## ON THE RELATIONSHIP BETWEEN TWO SINC-COLLOCATION METHODS FOR VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND AND THEIR FURTHER IMPROVEMENT\*

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Abstract. Two different Sinc-collocation methods for Volterra integral equations of the second kind have been independently proposed by Stenger and Rashidinia–Zarebnia. However, their relationship remains unexplored. This study theoretically examines the solutions of these two methods, and reveals that they are not generally equivalent, despite coinciding at the collocation points. Strictly speaking, Stenger's method assumes that the kernel of the integral is a function of a single variable, but this study theoretically justifies the use of his method in general cases, i.e., the kernel is a function of two variables. Then, this study rigorously proves that both methods can attain the same, root-exponential convergence. In addition to the contribution, this study improves Stenger's method to attain significantly higher, almost exponential convergence. Numerical examples supporting the theoretical results are also provided.

Key words. Sinc numerical method, tanh transformation, double-exponential transformation

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1. Introduction and summary. This paper is concerned with numerical solutions via Sinc numerical methods [14, 15] for Volterra integral equations of the second kind of the form

(1.1) 
$$u(t) - \int_{a}^{t} k(t, s)u(s) ds = g(t), \quad a \le t \le b.$$

Here, k(t, s) and g(t) are given continuous functions, and u(t) is the solution to be determined. The equations are often expressed symbolically as  $(\mathcal{I} - \mathcal{V})u = g$  by introducing Volterra integral operator  $\mathcal{V}: C([a, b]) \to C([a, b])$  as

$$\mathcal{V}[f](t) = \int_{a}^{t} k(t, s) f(s) \, \mathrm{d}s.$$

One of powerful tools in the Sinc numerical methods, especially for the target equations (1.1), is the Sinc indefinite integration [2, 4]. This provides an approximation formula for indefinite integral in the following form

$$\int_{a}^{t} F(s) ds \approx \sum_{j=-N}^{N} F(s_{j}) \omega_{j}(t),$$

where the weight  $\omega_j$  is a function depending on t, whereas the sampling point  $s_j$  is fixed, independent of t, even though the interval of the integral (a, t) depends on t. This is quite a unique feature, because not only  $\omega_j$  but also  $s_j$  should depend on t if a standard quadrature rule is used for approximating the indefinite integral. As another beautiful feature, the Sinc indefinite integration can attain exponential order of convergence, which significantly exceeds polynomial order of convergence. Leveraging

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these features, Muhammad et al. [5] considered the Sinc indefinite integration of  $\mathcal{V}$ , say  $\mathcal{V}_N$ , and numerical solution  $u_N$  that satisfies the following equation

$$(\mathcal{I} - \mathcal{V}_N)u_N = g.$$

The procedure to obtain the solution  $u_N$  is called the Sinc-Nyström method, which is described in section 3. Theoretical analysis has established that the method achieves a convergence rate of  $O(\exp(-\sqrt{\pi dN}))$  [10], where d indicates the size of the domain in which the solution u is analytic.

Another numerical solution for (1.1) via Sinc numerical methods was developed by Rashidinia and Zarebnia [12]. They derived their method following quite a standard collocation procedure based on the Sinc approximation (a function approximation formula), which also attains an exponential order of convergence. Let  $\mathcal{P}_N f$  denote the Sinc approximation of f. Then, as shown in this paper, the equation to be solved is written symbolically as

$$(\mathcal{I} - \mathcal{P}_N \mathcal{V}_N) w_N = \mathcal{P}_N g.$$

The procedure to obtain the solution  $w_N$  is called the Sinc-collocation method, which is described in section 4. Although a theoretical error analysis of the method was given [17], its convergence was not strictly proved.

Yet another numerical solution for (1.1) via Sinc numerical methods was developed by Stenger [13]. Although his method was introduced more than a decade before the above methods, it has received considerably less attention. This may be because the target equation of his method is not exactly (1.1). The interest of his method is in initial value problems

$$u'(t) = \tilde{k}(t)u(t) + \tilde{g}(t),$$
  
$$u(a) = u_a,$$

which can be reduced to a form of Volterra integral equations of the second kind as

(1.2) 
$$u(t) - \int_a^t \tilde{k}(s)u(s) \,\mathrm{d}s = g(t),$$

where  $g(t) = u_a + \int_a^t \tilde{g}(s) \, \mathrm{d}s$ . Because the kernel here  $(\tilde{k})$  is a function of a single variable, Stenger's method does not appear to cover the general case as (1.1). However, aside from theoretical justification, it is relatively evident that his method remains implementable even when the kernel is a function of two variables. Its numerical solution, say  $v_N$ , is determined in the following two steps: (i) obtain the Sinc-Nyström solution  $u_N$ , and (ii) apply the Sinc approximation to  $u_N$ . The step (i) implies that Stenger's method is based on the Sinc-Nyström method, but  $v_N$  is not equal to  $u_N$  because of the step (ii). The detailed procedure is described in section 4 (strictly speaking, it is the first time that the explicit procedure for the general case (1.1) is presented). Its convergence has been stated [13] assuming that the kernel is a function of a single variable.

As seen above, three numerical methods have been proposed based on the Sinc numerical methods: Sinc-Nyström method, Sinc-collocation method, and Stenger's method. Therefore, a question may naturally arise: what is the difference (or similarity), and which method is the best? The first objective of this study is to investigate this question from both theoretical and practical perspectives. This study first reveals that Stenger's method can be regarded as another Sinc-collocation method. Then, it

is shown that Stenger's method and Rashidinia–Zarebnia's method coincide at the collocation points, but they are not generally equivalent. Furthermore, this study shows that the convergence rate of the two Sinc-collocation methods is exactly the same:  $O(\sqrt{N}\exp(-\sqrt{\pi d\alpha N}))$ , where  $\alpha$  is the order of Hölder continuous with  $0 < \alpha \le 1$ . From an implementation perspective, Stenger's method is preferable, because it is simpler and easier to implement than the method by Rashidinia and Zarebnia.

Thus, we only have to compare two methods: Sinc-Nyström method and Stenger's Sinc-collocation method. Even when  $\alpha=1$ , the convergence rate of Stenger's method is slightly lower than that of the Sinc-Nyström method. However, numerical experiments indicate that the Sinc-Nyström method requires much computation time to obtain the same accuracy as Stenger's method. This is primarily because the basis functions of the Sinc-Nyström method include the sine integral (a special function), which requires a high computational cost. Based on this finding, we conclude that Stenger's method is preferable among the three methods described above.

The second objective of this study is to improve Stenger's method. In the aforementioned three methods, the tanh transformation

(1.3) 
$$t = \psi^{\text{SE}}(x) = \frac{b-a}{2} \tanh\left(\frac{x}{2}\right) + \frac{b+a}{2}$$

is employed in common to map  $\mathbb{R}$  onto the target interval (a, b). This is because Sinc numerical methods are originally defined over the entire real axis  $\mathbb{R}$ . Therefore, for the finite interval, a variable transformation such as (1.3) is required. This study aims to improve Stenger's method by replacing the tanh transformation with

(1.4) 
$$t = \psi^{\text{DE}}(x) = \frac{b-a}{2} \tanh\left(\frac{\pi}{2}\sinh x\right) + \frac{b+a}{2},$$

which is called the double-exponential (DE) transformation. The convergence rates of various methods via Sinc numerical methods have been improved by replacing the tanh transformation with the DE transformation [3, 15]. Specifically, the convergence rate of the Sinc-Nyström method was enhanced to  $O(\log(2dN)\exp(-\pi dN/\log(2dN))/N)$  through the replacement [10]. On the basis of the observation, this study develops a new Sinc-collocation method combined with the DE transformation. Furthermore, this study performs theoretical analysis of the proposed method and shows that its convergence rate is  $O(\exp(-\pi dN/\log(2dN/\alpha)))$ , which significantly exceeds that of Stenger's method. Although the rate is slightly lower than that of the Sinc-Nyström method combined with the DE transformation, numerical experiments indicate that the Sinc-Nyström method requires much computation time to obtain the same accuracy as the proposed method. This is similarly observed when comparing the Sinc-Nyström and Sinc-collocation methods combined with the tanh transformation.

The remainder of this paper is organized as follows. In section 2, as a preliminary, convergence theorems of the Sinc approximation and the Sinc indefinite integration are described. In section 3, the Sinc-Nyström methods developed by Muhammad et al. [5] are described, and their convergence theorems are stated. In section 4, the Sinc-collocation methods developed by Stenger [13] and Rashidinia–Zarebnia [12] are described. Subsequently, new theoretical results from this study are stated: (i) the two numerical solutions coincide at the collocation points but are not generally equivalent, and (ii) the two methods attain the same convergence rate  $O(\sqrt{N} \exp(-\sqrt{\pi d\alpha N}))$ . In section 5, a new Sinc-collocation method combined with the DE transformation is developed. Subsequently, its convergence theorem is stated claiming that the convergence rate is  $O(\exp(-\pi dN/\log(2dN/\alpha)))$ . In section 6, numerical experiments

are presented, where the DE-Sinc-collocation method demonstrates the best performance. In section 7, proofs for the new theorems presented in section 4 are provided. In section 8, proofs for the new theorems presented in section 5 are provided.

- 2. Preliminaries. This section summarizes the Sinc approximation and Sinc indefinite integration and their application with the aid of the tanh or DE transformation.
- **2.1.** Sinc approximation and Sinc indefinite integration. The Sinc numerical methods are generic names of numerical methods based on the *Sinc approximation*, expressed as

(2.1) 
$$F(x) \approx \sum_{j=-N}^{N} F(jh)S(j,h)(x), \quad x \in \mathbb{R},$$

where h is a mesh size appropriately chosen depending on N, and the basis function S(j,h) is the so-called Sinc function defined by

$$S(j,h)(x) = \begin{cases} \frac{\sin(\pi(x-jh)/h)}{\pi(x-jh)/h} & (x \neq jh), \\ 1 & (x = jh). \end{cases}$$

Integrating both sides of (2.1), we obtain an approximation formula called the *Sinc* indefinite integration as

(2.2)

$$\int_{-\infty}^{\xi} F(x) dx \approx \sum_{j=-N}^{N} F(jh) \int_{-\infty}^{\xi} S(j,h)(x) dx = \sum_{j=-N}^{N} F(jh)J(j,h)(\xi), \quad \xi \in \mathbb{R},$$

where J(j, h) is defined by

$$J(j,h)(x) = h\left\{\frac{1}{2} + \frac{1}{\pi}\operatorname{Si}\left[\frac{\pi(x-jh)}{h}\right]\right\},$$

where Si(x) is the sine integral defined by  $Si(x) = \int_0^x \{(\sin t)/t\} dt$ .

**2.2. SE-Sinc approximation and SE-Sinc indefinite integration.** To use the approximation formulas (2.1) and (2.2), the function F(x) must be defined over the entire real line  $\mathbb{R}$ . When the function f(t) is defined over the finite interval (a,b), a variable transformation is required to map  $\mathbb{R}$  onto (a,b). For the purpose, the tanh transformation  $t = \psi^{\text{SE}}(x)$  defined in (1.3) is widely employed. The change of variable  $(t = \psi^{\text{SE}}(x))$  enables us to apply (2.1) by setting  $F(x) = f(\psi^{\text{SE}}(x))$ . Introducing  $t_j^{\text{SE}} = \psi^{\text{SE}}(jh)$  and  $\phi^{\text{SE}}(t) = \{\psi^{\text{SE}}\}^{-1}(t)$ , we express the obtained formula as

(2.3) 
$$f(t) \approx \sum_{j=-N}^{N} f(t_j^{\text{SE}}) S(j,h) (\phi^{\text{SE}}(t)), \quad t \in (a,b).$$

This approximation is referred to as the SE-Sinc approximation in this paper. Similarly, applying  $s = \psi^{\text{SE}}(x)$  and setting  $F(x) = f(\psi^{\text{SE}}(x))$  in (2.2), we obtain

(2.4) 
$$\int_{a}^{t} f(s) ds = \int_{-\infty}^{\phi^{\text{SE}}(t)} f(\psi^{\text{SE}}(x)) \{\psi^{\text{SE}}\}'(x) dx$$

$$\approx \sum_{j=-N}^{N} f(t_{j}^{\text{SE}}) \{\psi^{\text{SE}}\}'(jh) J(j,h) (\phi^{\text{SE}}(t)), \quad t \in (a,b),$$

which is referred to as the SE-Sinc indefinite integration in this paper. If  $F(x) = f(\psi(x))$  is analytic on the strip complex domain

$$\mathcal{D}_d = \{ \zeta \in \mathbb{C} : |\operatorname{Im} \zeta| < d \}$$

for a positive constant d, then both approximations performs highly accurately. In other words, f(t) should be analytic on the transformed domain

$$\psi^{\text{SE}}(\mathcal{D}_d) = \{ z = \psi^{\text{SE}}(\zeta) : \zeta \in \mathcal{D}_d \},$$

which is a simply-connected domain. Convergence theorems of the two approximations were provided as follows.

THEOREM 2.1 (Stenger [13, Theorem 4.2.5]). Assume that f is analytic on  $\psi^{\text{SE}}(\mathcal{D}_d)$  for d with  $0 < d < \pi$ , and there exists constants K and  $\alpha$  such that

$$(2.5) |f(z)| \le K|z - a|^{\alpha}|b - z|^{\alpha}$$

holds for all  $z \in \psi^{\text{SE}}(\mathcal{D}_d)$ . Let N be a positive integer, and let h be selected by the formula

$$(2.6) h = \sqrt{\frac{\pi d}{\alpha N}}.$$

Then, there exists a constant C independent of N such that

$$\max_{t \in [a,b]} \left| f(t) - \sum_{j=-N}^{N} f(t_j^{\text{SE}}) S(j,h) (\phi^{\text{SE}}(t)) \right| \le C\sqrt{N} e^{-\sqrt{\pi d\alpha N}}.$$

THEOREM 2.2 (Okayama et al. [9, Theorem 2.9]). Assume that f is analytic on  $\psi^{\text{SE}}(\mathcal{D}_d)$  for d with  $0 < d < \pi$ , and there exists constants K and  $\alpha$  such that

$$(2.7) |f(z)| \le K|z - a|^{\alpha - 1}|b - z|^{\alpha - 1}$$

holds for all  $z \in \psi^{\text{SE}}(\mathcal{D}_d)$ . Let N be a positive integer, and let h be selected by the formula (2.6). Then, there exists a constant C independent of N such that

$$\max_{t \in [a,b]} \left| \int_a^t f(s) \, \mathrm{d}s - \sum_{j=-N}^N f(t_j^{\text{\tiny SE}}) \{\psi^{\text{\tiny SE}}\}'(jh) J(j,h) (\phi^{\text{\tiny SE}}(t)) \right| \le C \, \mathrm{e}^{-\sqrt{\pi d \alpha N}} \, .$$

**2.3. DE-Sinc approximation and DE-Sinc indefinite integration.** The SE-Sinc approximation (2.3) and SE-Sinc indefinite integration (2.4) employ the tanh transformation (1.3) to map  $\mathbb{R}$  onto the finite interval (a,b). The DE transformation (1.4) also plays the same role, and allows for the replacement of  $\psi^{\text{SE}}$  with  $\psi^{\text{DE}}$  in both formulas. On the basis of this idea, introducing  $t_j^{\text{DE}} = \psi^{\text{DE}}(jh)$  and  $\phi^{\text{DE}}(t) = \{\psi^{\text{DE}}\}^{-1}(t)$ , we can derive the following formulas

(2.8) 
$$f(t) \approx \sum_{j=-N}^{N} f(t_{j}^{\text{DE}}) S(j,h) (\phi^{\text{DE}}(t)), \quad t \in (a,b),$$

(2.9) 
$$\int_a^t f(s) \, \mathrm{d}s \approx \sum_{j=-N}^N f(t_j^{\text{DE}}) \{ \psi^{\text{DE}} \}'(jh) J(j,h) (\phi^{\text{DE}}(t)), \quad t \in (a,b),$$

which are referred to as the DE-Sinc approximation and DE-Sinc indefinite integration, respectively. For the formulas (2.8) and (2.9), f(t) should be analytic on the transformed domain

$$\psi^{\text{DE}}(\mathcal{D}_d) = \{ z = \psi^{\text{DE}}(\zeta) : \zeta \in \mathcal{D}_d \},$$

which forms a Riemann surface. Convergence theorems of the two approximations were provided as follows.

THEOREM 2.3 (Tanaka et al. [16, Theorem 3.1]). Assume that f is analytic on  $\psi^{\text{DE}}(\mathcal{D}_d)$  for d with  $0 < d < \pi/2$ , and there exists constants K and  $\alpha$  such that (2.5) holds for all  $z \in \psi^{\text{DE}}(\mathcal{D}_d)$ . Let N be a positive integer, and let h be selected by the formula

$$(2.10) h = \frac{\log(2dN/\alpha)}{N}.$$

Then, there exists a constant C independent of N such that

$$\max_{t \in [a,b]} \left| f(t) - \sum_{j=-N}^N f(t_j^{\mathrm{DE}}) S(j,h) (\phi^{\mathrm{DE}}(t)) \right| \le C \, \mathrm{e}^{-\pi dN/\log(2dN/\alpha)} \,.$$

THEOREM 2.4 (Okayama et al. [9, Theorem 2.16]). Assume that f is analytic on  $\psi^{\text{DE}}(\mathcal{D}_d)$  for d with  $0 < d < \pi/2$ , and there exists constants K and  $\alpha$  such that (2.7) holds for all  $z \in \psi^{\text{DE}}(\mathcal{D}_d)$ . Let N be a positive integer, and let h be selected by the formula (2.10). Then, there exists a constant C independent of N such that

$$\max_{t \in [a,b]} \left| \int_a^t f(s) \, \mathrm{d}s - \sum_{j=-N}^N f(t_j^{\mathrm{DE}}) \{\psi^{\mathrm{DE}}\}'(jh) J(j,h) (\phi^{\mathrm{DE}}(t)) \right|$$

$$\leq C \frac{\log(2dN/\alpha)}{N} \, \mathrm{e}^{-\pi dN/\log(2dN/\alpha)} \, .$$

**2.4.** Generalized SE/DE-Sinc approximation. In the convergence theorems of the SE/DE-Sinc approximation, the condition (2.5) is assumed. This condition requires f(t) to be zero at the endpoints t = a and t = b, which seems an impractical condition. To address this issue, using auxiliary functions

$$\omega_a(t) = \frac{b-t}{b-a}, \quad \omega_b(t) = \frac{t-a}{b-a},$$

and setting  $\tilde{f}_N^{\text{SE}}(t) = f(t) - f(t_{-N}^{\text{SE}})\omega_a(t) - f(t_N^{\text{SE}})\omega_b(t)$ , Stenger [13, 14] proposed to apply the SE-Sinc approximation to  $\tilde{f}_N^{\text{SE}}$ . If we define an approximation operator  $\mathcal{P}_N^{\text{SE}}: C([a,b]) \to C([a,b])$  as

(2.11) 
$$(\mathcal{P}_{N}^{\text{SE}}f)(t) = f(t_{-N}^{\text{SE}})\omega_{a}(t) + f(t_{N}^{\text{SE}})\omega_{b}(t) + \sum_{j=-N}^{N} \tilde{f}_{N}^{\text{SE}}(t_{j}^{\text{SE}})S(j,h)(\phi^{\text{SE}}(t)),$$

then the approximation is expressed as  $f \approx \mathcal{P}_N^{\text{SE}} f$ . This approximation is referred to as the generalized SE-Sinc approximation in this paper. Notably,  $\mathcal{P}_N^{\text{SE}}$  satisfies the interpolation property, that is,  $f(t_i^{\text{SE}}) = (\mathcal{P}_N^{\text{SE}} f)(t_i^{\text{SE}})$   $(i = -N, \ldots, N)$ .

Similarly, setting  $\tilde{f}_N^{\text{DE}}(t) = f(t) - f(t_{-N}^{\text{DE}})\omega_a(t) - f(t_N^{\text{DE}})\omega_b(t)$ , we may apply the DE-Sinc approximation to  $\tilde{f}_N^{\text{DE}}$ . If we define an approximation operator  $\mathcal{P}_N^{\text{DE}}: C([a,b]) \to C([a,b])$  as

$$(2.12) \qquad (\mathcal{P}_{N}^{\text{DE}}f)(t) = f(t_{-N}^{\text{DE}})\omega_{a}(t) + f(t_{N}^{\text{DE}})\omega_{b}(t) + \sum_{j=-N}^{N} \tilde{f}_{N}^{\text{SE}}(t_{j}^{\text{DE}})S(j,h)(\phi^{\text{DE}}(t)),$$

then the approximation is expressed as  $f \approx \mathcal{P}_N^{\text{DE}} f$ . This approximation is referred to as the generalized DE-Sinc approximation in this paper.  $\mathcal{P}_N^{\text{DE}}$  also satisfies the interpolation property, that is,  $f(t_i^{\text{DE}}) = (\mathcal{P}_N^{\text{DE}} f)(t_i^{\text{DE}})$  (i = -N, ..., N).

The convergence theorems of the two approximations are described using the following function spaces.

DEFINITION 2.5. Let  $\mathscr{D}$  be a bounded and simply-connected domain (or Riemann surface). Then,  $\mathbf{H}^{\infty}(\mathscr{D})$  denotes the family of functions f analytic on  $\mathscr{D}$  such that the norm  $||f||_{\mathbf{H}^{\infty}(\mathscr{D})}$  is finite, where

$$||f||_{\mathbf{H}^{\infty}(\mathscr{D})} = \sup_{z \in \mathscr{D}} |f(z)|.$$

DEFINITION 2.6. Let  $\alpha$  be a positive constant, and let  $\mathscr{D}$  be a bounded and simply-connected domain (or Riemann surface) that satisfies  $(a,b) \subset \mathscr{D}$ . Then,  $\mathbf{M}_{\alpha}(\mathscr{D})$  denotes the family of functions  $f \in \mathbf{H}^{\infty}(\mathscr{D})$  for which there exists a constant L such that for all  $z \in \mathscr{D}$ ,

$$|f(z) - f(a)| \le L|z - a|^{\alpha},$$
  

$$|f(b) - f(z)| \le L|b - z|^{\alpha}.$$

This function space  $\mathbf{M}_{\alpha}(\mathscr{D})$  only requires the Hölder continuity at the endpoints instead of the zero-boundary condition by (2.5). Convergence theorems of the two approximations were provided as follows. Here,  $\|\cdot\|_{C([a,b])}$  denotes the usual uniform norm over [a,b].

THEOREM 2.7 (Okayama [6, Theorem 3]). Assume that  $f \in \mathbf{M}_{\alpha}(\psi^{\text{SE}}(\mathscr{D}_d))$  for d with  $0 < d < \pi$ . Let N be a positive integer, and let h be selected by the formula (2.6). Then, there exists a constant C independent of N such that

$$||f - \mathcal{P}_N^{\text{SE}} f||_{C([a,b])} \le C\sqrt{N} e^{-\sqrt{\pi d\alpha N}}.$$

THEOREM 2.8 (Okayama [6, Theorem 6]). Assume that  $f \in \mathbf{M}_{\alpha}(\psi^{\mathrm{DE}}(\mathcal{D}_{d}))$  for d with  $0 < d < \pi/2$ . Let N be a positive integer, and let h be selected by the formula (2.10). Then, there exists a constant C independent of N such that

$$||f - \mathcal{P}_N^{\text{DE}} f||_{C([a,b])} \le C e^{-\pi dN/\log(2dN/\alpha)}$$

- 3. Sinc-Nyström methods. This section describes the Sinc-Nyström methods developed by Muhammad et al. [5]. The first method employs the tanh transformation (1.3) as a variable transformation, while the second method employs the DE transformation (1.4).
- **3.1. SE-Sinc-Nyström method.** By applying the SE-Sinc indefinite integration (2.4) to the integral in the given equation (1.1), we obtain an approximated equation as

(3.1) 
$$u_N^{\text{SE}}(t) = g(t) + \sum_{j=-N}^{N} k(t, t_j^{\text{SE}}) u_N^{\text{SE}}(t_j^{\text{SE}}) \{\psi^{\text{SE}}\}'(jh) J(j, h) (\phi^{\text{SE}}(t)).$$

The approximated solution  $u_N^{\text{SE}}$  is determined once the unknown coefficients  $u_N^{\text{SE}}(t_j^{\text{SE}})$  on the right-hand side are obtained. To this end, 2N+1 sampling points are set at  $t=t_i^{\text{SE}}$   $(i=-N,-N+1,\ldots,N)$  in (3.1) as (3.2)

$$u_N^{\text{SE}}(t_i^{\text{SE}}) = g(t) + \sum_{j=-N}^{N} k(t_i^{\text{SE}}, t_j^{\text{SE}}) u_N^{\text{SE}}(t_j^{\text{SE}}) \{\psi^{\text{SE}}\}'(jh) J(j, h)(ih), \quad i = -N, \dots, N,$$

which is a system of linear equations. This system is expressed in a matrix-vector form as follows. Let n = 2N + 1, let  $I_n$  be an identity matrix of order n, and let  $V_n^{\text{SE}}$  be  $n \times n$  matrix whose (i,j)-th element is

$$(V_n^{\text{SE}})_{ij} = k(t_i^{\text{SE}}, t_j^{\text{SE}}) \{\psi^{\text{SE}}\}'(jh) h \delta_{i-j}^{(-1)}, \quad i = -N, \dots, N, \quad j = -N, \dots, N,$$

where  $\delta_k^{(-1)} = (1/2) + \sigma_k$ , where  $\sigma_k$  is defined by

$$\sigma_k = \int_0^k \frac{\sin(\pi x)}{\pi x} dx = \frac{1}{\pi} \operatorname{Si}(\pi k).$$

Furthermore, let  $\boldsymbol{g}_n^{\mbox{\tiny SE}}$  and  $\boldsymbol{u}_n^{\mbox{\tiny SE}}$  be n-dimensional vectors defined by

$$\mathbf{g}_{n}^{\text{SE}} = [g(t_{-N}^{\text{SE}}), g(t_{-N+1}^{\text{SE}}), \dots, g(t_{N}^{\text{SE}})]^{\text{T}}, 
\mathbf{u}_{n}^{\text{SE}} = [u_{N}^{\text{SE}}(t_{-N}^{\text{SE}}), u_{N}^{\text{SE}}(t_{-N+1}^{\text{SE}}), \dots, u_{N}^{\text{SE}}(t_{N}^{\text{SE}})]^{\text{T}}.$$

Then, the system (3.2) is expressed as

$$(3.3) (I_n - V_n^{SE}) \boldsymbol{u}_n^{SE} = \boldsymbol{g}_n^{SE}.$$

By solving (3.3), we obtain the unknown coefficients  $u_n^{\text{SE}}$ , from which the approximated solution  $u_N^{\text{SE}}$  is determined by (3.1). This procedure is called the SE-Sinc-Nyström method. Its convergence theorem was provided as follows.

Theorem 3.1 (Okayama et al. [10, Theorem 3.4]). Let d be a positive constant with  $d < \pi$ . Assume that g,  $k(z,\cdot)$  and  $k(\cdot,w)$  belong to  $\mathbf{H}^{\infty}(\psi^{\text{SE}}(\mathscr{D}_d))$  for all z,  $w \in \psi^{\text{SE}}(\mathscr{D}_d)$ . Furthermore, assume that g,  $k(t,\cdot)$  and  $k(\cdot,s)$  belong to C([a,b]) for all t,  $s \in [a,b]$ . Let h be selected by the formula (2.6) with  $\alpha = 1$ . Then, there exists a positive integer  $N_0$  such that for all  $N \geq N_0$ , the coefficient matrix  $(I_n - V_n^{\text{SE}})$  is invertible. Furthermore, there exists a constant C independent of N such that for all  $N \geq N_0$ ,

$$||u - u_N^{\text{SE}}||_{C([a,b])} \le C e^{-\sqrt{\pi dN}}$$
.

**3.2. DE-Sinc-Nyström method.** Muhammad et al. [5] also considered another method by replacing  $\psi^{\text{SE}}$  with  $\psi^{\text{DE}}$  in the SE-Sinc-Nyström method. Applying the DE-Sinc indefinite integration (2.9) to the integral in the given equation (1.1), we obtain an approximated equation as

(3.4) 
$$u_N^{\text{DE}}(t) = g(t) + \sum_{j=-N}^{N} k(t, t_j^{\text{DE}}) u_N^{\text{DE}}(t_j^{\text{DE}}) \{\psi^{\text{DE}}\}'(jh) J(j, h) (\phi^{\text{DE}}(t)).$$

The approximated solution  $u_N^{\text{DE}}$  is determined once the unknown coefficients  $u_N^{\text{DE}}(t_j^{\text{DE}})$  on the right-hand side are obtained. To this end, 2N+1 sampling points are set at  $t=t_i^{\text{DE}}$   $(i=-N,-N+1,\ldots,N)$  in (3.4). This leads a system of linear equations

$$(3.5) (I_n - V_n^{\text{DE}}) \boldsymbol{u}_n^{\text{DE}} = \boldsymbol{g}_n^{\text{DE}},$$

where  $V_n^{\text{DE}}$  be  $n \times n$  matrix whose (i, j)-th element is

$$(V_n^{\rm DE})_{ij} = k(t_i^{\rm DE}, t_j^{\rm DE}) \{\psi^{\rm DE}\}'(jh) h \delta_{i-j}^{(-1)}, \quad i = -N, \dots, N, \quad j = -N, \dots, N,$$

and  $\boldsymbol{g}_n^{\text{\tiny DE}}$  and  $\boldsymbol{u}_n^{\text{\tiny DE}}$  be n-dimensional vectors defined by

$$\mathbf{g}_{n}^{\text{DE}} = [g(t_{-N}^{\text{DE}}), g(t_{-N+1}^{\text{DE}}), \dots, g(t_{N}^{\text{DE}})]^{\text{T}}, \mathbf{u}_{n}^{\text{DE}} = [u_{N}^{\text{DE}}(t_{-N}^{\text{DE}}), u_{N}^{\text{DE}}(t_{-N+1}^{\text{DE}}), \dots, u_{N}^{\text{DE}}(t_{N}^{\text{DE}})]^{\text{T}}.$$

By solving (3.5), we obtain the unknown coefficients  $\boldsymbol{u}_n^{\text{DE}}$ , from which the approximated solution  $\boldsymbol{u}_N^{\text{DE}}$  is determined by (3.4). This procedure is called the DE-Sinc-Nyström method. Its convergence theorem was provided as follows.

Theorem 3.2 (Okayama et al. [10, Theorem 3.5]). Let d be a positive constant with  $d < \pi/2$ . Assume that g,  $k(z, \cdot)$  and  $k(\cdot, w)$  belong to  $\mathbf{H}^{\infty}(\psi^{\mathrm{DE}}(\mathscr{D}_d))$  for all z,  $w \in \psi^{\mathrm{DE}}(\mathscr{D}_d)$ . Furthermore, assume that g,  $k(t, \cdot)$  and  $k(\cdot, s)$  belong to C([a, b]) for all t,  $s \in [a, b]$ . Let h be selected by the formula (2.10) with  $\alpha = 1$ . Then, there exists a positive integer  $N_0$  such that for all  $N \geq N_0$ , the coefficient matrix  $(I_n - V_n^{\mathrm{DE}})$  is invertible. Furthermore, there exists a constant C independent of N such that for all  $N \geq N_0$ ,

$$||u - u_N^{\text{DE}}||_{C([a,b])} \le C \frac{\log(2dN)}{N} e^{-\pi dN/\log(2dN)}$$
.

- 4. Existing Sinc-collocation methods. This section describes two different Sinc-collocation methods developed by Stenger [13] and Rashidinia and Zarebnia [12]. Both methods employ the tanh transformation (1.3) as a variable transformation, but their procedures are not identical.
- **4.1. Sinc-collocation method by Stenger.** As explained in section 1, Stenger derived his method for (1.2), where the kernel  $\tilde{k}$  is a function of a single variable. However, his method can be easily derived for (1.1) as follows. His method is closely related to the SE-Sinc-Nyström method, which is described in the previous section. First, solve the linear system (3.3) and obtain  $\boldsymbol{u}_n^{\text{SE}}$ . Then, his approximated solution  $\boldsymbol{v}_N^{\text{SE}}$  is expressed as the generalized SE-Sinc approximation of  $\boldsymbol{u}_N^{\text{SE}}$ , i.e.,

$$\begin{aligned} (4.1) \quad & v_N^{\text{SE}}(t) = \mathcal{P}_N^{\text{SE}}[u_N^{\text{SE}}](t) \\ & = \sum_{j=-N}^{N} \left\{ u_N^{\text{SE}}(t_j^{\text{SE}}) - u_N^{\text{SE}}(t_{-N}^{\text{SE}}) \omega_a(t_j^{\text{SE}}) - u_N^{\text{SE}}(t_N^{\text{SE}}) \omega_b(t_j^{\text{SE}}) \right\} S(j,h) (\phi^{\text{SE}}(t)) \\ & + u_N^{\text{SE}}(t_{-N}^{\text{SE}}) \omega_a(t) + u_N^{\text{SE}}(t_N^{\text{SE}}) \omega_b(t), \end{aligned}$$

where  $\mathcal{P}_N^{\text{SE}}$  is defined by (2.11).

The solution  $v_N^{\text{SE}}$  is also obtained by the standard collocation procedure as follows. Set the approximate solution  $v_N^{\text{SE}}$  as (4.1), where  $u_N^{\text{SE}}(t_j^{\text{SE}})$   $(j=-N,\ldots,N)$  are regarded as unknown coefficients. Substitute  $v_N^{\text{SE}}$  into the given equation (1.1), with approximating the Volterra integral operator  $\mathcal V$  by  $\mathcal V_N^{\text{SE}}$ , where

(4.2) 
$$\mathcal{V}_{N}^{\text{SE}}[f](t) = \sum_{j=-N}^{N} k(t, t_{j}^{\text{SE}}) f(t_{j}^{\text{SE}}) \{\psi^{\text{SE}}\}'(jh) J(j, h) (\phi^{\text{SE}}(t)),$$

which is the SE-Sinc indefinite integration of Vf. Then, setting n = 2N + 1 sampling points at  $t = t_i^{\text{SE}}$  (i = -N, -N + 1, ..., N), we obtain the same system of linear equations as (3.3). Thus, Stenger's method can be regarded as a collocation method utilizing the generalized SE-Sinc approximation, namely, SE-Sinc-collocation method.

- 4.2. Sinc-collocation method by Rashidinia and Zarebnia. Rashidinia and Zarebnia derived their method by the standard collocation procedure, but they considered their approximated solution  $w_N^{\rm RZ}$  in different manners in the following four cases.
  - (I) If u(a) = u(b) = 0, set  $w_N^{RZ}$  as

$$w_N^{\text{RZ}}(t) = \sum_{j=-N}^N c_j S(j,h) (\phi^{\text{SE}}(t)).$$

(II) If  $u(a) \neq 0$  and u(b) = 0, set  $w_N^{RZ}$  as

$$w_N^{\mathrm{RZ}}(t) = c_{-N} \omega_a(t) + \sum_{j=-N+1}^N c_j S(j,h) (\phi^{\mathrm{SE}}(t)). \label{eq:wN}$$

(III) If u(a) = 0 and  $u(b) \neq 0$ , set  $w_N^{RZ}$  as

$$w_N^{\text{RZ}}(t) = \sum_{j=-N}^{N-1} c_j S(j,h) (\phi^{\text{SE}}(t)) + c_N \omega_b(t).$$

(IV) If  $u(a) \neq 0$  and  $u(b) \neq 0$ , set  $w_N^{RZ}$  as

(4.3) 
$$w_N^{\text{RZ}}(t) = c_{-N}\omega_a(t) + \sum_{j=-N+1}^{N-1} c_j S(j,h)(\phi^{\text{SE}}(t)) + c_N \omega_b(t).$$

To obtain the unknown coefficients  $c_n = [c_{-N}, c_{-N+1}, \ldots, c_N]^T$ , where n = 2N + 1, they substituted  $w_N^{\text{RZ}}$  into the given equation (1.1), with approximating the Volterra integral operator  $\mathcal{V}$  by  $\mathcal{V}_N^{\text{SE}}$ . Then, setting n sampling points at  $t = t_i^{\text{SE}}$  ( $i = -N, -N + 1, \ldots, N$ ), they derived a system of linear equations in each of the four cases: (I)–(IV). For example, in the case (I), the resulting system is expressed as

$$(I_n - V_n^{\text{SE}})\boldsymbol{c}_n = \boldsymbol{g}_n^{\text{SE}}.$$

Particularly, in the case (IV), the resulting system is expressed as

$$(4.4) (E_n^{RZ} - V_n^{RZ})\boldsymbol{c}_n = \boldsymbol{g}_n^{SE},$$

where  $E_n^{\rm RZ}$  and  $V_n^{\rm RZ}$  are  $n \times n$  matrices defined by

$$E_{n}^{\rm RZ} = \begin{bmatrix} \omega_{a}(t_{-N}^{\rm SE}) & 0 & \cdots & 0 & \omega_{b}(t_{-N}^{\rm SE}) \\ \omega_{a}(t_{-N+1}^{\rm SE}) & 1 & 0 & \omega_{b}(t_{-N+1}^{\rm SE}) \\ \vdots & \vdots & \vdots & \vdots \\ \omega_{a}(t_{N-1}^{\rm SE}) & 0 & 1 & \omega_{b}(t_{N-1}^{\rm SE}) \\ \omega_{a}(t_{N}^{\rm SE}) & 0 & \cdots & 0 & \omega_{b}(t_{N}^{\rm SE}) \end{bmatrix},$$

$$V_{m}^{\rm RZ} = \begin{bmatrix} & \cdots & k(t_{-N}^{\rm SE}, t_{j}^{\rm SE}) \{\psi^{\rm SE}\}'(jh)h\delta_{-N-j}^{(-1)} & \cdots \\ \cdots & k(t_{-N+1}^{\rm SE}, t_{j}^{\rm SE}) \{\psi^{\rm SE}\}'(jh)h\delta_{-N+1-j}^{(-1)} & \cdots \\ \cdots & k(t_{N-1}^{\rm SE}, t_{j}^{\rm SE}) \{\psi^{\rm SE}\}'(jh)h\delta_{N-1-j}^{(-1)} & \cdots \\ \cdots & k(t_{N}^{\rm SE}, t_{j}^{\rm SE}) \{\psi^{\rm SE}\}'(jh)h\delta_{N-1-j}^{(-1)} & \cdots \\ \cdots & k(t_{N}^{\rm SE}, t_{j}^{\rm SE}) \{\psi^{\rm SE}\}'(jh)h\delta_{N-j}^{(-1)} & \cdots \end{bmatrix},$$

where  $\boldsymbol{p}_n^{\scriptscriptstyle{\mathrm{RZ}}}$  and  $\boldsymbol{q}_n^{\scriptscriptstyle{\mathrm{RZ}}}$  are n-dimensional vectors defined by

$$\boldsymbol{p}_{n}^{\mathrm{RZ}} = [\mathcal{V}_{N}^{\mathrm{SE}}[\omega_{a}](t_{-N}^{\mathrm{SE}}), \, \mathcal{V}_{N}^{\mathrm{SE}}[\omega_{a}](t_{-N+1}^{\mathrm{SE}}), \, \dots, \, \mathcal{V}_{N}^{\mathrm{SE}}[\omega_{a}](t_{N}^{\mathrm{SE}})]^{\mathrm{T}},$$
$$\boldsymbol{q}_{n}^{\mathrm{RZ}} = [\mathcal{V}_{N}^{\mathrm{SE}}[\omega_{b}](t_{-N}^{\mathrm{SE}}), \, \mathcal{V}_{N}^{\mathrm{SE}}[\omega_{b}](t_{-N+1}^{\mathrm{SE}}), \, \dots, \, \mathcal{V}_{N}^{\mathrm{SE}}[\omega_{b}](t_{N}^{\mathrm{SE}})]^{\mathrm{T}}.$$

This is the SE-Sinc-collocation method by Rashidinia and Zarebnia. In the case (I), the following error analysis was provided.

Theorem 4.1 (Zarebnia and Rashidinia [17, Theorem 3]). Let  $\alpha$  and d be positive constants with  $d < \pi$ . Assume that the solution u in (1.1) satisfies all the assumptions in Theorem 2.1. Furthermore, assume that  $k(t,\cdot)$  satisfies all the assumptions in Theorem 2.2 for all  $t \in [a,b]$ . Then, there exists a constant C independent of N such that

$$||u - w_N^{\text{RZ}}||_{C([a,b])} \le C||(I_n - V_n^{\text{SE}})^{-1}||_2 \sqrt{N} e^{-\sqrt{\pi d\alpha N}}.$$

However, this theorem does not prove the convergence of  $w_N^{\rm RZ}$ , because there exists an unestimated term  $\|(I_n - V_n^{\rm SE})^{-1}\|_2$ , which clearly depends on N. For the cases (II)–(IV), no error analysis has been provided thus far.

Moreover, in a practical situation, it is hard to determine whether u is zero or not at the endpoints. This is because the solution u is an unknown function to be determined. The idea to address the issue was presented for Fredholm integral equations [8]; set the approximate solution  $w_N^{\rm RZ}$  as (4.3) in any cases. In other words, we may treat the case (IV) as a general case. This idea can be employed for Volterra integral equations (1.1). Therefore, as a method by Rashidinia and Zarebnia, this study adopts the following procedure: (i) solve the linear system (4.4), and (ii) obtain the approximate solution by (4.3).

4.3. Main result 1: Relationship between the two methods and their convergence. Any relationship between Stenger's method  $(v_N^{\text{SE}})$  and Rashidinia–Zarebnia's method  $(w_N^{\text{RZ}})$  has not been investigated thus far. Furthermore, convergence of the two methods has not been rigorously proved. As a first contribution of this paper, we show the relationship between the two methods as follows. The proof is provided in subsection 7.1.

Theorem 4.2. Let  $v_N^{\text{SE}}$  be a function defined by (4.1), where  $\mathbf{u}_n^{\text{SE}}$  is determined by solving the linear system (3.3). Furthermore, let  $w_N^{\text{RZ}}$  be a function defined by (4.3), where  $\mathbf{c}_m$  is determined by solving the linear system (4.4). Then, it holds that

$$v_N^{\text{SE}}(t_i^{\text{SE}}) = w_N^{\text{RZ}}(t_i^{\text{SE}}), \quad i = -N, -N+1, \dots, N,$$

but generally  $v_N^{\text{SE}} \neq w_N^{\text{RZ}}$ .

Subsequently, we provide the convergence theorems of the two methods as follows. Their proofs are provided in subsections 7.2 and 7.3.

Theorem 4.3. Let  $\alpha$  and d be positive constants with  $\alpha \leq 1$  and  $d < \pi$ . Assume that all the assumptions on g and k in Theorem 3.1 are fulfilled. Furthermore, assume that g and  $k(\cdot, w)$  belong to  $\mathbf{M}_{\alpha}(\psi^{\text{SE}}(\mathcal{D}_d))$  for all  $w \in \psi^{\text{SE}}(\mathcal{D}_d)$ . Let h be selected by the formula (2.6). Then, there exists a positive integer  $N_0$  such that for all  $N \geq N_0$ , the coefficient matrix  $(I_n - V_n^{\text{SE}})$  is invertible. Furthermore, there exists a constant C independent of N such that for all  $N \geq N_0$ ,

$$||u - v_N^{\text{SE}}||_{C([a,b])} \le C\sqrt{N} e^{-\sqrt{\pi d\alpha N}}$$
.

Theorem 4.4. Assume that all the assumptions of Theorem 4.3 are fulfilled. Then, there exists a positive integer  $N_0$  such that for all  $N \geq N_0$ , the coefficient matrix  $(E_n^{\rm RZ} - V_n^{\rm RZ})$  is invertible. Furthermore, there exists a constant C independent of N such that for all  $N \geq N_0$ ,

$$||u - w_N^{\text{RZ}}||_{C([a,b])} \le C\sqrt{N} e^{-\sqrt{\pi d\alpha N}}$$
.

Remark 4.5. In view of Theorems 4.1 and 4.4, one might assume that Theorem 4.3 is proved by bounding  $\|(I_n - V_n^{\text{SE}})^{-1}\|_2$  uniformly for N. However, this is not the case; see section 7 for details.

Theorems 4.3 and 4.4 reveal that both methods achieve the same convergence rate. Therefore, users may prefer Stenger's method because the implementation of the method by Rashidinia and Zarebnia is rather complicated. This complication also causes difficulty in extension to the *system* of Volterra integral equations. For this reason, in the next section, we consider the improvement of Stenger's method.

- 5. Sinc-collocation method combined with the DE transformation. The SE-Sinc-collocation method described in the previous section employs the tanh transformation as a variable transformation. In this section, a new method is derived by replacing the tanh transformation with the DE transformation. Then, its convergence theorem is stated.
- **5.1. Derivation of the DE-Sinc-collocation method.** First, solve the linear system (3.5) and obtain  $\boldsymbol{u}_n^{\text{DE}}$ . Then, the approximated solution  $v_N^{\text{DE}}$  is expressed as the generalized DE-Sinc approximation of  $u_N^{\text{DE}}$ , i.e.,

(5.1)

$$\begin{split} v_N^{\mathrm{DE}}(t) &= \mathcal{P}_N^{\mathrm{DE}}[u_N^{\mathrm{DE}}](t) \\ &= \sum_{j=-N}^{N} \left\{ u_N^{\mathrm{DE}}(t_j^{\mathrm{DE}}) - u_N^{\mathrm{DE}}(t_{-N}^{\mathrm{DE}}) \omega_a(t_j^{\mathrm{DE}}) - u_N^{\mathrm{DE}}(t_N^{\mathrm{DE}}) \omega_b(t_j^{\mathrm{DE}}) \right\} S(j,h) (\phi^{\mathrm{DE}}(t)) \\ &+ u_N^{\mathrm{DE}}(t_{-N}^{\mathrm{DE}}) \omega_a(t) + u_N^{\mathrm{DE}}(t_N^{\mathrm{DE}}) \omega_b(t), \end{split}$$

where  $\mathcal{P}_N^{\text{DE}}$  is defined by (2.12). This procedure is referred to as the DE-Sinc-collocation method.

5.2. Main result 2: Convergence of the DE-Sinc-collocation method. In this paper, we show the convergence of the DE-Sinc-collocation method as follows. The proof is provided in section 8.

THEOREM 5.1. Let  $\alpha$  and d be positive constants with  $\alpha \leq 1$  and  $d < \pi/2$ . Assume that all the assumptions on g and k in Theorem 3.2 are fulfilled. Furthermore, assume that g and  $k(\cdot, w)$  belong to  $\mathbf{M}_{\alpha}(\psi^{\mathrm{DE}}(\mathcal{D}_d))$  for all  $w \in \psi^{\mathrm{DE}}(\mathcal{D}_d)$ . Let h be selected by the formula (2.10). Then, there exists a positive integer  $N_0$  such that for all  $N \geq N_0$ , the coefficient matrix  $(I_n - V_n^{\mathrm{DE}})$  is invertible. Furthermore, there exists a constant C independent of N such that for all  $N \geq N_0$ ,

$$||u - v_N^{\text{DE}}||_{C([a,b])} \le C e^{-\pi dN/\log(2dN/\alpha)}$$
.

Compared to Theorems 4.3 and 4.4, we see that the convergence rate given by this theorem is significantly improved.

6. Numerical experiments. This section presents numerical results for the following five methods: the SE/DE-Sinc-Nyström methods by Muhammad et al. [5], the SE-Sinc-collocation methods by Stenger [13] and Rashidinia–Zarebnia [12], and the DE-Sinc-collocation methods by this paper. The computation was performed on a MacBook Air computer with 1.7 GHz Intel Core i7 with 8 GB memory, running macOS Big Sur. The computation programs were implemented in the C programming language with double-precision floating-point arithmetic, and compiled with Apple clang version 13.0.0 with no optimization. Cephes Math Library was used for computation of the sine integral. LAPACK in Apple's Accelerate framework was used for computation of the system of linear equations. The source code for all programs is available at https://github.com/okayamat/sinc-colloc-volterra.

We consider the following two equations (taken from Rashidinia–Zarebnia [12, Example 4] and Polyanin–Manzhirov [11, Equation 2.1.45]):

(6.1) 
$$u(t) + \int_0^t tsu(s) ds = e^{-t^2} + \frac{t}{2}(1 - e^{-t^2}), \quad 0 \le t \le 1,$$

(6.2) 
$$u(t) - 6 \int_0^t (\sqrt{t} - \sqrt{s}) u(s) \, ds = 1 + \sqrt{t} - 2t\sqrt{t} - t^2, \quad 0 \le t \le 1,$$

whose solutions are  $u(t) = \mathrm{e}^{-t^2}$  and  $u(t) = 1 + \sqrt{t}$ , respectively. In the case of (6.1), the assumptions of Theorems 3.1, 4.3, and 4.4 are fulfilled with d = 3.14 and  $\alpha = 1$ , and those of Theorems 3.2 and 5.1 are fulfilled with d = 1.57 and  $\alpha = 1$ . In the case of (6.2), the assumptions of Theorems 3.1, 4.3, and 4.4 are fulfilled with d = 3.14 and  $\alpha = 1/2$ , and those of Theorems 3.2 and 5.1 are fulfilled with d = 1.57 and  $\alpha = 1/2$ . Therefore, those values were used for implementation. The errors were evaluated at 2048 equally spaced points over the given interval, and the maximum error among them was plotted on the graph in Figures 1 to 4.

From all figures, we can observe that the SE-Sinc-collocation methods by Stenger and Rashidinia–Zarebnia yield almost the same performance. This result coincides with Theorems 4.3 and 4.4. From Figure 1, we can observe that the SE/DE-Sinc-Nyström methods are slightly better than the SE/DE-Sinc-collocation methods with respect to N. This result coincides with Theorems 3.1, 3.2, 4.3, 4.4, and 5.1. However, Figure 2 shows that with respect to the computation time, the SE/DE-Sinc-collocation methods demonstrate significantly better performance than the SE/DE-Sinc-Nyström methods. This is because the SE/DE-Sinc-Nyström methods include a special function as well as given functions k and k0 in the basis functions of their approximate solutions. We note that the performance of the SE/DE-Sinc-collocation methods in Figure 3 reduced than that in Figure 1, which is due to the difference of k1.

- 7. Proofs for the theorems presented in section 4. In this section, we provide proofs for Theorems 4.2 and 4.3.
- **7.1. Proof of Theorem 4.2.** In addition to the given equation  $(\mathcal{I} \mathcal{V})u = g$ , let us consider the following three equations:

$$(7.1) (\mathcal{I} - \mathcal{V}_N^{\text{SE}}) u_N^{\text{SE}} = g,$$

(7.2) 
$$(\mathcal{I} - \mathcal{P}_N^{\text{SE}} \mathcal{V}_N^{\text{SE}}) v = \mathcal{P}_N^{\text{SE}} g,$$

$$(7.3) (\mathcal{I} - \mathcal{P}_N^{\text{RZ}} \mathcal{V}_N^{\text{SE}}) w = \mathcal{P}_N^{\text{RZ}} q,$$

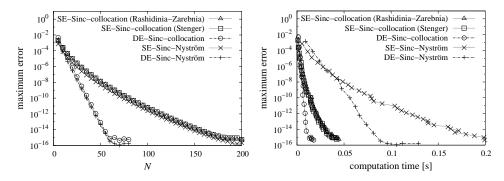


Fig. 1. Errors with respect to N for (6.1). Fig. 2. Errors with respect to the computation time for (6.1).

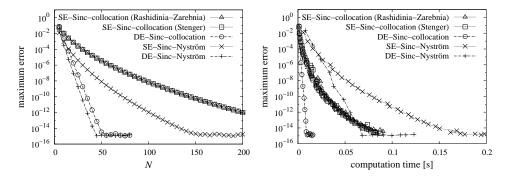


Fig. 3. Errors with respect to N for (6.2). Fig. 4. Errors with respect to the computation time for (6.2).

where  $\mathcal{V}_N^{\text{SE}}$  and  $\mathcal{P}_N^{\text{SE}}$  are defined by (4.2) and (2.11), respectively, and  $\mathcal{P}_N^{\text{RZ}}$  is defined by

(7.4) 
$$\mathcal{P}_{N}^{\text{RZ}}[f](t) = \sum_{j=-N+1}^{N+1} \left\{ f(t_{j}^{\text{SE}}) - \beta_{N}\omega_{a}(t_{j}^{\text{SE}}) - \gamma_{N}\omega_{a}(t_{j}^{\text{SE}}) \right\} S(j,h)(\phi^{\text{SE}}(t)) + \beta_{N}\omega_{a}(t) + \gamma_{N}\omega_{b}(t),$$

where  $\beta_N$  and  $\gamma_N$  are defined by

$$\beta_{N} = \frac{f(t_{-N}^{\text{SE}})\omega_{b}(t_{N}^{\text{SE}}) - f(t_{N}^{\text{SE}})\omega_{b}(t_{-N}^{\text{SE}})}{\omega_{a}(t_{-N}^{\text{SE}})\omega_{b}(t_{N}^{\text{SE}}) - \omega_{b}(t_{-N}^{\text{SE}})\omega_{a}(t_{N}^{\text{SE}})},$$

$$\gamma_{N} = \frac{f(t_{N}^{\text{SE}})\omega_{a}(t_{-N}^{\text{SE}}) - f(t_{-N}^{\text{SE}})\omega_{a}(t_{N}^{\text{SE}})}{\omega_{a}(t_{-N}^{\text{SE}})\omega_{b}(t_{N}^{\text{SE}}) - \omega_{b}(t_{-N}^{\text{SE}})\omega_{a}(t_{N}^{\text{SE}})}.$$

Remark 7.1. The denominator of  $\beta_N$  and  $\gamma_N$  is not zero because

$$\begin{split} \omega_a(t_{-N}^{\scriptscriptstyle{\mathrm{SE}}})\omega_b(t_N^{\scriptscriptstyle{\mathrm{SE}}}) - \omega_b(t_{-N}^{\scriptscriptstyle{\mathrm{SE}}})\omega_a(t_N^{\scriptscriptstyle{\mathrm{SE}}}) &= (1 - \omega_b(t_{-N}^{\scriptscriptstyle{\mathrm{SE}}}))\omega_b(t_N^{\scriptscriptstyle{\mathrm{SE}}}) - \omega_b(t_{-N}^{\scriptscriptstyle{\mathrm{SE}}})(1 - \omega_b(t_N^{\scriptscriptstyle{\mathrm{SE}}})) \\ &= \omega_b(t_N^{\scriptscriptstyle{\mathrm{SE}}}) - \omega_b(t_{-N}^{\scriptscriptstyle{\mathrm{SE}}}) \\ &= \tanh\left(\frac{Nh}{2}\right) \neq 0, \end{split}$$

provided that N is a positive integer and h > 0.

Because (7.1) is equivalent to (3.1), the solution of (7.1) is the approximate solution of the SE-Sinc-Nyström method. On (7.1), the following result was obtained.

LEMMA 7.2 (Okayama et al. [10, Lemma 6.7]). Assume that all the assumptions of Theorem 3.1 are fulfilled. Then, there exists a positive integer  $N_0$  such that for all  $N \geq N_0$ , (7.1) has a unique solution  $u_N^{\text{SE}} \in C([a,b])$ . Furthermore, there exists a constant C independent of N such that for all  $N \geq N_0$ ,

(7.5) 
$$||u - u_N^{\text{SE}}||_{C([a,b])} \le C||\mathcal{V}u - \mathcal{V}_N^{\text{SE}}u||_{C([a,b])}.$$

This lemma says that (7.1) has a unique solution for all sufficiently large N. Using this result, we show the following three things:

- (i) If (7.1) has a unique solution, then (7.2) has also a unique solution  $v = v_N^{\text{SE}}$ .
- (ii) If (7.1) has a unique solution, then (7.3) has also a unique solution  $w = w_N^{RZ}$ .
- (iii) The two solutions  $v_N^{\text{SE}}$  and  $w_N^{\text{RZ}}$  are not generally equivalent, but at the collocation points,  $v_N^{\text{SE}}(t_i^{\text{SE}}) = w_N^{\text{RZ}}(t_i^{\text{SE}}) \; (i = -N, \ldots, N)$  holds.

First, we show (i) as follows.

Lemma 7.3. The following two statements are equivalent:

- (A) Equation (7.1) has a unique solution  $u_N^{\text{SE}} \in C([a,b])$ .
- (B) Equation (7.2) has a unique solution  $v \in C([a, b])$ . Furthermore,  $v = v_N^{\text{SE}}$  holds.

*Proof.* First, let us show (A)  $\Rightarrow$  (B). Note that  $\mathcal{V}_N^{\text{SE}}\mathcal{P}_N^{\text{SE}}f = \mathcal{V}_N^{\text{SE}}f$  holds because of the interpolation property  $\mathcal{P}_N^{\text{SE}}[f](t_i^{\text{SE}}) = f(t_i^{\text{SE}})$  (i = -N, ..., N). Applying  $\mathcal{P}_N^{\text{SE}}$  on the both sides of (7.1), we have

$$\mathcal{P}_N^{\scriptscriptstyle{\mathrm{SE}}} u_N^{\scriptscriptstyle{\mathrm{SE}}} = \mathcal{P}_N^{\scriptscriptstyle{\mathrm{SE}}} (g + \mathcal{V}_N^{\scriptscriptstyle{\mathrm{SE}}} u_N^{\scriptscriptstyle{\mathrm{SE}}}) = \mathcal{P}_N^{\scriptscriptstyle{\mathrm{SE}}} (g + \mathcal{V}_N^{\scriptscriptstyle{\mathrm{SE}}} \mathcal{P}_N^{\scriptscriptstyle{\mathrm{SE}}} u_N^{\scriptscriptstyle{\mathrm{SE}}}),$$

which is equivalent to  $v_N^{\text{SE}} = \mathcal{P}_N^{\text{SE}}(g + \mathcal{V}_N^{\text{SE}}v_N^{\text{SE}})$  (recall that  $v_N^{\text{SE}} = \mathcal{P}_N^{\text{SE}}u_N^{\text{SE}}$ ). This equation implies that (7.2) has a solution  $v_N^{\text{SE}} \in C([a,b])$ .

Next, we show the uniqueness. Suppose that (7.2) has another solution  $\tilde{v} \in C([a,b])$ . Let us set a function  $\tilde{u}$  as  $\tilde{u} = g + \mathcal{V}_N^{\text{SE}} \tilde{v}$ . Because  $\tilde{v}$  is a solution of (7.2), we have

$$\tilde{v} = \mathcal{P}_N^{\text{SE}}(g + \mathcal{V}_N^{\text{SE}}\tilde{v}) = \mathcal{P}_N^{\text{SE}}\tilde{u},$$

from which it holds that

$$\tilde{u} = g + \mathcal{V}_N^{\text{SE}} \tilde{v} = g + \mathcal{V}_N^{\text{SE}} \mathcal{P}_N^{\text{SE}} \tilde{u} = g + \mathcal{V}_N^{\text{SE}} \tilde{u}.$$

This equation implies that  $\tilde{u}$  is a solution of (7.1). Because the solution of (7.1) is unique,  $\tilde{u} = u$  holds, from which we have  $\mathcal{P}_N^{\text{SE}} \tilde{u} = \mathcal{P}_N^{\text{SE}} u$ . Thus, we have  $\tilde{v} = v$ , which shows (B).

The above argument is reversible, which proves (B)  $\Rightarrow$  (A). Furthermore, in view of the proof above, we see  $v = v_N^{\text{SE}}$ , which is to be demonstrated.

Next, for showing (ii), we show the following result. The proof is omitted because it goes in the same way as that of Lemma 7.3.

Lemma 7.4. The following two statements are equivalent:

- (A) Equation (7.1) has a unique solution  $u_N^{\text{SE}} \in C([a, b])$ .
- (B) Equation (7.3) has a unique solution  $w \in C([a, b])$ . Furthermore,  $w = \mathcal{P}_N^{\text{RZ}} u_N^{\text{SE}}$  holds.

To show (ii) completely, we further have to show  $w = w_N^{\rm RZ}$ , which is done by the following result. Noting  $\mathcal{P}_N^{\rm RZ}[f](t_i^{\rm SE}) = f(t_i^{\rm SE})$   $(i = -N, \ldots, N)$ , we can prove this result following Atkinson [1, Sect. 4.3], and hence the proof is omitted.

Proposition 7.5. The following two statements are equivalent:

- (A) Equation (7.3) has a unique solution  $w \in C([a,b])$ .
- (B) Equation (4.4) has a unique solution  $\mathbf{c}_m \in \mathbb{R}^m$ . Furthermore,  $w = w_N^{\text{RZ}}$  holds.

From the above results (i) and (ii), we find that  $v_N^{\text{SE}} = \mathcal{P}_N^{\text{SE}} u_N^{\text{SE}}$  and  $w_N^{\text{RZ}} = \mathcal{P}_N^{\text{RZ}} u_N^{\text{SE}}$ . Using the interpolation property of  $\mathcal{P}_N^{\text{SE}}$  and  $\mathcal{P}_N^{\text{RZ}}$  as

$$\mathcal{P}_{N}^{\text{SE}}[u_{N}^{\text{SE}}](t_{i}^{\text{SE}}) = u_{N}^{\text{SE}}(t_{i}^{\text{SE}}) = \mathcal{P}_{N}^{\text{RZ}}[u_{N}^{\text{SE}}](t_{i}^{\text{SE}}), \quad i = -N, -N+1, \dots, N,$$

we have  $v_N^{\text{SE}}(t_i^{\text{SE}}) = w_N^{\text{RZ}}(t_i^{\text{SE}})$ . However, we note that  $\mathcal{P}_N^{\text{SE}}$  and  $\mathcal{P}_N^{\text{RZ}}$  is not generally equivalent. This can be observed by the limits  $t \to a$  and  $t \to b$  as

$$\lim_{t \to a} \mathcal{P}_N^{\text{SE}}[f](t) = f(t_{-N}^{\text{SE}}) \neq \beta_N = \lim_{t \to a} \mathcal{P}_N^{\text{RZ}}[f](t),$$
$$\lim_{t \to b} \mathcal{P}_N^{\text{SE}}[f](t) = f(t_N^{\text{SE}}) \neq \gamma_N = \lim_{t \to b} \mathcal{P}_N^{\text{RZ}}[f](t).$$

Thus, we obtain the claim of Theorem 4.2.

**7.2. Proof of Theorem 4.3.** The invertibility of  $(I_n - V_n^{\text{SE}})$  is already shown by Theorem 3.1. Thus, we concentrate on the analysis of the error of  $v_N^{\text{SE}}$ . Because  $v_N^{\text{SE}} = \mathcal{P}_N^{\text{SE}} u_N^{\text{SE}}$ , it holds that

$$u - v_N^{\text{SE}} = u - \mathcal{P}_N^{\text{SE}} u_N^{\text{SE}} = (u - \mathcal{P}_N^{\text{SE}} u) + \mathcal{P}_N^{\text{SE}} (u - u_N^{\text{SE}}),$$

which leads to

$$(7.6) \|u - v_N^{\text{SE}}\|_{C([a,b])} \le \|u - \mathcal{P}_N^{\text{SE}}u\|_{C([a,b])} + \|\mathcal{P}_N^{\text{SE}}\|_{\mathcal{L}(C([a,b]),C([a,b]))} \|u - u_N^{\text{SE}}\|_{C([a,b])}.$$

For the first term, we show  $u \in \mathbf{M}_{\alpha}(\psi^{\text{SE}}(\mathcal{D}_d))$ , from which we can use Theorem 2.7. For the purpose, the following theorem is useful.

THEOREM 7.6 (Okayama et al. [10, Theorem 3.2]). Let  $\mathscr{D} = \psi^{\text{SE}}(\mathscr{D}_d)$  or  $\mathscr{D} = \psi^{\text{DE}}(\mathscr{D}_d)$ . Assume that  $g, k(z,\cdot)$  and  $k(\cdot,w)$  belong to  $\mathbf{H}^{\infty}(\mathscr{D})$  for all  $z, w \in \mathscr{D}$ . Then, (1.1) has a unique solution  $u \in \mathbf{H}^{\infty}(\mathscr{D})$ .

Using this theorem, we can show the following result.

THEOREM 7.7. Let  $\alpha$  be a positive constant with  $\alpha \leq 1$ . Assume that all the assumptions of Theorem 7.6 are fulfilled. Furthermore, assume that g and  $k(\cdot, w)$  belong to  $\mathbf{M}_{\alpha}(\mathcal{D})$  for all  $w \in \mathcal{D}$ . Then, (1.1) has a unique solution  $u \in \mathbf{M}_{\alpha}(\mathcal{D})$ .

*Proof.* According to Theorem 7.6, (1.1) has a unique solution  $u \in \mathbf{H}^{\infty}(\mathcal{D})$ . Therefore, we only have to show the Hölder continuity of u at the endpoints. Using  $u = g + \mathcal{V}u$ , we have

$$\begin{split} &|u(b)-u(z)|\\ &=\left|\left(g(b)+\int_a^b k(b,w)u(w)\,\mathrm{d}w\right)-\left(g(z)+\int_a^z k(z,w)u(w)\,\mathrm{d}w\right)\right|\\ &\leq |g(b)-g(z)|+\left|\int_z^b k(b,w)u(w)\,\mathrm{d}w\right|+\left|\int_a^z \left\{k(b,w)-k(z,w)\right\}u(w)\,\mathrm{d}w\right|. \end{split}$$

From the Hölder continuity of g, the first term can be bounded by  $L_g|b-z|^{\alpha}$  for some constant  $L_g$ . From the boundedness of k and u, the second term can be bounded by  $L_{k,u}|b-z|$  for some constant  $L_{k,u}$ . Furthermore, from the boundedness of  $\mathscr{D}$  and  $\alpha \leq 1$ , we have  $|b-z| = |b-z|^{1-\alpha}|b-z|^{\alpha} \leq L_{\mathscr{D}}|b-z|^{\alpha}$  for some constant  $L_{\mathscr{D}}$ . From the Hölder continuity of k and boundedness of u, the third term can be bounded by  $\tilde{L}_{k,u}|b-z|^{\alpha}|z-a|$  for some constant  $\tilde{L}_{k,u}$ . Furthermore, from the boundedness of  $\mathscr{D}$ , we have  $|z-a| \leq \tilde{L}_{\mathscr{D}}$  for some constant  $\tilde{L}_{\mathscr{D}}$ . Thus, there exists a constant L such that  $|u(b)-u(z)| \leq L|b-z|^{\alpha}$ , which shows the Hölder continuity of u at z=b. The proof for the Hölder continuity at z=a is omitted because it follows the same method as that at z=b. This completes the proof.

From this theorem, we can use Theorem 2.7 for estimating the first term of (7.6) as

$$||u - \mathcal{P}_N^{\text{SE}} u||_{C([a,b])} \le C_1 \sqrt{N} e^{-\sqrt{\pi d\alpha N}}$$

for some constant  $C_1$ . For the second term, we use the following bound for the operator  $\mathcal{P}_N^{\text{SE}}$ .

LEMMA 7.8 (Okayama [7, Lemma 7.2]). Let  $\mathcal{P}_N^{\text{SE}}$  be defined by (2.11). Then, there exists a constant  $C_2$  independent of N such that

$$\|\mathcal{P}_N^{\text{SE}}\|_{\mathcal{L}(C([a,b]),C([a,b]))} \le C_2 \log(N+1).$$

The remaining term to be estimated in (7.6) is  $||u - u_N^{\text{SE}}||_{C([a,b])}$ . According to Lemma 7.2, it is estimated as (7.5). Because  $u \in \mathbf{H}^{\infty}(\psi^{\text{SE}}(\mathcal{D}_d))$ , u satisfies the assumptions of Theorem 2.2, from which we have

$$\|\mathcal{V}u - \mathcal{V}_N^{\text{SE}}u\|_{C([a,b])} \le C_3 e^{-\sqrt{\pi d\alpha N}}$$
.

Thus, there exists a constant  $C_4$  such that

$$||u - v_N^{\text{SE}}||_{C([a,b])} \le C_1 \sqrt{N} e^{-\sqrt{\pi d\alpha N}} + C_2 \log(N+1) C_3 e^{-\sqrt{\pi d\alpha N}}$$
$$\le C_4 \sqrt{N} e^{-\sqrt{\pi d\alpha N}}.$$

This completes the proof of Theorem 4.3.

**7.3. Proof of Theorem 4.4.** For Theorem 4.4, the invertibility of  $(E_n^{\rm RZ} - V_n^{\rm RZ})$  can be shown by combining Lemmas 7.2 and 7.4 and Proposition 7.5. Thus, we concentrate on the analysis of the error of  $w_N^{\rm RZ}$ . By the triangle inequality, we have

$$\begin{aligned} \|u(t) - w_N^{\text{RZ}}(t)\|_{C([a,b])} &\leq \|u(t) - v_N^{\text{SE}}(t)\|_{C([a,b])} + \|v_N^{\text{SE}}(t) - w_N^{\text{RZ}}(t)\|_{C([a,b])} \\ &= \|u(t) - v_N^{\text{SE}}(t)\|_{C([a,b])} + \|\mathcal{P}_N^{\text{SE}} u_N^{\text{SE}}(t) - \mathcal{P}_N^{\text{RZ}} u_N^{\text{SE}}(t)\|_{C([a,b])}. \end{aligned}$$

Because the first term is already estimated by Theorem 4.3, we estimate the second term. For the purpose, the following lemma is essential.

Lemma 7.9. Let  $\mathcal{P}_N^{\text{SE}}: C([a,b]) \to C([a,b])$  and  $\mathcal{P}_N^{\text{RZ}}: C([a,b]) \to C([a,b])$  be defined by (2.11) and (7.4), respectively. Then, there exists a constant C independent of N such that

$$\|\mathcal{P}_{N}^{\text{SE}} - \mathcal{P}_{N}^{\text{RZ}}\|_{\mathcal{L}(C([a,b]),C([a,b]))} \le \frac{C}{e^{Nh} - 1} \log(N + 1).$$

*Proof.* First, it holds for  $f \in C([a,b])$  that

$$\begin{split} & \mathcal{P}_N^{\text{SE}}[f](t) - \mathcal{P}_N^{\text{RZ}}[f](t) \\ & = -\sum_{j=-N}^{N} \left\{ \left( f(t_{-N}^{\text{SE}}) - \beta_N \right) \omega_a(t_j^{\text{SE}}) + \left( f(t_N^{\text{SE}}) - \gamma_N \right) \omega_b(t_j^{\text{SE}}) \right\} S(j,h) (\phi^{\text{SE}}(t)) \\ & + \left( f(t_{-N}^{\text{SE}}) - \beta_N \right) \omega_a(t) + \left( f(t_N^{\text{SE}}) - \gamma_N \right) \omega_b(t). \end{split}$$

Here, noting

$$|f(t_{-N}^{\text{SE}}) - \beta_N| = \frac{|f(t_N^{\text{SE}}) - f(t_{-N}^{\text{SE}})|}{e^{Nh} - 1} \le \frac{2||f||_{C([a,b])}}{e^{Nh} - 1},$$

$$|f(t_N^{\text{SE}}) - \gamma_N| = \frac{|f(t_{-N}^{\text{SE}}) - f(t_N^{\text{SE}})|}{e^{Nh} - 1} \le \frac{2||f||_{C([a,b])}}{e^{Nh} - 1},$$

and using  $\omega_a(t) + \omega_b(t) = 1$ , we have

$$\begin{split} & \|\mathcal{P}_{N}^{\text{SE}} - \mathcal{P}_{N}^{\text{RZ}}\|_{\mathcal{L}(C([a,b]),C([a,b]))} \\ & \leq \frac{2}{\mathrm{e}^{Nh} - 1} \left\{ \sum_{j=-N}^{N} \left( \omega_{a}(t_{j}^{\text{SE}}) + \omega_{b}(t_{j}^{\text{SE}}) \right) |S(j,h)(\phi^{\text{SE}}(t))| + \omega_{a}(t) + \omega_{b}(t) \right\} \\ & = \frac{2}{\mathrm{e}^{Nh} - 1} \left\{ \sum_{j=-N}^{N} |S(j,h)(\phi^{\text{SE}}(t))| + 1 \right\} \\ & \leq \frac{2}{\mathrm{e}^{Nh} - 1} \left\{ \frac{2}{\pi} (3 + \log N) + 1 \right\}, \end{split}$$

where the standard bound [13, Problem 3.1.5 (a)] is used for the last inequality. Thus, the claim follows.

From this lemma, substituting (2.6) into h, we estimate the second term as

$$\|(\mathcal{P}_N^{\text{SE}} - \mathcal{P}_N^{\text{RZ}})u_N^{\text{SE}}\|_{C([a,b])} \le \frac{C}{1 - \mathrm{e}^{-\sqrt{\pi d/\alpha}}} \log(N+1) \, \mathrm{e}^{-\sqrt{\pi dN/\alpha}} \, \|u_N^{\text{SE}}\|_{C([a,b])}.$$

Noting  $\alpha \in (0,1]$ , we obtain  $e^{-\sqrt{\pi dN/\alpha}} \le e^{-\sqrt{\pi d\alpha N}}$ . Furthermore,  $\log(N+1) \le \sqrt{N}$  holds. Hence, the proof is completed if  $\|u_N^{\text{SE}}\|_{C([a,b])}$  is uniformly bounded with respect to N. This is shown by the following estimate

$$||u_N^{\text{SE}}||_{C([a,b])} \le ||u - u_N^{\text{SE}}||_{C([a,b])} + ||u||_{C([a,b])}$$
.

From (7.5) and Theorem 2.2, we see that  $\|u - u_N^{\text{SE}}\|_{C([a,b])}$  converges to 0 as  $N \to \infty$ , and accordingly it is uniformly bounded. We also see that  $\|u\|_{C([a,b])}$  is bounded because u is continuous on [a,b] from the assumption (see Theorem 7.7). This completes the proof of Theorem 4.4.

**8. Proofs for the theorem presented in section 5.** In this section, we provide proofs for Theorem 5.1.

**8.1. Existence and uniqueness of the approximated equations.** In addition to the given equation  $(\mathcal{I} - \mathcal{V})u = g$ , let us consider the following two equations:

(8.1) 
$$(\mathcal{I} - \mathcal{V}_N^{\text{DE}}) u_N^{\text{DE}} = g,$$

(8.2) 
$$(\mathcal{I} - \mathcal{P}_N^{\text{DE}} \mathcal{V}_N^{\text{DE}}) v = \mathcal{P}_N^{\text{DE}} g,$$

where  $\mathcal{V}_{N}^{\text{DE}}$  is defined by

$$\mathcal{V}_{N}^{\text{DE}}[f](t) = \sum_{j=-N}^{N} k(t, t_{j}^{\text{DE}}) f(t_{j}^{\text{DE}}) \{\psi^{\text{DE}}\}'(jh) J(j, h) (\phi^{\text{DE}}(t)),$$

and  $\mathcal{P}_{N}^{\text{DE}}$  are defined by (2.12). Because (8.1) is equivalent to (3.4), the solution of (8.1) is the approximate solution of the DE-Sinc-Nyström method. On (8.1), the following result was obtained.

LEMMA 8.1 (Okayama et al. [10, Lemma 6.10]). Assume that all the assumptions of Theorem 3.2 are fulfilled. Then, there exists a positive integer  $N_0$  such that for all  $N \geq N_0$ , (8.1) has a unique solution  $u_N^{\text{DE}} \in C([a,b])$ . Furthermore, there exists a constant C independent of N such that for all  $N \geq N_0$ ,

(8.3) 
$$||u - u_N^{\text{DE}}||_{C([a,b])} \le C||\mathcal{V}u - \mathcal{V}_N^{\text{DE}}u||_{C([a,b])}.$$

On (8.2), we can show the following lemma in the same manner as Lemma 7.3 (hence, the proof is omitted).

Lemma 8.2. The following two statements are equivalent:

- (A) Equation (8.1) has a unique solution  $u_N^{DE} \in C([a,b])$ .
- (B) Equation (8.2) has a unique solution  $v \in C([a, b])$ . Furthermore,  $v = v_N^{\text{DE}}$  holds.

On the basis of the results, we proceed to analyze the error of  $v_N^{\text{DE}}$  next.

**8.2.** Proof of Theorem 5.1. In the same manner as (7.6), we have

$$(8.4) \|u - v_N^{\text{DE}}\|_{C([a,b])} \le \|u - \mathcal{P}_N^{\text{DE}}u\|_{C([a,b])} + \|\mathcal{P}_N^{\text{DE}}\|_{\mathcal{L}(C([a,b]),C([a,b]))} \|u - u_N^{\text{DE}}\|_{C([a,b])}.$$

For the first term, from Theorem 7.7, we can use Theorem 2.8 as

$$||u - \mathcal{P}_N^{\text{DE}} u||_{C([a,b])} \le C_1 e^{-\pi dN/\log(2dN/\alpha)}$$

for some constant  $C_1$ . For the second term, we use the following bound for the operator  $\mathcal{P}_N^{\text{DE}}$ .

LEMMA 8.3 (Okayama [7, Lemma 7.5]). Let  $\mathcal{P}_N^{\text{DE}}$  be defined by (2.12). Then, there exists a constant  $C_2$  independent of N such that

$$\|\mathcal{P}_N^{\text{DE}}\|_{\mathcal{L}(C([a,b]),C([a,b]))} \le C_2 \log(N+1).$$

The remaining term to be estimated in (8.4) is  $\|u - u_N^{\text{DE}}\|_{C([a,b])}$ . According to Lemma 8.1, it is estimated as (8.3). Because  $u \in \mathbf{H}^{\infty}(\psi^{\text{DE}}(\mathcal{D}_d))$ , u satisfies the assumptions of Theorem 2.4, from which we have

$$\|\mathcal{V}u - \mathcal{V}_N^{\mathrm{DE}}u\|_{C([a,b])} \le C_3 \frac{\log(2dN/\alpha)}{N} e^{-\pi dN/\log(2dN/\alpha)}.$$

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Thus, there exists a constant  $C_4$  such that

$$||u - v_N^{\text{DE}}||_{C([a,b])}$$

$$\leq C_1 e^{-\pi dN/\log(2dN/\alpha)} + C_2 \log(N+1)C_3 \frac{\log(2dN/\alpha)}{N} e^{-\pi dN/\log(2dN/\alpha)}$$

$$\leq C_4 e^{-\pi dN/\log(2dN/\alpha)}.$$

This completes the proof of Theorem 5.1.

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