# Topological Quantum Dark Matter via Global Anomaly Cancellation

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Standard Model (SM) with 15 Weyl fermions per family (lacking the 16th, the sterile right-handed neutrino  $\nu_B$ ) suffers from mixed gauge-gravitational anomalies tied to baryon number plus or minus lepton number  $\mathbf{B} \pm \mathbf{L}$  symmetry. Including  $\nu_B$  per family can cancel these anomalies, but when  $\mathbf{B} \pm \mathbf{L}$ symmetry is preserved as discrete finite subgroups rather than a continuous U(1), the perturbative local anomalies become nonperturbative global anomalies. In this work, we systematically enumerate these gauge-gravitational global anomalies involving discrete  $\mathbf{B} \pm \mathbf{L}$  that are enhanced from the fermion parity  $\mathbb{Z}_2^F$  to  $\mathbb{Z}_{2N}^F$ , with N=2,3,4,6,9, etc. The discreteness of  $\mathbf{B}\pm\mathbf{L}$  is constrained by multi-fermion deformations beyond-the-SM and the family number  $N_f$ . Unlike the free quadratic  $\nu_R$  Majorana mass gap preserving the minimal  $\mathbb{Z}_2^{\mathrm{F}}$ , we explore novel scenarios canceling ( $\mathbf{B} \pm \mathbf{L}$ )-gravitational anomalies while preserving the  $\mathbb{Z}_{2N}^{\mathrm{F}}$  discrete symmetries, featuring 4-dimensional interacting gapped topological orders (potentially with or without low-energy topological quantum field theory descriptions) or gapless sectors (e.g., conformal field theories). We propose symmetric anomalous sectors as quantum dark matter to cancel SM's global anomalies. We find the uniqueness of the family number at  $N_f = 3$ , such that when the representation of  $\mathbb{Z}_{2N}^{\mathrm{F}}$  from the faithful  $\mathbf{B} + \mathbf{L}$ for baryons at  $N=N_c=3$  is extended to the faithful  $\mathbf{Q}+N_c\mathbf{L}$  for quarks at  $N=N_cN_f=9$ , this symmetry extension  $\mathbb{Z}_{N_c=3} \to \mathbb{Z}_{N_cN_f=9} \to \mathbb{Z}_{N_f=3}$  matches with the topological order dark matter construction. Key implications include: (1) a 5th force mediating between SM and dark matter via discrete  $\mathbf{B} \pm \mathbf{L}$  gauge fields, (2) dark matter as topological order quantum matter with gapped anyon excitations at ends of extended defects, and (3) Ultra Unification and topological leptogenesis.

#### CONTENTS

I. Introduction and Summary	2
II. Discrete $\mathbf{B} + \mathbf{L}$ global anomaly and dark matter sector	5
III. Discrete $\mathbf{B} - \mathbf{L}$ global anomaly and dark matter sector	6
Acknowledgements	6
A. Anomaly of 3+1d Weyl fermion with discrete charge $q \in \mathbb{Z}_n$ 1. Anomaly Polynomial  2. Spin × U(1) symmetry  3. Spin × <sub><math>\mathbb{Z}_2^F</math></sub> U(1) <sup>F</sup> = Spin <sup>c</sup> symmetry  4. Spin × <sub><math>\mathbb{Z}_2^F</math></sub> $\mathbb{Z}_4^F$ symmetry  5. Spin × <sub><math>\mathbb{Z}_2^F</math></sub> $\mathbb{Z}_8^F$ symmetry  6. Spin × $\mathbb{Z}_3$ symmetry  7. Spin × <sub><math>\mathbb{Z}_2^F</math></sub> $\mathbb{Z}_6^F$ symmetry  8. Spin × <sub><math>\mathbb{Z}_2^F</math></sub> $\mathbb{Z}_{12}^F$ symmetry  9. Spin × $\mathbb{Z}_9$ symmetry  10. Spin × <sub><math>\mathbb{Z}_2^F</math></sub> $\mathbb{Z}_{18}^F$ symmetry	6 6 7 7 8 9 11 12 13 14
B. $\mathbb{Z}_9$ class topological invariants of Spin $\times \mathbb{Z}_3$ and the group extension $1 \to \mathbb{Z}_3 \to \mathbb{Z}_9 \to \mathbb{Z}_3 \to 1$	15
References	17

#### I. INTRODUCTION AND SUMMARY

Standard Model (SM), with 15 Weyl fermions per family and without the 16th Weyl fermion sterile right-handed neutrino  $\nu_R$ , suffers from the perturbative local mixed-gauge-gravitational anomalies [1–3] between the lepton number  ${\bf L}$  symmetry and gravitational background fields, in 3+1d or simply 4d spacetime. Namely these anomalies are computable via perturbative triangle Feynman diagrams  ${\bf U}(1)^3_{\bf L}$  and  ${\bf U}(1)_{\bf L}$ -gravity<sup>2</sup>, with the anomaly index coefficient  $-N_f+n_{\nu_R}$ , counting the difference between the family or generation number  $N_f$  (typically  $N_f=3$ ) and the total right-hand neutrino number  $n_{\nu_R}$ . See recent related expositions about this anomaly index  $-N_f+n_{\nu_R}$  for examples in [4–10]. However, because of the analogous Adler-Bell-Jackiw anomalies [11, 12] via the SM electroweak gauge instanton [13–16], instead of thinking of the classical lepton number  ${\bf L}$  symmetry, only the baryon number plus or minus lepton number  ${\bf B} \pm {\bf L}$  symmetries are physically meaningful quantum mechanical symmetries pertinent in the SM:<sup>1</sup>

- 1. For the  $\mathbf{B} \mathbf{L}$  symmetry, there is a full faithful  $\mathrm{U}(1)_{\mathbf{B}-\mathbf{L}}$  symmetry for the gauge-invariant baryons, or an unfaithful  $\mathrm{U}(1)_{\mathbf{Q}-N_c\mathbf{L}}$  symmetry for the gauge-invariant baryons (but  $\mathrm{U}(1)_{\mathbf{Q}-N_c\mathbf{L}}$  is faithful for the free quarks), survived under the SM electroweak gauge instanton, see Table I.
- 2. For the  $\mathbf{B} + \mathbf{L}$  symmetry, there is a full faithful  $\mathbb{Z}_{2N_f,\mathbf{B}+\mathbf{L}}$  symmetry for the gauge-invariant baryons or an unfaithful  $\mathbb{Z}_{2N_cN_f,\mathbf{Q}+N_c\mathbf{L}}$  symmetry for the gauge-invariant baryons (but  $\mathbb{Z}_{2N_cN_f,\mathbf{Q}+N_c\mathbf{L}}$  is faithful for the free quarks), survived under the SM electroweak gauge instanton, see Table I.

	$\bar{d}_R$	$l_L$	$q_L$	$\bar{u}_R$	$\bar{e}_R = e_L^+$	$\bar{\nu}_R = \nu_L$	$\phi_H$
SU(3)	3	1	3	3	1	1	1
SU(2)	1	2	2	1	1	1	2
$U(1)_Y$	1/3	-1/2	1/6	-2/3	1	0	1/2
$\mathrm{U}(1)_{ ilde{Y}}$	2	-3	1	-4	6	0	3
$U(1)_{\rm EM}$	1/3	0  or  -1	2/3  or  -1/3	-2/3	1	0	0
$U(1)_{\mathbf{B}-\mathbf{L}} = U(1)^{\mathbf{F}}$	-1/3	-1	1/3	-1/3	1	1	0
$U(1)_{\mathbf{Q}-N_c\mathbf{L}} = U(1)^{\mathrm{F}}$	-1	-3	1	-1	3	3	0
$\mathrm{U}(1)_X = \mathrm{U}(1)^\mathrm{F}$	-3	-3	1	1	1	5	-2
$\mathbb{Z}_{5,X}$	2	2	1	1	1	0	-2
$\mathbb{Z}_{4,X} = \mathbb{Z}_4^{\mathrm{F}}$	1	1	1	1	1	1	-2
$\mathbb{Z}_{8,X} = \mathbb{Z}_8^{\mathrm{F}}$	5	5	1	1	1	5	-2
$\mathbb{Z}_{2N_f=6,\mathbf{B}+\mathbf{L}} = \mathbb{Z}_6^{\mathrm{F}}$ for $N_f = 3$ ; or $\mathbb{Z}_2^{\mathrm{F}}, \mathbb{Z}_4^{\mathrm{F}}$ for $N_f = 1, 2$ (broken from $\mathrm{U}(1)_{\mathbf{B}+\mathbf{L}}$ ).	-1/3	1	1/3	-1/3	-1	-1	0
$ \begin{split} \mathbb{Z}_{2N_cN_f=18,\mathbf{Q}+N_c\mathbf{L}} &= \mathbb{Z}_{18}^{\mathrm{F}} \\ & \text{for } N_f = 3; \\ & \text{or } \mathbb{Z}_{6}^{\mathrm{F}}, \mathbb{Z}_{12}^{\mathrm{F}} \\ & \text{for } N_f = 1, 2 \\ & \text{(broken from U(1)}_{\mathbf{Q}+N_c\mathbf{L}}). \end{split} $	-1	3	1	-1	-3	-3	0
$\mathbb{Z}_2^{ ext{F}}$	1	1	1	1	1	1	0

TABLE I. Follow the convention in [9], we show the representations of quarks and leptons in terms of Weyl fermions in various internal symmetry groups. Each fermion is a spin- $\frac{1}{2}$  Weyl spinor  $\mathbf{2}_L$  representation of the spacetime symmetry group Spin(1,3). Each fermion is written as a left-handed particle  $\psi_L$  or a right-handed anti-particle i  $\sigma_2 \psi_R^*$ .

Although including the 16th Weyl fermion can cancel the SM's anomalies, when the  $\mathbf{B} \pm \mathbf{L}$  are preserved only as discrete finite subgroups instead of the conventional continuous U(1), the perturbative local anomalies become

We thank Yunqin Zheng for enlightening discussions on this issue.

<sup>&</sup>lt;sup>1</sup> In fact, the U(1)<sub>B-L</sub> is a faithful symmetry for the SM such that there exist gauge-invariant local operators that have a unit U(1) charge. The U(1)<sub>Q-N<sub>c</sub>L</sub> is not a faithful symmetry for the SM because there exists no gauge-invariant local operator that has a unit U(1) charge; but the gauge-invariant local operator has a minimal  $N_c = 3$  charge. But U(1)<sub>Q-N<sub>c</sub>L</sub> is a faithful symmetry for free quarks, because there exists a free quark of that unit U(1) charge.

Similarly, the  $\mathbb{Z}_{2N_f,\mathbf{B}+\mathbf{L}}$  is a faithful symmetry for the SM such that there exist gauge-invariant local operators that have a unit U(1) charge. The  $\mathbb{Z}_{2N_cN_f,\mathbf{Q}+N_c\mathbf{L}}$  is not a faithful symmetry for the SM because there exists no gauge-invariant local operator that has a unit  $\mathbb{Z}_{2N_cN_f,\mathbf{Q}+N_c\mathbf{L}}$  charge; but the gauge-invariant local operator has a minimal  $N_c=3$  charge. But  $\mathbb{Z}_{2N_cN_f,\mathbf{Q}+N_c\mathbf{L}}$  is a faithful symmetry for free quarks, because there exists a free quark of that unit  $\mathbb{Z}_{2N_cN_f,\mathbf{Q}+N_c\mathbf{L}}$  charge.

nonperturbative global anomalies,<sup>2</sup> there are alternative new scenarios that would cancel the ( $\mathbf{B} \pm \mathbf{L}$ )-gravitational anomaly but in such a novel way as to preserve the discrete ( $\mathbf{B} \pm \mathbf{L}$ )-symmetry, see Fig. 1.

In this work, we systematically enumerate these gauge-gravitational global anomalies involving discrete  $\mathbf{B} \pm \mathbf{L}$  that are enhanced from the fermion parity  $\mathbb{Z}_2^{\mathrm{F}}$  to  $\mathbb{Z}_{2N}^{\mathrm{F}}$ , with N=2,3,4,6,9, etc., see Appendix A. <sup>3</sup>

1. Phenomenologically, the discreteness of  $\mathbf{B} - \mathbf{L}$  is either imposed as a discretely dynamically gauged at the high-energy [27] or constrained by the allowed higher-order 2N-body multi-fermion beyond-the-Standard-Model (BSM) deformations. More precisely, we really require the X symmetry [28–30]

$$X \equiv 5(\mathbf{B} - \mathbf{L}) - \frac{2}{3}\tilde{Y} \equiv \frac{5}{N_C}(\mathbf{Q} - N_C \mathbf{L}) - \frac{2}{3}\tilde{Y}$$
(1)

with the properly integer quantized hypercharge  $\tilde{Y}$ . See Table I.

• The  $\mathbb{Z}_{4,X}$  has the advantage that all SM fermions have  $\mathbb{Z}_{4,X}$  charge 1, thus we can consider the 4-body BSM multi-fermion  $\mathbb{Z}_{4,X}$ -preserving deformations

$$\psi_q \psi_q \psi_l, \quad \psi_{\bar{q}} \psi_q \psi_{\bar{l}} \psi_l, \quad \psi_{\bar{q}} \psi_q \psi_{\bar{q}} \psi_q, \quad \psi_{\bar{l}} \psi_l \psi_{\bar{l}} \psi_l. \tag{2}$$

because their  $\mathbb{Z}_{4,X}$  charge is  $4 = 0 \mod 4$ .

• The  $\mathbb{Z}_{8,X}$  is less uniform so SM quarks and leptons carry different  $\mathbb{Z}_{8,X}$  charges. In addition to 4-body multi-fermion, one can consider 8-body multi-fermion deformations

$$(\psi_{\bar{q}}\psi_q\psi_{\bar{q}}\psi_q)(\psi_{\bar{q}}\psi_q\psi_{\bar{q}}\psi_q), \quad (\psi_{\bar{l}}\psi_l\psi_{\bar{l}}\psi_l)(\psi_{\bar{l}}\psi_l\psi_{\bar{l}}\psi_l), \quad (\psi_{\bar{q}}\psi_q\psi_{\bar{q}}\psi_q)(\psi_{\bar{l}}\psi_l\psi_{\bar{l}}\psi_l), \quad \dots$$

$$(3)$$

so that their  $\mathbb{Z}_{8,X}$  charge is  $0 \mod 8$ .

Crucially, without any of those BSM deformations at the electroweak SM energy scale, the  $\mathbf{B} - \mathbf{L}$  or X is preserved as a full U(1) symmetry.

2. The discreteness of  $\mathbf{B} + \mathbf{L}$  is constrained by the family number  $N_f$ , as faithful  $\mathbb{Z}_{2N_f,\mathbf{B}+\mathbf{L}}$  or an unfaithful  $\mathbb{Z}_{2N_cN_f,\mathbf{Q}+N_c\mathbf{L}}$  symmetry for the SM, for  $N_c=3$ . We will vary the family or generation number  $N_f=1,2,3,4,$  etc. to explore the uniqueness of  $N_f=3$ .

Unlike the free quadratic Majorana mass gap of  $\nu_R$  preserving only the minimal fermion parity  $\mathbb{Z}_2^F$  (see Fig. 1 (a) and (b)), the exotic  $\mathbb{Z}_{2N}^F$ -preserving new scenarios can contain highly interacting symmetric anomalous gapped topological orders (which inspiration originates from quantum condensed matter phenomena [31, 32]) or symmetric anomalous gapless sectors such as conformal field theories, see Fig. 1 (c). These anomalous topological orders may or may not have topological quantum field theory (TQFT) descriptions at low energy [33, 34]. The enumerations of possible anomaly cancellation scenarios are summarized in Fig. 2.

We propose introducing these symmetric anomalous sectors as quantum dark matter to cancel the SM's discrete gauge-gravitational global anomaly involving the discrete  $\mathbf{B} \pm \mathbf{L}$ . We identify some of the mathematical constraints of the topologically ordered quantum dark matter. The implications beyond the SM include:

- 1. The existence of the 5th force as the topological gauge force of discrete  $\mathbf{B} \pm \mathbf{L}$  gauge fields mediating between the SM and the dark matter.
- 2. The dark matter can partly contain topological quantum matter, such that the topologically ordered gapped excitations at the open ends of extended line and surface defects can have anyon statistics,
- 3. Ultra Unification [4–6] and topological leptogenesis [10].

<sup>&</sup>lt;sup>2</sup> See the recent modern systematic studies of nonperturbative global anomalies in the context of the Standard Model in Refs. [17–20]. For our terminology, we mean that:

<sup>•</sup> Perturbative local anomalies (e.g. [2, 3]) are detected by small (i.e. local) continuous symmetry transformation connected to the identity transformation, hence it is sometimes called the continuous anomaly.

<sup>•</sup> Non-perturbative global anomalies (e.g. [21–23]) are detected by large (i.e. global) discrete symmetry transformation disconnected from the identity transformation, hence it is sometimes called the discrete anomaly.

<sup>•</sup> Sometimes the discrete anomaly is also used differently to describe the anomaly associated with discrete symmetries. See pioneer work in [24–26]. In our case, we do have both the discrete global anomaly in both senses of (1) nonperturbative global anomalies, and (2) discrete  $\mathbf{B} \pm \mathbf{L}$  symmetries.

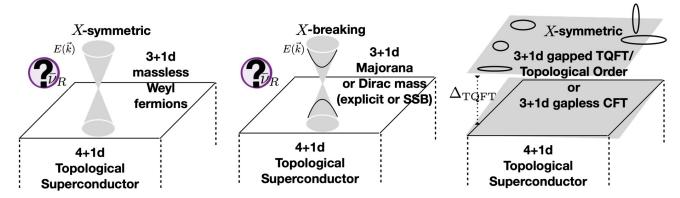
<sup>• &#</sup>x27;t Hooft anomalies are referred to the anomaly of the global symmetry (that cannot be consistently dynamically gauged).

<sup>•</sup> Adler-Bell-Jackiw anomalies are the violations of the global symmetry in the presence of some (other) dynamically gauge fields.

<sup>•</sup> Dynamical gauge anomalies are meant to be canceled with dynamical gauge fields, for the consistency under the name of anomaly cancellation

<sup>•</sup> Gauge or gravitational anomalies are anomalies associated with gauge or gravitational fields, that can be either background fields or dynamical fields. Bosonic or fermionic anomalies are anomalies associated with the anomalous boundary of one-higher dimensional bulk bosonic or fermionic Symmetry-Protected Topological states (SPTs).

The  $\mathbb{Z}_2^F$  is generated by  $(-1)^F$  with the fermion number F, so that  $((-1)^F)^2 = +1$ . The  $\mathbb{Z}_{2N}^F \supset \mathbb{Z}_2^F$  contains the fermion parity as a normal subgroup such that the quotient group  $\mathbb{Z}_{2N}^F/\mathbb{Z}_2^F = \mathbb{Z}_N$ . For  $\mathbb{Z}_{2N,X}$  as  $\mathbb{Z}_{2N}^F$ , we have the generator  $X^{2N} = +1$  as the cyclic group of order 2N.



 $k\in\mathbb{Z}_{16}$  class discrete Baryon - Lepton (B-L)  $\mathbb{Z}_{4,X}$  symmetric Topological Superconductor  $X^2=(-1)^F$  symmetric Atiyah-Patodi-Singer (APS) eta  $\eta$  invariant

- FIG. 1. Here we give a specific example for  $\mathbb{Z}_{16}$  class global anomaly cancellation for the mixed  $\mathbb{Z}_{4,X}$ -gravity anomalies between the SM and the BSM dark matter sector, known for the model of Ultra Unification [4–8]. Similar generalization for  $\mathbf{B} \pm \mathbf{L}$  symmetry can be analogously obtained, too.
- (a) A single Weyl fermion with a unit charge  $1 \in \mathbb{Z}_{4,X}$  can cancel an anomaly index  $v = 1 \in \mathbb{Z}_{16}$ . A single Weyl fermion can live on the boundary of 5d  $\mathbb{Z}_{16}$  class  $\mathbb{Z}_{4,X}$ -symmetric topological superconductor or SPTs in condensed matter, or known as Atiyah-Patodi-Singer eta invariant or invertible topological field theory (iTFT) cobordism invariant [17, 18, 20]. This can be the 16th Weyl fermion, the sterile right-handed neutrinos (denoted  $\bar{\nu}_R$  so to be left-handed), shown in cartoon.
- (b) A single Weyl fermion is equivalent to a Majorana fermion in 4d, which can obtain a Majorana mass gap, but that Majorana mass term breaks  $\mathbb{Z}_{4,X}$  down to the minimal fermion parity  $\mathbb{Z}_2^F$ .
- (c) The last scenario is inspired by the quantum condensed matter phenomena called the 2+1d symmetric anomalous boundary topological order on the boundary of 3+1d SPTs [31, 32]. Here we consider its one higher-dimensional generalization: the 3+1d symmetric anomalous boundary topological order on the boundary of 4+1d SPTs. There are gapped extended line or surfaces defects in the 3+1d topological order. The open ends of extended defects that carry anyon statistics. The last scenario could also include symmetric anomalous gapless sectors such as conformal field theories.

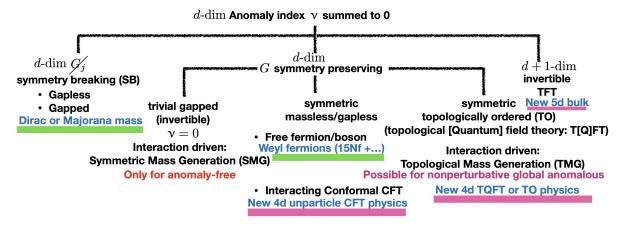


FIG. 2. The anomaly cancellation demands the total summation of the anomaly index  $\nu = 0$ . For perturbative local anomaly classified by an integer  $\mathbb{Z}$  class, the  $\nu = 0$  is strict. But for nonperturbative global anomaly classified by an integer  $\mathbb{Z}_n$  class, only the  $\nu = 0 \mod n$  is required. In addition to the familiar scenario of adding fermions (either gapless or gapped) to cancel the anomaly (marked in green), there are also novel scenarios (marked in pink) include adding interacting symmetric gapped topological order (TO) with or without low-energy topological quantum field theory (TQFT) desciption, or symmetric gapless conformal field theory (CFT), or an extra-dimensional bulk of invertible field theory. When the anomaly index is  $\nu = 0 \mod n$ , one can also use the Symmetric Mass Generation (SMG, e.g. see a review [35]) mechanism to move between different quantum phases.

Below we will mainly use the constraints obtained in [37, 38] or in 3+1d boundary topological order of 4+1d SPTs obtained in [33, 34] to comment about the hypothetical gapped or gapless topological quantum matter phases as dark matter sectors.

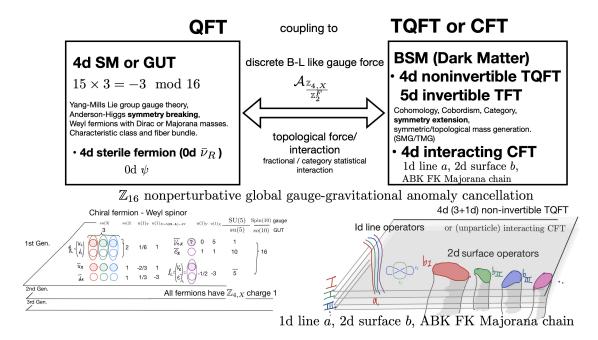


FIG. 3. A schematic picture on the nonperturbative global anomaly cancellation: how the quantum field theory (QFT) of the Standard Model (SM) can be coupled to the topological quantum field theory (TQFT) or conformal field theory (CFT) and others, via discrete gauge fields. Shown here Ref. [4–6] considers discrete X or  $\mathbf{B} - \mathbf{L}$  gauge field. But more generally, for the SM, this works consider discrete  $\mathbf{B} \pm \mathbf{L}$  gauge fields. This is a specific QFT coupling to a TQFT scenario, see more explorations on this topic in [36].

#### II. DISCRETE B + L GLOBAL ANOMALY AND DARK MATTER SECTOR

Based on the calculation done in Appendix A, we find the following results. (Since these anomalies listed below vanish for the 16-Weyl fermion SM, so we will only discuss the implications on the 15-Weyl fermion SM.) For  $\mathbb{Z}_{2N_f,\mathbf{B}+\mathbf{L}}$  (faithful for free quarks), at  $N_c=3$  color,

- 1.  $N_f = 1$ , there is no global anomaly for 15-Weyl fermion SM.
- 2.  $N_f = 2$ , the anomaly index of the  $N_f = 2$ -family SM (without  $2 \bar{\nu}_R$ ) is (2 mod 16). According to [34],
  - The (2 mod 16) corresponds to the 4d anomaly of beyond the group cohomology 5d SPT.
  - For  $\nu = N = 2$  or even  $\in \mathbb{Z}_{16}$ , we have a symmetric gapped  $\mathbb{Z}_4$  gauge theory TQFT.
  - For  $\nu = N/2 = 1$  or odd  $\in \mathbb{Z}_{16}$ , we find a non-TQFT symmetric gapped state via stacking lower-dimensional (2+1)d non-discrete-gauge-theory topological order (that has TQFT descriptions) inhomogeneously.
- 3.  $N_f = 3$ , the anomaly index of the  $N_f = 3$ -family SM (without  $3 \bar{\nu}_R$ ) is  $(3 \mod 9)$ . The  $(3 \mod 9)$  corresponds to the 4d anomaly of the group cohomology 5d SPT, see Appendix B.

For  $\mathbb{Z}_{2N_cN_f,\mathbf{Q}+N_c\mathbf{L}}$  (unfaithful for baryons, but faithful for free quarks or the ungauged SM without gauging the [SU(3)] color), at  $N_c=3$  color,

- 1.  $N_f = 1$ , the anomaly index is 0 mod 9 Thus, there is no needed to trivialize by  $\mathbb{Z}_{N_c=3}$ -extension via the construction in [39].
- 2.  $N_f = 2$ , the anomaly index of the 2-family SM (without  $2 \bar{\nu}_R$ ) is  $(-2 \mod 16, 0 \mod 9)$ , thus anomalous. The  $\mathbb{Z}_{N_c=3}$ -extension via the construction in [39] cannot trivialize this anomaly.
- 3.  $N_f = 3$ , the anomaly index of the  $N_f = 3$ -family SM (without 3  $\bar{\nu}_R$ ) is (0 mod 27, 0 mod 3).

• This shows the uniqueness of  $N_f = 3$  or a multiple of 3 such that the following group extension

$$1 \to \mathbb{Z}_{N_c=3} \to \mathbb{Z}_{N_cN_f=9} \to \mathbb{Z}_{N_f=3} \to 1, \tag{4}$$

can trivialize the previous (3 mod 9) anomaly for  $\mathbb{Z}_{2N_f=6,\mathbf{B}+\mathbf{L}}$ . Thus this also means that such a symmetry-extension can help to construct the symmetric gapped boundary topological order via the construction in [39]. See more discussions on the uniqueness of the  $\mathbb{Z}_3$  symmetry in Appendix B.

- Note that this  $N_f = 3$  family argument is different from the dimensional reduction argument via  $c_- = 24$  modular invariant in 2d CFT or framing anomaly free in 3d TQFT, and pure gravitational anomaly vanishing argument given in [40].
- Note that our  $\mathbb{Z}_3$  symmetry assignment here only relies on the  $\mathbf{B} + \mathbf{L}$  symmetry of the standard SM, thus crucially very different from the minimal-supersymmetric Standard Model (MSSM) previously discussed in [17, 41].
- There are other physical motivations to consider the  $\mathbb{Z}_{2N_cN_f=18,\mathbf{Q}+N_c\mathbf{L}}$  as a global or gauge symmetry in proton stability [8, 42] or in cosmological lithium problem [43].

#### III. DISCRETE B – L GLOBAL ANOMALY AND DARK MATTER SECTOR

We consider  $\mathbb{Z}_{2N,X}$  symmetry for the discrete  $\mathbf{B} - \mathbf{L}$  symmetry,

- 1.  $\mathbb{Z}_{4,X}$ : The anomaly index of the 3-family SM (without 3  $\bar{\nu}_R$ ) is (-3 mod 16). Using a  $\nu = \text{even} \in \mathbb{Z}_{16}$  index symmetric gapped  $\mathbb{Z}_4$  gauge theory TQFT [34] is not enough to cancel the SM's  $N_f = 3$  anomaly, either some  $\nu_R$ , or some 4d gapless vector, or some extra dimensional 5d bulk, or some non-TQFT kind of fracton topological order is needed.
- 2.  $\mathbb{Z}_{8,X}$ : Missing 3 right-handed neutrinos (not found in the SM), each has  $\mathbb{Z}_{8,X}$  charge q=5, thus the 3 right-handed neutrinos has  $3(-7 \mod 32, -1 \mod 2) = (11 \mod 32, -1 \mod 2)$ . The complementary anomaly index of the SM is (21 mod 32, -1 mod 2). Using a  $\nu=4k\in\mathbb{Z}_{32}$  index symmetric gapped  $\mathbb{Z}_4$  gauge theory TQFT [34] is not enough to cancel the SM's  $N_f=3$  anomaly, either some  $\nu_R$ , or some 4d gapless vector, or some extra dimensional 5d bulk, or some non-TQFT kind of fractor topological order is needed.

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### Appendix A: Anomaly of 3+1d Weyl fermion with discrete charge $q \in \mathbb{Z}_n$

In this Appendix, for a single Weyl fermion theory assigned with a discrete charge  $q \in \mathbb{Z}_n$  in 3+1d (or simply 4d) spacetime, we derive the anomaly index, and relate the index to a 4+1d Symmetry-Protected Topological state (SPTs), or a cobordism invariant (that is an invertible field theory [iFT] or an invertible topological field theory [iTFT]). We will follow the discussion in the Appendix of [9]. We will also require information from [18]. We will recombine the linear combination of the generators of the 3+1d anomaly (or 4+1d cobordism invariant) in [18] to derive the appropriate generator that can match with that of a charge q Weyl fermion.

Below we may simply denote the 3+1d spacetime as 4d, and 4+1d spacetime as 5d.

## 1. Anomaly Polynomial

We read the 4d Weyl fermion anomaly and its associated 5d invertible theory action  $S_5 = 2\pi \int_{M^5} I_5 \in 2\pi \mathbb{R}$  from the 6d anomaly polynomial  $I_6 = dI_5$  whose integration over a closed 6-manifold is valued in  $\mathbb{Z}$ , from the  $\hat{A}$  genus and the Chern character  $ch(\mathcal{E})$ ,

$$\hat{A} \operatorname{ch}(\mathcal{E}),$$
 (A1)

where  $\hat{A}$  and  $ch(\mathcal{E})$  are given as:

$$\hat{A} = 1 - \frac{p_1}{24} + \frac{7p_1^2 - 4p_2}{5760} + \dots, \tag{A2}$$

$$\operatorname{ch}(\mathcal{E}) = \operatorname{rank} \mathcal{E} + c_1(\mathcal{E}) + \frac{1}{2} \left( c_1^2(\mathcal{E}) - 2c_2(\mathcal{E}) \right) + \frac{1}{6} \left( c_1^3(\mathcal{E}) - 3c_1(\mathcal{E})c_2(\mathcal{E}) + 3c_3(\mathcal{E}) \right) + \dots \tag{A3}$$

For a single left-handed Weyl fermion of charge q, we take  $\mathcal{E}$  to be the complex line bundle associated with the corresponding representation of U(1). Hence, the fermionic 6d anomaly polynomial is integer quantized over a closed 6-manifold

$$\int_{M^6} I_{6,f} = \int_{M^6} [\hat{\mathbf{A}} \operatorname{ch}(\mathcal{E})]_6 = \int_{M^6} q^3 \frac{c_1^3}{6} - q \frac{c_1 p_1}{24} \in \mathbb{Z}, \tag{A4}$$

computed as the index of the 6d Dirac operator via Atiyah-Singer index theorem [2, 3]. The 5d invertible theory action is

$$S_5 = 2\pi \int_{M^5} I_5 = \int_{M^6} q^3 A \frac{c_1^2}{6} - q A \frac{p_1}{24} \in 2\pi \mathbb{R}.$$
 (A5)

Next, we will start with two compatible Spin  $\times$  U(1) and Spin  $\times_{\mathbb{Z}_2^F}$  U(1)  $\equiv$  Spin<sup>c</sup> structures for the fermion case with a U(1) symmetry, to match with the appropriate generators of the cobordism group  $\operatorname{Hom}(\Omega_6^{\operatorname{Spin}^c}, \mathbb{Z}) = \mathbb{Z}^2$  and  $\operatorname{Hom}(\Omega_6^{\operatorname{Spin}\times \mathrm{U}(1)}, \mathbb{Z}) = \mathbb{Z}^2$ . Then we consider the Weyl fermion with a charge q of a discrete subgroup  $\mathbb{Z}_n$  symmetry out of this U(1).

#### 2. Spin $\times$ U(1) symmetry

Assuming that  $\mathbb{Z}_2^{\mathrm{F}} \not\subset \mathrm{U}(1)$ , which means the spacetime-internal symmetry group structure is  $\mathrm{Spin} \times \mathrm{U}(1)$  structure, the 6d anomaly polynomial  $I_6$  above is in general a linear combination (over  $\mathbb{Z}$ , if the charge q is an integer) of the following two terms:

$$I^A := \frac{c_1^3}{6} - \frac{c_1 p_1}{24} \in \mathbb{Z}, \qquad I^B := c_1^3 \in \mathbb{Z}.$$
 (A6)

For a general charge q left-handed Weyl fermion, we have:

$$I_6 = qI^A + \frac{q^3 - q}{6}I^B \in \mathbb{Z}. \tag{A7}$$

 $I^A$  and  $I^B$  serve as the two generators of  $\mathrm{Hom}(\Omega_6^{\mathrm{Spin}}(\mathrm{BU}(1)),\mathbb{Z})\cong\mathbb{Z}\times\mathbb{Z}$  by considering their integrals over 6-manifolds representing the elements in the bordism group. The integer values follow the Atiyah-Singer index theorem for the Dirac operator. Note that  $(q^3-q)/6\in\mathbb{Z}$  for any  $q\in\mathbb{Z}$ . Instead of  $(I^A,I^B)$  as above, we can consider another pair related to it by a  $\mathrm{GL}(2,\mathbb{Z})$  transformation.

3. 
$$\operatorname{Spin} \times_{\mathbb{Z}_2^F} \operatorname{U}(1)^F = \operatorname{Spin}^c$$
 symmetry

Assuming  $\mathbb{Z}_2^{\mathrm{F}} \subset \mathrm{U}(1)$ , which means the spacetime-internal symmetry structure is  $\mathrm{Spin} \times_{\mathbb{Z}_2^{\mathrm{F}}} \mathrm{U}(1) \equiv \mathrm{Spin}^c$  structure (so that in particular q is necessarily odd for fermions), the general 6d anomaly polynomial is an integral linear combination of the following two terms:

$$I^{C} := \frac{c_1^3}{6} - \frac{c_1 p_1}{24} = \frac{(2c_1)^3 - (2c_1)p_1}{48} \in \mathbb{Z}, \qquad I^{D} := 4c_1^3 = \frac{(2c_1)^3}{2} \in \mathbb{Z}. \tag{A8}$$

For a general charge q left-handed Weyl fermion, we have:

$$I_6 = qI^C + \frac{q^3 - q}{24}I^D \in \mathbb{Z}.$$
 (A9)

 $I^C$  and  $I^D$  serve as the two generators of  $\operatorname{Hom}(\Omega_6^{\operatorname{Spin}^c}, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$ , whose integer values following the Atiyah-Singer index theorem for the Dirac operator.

Note that in this  $\mathrm{Spin}^c$  case,  $c_1$  is in general not a well-defined integer cohomology class, only  $2c_1$  is. This is because in general there is no globally well-defined U(1) bundle, only U(1)/ $\mathbb{Z}_2$  bundle, the first Chern class of which is  $c'_1 = 2c_1 \in \mathbb{Z}$ .

4. Spin 
$$\times_{\mathbb{Z}_2^F} \mathbb{Z}_4^F$$
 symmetry

Consider Spin  $\times_{\mathbb{Z}_2^F} \mathbb{Z}_4$  symmetry ( $\subset$  Spin symmetry), for an odd charge  $q \in \mathbb{Z}_4$  Weyl fermion theory in 4d, we like to match its 4d anomaly to a 5d bordism group index  $\Omega_5^{\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_4} = \mathbb{Z}_{16}$  or precisely a 5d cobordism group index  $\mathrm{Hom}(\Omega_5^{\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_4}, \mathrm{U}(1)) = \mathbb{Z}_{16}$ .

To derive the anomaly index formula  $\nu(q) \in \mathbb{Z}_{16}$ , here are some constraints,

- For charge- $q = 1 \in \mathbb{Z}_4$  Weyl fermion, we have an anomaly index:  $\nu = 1 \mod 16$ , following the Atiyah-Patodi-Singer eta invariant.
- For charge- $q = 3 = -1 \in \mathbb{Z}_4$  Weyl fermion, we have an anomaly index:  $\nu = -1 = 15 \mod 16$ , due to the additivity structure of the group of the anomaly index.
- There is a linear map between the 2-dimensional integral lattice generated by two generators  $(I^C, I^D)$  of  $\operatorname{Hom}(\Omega_6^{\operatorname{Spin}^c}, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$  and the 1-dimensional integral mod 16 lattice  $\operatorname{Hom}(\Omega_5^{\operatorname{Spin}^c}, \mathbb{Z}_2^{\mathbb{F}^{\mathbb{Z}_4}}, \mathrm{U}(1)) = \mathbb{Z}_{16}$ . Under that linear map, denote that  $I^C$  is mapped to  $\iota^C \in \mathbb{Z}_{16}$  and  $I^D$  is mapped to  $\iota^D \in \mathbb{Z}_{16}$ . Then we can solve  $\iota^C, \iota^D \in \mathbb{Z}_{16}$  by plugging the constraint of the anomaly index  $\nu(q) \in \mathbb{Z}_{16}$  with an odd  $q = 1, 3 \in \mathbb{Z}_4$ ; we have

$$\nu(q) = q\iota^{C} + \frac{q^{3} - q}{24}\iota^{D} \mod 16.$$

$$\nu(q = 1) = 1 = 1\iota^{C} + 0\iota^{D} \mod 16.$$

$$\nu(q = 3) = -1 = 3\iota^{C} + 1\iota^{D} \mod 16.$$

$$\Rightarrow \iota^{C} = 1 \mod 16, \quad \iota^{D} = -4 \mod 16.$$
(A10)

Finally, for an odd charge  $q \in \mathbb{Z}_4$  Weyl fermion, we obtain its anomaly index formula

$$\nu(q) = q\iota^C + \frac{q^3 - q}{24}\iota^D = q - 4\frac{q^3 - q}{24} = \frac{-q^3 + 7q}{6} \mod 16 \in \mathbb{Z}_{16}. \tag{A11}$$

Using the theory in [34], we find that the  $\mathbb{Z}_{16}$  class of 5d  $\mathbb{Z}_4^F$  fSPTs and  $C_2 \times \mathbb{Z}_2^F$  fSPTs has:

- For  $\nu = N = 2$  or even  $\in \mathbb{Z}_{16}$ , we have a symmetric gapped  $\mathbb{Z}_4$  gauge theory TQFT.
- For  $\nu = N/2 = 1$  or odd  $\in \mathbb{Z}_{16}$ , we find a non-TQFT symmetric gapped state via stacking lower-dimensional (2+1)d non-discrete-gauge-theory topological order (that has TQFT descriptions) inhomogeneously.

There are two applications for  $\mathbb{Z}_4^{\mathrm{F}}$  symmetry:

1. First application: The  $\mathbb{Z}_{4,X}$  symmetry as  $\mathbb{Z}_4^F$ . There are 3 missing right-handed neutrinos not found in the Standard Model, each of such  $\bar{\nu}_R$  has  $\mathbb{Z}_4^F$  charge 1, thus for 3 families of such  $\bar{\nu}_R$ , we have the total anomaly index

$$3(1 \mod 16) = (3 \mod 16).$$
 (A12)

<sup>&</sup>lt;sup>4</sup> For Spin<sup>c</sup>, the U(1)  $\supset \mathbb{Z}_2^F$  contains the fermion parity as a normal subgroup.

<sup>•</sup> For the original U(1) with  $c_1(U(1))$ , the gauge bundle constraint is  $w_2(TM) = 2c_1 \mod 2$ . In the original U(1), fermions have odd charges under U(1), while bosons have even charges under U(1). Call the original U(1) gauge field A, then  $c_1 = \frac{dA}{2\pi} \in \frac{1}{2}\mathbb{Z}$ .

<sup>•</sup> For the new U(1)' =  $\frac{\text{U}(1)}{\mathbb{Z}_2^F}$  with  $c_1(\text{U}(1)')$ , the gauge bundle constraint is  $w_2(TM) = c_1' = 2c_1 \mod 2$ . Call the new U(1)' gauge field A', then  $c_1' = \frac{\text{d}A'}{2\pi} = \frac{\text{d}(2A)}{2\pi} = 2c_1 \in 2\frac{1}{2}\mathbb{Z} = \mathbb{Z}$ .
• To explain why A' = 2A or  $c_1' = 2c_1$ , we look at the Wilson line operator  $\exp(iq' \oint A')$  and  $\exp(iq \oint A)$ . The original U(1) has charge transformation  $\exp(iq\theta)$  with  $\theta \in [0, 2\pi)$ , while the new U(1)' has charge transformation  $\exp(iq'\theta')$  with  $\theta' \in [0, 2\pi)$ . But the

<sup>•</sup> To explain why A'=2A or  $c_1'=2c_1$ , we look at the Wilson line operator  $\exp(\mathrm{i}\,q'\oint A')$  and  $\exp(\mathrm{i}\,q\oint A)$ . The original U(1) has charge transformation  $\exp(\mathrm{i}\,q'\theta)$  with  $\theta\in[0,2\pi)$ , while the new U(1)' has charge transformation  $\exp(\mathrm{i}\,q'\theta')$  with  $\theta'\in[0,2\pi)$ . But the U(1)'  $=\frac{\mathrm{U}(1)}{\mathbb{Z}_2^F}$ , so the  $\theta=\pi$  in the old U(1) is identified as  $\theta'=2\pi$  as a trivial zero in the new U(1)'. In the original U(1), the  $q\in\mathbb{Z}$  to be compatible with  $\theta\in[0,2\pi)$ . In the new U(1)', the original q is still allowed to have  $2\mathbb{Z}$  to be compatible with  $\theta\in[0,\pi)$ ; but the new  $q'=\frac{1}{2}q\in\mathbb{Z}$  and the new  $\theta'=2\theta\in[0,2\pi)$  are scaled accordingly. Since the new  $q'=\frac{1}{2}q\in\mathbb{Z}$ , we show the new A'=2A.

The complementary anomaly index of the 3-family SM (without 3  $\bar{\nu}_R$ ) is

$$(-3 \mod 16).$$
 (A13)

Using a  $\nu = \text{even} \in \mathbb{Z}_{16}$  index symmetric gapped  $\mathbb{Z}_4$  gauge theory TQFT [34] is not enough to cancel the SM's  $N_f = 3$  anomaly, either some  $\nu_R$ , or some 4d gapless vector, or some extra dimensional 5d bulk, or some non-TQFT kind of fracton topological order is needed.

2. Second application: The  $\mathbb{Z}_{2N_f=4,\mathbf{B}+\mathbf{L}}$  symmetry as  $\mathbb{Z}_4^F$  for  $N_f=2$ . There are 2 missing right-handed neutrinos not found in the  $N_f=2$ -family Standard Model, each of such  $\bar{\nu}_R$  has  $\mathbb{Z}_4^{\mathrm{F}}$  charge -1, thus for 3 families of such  $\bar{\nu}_R$ , we have the total anomaly index

$$2(-1 \mod 16) = (-2 \mod 16). \tag{A14}$$

The complementary anomaly index of the  $N_f=2$ -family SM (without  $2 \bar{\nu}_R$ ) is

$$(2 \mod 16).$$
 (A15)

Note that the 2 mod 16 corresponds to a 5d fermionic SPT which is beyond the bosonic group cohomology description. Using the theory in [34], for  $\nu = N = 2 \in \mathbb{Z}_{16}$ , we have a symmetric gapped fermionic  $\mathbb{Z}_4$  gauge theory TQFT.

In a one lower dimensional analogy, assume there is a 3d Pin<sup>+</sup> TQFT with anomaly described by the 4d effective action  $S_{4d} = -\nu(2\pi\eta/16)$  withan anomaly index  $-\nu \in \mathbb{Z}_{16} = \text{Hom}(\Omega_4^{\text{Pin}^+}, \text{U}(1))$ , such TQFTs were considered in

# 5. Spin $\times_{\mathbb{Z}_2^F} \mathbb{Z}_8^F$ symmetry

Consider Spin  $\times_{\mathbb{Z}_2^F} \mathbb{Z}_8$  symmetry ( $\subset$  Spin<sup>c</sup> symmetry), for an odd charge  $q \in \mathbb{Z}_8$  Weyl fermion theory in 4d, we like to match its 4d anomaly to a 5d bordism group index  $\Omega_5^{\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_8} = \mathbb{Z}_{32} \times \mathbb{Z}_2$  or precisely a 5d cobordism group index  $\mathrm{Hom}(\Omega_5^{\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_8}, \mathrm{U}(1)) = \mathbb{Z}_{32} \times \mathbb{Z}_2$ .

To derive the anomaly index formula  $(\nu_1(q), \nu_2(q)) \in \mathbb{Z}_{32} \times \mathbb{Z}_2$ , here are some constraints,

- For charge- $q = 1 \in \mathbb{Z}_8$  Weyl fermion, we have an anomaly index:  $\nu_1 = 1 \mod 32$ , following the Atiyah-Patodi-Singer eta invariant.
- For charge- $q=7=-1\in\mathbb{Z}_8$  Weyl fermion, we have an anomaly index:  $\nu_1=-1=31\mod 32$ , due to the additivity structure of the group of the anomaly index.
- There is a linear map between the 2-dimensional integral lattice generated by two generators  $(I^C, I^D)$  of  $q = 1, 7 \in \mathbb{Z}_8$ ; we have

$$\begin{array}{rcl} \mathbf{v}_{1}(q) &=& q \iota_{1}^{C} + \frac{q^{3} - q}{24} \iota_{1}^{D} \mod 32. \\ \mathbf{v}_{1}(q=1) &=& 1 = 1 \iota_{1}^{C} + 0 \iota_{1}^{D} \mod 32. \\ \mathbf{v}_{1}(q=7) &=& -1 = 7 \iota_{1}^{C} + 14 \iota_{1}^{D} \mod 32. \\ &\Rightarrow \iota_{1}^{C} = 1 \mod 32, \quad \iota_{1}^{D} = 4 \mod 32. \end{array} \tag{A16}$$

Plugging the constraint of the anomaly index  $\nu_2(q) \in \mathbb{Z}_2$  with an odd  $q = 1, 7 \in \mathbb{Z}_8$ ; we have

$$\nu_{2}(q) = q\iota_{2}^{C} + \frac{q^{3} - q}{24}\iota_{2}^{D} \mod 2.$$

$$\nu_{2}(q = 1) = 0 = 1\iota_{2}^{C} + 0\iota_{2}^{D} \mod 2.$$

$$\nu_{2}(q = 7) = 0 = 7\iota_{2}^{C} + 14\iota_{2}^{D} \mod 2.$$

$$\Rightarrow \iota_{2}^{C} = 0 \mod 2, \quad \iota_{2}^{D} = 0, \text{ or } 1 \mod 2.$$
(A17)

Note that  $\iota_2^D = 0$  or 1 cannot be solved directly here, but we expect that a nontrivial map thus  $\iota_2^D = 1$  which can be verified in the next step.

• Via Ref. [18] anomaly index formula (2.49), we are able to derive an appropriate linear combination of two generators in (2.49) to give rise to

$$\begin{aligned} (\mathbf{v}_{1}(q=3), \mathbf{v}_{2}(q=3)) &= (7 \mod 32, \quad 1 \mod 2), \\ (\mathbf{v}_{1}(q=5), \mathbf{v}_{2}(q=5)) &= (-7 \mod 32, \quad 1 \mod 2), \\ \mathbf{v}_{1}(q=3) &= 7 = 3\iota_{1}^{C} + 1\iota_{1}^{D} \mod 32, \\ \mathbf{v}_{1}(q=5) &= -7 = 5\iota_{1}^{C} + 5\iota_{1}^{D} \mod 32, \\ &\Rightarrow \iota_{1}^{C} = 1 \mod 32, \quad \iota_{1}^{D} = 4 \mod 32, \\ \mathbf{v}_{2}(q=3) &= 1 = 3\iota_{2}^{C} + 1\iota_{2}^{D} \mod 2, \\ \mathbf{v}_{2}(q=5) &= 1 = 5\iota_{2}^{C} + 5\iota_{2}^{D} \mod 2, \\ &\Rightarrow \iota_{2}^{C} = 0 \mod 2, \quad \iota_{2}^{D} = 1 \mod 2. \end{aligned}$$

$$\tag{A18}$$

Thus for an odd charge  $q \in \mathbb{Z}_8$  Weyl fermion, we obtain its anomaly index formula

$$(\nu_{1}(q), \nu_{2}(q)) \in \mathbb{Z}_{32} \times \mathbb{Z}_{2}$$

$$= (q\iota_{1}^{C} + \frac{q^{3} - q}{24}\iota_{1}^{D} \mod 32, q\iota_{2}^{C} + \frac{q^{3} - q}{24}\iota_{2}^{D} \mod 2)$$

$$= (\frac{q^{3} + 5q}{6} \mod 32, \frac{q^{3} - q}{24} \mod 2). \tag{A19}$$

For a charge q = 1, 3, 5, 7 Weyl fermion, we have a map to the anomaly index

$$q = 1, 3, 5, 7 \mapsto (\nu_1, \nu_2) = (1, 0), (7, 1), (-7, 1), (-1, 0).$$
 (A20)

Using the theory in [34], we find that the  $\mathbb{Z}_{32}$  class of  $\mathbb{Z}_8^F$  fSPTs and  $C_4 \times \mathbb{Z}_2^F$  fSPTs has:

- For  $\nu = N = 4 \in \mathbb{Z}_{32}$ , we have a symmetric gapped  $\mathbb{Z}_4$  gauge theory TQFT.
- For  $\nu = N/2 = 2 \in \mathbb{Z}_{32}$ , we find a non-TQFT symmetric gapped state via stacking lower-dimensional (2+1)d non-discrete-gauge-theory topological order (that has TQFT descriptions) inhomogeneously.
- For  $\nu = 1 \in \mathbb{Z}_{32}$ , we do not have either of symmetric gapped states.

We consider two applications for  $\mathbb{Z}_8^{\mathrm{F}}$  symmetry:

1. First application: The  $\mathbb{Z}_{8,X}$  symmetry as  $\mathbb{Z}_8^{\mathrm{F}}$ . Missing 3 right-handed neutrinos (not found in the SM), each has  $\mathbb{Z}_{8,X}$  charge q=5, thus the 3 right-handed neutrinos has

$$3(-7 \mod 32, 1 \mod 2) = (11 \mod 32, 1 \mod 2).$$
 (A21)

The complementary anomaly index of the SM is

$$(21 \mod 32, 1 \mod 2).$$
 (A22)

2. Second application: The  $\mathbb{Z}_{2N_f=8,\mathbf{B}+\mathbf{L}}^{\mathbf{F}}$  symmetry as  $\mathbb{Z}_{8}^{\mathbf{F}}$  with  $N_f=4$ . Missing right-handed neutrinos (not found in the SM), each has  $\mathbb{Z}_{2N_f=8,\mathbf{B}+\mathbf{L}}^{\mathbf{F}}$  charge  $q=-1=7\mod 8$ , thus the 4 right-handed neutrinos has

$$4(-1 \mod 32, 0 \mod 2) = (-4 \mod 32, 0 \mod 2).$$
 (A23)

The complementary anomaly index of the SM is

$$(4 \mod 32, \mod 2), \tag{A24}$$

thus anomalous. Note that the 4 mod 32 class of 5d  $\mathbb{Z}_8^F$  fermionic SPTs is still a beyond group cohomology SPTs. (The situation is different for  $N_f = 3$  which gives rise to a group cohomology SPTs, shown in Sec. A 7 and Sec. B.)

## 6. Spin $\times \mathbb{Z}_3$ symmetry

Consider Spin  $\times \mathbb{Z}_3$  symmetry ( $\subset$  Spin  $\times$  U(1) symmetry), for an integer charge  $q \in \mathbb{Z}_3$  Weyl fermion theory in 4d, we like to match its 4d anomaly to a 5d bordism group index  $\Omega_5^{\mathrm{Spin} \times \mathbb{Z}_3} = \mathbb{Z}_9$  or precisely a 5d cobordism group index  $\mathrm{Hom}(\Omega_5^{\mathrm{Spin} \times \mathbb{Z}_3}, \mathrm{U}(1)) = \mathbb{Z}_9$ .

To derive the anomaly index formula  $\nu(q) \in \mathbb{Z}_9$ , here are some constraints,

- For charge- $q = 1 \in \mathbb{Z}_3$  Weyl fermion, we have an anomaly index:  $\nu = 1 \mod 16$ , following the Atiyah-Patodi-Singer eta invariant.
- For charge- $q = 2 = -1 \in \mathbb{Z}_3$  Weyl fermion, we have an anomaly index:  $\nu = -1 = 8 \mod 9$ , due to the additivity structure of the group of the anomaly index.
- There is a linear map between the 2-dimensional integral lattice generated by two generators  $(I^A, I^B)$  of  $\operatorname{Hom}(\Omega_6^{\operatorname{Spin} \times \operatorname{U}(1)}, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$  and the 1-dimensional integral mod 9 lattice  $\operatorname{Hom}(\Omega_5^{\operatorname{Spin} \times \mathbb{Z}_3}, \operatorname{U}(1)) = \mathbb{Z}_9$ . Under that linear map, denote that  $I^A$  is mapped to  $\iota^A \in \mathbb{Z}_9$  and  $I^B$  is mapped to  $\iota^B \in \mathbb{Z}_9$ . Then we can solve  $\iota^A, \iota^B \in \mathbb{Z}_9$  by plugging the constraint of the anomaly index  $\mathbf{v}(q) \in \mathbb{Z}_9$  with  $q = 1, 2 \in \mathbb{Z}_3$ ; we have

$$\nu(q) = q\iota^{A} + \frac{q^{3} - q}{6}\iota^{B} \mod 9. 
\nu(q = 1) = 1 = 1\iota^{A} + 0\iota^{B} \mod 9. 
\nu(q = 2) = -1 = 2\iota^{A} + 1\iota^{B} \mod 9. 
\Rightarrow \iota^{A} = 1 \mod 9, \quad \iota^{B} = -3 \mod 9.$$
(A25)

Finally, for a charge  $q \in \mathbb{Z}_3$  Weyl fermion, we obtain its anomaly index formula

$$\nu(q) = q\iota^A + \frac{q^3 - q}{6}\iota^B = \frac{-q^3 + 3q}{2} \mod 9 \in \mathbb{Z}_9.$$
(A26)

For a charge q = 1, 2 Weyl fermion, we have a map to the anomaly index

$$q = 1, 2 \mapsto \mathbf{v} = 1, -1 \mod 9 \in \mathbb{Z}_9. \tag{A27}$$

We consider two applications for  $\mathbb{Z}_2^F \times \mathbb{Z}_3$  symmetry:

1. First application: The  $\mathbb{Z}_{2N_f=6,\mathbf{B}+\mathbf{L}}^{\mathbf{F}}$  symmetry as  $\mathbb{Z}_6^{\mathbf{F}}$  with  $N_f=3$ .

There are 3 missing right-handed neutrinos not found in the Standard Model, each of such  $\bar{\nu}_R$  has  $\mathbb{Z}_{2N_f=6}^{\mathrm{F}}$  charge -1=5, each has  $\mathbb{Z}_3$  charge 2,<sup>5</sup> thus for 3 families of such  $\bar{\nu}_R$ , we have the total anomaly index

$$3(-1 \mod 9) = (6 \mod 9). \tag{A28}$$

The complementary anomaly index of the 3-family SM is

$$(3 \mod 9). \tag{A29}$$

This result shows that  $N_f = 3$  is special which gives rise to a nontrivial  $\mathbb{Z}_3$  class of group cohomology SPTs, see Sec. B.

2. Second application: The  $\mathbb{Z}_{2N_cN_f=6,\mathbf{Q}+N_c\mathbf{L}}$  symmetry as  $\mathbb{Z}_6^{\mathrm{F}}$  for  $N_c=3$  and  $N_f=1$ . There are 1 missing right-handed neutrinos not found in the  $N_f=1$ -family Standard Model, each of such  $\bar{\nu}_R$  has  $\mathbb{Z}_6^{\mathrm{F}}$  charge -3=3 that has  $\mathbb{Z}_3$  charge 0, thus for 1 family of such  $\bar{\nu}_R$ , we have the total anomaly index

$$1(0 \mod 9) = (0 \mod 9). \tag{A30}$$

The complementary anomaly index of the  $N_f = 1$ -family SM (without a  $\bar{\nu}_R$ ) is

$$(0 \mod 9), \tag{A31}$$

still anomaly-free.

<sup>&</sup>lt;sup>5</sup> We can label  $n_6 \in \mathbb{Z}_6 = \mathbb{Z}_6^{\mathrm{F}} \supset \mathbb{Z}_2^{\mathrm{F}}$  in terms of a doublet  $(n_2^{\mathrm{F}}, n_3) \in \mathbb{Z}_2^{\mathrm{F}} \times \mathbb{Z}_3$ , such that the bosons have  $n_2^{\mathrm{F}} = 0$  and the fermions have  $n_2^{\mathrm{F}} = 1$ . In addition, without loss of generality, we assign the charge  $q = 1 \in \mathbb{Z}_6^{\mathrm{F}}$  fermion to the  $(n_2^{\mathrm{F}}, n_3) = (1, 1) \in \mathbb{Z}_2^{\mathrm{F}} \times \mathbb{Z}_3$ . These are enough to constrain the map in between as  $n_6 = 3n_2^{\mathrm{F}} - 2n_3$ . Thus,  $n_6 = 0, 1, 2, 3, 4, 5$  is mapped to  $(n_2^{\mathrm{F}}, n_3) = (0, 0), (1, 1), (0, 2), (1, 0), (0, 1), (1, 2)$ .

# 7. Spin $\times_{\mathbb{Z}_2^F} \mathbb{Z}_6^F$ symmetry

Consider Spin  $\times_{\mathbb{Z}_2^F} \mathbb{Z}_6^F$  symmetry ( $\subset$  Spin<sup>c</sup> symmetry), for an odd charge  $q \in \mathbb{Z}_6^F$  Weyl fermion theory in 4d, we like to match its 4d anomaly to a 5d bordism group index  $\Omega_5^{\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_6^F} = \Omega_5^{\mathrm{Spin} \times \mathbb{Z}_3} = \mathbb{Z}_9$  or precisely a 5d cobordism group index  $\mathrm{Hom}(\Omega_5^{\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_6^F}, \mathrm{U}(1)) = \mathrm{Hom}(\Omega_5^{\mathrm{Spin} \times \mathbb{Z}_3}, \mathrm{U}(1)) = \mathbb{Z}_9$ .

To derive the anomaly index formula  $\nu(q) \in \mathbb{Z}_9$ , here are some constraints,

- For charge- $q = 1 \in \mathbb{Z}_6$  Weyl fermion, we have an anomaly index:  $\nu = 1 \mod 9$ , following the Atiyah-Patodi-Singer eta invariant.
- For charge- $q = 5 = -1 \in \mathbb{Z}_6$  Weyl fermion, we have an anomaly index:  $\nu = -1 = 8 \mod 9$ , due to the additivity structure of the group of the anomaly index.
- There is a linear map between the 2-dimensional integral lattice generated by two generators  $(I^C, I^D)$  of  $\operatorname{Hom}(\Omega_6^{\operatorname{Spin}^c}, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$  and the 1-dimensional integral mod 9 lattice  $\operatorname{Hom}(\Omega_5^{\operatorname{Spin}^c}, \mathbb{U}(1)) = \mathbb{Z}_9$ . Under that linear map, denote that  $I^C$  is mapped to  $\iota^C \in \mathbb{Z}_9$  and  $I^D$  is mapped to  $\iota^D \in \mathbb{Z}_9$ . Then we can solve  $\iota^C, \iota^D \in \mathbb{Z}_9$  by plugging the constraint of the anomaly index  $\mathbf{v}(q) \in \mathbb{Z}_{16}$  with an odd  $q = 1, 5 \in \mathbb{Z}_6$ ; we have

$$\nu(q) = q\iota^{C} + \frac{q^{3} - q}{24}\iota^{D} \mod 9. 
\nu(q = 1) = 1 = 1\iota^{C} + 0\iota^{D} \mod 9. 
\nu(q = 5) = -1 = 5\iota^{C} + 5\iota^{D} \mod 9. 
\Rightarrow \iota^{C} = 1 \mod 9, \quad \iota^{D} = -3 \mod 9.$$
(A32)

Finally, for an odd charge  $q \in \mathbb{Z}_6$  Weyl fermion, we obtain its anomaly index formula

$$\nu(q) = q\iota^C + \frac{q^3 - q}{24}\iota^D = \frac{-q^3 + 9q}{8} \mod 9 \in \mathbb{Z}_9$$
(A33)

For a charge q = 1, 3, 5 Weyl fermion, we have a map to the anomaly index

$$q = 1, 3, 5 \mapsto \nu = 1, 0, -1 \mod 9 \in \mathbb{Z}_9.$$
 (A34)

We consider two applications for  $\mathbb{Z}_6^{\mathrm{F}}$  symmetry:

1. First application: The  $\mathbb{Z}_{2N_f=6,\mathbf{B}+\mathbf{L}}^{\mathrm{F}}$  symmetry as  $\mathbb{Z}_6^{\mathrm{F}}$  with  $N_f=3$ . There are 3 missing right-handed neutrinos not found in the Standard Model, each of such  $\bar{\nu}_R$  has  $\mathbb{Z}_{2N_f=6,\mathbf{B}+\mathbf{L}}^{\mathrm{F}}$  charge -1=5, thus for 3 families of such  $\bar{\nu}_R$ , we have the total anomaly index

$$3(-1 \mod 9) = (6 \mod 9). \tag{A35}$$

The complementary anomaly index of the 3-family SM (without 3  $\bar{\nu}_R$ ) is

$$(3 \mod 9). \tag{A36}$$

This result shows that  $N_f = 3$  is special which gives rise to a nontrivial  $\mathbb{Z}_3$  class of group cohomology SPTs, see Sec. B.

2. Second application: The  $\mathbb{Z}_{2N_cN_f=6,\mathbf{Q}+N_c\mathbf{L}}$  symmetry as  $\mathbb{Z}_6^{\mathrm{F}}$  for  $N_c=3$  and  $N_f=1$ . There are 1 missing right-handed neutrinos not found in the  $N_f=1$ -family Standard Model, each of such  $\bar{\nu}_R$  has  $\mathbb{Z}_6^{\mathrm{F}}$  charge -3=3, thus for 1 family of such  $\bar{\nu}_R$ , we have the total anomaly index

$$1(0 \mod 9) = (0 \mod 9). \tag{A37}$$

The complementary anomaly index of the  $N_f = 1$ -family SM (without a  $\bar{\nu}_R$ ) is

$$(0 \mod 9), \tag{A38}$$

still anomaly-free.

# 8. Spin $\times_{\mathbb{Z}_2^F} \mathbb{Z}_{12}^F$ symmetry

Consider Spin  $\times_{\mathbb{Z}_2^F} \mathbb{Z}_{12}^F$  symmetry ( $\subset$  Spin<sup>c</sup> symmetry), for an odd charge  $q \in \mathbb{Z}_{12}^F$  Weyl fermion theory in 4d, we like to match its 4d anomaly to a 5d bordism group index  $\Omega_5^{\operatorname{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{12}^F} = \Omega_5^{\operatorname{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_4^F \times \mathbb{Z}_3} = \mathbb{Z}_{16} \times \mathbb{Z}_9$  or precisely a 5d cobordism group index  $\operatorname{Hom}(\Omega_5^{\operatorname{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{12}^F}, \operatorname{U}(1)) = \operatorname{Hom}(\Omega_5^{\operatorname{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_4^F \times \mathbb{Z}_3}, \operatorname{U}(1)) = \mathbb{Z}_{16} \times \mathbb{Z}_9$ .

To derive the anomaly index formula  $(\nu_1(q), \nu_2(q)) \in \mathbb{Z}_{16} \times \mathbb{Z}_9$ , we follow the previous arguments with similar spectruments.

constraints.

For an odd charge  $q \in \mathbb{Z}_{12}$  Weyl fermion, We arrive at the map to the anomaly index

$$q = 1, 3, 5, 7, 9, 11$$
  

$$\mapsto (\mathbf{v}_1, \mathbf{v}_2) = (1, 1), (-1, 0), (1, -1), (-1, 1), (1, 0), (-1, -1) \in \mathbb{Z}_{16} \times \mathbb{Z}_9.$$
(A39)

We provide two applications for  $\mathbb{Z}_{12}^{F}$  symmetry:

1. First application: The  $\mathbb{Z}_{12,X}$  symmetry as  $\mathbb{Z}_{12}^{\mathrm{F}}$ . There are 3 missing right-handed neutrinos not found in the Standard Model, each of such  $\bar{\nu}_R$  has  $\mathbb{Z}_{12}^{\mathrm{F}}$  charge 5, thus for 3 families of such  $\bar{\nu}_R$ , we have the total anomaly index

$$3(1 \mod 16, -1 \mod 9) = (3 \mod 16, -3 \mod 9).$$
 (A40)

The complementary anomaly index of the 3-family SM (without 3  $\bar{\nu}_R$ ) is

$$(-3 \mod 16, 3 \mod 9)$$
. (A41)

2. Second application: The  $\mathbb{Z}_{2N_cN_f=12,\mathbf{Q}+N_c\mathbf{L}}$  symmetry as  $\mathbb{Z}_{12}^{\mathrm{F}}$  for  $N_c=3$  and  $N_f=2$ . Suppose there are 2 missing right-handed neutrinos not found in the  $N_f=2$  Standard Model, each such  $\bar{\nu}_R$  has  $\mathbb{Z}^{\mathrm{F}}_{2N_cN_f=12}$  charge -3 = 9, thus for 2 families of such  $\bar{\nu}_R$ , we have

$$2(1 \mod 16, \mod 9) = (2 \mod 16, \mod 9).$$
 (A42)

The complementary anomaly index of the 2-family SM (without  $2 \bar{\nu}_R$ ) is

$$(-2 \mod 16, 0 \mod 9),$$
 (A43)

thus anomalous. This result, up to the orientation of the charge, matches with eq. (A15), between the  $\mathbb{Z}_{2N_f=4,\mathbf{B}+\mathbf{L}}$  there and  $\mathbb{Z}_{2N_cN_f=12,\mathbf{Q}+N_c\mathbf{L}}$  here for  $N_f=2.6$ 

<sup>&</sup>lt;sup>6</sup> We can label  $n_{12} \in \mathbb{Z}_{12} = \mathbb{Z}_{12}^{F} \supset \mathbb{Z}_{2}^{F}$  in terms of a doublet  $(n_{4}^{F}, n_{3}) \in \mathbb{Z}_{4}^{F} \times \mathbb{Z}_{3}$ , such that the bosons have  $n_{4}^{F} = 0, 2$  and the fermions have  $n_{2}^{F} = 1, 3$ . In addition, without loss of generality, we assign the charge  $q = 1 \in \mathbb{Z}_{12}^{F}$  fermion to the  $(n_{4}^{F}, n_{3}) = (1, 1) \in \mathbb{Z}_{4}^{F} \times \mathbb{Z}_{3}$ . These are enough to constrain the map in between as  $n_{12} = -3n_{4}^{F} + 4n_{3}$ . Thus,  $n_{12} = 0, 2, 4, 6, 8, 10$  is mapped to  $(n_{4}^{F}, n_{3}) = (0, 0), (2, 2), (0, 1), (2, 0), (0, 2), (2, 1)$  while  $n_{12} = 1, 3, 5, 7, 9, 11$  is mapped to  $(n_{4}^{F}, n_{3}) = (1, 1), (3, 0), (1, 2), (3, 1), (1, 0), (3, 2)$ .

## 9. Spin $\times \mathbb{Z}_9$ symmetry

Consider Spin  $\times \mathbb{Z}_9$  symmetry ( $\subset$  Spin  $\times$  U(1) symmetry), for an integer charge  $q \in \mathbb{Z}_9$  Weyl fermion theory in 4d, we like to match its 4d anomaly to a 5d bordism group index  $\Omega_5^{{\rm Spin} \times \mathbb{Z}_9} = \mathbb{Z}_{27} \times \mathbb{Z}_3$  or precisely a 5d cobordism group index  ${\rm Hom}(\Omega_5^{{\rm Spin} \times \mathbb{Z}_9}, {\rm U}(1)) = \mathbb{Z}_{27} \times \mathbb{Z}_3$ .

- For charge- $q = 1 \in \mathbb{Z}_9$  Weyl fermion, we have an anomaly index:  $\nu_1 = 1 \mod 27$ , following the Atiyah-Patodi-Singer eta invariant.
- For charge- $q = 8 = -1 \in \mathbb{Z}_9$  Weyl fermion, we have an anomaly index:  $\mathbf{v}_1 = -1 = 26 \mod 27$ , due to the additivity structure of the group of the anomaly index.
- There is a linear map between the 2-dimensional integral lattice generated by two generators  $(I^A, I^B)$  of  $\operatorname{Hom}(\Omega_6^{\operatorname{Spin} \times \operatorname{U}(1)}, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$  and the 2-dimensional integral mod (27,3) lattice  $\operatorname{Hom}(\Omega_5^{\operatorname{Spin} \times \mathbb{Z}_9}, \operatorname{U}(1)) = \mathbb{Z}_{27} \times \mathbb{Z}_3$ . Under that linear map, denote that  $I^A$  is mapped to  $(\iota_1^A, \iota_2^A) \in \mathbb{Z}_{27} \times \mathbb{Z}_3$  and  $I^B$  is mapped to  $(\iota_1^B, \iota_2^B) \in \mathbb{Z}_{27} \times \mathbb{Z}_3$ . Then we can solve  $\iota_1^A, \iota_1^B \in \mathbb{Z}_{27}$  by plugging the constraint of the anomaly index  $\mathbf{v}_1(q) \in \mathbb{Z}_{27}$  with  $q = 1, 8 \in \mathbb{Z}_9$ ; we have

$$\begin{aligned}
\nu_1(q) &= q \iota_1^A + \frac{q^3 - q}{6} \iota_1^B \mod 27. \\
\nu_1(q = 1) &= 1 = 1 \iota_1^A + 0 \iota_1^B \mod 27. \\
\nu_1(q = 8) &= -1 = 8 \iota_1^A + 84 \iota_1^B \mod 27. \\
&\Rightarrow \iota_1^A = 1 \mod 27, \quad \iota_1^B = 6 \mod 27.
\end{aligned} \tag{A44}$$

Plugging the constraint of the anomaly index  $\nu_2(q) \in \mathbb{Z}_3$  with  $q = 1, 8 \in \mathbb{Z}_9$ ; we have

$$\begin{array}{rcl} \mathbf{v}_{2}(q) & = & q\iota_{2}^{A} + \frac{q^{3} - q}{6}\iota_{2}^{B} \mod 3. \\ \\ \mathbf{v}_{2}(q=1) & = & 0 = 1\iota_{2}^{A} + 0\iota_{2}^{B} \mod 3. \\ \\ \mathbf{v}_{2}(q=8) & = & 0 = 8\iota_{2}^{A} + 84\iota_{2}^{B} \mod 3. \\ \\ & \Rightarrow & \iota_{2}^{A} = 0 \mod 3, \quad \iota_{2}^{B} = 0, 1, \text{ or } 2 \mod 3. \end{array} \tag{A45}$$

Note that  $\iota_2^B = 0, 1, 2$  cannot be solved directly here, but we expect that a nontrivial map thus  $\iota_2^B = 1$  or 2 which can be verified in the next step.

• Via Ref. [18] anomaly index formula (2.31), we are able to derive an appropriate linear combination of two generators in (2.31) to give rise to

$$\begin{array}{rcl} (\mathbf{v}_1(q=3),\mathbf{v}_2(q=3)) &=& (0 \mod 27, \quad 2 \mod 3). \\ \mathbf{v}_2(q=3) &=& 2 = 3\iota_2^A + 4\iota_2^B \mod 3. \\ &\Rightarrow \iota_2^A = 0 \mod 3, \quad \iota_2^B = 2 \mod 3. \end{array} \tag{A46}$$

Thus for a charge  $q \in \mathbb{Z}_9$  Weyl fermion, we obtain its anomaly index formula

$$\begin{aligned} &(\mathbf{v}_{1}(q), \mathbf{v}_{2}(q)) \in \mathbb{Z}_{27} \times \mathbb{Z}_{3} \\ &= (q\iota_{1}^{A} + \frac{q^{3} - q}{6}\iota_{1}^{B} \mod 27, q\iota_{2}^{A} + \frac{q^{3} - q}{6}\iota_{2}^{B} \mod 3) \\ &= (q^{3} \mod 27, \frac{q^{3} - q}{3} \mod 3). \end{aligned}$$

$$\tag{A47}$$

For a charge  $q \in \mathbb{Z}_9$  Weyl fermion, we have a map to the anomaly index

$$q = 1, 2, 3, 4, 5, 6, 7, 8 \mapsto (\mathbf{v}_1, \mathbf{v}_2) = (1, 0), (8, 2), (0, 2), (10, 2), (17, 1), (0, 1), (19, 1), (26, 0). \tag{A48}$$

There are 3 missing right-handed neutrinos not found in the Standard Model, each such  $\bar{\nu}_R$  has  $\mathbb{Z}_{2N_cN_f=18}^{\mathrm{F}}$  charge -3=15, each has  $\mathbb{Z}_9$  charge 6, 7 thus for 3 families of such  $\bar{\nu}_R$ , we have

$$3(0 \mod 27, 1 \mod 3) = (0 \mod 27, 0 \mod 3).$$
 (A49)

<sup>&</sup>lt;sup>7</sup> We can label  $n_{18} \in \mathbb{Z}_{18} = \mathbb{Z}_{18}^{F} \supset \mathbb{Z}_{2}^{F}$  in terms of a doublet  $(n_{2}^{F}, n_{9}) \in \mathbb{Z}_{2}^{F} \times \mathbb{Z}_{9}$ , such that the bosons have  $n_{2}^{F} = 0$  and the fermions have  $n_{2}^{F} = 1$ . In addition, without loss of generality, we assign the charge  $q = 1 \in \mathbb{Z}_{18}^{F}$  fermion to the  $(n_{2}^{F}, n_{9}) = (1, 1) \in \mathbb{Z}_{2}^{F} \times \mathbb{Z}_{9}$ . These are enough to constrain the map in between as  $n_{18} = 9n_{2}^{F} - 8n_{9}$ . Thus,  $n_{18} = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17$  is mapped to  $(n_{2}^{F}, n_{9}) = (0, 0), (1, 1), (0, 2), (1, 3), (0, 4), (1, 5), (0, 6), (1, 7), (0, 8), (1, 0), (0, 1), (1, 2), (0, 3), (1, 4), (0, 5), (1, 6), (0, 7), (1, 8)$ .

The complementary anomaly index of the 3-family SM (without 3  $\bar{\nu}_R$ ) is

$$(0 \mod 27, 0 \mod 3),$$
 (A50)

thus anomaly-free.

10. Spin 
$$\times_{\mathbb{Z}_2^F} \mathbb{Z}_{18}^F$$
 symmetry

Consider Spin  $\times_{\mathbb{Z}_2^{\mathrm{F}}} \mathbb{Z}_{18}^{\mathrm{F}}$  symmetry ( $\subset$  Spin<sup>c</sup> symmetry), for an odd charge  $q \in \mathbb{Z}_1 8^{\mathrm{F}}$  Weyl fermion theory in 4d, we like to match its 4d anomaly to a 5d bordism group index  $\Omega_5^{\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{18}^F} = \Omega_5^{\mathrm{Spin} \times_{\mathbb{Z}_9}} = \mathbb{Z}_{27} \times \mathbb{Z}_3$  or precisely a 5d cobordism group index  $\mathrm{Hom}(\Omega_5^{\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{18}}, \mathrm{U}(1)) = \mathrm{Hom}(\Omega_5^{\mathrm{Spin} \times \mathbb{Z}_9}, \mathrm{U}(1)) = \mathbb{Z}_{27} \times \mathbb{Z}_3$ .

To derive the anomaly index formula  $(\nu_1(q), \nu_2(q)) \in \mathbb{Z}_{27} \times \mathbb{Z}_3$ , we follow the previous arguments with similar spectralists.

constraints.

For an odd charge  $q \in \mathbb{Z}_{18}$  Weyl fermion, We arrive at the map to the anomaly index

$$\begin{array}{l} q=1,3,5,7,9,11,13,15,17 \\ \mapsto (\mathbf{v}_1,\mathbf{v}_2)=(1,0),(0,2),(17,1),(19,1),(0,0),(8,2),(10,2),(0,1),(26,0). \end{array} \tag{A51}$$

There are 3 missing right-handed neutrinos not found in the Standard Model, each such  $\bar{\nu}_R$  has  $\mathbb{Z}^{\mathrm{F}}_{2N_cN_f=18}$  charge -3 = 15, thus for 3 families of such  $\bar{\nu}_R$ , we have

$$3(0 \mod 27, 1 \mod 3) = (0 \mod 27, 0 \mod 3).$$
 (A52)

There is an ambiguity of the first mod 27 index, where the ambiguity comes  $9k \mod 27$  for  $k \in \mathbb{Z}$  thus at 0 mod 9; but the outcome choice would not affect our  $N_f = 3$  anomaly, because  $9N_f = 27 = 0 \mod 27$ . The complementary anomaly index of the 3-family SM (without 3  $\bar{\nu}_R$ ) is

$$(0 \mod 27, 0 \mod 3),$$
 (A53)

thus anomaly-free.

# Appendix B: $\mathbb{Z}_9$ class topological invariants of $\mathrm{Spin} \times \mathbb{Z}_3$ and the group extension $1 \to \mathbb{Z}_3 \to \mathbb{Z}_9 \to \mathbb{Z}_3 \to 1$

Here we demonstrate the bordism  $\Omega_5^{\mathrm{Spin} \times \mathbb{Z}_3} = \mathbb{Z}_9$  or the cobordism  $\mathrm{Hom}(\Omega_5^{\mathrm{Spin} \times \mathbb{Z}_3}, \mathrm{U}(1)) = \mathbb{Z}_9$  class topological invariants of  $\mathrm{Spin} \times \mathbb{Z}_3$  symmetry; such that the magical  $\mathbb{Z}_9$  class forms a group extension:

$$1 \to \mathbb{Z}_3 \to \mathbb{Z}_9 \to \mathbb{Z}_3 \to 1, \tag{B1}$$

or more schematically

$$1 \to (\mathbb{Z}_3)_{\text{group cohomology}} \to (\mathbb{Z}_9)_{\text{full cobordism class}} \to (\mathbb{Z}_3)_{\text{beyond group cohomology}} \to 1.$$
 (B2)

The normal  $\mathbb{Z}_3$  is from the group cohomology bosonic SPT class in  $H^5(B\mathbb{Z}_3, U(1)) = \mathbb{Z}_3$  tensor product with the free fermion SPT class. The quotient  $\mathbb{Z}_9/\mathbb{Z}_3 = \mathbb{Z}_3$  is from the beyond group cohomology SPT class.

To demonstrate the above statement, we shall embed  $\operatorname{Spin} \times \mathbb{Z}_3 \subset \operatorname{Spin} \times \operatorname{U}(1)$  first, then change the U(1) gauge field to a  $\mathbb{Z}_3$  gauge field.

We take the Spin  $\times$  U(1) structure's 6d anomaly polynomial  $I_6$  above in eq. (A6) with q=1 Weyl fermion gives the following 6d anomaly polynomial's invertible field theory

$$\exp(ik\theta \int_{M^6} \frac{c_1^3}{6} - \frac{c_1 p_1}{24})$$
 (B3)

with  $k \in \mathbb{Z}$ , while  $\theta \in [0, 2\pi)$ , and  $\int_{M^6} \frac{c_1^3}{6} - \frac{c_1 p_1}{24} \in \mathbb{Z}$  on a closed  $M^6$  of the given structure. The 5d manifold at the interface of jumping  $\theta = 0$  to  $\theta = 2\pi$  gives a 5d invertible field theory (as a 5d SPTs):

$$\exp(ik \int_{M^5} A \frac{c_1^2}{6} - A \frac{p_1}{24}). \tag{B4}$$

Now we redefine the U(1) gauge field A as a  $\mathbb{Z}_3$  gauge field  $\tilde{A} \in \mathbb{Z}_3$  with the following replacement:

$$A \mapsto \frac{2\pi}{3}\tilde{A}.$$

$$c_1 = \frac{\mathrm{d}A}{2\pi} \mapsto \frac{\mathrm{d}\tilde{A}}{3} \equiv \beta_{(3,3)}\tilde{A}.$$
(B5)

The  $\beta_{(n,m)}: H^*(-,\mathbb{Z}_m) \mapsto H^{*+1}(-,\mathbb{Z}_n)$  is the Bockstein homomorphism associated with the extension  $\mathbb{Z}_n \stackrel{\cdot m}{\to} \mathbb{Z}_{nm} \to \mathbb{Z}_m$ . Thus we get the 5d topological invariant of the Spin  $\times \mathbb{Z}_3$  as:

$$\exp\left(i2\pi k \int_{M^5} \left(\frac{1}{18}\tilde{A}(\beta_{(3,3)}\tilde{A})(\beta_{(3,3)}\tilde{A}) - \frac{1}{3\cdot 24}\tilde{A}p_1\right)\right). \tag{B6}$$

Now when k = 3, we get

$$\exp\left(i2\pi \int_{M^5} \left(\frac{1}{6}\tilde{A}(\beta_{(3,3)}\tilde{A})(\beta_{(3,3)}\tilde{A}) - \frac{1}{24}\tilde{A}p_1\right)\right) = \exp\left(i2\pi \int_{M^5} \left(\frac{1}{6}\tilde{A}(\beta_{(3,3)}\tilde{A})(\beta_{(3,3)}\tilde{A})\right)\right),\tag{B7}$$

while the equality relies on a Theorem of Tomonaga [52] such that  $\int_{M^5} \tilde{A} p_1 = 0 \mod 24$  on the Spin  $\times \mathbb{Z}_3$  manifold. Next, we show that this k = 3 class indeed is the generator of the group cohomology bosonic SPT class in  $H^5(\mathbb{BZ}_3, \mathbb{U}(1)) = \mathbb{Z}_3$  tensor product with the free fermion SPT class, such that the 3 layers of eq. (B7) becomes a trivial class. Namely, when  $k = 3 \cdot 3 = 9$ , we get

$$\exp\left(i\pi \int_{M^5} \tilde{A}(\beta_{(3,3)}\tilde{A})(\beta_{(3,3)}\tilde{A})\right) = \exp\left(i\pi \int_{M^5} \tilde{A}(4\beta_{(3,3)}\tilde{A})(\beta_{(3,3)}\tilde{A})\right) = 1,$$
(B8)

which can only read the  $0,1 \mod 2$  in  $\int_{M^5} (\tilde{A}(\beta_{(3,3)}\tilde{A})(\beta_{(3,3)}\tilde{A}) \mod 2$ . Now, we use the conflict between the mod 2 index built out of the mod 3 gauge field to argue that this is a trivial class of SPTs. Note that  $3(\beta_{(3,3)}\tilde{A})=0 \mod 3$  and  $(\beta_{(3,3)}\tilde{A})=4(\beta_{(3,3)}\tilde{A})=\mod 3$  used in the first equality for the  $\mathbb{Z}_3$  valued gauge field, while the second equality uses  $2(2\pi\mathbb{Z})=0 \mod 2\pi$  thus eq. (B8) is a trivial class SPTs.

We may simply consider k=3 case as  $\exp\left(i2\pi\int_{M^5}(\frac{1}{3}\tilde{A}(\beta_{(3,3)}\tilde{A})(\beta_{(3,3)}\tilde{A}))\right)$  tensor product with a trivial gapped fermionic state. Then the k=9 also generates a +1 as a trivial SPTs, This concludes the proof of the group extension in eq. (B2) as the  $\mathbb{Z}_9$  classification of 5d Spin  $\times \mathbb{Z}_3$  topological invariants thus also 5d fermionic SPTs.

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