Concentration around a stable equilibrium for the non-autonomous Φ^4_3 model

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Abstract

We consider time-dependent singular stochastic partial differential equations on the three-dimensional torus. These equations are only well-posed after one adds renormalization terms. In order to construct a well-defined notion of solution, one should put the equation in a more general setting, like the one of regularity structures. In this article, we consider the alternative paradigm of paracontrolled distributions, and get concentration results around a stable deterministic equilibrium for solutions of non-autonomous generalizations of the (Φ^4_3) model.

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1 Introduction

Since Hairer's landmark aticles on regularity structures [1, 2], numerous articles have been written to solve existence and uniqueness problems for important singular stochastic partial differential equations (SPDEs). Singular SPDEs are characterized by the presence of undefined products as soon as one tries to solve them using regular methods. Indeed, the spacetime white noise ξ which appears in many SPDEs has the regularity of a distribution, and we expect solutions to a lot of stochastic PDEs to also have the regularity of a distribution: all non-linerarities in the equation are hence undefined, and consequently the equation itself. A classical example of singular SPDE is the (Φ_3^4) model which is written

$$\partial_t \varphi = \Delta \varphi - \varphi^3 + \xi$$

where we have formally $\varphi:(t,x)\in[0,T]\times\mathbb{T}^3\longmapsto \varphi(t,x)\in\mathbb{R}$, the undefined term in the equation being φ^3 .

To solve this conceptual difficulty, Hairer opts for a very local study of subcritical SPDEs. Generalizing Taylor series to non-continuous functions by adding to monomials new symbols depicting singular stochastic distributions, he first computes what the generalized Taylor expansion of a solution would look like at each point. Interpreting then this generalized Taylor expansion in each point as a function taking values in an algebraic space named regularity structure, he proves that this function solves an abstract fixed point problem, which admits a unique solution. Finally, he builds a reconstruction operator which maps an abstract function to a space-time distribution, whose generalized Taylor expansion at each point is given by the abstract function. We can then verify that this new distribution solves the SPDE in a particular sense, meaning after we add to the equation new diverging terms named "renormalization terms".

While Hairer's method is extremely powerful for existence and uniqueness results, it is an abstract method which is extremely general, making it less useful when one is studying quantitative properties of the solutions. In parallel with Hairer's work, Kupiainen developed a method to study SPDEs based on Wilsonian renormalization group analysis [3], while Gubinelli, Imkeller and Perkowski put forward the theory of paracontrolled distributions in [4], which was used in [5] to get local-in-time well-posedness results for a specific SPDE. The three methods presented have the same notion of solutions, but for the problem we study, the theory of paracontrolled distributions is the handiest one. Indeed for the (Φ_3^4) model, it gives an almost explicit expression of the solutions, and since we are interested in concentration around an equilibrium results, we prefer this to abstract writings of solutions.

Our work is at the crossroads of two series of papers. In [6, 7], the authors study concentration results around an equilibrium for non-autonomous generalizations of the (Φ_d^4) problem in dimension d=1 and d=2, and describe some bifurcation phenomena between equilibrium branches. In these dimensions, the equation is not singular (d=1) or the solution is a distribution but it is "almost" a continuous function (d=2), and therefore we can guess what renormalization terms look like without introducing new theories. In [8, 9], the authors study the (Φ_3^4) model, with a particularly useful diagrambased formalism. More precisely, in [8], Mourrat and Weber get an a priori bound for the (Φ_3^4) model on the torus which is independent of the initial condition while rewriting the paracontrolled calculus associated with solving (Φ_3^4) in the new formalism. In [9], in collaboration with Xu, they compute explicit bounds on the moments of the Hölder norm of the standard stochastic integrals that are needed to study (Φ_3^4) , stochastic integrals we will call symbols from now on.

In this article, we are interested in getting concentration results around a deterministic equilibrium for the solution of a non-autonomous generalization of (Φ_d^4) in dimension d=3. To our knowledge, this problem has not been studied yet, the theory of singular SPDEs being a still young and mainly qualitative theory, especially when one considers non-autonomous equations. The main result of this article is that for ϕ a solution to a non-autonomous generalized (Φ_3^4) equation where the white noise ξ is multiplied against a constant $\sigma \ll 1$, denoting $\overline{\phi}$ an attractive solution to the associated deterministic PDE with the same initial condition as ϕ , we have for any $\varepsilon > 0$ that $\sup_{t \in [0,T]} \|\phi(t) - \overline{\phi}(t)\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}}$ is close to 0 with a high probability. We emphasize that the existence of an attractive deterministic solution is not guaranteed, and that the regime $\sigma \approx 1$ also deserves to be considered, but we highlight that concentration results around stable equilibrium are the first step towards the study of scenarii involving bifurcations.

In order to prove the main concentration results, we will combine several technical tools found in our different references. Here are some insights on the mathematical ideas behind our work.

• First, we underscore that the theory of paracontrolled distributions is formulated in the framework of Besov spaces, and that we will identify the Hölder space \mathcal{C}^{α} with the Besov space $\mathcal{B}^{\alpha}_{\infty,\infty}$ (see Appendix). While this definition of the Hölder spaces seems complicated at first glance since we decompose a function f into an infinite sum of "blocks" $\delta_k f$, it is particularly relevant when f is a random function we want to get concentration results on. Indeed, to control the tail of a random variable one can control its moments, and Proposition A.4 proves that we can bound the p-th moment of $||f||_{\mathcal{C}^{\alpha}}$ with bounds on the p-th

moments of the $\|\delta_k f\|_{L^p}$. The blocks being always smooth while f is often only a distribution, computations are simpler with blocks than with f.

- Second, all the symbols that we use in this article can be expressed with stochastic integrals only involving independent copies of the stochastic white noise ξ . This allows us to use the theory of Wiener chaos and especially Nelson's inequality (Proposition A.11) to control the p-th moment of key random variables with bounds on their second moments. These kinds of bounds are precisely the ones proven in [9].
- Third, a key idea behind the resolution of (Φ_3^4) using paracontrolled distributions is to to regularize previously undefined products by tradingoff temporal regularity for spatial regularity in some equations. Therefore, random temporal Hölder constants appear naturally in our computations, and to get concentration results of solutions of (Φ_3^4) we first need concentration results on some random temporal Hölder constants. Besides, concentration results on $\sup_{t \in [0,T]} ||f(t)||$ for f a symbol are directly implied by concentration results on the temporal Hölder constant of f (since we know the initial condition), making the last ones even more useful in our context. Computing the moments of a Hölder constant is however a non-trivial task, and we cannot directly use the results found of [9] to do this. To overcome this difficulty, we invoke the Garsia-Rodemich-Rumsey inequality proved in [10] to bound the supremum in the definition of the Hölder constant with an integral involving terms we can control. This technical trick reveals itself to be extremely powerful when combined with the other tools presented above, and is the main original idea of this article.
- Fourth, we adapt the formalism of [8] to the non-autonomous context in order to write the solution of non-autonomous (Φ⁴₃) as the sum of a singular part made of symbols and a regular one solution to an explicit equation. We can then combine technical Lemmas found in the Appendix of [8] with the method developed in [7] to transform concentration results on symbols into concentration results on the solution of an equation involving these symbols. We underscore that the results we eventually get are stronger than the strong deviations ones found for instance in [2], since the concentration results we will get on the regular part is for Hölder norms of strictly positive exponents.

The work is divided as follows for the main parts of the article: in the second section, we introduce the main notations, and state the main concentration theorems, while in the third section we prove them. With regards to the appendix, it recalls definitions and important results in the theory of Besov spaces, paracontrolled calculus and Wiener chaos.

2 Setting and main results

2.1 Notations

- \mathcal{H}_k for $k \geq 0$: the k-th homogeneous Wiener chaos.
- $\mathcal{H}_{\leq k} := \bigoplus_{i=0}^k \mathcal{H}_k$ for $k \geq 0$: the k-th inhomogeneous Wiener chaos.
- Π_k for $k \geq 0$: the projection on \mathcal{H}_k .
- $(e^{t\Delta})_{t>0}$: the heat semigroup.
- $(\mathcal{C}^{\alpha}, \|\cdot\|_{\mathcal{C}^{\alpha}}) := (\mathcal{C}^{\alpha}(\mathbb{T}^3), \|\cdot\|_{\mathcal{C}^{\alpha}})$ for $\alpha \in \mathbb{R}$: the space of functions on the 3-dimensional torus which are α -Hölder (see Appendix A.1 for a definition of Hölder spaces of negative exponents).
- $(\mathcal{C}^{\alpha}_{\mathfrak{s}}, \|\cdot\|_{\mathcal{C}^{\alpha}})$ for $\alpha \in \mathbb{R}$: the space of functions on $\mathbb{R} \times \mathbb{T}^3$ which are α -Hölder is space and $\frac{\alpha}{2}$ -Hölder in time.
- $|\tau|$ for a processus $(\tau(t))_{t\geq 0}$: the greatest $\alpha\in\mathbb{R}$ such that $\tau(t)\in\mathcal{C}^{\beta}$ for all $\beta<\alpha$.
- ξ : the space-time white noise.
- $(f,g) := \int_{\mathbb{T}^3} f(x)g(x)dx$ for $f,g \in L^2(\mathbb{T}^3)$.
- \oplus : $(f,g) \in \mathcal{C}^{\alpha} \times \mathcal{C}^{\beta} \mapsto f \oplus g \in \mathcal{C}^{\gamma}$ for specific triples (α,β,γ) : the resonant product (see Appendix A.2 for a precise definition).
- \otimes : $(f,g) \in \mathcal{C}^{\alpha} \times \mathcal{C}^{\beta} \mapsto f \otimes g \in \mathcal{C}^{\gamma}$ for specific triples (α,β,γ) : the paraproduct.
- \otimes : $(f,g) \in \mathcal{C}^{\alpha} \times \mathcal{C}^{\beta} \mapsto f \otimes g = g \otimes f \in \mathcal{C}^{\gamma}$ for specific triples (α,β,γ) : the paraproduct with inverted arguments.
- $[\odot, \odot] : (f, g, h) \in \mathcal{C}^{\alpha} \times \mathcal{C}^{\beta} \times \mathcal{C}^{\gamma} \longmapsto (f \odot g) \odot h f(g \odot h) \in \mathcal{C}^{\delta}$ for specific quadruples $(\alpha, \beta, \gamma, \delta)$: the commutator between \odot and \odot .
- $[e^{t\Delta}, \odot] : (f, g) \in \mathcal{C}^{\alpha} \times \mathcal{C}^{\beta} \longmapsto e^{t\Delta}(f \odot g) f \odot (e^{t\Delta}g) \in \mathcal{C}^{\gamma}$ for specific triples (α, β, γ) : the commutator between $e^{t\Delta}$ and \odot .

2.2 Setting and main theorems

We are considering the following general equation in dimension 3:

$$\begin{cases}
\partial_t \varphi = \Delta \varphi + F(t, \varphi) + \sigma \xi \\
\varphi(0) = \varphi_0
\end{cases}$$
(1)

where F is a third degree polynomial with non-constant coefficients

$$F(t,\varphi) = a_3(t)\varphi^3 + a_2(t)\varphi^2 + a_1(t)\varphi + a_0(t).$$

Here we will assume that $a_0, a_1, a_2, a_3 : [0, T] \to \mathbb{R}$ are differentiable (and therefore continuous) and that $a_3 < 0$ (and therefore bounded away from 0). Since we are interested in concentration results around an equilibrium for potentially large times T, we will make the dependance in T of all constants explicit.

Remark 2.1. The degree of F has to be odd for the solution of the associated PDE not to explode, and the equation is no longer subcritical if the degree is greater or equal to five. Since there is no known theory to solve non-subcritical SPDE, our setting is the most general we can expect.

Since the computations needed to get the results we want are sometimes long, we want to simplify right now our expressions. For instance, we can replace the term a_3 by -1. Indeed, writing $\varphi = b(t)\psi$, we get

$$b(t)\partial_t \psi = b(t)\Delta \psi + a_3(t)b(t)^3 \psi^3 + a_2(t)b(t)^2 \psi^2 + (a_1(t)b(t) - b'(t))\psi + a_0(t) + \sigma \xi$$

so, taking $b = \frac{1}{\sqrt{-a_3}}$ which is differentiable, and dividing the equation by b(t) we get

$$\partial_t \psi = \Delta \psi - \psi^3 + b_2(t)\psi^2 + b_1(t)\psi + b_0(t) + \sigma \xi^b$$

where $b_2(t)=a_2(t)b(t),\ b_1(t)=\frac{a_1(t)b(t)-b'(t)}{b(t)},\ b_0=\frac{a_0}{b}$ and $\xi^b=\frac{1}{b}\xi$. This new gaussian white noise is not homogenous in time, but since b is positive, bounded away from 0 and continuous on $[0,T],\ \xi^b$ has essentially the same properties as ξ . For instance, for $\varphi_1,\varphi_2\in L^2(\mathbb{R}\times\mathbb{T}^3)$ of support included in $[0,T]\times\mathbb{T}^3$, we have

$$\mathbb{E}((\varphi_1, \xi^b)(\varphi_2, \xi^b)) = \int_{[0,T] \times \mathbb{T}^3} \frac{1}{b(t)^2} \varphi_1(t, x) \varphi_2(t, x) dt dx.$$

We will therefore drop the *b*-notation in ξ^b to simplify writing. We will now get rid of the constant term and take an initial condition equal to 0. Indeed, for $\phi = \psi - \overline{\phi}$ where $\overline{\phi}$ is a (deterministic) solution of

$$\begin{cases} \frac{\partial_t \overline{\phi} = \Delta \overline{\phi} + F(t, \overline{\phi})}{\overline{\phi}(0) = \frac{1}{b(0)} \varphi_0} \end{cases}$$

we have that

$$\partial_t \phi = \Delta \phi + (F(t, \overline{\phi} + \phi) - F(t, \overline{\phi})) + \sigma \xi$$

therefore, since $\phi \in \mathbb{R} \mapsto F(t, \overline{\phi} + \phi) - F(t, \overline{\phi})$ is a polynomial in ϕ equal to 0 in $\phi = 0$, we get that

$$\begin{cases} \partial_t \phi = \Delta \phi + \tilde{F}(t, \phi) + \sigma \xi \\ \phi(0) = \overline{0} \end{cases}$$
 (2)

with $\tilde{F}(t,\phi) = -\phi^3 + f_2(t)\phi^2 + f_1(t)\phi$. We now take $\sigma > 0$ and write $a := f_1$.

Assumption. From now on we assume that a < 0, since it is the most relevant case (see Section 2.3).

Remark 2.2. The hypothesis a < 0 is not necessary to get the concentration results we will prove in this article, but it has a massive impact on the T-dependence of the constants in it. Indeed, if we assume that a may take positive values, we will have to bound terms like $\exp(\int_s^t a(u) du)$ by $\exp(T \times \sup a)$, and therefore all constants appearing in our concentration results will exponentially depend of T, making them far less relevant for large times T.

The solutions of stochastic PDEs are often only random distributions and instead of studying them directly, we write them as the sum of a regular part and of some specific random distributions that we represent with symbols. All the symbols we will need in our computations can be built from $^{\dagger}_n$ solution of

$$\left\{ \begin{array}{c} \partial \mathring{\mathbf{1}}_n = \Delta \mathring{\mathbf{1}}_n + a(t) \mathring{\mathbf{1}}_n + \sigma \xi_n \\ \mathring{\mathbf{1}}_n(0) = \overline{0} \end{array} \right. ,$$

where ξ_n is a regularized white noise such that ξ_n is smooth and $\xi_n \to \xi$. Here $\mathring{}_n$ is a smooth space-time function, but taking the limit $n \to \infty$ we get $\mathring{} = \lim \mathring{}_n$ an irregular space-time distribution. We will now construct the other symbols, but in order to do that we need first to introduce elements of paracontrolled calculus. The main tool of paracontrolled calculus is the decomposition of the standard product into three bilinear operators. More precisely, for two functions f,g satisfying certain regularity hypotheses we have

$$fg = f \otimes g + f \otimes g + f \otimes g$$

with \odot and \odot the paraproducts, and \odot the resonant product, that are all expressed using the decomposition of functions in Hölder spaces into infinite sums of Paley-Littlewood blocks. The technical details are left for the Appendix, but we want to underscore that \odot , \odot and \odot are relevant for our study because they interact particularly well with one another, so we can use technical results such as Proposition A.7 to give meaning to products that are supposed to be undefined.

$$\uparrow(t) := \lim_{n \to +\infty} \uparrow_{n}$$

$$\checkmark(t) := \lim_{n \to \infty} \checkmark_{n}(t) = \lim_{n \to +\infty} \uparrow_{n}(t)^{2} - c_{n}(t)$$

$$\Upsilon(t) := \lim_{n \to \infty} I(\uparrow_{n}^{2} - c_{n})(t)$$

$$\Psi(t) := \lim_{n \to +\infty} \Psi_{n} = \lim_{n \to +\infty} I(\uparrow_{n}^{3} - 3c_{n}\uparrow)(t)$$

$$\overset{\bullet}{\checkmark}(t) := \lim_{n \to +\infty} \overset{\bullet}{\checkmark}_{n}(t) \ominus \overset{\bullet}{\uparrow}_{n}(t)$$

$$\overset{\bullet}{\checkmark}(t) := \lim_{n \to +\infty} \overset{\bullet}{\checkmark}_{n}(t) = \lim_{n \to \infty} I(\overset{\bullet}{\checkmark}_{n})(t) \ominus \overset{\bullet}{\checkmark}_{n}(t) - 2c'_{n}(t)$$

$$\overset{\bullet}{\checkmark}(t) := \lim_{n \to \infty} \overset{\bullet}{\checkmark}_{n}(t) \ominus \overset{\bullet}{\checkmark}_{n}(t) - 6c'_{n}(t) \overset{\bullet}{\uparrow}_{n}$$

with $c_n(t) := \mathbb{E}(\mathbf{1}_n(t)^2) \propto \sigma^2$ and $c'_n(t) := \mathbb{E}(I(\mathbf{1}_n(t)) \otimes \mathbf{1}_n(t)) \propto \sigma^4$, and where we have for $\alpha(t,u) = \int_u^t a(s) ds$

$$I(f)(t) = \int_0^t e^{\alpha(t,u)} e^{(t-u)\Delta} f(u) du,$$

the solution to $(\partial_t - \Delta - a(t))g = f(t)$ with initial condition g(0) = 0.

Remark 2.3. We will sometimes need in our computations $\mathfrak{V}_n = \mathring{1}_n^3 - 3c_n$, but we don't put its limit in \mathcal{T} because it is very irregular in time and we cannot therefore give meaning to " $\mathring{\mathbf{V}}(t)$ " for $t \in [0,T]$.

Finally, we introduce for all these objects their chaos decomposition (CD). The precise definitions are left for the Appendix, but in few words we can write all our symbols as finite sums of space-time distributions called Wiener chaos, that are homogeneous in σ . The largest $k \in \mathbb{N}$ such that $\tau \in \mathcal{T}$ admits a component in the Wiener chaos of order k is denoted n_{τ} and always corresponds to the number of leaves of τ . We have the following table, with all information inside found in [9]:

$\tau(t)$	$^{\dagger}(t)$	v (t)	$Y^{\bullet}(t)$	Ψ (t)	(t)	(t)	* (t)
au	-1/2	-1	1	1/2	0	0	-1/2
$^{\mathrm{CD}}$	\mathcal{H}_1	\mathcal{H}_2	\mathcal{H}_2	\mathcal{H}_3	$\mathcal{H}_2 \bigoplus \mathcal{H}_4$	$\mathcal{H}_2 \bigoplus \mathcal{H}_4$	$\mathcal{H}_1 \bigoplus \mathcal{H}_3 \bigoplus \mathcal{H}_5$
$n_{ au}$	1	2	2	3	4	4	5

Remark 2.4. If we consider a space-time setting as in [2], we can only expect \uparrow to be in $C_{\mathfrak{s}}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3 \times \mathbb{R})$, so its time regularity is $C^{-\frac{1}{4}-\frac{\varepsilon}{2}}$ and the notation " \uparrow (t)" is a priori meaningless. It is thanks to the computations made in [9] that we know that " \uparrow (t)" has a meaning as an element of $C^{-\frac{1}{2}-\varepsilon}$.

Now that we have all the definitions needed, we will state our first concentration result on the spatial norm of the symbols of \mathcal{T} .

Theorem 2.5. For all $\tau \in \mathcal{T}$, $\lambda \in (0,1)$ and $\alpha < |\tau| - \lambda$, there exists $c_{\tau} := c_{\tau}(\lambda), d_{\tau} := d_{\tau}(\lambda) > 0$ independent of T such that for $k \leq n_{\tau}$ and h > 0

$$\mathbb{P}\left(\sup_{t\in[0,T]}\|\Pi_k\tau(t)\|_{\mathcal{C}^{\alpha}}>h^k\right)\leq d_{\tau}\exp\left(-\frac{c_{\tau}}{T^{\frac{\lambda}{k}}}\frac{h^2}{\sigma^2}\right).$$

Remark 2.6. We need to put Π_k in the formula because the scaling in σ is not the same for the different components of τ in the Wiener chaos decomposition. If we fix σ and only consider a variable h, a similar inequality would hold without the Π_k . Since in this article we are interested in the case $\sigma \ll h$, we prefer to use this formulation.

Remark 2.7. While c_{τ} and d_{τ} do not depend of T, they do depend of $a_{+} = -\sup a$ and $a_{-} = -\inf a$.

Now that we have introduced the symbols of \mathcal{T} and established that we have a strong control on their norms, we can go back to our main equation. A key remark of singular SPDE theory, is that there is no solution to our equation in the classical sense. Indeed, knowing that the space regularity of ξ is $\mathcal{C}^{-\frac{5}{2}-\varepsilon}$, Schauder's theory gives us that the expected space regularity of the solution to (2) is $\mathcal{C}^{-\frac{1}{2}-\varepsilon}$ making all the non-linear terms undefined. This underscores the limits of the standard formalism when describing phenomena associated with stochastic PDEs, and we now have to introduce the procedure called renormalization.

Instead of saying that ϕ is the solution of (2), we say that $\phi = \lim_{n \to +\infty} \phi_n$ where ϕ_n is the unique smooth solution of a "renormalized equation" where additional (diverging with n) terms are added, here highlighted in red:

$$\begin{cases} \partial_t \phi_n = \Delta \phi_n - (\phi_n^3 - 3c_n(t)\phi) + f_2(t)(\phi_n^2 - c_n(t)) + a(t)\phi_n + \sigma \xi_n \\ \phi_n(0) = \overline{0} \end{cases}$$
(3)

Then, considering $\theta_n = \phi_n - {}^{\dagger}_n$, we have

$$\begin{split} \partial_t \theta_n &= \Delta \theta_n - [(\theta_n + \mathring{\uparrow}_n)^3 - 3c_n(t)(\theta_n + \mathring{\uparrow}_n)] + f_2(t)[(\theta + \mathring{\uparrow}_n)^2 - c_n(t)] + a(t)\theta \\ &= \Delta \theta_n - \theta_n^3 - 3\mathring{\uparrow}_n \theta_n^2 - 3(\mathring{\uparrow}_n^2 - c_n(t))\theta_n - (\mathring{\uparrow}_n^3 - 3c_n(t)\mathring{\uparrow}_n) \\ &+ f_2(t)\theta_n^2 + 2f_2(t)\theta_n\mathring{\uparrow}_n + f_2(t)(\mathring{\uparrow}_n^2 - c_n(t)) + a(t)\theta_n \\ &= \Delta \theta_n - \theta_n^3 - 3\mathring{\uparrow}_n \theta_n^2 - 3 \mathring{\checkmark}_n \theta_n - \mathring{\checkmark}_n \\ &+ f_2(t)\theta_n^2 + 2f_2(t)\theta_n\mathring{\uparrow}_n + f_2(t) \mathring{\checkmark}_n + a(t)\theta_n. \end{split}$$

Contrary to (3), all the terms in this equation admit a limit when n goes to infinity, so for $\theta := \lim_{n \to +\infty} \theta_n$, we have

$$(\partial_t - \Delta - a(t))\theta = -\theta^3 - 3 \cdot \theta^2 - 3 \cdot \theta - \cdot + f_2(t)\theta^2 + 2f_2(t)\theta \cdot + f_2(t) \cdot \nabla$$

We here face a new difficulty, since the term Ψ is in $C_{\mathfrak{s}}^{-\frac{3}{2}-\varepsilon}$, so Schauder's theory predicts that the solution of this equation would be at most of space regularity $C^{\frac{1}{2}-\varepsilon}$, making the product $\Psi \theta$ undefined. We therefore use a similar method as before by adding a second renormalization constant $c'_n(t)$ in the equation of θ_n and looking at the equation solved by $u_n = \theta_n + \Psi_n$ before taking the limit (we here use that $(\partial_t - \Delta - a(t))\Psi = \Psi$ to make the irregular term Ψ disappear).

Remark 2.8. We added a first renormalization term when we tried to define θ and a second one we tried to define u. If we go back to the equation of ϕ_n , the first term is equal to $+3c_n(t)\phi_n - f_2(t)c_n(t)$, while we will see that the second one is equal to

$$-6c'_n(t)(3^{\dagger}_n + 3\theta_n) = -6c'_n(3^{\dagger}_n + 3(\phi_n - {}^{\dagger}_n)) = -18c'_n(t)\phi_n.$$

Eventually the renormalized equation solved by ϕ_n is

$$\begin{cases} \partial_t \phi_n = \Delta \phi_n - \phi_n^3 + f_2(t)\phi_n^2 + a(t)\phi_n + (3c_n(t) - 18c_n'(t))\phi_n - f_2(t)c_n(t) + \sigma \xi \\ \phi(0) = \overline{0} \end{cases}$$
(4)

where we can show that $c_n(t) \approx n$ and $c'_n(t) \approx \ln(n)$.

The technical details are left to Section 3.3.1, but we can show that space regularity of $u := \lim u_n$ should be $\mathcal{C}^{1-\varepsilon}$. This is still not enough to give meaning to all terms in the equation solved by u, since there are in it products of the form ∇u , and the space regularity of ∇ is only $\mathcal{C}^{-1-\varepsilon}$. To solve this last difficulty, we need to decompose some products into their paraproducts and resonant product components, and following exactly the same method as in [8], we find that we can write $u = v + w + 3I(\nabla)$ with (u, w) solution of

$$\begin{cases}
(\partial_t - \Delta - a(t))v &= F(v+w), \\
(\partial_t - \Delta - a(t))w &= G(v,w),
\end{cases}$$
(5)

for F and G specific functions. Technical details are again left to Section 3.3.1, but we stress right now that all terms in (5) are well-defined and that, contrary to the other equations above, (5) is well-posed.

Remark 2.9. The term $+3I(\checkmark)$ in the decomposition of u is not usually met when dealing with (Φ_3^4) and cannot be found in [8]. We subtracted it from the w of [8] because it has a component in the Wiener chaos of order 1, which creates technical difficulties with regards to the σ -homogeneity in some proofs (see Remark 3.8).

Now that we have rigorously defined functions v and w, we can state the main result of this article.

Theorem 2.10. For $\varepsilon > 0$ and $\lambda \in (0, \frac{\varepsilon}{3} \wedge 1)$, there exists $h_0 > 0$, and $C := C(\lambda), D := D(\lambda) > 0$ independent of T such that for all $h \in (0, h_0)$ we have

$$\mathbb{P}\left(\sup_{t\in[0,T]}\max(\|v(t)\|_{\mathcal{C}^{1-2\varepsilon}},\|w(t)\|_{\mathcal{C}^{\frac{3}{2}-2\varepsilon}})\geq h\right)\leq D\exp\left(-\frac{C}{(1+T)^{\lambda}}\frac{h^2}{\sigma^2}\right).$$

Here we recall that for φ solution of the renormalized version of (1) (see Remark 2.8), we have $\varphi = b(t) \left[\overline{\phi} + \overline{} - \overline{} + 3I(\overline{}) + (v+w) \right]$.

Remark 2.11. Here † , $\overset{\bullet}{\mathbf{Y}}$ and $\overset{\bullet}{\mathbf{Y}}$ are defined with the modified noise ξ^b .

2.3 An application: the non-autonomous (Φ_3^4) model

We consider the equation

$$\begin{cases}
 \partial_t \varphi = \Delta \varphi - \varphi^3 + \gamma(t)\varphi + \xi \\
 \varphi(0) = \varphi_0
\end{cases}$$
(6)

which is (1) with $F(t,\varphi) = -\varphi^3 + \gamma(t)\varphi$. We have therefore b = 1, and we can decompose a solution φ of (6) as

$$\varphi = \overline{\phi} + \phi$$

where $\overline{\phi}$ is the solution of the deterministic equation

$$\begin{cases}
\frac{\partial_t \overline{\phi} = \Delta \overline{\phi} - \overline{\phi}^3 + \gamma(t) \overline{\phi} \\
\overline{\phi}(0) = \varphi_0
\end{cases}$$
(7)

and ϕ is the solution of

$$\begin{cases} \partial_t \phi = \Delta \phi + \tilde{F}(t, \phi) + \sigma \xi \\ \phi(0) = 0 \end{cases}$$

with $\tilde{F}(t,\phi) = -\phi^3 + f_2(t)\phi^2 + f_1(t)\phi$. We will look for stable, constant in space solutions to (7), that is to say applications $\overline{\phi}: [0,T] \to \mathbb{R}$ such that

$$\begin{cases} \partial_t \overline{\phi}(t) = -\overline{\phi}(t)^3 + \gamma(t)\overline{\phi}(t) \\ \gamma(t) - 3\overline{\phi}(t)^2 < 0 \end{cases}$$
 for all $t \in [0, T]$ (8)

the second condition being a rewriting of $a := f_1 < 0$. There are two noteworthy particular cases:

- If $\gamma(t) < 0$ for all $t \in [0,T]$, the condition $\gamma(t) 3\overline{\phi}(t)^2 < 0$ is always verified. Therefore, for any constant in space initial condition, we have that $\overline{\phi}$ is a stable solution.
- If $\gamma(t) > 0$ for all $t \in [0,T]$ and γ is decreasing, we can check that $\phi_+(t) = \sqrt{\gamma(t)/3}$ is a strict subsolution to (7) and $\phi_-(t) = -\sqrt{\gamma(t)/3}$ is a strict supersolution. Therefore, if $\overline{\phi}(0) > \phi_+(0)$ (respectively $\overline{\phi}(0) < \phi_-(0)$), we have that $\overline{\phi}(t) > \phi_+(t)$ for all $t \in [0,T]$ (respectively $\overline{\phi}(t) < \phi_-(t)$ for all $t \in [0,T]$), which means that $\gamma(t) 3\overline{\phi}(t)^2 < 0$. Eventually, for any constant in space initial condition taken outside of $[-\sqrt{\gamma(0)/3}, \sqrt{\gamma(0)/3}]$, we have that $\overline{\phi}$ is a stable solution.

For such a stable, constant in space solution $\overline{\phi}$ and for $\sigma \ll 1$, Theorem 2.10 gives us that v+w is close to 0 for $t \in [0,T]$ and likewise, Theorem 2.5 gives us that \uparrow , $\ref{posterior}$ and $I(\ref{posterior})$ are close to 0 for $t \in [0,T]$. Eventually we have that $\varphi - \overline{\phi} = \uparrow - \ref{posterior} + 3I(\ref{posterior}) + (v+w)$ is close to 0 for $t \in [0,T]$.

Remark 2.12. Since $I(\stackrel{\bullet}{\checkmark})$ is not in \mathcal{T} we cannot directly use Theorem 1 on it, but since we proved that $\|\stackrel{\bullet}{\checkmark}\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}}$ has a gaussian tail, Schauder's estimate implies that $\|I(\stackrel{\bullet}{\checkmark})\|_{\mathcal{C}^{\frac{3}{2}-2\varepsilon}}$ also has a gaussian tail.

3 Proof of the main theorems

3.1 A note on bounds from [9]

We must first notice that we cannot directly use the inequalities proven in [9] since we are not exactly in the same setting. Indeed, in [9] the authors consider formally for $P_{t-s} = e^{-(t-s)}e^{(t-s)\Delta}$

$$\tilde{\dagger}(t) = \int_{-\infty}^{t} P_{t-s}(\xi(s)) ds,$$

that is an "ancient solution" to the homogeneous in time stochastic heat equation. Here, we consider for $P(t,s)=e^{\int_s^t a(u)\mathrm{d}u}e^{(t-s)\Delta}$

$$^{\dagger}(t) = \int_{0}^{t} P(t, s)(\xi(s)) ds,$$

that is a solution to the non-autonomous heat equation equal to 0 in 0. We can however prove that bounds of the type [9, (3.13)&(3.15)] are still valid in our setting, and that constants in them are independent of T. It would be tedious to prove this for all seven symbols, so we will prove it for $^{\uparrow}$. The reader can convince themself that those bounds work for all other symbols.

We have for $t \in [0, T]$ and $\omega \in \mathbb{Z}^3$

$$\hat{\mathbf{f}}(t,\omega) = \int_0^t \hat{P}(t,s) dW(s,\omega)$$
$$= \int_0^t e^{\alpha(t,s)} e^{-(t-s)4\pi^2|\omega|^2} dW(s,\omega)$$

Then, writing $a_{+} = -\sup_{[0,T]} a > 0$, we have that $a + a_{+} \leq 0$ and therefore

$$\mathbb{E}[|\hat{\mathbf{f}}(t,\omega)|^2] = \int_0^t \hat{P}(t,s)^2 ds$$

$$= \int_0^t e^{2\int_s^t [a(u) + a_+] du} e^{-2(t-s)(a_+ + 4\pi^2 |\omega|^2)} ds$$

$$\leq \int_0^t e^{-2(t-s)(a_+ + 4\pi^2 |\omega|^2)} ds$$

$$\leq \frac{1}{2(a_+ + 4\pi^2 |\omega|^2)}$$

This is exactly the estimate [9, (3.15)] we want and it only depends of the values that a takes, being therefore independent of T. If we now look at

temporal differences we have for $0 \le s \le t \le T$

$$\begin{split} \mathbb{E}[|\hat{\mathbf{f}}(t,\omega) - \hat{\mathbf{f}}(s,\omega)|^2] &\leq \int_0^s |\hat{P}(t,u) - \hat{P}(s,u)|^2 \mathrm{d}u \\ &\leq \int_0^s |e^{\alpha(t,u)}e^{-(t-u)4\pi^2|\omega|^2} - e^{\alpha(s,u)}e^{-(s-u)4\pi^2|\omega|^2}|^2 \mathrm{d}u \\ &\leq 2\int_0^s |e^{\alpha(s,u)}e^{-(t-u)4\pi^2|\omega|^2} - e^{\alpha(s,u)}e^{-(s-u)4\pi^2|\omega|^2}|^2 \mathrm{d}u \\ &\quad + 2\int_0^s |(e^{\alpha(t,s)} - 1)e^{\alpha(s,u)}e^{-(t-u)(4\pi^2|\omega|^2)}|^2 \mathrm{d}u \\ &\leq 2\int_0^s |e^{-(t-u)(a_+ + 4\pi^2|\omega|^2)} - e^{-(s-u)(a_+ + 4\pi^2|\omega|^2)}|^2 \mathrm{d}u \\ &\quad + 2\int_0^s |(e^{\alpha(t,s)} - 1)e^{-(t-u)(a_+ + 4\pi^2|\omega|^2)}|^2 \mathrm{d}u \\ &\quad \leq 2|e^{-(t-s)(a_+ + 4\pi^2|\omega|^2)} - 1|^2\int_0^s e^{-2(s-u)(a_+ + 4\pi^2|\omega|^2)} \mathrm{d}u \\ &\quad + 2|e^{\alpha(t,s)} - 1|^2\int_0^s e^{-2(t-u)(a_+ + 4\pi^2|\omega|^2)} \mathrm{d}u \end{split}$$

We then use the standard inequalities $|1 - e^{-x}| \le 1 \land x \le 1 \land x^{\frac{\lambda}{2}}$ for $x \ge 0$ and $\lambda \in (0,1)$. Denoting $a_{-} = -\inf a$ we have

$$\mathbb{E}[|\hat{\mathbf{1}}(t,\omega) - \hat{\mathbf{1}}(s,\omega)|^{2}] \leq 2(1 \wedge (t-s)^{\lambda} (a_{+} + 4\pi^{2}|\omega|^{2})^{\lambda}) \int_{0}^{s} e^{-2(s-u)(a_{+} + 4\pi^{2}|\omega|^{2})} du$$

$$+ 2(1 \wedge (a_{-})^{\lambda} (t-s)^{\lambda}) \int_{0}^{s} e^{-2(t-u)(a_{+} + 4\pi^{2}|\omega|^{2})} du$$

$$\leq 2(t-s)^{\lambda} ((a_{+} + 4\pi^{2}|\omega|^{2})^{\lambda} + (a_{-})^{\lambda}) \frac{1}{2(a_{+} + 4\pi^{2}|\omega|^{2})}$$

$$\leq C_{1}(t-s)^{\lambda} \langle \omega \rangle^{-2+2\lambda}$$

where $\langle \omega \rangle = \sqrt{1 + |\omega|^2}$ and $C_1 := C(a_+, a_-)$ is independent of λ since we can bound terms like $(a_-)^{\lambda}$ by $(1 + a_-)$. This is exactly the estimate [9, (3.17)] for ¶ . In fact, it is better than [9, (3.17)], since in [9] they ask for |t-s| to be less than 1, while we have an inequality that does not depend of T and is uniform in $0 \le s \le t \le T$.

3.2 Proof of Theorem 2.5

We first underscore that the constant in front of ξ in the definition of \dagger is not 1 but σ , so all occurrences of ξ in equations are replaced by $\xi^{\sigma} = \sigma \xi$ and the iterated stochastic integrals are against an element of the form $\xi^{\sigma}(\mathrm{d}z_1)\cdots\xi^{\sigma}(\mathrm{d}z_k) = \sigma^k\xi(\mathrm{d}z_1)\cdots\xi(\mathrm{d}z_k)$. Therefore, writing

$$\tau = \sum_{\ell=0}^{n_{\tau}} T_{\ell},$$

with T_{ℓ} a process in \mathcal{H}_{ℓ} , we have $T_{\ell} = \sigma^{\ell} \tilde{T}_{\ell}$ with \tilde{T}_{ℓ} independent of σ . We write the computations that follow with τ in order not to overload the presentation, but they trivially work for all \tilde{T}_{ℓ} .

Proposition 3.1. Let $\tau \in \mathcal{T}$, and $\beta < |\tau| - \lambda$ for $\lambda \in (0,1)$. Then there exist $\ell_{\tau} := \ell_{\tau}(\lambda)$ and $m_{\tau} := m_{\tau}(\lambda)$ independent of T such that for $\gamma_0 := \gamma_0(T,\lambda) \in (\frac{\lambda}{4},\frac{\lambda}{2})$, $k \leq n_{\tau}$ and all $\gamma \in (\gamma_0,\frac{\lambda}{2})$ we have

$$\mathbb{P}\left([\Pi_k \tau]_{\beta,\gamma} > h^k\right) \le m_\tau \exp\left(-\ell_\tau \frac{h^2}{\sigma^2}\right),\,$$

where $[f]_{\beta,\gamma} = \sup_{0 \le s < t \le T} \frac{\|f(t) - f(s)\|_{\mathcal{C}^{\beta}}}{|t - s|^{\gamma}}$

The proof relies heavily on the Garsia-Rodemich-Rumsey Lemma (see [10, Lemma 1.1]).

Lemma 3.2. Let $(\mathcal{B}, |\cdot|)$ be a Banach space, $f: [0,1] \to \mathcal{B}$ a continuous function, $\Psi: \mathbb{R}_+ \to \mathbb{R}_+$ a strictly increasing function with $\Psi(+\infty) = +\infty$ and $p: [0,1] \to [0,1]$ strictly increasing with p(0) = 0, such that

$$\int_0^1 \int_0^1 \Psi\left[\frac{|f(x) - f(y)|}{p(|x - y|)}\right] \mathrm{d}x \mathrm{d}y =: B < +\infty.$$

Then, for all $s, t \in [0, 1]$, we have

$$|f(t) - f(s)| \le 8 \int_0^{|t-s|} \Psi^{-1} \left(\frac{4B}{u^2}\right) dp(u)$$

Proof. (Proposition 3.1)

We take $\Psi(u) = |u|^p$ and $p(u) = |u|^{\gamma' + \frac{1}{p}}$ in Lemma 3.2, where $\gamma' \in (\gamma, \frac{\lambda}{2})$. We then get that

$$|f(t) - f(s)| \le 8 \int_0^{|t-s|} \frac{(4B)^{\frac{1}{p}}}{u^{\frac{2}{p}}} (\gamma' + \frac{1}{p}) u^{\gamma' - 1 + \frac{1}{p}} du$$

$$\le 8 \cdot 4^{\frac{1}{p}} (\gamma' + \frac{1}{p}) B^{\frac{1}{p}} \int_0^{|t-s|} u^{\gamma' - \frac{1}{p} - 1} du$$

$$\le 8 \cdot 4^{\frac{1}{p}} \frac{\gamma' + \frac{1}{p}}{\gamma' - \frac{1}{p}} |t - s|^{\gamma' - \frac{1}{p}} \left(\int_0^1 \int_0^1 \frac{|f(x) - f(y)|^p}{|x - y|^{\gamma' p + 1}} dx dy \right)^{\frac{1}{p}}$$

The only term involving t, s on the right-hand side is $|t-s|^{\gamma'-\frac{1}{p}}$ so we have

$$\sup_{0 \le s < t \le 1} \frac{|f(t) - f(s)|^p}{|t - s|^{\gamma'p - 1}} \le \left(8 \cdot 4^{\frac{1}{p}} \frac{\gamma' + \frac{1}{p}}{\gamma' - \frac{1}{p}}\right)^p \int_0^1 \int_0^1 \frac{|f(x) - f(y)|^p}{|x - y|^{\gamma'p + 1}} \mathrm{d}x \mathrm{d}y.$$

This inequality is noteworthy, because it says that a control on the integral of a quantity implies a control on a closely related supremum. Let us now

apply it to the case $f(t) = \tau(T \cdot t)$ where we take $(\mathcal{B}, |\cdot|) = (\mathcal{C}^{\beta}, ||\cdot||_{\mathcal{C}^{\beta}})$. We have

$$\sup_{0 \le s < t \le 1} \frac{\|\tau(t \cdot T) - \tau(s \cdot T)\|_{\mathcal{C}^{\beta}}^{p}}{|t - s|^{\gamma'p - 1}} \\
\le \left(8 \cdot 4^{\frac{1}{p}} \frac{\gamma' + \frac{1}{p}}{\gamma' - \frac{1}{p}}\right)^{p} \left(\int_{0}^{1} \int_{0}^{1} \frac{\|\tau(T \cdot x) - \tau(T \cdot y)\|_{\mathcal{C}^{\beta}}^{p}}{|x - y|^{\gamma'p + 1}} dx dy\right)$$

therefore, doing a change of variable we get

$$\sup_{0 \le s < t \le T} \frac{\|\tau(t) - \tau(s)\|_{\mathcal{C}^{\beta}}^{p}}{|t - s|^{\gamma'p - 1}} T^{\gamma'p - 1}$$

$$\le \left(8 \cdot 4^{\frac{1}{p}} \frac{\gamma' + \frac{1}{p}}{\gamma' - \frac{1}{p}}\right)^{p} \left(\int_{0}^{T} \int_{0}^{T} \frac{\|\tau(x) - \tau(y)\|_{\mathcal{C}^{\beta}}^{p}}{|x - y|^{\gamma'p + 1}} \mathrm{d}x \mathrm{d}y\right) \frac{1}{T^{2}} T^{\gamma'p + 1}$$

and we can simplify the terms in T

$$\sup_{0 \le s < t \le T} \frac{\|\tau(t) - \tau(s)\|_{\mathcal{C}^{\beta}}^{p}}{|t - s|^{\gamma'p - 1}} \le \left(8 \cdot 4^{\frac{1}{p}} \frac{\gamma' + \frac{1}{p}}{\gamma' - \frac{1}{p}}\right)^{p} \left(\int_{0}^{T} \int_{0}^{T} \frac{\|\tau(x) - \tau(y)\|_{\mathcal{C}^{\beta}}^{p}}{|x - y|^{\gamma'p + 1}} dx dy\right)$$
(9)

Since the left-hand side is essentially the p-th power of the random quantity we want to control (we just have to replace $\gamma'p-1$ by γp), (9) implies that to have an estimate on the p-th moment of the Hölder constant, we just need an estimate on $\mathbb{E} \frac{\|\tau(t)-\tau(s)\|_{\mathcal{C}^{\beta}}^p}{\|x-y\|^{\gamma'p+1}}$.

To get these estimates, we will use that $\tau: t \mapsto \tau(t)$ is in the n_{τ} -th inhomogeneous Wiener chaos (and therefore $\delta_k \tau(t)$ also) where n_{τ} is the number of leaves of τ . In other words, we have that $(\tau(t), \phi) \in \mathcal{H}_{\leq n_{\tau}}$ for all smooth functions ϕ on \mathbb{T}^3 and $t \in \mathbb{R}$, with furthermore an expectation equal to 0. Since $\delta_k \tau(t)$ is a continuous function for all $k \geq -1$ and $t \in [0, T]$, we have eventually that $\delta_k \tau(t, x) \in \mathcal{H}_{\leq n_{\tau}}$ for all $(t, x) \in [0, T] \times \mathbb{T}^3$. Hence, using first Fubini's theorem and then Nelson's estimate (Proposition A.11), we have

$$\mathbb{E}[\|\delta_k \tau(t) - \delta_k \tau(s)\|_{L^p}^p] \le \sup_{z \in \mathbb{T}^3} \mathbb{E}[|\delta_k \tau(t, z) - \delta_k \tau(s, z)|^p]$$

$$\le \sup_{z \in \mathbb{T}^3} C_{n_\tau}^p (p - 1)^{\frac{n_\tau p}{2}} \left(\mathbb{E}[|\delta_k \tau(t, z) - \delta_k \tau(s, z)|^2] \right)^{\frac{p}{2}}.$$

Then, we use the estimates proven in [9] and the proof of [9, Proposition 3.6] to get that for $\alpha = |\tau| - \lambda$ and all $(t, z) \in [0, T] \times \mathbb{T}^3$

$$\mathbb{E}[|\delta_k \tau(t,z) - \delta_k \tau(s,z)|^2] \le C_1 |t-s|^{\lambda} 2^{-2k\alpha},\tag{10}$$

so eventually

$$\mathbb{E}[\|\delta_k \tau(t) - \delta_k \tau(s)\|_{L^p}^p] \le |t - s|^{\frac{\lambda p}{2}} C_{n-1}^p C_1^{\frac{p}{2}} (p-1)^{\frac{n_T p}{2}} 2^{-kp\alpha}.$$

Since $\beta < \alpha$, Proposition A.4 gives us that, for $p > \frac{3}{\alpha - \beta} + 1$,

$$\mathbb{E}[\|\tau(t) - \tau(s)\|_{\mathcal{C}^{\beta}}^{p}] \leq C_{0}^{p} \sup_{k \geq -1} 2^{kp\alpha} |t - s|^{\frac{\lambda p}{2}} C_{n_{\tau}}^{p} C_{1}^{\frac{p}{2}} (p - 1)^{\frac{n_{\tau}p}{2}} 2^{-kp\alpha} \\
\leq |t - s|^{\frac{\lambda p}{2}} C_{0}^{p} C_{n_{\tau}}^{p} C_{1}^{\frac{p}{2}} (p - 1)^{\frac{n_{\tau}p}{2}}.$$
(11)

Remark 3.3. The estimate needed to get (10) for \forall is not explicitly proven in [9], but it is a trivial consequence of the proof of the estimate for \forall .

Since $\gamma < \gamma' < \frac{\lambda}{2}$ and 0 < |t - s| < T we have

$$T^{\gamma'p+1-\frac{\lambda p}{2}}|t-s|^{\frac{\lambda p}{2}} \le |t-s|^{\gamma'p+1}$$
 and $|t-s|^{\gamma'p-1} \le T^{\gamma'p-1-\gamma p}|t-s|^{\gamma p}$. (12)

Choosing $L \ge 8 \cdot 4^{\frac{1}{p}} \frac{\frac{\lambda}{2} + \frac{1}{p}}{\frac{\lambda}{4} - \frac{1}{p}} \ge 8 \cdot 4^{\frac{1}{p}} \frac{\gamma' + \frac{1}{p}}{\gamma' - \frac{1}{p}}$, for $p > \frac{4}{\lambda} + 1 > \frac{1}{\gamma'} + 1$, we have from (9), (11) and (12)

$$\mathbb{E}\left[\sup_{s < t} \frac{\|\tau(t) - \tau(s)\|_{\mathcal{C}^{\beta}}^{p}}{|t - s|^{\gamma p}}\right] \le T^{\gamma' p - 1 - \gamma p - (\gamma' p + 1 - \frac{\lambda p}{2})} L^{p} \int_{0}^{T} \int_{0}^{T} \mathbb{E}\frac{\|\tau(x) - \tau(y)\|_{\mathcal{C}^{\beta}}^{p}}{|x - y|^{\frac{\lambda p}{2}}} dx dy \\
\le T^{\frac{\lambda p}{2} - \gamma p} L^{p} C_{0}^{p} C_{n_{\tau}}^{p} C_{1}^{\frac{p}{2}} (p - 1)^{\frac{n_{\tau} p}{2}}.$$
(13)

The left-hand side is exactly $\mathbb{E}([\tau]_{\beta,\gamma}^p)$, and these bounds on the moments of the Hölder constant allow us to control its tail. It is here that we finally define γ_0 , taking it such that $T^{\frac{\lambda}{2}-\gamma_0} < 2$ so that $T^{\frac{\lambda p}{2}-\gamma p} < 2^p$. Finally, using that all considered quantities are positive, we observe that for v > 0

$$\mathbb{E}[\exp(v[\tau]_{\beta,\gamma}^{\frac{2}{n_{\tau}}})] = \mathbb{E}\left[\sum_{k=0}^{+\infty} \frac{v^{k}[\tau]_{\beta,\gamma}^{\frac{2k}{n_{\tau}}}}{k!}\right]$$

$$\leq \sum_{k=0}^{+\infty} \frac{v^{k}\mathbb{E}([\tau]_{\beta,\gamma}^{\frac{2k}{n_{\tau}}})}{k!}$$

$$\leq \sum_{k=0}^{p_{0}-1} \frac{v^{k}\mathbb{E}([\tau]_{\beta,\gamma}^{\frac{2k}{n_{\tau}}})}{k!} + \sum_{k=p_{0}}^{+\infty} \frac{v^{k}2^{\frac{2k}{n_{\tau}}}L^{\frac{2k}{n_{\tau}}}C_{0}^{\frac{2k}{n_{\tau}}}C_{n_{\tau}}^{\frac{2k}{n_{\tau}}}C_{1}^{\frac{2k}{n_{\tau}}}(\frac{2k}{n_{\tau}}-1)^{k}}{k!}$$

where $p_0 = \max(\frac{3}{\alpha - \beta} + 1, \frac{4}{\lambda} + 1)$ so that we can use (11) and (13). Since $\mathbb{E}([\tau]_{\beta,\gamma}^{\frac{2p_0}{n_{\tau}}})$ is bounded uniformly in T, the first sum is finite and the number of terms in it is independent of T, we know that it converges to a quantity independent of T. For the second one, writing $K_0 = 2^{\frac{2}{n_{\tau}}} L^{\frac{2}{n_{\tau}}} C_0^{\frac{2}{n_{\tau}}} C_{n_{\tau}}^{\frac{1}{n_{\tau}}} C_1^{\frac{1}{n_{\tau}}}$, we have using Stirling's formula that

which converges exponentially fast towards 0 for $v=\ell_{\tau}$ where we have $\ell_{\tau}:=\frac{1}{2}\frac{n_{\tau}}{2e}2^{-\frac{2}{n_{\tau}}}L^{-\frac{2}{n_{\tau}}}C_{0}^{-\frac{2}{n_{\tau}}}C_{n_{\tau}}^{-\frac{1}{n_{\tau}}}C_{1}^{-\frac{1}{n_{\tau}}}>0.$

Remark 3.4. We emphasize that ℓ_{τ} still depends of λ since our definition of L involves λ .

To conclude we just have to use Markov's inequality. We have that $\Pi_k \tau = \sigma^k \tilde{T}_k$ so

$$\begin{split} \mathbb{P}([\Pi_k \tau]_{\beta,\gamma} > h^k) &= \mathbb{P}([\sigma^k \tilde{T}_k(t)]_{\beta,\gamma} > h^k) \\ &\leq \mathbb{P}([\tilde{T}_k(t)]_{\beta,\gamma} > \frac{h^k}{\sigma^k}) \\ &\leq \mathbb{P}\left(\exp(\ell_\tau [\tilde{T}_k(t)]_{\beta,\gamma}^{\frac{2}{k}}) > \exp(\ell_\tau \frac{h^2}{\sigma^2})\right) \\ &\leq \exp(-\ell_\tau \frac{h^2}{\sigma^2}) \mathbb{E}(\exp(\ell_\tau [\tilde{T}_k(t)]_{\beta,\gamma}^{\frac{2}{k}})) \end{split}$$

We then take $m_{\tau} = \mathbb{E}(\exp(\ell_{\tau}[\tilde{T}_k(t)]_{\beta,\gamma}^{\frac{2}{k}}))$ which is finite and independent of T and we are done.

Corollary 3.5. Let $\tau \in \mathcal{T}$, and $\beta < |\tau| - \lambda$ for $\lambda \in (0,1)$. Then there exist $\ell'_{\tau} := \ell'_{\tau}(\lambda)$ and $m'_{\tau} := m'_{\tau}(\lambda)$ independent of T such that for $k \leq n_{\tau}$ we have

$$\mathbb{P}\left(\left[\Pi_k \tau\right]_{\beta, \frac{\lambda}{2}} > h^k\right) \le m'_{\tau} \exp\left(-\ell'_{\tau} \frac{h^2}{\sigma^2}\right).$$

Proof. We just have to use Proposition 3.1 for λ and $\lambda' \in (\lambda, |\tau| - \beta)$ where the upper bound gives us that $\beta < |\tau| - \lambda'$. There exists $\gamma_0(T, \lambda) < \frac{\lambda}{2}$ and $\tilde{\gamma}_0(T, \lambda') < \frac{\lambda'}{2}$ such that we have the wanted property uniformly in $\gamma_1 \in (\gamma_0, \frac{\lambda}{2})$ and $\gamma_2 \in (\tilde{\gamma}_0, \frac{\lambda'}{2})$, so we take $\gamma_1 < \frac{\lambda}{2} < \gamma_2$ and then use that

$$[\Pi_k \tau]_{\beta, \frac{\lambda}{2}} \le [\Pi_k \tau]_{\beta, \gamma_1, +} [\Pi_k \tau]_{\beta, \gamma_2}$$

to conclude. \Box

Theorem 2.5 is a direct consequence of Corollary 3.5. Indeed, for $\alpha < |\tau| - \lambda$ where we can choose λ arbitrarily small, using that $\tau(0) = 0$ we have

$$\mathbb{P}\left(\sup_{t\in[0,T]}\|\Pi_k\tau(t)\|_{\mathcal{C}^{\alpha}} > h^k\right) = \mathbb{P}\left(\sup_{t\in[0,T]}\|\Pi_k\tau(t) - \Pi_k\tau(0)\|_{\mathcal{C}^{\alpha}} > h^k\right) \\
\leq \mathbb{P}\left(T^{\frac{\lambda}{2}}[\Pi_k\tau]_{\alpha,\frac{\lambda}{2}} > h^k\right) \\
\leq m'_{\tau}\exp\left(-\frac{\ell'_{\tau}}{(T^{\frac{\lambda}{2}})^{\frac{2}{k}}}\frac{h^2}{\sigma^2}\right) \\
\leq m'_{\tau}\exp\left(-\frac{\ell'_{\tau}}{(T^{\lambda})^{\frac{1}{k}}}\frac{h^2}{\sigma^2}\right)$$

and the proof is complete for $d_{\tau} = m'_{\tau}$ and $c_{\tau} = \ell'_{\tau}$

3.3 Proof of Theorem 2.10

3.3.1 Explicit expression of (v, w)

We start by doing the computations mentioned in the end of Section 2.2. They are essentially the same as the ones found in [8], but with extra terms due to f_2 . We need to renormalize a second time our equation, and we therefore write that $\theta := \lim_{n \to +\infty} \theta_n$ where θ_n is the unique solution with initial condition 0 of the equation

$$(\partial_t - \Delta - a(t))\theta_n = -\theta_n^3 - 3 \mathring{\uparrow}_n \theta_n^2 - 3 \mathring{\checkmark}_n \theta_n - \mathring{\checkmark}_n + f_2(t)\theta_n^2 + 2f_2(t)\theta_n^{\dagger} + f_2(t) \mathring{\checkmark}_n - 6c'_n(t)(3\mathring{\uparrow}_n + 3\theta_n).$$

We then take $u_n = \theta_n + \stackrel{\bullet}{\Upsilon}_n$ and decompose the product $\stackrel{\bullet}{\nabla}_n (u_n - \stackrel{\bullet}{\Upsilon}_n)$ into its paraproducts and resonant product parts. Since $(\partial_t - \Delta - a(t)) \stackrel{\bullet}{\Upsilon}_n = \stackrel{\bullet}{\nabla}_n$ we have

$$(\partial_{t} - \Delta - a(t))u_{n}$$

$$= -(u_{n} - \stackrel{\checkmark}{\Psi}_{n})^{3} - 3(u_{n} - \stackrel{\checkmark}{\Psi}_{n})^{2} \stackrel{\dagger}{\uparrow}_{n} - 3(u_{n} - \stackrel{\checkmark}{\Psi}_{n}) \stackrel{\checkmark}{\Psi}_{n}$$

$$+ f_{2}(t)(u_{n} - \stackrel{\checkmark}{\Psi}_{n})^{2} + 2f_{2}(t)(u_{n} - \stackrel{\checkmark}{\Psi}_{n}) \stackrel{\dagger}{\uparrow} + f_{2}(t) \stackrel{\checkmark}{\Psi}_{n}$$

$$- 6c'_{n}(t)(3 \stackrel{\dagger}{\uparrow}_{n} + 3(u - \stackrel{\checkmark}{\Psi}_{n}))$$

$$= -u_{n}^{3} - 3(u_{n} \odot \stackrel{\checkmark}{\Psi}_{n} + 6c'_{n}(t)(u_{n} - \stackrel{\checkmark}{\Psi}_{n})) + 3(\stackrel{\checkmark}{\Psi}_{n} \odot \stackrel{\checkmark}{\Psi}_{n} - 6c'_{n}(t) \stackrel{\dagger}{\uparrow}_{n})$$

$$- 3(u_{n} - \stackrel{\checkmark}{\Psi}_{n}) \odot \stackrel{\checkmark}{\Psi}_{n} - 3(u_{n} - \stackrel{\checkmark}{\Psi}_{n}) \odot \stackrel{\checkmark}{\Psi}_{n} + Q^{n}(u_{n})$$

$$= -u_{n}^{3} - 3(u_{n} \odot \stackrel{\checkmark}{\Psi}_{n} + 6c'_{n}(t)(u_{n} - \stackrel{\checkmark}{\Psi}_{n})) + 3 \stackrel{\checkmark}{\Psi}_{n}$$

$$- 3(u_{n} - \stackrel{\checkmark}{\Psi}_{n}) \odot \stackrel{\checkmark}{\Psi}_{n} - 3(u_{n} - \stackrel{\checkmark}{\Psi}_{n}) \odot \stackrel{\checkmark}{\Psi}_{n} + Q^{n}(u_{n})$$

where we have $Q^{n}(u_{n}) = q_{2}^{n}(t)u_{n}^{2} + q_{1}^{n}(t)u_{n} + q_{0}^{n}(t)$ with:

$$q_0^n(t) = (\mathring{\P}_n)^3 - 3 \mathring{\P}_n (\mathring{\P}_n)^2 + f_2(t) (\mathring{\P}_n)^2 - 2f_2(t) \mathring{\P}_n \mathring{\P}_n + f_2(t) \mathring{\P}_n$$

$$q_1^n(t) = -3 (\mathring{\P}_n)^2 + 6 \mathring{\P}_n \mathring{\P}_n - 2f_2(t) \mathring{\P}_n + 2f_2(t) \mathring{\P}_n$$

$$q_2^n = -3 \mathring{\P}_n + 3 \mathring{\P}_n + f_2(t)$$

Remark 3.6. The term $u_n
otin
v_n$ does not admit a limit since, as we said above, u
v is not well-defined: that is precisely why we need to compensate it with the diverging renormalization term $+6c'_n(t)(u_n -
v_n)$. We will not explain how the renormalization procedure works here, since details can be found in [8], but we emphasize that it generates new v_n terms to be put in q_0 and q_1 .

Following exactly the same method as in [8] (the term with m is here replaced with a(t) and put in the left-hand side instead of the right-hand side) and putting the new $+f_2(t) \checkmark_n$ term of $q_0(t)$ in the equation of v_n , we find that we can write $u_n = v_n + w_n + 3I(\checkmark_n)$ such that when n goes to infinity the pair (v, w) satisfies (5)

$$\begin{cases}
(\partial_t - \Delta - a(t))v &= F(v+w), \\
(\partial_t - \Delta - a(t))w &= G(v,w),
\end{cases}$$
(5)

where the explicit expression of F and G are

$$\begin{split} F(v+w) &:= -3(v+w-\overset{\P^\bullet}{\downarrow}) \otimes \overset{\P^\bullet}{\searrow} + f_2(t) \overset{\P^\bullet}{\searrow}, \\ G(v,w) &:= -(v+w)^3 - 3\mathrm{com}(v,w) \\ &- 3w \otimes \overset{\P^\bullet}{\searrow} - 3(v+w-\overset{\P^\bullet}{\searrow}) \otimes \overset{\P^\bullet}{\searrow} + P(v+w). \end{split}$$

where $P(X) = d_2(t)X^2 + d_1(t)X + d_0(t)$ is a random polynomial of coefficients:

$$d_{0}(t) = (\overset{\mathbf{Y}}{\mathbf{Y}})^{3} - 9\overset{\mathbf{Y}}{\mathbf{Y}}\overset{\mathbf{Y}}{\mathbf{Y}} + f_{2}(t)(\overset{\mathbf{Y}}{\mathbf{Y}})^{2} - 2f_{2}(t)[\overset{\mathbf{Y}}{\mathbf{Y}} + \overset{\dagger}{\mathbf{Y}} \otimes \overset{\mathbf{Y}}{\mathbf{Y}}]$$

$$- 3\left[\overset{\dagger}{\mathbf{Y}} \otimes (\overset{\mathbf{Y}}{\mathbf{Y}})^{2} + \overset{\dagger}{\mathbf{Y}} \otimes [\overset{\mathbf{Y}}{\mathbf{Y}} \otimes \overset{\mathbf{Y}}{\mathbf{Y}}] + 2\overset{\mathbf{Y}}{\mathbf{Y}}\overset{\mathbf{Y}}{\mathbf{Y}} + 2[\otimes, \otimes](\overset{\mathbf{Y}}{\mathbf{Y}}, \overset{\mathbf{Y}}{\mathbf{Y}}, \overset{\mathbf{Y}}{\mathbf{Y}})\right]$$

$$d_{1}(t) = 6\left[\overset{\mathbf{Y}}{\mathbf{Y}} \otimes \overset{\dagger}{\mathbf{Y}} + \overset{\mathbf{Y}}{\mathbf{Y}}\right] - 3(\overset{\mathbf{Y}}{\mathbf{Y}})^{2} + 9\overset{\mathbf{Y}}{\mathbf{Y}} - 2f_{2}(t)\overset{\mathbf{Y}}{\mathbf{Y}} + 2f_{2}(t)\overset{\dagger}{\mathbf{Y}}$$

$$d_{2}(t) = -3\overset{\dagger}{\mathbf{Y}} + 3\overset{\mathbf{Y}}{\mathbf{Y}} + f_{2}(t)$$

and $com(v, w) = com_1(v, w) \oplus \checkmark + com_2(v + w)$ where:

$$\begin{cases}
com_1(v, w)(t) &= v(t) + 3[(v + w - \overset{\checkmark}{\Psi})](t) \\
com_2(v + w) &= [\circlearrowleft, \circlearrowleft](-3(v + w - \overset{\checkmark}{\Psi}), \overset{\checkmark}{\Psi}, \overset{\checkmark}{\Psi})
\end{cases}$$

Remark 3.7. The removal of the term $+3I(\checkmark)$ from the definition of w found in [8] implies the disappearance of the term \checkmark from the definition of d_0 .

3.3.2 Proof of Theorem 2.10

We write C for a universal constant independent of T and σ to simplify the notations. In what follows, if we do not explicitly write the temporal variable, it means that the result is true for all $s \in [0, T]$ and that the constants involved in the computations are uniform in time. Since proving Theorem 2.10 for a specific $\varepsilon > 0$ immediately implies the result for all $\varepsilon' \geq \varepsilon$, we assume from now on that $\varepsilon \in (0, \frac{1}{16})$.

Our first goal is to deduce a priori bounds on $\|v\|_{\mathcal{C}^{1-2\varepsilon}}$ and $\|w\|_{\mathcal{C}^{\frac{3}{2}-2\varepsilon}}$ from the explicit expressions of F and G. Using Schauder's estimate (Proposition

A.5), the inequality $\sup a = -a_+ < 0$ and the fact that $\alpha \mapsto \|\cdot\|_{\mathcal{C}^{\alpha}}$ is non-decreasing, we directly get that

$$||v(t)||_{\mathcal{C}^{1-2\varepsilon}} = ||\int_{0}^{t} e^{\alpha(t,u)} e^{(t-u)\Delta} F(v+w)(u) du||_{\mathcal{C}^{1-2\varepsilon}}$$

$$\leq \int_{0}^{t} e^{\alpha(t,u)} ||e^{(t-u)\Delta} F(v+w)(u)||_{\mathcal{C}^{1-2\varepsilon}} du$$

$$\leq \int_{0}^{t} e^{-a_{+}(t-u)} C(t-u)^{\frac{-1-\varepsilon-(1-2\varepsilon)}{2}} ||F(v+w)(u)||_{\mathcal{C}^{-1-\varepsilon}} du$$

$$\leq \int_{0}^{t} e^{-a_{+}(t-u)} C(t-u)^{\frac{-1-\varepsilon-(1-2\varepsilon)}{2}} ||F(v+w)(u)||_{\mathcal{C}^{1-\varepsilon}} du$$

$$\leq C \sup_{s \in [0,t]} ||F(v+w)(s)||_{\mathcal{C}^{-1-\varepsilon}} \int_{0}^{t} e^{-a_{+}u} u^{-1+\frac{\varepsilon}{2}} du$$

$$\leq C \sup_{s \in [0,t]} ||F(v+w)(s)||_{\mathcal{C}^{-1-\varepsilon}}$$

since $u \mapsto e^{-a_+ u} u^{-1 + \frac{\varepsilon}{2}}$ is integrable, and we have therefore C independent of T. We have then using Proposition A.6 that

$$||F(v+w)||_{\mathcal{C}^{-1-\varepsilon}} \leq 3||(v+w-\Psi) \otimes V||_{\mathcal{C}^{-1-\varepsilon}} + ||f_2||_{\infty} ||V||_{\mathcal{C}^{-1-\varepsilon}}$$

$$\leq C||v+w-\Psi||_{\mathcal{C}^{\frac{1}{2}-\varepsilon}} \times ||V||_{\mathcal{C}^{-1-\varepsilon}} + ||f_2||_{\infty} ||V||_{\mathcal{C}^{-1-\varepsilon}}$$

$$\leq C(||v||_{\mathcal{C}^{1-2\varepsilon}} + ||w||_{\mathcal{C}^{\frac{3}{2}-2\varepsilon}} + ||\Psi||_{\mathcal{C}^{\frac{1}{2}-\varepsilon}} + ||f_2||_{\infty}) ||V||_{\mathcal{C}^{-1-\varepsilon}}.$$

Likewise, we have

$$||w(t)||_{\mathcal{C}^{\frac{3}{2}-2\varepsilon}} \le C \sup_{s \in [0,t]} ||G(v,w)(s)||_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}}.$$

Since the expression of G is more complex than the one of F, we will study its five terms one after the other. For the cubic term $(v+w)^3$ we have

$$\| - (v+w)^3 \|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} \le \| (v+w)^3 \|_{\mathcal{C}^{1-2\varepsilon}}$$

$$\le C \| v+w \|_{\mathcal{C}^{1-2\varepsilon}}^3$$

$$\le C (\| v \|_{\mathcal{C}^{1-2\varepsilon}}^3 + \| w \|_{\mathcal{C}^{\frac{3}{2}-2\varepsilon}}^3).$$

If we consider the resonant product of \checkmark and w we have, thanks to Proposition, A.6

$$\begin{split} \|3w & \, \lozenge \, \, \bigvee \|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} \leq \|3w \, \circledcirc \, \, \bigvee \|_{\mathcal{C}^{\frac{1}{2}-3\varepsilon}} \\ & \leq C \|w\|_{\mathcal{C}^{\frac{3}{2}-2\varepsilon}} \| \, \bigvee \|_{\mathcal{C}^{-1-\varepsilon}}. \end{split}$$

For the fourth term we have likewise

$$\begin{split} \|3(v+w-\overset{\P}{\Psi})\otimes \mathsf{V}\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} &\leq C\|v+w-\overset{\P}{\Psi}\|_{\mathcal{C}^{\frac{1}{2}-\frac{\varepsilon}{2}}}\|\mathsf{V}\|_{\mathcal{C}^{-1-\frac{\varepsilon}{2}}} \\ &\leq C(\|\overset{\P}{\Psi}\|_{\mathcal{C}^{\frac{1}{2}-\frac{\varepsilon}{2}}} + \|v\|_{\mathcal{C}^{1-2\varepsilon}} + \|w\|_{\mathcal{C}^{\frac{3}{2}-2\varepsilon}})\|\mathsf{V}\|_{\mathcal{C}^{-1-\frac{\varepsilon}{2}}}. \end{split}$$

With regards to the polynomial P we have

$$\begin{split} \|P(v+w)\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} &\leq C \left[\|d_0\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} + \|d_1\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} \|v+w\|_{\mathcal{C}^{1-2\varepsilon}} \right. \\ & + \|d_2\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} \|(v+w)^2\|_{\mathcal{C}^{1-2\varepsilon}} \right] \\ &\leq C \left[\|d_0\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} + \|d_1\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} (\|v\|_{\mathcal{C}^{1-2\varepsilon}} + \|w\|_{\mathcal{C}^{\frac{3}{2}-2\varepsilon}}) \right. \\ & + \|d_2\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} (\|v\|_{\mathcal{C}^{1-2\varepsilon}}^2 + \|w\|_{\mathcal{C}^{\frac{3}{2}-2\varepsilon}}^2) \right] \end{split}$$

The term involving com is more complex. We have on the one hand, thanks to Proposition A.7, that

$$\begin{split} &\| \mathrm{com}_{2}(v+w) \|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} \\ &\leq \| \mathrm{com}_{2}(v+w) \|_{\mathcal{C}^{\frac{1}{2}-\varepsilon}} \\ &\leq C(\|v+w-\mathring{\Upsilon}\|_{\mathcal{C}^{\frac{1}{2}-\frac{1}{3}\varepsilon}} \| \mathring{\Upsilon}\|_{\mathcal{C}^{1-\frac{1}{3}\varepsilon}} \| \mathring{\nabla}\|_{\mathcal{C}^{-1-\frac{1}{3}\varepsilon}}) \\ &\leq C\left(\|v\|_{\mathcal{C}^{1-2\varepsilon}} + \|w\|_{\mathcal{C}^{\frac{3}{2}-2\varepsilon}} + \| \mathring{\Upsilon}\|_{\mathcal{C}^{\frac{1}{2}-\frac{1}{3}\varepsilon}} \right) \| \mathring{\Upsilon}\|_{\mathcal{C}^{1-\frac{1}{3}\varepsilon}} \| \mathring{\nabla}\|_{\mathcal{C}^{-1-\frac{1}{3}\varepsilon}} \end{split}$$

For com_1 on the other hand, integrating (5) we have that

$$v(t) = -3 \int_0^t e^{\alpha(t,s)} e^{(t-s)\Delta} [(v+w- \overset{\bullet}{\lor}) \otimes \overset{\bullet}{\lor}](s) ds$$

and therefore

$$com_1(v, w)(t) = -3 \int_0^t e^{\alpha(t, s)} e^{(t - s)\Delta} [(v + w - \overset{\bullet}{\mathbf{Y}}) \otimes \overset{\bullet}{\mathbf{Y}}](s) ds$$
$$+ 3(v + w - \overset{\bullet}{\mathbf{Y}})(t) \otimes \overset{\bullet}{\mathbf{Y}}(t).$$

The computations associated with com_1 are the most subtle of this Part, and we recall from [8] that the motivation behind the definition of $\operatorname{com}_1(v,w)$ is that we expect it to be a bit more regular than v so that $\operatorname{com}_1(v,w) \odot \checkmark$ is well defined (while $v \odot \checkmark$ is not). To understand why this is the case, we have to rewrite $\operatorname{com}_1(v,w)(t) = A_t + B_t$ where the couple $(A,B) \neq (v,3(v+w-\Lsh)) \odot \Lsh)$ is to be defined in the next paragraphs. We finally emphasize that while the inequalities above were purely spatial and uniformly true in $t \in [0,T]$, we will get for com_1 inequalities where t is explicitly present. We first rewrite

$$v(t) = -3 \int_0^t e^{\alpha(t,s)} e^{(t-s)\Delta} [(v+w-\overset{\bullet}{\mathbf{V}}) \otimes \overset{\bullet}{\mathbf{V}}](s) ds$$
$$= -3 \int_0^t e^{\alpha(t,s)} [e^{(t-s)\Delta}, \otimes] ((v+w-\overset{\bullet}{\mathbf{V}}), \overset{\bullet}{\mathbf{V}})(s) ds$$
$$-3 \int_0^t e^{\alpha(t,s)} [(v+w-\overset{\bullet}{\mathbf{V}}) \otimes e^{(t-s)\Delta} \overset{\bullet}{\mathbf{V}}](s) ds$$

where $[e^{(t-s)\Delta}, \otimes](f,g) = e^{(t-s)\Delta}(f \otimes g) - f \otimes (e^{(t-s)\Delta}g)$. From now on, we denote by A_t the first term on the left-hand side. We now recall that $\Upsilon(t) = \int_0^t e^{\alpha(t,s)} e^{(t-s)\Delta} \Upsilon(s) ds$, so that, writing $\delta_{s,t} f = f(s) - f(t)$, we have

$$-3\int_{0}^{t} [(v+w-\Psi) \otimes e^{\alpha(t,s)}e^{(t-s)\Delta} \nabla](s)ds + 3(v+w-\Psi) \otimes Y(t)$$

$$= -3\int_{0}^{t} [(v+w-\Psi) \otimes e^{\alpha(t,s)}e^{(t-s)\Delta} \nabla](s)ds$$

$$+3(v+w-\Psi)(t) \otimes \left[\int_{0}^{t} e^{\alpha(t,s)}e^{(t-s)\Delta} \nabla(s)ds\right]$$

$$= -3\int_{0}^{t} [[\delta_{s,t}(v+w-\Psi)] \otimes e^{\alpha(t,s)}e^{(t-s)\Delta} \nabla(s)]ds$$

$$=: B_{t}$$

Combining both writings, we check that $com_1(v, w)(t) = A_t + B_t$. Therefore, using Proposition A.9 to bound the norm of A_t and Proposition A.6 combined with Proposition A.5 to bound the one of B_t , we get

$$\begin{split} &\| \text{com}_{1}(v,w)(t) \|_{\mathcal{C}^{1+2\varepsilon}} \\ &\leq \|A_{t}\|_{\mathcal{C}^{1+2\varepsilon}} + \|B_{t}\|_{\mathcal{C}^{1+2\varepsilon}} \\ &\leq C \sup_{s \in [0,t]} \|(v+w-\overset{\bullet}{\mathbf{Y}})(s)\|_{\mathcal{C}^{\frac{1}{2}-\varepsilon}} \|\overset{\bullet}{\mathbf{Y}}(s)\|_{-1-\varepsilon} \\ &\quad + C \int_{0}^{t} \|\delta_{s,t}(v+w-\overset{\bullet}{\mathbf{Y}})\|_{L^{\infty}} \|e^{\alpha(t,s)}e^{(t-s)\Delta} \overset{\bullet}{\mathbf{Y}}(s)\|_{\mathcal{C}^{1+2\varepsilon}} \mathrm{d}s \\ &\leq C \sup_{s \in [0,t]} \|(v+w-\overset{\bullet}{\mathbf{Y}})(s)\|_{\mathcal{C}^{\frac{1}{2}-\varepsilon}} \|\overset{\bullet}{\mathbf{Y}}(s)\|_{\mathcal{C}^{-1-\varepsilon}} \\ &\quad + C \int_{0}^{t} \|\delta_{s,t}(v+w-\overset{\bullet}{\mathbf{Y}})\|_{L^{\infty}} e^{-a_{+}(t-s)}(t-s)^{\frac{-1-\varepsilon-1-2\varepsilon}{2}} \|\overset{\bullet}{\mathbf{Y}}(s)\|_{\mathcal{C}^{-1-\varepsilon}} \mathrm{d}s \end{split}$$

We then use the temporal regularity of v, w and Ψ to argue that

$$\|\delta_{s,t}(v+w-\Psi)\|_{L^{\infty}} \leq \tilde{C}(t-s)^{\frac{1}{8}},$$

so the integral above converges since $\varepsilon < \frac{1}{16}$ and we have

$$\begin{aligned} &\| \text{com}_{1}(v, w)(t) \|_{\mathcal{C}^{1+2\varepsilon}} \\ &\leq C (\sup_{s \in [0, t]} [\| v(s) \|_{\mathcal{C}^{1-2\varepsilon}} + \| w(s) \|_{\mathcal{C}^{\frac{3}{2}-2\varepsilon}} + \|^{\P^{\bullet}}(s) \|_{\mathcal{C}^{\frac{1}{2}-\varepsilon}}] \|^{\P^{\bullet}}(s) \|_{\mathcal{C}^{-1-\varepsilon}} \\ &+ \tilde{C} \sup_{s \in [0, t]} \|^{\P^{\bullet}}(s) \|_{\mathcal{C}^{-1-\varepsilon}}) \end{aligned}$$

This quantity being finite, $com_1(v, w)(t)$ belongs to $\mathcal{C}^{1+2\varepsilon}$ and the product $com_1(v, w)(t) \ominus \mathcal{V}(t)$ is therefore well-defined. Besides, Proposition A.6 gives

us that

$$\| \operatorname{com}_{1}(v, w) \odot \checkmark(t) \|_{\mathcal{C}^{\varepsilon}}$$

$$\leq C \| \operatorname{com}_{1}(v, w)(t) \|_{\mathcal{C}^{1+2\varepsilon}} \| \checkmark(t) \|_{\mathcal{C}^{-1-\varepsilon}}$$

$$\leq C \left\{ \sup_{s \in [0,t]} \left[\| v(s) \|_{\mathcal{C}^{1-2\varepsilon}} + \| w(s) \|_{\mathcal{C}^{\frac{3}{2}-2\varepsilon}} + \| \checkmark(s) \|_{\mathcal{C}^{\frac{1}{4}-\varepsilon}} \right] \| \checkmark(s) \|_{\mathcal{C}^{-1-\varepsilon}} + \tilde{C} \sup_{s \in [0,t]} \| \checkmark(s) \|_{\mathcal{C}^{-1-\varepsilon}}^{2} \right\}.$$

$$(14)$$

Combining all the inequalities above, we get a priori bounds on $||v||_{\mathcal{C}^{1-2\varepsilon}}$ and $||w||_{\mathcal{C}^{\frac{3}{2}-2\varepsilon}}$ involving both these two norms and the spatial norms of the different symbols of \mathcal{T} . Now, we would like to combine Theorem 2.5, that proves concentration for all the symbols of \mathcal{T} , with these inequalities in order to obtain concentration for v and w. However, we stress that (14) introduces an additional difficulty with the constant \tilde{C} . Indeed, \tilde{C} is not deterministic but random, and to get a Gaussian tail for v and w, we need to prove that \tilde{C} also has a Gaussian tail. We have for $t \leq t_0$ that

$$\tilde{C} = [(v+w-{\overset{\P^\bullet}{\mathbf{v}}})_{|[0,t]}]_{0,\frac{1}{8}} \leq [v_{|[0,t_0]}]_{0,\frac{1}{8}} + [w_{|[0,t_0]}]_{0,\frac{1}{8}} + [\overset{\P^\bullet}{\mathbf{v}}]_{0,\frac{1}{8}},$$

where we recall that the definition of $[f]_{\beta,\gamma}$ is in Proposition 3.1. Thanks to Corollary 3.5, we already now that $[\stackrel{\bullet}{\Upsilon}]_{0,\frac{1}{8}}$ has a gaussian tail. For the other terms, we will use a trick found in [8] to get inequalities on $[v]_{0,\frac{1}{8}}$ and $[w]_{0,\frac{1}{8}}$ similar to the ones we got on $||v||_{\mathcal{C}^{1-2\varepsilon}}$ and $||w||_{\mathcal{C}^{\frac{3}{2}-2\varepsilon}}$. We can then prove concentration on all those terms at the same time, and conclude. Using that we can write

$$v(t) = \int_0^t e^{\alpha(t,u)} e^{(t-u)\Delta} F(v+w)(u) du,$$

we have for $0 \le s < t \le T$ that

$$v(t) - v(s) = \int_{s}^{t} e^{\alpha(t,u)} e^{(t-u)\Delta} F(v+w)(u) du$$
$$+ (e^{\alpha(t,s)} - 1) \int_{0}^{s} e^{\alpha(s,u)} e^{(t-u)\Delta} F(v+w)(u) du$$
$$+ (e^{(t-s)\Delta} - \operatorname{Id}) \int_{0}^{s} e^{\alpha(s,u)} e^{(s-u)\Delta} F(v+w)(u) du$$

We then try to bound the L^{∞} norm of each of these three terms. For the

first one we have

$$\begin{split} &\| \int_{s}^{t} e^{\alpha(t,u)} e^{(t-u)\Delta} F(v+w)(u) du \|_{L^{\infty}} \\ &\leq \int_{s}^{t} e^{\alpha(t,u)} \| e^{(t-u)\Delta} F(v+w)(u) \|_{L^{\infty}} du \\ &\leq C \int_{s}^{t} e^{-a_{+}(t-u)} (t-u)^{-\frac{1+\varepsilon}{2}} \| F(v+w)(u) \|_{\mathcal{C}^{-1-\varepsilon}} du \\ &\leq C \sup_{u \in [0,t]} \| F(v+w)(u) \|_{\mathcal{C}^{-1-\varepsilon}} \int_{s}^{t} e^{-a_{+}(t-u)} (t-u)^{-\frac{1+\varepsilon}{2}} du \\ &\leq C \sup_{u \in [0,t]} \| F(v+w)(u) \|_{\mathcal{C}^{-1-\varepsilon}} 1 \wedge (t-s)^{1-\frac{1+\varepsilon}{2}} \\ &\leq C \sup_{u \in [0,t]} \| F(v+w)(u) \|_{\mathcal{C}^{-1-\varepsilon}} (t-s)^{\frac{1}{8}}. \end{split}$$

where C is independent of T. We then consider the second term using the same inequalities as in Section 3.1

$$\begin{split} &\|(e^{\alpha(t,s)}-1)\int_{0}^{s}e^{\alpha(s,u)}e^{(t-u)\Delta}F(v+w)(u)\mathrm{d}u\|_{L^{\infty}} \\ &\leq |e^{\alpha(t,s)}-1|\int_{0}^{s}e^{-a_{+}(s-u)}\|e^{(t-u)\Delta}F(v+w)(u)\|_{L^{\infty}} \\ &\leq C\sup_{u\in[0,t]}\|F(v+w)(u)\|_{\mathcal{C}^{-1-\varepsilon}}(1\wedge a_{-}(t-s))\int_{0}^{s}e^{-a_{+}(t-u)}(t-u)^{-\frac{1+\varepsilon}{2}}\mathrm{d}u \\ &\leq C\sup_{u\in[0,t]}\|F(v+w)(u)\|_{\mathcal{C}^{-1-\varepsilon}}(1\wedge a_{-}(t-s)) \\ &\leq C\sup_{u\in[0,t]}\|F(v+w)(u)\|_{\mathcal{C}^{-1-\varepsilon}}(t-s)^{\frac{1}{8}} \end{split}$$

Finally, for the third term, we use the second inequality of Proposition A.5 to control the operator $(e^{(t-s)\Delta} - \text{Id})$

$$\begin{aligned} &\|(e^{(t-s)\Delta} - \operatorname{Id}) \int_{0}^{s} e^{\alpha(s,u)} e^{(s-u)\Delta} F(v+w)(u) du\|_{L^{\infty}} \\ &\leq (t-s)^{\frac{1}{4}-0} \|\int_{0}^{s} e^{\alpha(s,u)} e^{(s-u)\Delta} F(v+w)(u) du\|_{\mathcal{C}^{\frac{1}{4}}} \\ &\leq C(t-s)^{\frac{1}{8}} \int_{0}^{s} e^{-a_{+}(s-u)} (s-u)^{-\frac{1+\varepsilon+\frac{1}{4}}{2}} \|F(v+w)(u)\|_{\mathcal{C}^{-1-\varepsilon}} du \\ &\leq C(t-s)^{\frac{1}{8}} \sup_{u \in [0,t]} \|F(v+w)(u)\|_{\mathcal{C}^{-1-\varepsilon}} \int_{0}^{s} e^{-a_{+}(s-u)} (s-u)^{-\frac{1+\varepsilon+\frac{1}{4}}{2}} \\ &\leq C(t-s)^{\frac{1}{8}} \sup_{u \in [0,t]} \|F(v+w)(u)\|_{\mathcal{C}^{-1-\varepsilon}} \end{aligned}$$

Dividing by $(t-s)^{\frac{1}{8}}$ on both sides of the three inequalities and taking the supremum in $0 \le s < t \le t_0$, we eventually get

$$[v_{|[0,t_0]}]_{0,\frac{1}{8}} \le C \sup_{u \in [0,t_0]} ||F(v+w)(u)||_{\mathcal{C}^{-1-\varepsilon}}, \tag{15}$$

where we already have a bound on $\sup_{u \in [0,t]} ||F(v+w)(u)||_{\mathcal{C}^{-1-\varepsilon}}$. Likewise, following exactly the same strategy, we get

$$[w_{|[0,t_0]}]_{0,\frac{1}{8}} \le C \sup_{u \in [0,t_0]} \|G(v+w)(u)\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}}.$$
 (16)

The main idea in what follows is that if we assume that our symbols stay close to 0 on an interval of [0,T], it will imply that v and w also stay close to 0 on this interval. Let us indeed take h < 1 (so that $h^n \le h^2$ if $n \ge 2$) and a given time t_0 , and let us assume that

$$\max(\sup_{s\in[0,t_0]}\|v(s)\|_{\mathcal{C}^{1-2\varepsilon}},\sup_{s\in[0,t_0]}\|w(s)\|_{\mathcal{C}^{\frac{3}{2}-2\varepsilon}},[v_{|[0,t_0]}]_{0,\frac{1}{8}},[w_{|[0,t_0]}]_{0,\frac{1}{8}},[\overset{\P}{\Psi}]_{0,\frac{1}{8}})\leq h.$$
(17)

and that for all symbols $\tau \in \mathcal{T}$ and $k \leq n_{\tau}$

$$\sup_{s \in [0, t_0]} \|\Pi_k \tau(s)\|_{\mathcal{C}^{\alpha_\tau}} \le h^k. \tag{18}$$

where $\alpha_{\tau} = |\tau| - \frac{\varepsilon}{3}$. The crucial observation is that under (17) and (18), combining all the inequalities we got above on the components of F and G, we have

$$\sup_{s \in [0,t_0]} \|F(v+w)(s)\|_{\mathcal{C}^{-1-\varepsilon}} \le Ch^2 \quad \text{ and } \quad \sup_{s \in [0,t_0]} \|G(v+w)(s)\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} \le Ch^2$$

Remark 3.8. The second bound would not hold if we hadn't removed $3I(\checkmark)$ from the definition of w. Indeed, the term d_0 would therefore contain a $3\checkmark$ whose component in the first Wiener chaos is only bounded by an h (and not h^2) in our hypothesis.

Now, we can finally end the proof of Theorem 2.10. Let us take $h_0 \in (0,1)$ such that $Kh_0^2 < h_0$ for $K = \mathbb{C}^2$. Then, considering for $h \in (0,h_0)$

$$\kappa = \inf \left\{ t > 0, \, \max \{ \sup_{s \in [0,t]} \|v(s)\|_{\mathcal{C}^{1-2\varepsilon}}, \, \sup_{s \in [0,t]} \|w(s)\|_{\mathcal{C}^{\frac{3}{2}-2\varepsilon}}, \\ [v_{|[0,t]}]_{0,\frac{1}{\aleph}}, [w_{|[0,t]}]_{0,\frac{1}{\aleph}} \} > h \right\} \wedge T$$

we get

$$\{\kappa < T\} \cap \{\forall \tau \in \mathcal{T}, \, \forall k \le n_{\tau}, \, \sup_{s \le \kappa} \|\Pi_k \tau(s)\|_{\mathcal{C}^{\alpha_{\tau}}} \le h^k\} \cap \{[{}^{\bullet \uparrow}]_{0, \frac{1}{8}} \le h\} = \emptyset$$

Therefore, using Theorem 2.5 and Corollary 3.5, we get that

$$\mathbb{P}(\kappa < T) \leq \mathbb{P}(\exists (\tau, k) \in (\mathcal{T}, \llbracket 1, n_{\tau} \rrbracket), \sup_{s \in [0, T]} \|\Pi_{k} \tau(s)\|_{\mathcal{C}^{\alpha_{\tau}}} > h^{k})
+ \mathbb{P}([\overset{\bullet}{\Psi}]_{0, \frac{1}{8}} > h)
\leq \sum_{\tau \in T} \sum_{k=1}^{n_{\tau}} \mathbb{P}(\sup_{s \in [0, T]} \|\Pi_{k} \tau(s)\|_{\mathcal{C}^{\alpha_{\tau}}} > h^{k})
+ \mathbb{P}([\overset{\bullet}{\Psi}]_{0, \frac{1}{8}} > h^{3})
\leq 5 \operatorname{Card}(\mathcal{T}) D' \exp(-\frac{C'}{\max(T^{\frac{\lambda}{5}}, T^{\lambda})} \frac{h^{2}}{\sigma^{2}}) + m'_{\overset{\bullet}{\Psi}} \exp(-\ell'_{\overset{\bullet}{\Psi}} \frac{h^{2}}{\sigma^{2}})$$

since $T^{\frac{\lambda}{k}} \leq \max(T^{\frac{\lambda}{5}}, T^{\lambda})$. Here we may take $D' = \sup_{\tau \in \mathcal{T}} d_{\tau}(\lambda)$ et $C' = \inf_{\tau \in \mathcal{T}} c_{\tau}(\lambda)$. We then take $D = 5\operatorname{Card}(\mathcal{T})D' + m'_{\P^{\bullet}}$ and $C = \min(C', \ell'_{\P^{\bullet}})$ and Theorem 2.10 is proven.

A Technical tools

A.1 Decomposition into Paley-Littlewood blocks

There exists $\tilde{\chi}, \chi \in \mathcal{C}_c^{\infty}(\mathbb{R}^d)$ taking values in [0,1] with

Supp
$$\tilde{\chi} \subset B(0, \frac{4}{3})$$
, Supp $\chi \subset B(0, \frac{8}{3}) \setminus B(0, \frac{3}{4})$

and such that

$$\tilde{\chi}(\zeta) + \sum_{k=0}^{\infty} \chi(\frac{\zeta}{2^k}) = 1, \quad \forall \zeta \in \mathbb{R}^d.$$

We furthermore assume that $\tilde{\chi}$ and χ are radially symmetric. We write

$$\chi_{-1} = \tilde{\chi}, \qquad \chi_k(\cdot) := \chi(\frac{\cdot}{2^k}) \quad k \ge 0$$

These objects allows us to define $\|\cdot\|_{\mathcal{C}^{\alpha}}$ for all $\alpha \in \mathbb{R}$ in a way that is consistent with the common definition for $\alpha \in (0,1)$. Indeed, writing for $f \in \mathcal{C}^{\infty}(\mathbb{T}^d)$ and $k \geq -1$

$$\delta_k f := \mathcal{F}^{-1}(\chi_k \hat{f}),$$

we define the norm of $\|\cdot\|_{\mathcal{C}^{\alpha}}$ by

$$||f||_{\mathcal{C}^{\alpha}} := \sup_{k \ge -1} 2^{\alpha k} ||\delta_k f||_{L^{\infty}}.$$

This quantity is finite for all $f \in \mathcal{C}^{\infty}(\mathbb{T}^d)$, and the space $\mathcal{C}^{\alpha}(\mathbb{T}^d)$ is therefore defined as the completion of \mathcal{C}^{∞} for this norm.

Remark A.1. We do not have necessarily that $||f||_{\mathcal{C}^{\alpha}} < +\infty$ implies that $f \in \mathcal{C}^{\alpha}$ but we can verify that if a distribution f satisfies $||f||_{\mathcal{C}^{\alpha}} < +\infty$, then $f \in \mathcal{C}^{\beta}$ for all $\beta < \alpha$.

We now state one of the most crucial Lemmas of the theory of Besov spaces. It corresponds to [11, Lemma 2.2], we will here only state the first part.

Lemma A.2. (Bernstein's Lemma) Let B be the unit ball. There exists a constant C such that for any $n \geq 0$, any couple $(p,q) \in [1,+\infty]^2$ with $q \geq p \geq 1$ and any function u of L^p we have

$$Supp \ \hat{u} \subset \lambda B \ \Rightarrow \ \sup_{|\alpha|=n} \|\partial^{\alpha} u\|_{L^{q}} \leq C^{n+1} \lambda^{n+d(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^{p}}$$

Remark A.3. The theorem is generally written for $L^p = L^p(\mathbb{R}^d)$, but since it is a direct consequence of Young's inequality, and since the Fourier transform maps functions on \mathbb{T}^d to functions on $\mathbb{Z}^d \subset \mathbb{R}^d$, it also works for $L^p = L^p(\mathbb{T}^d)$.

In the setting of Paley-Littlewood theory, we take $B = B(0, \frac{8}{3})$, $\lambda = 2^n$ and $q = +\infty$ and we get for n = 0 that

$$\|\delta_k u\|_{L^{\infty}} \le C2^{\frac{dk}{p}} \|\delta_k u\|_{L^p} \tag{19}$$

for all $k \geq -1$, $p \geq 1$ and $u \in L^1(\mathbb{T}^d)$. This inequality is key to prove the following boundedness criterion:

Proposition A.4. Let $\beta < \alpha$. There exists C_0 such that for $p > \frac{d}{\alpha - \beta} + 1$ we have for every random distribution f on \mathbb{T}^d that

$$\mathbb{E}(\|f\|_{\mathcal{C}^{\beta}}^p) \le C_0^p \sup_{k \ge -1} 2^{\alpha k p} \mathbb{E}[\|\delta_k f\|_{L^p}^p]$$

Proof. By definition of the \mathcal{C}^{β} norm and then using (19) we have

$$||f||_{\mathcal{C}^{\beta}}^{p} = \sup_{k \ge -1} 2^{\beta kp} ||\delta_{k}||_{L^{\infty}}^{p} \le C^{p} \sup_{k \ge -1} 2^{k(\beta p + d)} ||\delta_{k} f||_{L^{p}}^{p}$$

To take the expectation of $\|\delta_k f\|_{L^p}^p$ directly, we enlarge the supremum on the right-hand side to a sum and we get

$$\mathbb{E}\|f\|_{\mathcal{C}^{\beta}}^{p} \leq C^{p} \sum_{k \geq -1} 2^{k(\beta p + d)} \mathbb{E}\|\delta_{k} f\|_{L^{p}}^{p} = C^{p} \sum_{k \geq -1} 2^{kp(\beta + \frac{d}{p} - \alpha)} 2^{\alpha kp} \mathbb{E}\|\delta_{k} f\|_{L^{p}}^{p}.$$

The sum $\sum_{k\geq -1} 2^{kp(\beta+\frac{d}{p}-\alpha)}$ can be bounded uniformly in $p>\frac{d}{\alpha-\beta}+1$. So taking C_0 slightly larger than C we have the result.

We now state Schauder's estimates which are extremely useful when we are dealing with the heat semigroup.

Proposition A.5 (Proposition A.13 of [8]). Let $\alpha, \beta \in \mathbb{R}$:

• If $\alpha \geq \beta$ there exists C > 0 such that uniformly in t > 0 and $f \in C^{\beta}$ we have

$$||e^{t\Delta}f||_{\mathcal{C}^{\alpha}} \le Ct^{\frac{\beta-\alpha}{2}}||f||_{\mathcal{C}^{\beta}}$$

• If $0 \le \beta - \alpha \le 2$ there exists C > 0 such that uniformly in $t \ge 0$ and $f \in \mathcal{C}^{\beta}$ we have

$$\|(Id - e^{t\Delta})f\|_{\mathcal{C}^{\alpha}} \le Ct^{\frac{\beta - \alpha}{2}} \|f\|_{\mathcal{C}^{\beta}}$$

A.2 Paracontrolled calculus

As we said above, the main tool of paracontrolled calculus is the decomposition of the standard product of functions into three different objects we will now describe explicitly:

$$fg = \sum_{k < l-1} \delta_k f \delta_l g + \sum_{|k-l| < 1} \delta_k f \delta_l g + \sum_{k > l+1} \delta_k f \delta_l g.$$

We write the first term $f \otimes g$, the second one $f \otimes g$ and the third one $f \otimes g$. The paraproducts $f \otimes g$ and $f \otimes g = g \otimes f$ are defined for all functions f and g with at least (negative) Hölder regularity. The resonant product $f \otimes g$ is only defined for functions with "compensating" Hölder regularity, and is the reason why the standard product is often not defined. More precisely, both for the paraproducts and the resonant product we have the following mappings:

Proposition A.6 (Proposition A.7 of [8]). Let $\alpha, \beta \in \mathbb{R}$:

- If $\alpha + \beta > 0$, then the mapping $(f, g) \mapsto f \ominus g$ extends to a continuous bilinear map from $\mathcal{C}^{\alpha} \times \mathcal{C}^{\beta}$ to $\mathcal{C}^{\alpha+\beta}$.
- The mapping $(f,g) \mapsto f \otimes g$ extends to a continuous bilinear map from $L^{\infty} \times \mathcal{C}^{\beta}$ to \mathcal{C}^{β} .
- If $\alpha < 0$, then the mapping $(f,g) \mapsto f \otimes g$ extends to a continuous bilinear map from $\mathcal{C}^{\alpha} \times \mathcal{C}^{\beta}$ to $\mathcal{C}^{\alpha+\beta}$.
- If $\alpha < 0 < \beta$ and $\alpha + \beta > 0$ then the mapping $(f, g) \mapsto fg$ extends to a bilinear map from $\mathcal{C}^{\alpha} \times \mathcal{C}^{\beta} \to \mathcal{C}^{\alpha}$.

Now we will state two technical lemmas that illustrate that the resonant product, the paraproduct and the heat semigroup have powerful interactions with one another. We start with the commutator of \odot and \odot .

Proposition A.7 (Proposition A.9 of [8]). Let $\alpha < 1$ and $\beta, \gamma \in \mathbb{R}$ such that $\beta + \gamma < 0$ and $\alpha + \beta + \gamma > 0$. Then the mapping

$$[\odot, \odot] : (f, g, h) \longmapsto (f \odot g) \odot h - f(g \odot h)$$

extends to a continuous trilinear map from $C^{\alpha} \times C^{\beta} \times C^{\gamma} \to C^{\alpha+\beta+\gamma}$.

Remark A.8. As said in Section 2, this result is quite remarkable because the resonant product g
in h is not supposed to be defined for $\beta + \gamma < 0$.

Finally we consider the commutator of $e^{t\Delta}$ and \otimes .

Proposition A.9 (Proposition A.16 of [8]). Let $\alpha < 1$, $\beta \in \mathbb{R}$, $\gamma \geq \alpha + \beta$. For every $t \geq 0$ define

$$[e^{t\Delta}, \odot] : (f, g) \mapsto e^{t\Delta}(f \odot g) - f \odot (e^{t\Delta}g).$$

There exists $C < +\infty$ such that, uniformly over t > 0,

$$||[e^{t\Delta}, \odot](f, g)||_{\mathcal{C}^{\gamma}} \le Ct^{\frac{\alpha+\beta-\gamma}{2}} ||f||_{\mathcal{C}^{\alpha}} ||g||_{\mathcal{C}^{\beta}}.$$

A.3 Wiener chaos and Nelson estimate

Let us consider ξ the space-time white noise, ξ is a distribution and therefore $\xi(t,x)$ is not well-defined. We can however define for all $\varphi \in L^2(\mathbb{R} \times \mathbb{T}^d)$ the quantity $\int_{\mathbb{R} \times \mathbb{T}^d} \varphi(z) \xi(\mathrm{d}z) = \xi(\varphi)$ that verifies

$$\mathbb{E}[\xi(\varphi)^2] = \|\varphi\|_{L^2(\mathbb{R} \times \mathbb{T}^d)}^2.$$

We can furthermore define iterated Wiener-Itô integrals based on ξ written $\xi^{\otimes k}(\varphi)$ for $k \geq 1$ and $\varphi \in L^2((\mathbb{R} \times \mathbb{T}^d)^k)$. We usually write

$$\xi^{\otimes k}(\varphi) = \int_{(\mathbb{R} \times \mathbb{T}^d)^k} \varphi(z_1, \dots, z_k) \xi(\mathrm{d}z_1) \cdots \xi(\mathrm{d}z_k).$$

We then consider

$$\mathcal{H}_k := \{ \xi^{\otimes}(\varphi), \ \varphi \in L^2((\mathbb{R} \times \mathbb{T}^d)^k) \}$$

which is the k-th homogeneous Wiener chaos with $\mathcal{H}_0 = \mathbb{R}$. Since we have

$$L^2(\Omega, \mathcal{F}, \mathbb{P}) = \bigoplus_{k=0}^{+\infty} \mathcal{H}_k$$

we can for all $k \geq 0$ consider Π_k the projection on \mathcal{H}_k . We have the following property:

Lemma A.10. For each $n \in \mathbb{N}$, the closure in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ of the linear span of the set

$$\{\xi(\varphi_1)\cdots\xi(\varphi_k),\ k\leq n,\ \varphi_1,\ldots,\varphi_k\in L^2(\mathbb{R}\times\mathbb{T}^d)\}$$

coincides with

$$\mathcal{H}_{\leq n} := \bigoplus_{k=0}^{n} \mathcal{H}_{k}.$$

Let us now state Nelson's estimate which is a key inequality when one deals with Wiener chaos.

Proposition A.11. For every $n \ge 1$, there exists a constant $C_n < +\infty$ such that for every $X \in \mathcal{H}_{\leq n}$ and $p \ge 2$ we have

$$\mathbb{E}[|X|^p]^{\frac{1}{p}} \le C_n(p-1)^{\frac{n}{2}} \mathbb{E}(X^2)^{\frac{1}{2}}$$

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