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Piecewise output feedback control for affine systems with disturbances based on linear temporal logic specifications

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Abstract: In the paper, we investigate the problem of finding a piecewise output feedback control law for an uncertain affine system such that the resulting closed-loop output satisfies a desired linear temporal logic (LTL) specification. A two-level hierarchical approach is proposed to solve the problem in a triangularized output space. In the lower level, we explore whether there exists a robust output feedback control law to make the output starting in a simplex either remains in it or leaves via a specific facet. In the higher level, for the triangularization, we construct the transition system according to the reachability relationship obtained in the lower level and search for feasible paths that meet the LTL specification. The control approach is then applied to solve a motion planning problem.

Keywords: Reachability; Piecewise output feedback control; Affine systems; Linear temporal logic

1 Introduction

Reachability problems of hybrid systems have been extensively studied in the last two decades [1∼3]. Recently, these problems have attracted renewed attention, partly because improvements in computational capabilities have been made and algorithms are possible to be implemented for practical interests [4∼6]. Another reason for the renewed interest in these problems is the emergence of some promising new ideas [7∼13] in piecewise affine hybrid systems.

A piecewise affine hybrid system consists of a partition of the state space and a collection of affine dynamics valid on each region. It has been proven that many physical systems can be approximated by this system with suitable precision. In recent years, this kind of systems has been studied by a number of researchers in various fields for different applications, e.g., robots, gene networks, power systems, and chemical processes [14∼21].

The temporal logic reachability of a hybrid system considers specifications of the form "trajectory moves from an initial region I to a target region G while staying within a region R and avoiding an unsafe region U" (see [22] for more details and a set of relevant references therein). It is originally developed for specifying and verifying the correctness of computer programs [23]. However, due to their resemblance to natural language, their expressivity, and the existence of off-the-shelf algorithms for model checking, temporal logic is introduced to reachability problems of hybrid systems. In this setting, symbolic control approaches are used to compute a partition of the state space, design a path at the discrete level, and refine it using a local continuous controller in each domain of the partition [19, 24].

Motivated by the motion planning problems [17, 19] and formal analysis of hybrid systems [8, 9], we study the problem of finding a piecewise output feedback control law for an uncertain affine system to meet a desired linear temporal logic (LTL) specification. We consider a set of affine systems valid on a set of polytopes for which the corresponding outputs lie in simplices. The affine systems are affected by additional disturbances. This kind of disturbances may come from the approximation of a nonlinear system or parametric uncertainty when modeling [4, 5, 10]. In the paper, we focus on output feedback control rather than state feedback control. Output feedback control takes less measurement information and requires less implementation cost as some states may be costly to measure. For example, for the motion planning problem, the position of a robot is more concerned rather than its velocity and acceleration (i.e., "Is it in position?"). To our knowledge, little attention has been paid to the reachability problem using output feedback control due to some technical challenges. Some analysis tools (e.g., the comparison lemma and Nagumo's theorem) often used in state reachability analysis [25, 26] cannot be directly applied.

In the paper, a two-level approach is proposed to solve the problem using output feedback control. First, for each affine system whose output is in a simplex, we determine whether there is an output feedback control law to make the output either leave a simplex and enter an adjacent simplex in finite time or remain in a simplex forever for a set of allowable disturbances. Next, we construct a transition system based on the former results and search for feasible paths that satisfy the LTL specification. In this way, a robust piecewise affine output feedback controller is obtained to solve the problem.

The contribution of this work is threefold. First, we extend reachability results for affine systems from the state space to the output space. Second, we introduce LTLs to describe a system specification. Then, a supervisory procedure is proposed to search for feasible paths satisfying the desired specification. The formulation makes us possible to derive an analytical solution by formal analysis. Third, the results obtained in the paper allow for a set of disturbances. Hence, these results can be applied to address certain problems for nonlinear systems by hybridization methods [10].

The paper is organized as follows. In the next section, we

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formulate the problem after giving a few necessary notions. Then, in Section 3, a robust local control strategy and a supervisory control law are presented. A simulation result is given in Section 4. The paper ends with concluding remarks and brief discussions of future research in Section 5.

2 Preliminaries

In this section, we first present some background materials and then introduce the problem.

2.1 Piecewise affine hybrid systems

Notations $\mathbb{R}$ and $\mathbb{N}$ are used to represent the set of real numbers and natural numbers, respectively.

We use $\mathrm{co}(v_1, \cdots, v_n)$ to denote the convex hull of points $v_1, \cdots, v_n$. Let $\mathcal{P}$ be an n-dimensional polytope in $\mathbb{R}^n$. It can be written as the intersection of e half spaces, where e is the least number required. That is,

$$\mathcal{P} = \bigcap_{i=1}^{e} \{x \in \mathbb{R}^n | n_i^{\mathrm{T}} x \leqslant \gamma_i\}, \tag{1}$$

where n_i is a unit normal vector and γ_i is a constant, $i = 1, \cdots, e$. The set $\{x \in \mathbb{R}^n | n_i^{\mathrm{T}} x = \gamma_i\}$ is called a supporting hyperplane of $\mathcal{P}$. A facet of polytope $\mathcal{P}$ is the intersection of $\mathcal{P}$ with one of its supporting hyperplanes, which is of $(n-1)$-dimension. That is,

$$\mathcal{F}_i = \{x \in \mathcal{P} | n_i^{\mathrm{T}} x = \gamma_i\}, \ \ i = 1, \cdots, e.$$

Alternatively, the polytope $\mathcal{P}$ can be viewed as the convex hull of its vertices. We denote $\mathrm{vert}(\mathcal{P})$ the set of vertices of $\mathcal{P}$. A simplex is an n-dimensional polytope with $n+1$ vertices. We label the vertices of a simplex $v_1, \cdots, v_{n+1}$, and label its facets $\mathcal{F}_1, \cdots, \mathcal{F}_{n+1}$ such that $v_i \notin \mathcal{F}_i$.

Let $\mathscr{P} = \{\mathcal{P}_i, i = 1, \cdots, p\}$ be a finite collection of n-dimensional polytopes in $\mathbb{R}^n$ such that $\mathcal{P}_i \cap \mathcal{P}_j$, $i \neq j$, is a common (possibly empty) face of $\mathcal{P}_i$ and $\mathcal{P}_j$. Consider a piecewise affine hybrid system $\Sigma = \{\Sigma_i, i = 1, \cdots, p\}$. The dynamics of each subsystem Σ_i is defined on each polytope $\mathcal{P}_i \subset \mathbb{R}^n$ for which the output is in a simplex $\mathcal{S}_i \subset \mathbb{R}^m$. That is, whose state evolves on polytopes $\mathcal{P}_i \subset \mathbb{R}^n$ and whose output within a simplex $\mathcal{S}_i \subset \mathbb{R}^m$,

$$\Sigma_i: \begin{cases} \dot{x}(t) = A_i x(t) + a_i + B_i u(t) + \delta_i(t), \\ y(t) = C_i x(t) + d_i, \end{cases} \tag{2}$$

where $A_i \in \mathbb{R}^{n \times n}, B_i \in \mathbb{R}^{n \times s}$, and $C_i \in \mathbb{R}^{m \times n}$ are the system, input, and output matrix, respectively; $a_i \in \mathbb{R}^n$ and $d_i \in \mathbb{R}^m$ are affine terms (constant vectors); and $\delta_i(t)$ is a bounded time-varying disturbance, i.e., $\|\delta_i(t)\| \leqslant \Delta_i$.

2.2 Transition systems and LTL

A transition system is a tuple $T = (Q, Q_0, \rightarrow, \Pi, \models)$, where Q is a set of states, $Q_0 \subset Q$ is a set of initial states, $\rightarrow \subset Q \times Q$ is a transition relation, Π is a finite set of atomic propositions, and $\models \subset Q \times \Pi$ is a satisfaction relation.

In the paper, we assume that the transition system is finite (that is, the state set Q is finite). For any proposition $\pi \in \Pi$, we define $[\,\pi\,] = \{q \in Q \big| q \models \pi\}$ as the set of states satisfying it. Conversely, for any state $q \in Q$, let $\Pi_q = \{\pi \in \Pi \big| q \models \pi\}$ ($\Pi_q \in 2^{\Pi}$) denote the set of all atomic propositions satisfied at q. A path of a transition system T starting from q is an infinite sequence $r = r(1)r(2)r(3)\cdots$ with the property that $r(1) = q$, $r(i) \in Q$, and $(r(i), r(i+1)) \in \rightarrow$ for all $i \geqslant 1$. A path $r = r(1)r(2)r(3)\cdots$ defines a word $w = w(1)w(2)w(3)\cdots$, where $w(i) = \Pi_{r(i)}$.

A LTL formula over a set of atomic proportions $\Pi = \{\pi_1, \cdots, \pi_N\}$ is recursively defined as follows (syntax):

1) Every atomic proposition π_i is a formula.

2) If ϕ_1 and ϕ_2 are formulas, then $\phi_1 \vee \phi_2$, $\neg\phi_1$, $\phi_1 \mathcal{U} \phi_2$ are also formulas.

The semantics of LTL formulas are given over words of a transition system T. The satisfaction of a formula ϕ at position $i \in \mathbb{N}$ of a word w, denoted by $w(i) \models \phi$, is defined recursively as follows:

1) $w(i) \models \pi$, if $\pi \in w(i)$;

2) $w(i) \not\models \pi$, if $\pi \notin w(i)$;

3) $w(i) \models \neg\phi$, if $w(i) \not\models \phi$;

4) $w(i) \models \phi_1 \vee \phi_2$, if $w(i) \models \phi_1$ or $w(i) \models \phi_2$;

5) $w(i) \models \phi_1 \mathcal{U} \phi_2$, if there exists a $j \geqslant i$ such that $w(j) \models \phi_2$ and for all k, $(i \leqslant k < j,)$ we have $w(k) \models \phi_1$.

w satisfies an LTL formula ϕ, written as $w \models \phi$, if $w(1) \models \phi$.

The boolean constants $\top$ and $\perp$ are defined as $\top = \pi \vee \neg\pi$ and $\perp = \neg\top$, respectively. Given negation $\neg$ and disjunction $\vee$, we can define conjunction $\wedge$, implication $\Rightarrow$, and equivalence $\Leftrightarrow$. Furthermore, we can also derive additional temporal operators such as eventuality $\diamond\phi = \top\mathcal{U}\phi$ and safety $\Box\phi = \neg \diamond \neg\phi$.

2.3 Problem formulation

In this paper, we consider the output trajectory of a piecewise affine system (2) evolving in a polyhedral set, which may not be convex. The goal of this paper is to devise an output feedback control law $u(y)$ so that all output trajectories satisfy a given LTL formula. The formula is built from a finite number of propositions, which label areas of interest in the polyhedral set such as unsafe regions or target areas. Finally, the problem is stated as follows.

Problem 1 Given an LTL formula ϕ and a piecewise affine hybrid system Σ with its dynamics defined in (2), devise an output feedback control $u(y)$ so that all output trajectories satisfy the formula ϕ.

3 Control synthesis

3.1 Local control strategy

In this subsection, we look at one location of the piecewise affine system, i.e., $(\Sigma_i, \mathcal{P}_i, \mathcal{S}_i)$. For simplicity, we drop the subscript and rewrite it as

$$\begin{cases} \dot{x}(t) = Ax(t) + a + Bu(t) + \delta(t), \\ y(t) = Cx(t) + d, \end{cases} \tag{3}$$

where $x \in \mathcal{P}, y \in \mathcal{S}$, and $\|\delta(t)\| \leqslant \Delta$. Without loss of generality, we assume $\mathrm{rank}(C) = m$.

We then study two local control problems and their solutions, which will be used for constructing a finite transition system and verifying LTL specifications.

Problem 2 Find, if possible, an output feedback control law $u(t) = Fy(t) + g$, where $F \in \mathbb{R}^{s \times m}$ and $g \in \mathbb{R}^s$, such that all the output trajectories starting from $\mathcal{S}$ exit $\mathcal{S}$ in finite time through a specified facet (denoted by $\mathcal{F}_1$).

Another problem is to find an output feedback control law such that the output trajectories remain in $\mathcal{S}$.

Problem 3 Find, if possible, an output feedback control law $u(t) = Fy(t) + g$ such that all the output trajectories starting from $\mathcal{S}$ remain in $\mathcal{S}$ forever.

Some similar problems are studied in [25] and then [26], in which state feedback control is considered. Here, we study output feedback control for affine systems with disturbances. The synthesis of local controllers solving Problems 2 and 3 requires some preliminary results. First, we denote the uncontrolled affine system of (3) by

$$\begin{cases} \dot{x}(t) = Ax(t) + a + \delta(t), \\ y(t) = Cx(t) + d, \end{cases} \quad x \in \mathcal{P} \text{ and } y \in \mathcal{S}. \qquad (4)$$

Next, for any point $y \in \mathcal{S}$, let

$$\mathcal{R}(y) = \{x \in \mathcal{P} | Cx + d = y\}.$$

Notice that $\mathcal{R}(y)$ is an $(n-m)$-dimensional polytope and is the set of states whose corresponding output is y. Similarly, for a set $\mathcal{A}$, $\mathcal{R}(\mathcal{A})$ is defined by $\bigcup_{y \in \mathcal{A}} \mathcal{R}(y)$.

Let $x(t, x_0)$ be the state trajectory of (4) starting at x_0. Let $Y(t, y_0)$ denote the set of output trajectories of (4) with any initial state x_0 satisfying $Cx_0 + d = y_0$, i.e.,

$$Y(t, y_0) := \{Cx(t, x_0) + d | x_0 \in \mathcal{R}(y_0)\}.$$

Notation $y(t, y_0)$ is used to represent an element of $Y(t, y_0)$. When the initial output y_0 can be ignored, we just use the notations $Y(t)$ and $y(t)$.

Define a set-valued map $\mathcal{G} : \mathbb{R}^m \to \mathbb{R}^m$ with

$$\mathcal{G}(y) = \{C(Ax + a) | x \in \mathcal{R}(y)\}.$$

The set $\mathcal{G}(y)$ denotes all possible directions at y in the output space without disturbances. In what follows, for any set $\mathcal{A} \subset \mathbb{R}^m$, we use $\mathcal{G}(\mathcal{A})$ to denote the set $\{C(Ax + a) | x \in \mathcal{R}(\mathcal{A})\}$.

Before presenting the solutions, we introduce a lemma on the existence and uniqueness of affine feedback control when control inputs u_i at vertices v_i are known.

Lemma 1 [26] Consider two sets of points $\{v_1, \cdots, v_{m+1}\}$ in $\mathbb{R}^m$ and $\{u_1, \cdots, u_{m+1}\}$ in $\mathbb{R}^s$. Suppose that $v_1, \cdots, v_{m+1}$ are affinely independent (i.e., the vectors $(v_2 - v_1), \cdots, (v_{m+1} - v_1)$ are linearly independent). Then, there exist a unique matrix $F \in \mathbb{R}^{s \times m}$ and a unique vector $g \in \mathbb{R}^s$ such that for each v_i, we have $u_i = Fv_i + g$.

This lemma states that the matrix F and g can be calculated from the following equation:

$$[F \;\; g] = [u_1 \; \cdots \; u_{m+1}] \begin{bmatrix} v_1 \; v_2 \; \cdots \; v_{n+1} \\ 1 \;\; 1 \;\; \cdots \;\; 1 \end{bmatrix}^{-1}.$$

The following lemma gives a condition that all output trajectories will leave a simplex in finite time.

Lemma 2 Consider affine system (4). If there is a vector $\xi \in \mathbb{R}^m$ such that $\xi^{\mathrm{T}} h < -\|\xi^{\mathrm{T}} C\| \varDelta$ for all $h \in \mathcal{G}(\mathcal{S})$, then all output trajectories starting from $\mathcal{S}$ leave $\mathcal{S}$ in finite time.

Proof Suppose by contradiction that there is an output trajectory $y(t, y_0)$ remaining in $\mathcal{S}$ forever. Since $\xi^{\mathrm{T}} h < -\|\xi^{\mathrm{T}} C\| \varDelta$ for all $h \in \mathcal{G}(\mathcal{S})$ and $\mathcal{G}(\mathcal{S})$ is compact, it follows that there is an $\epsilon > 0$ such that $\xi^{\mathrm{T}} h < -\|\xi^{\mathrm{T}} C\| \varDelta - \epsilon$ for all $h \in \mathcal{G}(\mathcal{S})$. Notice that

$$\xi^{\mathrm{T}} C\delta(t) \in [-\|\xi^{\mathrm{T}} C\| \varDelta, \|\xi^{\mathrm{T}} C\| \varDelta].$$

Hence, we have

$$\begin{aligned} \xi^{\mathrm{T}} \dot{y}(t, y_0) &= \xi^{\mathrm{T}}(C(Ax(t, x_0) + a + \delta(t)) + d) \\ &= \xi^{\mathrm{T}}(C(Ax(t, x_0) + a) + d) + \xi^{\mathrm{T}} C\delta(t) \\ &< -\|\xi^{\mathrm{T}} C\| \varDelta - \epsilon + \|\xi^{\mathrm{T}} C\| \varDelta \\ &< -\epsilon \end{aligned}$$

for all t and for all $x_0 \in \mathcal{R}(y_0)$. Therefore, there is a nonzero positive speed of the output trajectory in the direction $-\xi$. That is, the output trajectory $y(t, y_0)$ eventually leaves $\mathcal{S}$, a contradiction.

A facet $\mathcal{F}_j$ is called restricted or blocked if $n_j^{\mathrm{T}} h < -\|n_j^{\mathrm{T}} C\| \varDelta$ for all $h \in \mathcal{G}(\mathcal{F}_j)$. The lemma below states that no output trajectory can leave $\mathcal{S}$ from a restricted facet.

Lemma 3 Let $\mathcal{F}_j$ be a facet of $\mathcal{S}$ with the unit normal vector n_j pointing out of $\mathcal{S}$. For affine system (4), if $n_j^{\mathrm{T}} h < -\|n_j^{\mathrm{T}} C\| \varDelta$ for all $h \in \mathcal{G}(\mathcal{F}_j)$, then no output trajectory leaves $\mathcal{S}$ from $\mathcal{F}_j$.

Proof Without loss of generality, we consider $j = 1$ and assume that $\mathcal{F}_1$ lies in the subspace $\{y \in \mathbb{R}^m | n_1^{\mathrm{T}} y = 0\}$, and $\mathcal{S}$ is in the half subspace $\{y \in \mathbb{R}^m | n_1^{\mathrm{T}} y \leqslant 0\}$.

First, we claim that if $\|\varDelta\| = 0$, then no output trajectory can escape through $\mathcal{F}_1$. It can be checked that $n_1^{\mathrm{T}} \mathcal{G}(y)$ is upper semicontinuous (For any $\epsilon_1 > 0$, there exists

$$\delta = \frac{\epsilon_1}{2\|n_1^{\mathrm{T}} CA\|}$$

such that $\forall y' \in \mathcal{B}(y, \delta)$, $n_1^{\mathrm{T}} \mathcal{G}(y') \subset \mathcal{B}(n_1^{\mathrm{T}} \mathcal{G}(y), \epsilon_1)$). Besides, since $\mathcal{F}_1$ is compact and $n_1^{\mathrm{T}} h < 0$ for all $h \in G(\mathcal{F}_1)$, there is a neighborhood of $\mathcal{F}_1$, $\mathcal{B}(\mathcal{F}_1)$, such that $n_1^{\mathrm{T}} h \leqslant 0$ for all $h \in \mathcal{G}(\mathcal{B}(\mathcal{F}_1))$. Let $\mathcal{W} = \mathcal{B}(\mathcal{F}_1) \cap \{y \in \mathbb{R}^m | n_1^{\mathrm{T}} y \geqslant 0\}$. Moreover, suppose, by contradiction that, there is one output trajectory that leaves through $\mathcal{F}_1$, i.e., $y(0) \in \mathcal{F}_1$ and $y(t_1) \in \mathcal{W}$ for some $t_1 > 0$. Then, it follows that

$$y_1(t) = n_1^{\mathrm{T}} y(t) = \int_0^t n_1^{\mathrm{T}} \dot{y}(t) \mathrm{d}t \leqslant 0$$

since $\dot{y}(t) \in \mathcal{G}(\mathcal{W})$ for $t \in [0, t_1]$, a contradiction. See Fig. 1.

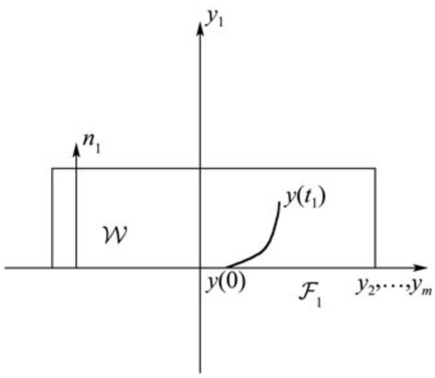

Fig. 1 An output trajectory leaves $\mathcal{S}$ via $\mathcal{F}_1$.

When taking disturbances into account as in the proof of Lemma 2, notice that

$$n_1^{\mathrm{T}} C\delta(t) \in [-\|n_1^{\mathrm{T}} C\| \varDelta, \|n_1^{\mathrm{T}} C\| \varDelta],$$
$$n_1^{\mathrm{T}} \dot{y} = n_1^{\mathrm{T}}(h + C\delta) < 0$$

for all $h \in \mathcal{G}(y)$. It follows that $n_1^{\mathrm{T}} h < -\|n_1^{\mathrm{T}} C\| \varDelta$ for all $h \in \mathcal{G}(y)$. Thus, the conclusion follows.

For a facet $\mathcal{F}_j$ and a vertex $v_i \in \mathrm{vert}(\mathcal{F}_j)$, we define

$$\beta_{ij}^- = \min_{x \in \mathcal{R}(v_i)} n_j^{\mathrm{T}} CAx, \;\; \beta_{ij}^+ = \max_{x \in \mathcal{R}(v_i)} n_j^{\mathrm{T}} CAx.$$

Then, the restriction conditions in Lemma 3 can be checked by only looking at the vertices of facet $\mathcal{F}_j$.

Lemma 4 For affine system (4), a facet $\mathcal{F}_j$ is restricted if $\beta_{ij}^+ + n_j^{\mathrm{T}} Ca < -\|n_j^{\mathrm{T}} C\| \varDelta$, $\forall v_i \in \mathrm{vert}(\mathcal{F}_j)$.

Proof Denote the vertices of $\mathcal{F}_j$ by $v_1, \cdots, v_k$. For any $y \in \mathcal{F}_j$, it can be written as a convex combination of $v_1, \cdots, v_k$, i.e., $y = \alpha_1 v_1 + \cdots + \alpha_k v_k$ for $\alpha_1, \cdots, \alpha_k$ satisfying $\alpha_1 + \cdots + \alpha_k = 1$ and $0 \leqslant \alpha_i \leqslant 1$. For any $x \in \mathcal{R}(y)$, where $y \in \mathcal{F}_j$, there are points $x_1 \in \mathcal{R}(v_1), \cdots,$

$x_k \in \mathcal{R}(v_k)$ such that x is the same convex combination of $x_1, \cdots, x_k$, i.e., $x = \alpha_1 x_1 + \cdots + \alpha_k x_k$. For $h \in \mathcal{G}(y)$, there exists an $x \in \mathcal{R}(y)$ satisfying $h = C\dot{x} = C(Ax + a)$. Therefore,

$$\begin{aligned} n_j^{\mathrm{T}} h &= n_j^{\mathrm{T}}[CA(\alpha_1 x_1 + \cdots + \alpha_k x_k) + Ca] \\ &= \alpha_1 n_j^{\mathrm{T}} C(Ax_1 + a) + \cdots + \alpha_k n_j^{\mathrm{T}} C(Ax_k + a). \end{aligned}$$

Every term, $n_j^{\mathrm{T}} C(Ax_i + a)$, $i = 1, \cdots, k$, is less than $-\|n_j^{\mathrm{T}} C\|\Delta$. Thus, $n_j^{\mathrm{T}} h < -\|n_j^{\mathrm{T}} C\|\Delta$. Therefore, the facet $\mathcal{F}_j$ is restricted by the definition.

Now, by virtue of Lemmas 2 and 4, we are ready to present the solutions to Problems 2 and 3.

Theorem 1 Problem 2 is solvable if there are vectors $u_1, \cdots, u_{m+1}$ in $\mathbb{R}^s$ such that

$$\beta_{i1}^{-} + n_1^{\mathrm{T}}(Ca + CBu_i) > \|n_1^{\mathrm{T}} C\|\Delta, \ \ \forall v_i \in \mathrm{vert}(\mathcal{S}), \tag{5}$$

and for $j = 2, \cdots, m+1$,

$$\beta_{ij}^{+} + n_j^{\mathrm{T}}(Ca + CBu_i) < -\|n_j^{\mathrm{T}} C\|\Delta, \ \ \forall v_i \in \mathrm{vert}(\mathcal{F}_j). \tag{6}$$

Proof By Lemma 1, a matrix F and a vector g can be constructed uniquely when $u_1, \cdots, u_m$ are solved from (5) and (6). Taking these F and g as output feedback, we have that the closed loop system is still affine. Then, consider the closed loop system. For $h \in \mathcal{G}(v_i)$, $x \in \mathcal{R}(v_i)$, we have

$$\begin{aligned} n_1^{\mathrm{T}} h &= n_1^{\mathrm{T}}(CAx + Ca + CBu_i) \\ &> \beta_{i1}^{-} + n_1^{\mathrm{T}}(Ca + CBu_i) \\ &> \|n_1^{\mathrm{T}} C\|\Delta. \end{aligned}$$

Therefore, by the convex argument and Lemma 2, it follows that all output trajectories will leave the polytope. Furthermore, by Lemma 4, condition (6) ensures that all other facets are restricted. Combining these two facts, all output trajectories will leave $\mathcal{S}$ only via $\mathcal{F}_1$, and the conclusion follows.

Remark 1 Notice that conditions (5) and (6) can be written as a set of linear inequalities. And solvability can be determined by the well-known Farkas' Lemma [27], which is polynomial hard.

Similarly, we have the following condition for the solvability of Problem 3.

Theorem 2 Problem 3 is solvable if there are vectors $u_1, \cdots, u_{m+1}$ in $\mathbb{R}^s$ such that for $j = 1, \cdots, m+1$,

$$\beta_{ij}^{+} + n_j^{\mathrm{T}}(Ca + CBu_i) < -\|n_j^{\mathrm{T}} C\|\Delta, \ \ \forall v_i \in \mathrm{vert}(\mathcal{F}_j). \tag{7}$$

Proof The conclusion follows since every facet is restricted by (7) and Lemma 4.

Remark 2 Although, in this part, we consider the problems in a simplex, there is no difficulty to extend the results when the set is a polytope instead of a simplex. In other words, for Problems 2 and 3, $\mathcal{S}$ can be a polytope, and the results need only a few modifications.

3.2 Determining feasible paths

Next, we come to find the solution to Problem 1 along the following three main steps.

Step 1 Construct a transition system.

We construct an associated transition system $T = (Q, Q_0, \rightarrow, \Pi, \models)$ of $\Sigma = \{(\Sigma_i, \mathcal{P}_i, \mathcal{S}_i) : i = 1, \cdots, p\}$ as follows:

1) $Q = \{\mathcal{S}_1, \cdots, \mathcal{S}_p\}$;

2) $Q_0 = \{\mathcal{S}_i\}$ if $y_0 \in \mathcal{S}_i$;

3) For two simplices $\mathcal{S}_i$, $\mathcal{S}_j$ with a common facet $\mathcal{F}$, if Problem 2 is solvable for $\mathcal{S}_i$ and $\mathcal{F}$, then let $(\mathcal{S}_i, \mathcal{S}_j) \in \rightarrow$. On the other hand, if Problem 3 is solvable for simplex $\mathcal{S}_i$, then $(\mathcal{S}_i, \mathcal{S}_i) \in \rightarrow$.

4) $\Pi = \{\pi_1, \cdots, \pi_M\}$, where each proposition π_i, $i = 1, \cdots, M$, denotes a region of interest, and $[\pi_i] \in 2^Q$.

5) $\mathcal{S}_i \models \pi_j \in \Pi$ if $\mathcal{S}_i \in [\pi_j]$.

Step 2 Determine feasible paths.

For the transition system T we just obtained, determine feasible paths that satisfy the given LTL formula ϕ. This is a well-studied problem in model checking [22, 23, 28] and there are many simulation tools, e.g., NuSMV [29]. We proceed along the following route to determine feasible paths:

1) Construct an automaton (also known as Buchi automaton) for the LTL formula ϕ and denote it by A_ϕ. The automaton has the property that it encodes precisely the paths that satisfy the LTL formula ϕ.

2) Combine the automaton A_ϕ and the transition system T. The combination operation results in a transition system $A_\phi \times T$ whose paths are paths for both the automaton A_ϕ and the transition system T.

3) Find a path starting from a state derived from q_0 in the combined transition system $A_\phi \times T$. Such a path (if exits) can be interpreted as a path in T starting at q_0, which satisfies ϕ.

Step 3 Design output feedback control laws.

Let r be a path of T that satisfies ϕ. Then, for any two successive states $\mathcal{S}_i, \mathcal{S}_j \in r$, design local control $u_{ij} = F_{ij} y + g_{ij}$ according to Theorem 1. For an infinitely repeated state $\mathcal{S}_i \in r$, design local control $u_{ii} = F_{ii} y + g_{ii}$ according to Theorem 2.

In this way, we can obtain a piecewise affine output feedback control law for Problem 1. However, it is worth pointing out that our approach for LTL reachability (Problem 1) is conservative due to the following reason. Theorems 1 and 2 give only sufficient conditions for local reachability and set invariance. Therefore, it may fail to find a transition relation between two adjacent simplices, or it may fail to find a self loop at some state of the transition system. However, as shown in the following section, the method allows us to synthesize controllers that are fully automated and computationally effective.

4 Simulation

In this section, we present a simulation to illustrate our results. This problem is adapted from the motion planning problem of mobile robots. Consider a polygonal environment in Fig. 2. It is expected that a robot goes to room R_1, R_2, R_3, and R_4 (polygons of dashed lines in Fig. 2) in order, and then remains in R_1 thereafter. Meanwhile, it is desired that it shall avoid obstacles $O_1, \cdots, O_6$. We assume that the dynamics of the robot has been linearized and it is given as

$$\begin{cases} \dot{x} = Ax + Bu + a + \delta, \\ y = Cx + d, \end{cases}$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \ B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \ a = d = 0.$$

$u = [u_1 \ \ u_2]^{\mathrm{T}}$ is the control input, $x = [x_1 \ \ x_2 \ \ x_3 \ \ x_4]^{\mathrm{T}}$ is the robot's state, the output $y = [y_1 \ \ y_2]^{\mathrm{T}}$ is the position on the plane, (x_3, x_4) is its velocity, and $\|\delta\| \leqslant 0.1$. The robot is constrained in the region $\{0 \leqslant x_1 \leqslant 9, -9 \leqslant x_2 \leqslant -1\}$. Its velocity lies in $\{-30 \leqslant x_3 \leqslant 30, -30 \leqslant x_3 \leqslant 30\}$ due to physical constraints. Thus, the polytope describing the state constraints is given by $\mathcal{P} = \{0 \leqslant x_1 \leqslant 9, -9 \leqslant x_2 \leqslant -1, -30 \leqslant x_3 \leqslant 30, -30 \leqslant x_4 \leqslant 30\}$.

Propositions π_1, π_2, π_3, and π_4 are used to denote room R_1, R_2, R_3, and R_4, respectively. Proposition π_0 denotes the obstacles $O = O_1 \cup \cdots \cup O_6$. The LTL specification is then given as

$$\neg\pi_0 \wedge \diamond(\pi_1 \wedge \diamond(\pi_2 \wedge \diamond(\pi_3 \wedge \diamond(\pi_4 \wedge \diamond\Box\pi_1)))).$$

Applying our approach to the specific problem, we obtain a piecewise output feedback control such that resulting paths satisfy the LTL specification we give. Taking S_6 for an example, an output feedback control law making the output trajectories leave S_6 and enter S_7 is $u = F_6 y + g_6$, where

$$F_6 = \begin{bmatrix} -168.3315 & -51.0220 \\ -106.4808 & -42.1862 \end{bmatrix} \text{ and } g_6 = \begin{bmatrix} 949.9340 \\ 300.9193 \end{bmatrix}.$$

An output feedback control law making R_1 positive invariant is $u = F_1 y + g_1$, where

$$F_1 = \begin{bmatrix} -98.8542 & 35.8610 \\ 8.8542 & -125.8610 \end{bmatrix} \text{ and } g_1 = \begin{bmatrix} 161.7221 \\ -221.7221 \end{bmatrix}.$$

A simulated trajectory starting at

$$[1.0000, \ -1.6667 \ \ 0.7094 \ \ 0.7547]^{\mathrm{T}} \ (\text{in } R_1)$$

is shown in Fig. 2, which satisfies the LTL specification.

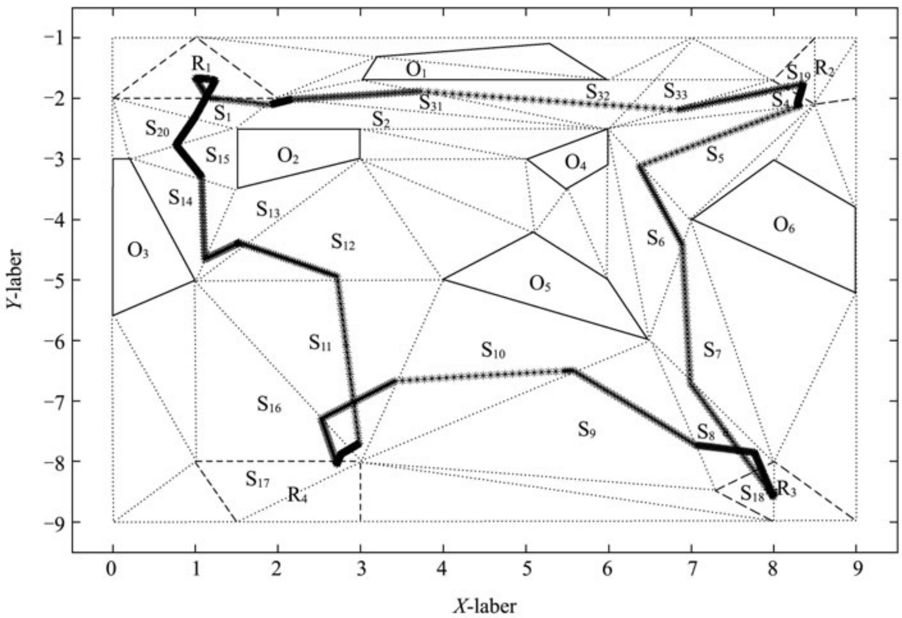

Fig. 2 The output trajectory of the robot.

5 Conclusions

In this paper, we present an algorithmic approach to an LTL motion planning problem. Our technique is based on two main parts, namely, a local output feedback strategy and a supervisory control for LTL specifications. The supervisory control is performed in a transition system, done by model checking. The local control is based on the reachability and invariance problem on simplices. The resulting control law is piecewise affine output feedback that is valid on the corresponding simplices. There are several possible extensions for this work. First, one could use the hybridization method to approximate a nonlinear system into a piecewise affine system with disturbances, that is, manipulating nonlinear systems. Second, one could investigate necessary and sufficient conditions for local control in both simplices and polytopes.

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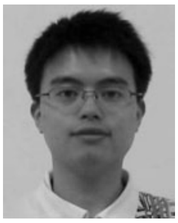

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