REAL AND BI-LIPSCHITZ VERSIONS OF THE THEOREM OF NOBILE

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ABSTRACT. The famous Theorem of Nobile says that a pure dimensional complex analytic set X is analytically smooth if, and only if, its Nash transformation $\eta\colon \mathcal{N}(X)\to X$ is an analytic isomorphism. This result was proven in 1975 and since then, as far as the author knows, no answer has been given to the real case, even more so when one only asks for C^k smoothness. In this paper, we prove the real version of the Theorem of Nobile asking only C^k smoothness, i.e., we prove that for a pure dimensional real analytic set X the following statements are equivalent:

- (1) X is a real analytic (resp. $C^{k+1,1}$) submanifold;
- (2) the mapping $\eta \colon \mathcal{N}(X) \to X$ is a real analytic (resp. $C^{k,1}$) diffeomorphism;
- (3) the mapping $\eta \colon \mathcal{N}(X) \to X$ is a C^{∞} (resp. $C^{k,1}$) diffeomorphism;
- (4) X is a C^{∞} (resp. $C^{k+1,1}$) submanifold.

In this paper, we also prove the bi-Lipschitz version of the Theorem of Nobile. More precisely, we prove that X is analytically smooth if and only if its Nash transformation $\eta\colon \mathcal{N}(X)\to X$ is a homeomorphism that locally bi-Lipschitz.

1. Introduction

Given a pure d-dimensional \mathbb{K} -analytic set X in \mathbb{K}^n , $\mathcal{N}(X)$ is the closure in $X \times Gr_{\mathbb{K}}(d,n)$ of the graph of the **Gauss mapping** $\nu \colon X \setminus \operatorname{Sing}(X) \to Gr_{\mathbb{K}}(d,n)$ given by $\nu(x) = T_x X$, where \mathbb{K} is \mathbb{C} or \mathbb{R} and $Gr_{\mathbb{K}}(d,n)$ is the Grassmannian of d-dimensional \mathbb{K} -linear subspaces in \mathbb{K}^n . The set $\mathcal{N}(X)$ is called the **Nash transformation of** X and the projection $\eta \colon \mathcal{N}(X) \to X$ is called the **Nash mapping of** X. Sometimes we also call $\eta \colon \mathcal{N}(X) \to X$ the Nash transformation of X. The notion of Nash transformation trivially extends to algebraic varieties over a field of arbitrary characteristic.

An important problem in the resolution of singularities related to Nash transformations is the following conjecture:

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Conjecture 1. A finite succession of Nash transformations resolves the singularities of X.

This problem was proposed by Nash in a private communication to Hironaka in the early sixties (cf. [18, p. 412]) and it was also posed by Semple in [17]. This is why we call it Nash-Semple's conjecture.

It is not the purpose of this article to discuss this conjecture, but it is important to say that a relevant partial answer to the above conjecture was given by Spivakovsky in [18]. He showed that in the case of surfaces, the normalised Nash transformations (i.e. Nash transformations followed by normalizations) resolve the singularities. Recently, in the preprint [1] was given a very important partial answer to Nash-Semple's conjecture. The authors in [1] presented counterexamples for this conjecture for any dimension $d \geq 4$. Thus, Nash-Semple's conjecture is only an open problem in dimensions 2 and 3. However, there are still several important families of analytic sets for which this conjecture is an open problem, for example, hypersurfaces in all dimensions $d \geq 2$.

Probably the most basic and fundamental result related to the Nash transformation and Nash-Semple's conjecture is the following result proved by Nobile [11]:

Theorem 1.1 (Theorem of Nobile). Let X be a pure r-dimensional analytic subset in \mathbb{C}^n . Then, X is smooth if and only if $\eta \colon \mathcal{N}(X) \to X$ is an (analytic) isomorphism.

In fact, the precise statement of the Theorem of Nobile is the following: Let k be an algebraically closed field of characteristic zero (resp. $k = \mathbb{C}$), X be a pure r-dimensional algebraic (resp. analytic) variety over k. Then, X is smooth if and only if $\eta \colon \mathcal{N}(X) \to X$ is an isomorphism.

The proof in [11] shows that it suffices to prove the theorem in the analytic case with $k = \mathbb{C}$, as stated in Theorem 1.1. The proof presented in [11] is an analytic proof and systematically used the complex structure. A more algebraic proof of the Theorem of Nobile is due to Teissier [19] (cf. [12]). See more about the Theorem of Nobile in the recent notes [12].

The version of Theorem of Nobile in positive characteristic does not work. Indeed, it is shown in [11] that for the cusp $X: x^2 = y^3$, working in characteristic two, the Nash mapping $\eta: \mathcal{N}(X) \to X$ is an isomorphism, although, of course, X is singular. Recently, in the article [4] the authors showed that if X is a normal variety of any dimension, $\eta: \mathcal{N}(X) \to X$ is an

isomorphism if and only if X is smooth, even if the base field has positive characteristic. More about the Theorem of Nobile can be found in [12].

However, as far as the author knows, there is no proof for the real version of this theorem, even more so when one only asks for C^k smoothness. So, it is natural to ask: Does the real version of the Theorem of Nobile hold true?

Another natural problem is whether the analytic diffeomorphism condition of the Nash mapping can be weakened.

Recall that we have a distance defined in $Gr_{\mathbb{R}}(d, n)$ as follows: Given two linear subspaces L_1 and L_2 in $Gr_{\mathbb{R}}(d, n)$,

$$\begin{array}{lcl} \delta(L_1,L_2) & = & d_H(S_1,S_2) \\ & = & \max\{\sup_{a \in S_1} d(a,S_2), \sup_{b \in S_2} d(b,S_1)\}, \end{array}$$

where $S_i = L_i \cap \mathbb{S}^{n-1}$, i = 1, 2 and $d(a, S) = \inf\{||a - x||; x \in S\}$.

In this paper, we prove that it is enough to ask that the Nash mapping is locally bi-Lipschitz.

Theorem 3.4. Let $X \subset \mathbb{C}^n$ be a pure dimensional complex analytic set. Then, X is smooth if and only if $\eta \colon \mathcal{N}(X) \to X$ is a homeomorphism that is locally bi-Lipschitz.

We also prove a real version of Theorem of Nobile, when one only asks for $C^{k,1}$ smoothness. Indeed, we prove the following result:

Theorem 3.5. Let $X \subset \mathbb{R}^n$ be a pure dimensional real analytic set, and let k be a nonnegative integer number. Then, X is a $C^{k+1,1}$ submanifold if and only if the mapping $\eta \colon \mathcal{N}(X) \to X$ is a homeomorphism such that η^{-1} is $C^{k,1}$ smooth.

Consequently, we also obtain the following real version of Theorem of Nobile.

Theorem 3.6. Let $X \subset \mathbb{R}^n$ be a pure dimensional real analytic set. Then the following statements are equivalent:

- (1) X is a real analytic submanifold;
- (2) the mapping $\eta \colon \mathcal{N}(X) \to X$ is a real analytic diffeomorphism;
- (3) the mapping $\eta: \mathcal{N}(X) \to X$ is a C^{∞} diffeomorphism;
- (4) X is a C^{∞} submanifold.

In fact, we obtain a stronger result. Here, we present a proof that works even in the case of sets that are locally definable in a polynomially bounded o-minimal structure under small assumptions. For a subset $X \subset \mathbb{R}^n$, which is a d-dimensional set definable in an o-minimal structure, we say that $C_3(X)$ and $C_4(X)$ coincide linearly, if for any $p \in X$ such that $C_4(X,p)$ is a d-dimensional linear subspace, then $C_3(X,p) = C_4(X,p)$.

We prove the following more general result:

Theorem 3.1. Let $X \subset \mathbb{R}^n$ be a locally closed set that is pure dimensional and locally definable in a polynomially bounded o-minimal structure on \mathbb{R} . Suppose that $C_3(X)$ and $C_4(X)$ coincide linearly. Then, for a fixed $k \in \mathbb{N}$, the mapping $\eta \colon \mathcal{N}(X) \to X$ is a homeomorphism such that η^{-1} is $C^{k,1}$ smooth if and only if X is a $C^{k+1,1}$ submanifold. Moreover, the mapping $\eta \colon \mathcal{N}(X) \to X$ is a C^{∞} diffeomorphism if and only if X is a C^{∞} submanifold.

In order to know more about o-minimal geometry, see, for instance, [2] and [3].

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2. Preliminaries

All the subsets of \mathbb{R}^n or \mathbb{C}^n considered in the paper are supposed to be equipped with the Euclidean distance. When we consider other distances, it is clearly emphasised.

2.1. O-minimal structures.

Definition 2.1. A structure on \mathbb{R} is a collection $S = \{S_n\}_{n \in \mathbb{Z}_{>0}}$ where each S_n is a set of subsets of \mathbb{R}^n , satisfying the following axioms:

- 1) All algebraic subsets of \mathbb{R}^n are in \mathcal{S}_n ;
- 2) For every n, S_n is a Boolean subalgebra of the powerset of \mathbb{R}^n ;
- 3) If $A \in \mathcal{S}_m$ and $B \in \mathcal{S}_n$, then $A \times B \in \mathcal{S}_{m+n}$.
- 4) If $\pi: \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first n coordinates and $A \in \mathcal{S}_{n+1}$, then $\pi(A) \in \mathcal{S}_n$.

An element of S_n is called **definable in** S. The structure S is said **o-minimal** if it satisfies the following condition:

5) The elements of S_1 are precisely finite unions of points and intervals.

Definition 2.2. A mapping $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$ is called **definable in** S if its graph is an element of S_{n+m} .

A structure S is said **polynomially bounded** if it satisfies the following condition:

6) For every function $f: \mathbb{R} \to \mathbb{R}$ that is definable in \mathcal{S} , there exists a positive integer number N such that $|f(t)| \leq t^N$ for all sufficiently large positive t.

We say that a set $X \subset \mathbb{R}^n$ is **locally definable in** S if for any $x \in \overline{X}$ there is a neighbourhood $U \subset \mathbb{R}^n$ of x such that $X \cap U$ is definable in S.

Throughout this paper, we fix a polynomially bounded o-minimal structure S on \mathbb{R} .

In the sequel, the adjective **definable** denotes definable in S.

We say that a set $X \subset \mathbb{R}^n$ is **locally definable** if for any $x \in \overline{X}$ there is a neighbourhood $U \subset \mathbb{R}^n$ of x such that $X \cap U$ is definable.

Remark 2.3. If $f: [0, \epsilon) \to \mathbb{R}$ is a continuous and definable function that is not identically zero close to 0, then there is a number $\alpha \geq 0$ and $c \neq 0$ such that $f(t) = ct^{\alpha} + o(t^{\alpha})$, where $g(t) = o(t^{\alpha})$ means $\lim_{t \to 0^{+}} \frac{g(t)}{t^{\alpha}} = 0$ (see [8, Proposition]). Thus, we define the **order of** f at 0 by $\operatorname{ord}_{0} f = \alpha$.

2.2. Tangent cones.

Definition 2.4. Let $X \subset \mathbb{R}^n$ be a subset and let $p \in \overline{X}$. We say that $v \in \mathbb{R}^n$ is a tangent vector of X at p if there are a sequence $\{x_j\}_{j\in\mathbb{N}} \subset X$ and a sequence of positive real numbers $\{t_j\}_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} t_j = 0$ and $\lim_{j\to\infty} \frac{1}{t_j}(x_j - p) = v$. The set of all tangent vectors of X at p is denoted by $C_3(X,p)$ and is called the tangent cone of X at p.

Remark 2.5. Let $X \subset \mathbb{C}^n$ be a complex analytic (resp. algebraic) set and $x_0 \in X$. In this case, $C(X, x_0)$ (resp. $C(X, \infty)$) is the zero set of a set of complex homogeneous polynomials, as we can see in [20, Theorem 4D]. In particular, $C(X, x_0)$ (resp. $C(X, \infty)$) is the union of complex lines passing through the origin $0 \in \mathbb{C}^n$.

Definition 2.6. Given a pure d-dimensional definable set $X \subset \mathbb{R}^n$ and $p \in X$, $C_4(X,p)$ is the union of the d-dimensional linear subspaces $T \subset \mathbb{R}^n$ such that there is a sequence $\{x_j\}_{j\in\mathbb{N}} \subset X \setminus Sing_1(X)$ satisfying $\lim_{j\to+\infty} T_{x_j}X = T$, where $Sing_1(X)$ denotes the set of points $x \in X$ such that X is not a C^1 submanifold around x.

Note that $C_3(X,p) \subset C_4(X,p)$ for all $p \in X$.

2.3. Multiplicity and degree of real sets. This subsection is closely related to the subsection with the same title in [6].

Let $X \subset \mathbb{R}^n$ be a d-dimensional real analytic set with $0 \in X$ and

$$X_{\mathbb{C}} = V(\mathcal{I}_{\mathbb{R}}(X,0)),$$

where $\mathcal{I}_{\mathbb{R}}(X,0)$ is the ideal in $\mathbb{C}\{z_1,\ldots,z_n\}$ generated by the complexifications of all germs of real analytic functions that vanish on the germ (X,0). We know that $X_{\mathbb{C}}$ is a germ of a complex analytic set and $\dim_{\mathbb{C}} X_{\mathbb{C}} = \dim_{\mathbb{R}} X$. Then, for a linear projection $\pi: \mathbb{C}^n \to \mathbb{C}^d$ such that $\pi^{-1}(0) \cap C(X_{\mathbb{C}},0) = \{0\}$, there exists an open neighbourhood $U \subset \mathbb{C}^n$ of 0 such that $\#(\pi^{-1}(x) \cap (X_{\mathbb{C}} \cap U))$ is constant for a generic point $x \in \pi(U) \subset \mathbb{C}^d$. This number is the multiplicity of $X_{\mathbb{C}}$ at the origin and is denoted by $m(X_{\mathbb{C}},0)$.

Definition 2.7. With the above notation, we define the multiplicity of X at the origin by $m(X,0) := m(X_{\mathbb{C}},0)$.

One can learn more about multiplicity of real analytic sets in [16].

Remark 2.8. It follows from the works in [16] and [6] that if $X \subset \mathbb{R}^n$ is a pure d-dimensional real analytic set and $\pi \colon \mathbb{R}^n \to \mathbb{R}^d$ is a projection such that $\pi^{-1}(0) \cap C(X,0) = \{0\}$, then there are an open neighbourhood $U \subset \mathbb{R}^n$ of 0 and a subanalytic set $\sigma \subset \mathbb{R}^d$ such that $\dim_{\mathbb{R}} \sigma < d$ and $m(X,0) \equiv \#(\pi^{-1}(t) \cap U) \pmod{2}$ for all small enough $t \in \mathbb{R}^d \setminus \sigma$, where $\#(\pi^{-1}(t) \cap U)$ denotes the cardinality of $\pi^{-1}(t) \cap U$.

3. Main results

3.1. Definable version of the Theorem of Nobile.

Theorem 3.1. Let $X \subset \mathbb{R}^n$ be a locally closed set that is pure d-dimensional and locally definable in a polynomially bounded o-minimal structure on \mathbb{R} . Suppose that $C_3(X)$ and $C_4(X)$ coincide linearly. Then, for a fixed nonnegative integer number k, the mapping $\eta \colon \mathcal{N}(X) \to X$ is a homeomorphism such that η^{-1} is $C^{k,1}$ smooth if and only if X is a $C^{k+1,1}$ smooth submanifold. Moreover, the mapping $\eta \colon \mathcal{N}(X) \to X$ is a C^{∞} diffeomorphism if and only if X is a C^{∞} submanifold.

Proof. It is clear that if X is a $C^{k+1,1}$ smooth submanifold, then $\eta \colon \mathcal{N}(X) \to X$ is a homeomorphism such that η^{-1} is $C^{k,1}$. Moreover, if X is a C^{∞} smooth submanifold, then $\eta \colon \mathcal{N}(X) \to X$ is a C^{∞} smooth diffeomorphism.

Reciprocally, assume that the mapping $\eta \colon \mathcal{N}(X) \to X$ is a homeomorphism such that η^{-1} is $C^{k,1}$ smooth.

Firstly, we assume k = 0. In this case, $\eta \colon \mathcal{N}(X) \to X$ is a homeomorphism that is locally bi-Lipschitz.

Since $\eta: \mathcal{N}(X) \to X$ is in particular a bijection, we obtain that $C_4(X, p)$ is a d-dimensional linear subspace for all $p \in X$. Since $C_3(X)$ and $C_4(X)$ coincide linearly, then $C_3(X, p)$ is a d-dimensional linear subspace for all $p \in X$ and continuously varies in p. For simplicity, in this case, for each $p \in X$, we denote $C_3(X, p)$ by T_pX .

Claim 1. X is a C^1 submanifold.

Proof of Claim 1. Suppose on the contrary that X is not a C^1 submanifold. By [7, Theorem 4.4], there exists $p \in X$ such that there is no neighbourhood of p in X to which the restriction to X of the orthogonal $\pi_p \colon \mathbb{R}^n \to T_x X$ is injective. We may assume that p coincides with the origin 0 and we choose the linear coordinates (x, y) of \mathbb{R}^n such that $T_p X = \{(x, y); y = 0\} = \mathbb{R}^d \times \{0\} \cong \mathbb{R}^d$, where $k = \dim X$. With this identification, π_p now becomes the orthogonal projection to the first k coordinates $\pi \colon \mathbb{R}^n \to \mathbb{R}^d$.

By [7, Lemma 4.1], there is an open neighbourhood U of 0 in X such that $\pi|_U$ is an open map. By shrinking U, if necessary, we assume that $\pi(U) = \mathbf{B}^d(0,r)$ and for every $p \in U$, $\delta(T_pX, T_0X) < \frac{1}{2}$ and, in particular, T_pX is not orthogonal to T_0X . Since X is locally closed, we may assume that r was taken small enough so that $X \cap \overline{\mathbf{B}^n(0,r)} \cap X$ is a compact set and $\pi|_U^{-1}(0) = \{0\}$.

Let $N = \sup_{x \in \mathbf{B}^d(0,r)} \#\pi|_U^{-1}(x)$ and $S = \{x \in \mathbf{B}^d(0,r) : \#\pi|_U^{-1}(x) = N\}$, where $\#\pi|_U^{-1}(x)$ denotes the cardinality of $\pi|_U^{-1}(x)$. By shrinking r, if necessary, we assume that $0 \in \overline{S}$. Given $x_0 \in S \cap \mathbf{B}^d(0,r/2)$, let $t = \sup\{s \le r/2; \#\pi|_U^{-1}(x) = N \text{ for all } x \in \mathbf{B}^d(x_0,s)\}$. We have that S is an open set, $\pi_U^{-1}(\mathbf{B}^d(x_0,t))$ has exactly N connected components, say X_1, \ldots, X_N , and each X_i is the graph of a C^1 definable mapping $f_i \colon \mathbf{B}^d(x_0,t) \to \mathbb{R}^{n-d}$. By the assumptions on the tangent cones, we also have that each f_i has bounded derivative, and thus it is a Lipschitz mapping and has a Lipschitz extension $\bar{f}_i \colon \overline{\mathbf{B}^d(x_0,t)} \to \mathbb{R}^{n-d}$. Note that each \bar{f}_i is also a definable mapping. Thus, there is $x \in \overline{\mathbf{B}^d(x_0,t)}$ such that $\|x-x_0\| = t$ and $\#\pi|_U^{-1}(x) < N$. This implies that there are i and j such that $\bar{f}_i(x) = \bar{f}_j(x)$.

Let $\lambda \geq 1$ be a number such that $||Df_i|| \leq \lambda$ and $||Df_i|| \leq \lambda$ on S.

There is a constant \tilde{K} such that $\delta(T_{(z,f_i(z))}X_i, T_{(z,f_j(z))}X_j) \geq \tilde{K} \| (Df_i)_z - (Df_j)_z \|$ for all $z \in S$, where $\| (Df_i)_z - (Df_j)_z \| := \sup\{ \| (Df_i)_z(u) - (Df_j)_z(u) \|; u \in \mathbb{R}^d, \| u \| \leq 1 \}.$

In fact, for a fixed $z \in S$, let $S_i = T_{(z,f_i(z))}X \cap \mathbb{S}^{n-1}$ and $S_j = T_{(z,f_j(z))}X \cap \mathbb{S}^{n-1}$. Then, we have

$$\begin{split} \delta(T_{(z,f_i(z))}X,T_{(z,f_j(z))}X) &= d_H(S_i,S_j) \\ &= \max\{\sup_{a \in S_i} d(a,S_j), \sup_{b \in S_j} d(b,S_i)\}, \end{split}$$

and thus for any $u \in \mathbb{R}^d \setminus \{0\}$, there is $v \in \mathbb{R}^d \setminus \{0\}$ such that

$$\delta(T_{(z,f_i(z))}X,T_{(z,f_j(z))}X) \geq \left\| \frac{(u,(Df_i)_z(u))}{\|(u,(Df_i)_z(u))\|} - \frac{(v,(Df_j)_z(v))}{\|(v,(Df_j)_z(v))\|} \right\|.$$

Denoting $\tilde{u} = \frac{u}{\|(u,(Df_i)_z(u))\|}$ and $\tilde{v} = \frac{v}{\|(v,(Df_j)_z(v))\|}$, we have

$$\delta(T_{(z,f_{i}(z))}X,T_{(z,f_{j}(z))}X) \geq \|(\tilde{u},(Df_{i})_{z}(\tilde{u})) - (\tilde{v},(Df_{j})_{z}(\tilde{v}))\|
\geq \frac{\sqrt{2}}{2}(\|\tilde{u} - \tilde{v}\| + \|(Df_{i})_{z}(\tilde{u}) - (Df_{j})_{z}(\tilde{v})\|)
\geq \frac{\sqrt{2}}{2}(\|\tilde{u} - \tilde{v}\| + \frac{1}{\lambda}\|(Df_{i})_{z}(\tilde{u}) - (Df_{j})_{z}(\tilde{v})\|)
\geq \frac{\sqrt{2}}{2}\|\tilde{u} - \tilde{v}\| + \frac{\sqrt{2}}{2\lambda}\|(Df_{i})_{z}(\tilde{u}) - (Df_{j})_{z}(\tilde{u})\|
- \frac{\sqrt{2}}{2\lambda}\|(Df_{j})_{z}(\tilde{u} - \tilde{v})\|
\geq \frac{\sqrt{2}}{2\lambda}\|(Df_{i})_{z}(\tilde{u}) - (Df_{j})_{z}(\tilde{u})\|.$$

Note that $\|\tilde{u}\| \geq \frac{1}{\sqrt{1+\lambda^2}}$. Therefore,

(1)
$$\delta(T_{(z,f_i(z))}X,T_{(z,f_j(z))}X) \geq \frac{\sqrt{2}}{2\lambda\sqrt{1+\lambda^2}}\|(Df_i)_z-(Df_j)_z\|.$$

Let $\gamma \colon [0,\epsilon) \to \overline{\mathbf{B}^d(x_0,t)}$ be the arc given by $\gamma(t) = x + tu$, where $u = \frac{x_0 - x}{\|x_0 - x\|}$. We have $\gamma((0,\epsilon)) \subset \mathbf{B}^d(x_0,t)$ and $\|\gamma'(t)\| = \|u\| = 1$ for all $t \in [0,\epsilon)$. Let $\gamma_i(t) = (\gamma(t), f_i(\gamma(t)))$ and $\gamma_j(t) = (\gamma(t), f_j(\gamma(t)))$. For simplicity, denote $g(t) = f_i \circ \gamma(t) - f_j \circ \gamma(t)$.

By the inequality 1,

$$\delta(T_{\gamma_i(t)}X, T_{\gamma_j(t)}X) \ge \frac{\sqrt{2}}{2\lambda\sqrt{1+\lambda^2}} \|(Df_i)_{\gamma(t)}(\gamma'(t)) - (Df_j)_{\gamma(t)}(\gamma'(t))\|$$

for all $t \in (0, \epsilon)$. Then

$$\delta(T_{\gamma_i(t)}X, T_{\gamma_j(t)}X) \ge \frac{\sqrt{2}}{2\lambda\sqrt{1+\lambda^2}} \|g'(t)\|$$

for all $t \in [0, \epsilon)$, and thus

$$\operatorname{ord}_0 \delta(T_{\gamma_i(t)}X, T_{\gamma_j(t)}X) \le \operatorname{ord}_0 \|g'(t)\| = \operatorname{ord}_0 \|g(t)\| - 1.$$

Thus, since η is a local bi-Lipschitz homeomorphism, we have

$$b := \operatorname{ord}_{0} \| \gamma_{i}(t) - \gamma_{j}(t) \| = \operatorname{ord}_{0} \| f_{i} \circ \gamma(t) - f_{i} \circ \gamma(t) \|$$

$$= \operatorname{ord}_{0} d(\eta^{-1}(\gamma_{i}(t)), \eta^{-1}(\gamma_{j}(t)))$$

$$= \min \{ \operatorname{ord}_{0} \| \gamma_{i}(t) - \gamma_{j}(t) \|, \operatorname{ord}_{0} \delta(T_{\gamma_{i}(t)}X_{i}, T_{\gamma_{j}(t)}X_{j}) \}$$

$$\leq \min \{ b, \operatorname{ord}_{0} \| g'(t) \| \}$$

$$= \min \{ b, b - 1 \} = b - 1,$$

which is a contradiction. Therefore X is a C^1 submanifold.

Claim 2. X is a $C^{1,1}$ submanifold.

Proof of Claim 2. This follows by using the Plücker embedding, but we present a direct proof here.

Since we are dealing with a local problem, we may assume that X is the graph of a C^1 mapping $f: \mathbf{B}^d(0,r) \to \mathbb{R}^{n-d}$, for some r > 0. By shrinking r > 0, if necessary, we may assume that f is λ -Lipschitz. We assume that $\lambda \geq 1$. Then, for $S_1 = T_{(z,f(z))}X \cap \mathbb{S}^{n-1}$ and $S_2 = T_{(w,f(w))}X \cap \mathbb{S}^{n-1}$, we have

$$\delta(T_{(z,f(z))}X, T_{(w,f(w))}X) = d_H(S_1, S_2)$$

$$= \max\{\sup_{a \in S_1} d(a, S_2), \sup_{b \in S_2} d(b, S_1)\},$$

and thus for any $u \in \mathbb{R}^d \setminus \{0\}$, there is $v \in \mathbb{R}^d \setminus \{0\}$ such that

$$\delta(T_{(z,f(z))}X,T_{(w,f(w))}X) \geq \left\| \frac{(u,Df_z(u))}{\|(u,Df_z(u))\|} - \frac{(v,Df_w(v))}{\|(v,Df_w(v))\|} \right\|.$$

Denoting $\tilde{u} = \frac{u}{\|(u, Df_z(u))\|}$ and $\tilde{v} = \frac{v}{\|(v, Df_w(v))\|}$, we have

$$\begin{split} \delta(T_{(z,f(z))}X,T_{(w,f(w))}X) & \geq & \|(\tilde{u},Df_{z}(\tilde{u})) - (\tilde{v},Df_{w}(\tilde{v}))\| \\ & \geq & \frac{\sqrt{2}}{2}(\|\tilde{u}-\tilde{v}\| + \|Df_{z}(\tilde{u}) - Df_{w}(\tilde{v})\|) \\ & \geq & \frac{\sqrt{2}}{2}(\|\tilde{u}-\tilde{v}\| + \frac{1}{\lambda}\|Df_{z}(\tilde{u}) - Df_{w}(\tilde{v})\|) \\ & \geq & \frac{\sqrt{2}}{2}(\|\tilde{u}-\tilde{v}\| + \frac{1}{\lambda}(\|Df_{z}(\tilde{u}) - Df_{w}(\tilde{u})\| - \|Df_{w}(\tilde{u}-\tilde{v})\|)) \\ & \geq & \frac{\sqrt{2}}{2\lambda}\|Df_{z}(\tilde{u}) - Df_{w}(\tilde{u})\|. \end{split}$$

Note that $\|\tilde{u}\| \geq \frac{1}{\sqrt{1+\lambda^2}}$. Therefore,

$$\delta(T_{(z,f(z))}X,T_{(w,f(w))}X) \geq \frac{\sqrt{2}}{2\lambda\sqrt{1+\lambda^2}}\|Df_z - Df_w\|,$$

where $||Df_z - Df_w|| = \sup\{||Df_z(u) - Df_w(u)||; u \in \mathbb{R}^d, ||u|| \le 1\}$. By hypothesis, there is a constant K > 0 such that $\delta(T_{(z,f(z))}X, T_{(w,f(w))}X) \le K||(z,f(z)) - (w,f(w))||$. Then,

$$||Df_{z} - Df_{w}|| \leq K\lambda\sqrt{2 + 2\lambda^{2}}||(z, f(z)) - (w, f(w))||$$

$$\leq K\lambda\sqrt{2 + 2\lambda^{2}}(||z - w|| + ||f(z) - f(w)||)$$

$$\leq K\lambda\sqrt{2 + 2\lambda^{2}}(1 + \lambda)||z - w||.$$

Therefore, Df is Lipschitz, which shows that X is a $C^{1,1}$ submanifold. \square

Now, we assume that $k \geq 1$. By the first part of this proof, X is $C^{1,1}$ smooth.

Since our problem is a local problem, we may assume that X is the graph of a $C^{1,1}$ smooth mapping $h: B \to \mathbb{R}^{n-d}$ and such that h is a definable mapping. Let us write $h = (h_1, ..., h_d)$.

We are going to show that h is $C^{k+1,1}$. This follows from the following claim:

Claim 3. If h is $C^{s,1}$ smooth for some $1 \le s \le k$, then h is $C^{s+1,1}$ smooth.

Proof of Claim 3. Assume that h is $C^{s,1}$ smooth for some $1 \leq s \leq k$. Then X is a $C^{s,1}$ submanifold. Since $\eta \colon \mathcal{N}(X) \to X$ is a $C^{k,1}$ smooth diffeomorphism and $k \geq s$, then $\mathcal{N}(X)$ is a $C^{s,1}$ submanifold as well, and thus $\nu \colon X \to Gr_{\mathbb{R}}(d,n)$ is $C^{s,1}$ smooth. By using the Plücker embedding $p \colon Gr_{\mathbb{R}}(d,n) \to \mathbb{P}(\bigwedge^d \mathbb{R}^n)$, where $\bigwedge^d \mathbb{R}^n$ is the d-th exterior power of \mathbb{R}^n and $\mathbb{P}(\bigwedge^d \mathbb{R}^n)$ is the projectivization of $\bigwedge^d \mathbb{R}^n$, we see that 1 and each partial derivative $\frac{\partial h_i}{\partial x_j}$ are coordinates of $(id \times p) \circ \eta^{-1}(x,h(x))$, and thus $\frac{\partial h_i}{\partial x_j}$ is $C^{s,1}$ smooth. Therefore, h is $C^{s+1,1}$ smooth.

Since h is $C^{1,1}$ smooth, by using recursively Claim 3, h is $C^{k+1,1}$ smooth. Moreover, if $\eta \colon \mathcal{N}(X) \to X$ is a C^{∞} diffeomorphism, then by Claim 3, h is $C^{k,1}$ smooth for all positive integer k. Therefore, h is C^{∞} smooth, and thus X is a C^{∞} submanifold.

Example 3.2. Let $X = \{(x,y) \in \mathbb{C}^2; x^2 = y^3\}$. We have that $\eta \colon \mathcal{N}(X) \to X$ is a bi- α -Hölder homeomorphism for some $\alpha \in (0,1)$. However, X is not a C^1 submanifold.

Example 3.3. Let $X = \{(x,y) \in \mathbb{R}^2; y \geq 0\}$. We have that $\eta \colon \mathcal{N}(X) \to X$ is a C^{∞} diffeomorphism. Indeed, $\eta^{-1}(x) = (x,\mathbb{R}^2)$ for all $x \in X$. However, X is not a C^1 submanifold.

3.2. Metric version of the Theorem of Nobile.

Theorem 3.4. Let $X \subset \mathbb{C}^n$ be a pure dimensional complex analytic set. Then, X is smooth if and only if $\eta \colon \mathcal{N}(X) \to X$ is a homeomorphism that is locally bi-Lipschitz.

Proof. It is clear that if X is smooth, then $\eta: \mathcal{N}(X) \to X$ is a homeomorphism that is locally bi-Lipschitz.

Reciprocally, assume that $\eta \colon \mathcal{N}(X) \to X$ is a homeomorphism that is locally bi-Lipschitz. By Theorem 3.1, X is a $C^{1,1}$ submanifold. By a result proved by Milnor in the introduction of [9], X is smooth. This also follows from the Lipschitz Regularity Theorem [15, Theorem 4.2] (see also [14]). \square

Note that the condition that $\eta \colon \mathcal{N}(X) \to X$ is locally bi-Lipschitz cannot be dropped. Indeed, if $X = \{(x,y) \in \mathbb{C}^2; x^2 = y^3\}$, then $\eta \colon \mathcal{N}(X) \to X$ is a homeomorphism, but X is not smooth at 0.

3.3. Real version of the Theorem of Nobile.

Theorem 3.5. Let $X \subset \mathbb{R}^n$ be a pure d-dimensional real analytic set, and let k be a nonnegative integer number. Then X is a $C^{k+1,1}$ submanifold if, and only if, the mapping $\eta \colon \mathcal{N}(X) \to X$ is a homeomorphism such that η^{-1} is $C^{k,1}$ smooth.

Proof. It is clear that if X is a $C^{k+1,1}$ submanifold, then $\eta \colon \mathcal{N}(X) \to X$ is a homeomorphism such that η^{-1} is $C^{k,1}$ smooth.

Reciprocally, assume that $\eta \colon \mathcal{N}(X) \to X$ is a homeomorphism such that η^{-1} is $C^{k,1}$ smooth.

Since $\eta: \mathcal{N}(X) \to X$ is in particular a bijection, we obtain that $C_4(X, p)$ is a d-dimensional linear subspace for all $p \in X$. In particular, $C_4(X, p)$ continuously varies on p.

We are going to prove that $C_3(X,p) = C_4(X,p)$ for all $p \in X$. Suppose by contradiction that this is not true, that is, suppose that there is $p \in X$ such that $C_3(X,p) \subsetneq C_4(X,p)$. Then $W := int(C_4(X,p) \setminus C_3(X,p))$ is not an empty cone.

We may assume that p = 0 and we choose linear coordinates (x, y) in \mathbb{R}^n such that $C_4(X, 0) = \{(x, y) \in \mathbb{R}^n; y = 0\}$. Let $\pi \colon \mathbb{R}^n \to C_4(X, 0) \cong \mathbb{R}^d$ be the orthogonal projection. Since X is a real analytic set, the multiplicity mod 2 is well-defined. Let U be a sufficiently small neighbourhood of 0 in X such that there is a subanalytic set $\sigma \subset C_4(X, 0)$ such that $\#(\pi^{-1}(v) \cap U)$ (mod 2) is constant for all sufficiently small $v \in C_4(X, 0) \setminus \sigma$.

For $v \in W$ and a sufficiently small neighbourhood U of 0 in X, we have that $\pi^{-1}(tv) \cap U = \emptyset$ for all sufficiently small t > 0. This shows that the multiplicity of X at the origin is $0 \pmod 2$. Since $C_4(X,q)$ continuously varies on q, by shrinking U, if necessary, we may assume that $\delta(C_4(X,0), C_4(X,q)) < 1/3$ for all $q \in U$. In particular, for each $q \in U$, the restriction of π to $C_4(X,q)$ is a linear isomorphism. Therefore, $\pi(U \setminus \operatorname{Sing}(X))$ is an open subset of $C_4(X,0)$. We may assume that $U = X \cap \mathbf{B}^n(0,r)$ for some r > 0.

Let $B = \pi(U)$, $B' = B \setminus \pi(\operatorname{Sing}(X))$, $U' = \pi^{-1}(B') \cap U$, $N = \sup_{x \in B'} \#\pi|_{U'}^{-1}(x)$ and $S = \{x \in B' : \#\pi|_{U}^{-1}(x) = N\}$. By shrinking r, if necessary, we assume that $0 \in \overline{S}$ and $\pi^{-1}(0) \cap U = \{0\}$. We have that S is an open set, $\pi_{U}^{-1}(S)$ has exactly N connected components, say $X_1, ..., X_N$, and for each i, X_i is the graph of a C^1 mapping $f_i \colon S \to \mathbb{R}^{n-d}$. By the assumptions on the tangent cones, we also have that each f_i has bounded derivative. Let $\lambda \geq 1$ be a number such that $\|Df_i\| \leq \lambda$ in S for all $i \in \{1, ..., N\}$.

Since $0 \in \overline{S}$, there is a definable C^1 smooth arc $\gamma \colon [0, \epsilon) \to \overline{S}$ such that $\gamma(0) = x$, $\gamma((0, \epsilon)) \subset S$ and $\|\gamma(t)\| = t$ for all $t \in [0, \epsilon)$. Let $\tilde{\gamma}_i(t) = (\gamma(t), f_i(\gamma(t)))$ and $\tilde{\gamma}_j(t) = (\gamma(t), f_j(\gamma(t)))$.

Since $m(X,0) = 0 \pmod{2}$, we have $N \geq 2$ and, in particular, $\pi|_U$ is not injective.

By proceeding as in the proof of Theorem 3.1, there is a constant \tilde{K} such that $\delta(T_{(z,f_i(z))}X_i,T_{(z,f_j(z))}X_j) \geq \tilde{K}\|(Df_i)_z - (Df_j)_z\|$ for all $z \in S$. Then, $\delta(T_{\gamma_i(t)}X_i,T_{\gamma_j(t)}X_j) \geq \tilde{K}\|(f_i \circ \gamma)'(t) - (f_j \circ \gamma)'(t)\|$ for all $t \in [0,\epsilon)$.

Since η is, in particular, a local bi-Lipschitz homeomorphism, by proceeding in the same way as in the proof of Theorem 3.1, we obtain

$$\operatorname{ord}_0 \|\gamma_i(t) - \gamma_j(t)\| \leq \operatorname{ord}_0 \|\gamma_i(t) - \gamma_j(t)\| - 1,$$

which is a contradiction. Therefore, $C_3(X,p) = C_4(X,p)$ for all $p \in X$. In particular, $C_3(X)$ and $C_4(X)$ coincide linearly. By Theorem 3.1, X is a $C^{k+1,1}$ submanifold.

Consequently, we obtain the following real version of the Theorem of Nobile.

Theorem 3.6. Let $X \subset \mathbb{R}^n$ be a pure d-dimensional real analytic set. Then the following statements are equivalent:

- (1) X is a real analytic submanifold;
- (2) the mapping $\eta \colon \mathcal{N}(X) \to X$ is a real analytic diffeomorphism;
- (3) the mapping $\eta: \mathcal{N}(X) \to X$ is a C^{∞} diffeomorphism;

(4) X is a C^{∞} submanifold.

Proof. It is clear that $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$.

Let us prove (3) \Rightarrow (4). So, assume that $\eta: \mathcal{N}(X) \to X$ is a C^{∞} diffeomorphism. In particular, η^{-1} is $C^{k,1}$ smooth for any nonnegative integer k. By Theorem 3.5 and the proof of Theorem 3.1, X is locally the graph of a mapping a $h: B \to \mathbb{R}^{n-d}$ that is $C^{k+1,1}$ smooth for all nonnegative integer k. So, h is smooth C^{∞} and, therefore, X is a C^{∞} submanifold.

The implication $(4) \Rightarrow (1)$ follows from [5, Proposition 1.1].

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