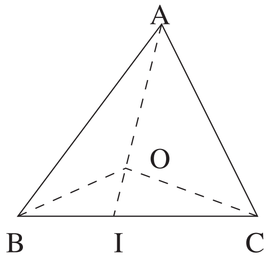

FIGURE 228

3°. *Point O is inside the triangle* (Fig. 228). We extend OA to its intersection I with BC. Triangle ABC is equal (1°) to the sum of the triangles ABI, ACI, which can in turn be decomposed into $AOB + BOI$, $AOC + COI$. Now the triangles AOB, AOC are additive triangles, and the triangles BOI, COI together give a third additive triangle BOC.

4°. *Point O is outside triangle ABC, but inside one of its angles,* for example $\widehat{A}$. If I is the intersection of OA and BC, the sum of the two additive triangles AOB, AOC can be replaced (1°) by the sum of the four triangles AIB, AIC, OIB, OIC. The sum of the first two of these is ABC, and the sum of the other two is OBC. Thus we have

$$AOB + AOC = ABC + OBC,$$

or

$$OAB + OAC - OBC = ABC,$$

which is the required relation, since triangle OBC is subtractive.

5°. *Point O is in a vertical angle corresponding to one of the angles of the triangle,* for instance $\widehat{A}$ (Fig. 230). In this case A is inside triangle OBC, so that (3°)

$$OBC = ABC + OAB + OAC,$$

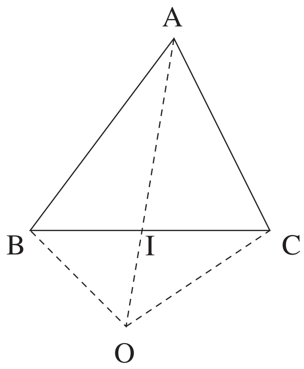

FIGURE 229

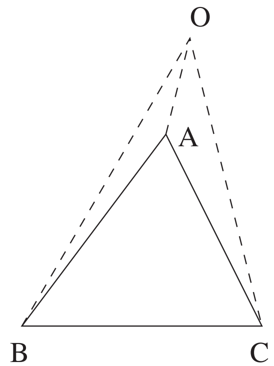

FIGURE 230

or

$$OBC - OAB - OAC = ABC,$$

which is the required relation because triangle OBC is additive, and the other two subtractive.

316. Now consider an arbitrary polygon $ABCDE$ (Fig. 231) and an arbitrary point O in its plane. Joining O to the vertices, we form triangles with vertex O and the various sides as bases. Each of these triangles will be considered additive or subtractive depending on whether it is on the same side as the polygon,[47] relative to the common side.

THEOREM. *Consider a polygon, decomposed in an arbitrary manner into a number n of triangles, and a point O of the plane, joined to all its vertices. The difference S between the sum of the areas of the additive triangles with vertex O and the sum of the areas of the subtractive triangles* (or just the first sum, if the second one does not exist) *is equal to the sum Σ of the areas of the n triangles into which the polygon was decomposed.*

[47]When we talk of the position of the polygon, relative to one of its sides, we refer to the region which is immediately next to this side. For instance, triangle OAB (Fig. 231) is additive because it is on the same side of AB as the area shaded in the figure.

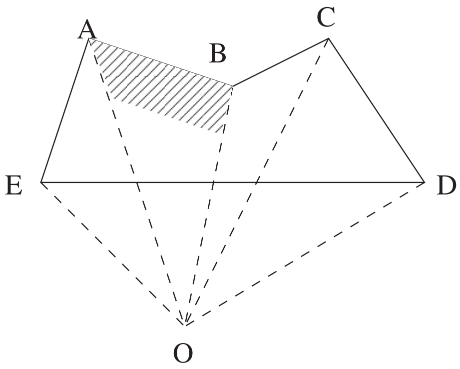

FIGURE 231

If the theorem is true for two adjacent polygons P, P' decomposed into triangles, it is also true for their sum P''. Indeed, we can distinguish two parts in the perimeters of P and P': the sides (or parts of sides) which are not common to both polygons, but which form part of the perimeter of P'', together with their additive and subtractive triangles,[48] and the common sides, or parts of sides. To these common sides correspond triangles which are additive for P or P', and subtractive for the other (since P, P' are situated on different sides of the common side) and which will therefore disappear when calculating the sum of the quantities S relative to these two polygons. Thus this sum is indeed the quantity S formed with polygon P''. On the other hand, the quantity Σ for P'' is obviously the sum of the analogous quantities for P and P', so that the equality of the quantities S and Σ is true for this third polygon P'' if it is true for the first two.

We can now consider the proof finished, because the theorem has been proved when $n = 1$ (in which case it reduces to the preceding theorem) and, if it is true for some value of n, it will be true for the following value $n + 1$ (since a polygon composed of $n + 1$ triangles is the sum of a triangle and a polygon composed of n triangles).

COROLLARIES. *The quantity S is independent of the choice of the point O,* since the quantity Σ does not depend on this choice.

Likewise, *the quantity Σ is independent of the decomposition of the polygon into triangles.*

317. The common value of the numbers S and Σ will be called the *area* of the polygon.

Two congruent polygons have the same area since they can be decomposed into pairs of congruent triangles; also, the preceding proof shows that, *when two polygons are adjacent, the area of their sum is equal to the sum of the partial areas.*

In other words, *the areas thus defined possess properties I and II.*

[48]We may have to divide a side of the polygon P into several segments, some of which will be shared with P', and some not, and to replace the triangle with vertex O and this segment as base with 'partial' triangles whose bases are the smaller segments. This will not change the quantity S, since these partial triangles add up to the total triangle (by the preceding theorem). We can do the same with P' and P''.

318. It follows that *it is impossible to decompose a polygon into parts which, assembled differently (but so that they are still adjacent to each other), form a polygon that lies inside the original one.*

This is true because the new polygon would have the same area as the first.

This proposition is not proved by the theory given in the text, since the existence of areas is taken as a postulate.

319. We have not yet discussed the choice of the constant k. It is clear that changing this number amounts to changing all areas into proportional areas. We have already remarked (**244**) that such a change would not affect its two fundamental properties.

We can then determine k in such a way that the square with a side equal to the unit of length has unit area. Now this square is composed of two triangles, each of which has both a base and an altitude equal to the unit length. Its area is $2k$, so that we choose $k = \frac{1}{2}$. The area of any triangle will then be half the product of its base and altitude. The areas determined in this way are then just those which we learned how to calculate in the text.[49]

[49]The arguments given in the text show that the method we have used earlier to calculate areas is the only one which has the preceding properties, while at the same time satisfying the condition that the square with unit side has unit area.

Miscellaneous Problems and Problems Proposed in Various Contests[50]

Exercise 343. If A, B, C, D are four points on a circle (in this order), and if a, b, c, d are the midpoints of arcs $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CD}, \overset{\frown}{DA}$, show that lines ac, bd are perpendicular.

Exercise 344. We take points D, E, F on sides BC, CA, AB of a triangle, and construct circles AEF, BFD, and CDE. Prove that:

1°. These three circles are concurrent at a point O;

2°. If an arbitrary point P in the plane is joined to A, B, C, then the new points a, b, c where PA, PB, PC intersect these circles belong to a circle passing through O and P.

Exercise 345. With each side of a cyclic quadrilateral $ABCD$ as a chord, we draw an arbitrary circular segment. The four new points A', B' ,C', D' where each of these four circles S_1, S_2, S_3, S_4 intersects the next are also the vertices of a cyclic quadrilateral.

Exercise 346. Two circles S_1, S_2 intersect at A, A'; S_2 and a third circle S_3 intersect at B, B'; S_3 and a fourth circle S_4 at C, C'; and S_4, S_1 at D, D'. A condition for quadrilateral $ABCD$ (and, by the previous exercise, quadrilateral $A'B'C'D'$) to be cyclic is that the angle between S_1 and S_2, plus the angle between S_3 and S_4 (these angles being taken with an appropriate orientation[51]) be the same as the angle between S_2 and S_3 plus the angle between S_4 and S_1.

Exercise 347. Consider four circles S_1, S_2, S_3, S_4, and two common tangents of the same kind[52] (that is, both internal or both external) α, α' for S_1, S_2; β, β' for S_2, S_3; γ, γ' for S_3, S_4; δ, δ' for S_4, S_1. If there is a circle tangent to α, β, γ, δ, then there is also a circle tangent to α', β', γ', δ'. A condition for the existence of such circles is that the sum of the lengths of two of these tangents (between their points of contact) be equal to the sum of the other two.

[50]Exercises 349, 350, 353, 354, 384, 386, 387, 393, 394, 400, 404, 413 are taken from the General Competition of Lycées and Colleges. Exercises 365, 374, 397, 406, 409, 412, 421 were taken from the contest of the Assembly of the mathematical sciences. We have not felt obligated to give these problems in the form in which they were originally proposed; we have, in particular, made certain changes in their formulation to correspond to other exercises given in the rest of this work.

[51]One should try to orient the angles, using, as needed, the conventions of Trigonometry (*Leçons* de Bourlet, book I, chapter I) in order to give a proof which applies for all possible cases of the figure.

[52]There are more restrictions than the author here indicates to the choice of tangents. See solution for details. – transl.

Exercise 348. Given an arbitrary pentagon, we construct the circles circumscribing the triangles formed by three consecutive sides (extended if necessary). Show that the five points (other than the vertices of the pentagon) where each circle intersects the next are concyclic. (Exercise 106.)

Exercise 349. Two congruent triangles ABC, abc are given. Find the locus of points O with the following property: when triangle ABC is rotated about center O until AB occupies a new position $a'b'$ parallel to ac, the new position b' of B is on the line OC. Also find, under these conditions, the loci of the points a', b', c'.

Exercise 350. Let A', B', C' be the reflections of the intersection point of the altitudes of a triangle in its sides BC, CA, AB. Let points M, N be the intersections of line $B'C'$ with AC and AB respectively; let points P, Q be the intersections of $C'A'$ with BA, BC; and let points R, S be the intersections of $A'B'$ with CB, CA. Show that lines MQ, NR, PS are concurrent. (Their point of intersection is just the point of intersection of the altitudes of the triangle ABC.)

Exercise 351. Inscribe a trapezoid in a given circle, knowing its altitude and the sum or difference of the bases.

Exercise 352. Let AB be a diameter of a given circle. A circle CMD, with center A, intersects the first one at C and D, and M is an arbitrary point on this circle. Let points N, P, Q be the intersections of lines BM, CM, DM respectively with the original circle. Then:

1°. $MPBQ$ is a parallelogram.

2°. MN is the geometric mean of NC and ND.

Exercise 353. Given an isosceles triangle OAB (in which $OA = OB$), we draw a variable circle with center O, and two tangents from A, B to this circle which do not intersect on the altitude of the triangle.

1°. Find the locus of the intersection M of these two tangents.

2°. Show that the product of MA and MB is equal to the difference of the squares of OM and OA.

3°. Find the locus of point I on MB such that $MI = MA$.

Exercise 354. On base BC of an arbitrary triangle ABC, we take any point D. Let O, O' be the circumcenters of triangles ABD, ACD.

1°. Show that the ratio of the radii of the two circles is constant.

2°. Determine the position of D for which these radii are as small as possible.

3°. Show that triangle AOO' is similar to triangle ABC.

4°. Find the locus of the point M which divides segment OO' in a given ratio; examine the special case when M is the projection of A onto OO'.

Exercise 355. An angle of fixed size rotates around a common point of two circles O, O'. Its sides intersect the two circles again at M, M' respectively. Find the locus of points which divide MM' in a given ratio. More generally, find the locus of the vertex of a triangle with base MM', and similar to a given triangle.

Exercise 356. If five lines A, B, C, D, E are such that two of them, for instance A and B, are divided in the same ratio by the other three, then any two of them are divided in the same ratio by the other three. (The proof must distinguish two cases: of the two new lines to which we want to apply it, one line may or may not be one of the original lines.)

Exercise 357. Let a, b, c be the three sides of a triangle, and x, y, z the distances from a point in the plane to these three sides. If this point is on the circumscribed circle, one of the ratios $\frac{a}{x}$, $\frac{b}{y}$, $\frac{c}{z}$ is equal to the sum of the other two, and conversely.

Exercise 358. Given a line segment AB, and a point C on this segment, find the locus of the points of intersection of a variable circle passing through A, B with a line joining point C to the intersection of the tangents at A, B to this circle.

Exercise 359. On the extension of a fixed diameter of a circle O, we take a variable point, from which we draw a tangent to the circle. Find the locus of the point P on this tangent such that $PM = MO$ (see **92**).

Exercise 360. From a point M in the plane of a rectangle we drop perpendiculars to the sides, the first one intersecting two opposite sides at P, Q, and the second intersecting the other two sides at R, S.

1°. For any M, show that the intersection H of PR and QS is on a fixed line, and the intersection K of PS and QR is on another fixed line;

2°. Show that the bisector of angle $\widehat{HMK}$ is parallel to a side of the rectangle;

3°. Find point M, knowing points H and K;

4°. This last problem has two solutions M, M'. Show that the circle with diameter MM' is orthogonal to the circumscribed circle of the rectangle.

5°. Find the locus of points M such that PR is perpendicular to QS.

Exercise 361. From vertices B and C of triangle ABC we draw two lines BB', CC' (where B' is on side AC and C' is on side AB), such that $BB' = CC'$. Show that the two angles $\widehat{CBB'}$, $\widehat{B'BA}$ into which BB' divides angle $\widehat{B}$ cannot both be less than or both be greater than the corresponding angles $\widehat{BCC'}$, $\widehat{C'CA}$ into which CC' divides angle C (that is, we cannot at the same time have $\widehat{CBB'} > \widehat{BCC'}$ and $\widehat{B'BA} > \widehat{C'CA}$).

(Form parallelogram $BB'CF$, in which B, C are two opposite vertices, and, drawing $C'F$, compare the angles it determines at C' to those it determines at F.)

A triangle which has two equal angle bisectors is isosceles.

Exercise 361b. In any triangle, the greater side corresponds to the smaller angle bisector (take the difference of the squares of the bisectors given by the formula in **128b**, and factor out the difference of the corresponding sides).

Exercise 362. Of all the triangles inscribed in a given triangle, which has the minimum perimeter?

Exercise 362b. In a quadrilateral $ABCD$, inscribe a quadrilateral $MNPQ$ with minimum perimeter. Show that the problem does not have a proper solution (that is, one which is a true quadrilateral) unless the given quadrilateral is cyclic.

But if $ABCD$ is cyclic, there exist infinitely many quadrilaterals $MNPQ$ with the same perimeter, which is smaller than the perimeter of any other quadrilateral inscribed in $ABCD$. This perimeter is the fourth proportional for the radius of the circle $ABCD$ and the two diagonals AC, BD.

What additional condition must $ABCD$ satisfy in order that the quadrilaterals $MNPQ$ found this way will also be also cyclic? For this case, find the locus of the centers of their circumscribed circles.

Exercise 363. Show that the point obtained in Exercise 105, if it is inside the triangle, is such that the sum of its distances to the three vertices is as small as possible (Exercise 269). Evaluate this sum. (Its square is half the sum of the squares of the three sides, plus $2\sqrt{3}$ times the area.)

What happens when the point is outside the triangle?

(This circumstance occurs when one of the angles, for instance $\widehat{A}$, is greater than $120°$. Ptolemy's theorem gives the ratio of the sum $AB + AC$ to the segment AI intercepted by the circumscribed circle on the bisector of angle $\widehat{A}$. Applying the theorem of **237b** to quadrilateral $BCMI$, it will be seen that the sum $MA + MB + MC$ is minimal when point M coincides with A.)

Exercise 364. More generally, find a point such that the sum of its distances to the three vertices of a triangle, multiplied by given positive numbers ℓ, m, n, is minimal. We assume first that the three given numbers can represent the sides of a triangle.

(Let this triangle be T, and let its angles be α, β, γ. At A, using sides AB, AC respectively, we construct two angles $\widehat{BAC'}$, $\widehat{CAB'}$ equal to α. Likewise, at B, using sides BC, BA we construct angles $\widehat{CBA'}$, $\widehat{ABC'}$ equal to β, and at C, using sides CA, CB, we construct $\widehat{ACB'}$, $\widehat{BCA'}$ equal to γ. All of these angles are exterior to the triangle. Lines AA', BB', CC' intersect at a point O, which is the required point if it is inside the triangle. If it is not, and also in the case in which the three given numbers are not proportional to the sides of a triangle, the minimum is achieved at one of the vertices of triangle ABC.

In the first case, where the minimum is not at a vertex, the square of the minimum is the sum of

$$\ell^2(b^2 + c^2 - a^2) + m^2(c^2 + a^2 - b^2) + n^2(a^2 + b^2 - c^2)$$

plus the product of the areas of triangles T and ABC.)

Exercise 365. We divide each side of a triangle into segments proportional to the squares of the adjacent sides, then join each division point to the corresponding vertex. Show that:

1°. The three lines obtained in this way are concurrent;

2°. That this is precisely the point that would be obtained in Exercise 197, taking the point O to be the center of mass of the triangle;

3°. That this point is the center of mass of the triangle PQR formed by its projections on the sides of the original triangle.

Exercise 366. In a given triangle, inscribe a triangle such that the sum of the squares of its sides is minimal. (Assuming that this minimum exists, show that it can only be the triangle PQR of the preceding exercise.)

Conclude that the point O' (in the preceding exercise) is the one such that the sum of the squares of its distances to the three sides is the smallest possible (Exercises 137, 140).

More generally, in a given triangle, inscribe a triangle such that the squares of its sides, multiplied by given numbers, yield the smallest possible sum.

Exercise 367. In a given circle, inscribe a triangle such that the sum of the squares of its sides, multiplied by three given numbers, is as large as possible.

Exercise 368. A necessary and sufficient condition for the existence of a solution to Exercise 127 (a point whose distances to the three vertices of a triangle ABC are proportional to three given numbers m, n, p) is that there exist a triangle with sides $m \cdot BC$, $n \cdot CA$, $p \cdot AB$.

Exercise 369. We join the vertices of a triangle ABC to points D, E, F so that the segments AD, BE, CF are equal. Through an interior point O of the triangle we draw segments OD', OE', OF', parallel to these, with D', E', F' on the corresponding sides. Show that the sum of these segments is constant, no matter what point is chosen for O.

Exercise 370. When three lines are concurrent, there always exist numbers such that the distance of an arbitrary point in the plane to one of them is equal to the sum or the difference of its distances to the other two, multiplied by these numbers. Formulate the result in a manner entirely independent of the position of the point by an appropriate convention for the signs of the segments.

Conversely, the sum or difference of the distances of an arbitrary point M in the plane to two fixed lines, multiplied by given numbers, is proportional to the distance from M to a certain fixed line, passing through the intersection of the first two.

Exercise 371. Find the locus of points such that the sum of their distances to n given lines, taken with appropriate signs and multiplied by given numbers, is constant; in other words, the locus of points such that the areas of the triangles with a vertex at the point, and with n given segments as bases, have a constant algebraic sum. (The preceding exercise provides a solution of the problem for n lines, provided that we know how to solve it for $n - 1$ lines.) Deduce that the midpoints of the three diagonals of a complete quadrilateral are collinear.

Exercise 371b. The three circles whose diameters are the diagonals of a complete quadrilateral have the same radical axis. This axis passes through the intersection of the altitudes of each of the four triangles formed by three sides of the quadrilateral.

Exercise 372. The opposite sides of a complete quadrilateral, and its diagonals, form a set of three angles such that the polars of any point O in the plane relative to these three angles are concurrent. (Transform by reciprocal polars, and take O as the center of the directing circle.)

These same lines intercept three segments on an arbitrary transversal such that a segment which divides two of them harmonically (if it exists) also divides the third harmonically.

Three segments with this property are said to be *in involution.*

Exercise 373. The Simson line (Exercise 72) which joins the feet of the perpendiculars from a point P on the circumscribed circle of a triangle to the three sides, divides the segment joining P with the intersection H of the altitudes of the triangle into equal parts. (Prove, using Exercise 70, that the points symmetric to P relative to the three sides, are on a line passing through P.)

Deduce from this, and from Exercise 106 that the points of concurrence of the altitudes of four triangles formed by four lines, taken three at a time, are collinear.

Exercise 374. We fix points A, B, P, P' on a circle S, and let C be a variable point on the same circle. Show that the intersection M of the Simson lines for P and P', with respect to the triangle ABC, describes a circle S'.

Find the locus of the center of S' when A, B remain fixed, and P, P' move along S so that the distance PP' remains constant.

Also find the locus of the point M when the points A, B, C are fixed, and P, P' are variable and diametrically opposite.

Exercise 375. Find the locus of the midpoints of a triangle inscribed in a fixed circle, whose altitudes pass through a fixed point.

Exercise 376. We transform the nine-point circle (Exercise 101) of a triangle by an inversion whose pole is the midpoint of a side, and with a power equal to the power of the pole relative to the inscribed circle or equivalently (Exercise 90b), relative to the escribed circle corresponding to that side. Show that the line which is the transform of the circle is precisely the common tangent of these two circles, other than the sides of the triangle. It follows that the nine-point circle is tangent to the inscribed circle and to the escribed circles.

Exercise 377. If R and r (respectively) are the radii of the circumscribed and inscribed circles in a triangle, and if d is the distance between their centers, show that $d^2 = R^2 - 2Rr$ (Use Exercise 103 and **126**.)[53]

Conversely, if the radii of two circles and the distance between their centers satisfy this relation, we can inscribe infintely many triangles in one circle that are also circumscribed about the other.

Obtain analogous results replacing the inscribed circle by an escribed circle.

Exercise 378. In any triangle ABC:

1°. The line joining the projection of B onto the bisector of $\widehat{C}$ with the projection of C onto the bisector of $\widehat{B}$ is precisely the chord joining the points of contact E, F (Fig. 94, Exercise 90b) of the inscribed circle with sides AC, AB;

2°. The line joining the projection of B onto the bisector of $\widehat{C}$ with the projection of C onto the bisector of the exterior angle at B is the chord of contact E_3F_3 of the escribed circle for angle $\widehat{C}$ with these same sides;

3°. The line joining the projection of B onto the bisector of the exterior angle at C with the projection of C onto the bisector of the exterior angle at B is the chord of contact E_1F_1 of the escribed circle for angle $\widehat{A}$ with these same sides;

[53] For another solution to this problem, see Exercise 411.

4°. The projections of A onto the bisectors of the interior and exterior angles at B and C are on the same line parallel to BC, and their consecutive distances are equal to $p - c$, $p - a$, $p - b$;

5°. The six points obtained by projecting each of the vertices A, B, C onto the exterior angle bisectors at the other two vertices belong to the same circle (this reduces to Exercise 102). This circle is orthogonal to the escribed circles. Its center is the same as the center of the circumscribed circle of $A'B'C'$, the triangle whose vertices are the midpoints of the sides of ABC. Its radius is equal to the hypotenuse of the right triangle whose legs are the radius of the inscribed circle and the semiperimeter of triangle $A'B'C'$. There are three analogous circles, each of which passes through two projections onto exterior angle bisectors, and four projections onto interior angle bisectors.

Exercise 379. Each escribed circle of triangle ABC is tangent to the extensions of two of its sides. We draw the lines through pairs of these points of contact. Show that these three lines form a triangle whose vertices are on the altitudes of the original triangle, and the intersection of these altitudes is the center of the circumscribed circle for the new triangle.

Exercise 380. Suppose we know (a) the point O corresponding to itself (**150b**) in two similar figures F, F' with the same orientation, and (b) a triangle T similar to the triangle formed by this point with two other corresponding points. Suppose we also know the point O' corresponding to itself (**150b**) in similar figures F', F'' with the same orientation, and a triangle T' similar to the triangle formed by this point with two other corresponding points. Construct the point O_1 corresponding to itself (**150b**) in the similar figures F, F'', and a triangle T_1 similar to the triangle formed by this point with two other corresponding points.

(Place triangles T, T' so that they have a common side $\omega\alpha'$, and denote by α, α'' the vertices opposite this side in the two triangles. Then T_1 is the triangle $\omega\alpha\alpha''$. Take the inverses A, A', A'' of α, α', α'' with respect to the pole ω: then triangle O_1OO' must be similar to $A_1A''A$.)

Exercise 381. Construct a polygon knowing the vertices of the triangles whose bases are its various sides, and similar to given triangles.

(The preceding exercise allows a reduction of one in the number of sides of the required polygon. One can continue until there are only two vertices to determine.)

When is the problem impossible or underdetermined?

Exercise 382. Let ABC be a triangle, and let O, a, b, c be four arbitrary points. Construct (a) a triangle BCA' similar to triangle bcO and with the same orientation (B, C being the points corresponding to b, c); (b) a triangle CAB' similar to caO with base CA, and (c) a triangle ABC' similar to abO with base AB. Show that triangle $A'B'C'$ is similar to, but with opposite orientation from, the triangle whose vertices are the inverses of the points a, b, c with pole O.

Exercise 383. On two given segments as chords, construct circular arcs subtending the same arbitrary angle V. Show that, as V varies, the radical axis of the two circles turns about a fixed point. (This point can be considered to be determined by the fact that the triangles with these vertices and with the two segments as bases are equivalent, and they have the same angle at the common vertex.)

Exercise 384. A quadrilateral $ABCD$ (a *kite* or *rhomboid*) is such that the adjacent sides AD, AB are equal, and the other two sides are equal as well. Show that this quadrilateral is circumscribed about two circles.[54] Find the locus of the centers of these circles if the quadrilateral is articulated, one of its sides remaining fixed.

Exercise 385. More generally, if a quadrilateral $ABCD$ has an inscribed circle, and is articulated while side AB remains fixed, then it has an inscribed circle in all its positions (Exercise 87). Find the locus of the center O of the circumscribed circle.

(To make the situation definite, assume the inscribed circle is inside the polygon, and lay off lengths $AE = AD$ (in the direction of AB) and $BF = BC$ (in the direction of BA), both on side AB. Using Exercise 87, reduce the question to Exercise 257.)

Show that the ratio of the distances from O to two opposite vertices remains constant.

Exercise 386. Given four fixed points A, B, C, D on a circle, take an arbitrary point P in the plane, and denote by Q the second intersection point of the circles PAB and PCD. Find the locus of Q as P moves on a line or on a circle. Find the locus of points P such that Q coincides with P.

Exercise 387. We join the vertices of a square $ABCD$ with an arbitrary point P in the plane. Let A', B', C', D' be the second points of intersection of these four lines with the circle circumscribed about $ABCD$. Show that $A'B' \times C'D' = A'D' \times B'C'$.

Conversely, let $A'B'C'D'$ be a cyclic quadrilateral such that $A'B' \times C'D' = A'D' \times B'C'$. Find a point P such that the lines PA', PB', PC', PD' intersect the circumscribed circle in the vertices of a square.

(This is a particular case of Exercise 270b, 5^o. However, the problem here admits of two solutions, while there is only one in the general case. What is the reason for this difference?)

Exercise 388. More generally, find an inversion which transforms the vertices A', B', C', D' of a cyclic quadrilateral into the vertices of a rectangle.

Show that the poles are the limit points (Exercise 152) of the inscribed circle and of the third diagonal of quadrilateral $A'B'C'D'$.

Exercise 389. Still more generally, find an inversion which transforms four given points into the vertices of a parallelogram.

Exercise 390. Given two circles and a point A, find an inversion in which the point corresponding to A is a center of similarity of the transformed circles.

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[54] That is, there are two circles which are tangent to all four lines along which the sides of the quadrilateral lie. The solution makes this clearer.–transl.

Exercise 391. A variable point M on a circle is joined to two fixed points A, B. The two lines intersect the circle again at P, Q. Denote by R the second intersection of the circle with the parallel to AB passing through M. Show that line QR intersects AB at a fixed point.

Use this result to find a method of inscribing a triangle in a given circle with two sides passing through given points, while the third is parallel to a given direction; or such that the three sides pass through given points. (These two questions are easily reduced to each other, and to Exercise 115). Solve the analogous problem for a polygon with an arbitrary number of sides. (Another method is proposed in Exercise 253b.)

Exercise 392. About a given circle, circumscribe a triangle whose vertices belong to given lines.

Exercise 393. Given two points A, B on a line, we draw two variable circles tangent to the line at these points, and also tangent to each other. Show that the second common tangent $A'B'$ to these two circles is tangent to a fixed circle, and find the locus of the midpoint of $A'B'$.

Exercise 394. Two variable circles C, C_1 are tangent at a point M, and tangent to a given circle at given points A, B.

1°. Find the locus of M.

2°. Find the locus of the second center of similarity of C, C_1.

3°. To each point N of the preceding locus there correspond two pairs of circles C, C_1; C', C_1' satisfying the given conditions, and therefore two points of tangency M, M'.

Find the locus of the center of the circle circumscribing NMM', the locus of the circle inscribed in this triangle, and the locus of the intersection of its altitudes. Each common point of pairs of these loci belongs to the third.

Exercise 395. Two circles C, C' meet at A, and a common tangent meets them at P, P'. If we circumscribe a circle about triangle APP', show that the angle subtended by PP' at the center of this circumscribed circle is equal to the angle between circles C, C', and that the radius of this circle is the mean proportion between the radii of circles C, C' (which implies the result of Exercise 262, 3°). Show that the ratio $\frac{AP}{AP'}$ is the square root of the ratio of these two radii.

Exercise 396. What are necessary and sufficient conditions which four circles A, B; C, D must satisfy in order that they can be transformed by inversion so that the figure formed by the first two is congruent to that formed by the second two? (Using the terminology introduced in Note A, **289, 294**, what are the *invariants*, under the group of inversions, of the figure formed by two circles?)

1°. If circles A, B have a common point, it is necessary and sufficient that the angle of these two circles equal the angle of C, D; or, which is the same (by the preceding exercise), that the ratio of the common tangent to the geometric mean of the radii be the same in both cases;

2°. If circles A, B have no common point, it is necessary and sufficient that the ratio of the radii of the concentric circles into which they can be transformed by an inversion (Exercise 248) be the same as the ratio of the radii of the concentric circles into which C, D can be transformed by an inversion (generally, a different inversion from the first). (Using the language of Note A, it is necessary and sufficient that the figures (A, B) and (C, D) have the same *reduced form* under inversion.)

This result can also be expressed as follows: the cross ratio (**212**) of the intersection points of A, B with any of their common orthogonal circles is constant, and the same is true of the cross ratio of two of these points and the limit points. The required condition is that this ratio have the same value for the circles C, D as for A, B.

Finally, if r, r' are the radii of A, B, and d is the distance between their centers, the quantity $\frac{d^2-r^2-r'^2}{rr'}$ must have the same value as the corresponding value calculated for the circles C, D.

We could also express this by saying that if the circles A, B have a common tangent (for example, a common external tangent) of length t, and the same is true for C, D, then the ratio $\frac{t}{\sqrt{rr'}}$ must be the same in the two cases.

Exercise 397. We are given two points A, A' and two lines D, D' parallel to, and at equal distance from, AA'.

1°. Show that for every point P on D there corresponds a point P' on D' such that line PP' is tangent to the two circles PAA', $P'AA'$.

2°. Prove that the product of the distances from A, A' to line PP' is constant.

3°. Find the locus of the projection of A onto PP'.

4°. Find the point P such that the line PP' passes through a given point Q.

5°. Show that the angle of the circles PAA', $P'AA'$, and the angle $\widehat{PAP'}$, are constant.

Exercise 398. Let C be a circle with diameter AB, and D a line perpendicular to this diameter, which intersects C. Let c, c' be the circles whose diameters are the two segments into which D divides AB. We draw a circle tangent to C, c, D, and another circle tangent to C, c', D. Show that these two circles are equal, and that their common radius is one-fourth the geometric mean of the radii of C, c, c'.

Exercise 399 (the Greek *Arbelos*). Let A, B be two tangent circles. Let C be a circle tangent to the first two; let C_1 be a circle tangent to A, B, C; let C_2 be a circle tangent to A, B, C_1; let C_3 be tangent to A, B, C_2, ...; and let C_n be a circle tangent to A, B, C_{n-1}. Consider the distance from any of the centers of C, $C_1, \ldots, C_n$ to the line of centers of A, B, and the ratio of this distance to the diameter of the corresponding circle. Show that this ratio varies by one unit in passing from any circle to the next one, at least in the case in which they are exterior (which always happens when the circles A, B are tangent internally),and when the centers of these circles all lie on the same side of line AB. Show how this statement must be modified when two consecutive circles C_{n-1}, C_n are tangent internally. (*Arbelos* is a Greek word meaning *sickle.*)[55]

Exercise 400. Let A, B, C be three circles with centers at the vertices of a triangle, each pair of which are tangent externally (Exercise 91). Draw the circle externally tangent to these three circles, and also the circle internally tangent to these three circles. Calculate the radii of these circles knowing the sides a, b, c of the triangle (preceding exercise, Exercise 301).

[55]This note is Hadamard's own. The usual translation of *arbelos* is *shoemaker's knife*. But see for instance Harold P. Boas, *Reflections On the Arbelos*, American Mathematical Monthly, 113, no. 3 (March 2006), 236–249. –transl.

Exercise 401. Given three circles with centers A, B, C and radii a, b, c, let H be the radical center of the circles concentric with the first and with radii $a+h$, $b+h$, $c+h$, and let N be taken on AH such that $\frac{AN}{AH} = \frac{a}{a+h}$. Show that, as h varies, the points H and N describe two straight lines, the first of which passes through the centers of the circles tangent to the three given circles (with contacts of the same kind), and the second passes through the points of contact of these circles with the circle A.

Give an analogous statement which will allow us to find the circles which have different kinds of contact with the circles A, B, C.

Exercise 402. Find a circle which intersects four given circles at equal angles.

Exercise 402b. Find a circle which intersects three given circles at given angles. (We know (Exercise 256) the angle at which the required circle intersects any of the circles which have the same radical center as the given circles. Among these, determine (Note C, **311**) three for which this angle is zero, so as to reduce the problem to the problem of tangent circles; or two[56] for which this angle is right, thus reducing the question to Exercise 259.)

Exercise 403. Given three circles, find a fourth whose common tangents with the first have given lengths. (Reduce this to the preceding problem by drawing, through each point of contact of these common tangents, a circle concentric to the corresponding given circle.)

Exercise 404. We are given a circle, two points A, A' on this circle, and a line D. Show that this line contains points I, I' with the following property: if P, P' denote the intersections of D with the lines joining A, A' with a variable point M on the circle, the product $IP \cdot I'P'$ remains constant; that is, independent of the position of M.

Exercise 405. In the preceding problem, assume that line D does not intersect the circle. Show that on each side of the line there is a point at which segment PP' subtends a constant angle (Exercise 278).

Exercise 406. We are given circles S, Σ with no points in common, with centers O, ω, and with radii R, ρ. We consider the circles C which are tangent to S and orthogonal to Σ.

1°. Show that all of these circles are tangent to a second fixed circle;

2°. We denote by M, M' the common points of any such circle C with Σ and, through a fixed point A on $O\omega$, we draw lines parallel to the bisectors of angles $\widehat{O\omega M}$, $\widehat{O\omega M'}$. We extend these lines to their intersection points P, P' with a fixed line D, perpendicular to $O\omega$. Show that there exist two points X, X' such that the lines XP, $X'P'$ are always perpendicular;

3°. Show that there exist two points at which segment PP' subtends a constant angle (preceding exercise).

[56]The problem considered in **311** does not always have a solution, since the point α mentioned there might be inside the given circles. This situation can actually occur in this problem, even when the problem has a solution. One should show that this inconvenience can always be avoided by an appropriate combination of the two methods we indicate.

4°. Consider a position C_1 of the circle C, intersecting Σ at M, M', and a second position C_2 intersecting Σ at M' and a third point M'', then C_3 intersecting Σ in M'', M''', etc. Find a condition for the circle C_{n+1} to coincide with C_1.

(If d is the distance $O\omega$, the right triangle whose hypotenuse is $d^2 - R^2 - \rho^2$ (or $R^2 + \rho^2 - d^2$) and a leg equal to $2R\rho$, must have an acute angle equal to half the central angle of a regular (convex or star) polygon whose number of sides is n or a divisor of n.)

If the circles S, Σ have a common point, the limiting position of points M, M', M'', M''' will be their points of intersection.

Exercise 407. The perpendicular dropped from the intersection of the diagonals of a cyclic quadrilateral to the line which joins this point to the center of the circumscribed circle is divided into equal parts by the opposite sides of the quadrilateral. (Apply the remark in **211**.)

Exercise 408. Given two circles C, C', and two lines which intersect them, the circle which passes (Exercise 107b) through the intersections of the arcs intercepted on C with the chords of the arcs intercepted on C' has the same radical axis as C, C' (use Exercise 149).

Exercise 409. We are given two concentric circles S, C and a third circle C_1. The locus of the centers of the circles orthogonal to C, and such that their radical axis with C_1 is tangent to S, is a circle S_1, concentric with C_1.

Conversely, the locus of the centers of the circles orthogonal to C_1, and such that their radical axis with C is tangent to S_1, is the circle S.

Exercise 410. With each point of a given circle C as center, we draw circles whose radii have a given ratio to the distance of this center to a given point A in the plane (or, more generally, with a given ratio to the tangent from this point to another fixed circle). Show that there exists a point P which has the same power with respect to all of these circles.

The radical axis of each of these circles with the circle C is tangent to a fixed circle with center P.

Exercise 411. Through an arbitrary point of a circle C, we draw tangents to a circle C'. Show that the line joining the new intersections of these tangents with the circle C is tangent to a fixed circle (reduce this to the preceding exercise). This circle has the same radical axis as C, C'.

Calculate the radius of this new circle, and the distance from its center to the center of C, knowing the radii of the given circles, and the distance between their centers.

Deduce from this a solution to Exercise 377.

Exercise 412. We are given an angle $\widehat{AOB}$ and a point p.

1°. Find a point M on side OA such that the two circles C, C' tangent to OB and passing through points M, P intersect at a given angle;

2°. Study the variation of the angle between C, C' as M moves on OA;

3°. Let Q, Q' be the points (other than M) where these circles intersect side OA. Show that the circle through P, Q, and Q' is tangent to a fixed line as M moves on OA (this reduces to the preceding exercise).

Exercise 413. We are given two parallel lines, and a common perpendicular which intersects them in A, B. Points C, D are taken on these lines so that trapezoid $ABCD$ has an area equal to the area of a given square. Denote by H the projection on CD of the midpoint of AB. Find the locus of H. (One must distinguish two cases, according as the points C, D are on the same side or opposite sides of the common perpendicular.)

Exercise 413b. If four circles are inscribed in the same angle, or in the corresponding vertical angle, and they are also tangent to a fifth circle, then their radii r_1, r_2, r_3, r_4 are in proportion. (Observe that these circles can be arranged in pairs which correspond to each other in the same inversion.)

Exercise 414. Another solution to Exercise 329 (to draw through a given point inside an angle a secant which forms, with the sides of the angle, a triangle with given area): Construct a parallelogram with a vertex at the given point, and two sides on the sides of the angle. This parallelogram cuts two 'partial' triangles from the required triangle, whose sum is known. The problem then reduces to that of Exercise 216.

Exercise 415. Construct a triangle knowing an angle, the perimeter, and the area (Exercises 90b, 299). Among all triangles with a given angle and given perimeter, which has the largest area?

Exercise 416. Construct a triangle knowing a side, the perimeter, and the area (construct the figure formed by the inscribed circle and an escribed circle).

Of all triangles with a given side and given area, which one has the smallest perimeter?

Of all triangles with a common side and given perimeter, which is the largest in area?

Of all triangles with the same perimeter, which is the largest in area?[57]

Exercise 417. Construct a quadrilateral knowing the four sides and the area.

(Let $ABCD$ be the required quadrilateral, so that $AB = a$, $BC = b$, $CD = c$, $DA = d$. Let ABC_1 be a triangle exterior to this quadrilateral, equivalent to ADC, and with an angle $\widehat{C_1BA} = \widehat{ADC}$. Prove successively that we know the following quantities:

1°. side BC_1;

2°. the difference of the squares of AC and AC_1;

3°. the projection of CC_1 onto AB; finally,

4°. with the help of the known area we will know the projection of CC_1 on a perpendicular to AB. This will allow us, once the segment $AB = a$ is placed anywhere, to construct a segment equal and parallel to CC_1, and therefore to complete the required construction.)

If the problem can be solved, it generally has two solutions. For these two quadrilaterals, the triangle considered in Exercise 270b has the same shape, and consequently (Exercise 270b, 5°), these quadrilaterals can be considered inverse to each other.

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[57] See the algebraic solution to the same problem in *Leçons d'Algèbre Élémentaire*, Bourlet, n^o 127, page 423.

Given a quadrilateral, construct another, not equal to the first, but with equal corresponding sides and equal area.

Of all the quadrilaterals with given sides, the largest is the one which is cyclic.

Exercise 417b. Given a quadrilateral with sides a, b, c, d, diagonals e, f, and area S, we have

$$4e^2f^2 = (a^2 + c^2 - b^2 - d^2)^2 + 16S^2.$$

The angle $\widehat{V}$ of the diagonals is given by

$$\tan \widehat{V} = \frac{4S}{a^2 + c^2 - b^2 - d^2}.$$

Deduce from this a solution of the preceding exercise. (Having fixed the position of one side, each of the remaining two vertices will be the intersection of two circles.)

Exercise 418. Construct a cyclic quadrilateral knowing its sides.

Exercise 418b. Among all polygons with the same number of sides, and the same perimeter, the largest is the regular polygon. (Assuming that a polygon of maximum area exists, use the preceding exercises and Exercise 331 to show that this polygon must be regular.)

The result can be restated as follows: If S is the area of a polygon, and p its perimeter, the ratio $\frac{S}{p^2}$ is larger for a regular polygon than for an irregular polygon with the same number of sides.

Exercise 419. Among all the closed curves of same length, the circle is the one whose interior has the largest area. (Consider the ratio $\frac{S}{p^2}$ for a polygon inscribed in a circle and a polygon inscribed in a curve of the same length, the number of sides being the same in the two cases. Let the number of sides of the polygon increase indefinitely.)

Exercise 419b. Let O be the intersection of the diagonals of a quadrilateral $ABCD$, and let O_1, O_2, O_3, O_4 be the centers of the circles OAB, OBC, OCD, ODA; these four centers are the vertices of a parallelogram P.

1°. If this parallelogram is known, then the area and the diagonals of the quadrilateral are determined;

2°. If points O_1, O_2, O_3, O_4 are given, and point O moves on a line Δ, the vertices of the quadrilateral move on the sides of a parallelogram P'. Study the change in this parallelogram as line Δ varies. Find the positions of Δ for which its area is maximum;

3°. Construct a quadrilateral $ABCD$ knowing its angles and the parallelogram P; or, knowing P and the ratios $\frac{AB}{AD}$, $\frac{CB}{CD}$. Discuss.

Exercise 420. The radii of the circles circumscribing (Exercise 66) the quadrilaterals determined by the bisectors of the angles (resp., exterior angles) of a quadrilateral have the ratio $\frac{a+c-b-d}{a+c+b+d}$, where a, b, c, d are the sides of the given quadrilateral, taken in their natural order.

Exercise 420b. The opposite sides of a cyclic quadrilateral are extended to their intersections E, F, and the bisectors of the angles thus formed are drawn. Show

1°. that these bisectors intersect on the line joining the midpoints of the diagonals of the quadrilateral, and divide this segment into a ratio equal to the ratio of the diagonals;

2°. that these lines also bisect the angles subtended by this segment at E and F;

3°. that these bisectors intersect the sides of the quadrilateral in four points (other than E , F) which are the vertices of a rhombus. The sides of the rhombus are parallel to the diagonals of the quadrilateral, and their length is the fourth proportional to the lengths of these diagonals and their sum;

4°. analogous statements for the bisectors of the angles formed by extending the opposite sides of the quadrilateral, one side up to their point of intersection, the other past this point.

5°. that the ratio of EF to the segment joining the midpoints of the diagonals is the same as the ratio of twice the product of these diagonals and the difference of their squares. Calculate EF knowing the sides of the quadrilateral.

Exercise 421. Let O be a point inside triangle $A_1A_2A_3$, and let (k_1'), (k_2'), (k_3') be the circles inscribed in the triangles A_2A_3O, A_3A_1O, A_1A_2O.

1°. If (k_1) is any circle concentric with (k_1'), one can find (k_2) concentric with (k_2') and (k_3) concentric with (k_3') such that (k_2), (k_3) intersect at a point N_1 on A_1O, (k_3), (k_1) intersect at a point N_2 on A_2O, and (k_3), (k_1) at a point N_3 on A_3O.

2°. Circle (k_1) intersects A_2A_3 at two points m_1, n_1 such that $A_3n_1 = A_3N_3$, $A_2m_1 = A_2N_2$; Circle (k_2) intersects A_3A_1 at two points ℓ_2, n_2 such that $A_3n_2 = A_3N_3$, $A_1\ell_2 = A_1N_1$.

3°. As the radius of (k_1) varies, the radii of (k_2), (k_3) vary so that the properties in 1° remain true. The points P_1, P_2, P_3, where pairs of these circles intersect (other than N_1, N_2, N_3) move along the common tangents (t_1), (t_2), (t_3) to the pairs of circles $(k_2'),(k_3')$; $(k_3'),(k_1')$; $(k_1'),(k_2')$ respectively. These three lines are concurrent at a point obtained from O by the construction indicated in Exercise 197, with triangle ABC in that exercise here replaced by the triangle formed by the centers of (k_1'), (k_2'), (k_3').

4°. Quadrilateral $P_2P_3\ell_2\ell_3$ is cyclic (Exercise 345) and its circumscribed circle (x_1') intersects sides A_2A_1, A_1A_3 and lines (t_2), (t_3) at equal angles; likewise, P_3, P_1, m_3, m_1 are on a circle (x_2') intersecting A_2A_3, A_2A_1, $(t_3),(t_1)$ at equal angles, and P_1, P_2, n_1, n_2 are on a circle (x_3') intersecting A_3A_1, A_3A_2, $(t_1),(t_2)$ at equal angles.

The center of (x_1') remains fixed as the radii of $(k_1),(k_2),(k_3)$ vary as in 3°; the line which joins it to the center of (k_1') passes through the intersection of $(t_1),(t_2),(t_3)$. Similar statements for the centers of $(x_2'),(x_3')$.

There exists a circle (x_1) tangent to A_2A_1, A_1A_3, $(t_2),(t_3)$, a circle (x_2) tangent to A_2A_3, A_2A_1, $(t_3),(t_1)$, and a circle (x_3) tangent to A_3A_1, A_3A_2, $(t_1),(t_2)$.

5°. The intersection of m_1P_3, n_1P_2 is on the radical axis of (x_2'), (x_3'). As the radii of (k_1), (k_2), (k_3) vary, it describes the line joining the intersection of (t_1), (t_2), (t_3) with the point of contact of (k_1') with A_2A_3. A condition for (x_2), (x_3) to be tangent is that this line be (t_1) itself.

Exercise 421b. Through a given point A in the plane, construct a line on which two given circles C, C' intercept equal chords MN, $M'N'$. (If the notation is such that these segments are in the same sense, one should look for the common midpoint of MN' and NM'.) More generally, draw a line through A such that the chords intercepted by C, C' have a given ratio k. (Use Exercise 149.) Is the maximum number of solutions the same for $k \neq 1$ as it is for $k = 1$?

Exercise 422 (Morley's theorem). We divide each angle of a triangle ABC into three equal parts by lines AS, AT (so that $\widehat{CAS} = \widehat{SAT} = \widehat{TAB}$); BT, BR (so that $\widehat{ABT} = \widehat{TBR} = \widehat{RBC}$); CR, CS (so that $\widehat{BCR} = \widehat{RCS} = \widehat{SCA}$). The lines through B, C and closest to BC intersect in R; the ones from C, A and closest to CA intersect in S, and the ones from A, B and closest to AB intersect in T. The three points R, S, T obtained this way are the vertices of an equilateral triangle.

(Extending BT, CS to their intersection I, we form $30°$ angles, one on either side of RI. We extend their sides to intersect BT, CS at T', S' respectively. Triangle $RT'S'$ is equilateral: it suffices to show that lines AT', AS' divide $\widehat{BAC}$ into three equal parts. To show this, denote by B', C' the points symmetric to R in lines BT, CS, respectively. Then show that $C'T'S'B'$ is a regular broken line (**160**), and that the circumscribed circle passes through A.)

Appendix: Malfatti's Problem

The principles established in the text (**227–236**) and in Note C, concerning tangent circles and circles *isogonal* to two given circles (that is, intersecting them at equal angles) allow us to present a solution to a famous problem, due to Malfatti, and stated as follows:

Given three lines (a_1), (a_2), (a_3) *in a plane,*[58] *find three circles* (x_1), (x_2), (x_3) *each pair of which are tangent and, in addition,* (x_1) *is tangent to* (a_2, a_3), (x_2) *is tangent to* (a_3, a_1), *and* (x_3) *is tangent to* (a_1, a_2).[59]

The solution was first given by Steiner, and we will follow an exposition by Schröter. It involves the property established in **228**.

We have seen there that the circles Σ which are tangent to two given circles C, C' or, more generally, which are isogonal to C, C', are divided into two series which can be characterized (in the case where the inversions which change C into C' have positive power) by saying that the circles Σ of the first series are orthogonal to a certain circle Γ, and those in the second series are orthogonal to an analogous circle Γ_1.

The circles Γ, Γ_1 are, as we said, analogous to the bisectors of the angles formed by two lines, and we will call them the *bisecting circles* of C and C'.

One of these bisectors reduces to a line when C and C' are equal (**225**), and only in this case.

When the circles C, C' are tangent, one series of isogonal circles is obviously formed by the circles passing through their point of contact. The corresponding bisecting circle is reduced to that point. In this case, one cannot define an inversion which transforms C into C', since the power of this inversion would have to be zero. In what follows, when we talk about circles isogonal to two tangent circles C, C', we do not refer to those just mentioned, but to the *proper* series, whose circles do not pass through the same point, and are preserved by an inversion (whose power is not zero).

If an inversion transforms two circles C, C' into C_1, C'_1, it also transforms the bisecting circles of C, C' into those of C_1, C'_1. Indeed, the definition of the bisecting circles is based solely on the notion of the angle of two circles, and this is not changed by inversion.

[58] We state the problem as if the three lines form a triangle with vertices A_1, A_2, A_3, which is the general case. The solution does not change if, for example, two of the lines are parallel.

[59] We suppose that the nine points of contact (those of the required circles with the given lines, and those of each pair of circles) are distinct. In other words, we do not consider the situation when, for instance, (a_1) is tangent at the point of contact of (x_2) and (x_3) (a situation which could, however, be treated with the same methods).

* * *

Let us now consider circles (ξ_1), (ξ_2), (ξ_3) with centers ξ_1, ξ_2, ξ_3 and pairwise tangent (externally, to be definite) at points π_1, π_2, π_3 (see *fig.* 232). Join $\pi_2\pi_3$, and let μ_1, ν_1 be the new points where this line intersects (ξ_2), (ξ_3), respectively.

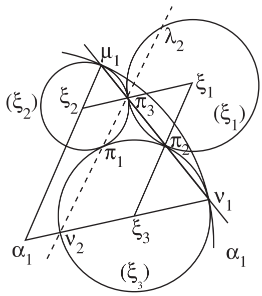

FIGURE 232

Through the points μ_1, ν_1 we can draw a new circle (α_1), which is tangent to (ξ_2), (ξ_3) at these points, and whose center α_1 is at the intersections of the radii $\mu_1\xi_2$, $\nu_1\xi_3$. *The radius of this circle is the sum of the radii of the given circles.*

To see this, observe that $\mu_1\xi_2$ is parallel to $\xi_1\xi_3$ (because π_3 is a center of similarity for (ξ_1), (ξ_2)) and $\nu_1\xi_3$ is parallel to $\xi_1\xi_2$ (for an analogous reason). In parallelogram $\alpha_1\xi_2\xi_1\xi_3$, the equality of the opposite sides yields the indicated value for both $\alpha_1\mu_1$ and $\alpha_1\nu_1$.

Thus, if we also join points π_1, π_3 with a line which intersects (ξ_1) again at λ_2 and (ξ_3) at ν_2, *the circle* (α_2) *which is tangent at these points to* (ξ_1), (ξ_2), *respectively, is equal to* (α_1), since the preceding argument can be applied to it as well.

It is in fact be easy to see that the equality of the circles (α_1), (α_2) would still hold if the contacts of (ξ_1), (ξ_2), (ξ_3) were not all exterior.[60]

We deduce immediately the following lemma, on which our subsequent reasoning will be based.

LEMMA. *If three circles* (x_1), (x_2), (x_3) *are pairwise tangent*[61] *at* P_1, P_2, P_3, *let* (a_1) *be a circle tangent to* (x_2), (x_3) *at* m_1, n_1, *and let* (a_2) *be a circle tangent to* (x_1), (x_3) *at* ℓ_2, n_2, *so that points* m_1, n_1, P_2, P_3 *are* (**227**, **224**)[62] *on a circle* (k_1), *and the points* ℓ_2, n_2, P_1, P_3 *are on a circle* (k_2).

Then the common point N_3 *(other than* P_3*) of* (k_1), (k_2) *is on the bisecting*[63] *circle* (g_3) *of* $(a_1), (a_2)$.

[60] In this case, two of the circles (ξ) would have to be tangent externally to each other, and internally to the third. Each circle (α) would have a radius equal to the difference between the radius of the large circle and the sum of the radii of the small ones.

[61] Each point P is the point of contact of the circles (x) with indices different from its own.

[62] In order to apply the proposition of **224**, we should generally make sure that (a_1), (x_1) belong to the same series of tangent circles to (x_2), (x_3), but in our situation there is only one proper series.

[63] Circle (x_3) belongs to one of two series of circles tangent to (a_1), (a_2); we are referring here to the bisecting circle corresponding to this series of circles.

Indeed, transform the figure by inversion, taking N_3 as the pole. Circles (x_1), (x_2), (x_3) will give us new circles (ξ_1), (ξ_2), (ξ_3); (a_1), (a_2) will give circles (α_1), (α_2), while (k_1), (k_2) become two lines. We are led to the figure considered above, and we have just established that (α_1), (α_2) are equal. Their bisecting circle will thus be a line, so that the bisector of (a_1), (a_2) will pass through N_3. QED

Moreover, *the circle through n_1, N_3, n_2 is orthogonal to (x_3), (a_1), (a_2), and (g_3).*

Circle (k_1) intersects (a_1), (g_3) at equal angles, and likewise, (k_2) intersects (a_2), (g_3) at equal angles.

The angle made by circle (k_1) with (g_3) at N_3 is of the opposite sense from its angle with (a_1) at (n_1).

All of this is easily seen on a figure transformed by inversion with pole N_3, on which figure the points ν_1, ν_2 will be diametrically opposite on the circle (ξ_3).

* * *

Having established these points, we can now attack the proposed problem.

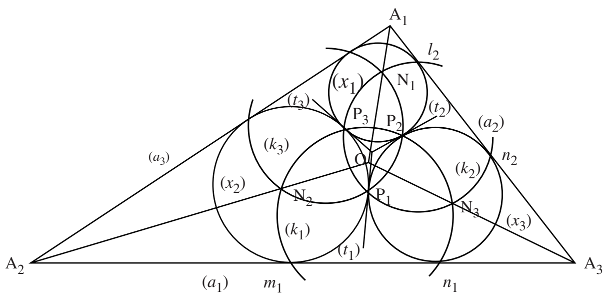

FIGURE 233

I. As above, denote by P_1, P_2, P_3 the points of contact of the required circles (x_1), (x_2), (x_3) (Fig. 233). We can apply the lemma just proved to these circles, with circles (a_1), (a_2) replaced by the given lines. We then know

1°. that there is a circle (k_1) passing through points P_2, P_3 and the points of contact m_1, n_1 of (x_2), (x_3) with (a_1); a circle (k_2) through P_3, P_1 and the points of contact of (x_3), (x_1) with (a_2); and a circle (k_3) through P_1, P_2 and the points of contact of (x_1), (x_2) with (a_3).

2°. that, for instance, (k_1), (k_2) intersect at a point N_3 (other than P_3) on the bisector (g_3) of the angle between (a_1) and (a_2), which here replaces the bisecting circle. (If, as we are now assuming, the circles (x) are all inside the triangle, we are dealing with the bisector of the interior angle.) Similarly, circles (k_2), (k_3) intersect at a point N_1 on the bisector (g_1) of the angle between (a_2) and (a_3), and circles (k_3), (k_1) intersect at a point N_2 on the bisector (g_2) of the angle between (a_3) and (a_1).

3°. that (k_1) intersects (a_1), (g_3) at equal angles.

We can observe immediately that, for an analogous reason, the angle made by circle (k_1) with (g_2) is equal to the angles made by the same circle with (a_1) or with (g_3).

It is also equal to the angle made by (k_1) with circle (x_2) at point m_1 (where this circle is tangent to (a_1)) or at the point P_3 (which is the same, but with the opposite orientation).[64]

II. Denote by (t_1), (t_2), (t_3) the common tangents to the required circles at the points of contact P_1, P_2, P_3, respectively. We see that line (t_3) intersects (k_1) at the same angle as (g_3) since this angle, measured at P_3, has the opposite sense from the analogous angle of (k_1) and (g_3) at N_3, as can easily be seen from what we have said so far.

Point P_3 is obviously symmetric to N_3 with respect to the line of centers of (k_1), (k_2). Thus (t_3) is symmetric to (g_3) with respect to the same line (since it forms the same angle with (k_1), but in the opposite sense).

For the same reason, (t_2) intersects (k_1) at the same angle, as it is symmetric to (g_2) with respect to the line of centers of (k_3), (k_1).

It is clear, however, that all the lines intersecting a given circle at a given angle are tangent to a second circle, concentric with the first. Therefore, *the five lines (a_1), (g_2), (g_3), (t_2), (t_3) are tangent to a circle*[65] *(k_1') concentric with (k_1).*

This last result is equivalent to a solution of the problem. Circle (k_1') can be considered to be known, since we know three of the five tangents (namely, (a_1), (g_2), (g_3)). Similar considerations apply to the analogous circles (k_2'), (k_3') which are concentric to (k_2), (k_3), and we arrive at the following construction:

Let O be the center of a circle tangent to the three given lines, and let (g_1), (g_2), (g_3) be the lines joining the vertices of the triangle formed by these lines to O. We draw a circle (k_1') tangent to the lines (a_1), (g_2), (g_3), a circle[66] *(k_2') tangent to the lines (a_2), (g_3), (g_1), and a circle (k_3') tangent to the lines (a_3), (g_1), (g_2).*

Let (t_1) be the second common tangent of (k_2'), (k_3'), symmetric to (g_1) with respect to the line of the centers of these circles. Let (t_2) be the common tangent to (k_3'), (k_1') symmetric to (g_2) with respect to the line of their centers. Let (t_3) be the common tangent to (k_1'), (k_2') symmetric to (g_3) with respect to the line of their centers.

The first circle (x_1) is tangent[67] *to lines (a_2), (a_3), (t_2), (t_3); likewise, (x_2) is tangent to (a_3), (a_1), (t_3), (t_1), and (x_3) is tangent to (a_1), (a_2), (t_1), (t_2).*

---

[64]To determine the orientation of the angles under consideration, we rely on the fact that two circles (or a circle and a line) which have two common points form, at these two points, equal angles of opposite orientation, which is obvious, as these angles are symmetric relative to a line.

[65]In order not to make the figure too complicated, we have not drawn in circle (k_1') or the analogous circles (k_2'), (k_3').

[66]There exist four circles tangent to (a_1), (g_2), (g_3). If one wants the circles (x) to be inside the given triangle, one must obviously choose the circle inscribed in the triangle formed by these lines. If not, any of these four circles can be chosen, but the other two analogous circles are then determined: indeed, we see (by performing another inversion relative to the pole N_3) that the chords N_3n_1, N_3n_2 must be symmetric with respect to (g_3), and the same must be true of the bisectors of the angle between (a_1) and (g_3) and the angle between (a_2) and (g_3), which pass through the centers of (k_1) and (k_2) respectively.

[67]Its center is on line (g_1) (and not on the bisector perpendicular to this line); one can show, by assigning directions to the lines (g), (t), that it is also on the bisector of the angle formed by (t_2), (t_3) which passes through the center of (k_1).

Lines (g_1), (g_2), (g_3) can be the bisectors of any of the angles formed by the given lines, provided that these bisectors intersect.[68]

It remains to prove that the construction indicated above does effectively produce circles solving the problem. We will not establish this fact, which was treated by Petersen.[69]

* * *

It is remarkable that the preceding solution extends to the case in which (a_1), (a_2), (a_3) are arbitrary circles, instead of being lines.[70]

In this case (g_1), (g_2), (g_3) will be the bisecting circles of $(a_2), (a_3)$; $(a_3), (a_1)$; $(a_1), (a_2)$, respectively.[71] Using this fact, all the reasoning involving only the circles (a_1), (a_2), (a_3), (g_1), (g_2), (g_3), (k_1), (k_2), (k_3) (whose definitions are not changed), that is, all the reasoning done until Part II, remain true without modification.[72]

In order to apply the results of this reasoning to actual situations, we must state what is meant by (t_1), (t_2), (t_3).

To do this, let (C) be the circle orthogonal to (a_1), (a_2), (a_3) (assuming, to be definite, that it exists).[73] We denote by (t_1) the circle orthogonal to (C), and tangent to (x_2) and (x_3) at their point of contact P_3.

Similarly, let (t_2), (t_3) be two circles orthogonal to (C), and each tangent to two of the given circles at their point of tangency.

---

[68]This follows from the fact that the points of contact of the required circles must be external, and the lines (a) must be common external tangents. (See an analogous remark later in this note for the case when (a_1), (a_2), (a_3) are replaced by circles.)

[69]Crelle's Journal, vol. 89, pp. 130–135. Clearly one must establish first that the lines (a_2), (a_3), (t_2), (t_3) are tangent to the same circle. The reader who has solved Exercise 422 will convince himself that this is indeed the case; that this is so even when O is any point, not necessarily the center of the inscribed circle; and that the lines (t_1), (t_2), (t_3) are concurrent.

On the other hand, the fact that these circles are tangent depends on the particular position of the point O (see Petersen's work). We must remark (limiting ourselves to the case in which the circles (x) are inside the triangle) that the existence of a solution is obvious by continuity. Indeed, draw an arbitrary circle (x_1) tangent to (a_2) , (a_3) and inside the triangle: there exists a circle (x_2) inside the triangle and tangent to (a_1), (a_3), (x_1), and a circle (x_3) tangent to (a_1), (a_2), (x_1).

These circles (x_2), (x_3) clearly intersect in two points when (x_1) is very small, and do not intersect at all when (x_1) is the inscribed circle in the triangle. Thus, for some value of the radius of (x_1), the circles (x_2), (x_3) will be tangent.

[70]The new problem would immediately reduce to the first if circles (a_1), (a_2), (a_3) had a common point S (an inversion with pole S would show this). But the solution must be considered differently if this is not the case.

[71]Among the six possible circles bisecting (a_1), (a_2), (a_3) in pairs, we must choose three for (g_1), (g_2), (g_3) which have (Exercise 276) the same radical axis. Assume, to be definite, that the required circles are tangent externally. Then circle (a_1) will have contacts of the same kind with (x_2), (x_3) as (x_3) itself , since there is only one proper series of common tangent circles to these two. Likewise, (a_2) has contacts of the same kind with (x_3), (x_1), and (a_3) with (x_1), (x_2). By considering the various kinds of contacts possible, we will be able to verify that the circles (g) satisfy the relation indicated.

The case where two of the circles are tangent internally can be reduced to the preceding one by inversion (and this discussion can thus be simplified).

[72]It follows that the bisecting circles in question exist whenever there is a solution to the problem. For instance, (g_1) must have its center at one of the centers of similarity of (a_2), (a_3), and must pass through N_3.

[73]This circle is reduced to one point in the case of footnote IV [i.e. when the three given circles have a common point. –transl.].

The reasoning developed above shows that circle (k_1) forms equal angles[74] with (x_2), (x_3), and with (a_1), (g_2), (g_3).

The last three are orthogonal to (C).

It follows (see Note C, especially **310**) that any circle (k_1'), having the same radical axis[75] as (k_1), (C), is also isogonal to (a_1), (g_2), (g_3), and that if it is tangent to one of them (**311**, construction) it will also be tangent to the others. Circle (t_3) is in fact the image[76] of (g_3) under the inversion which does not change (C), (k_1), or (k_2). The same is true if we replace (k_1), (k_2) by (k_1'), (k_2').

Thus the construction is as follows:

Let (a_1), (a_2), (a_3) be given circles, and let (g_1), (g_2), (g_3) be bisecting circles of $(a_2), (a_3)$; $(a_3), (a_1)$; $(a_1), (a_2)$ respectively, these circles being chosen so as to have the same radical axis. Let (C) be the circle orthogonal to the given circles

Determine circle (k_1') tangent to (a_1), (g_2), (g_3), circle (k_2') tangent to (a_2), (g_3), (g_1), and circle (k_3') tangent to (a_3), (g_1), (g_2)[77]; or, more generally[78], determine circle (k_1'') isogonal to (a_1), (g_2), (g_3), circle (k_2'') isogonal to (a_2), (g_3), (g_1), and circle (k_3'') isogonal to (a_3), (g_1), (g_2).

Let (t_1) be the image of (g_1) under the inversion which does not change (C), (k_2''), or (k_3''); let (t_2) be the image of (g_2) under the inversion which does not change (C), (k_3''), or (k_1''); let (t_3) be the image of (g_3) under the inversion which does not change (C), (k_1''), or (k_2'').

Of the required circles, the first is tangent to (a_2), (a_3), (t_2), (t_3); the second is tangent to (a_3), (a_1), (t_3), (t_1); and the third is tangent to (a_1), (a_2), (t_1), (t_2).

It is easy to verify that what we have said remains true when the circle (C) does not exist, when the radical center I of (a_1), (a_2), (a_3) is interior to these circles. We just replace the words "circle orthogonal to (C)" by "circle relative to which the power of I is the same as its power relative to (a_1), (a_2), (a_3)", and we say that a circle (k_1') has 'the same radical axis as (k_1), (C)' if every circle orthogonal to (k_1), such that the power of I is the same as that with respect to (a_1), is also orthogonal to (k_1').

[74]With the same remark as before concerning the orientation of these equal angles.

[75]It is clear that, unlike the first case [where the (a) are lines – transl.], the circle (k_1') is not generally concentric with (k_1).

[76]This follows as before from the fact that this image satisfies all the conditions defining (t_3), including its intersection with (g_1) and the size of the angle at that point, as well as concerning its orthogonality to C. The inversion involved here has as its pole the radical center of the circles under consideration.

[77]As before, we can take (k_1') to be any of the circles tangent to (a_1), (g_2), (g_3). Once this choice is made, it determines the series of circles isogonal to (a_2), (g_3), (g_1) to which (k_2'), (k_2'') belong.

[78]It is important to be aware of this generalization. It can indeed happen that the circle (k_1') does not exist (when the construction of **311** is impossible), even though our problem does have solutions.

This is a book in the tradition of Euclidean synthetic geometry written by one of the twentieth century's great mathematicians. The original audience was pre-college teachers, but it is useful as well to gifted high school students and college students, in particular, to mathematics majors interested in geometry from a more advanced standpoint.

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