

Output Feedback Controller Synthesis for Piecewise-Affine Systems with Multiple Equilibria

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Abstract

This work builds on the stability analysis of piecewise-affine systems reported in [1] and extends it to obtain a new synthesis tool for output feedback controllers. The proposed technique relies on formulating the search for a piecewise-quadratic Lyapunov function and a piecewise-affine controller as a *Bilinear Matrix Inequality*. This can be solved iteratively as a set of two convex optimization problems involving linear matrix inequalities which can be solved numerically very efficiently. A key point in this design technique is that it can be used to design controllers with different structures depending on the number of constraints that are added. In particular, it is shown that a controller with the structure of a regulator and estimator can be designed so that switching based on state estimates rather than on the output can be performed. It is also shown that many other desired features can be included in the design, such as boundedness of the control signals. Furthermore, the applicability of this design method to systems with multiple equilibrium points is shown in simulation examples.

1 Introduction

Over the last few years promising new methods have emerged for the analysis of piecewise-linear and piecewise-affine systems. These systems represent a powerful model class for nonlinear systems. They arise naturally from linear dynamics in the presence of saturations or as simple examples of hybrid systems where the continuous dynamics within the different discrete states is linear or affine (see [2]). Piecewise-affine systems can also be used to approximate other nonlinear systems.

The analysis methods reported in [1], [3] and [4] rely on reformulating the analysis problem in terms of certain convex optimization programs involving *Linear Matrix Inequalities* (LMIs), which are then solved numerically. The theory of LMIs applied to System and Control Theory (up to 1994) is covered in [5] and a complete description of the new interior-point algorithms used to solve these convex programs can be found in [6].

Synthesis has also been attempted for piecewise-linear systems based on the analysis methods described in the last paragraph. In [1], piecewise-linear state feedback

is formulated as a convex optimization problem involving the search for a globally quadratic Lyapunov function. This feedback control is not suited for systems with multiple equilibria. In [7] an integral piecewise-quadratic performance index is suggested for piecewise-affine systems. Lower and upper bounds are computed for the control law that optimizes this performance index. However, the control law is not computed. In [8] a piecewise-affine state feedback is computed using a globally quadratic Lyapunov function for uncertain piecewise-affine discrete-time systems. Again, this approach does not handle either output feedback or systems with multiple equilibria.

None of these previous works provide a systematic synthesis method that can include features such as continuity of the input signals or can handle multiple equilibria. Furthermore, these synthesis methods do not address the output feedback problem. This paper presents an iterative method that can be used to design state and output feedback controllers with various constraints on the continuity and smoothness of the Lyapunov Function and control signals. To the best of the authors' knowledge it is the first time that an output feedback controller is designed for piecewise-affine systems using Convex Optimization Methods. An interesting recent work that addresses estimation for hybrid systems from a different perspective can be found in [9].

This paper is organized as follows: we start by presenting the system description to be used, the class of Lyapunov functions considered and conditions for open loop stability analysis. These sections follow closely the work reported in [1]. We then describe dynamic output feedback synthesis as the result of solving *Bilinear Matrix Inequalities* (BMIs, see [10]) using the V-K iteration method [11]. See [12] for an exposition of piecewise-affine state feedback. Finally, the applicability of this design method to systems with multiple equilibria is shown in simulation examples.

2 System Description

The method that will be studied is suited for controlling systems whose dynamics partition the state space R^n into a finite number of closed polytopic regions. We will therefore assume that such a partition exists with polytopic cells $\mathcal{R}_i, i \in \mathcal{I}$. Following [1] each cell is con-

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structured as the intersection of a finite number p_i of half spaces given by the following affine inequalities:

$$\mathcal{R}_i = \{x \mid h_{ij}^T x < g_{ij}, j = 1, \dots, p_i\}.$$

Thus each cell can be characterized by the vector inequality

$$H_i^T x - g_i < 0 \quad (1)$$

where

$$H_i = [h_{i1} \ h_{i2} \ \dots \ h_{ip_i}], \quad g_i = [g_{i1} \ g_{i2} \ \dots \ g_{ip_i}]^T.$$

Within each cell the dynamics is piecewise-affine of the form

$$\dot{x}(t) = A_i x(t) + b_i + B_i u(t), \quad (2)$$

$$y = C_i x. \quad (3)$$

We also assume as in [1] that we have a parametric description of the boundaries as

$$\overline{\mathcal{R}_i} \cap \overline{\mathcal{R}_j} \subseteq \{l_{ij} + F_{ij} z \mid z \in \mathbb{R}^{n-1}\} \quad (4)$$

for $i = 1, \dots, M$, $j = i+1, \dots, M$ and for some $F_{ij} \in \mathbb{R}^{n \times (n-1)}$ (full rank) and $l_{ij} \in \mathbb{R}^n$ (see Figure 1).

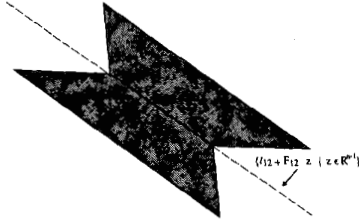


Figure 1: Polytopic Regions and Boundaries

3 Piecewise-Quadratic Lyapunov Functions

The Lyapunov functions considered in this work are of the form

$$V(x) = \sum_{i=1}^M \beta_i(x) V_i(x), \quad V(x) > 0, \quad V \text{ is continuous}, \quad (5)$$

$$V_i(x) = (x^T P_i x + 2q_i^T x + r_i), \quad (6)$$

where $P_i = P_i^T \in \mathbb{R}^{(n \times n)}$, $q_i \in \mathbb{R}^n$, $r_i \in \mathbb{R}$ and

$$\beta_i(x) = \begin{cases} 1, & x \in \mathcal{R}_i \\ 0, & \text{otherwise} \end{cases}, \quad (7)$$

for $i = 1, \dots, M$. Since $V(x)$ is piecewise-quadratic, $V > 0$ also implies that V is radially unbounded, i.e., $V(x) \rightarrow +\infty$ as $\|x\| \rightarrow \infty$. The expression for the Lyapunov function in each region can be recast as

$$V_i(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} P_i & q_i \\ q_i^T & r_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \bar{x}^T \bar{P}_i \bar{x}. \quad (8)$$

The Lyapunov function must be continuous across the boundaries of the cells. Using the boundary description (4), continuity is enforced for each region i by

$$F_{ij}^T (P_i - P_j) F_{ij} = 0,$$

$$F_{ij}^T (P_i - P_j) l_{ij} + F_{ij}^T (q_i - q_j) = 0,$$

$$l_{ij}^T (P_i - P_j) l_{ij} + 2(q_i - q_j)^T l_{ij} + (r_i - r_j) = 0, \quad (9)$$

for $j = i+1, \dots, M$. Continuous differentiability can also be enforced as discussed in [12] but it will not be needed for our examples and therefore it will not be discussed.

4 Open Loop Stability Analysis

This section presents sufficient conditions for the open loop stability of system (2). To enforce that the Lyapunov function (5) must decay at least at a rate α_i within each region i we search for the existence of $P_i = P_i^T > 0$ such that

$$x \in \mathcal{R}_i \Rightarrow \begin{cases} V_i(x) > 0 \\ \frac{d}{dt} V_i(x) < -\alpha_i V_i(x) \end{cases} \quad (10)$$

For all $x \in \mathcal{R}_i$ we have

$$\frac{d}{dt} V_i(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} A_i^T P_i + P_i A_i & P_i b_i + A_i^T q_i \\ b_i^T P_i + q_i^T A_i & 2b_i^T q_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}.$$

Using the polytopic description of the cells (1) and the S -procedure (see [5]), it can be shown that (see [1] or [12] for details) the conditions for open-loop stability are equivalent to the existence of P_i , λ_{ij} and τ_{ij} satisfying the LMIs

$$\lambda_i > 0, \quad \tau_i > 0, \quad i = 1, \dots, M$$

$$\begin{bmatrix} P_i & q_i + H_i \lambda_i \\ q_i^T + \lambda_i^T H_i^T & r_i - 2g_i^T \lambda_i \end{bmatrix} > 0, \quad (11)$$

$$\begin{bmatrix} A_i^T P_i + P_i A_i & P_i b_i + A_i^T q_i - H_i \tau_i \\ +\alpha_i P_i & +\alpha_i q_i \\ b_i^T P_i + q_i^T A_i - \tau_i^T H_i^T & 2(g_i^T \tau_i + b_i^T q_i) \\ +\alpha_i q_i^T & +\alpha_i r_i \end{bmatrix} < 0. \quad (12)$$

Two remarks should be made at this point. First, when \mathcal{R}_i is a slab, a (degenerate) ellipsoid of the form $\|E_{i1}x + f_{i1}\| \leq 1$ can be found that approximates \mathcal{R}_i exactly. If $\mathcal{R}_i = \{x \mid d_1 \leq c^T x \leq d_2\}$, then we can take $E_{i1} = 2c^T / (d_2 - d_1)$ and $f_{i1} = -(d_2 + d_1) / (d_2 - d_1)$. In this case, the alternative LMIs presented in [1] should be used. Second, the equilibrium points of the piecewise-affine system should be the local minima of any Lyapunov function candidate. Therefore, if $x_{eq,i} \in \mathcal{R}_i$ is an equilibrium point we must have $x_{eq,i} = -P_i^{-1} q_i$. In other words, for $x \in \mathcal{R}_i$

$$V(x) = (x - x_{eq,i})^T P_i (x - x_{eq,i}) + r_i.$$

Replacing q_i by $-P_i x_{eq,i}$ and r_i by $r_i + x_{eq,i}^T P_i x_{eq,i}$ in the LMIs (11) and (12), as well as in (9), gives a new set of conditions. These conditions are more favorable from a numerical point of view because they remove some problems with strict infeasibility of the LMIs experienced with numerical packages that only solve strict LMIs.

5 Dynamic Output Feedback Controller Synthesis

We now seek an output feedback controller with state space representation in each region of the form

$$\begin{aligned} \dot{x}_c(t) &= A_{c_i} x_c + L_i y + b_{c_i}, \\ u &= K_i x_c + m_i, \end{aligned} \quad (13)$$

Using the augmented state $\tilde{x} = [x^T \ x_c^T]^T$ we can rewrite the state equations of both the system and the controller in the form

$$\dot{\tilde{x}}(t) = \tilde{A}_i \tilde{x} + \tilde{b}_i, \quad (14)$$

where

$$\tilde{A}_i = \begin{bmatrix} A_i & B_i K_i \\ L_i C_i & A_{c_i} \end{bmatrix}, \quad \tilde{b}_i = \begin{bmatrix} b_i + B_i m_i \\ b_{c_i} \end{bmatrix}.$$

The Lyapunov function for the augmented system is

$$V(\tilde{x}) = \sum_{i=1}^M \beta_i(\tilde{x}) V_i(\tilde{x}) = \sum_{i=1}^M \beta_i(\tilde{x}) \left(\tilde{x}^T \tilde{P}_i \tilde{x} + 2\tilde{q}_i^T \tilde{x} + \tilde{r}_i \right), \quad (15)$$

where $\tilde{P}_i = \tilde{P}_i^T \in \mathbb{R}^{(2n \times 2n)}$, $\tilde{q}_i \in \mathbb{R}^{2n}$, $\tilde{r}_i \in \mathbb{R}$ for $i = 1, \dots, M$ and β_i is defined as in (7). The continuity conditions (9) are now rewritten as

$$\begin{aligned} \tilde{F}_{ij}^T (\tilde{P}_i - \tilde{P}_j) \tilde{F}_{ij} &= 0, \\ \tilde{F}_{ij}^T (\tilde{P}_i - \tilde{P}_j) \tilde{l}_{ij} + \tilde{F}_{ij}^T (\tilde{q}_i - \tilde{q}_j) &= 0, \\ \tilde{l}_{ij}^T (\tilde{P}_i - \tilde{P}_j) \tilde{l}_{ij} + 2(\tilde{q}_i - \tilde{q}_j)^T \tilde{l}_{ij} + (\tilde{r}_i - \tilde{r}_j) &= 0, \end{aligned} \quad (16)$$

where

$$\tilde{F}_{ij} = \begin{bmatrix} F_{ij} & 0 \\ 0 & I \end{bmatrix}, \quad \tilde{l}_{ij} = [l_{ij}^T \ 0]^T.$$

To enforce continuity of the control inputs at the boundaries we include the actuator dynamics into the plant (see [12] for a discussion of alternative constraints to enforce continuity). We will assume a first order actuator dynamics described by

$$\dot{u} = -\tau u + \tau u_c, \quad u_c = K_i x_c + m_i. \quad (17)$$

The order of the system dynamics will then increase by one state for each input. Including these first order actuator dynamics we can solve for the optimal τ as a parameter in the optimization problem. From now on, the control input to the augmented plant will be denoted by u_c .

Replacing now A_i by \tilde{A}_i , b_i by \tilde{b}_i , P_i by \tilde{P}_i , q_i by \tilde{q}_i and r_i by \tilde{r}_i in expressions (11), (12) we obtain the following inequalities, respectively:

$$\begin{bmatrix} \tilde{P}_i & \tilde{q}_i + \tilde{H}_i \lambda_i \\ \tilde{q}_i^T + \lambda_i^T \tilde{H}_i^T & \tilde{r}_i - 2\tilde{g}_i^T \lambda_i \end{bmatrix} > 0, \quad (18)$$

$$\begin{bmatrix} \tilde{A}_i^T \tilde{P}_i + \tilde{P}_i \tilde{A}_i & \tilde{P}_i \tilde{b}_i + \tilde{A}_i^T \tilde{q}_i - \tilde{H}_i \tau_i \\ +\alpha_i \tilde{P}_i & +\alpha_i \tilde{q}_i \\ \tilde{b}_i^T \tilde{P}_i + \tilde{q}_i^T \tilde{A}_i - \tau_i^T \tilde{H}_i^T & 2(\tilde{g}_i^T \tau_i + \tilde{b}_i^T \tilde{q}_i) \\ +\alpha_i \tilde{q}_i^T & +\alpha_i \tilde{r}_i \end{bmatrix} < 0, \quad (19)$$

where

$$\tilde{H}_i = [H_i^T \ 0]^T, \quad \tilde{g}_i = g_i, \quad \tilde{E}_{ij} = [E_{ij} \ 0], \quad \tilde{f}_{ij} = f_{ij}.$$

To specify that the closed loop equilibrium point for each region be at some desired point x_{des} , we add the constraint

$$\tilde{A}_i x_{des} + \tilde{b}_i = 0. \quad (20)$$

This will enable us to specify that the closed loop system has only one equilibrium point by placing the other equilibrium points outside their respective regions. This closed loop equilibrium point can then be made globally asymptotically stable by the output feedback.

The inequalities above are bilinear matrix inequalities (BMIs, see [10]). To solve them we can use the V-K iteration method [11]. More specifically, our optimization problem can be described as follows:

V-Step: given a fixed controller and fixed α_i , we solve the following feasibility problem to find $V(\tilde{x})$

$$\begin{aligned} \text{find } & \tilde{P}_i, \tilde{q}_i, \tilde{r}_i, \\ \text{s.t. } & (16), (18), (19) \\ & \lambda_{ij} > 0, \tau_{ij} > 0, j = 1, \dots, m_i, \quad i = 1, \dots, M. \end{aligned} \quad (19)$$

K-Step: We then fix the parameters \tilde{P}_i , \tilde{q}_i , \tilde{r}_i and solve the following optimization problem

$$\begin{aligned} \text{maximize } & \min_i \alpha_i \\ \text{s.t. } & (18), (19), (20) \\ & \lambda_{ij} > 0, \tau_{ij} > 0, \alpha_i > l_0 \geq 0, j = 1, \dots, m_i, \\ & -l_1 \leq K_i \leq l_1, -l_2 \leq m_i \leq l_2, i = 1, \dots, M. \end{aligned} \quad (20)$$

Using slack variables the K-Step can be rewritten with a linear objective. Note that constraint (20) must be included if we want the equilibrium points to be fixed during the iterative procedure. This optimization corresponds to an iterative loop that is repeatedly executed until there is no major improvement in the cost relative to the previous iteration, or until an unfeasible solution of the LMIs is found. The fixed α_i in the V-Step should be the ones obtained in the previous iteration as a solution to the K-Step. Also, the constraint that the sum of the α_i should be higher than the one obtained in the previous iteration should be included in the K-Step.

This approach does not allow to switch based on state estimates and to use it we must assume that the switching of the controllers is only driven by the output. However, it is possible for the system dynamics to change according to internal states which are not necessarily measured. Thus, to be able to switch based on state estimates we

need to design a controller with the structure of a regulator and an estimator. To this aim, start by performing the change of coordinates $z = x - x_{des}$ and rewrite the dynamics as

$$\dot{z}(t) = A_i z + B_i u_c + A_i x_{des} + b_i. \quad (22)$$

Let us now define a new output as $\tilde{y} = y - C_i x_{des}$ and the control input to be used as $u_c = \tilde{u} + m_i$. Based on (22) we define the regulator/estimator dynamics in each region in the form

$$\begin{aligned} \dot{\hat{z}}(t) &= A_i \hat{z} + B_i u_c + A_i x_{des} + b_i + L_i (\tilde{y} - C_i \hat{z}), \\ \tilde{u} &= K_i \hat{z}. \end{aligned} \quad (23)$$

The augmented dynamics can then still be written in the form (15) with

$$A_{c_i} = A_i + B_i K_i - L_i C_i, \quad (24)$$

$$b_{c_i} = B_i m_i + b_i + (A_i - L_i C_i) x_{des}. \quad (25)$$

The additional constraints (24), (25) should therefore be included in the K-Step to obtain a regulator/estimator structure. Notice that constraint (21) simplifies to

$$A_i x_{des} + b_i + B_i m_i = 0 \quad (26)$$

$$b_{c_i} + L_i C_i x_{des} = 0 \quad (27)$$

with $\hat{z}_{des} = 0$, where \hat{z}_{des} stands for the controller equilibrium states, which must be zero since they correspond to the desired estimated errors \hat{z} . Notice also that (26) has to be verified or otherwise we do not have $x \rightarrow x_{des}$ as $t \rightarrow \infty$ according to (22) and the definition of u_c . Solving the regulator/estimator synthesis problem by searching for K_i and L_i , $i = 1, \dots, M$ using the V-K iteration method basically extends the results in [5] and [13] to the case of piecewise-affine systems.

6 Initial Controller Design

To design the initial controller of the V-K iteration for each region, i.e, for finding initial values for A_{c_i} , L_i , b_{c_i} , K_i and m_i we use LQG. Assume the plant dynamics has Gaussian process noise $\eta(t)$ and measurement noise $\nu(t)$ with zero mean and uncorrelated, i.e, $E[\eta] = E[\nu] = 0$, $E[\eta\nu^T] = 0$. The state equations are then given by

$$\begin{aligned} \dot{x}(t) &= A_i x + B_i u_c + b_i + \eta, \\ y &= C_i x + \nu. \end{aligned} \quad (28)$$

Defining the augmented noise vector $\sigma = [\eta^T \ \nu^T]^T$, suppose the noise covariance matrix is given by $E[\sigma\sigma^T] = W$, where

$$W = \begin{bmatrix} W_\eta & 0 \\ 0 & W_\nu \end{bmatrix}.$$

From (28) the dynamics of $z = x - x_{des}$ are

$$\begin{aligned} \dot{z}(t) &= A_i z + A_i x_{des} + B_i u_c + b_i + \eta, \\ y &= C_i z + C_i x_{des} + \nu, \end{aligned} \quad (29)$$

As before, we define a new output $\tilde{y} = y - C_i x_{des} = C_i z + \nu$ and the control input to be used as $u_c = \tilde{u} + m_i$. Inserting the expression for the control input into (29) and rearranging we get

$$\dot{z}(t) = A_i z + B_i \tilde{u} + A_i x_{des} + b_i + B_i m_i + \eta. \quad (30)$$

Now if the constraint (26) is verified then the dynamics (29) with the new output \tilde{y} can be rewritten as

$$\begin{aligned} \dot{z}(t) &= A_i z + B_i \tilde{u} + \eta, \\ \tilde{y} &= C_i z + \nu. \end{aligned} \quad (31)$$

For this system a controller with dynamics

$$\begin{aligned} \dot{x}_c(t) &= A_{c_i} x_c + L_i \tilde{y}, \\ \tilde{u} &= K_i x_c. \end{aligned} \quad (32)$$

can then be designed using LQG with the noise covariance matrix of the augmented noise vector given by W . This design will give us the parameters A_{c_i} , L_i and K_i . The control input to the original problem is then given by $u_c = \tilde{u} + m_i$ where the remaining controller parameter m_i is computed from (26) assuming x_{des} was chosen such that there is a solution to this equation. Finally, making the substitution $\tilde{y} = y - C_i x_{des}$ in (32) the state space representation of the controller is

$$\begin{aligned} \dot{x}_c(t) &= A_{c_i} x_c + L_i y + b_{c_i}, \\ u_c &= K_i x_c + m_i, \end{aligned} \quad (33)$$

where b_{c_i} is given by (27). Notice that from (31) and (32), the closed loop dynamics can be recast in the form

$$\begin{bmatrix} \dot{z} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A_i & B_i K_i \\ L_i C_i & A_{c_i} \end{bmatrix} \begin{bmatrix} z \\ x_c \end{bmatrix} + \begin{bmatrix} \eta \\ L_i \nu \end{bmatrix}, \quad (34)$$

i.e, the affine terms cancel out in the z coordinates and therefore, without noise, both $z = x - x_{des}$ and x_c converge to zero in steady state, as desired.

7 Examples

In the following examples¹ we consider the tunnel diode circuit taken from [1] and [15] and shown in Figure 2. With time expressed in 10^{-10} seconds, the inductor current expressed in milliAmps and the capacitor voltage expressed in Volts, the dynamics of the circuit are

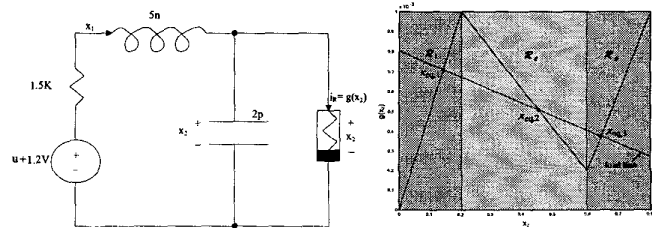


Figure 2: Circuit and Tunnel Diode Characteristic

¹These examples were carried out using the semidefinite program solver package `sdpso1` [14].

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -30 & -20 \\ 0.05 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 24 \\ -50g(x_2) \end{bmatrix} + \begin{bmatrix} 20 \\ 0 \end{bmatrix} u.$$

Following [1] the characteristic of the tunnel diode is considered to be piecewise-linear as shown in Figure 2. The switching conditions are described by

$$i = \begin{cases} 1, & x_2 < 0.2 \\ 2, & 0.2 \leq x_2 < 0.6 \\ 3, & x_2 \geq 0.6 \end{cases}$$

The polytopic regions are defined as

$$\mathcal{R}_i = \{x \mid H_i^T x - g_i < 0\}, \quad i = 1, \dots, 3,$$

$$H_1^T = [0 \quad 1], \quad H_2^T = [0 \quad -1], \quad H_3^T = [-0.2 \quad 0.6],$$

$$g_1 = 0.2, g_2 = -0.6, g_3 = -0.6$$

See [1] for a feasible solution for open loop stability.

Example 1: Assume that we want to stabilize $x_{eq}^3 = x_3^0$ but we only have access to the voltage, i.e. x_2 . The initial output feedback controller is designed using LQG and the technique described in Section 6. The state/input weighting matrix \tilde{Q} and the covariance matrix W were designed for a plant model including a first order actuator dynamics and are given by $\tilde{Q} = \text{diag}[Q, R, R]$, $W = \text{diag}[10^{-7}I, 0.0625]$, where $Q = \text{diag}[4, 4]$ and $R = 1$. The equilibrium points were picked to be at

$$x_1 = \begin{bmatrix} 1.25 \\ 0.25 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1.4 \\ 0 \end{bmatrix}, \quad x_3 = x_3^0 = \begin{bmatrix} 0.37 \\ 0.64 \end{bmatrix},$$

and the initial value of the filter bandwidth was picked to be $\tau = 1000$. The controller after the first iteration (subjected to bounds $l_0 = 0.01$, $l_1 = l_2 = 100$, and all entries in the different matrices bounded between $-l_1$ and l_1) gives the results shown in Figures 3 and 5. Note that

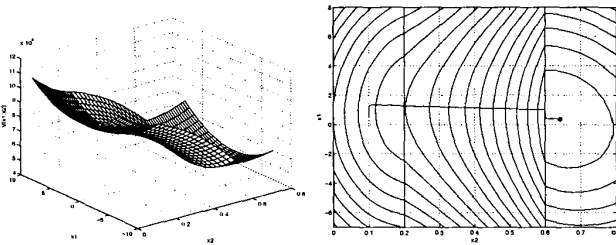


Figure 3: 1x1 Block of Lyapunov Function for First Controller (Example 1)

the Lyapunov function depends on 6 variables. Therefore, for visualization purposes, we have just plotted the 1x1 block of this function, i.e. all the terms that depend only on the plant state plus the independent terms \tilde{r}_i .²

²Note that this 1x1 block does not need to be continuous; continuity is only enforced for the global Lyapunov function.

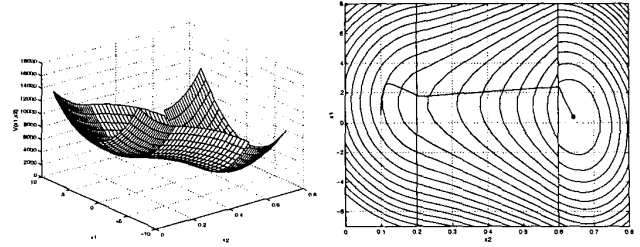


Figure 4: 1x1 Block of Lyapunov Function for Last Controller (Example 1)

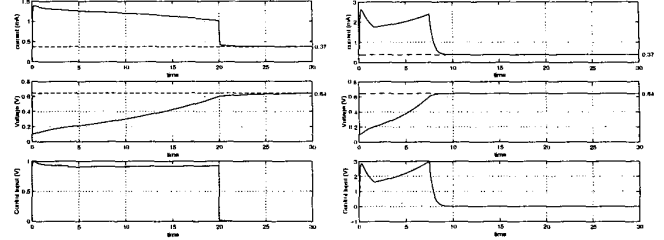


Figure 5: Time Response for First and Last Controllers with $x_{10} = 0.5[mA]$, $x_{20} = 0.1[V]$

The controller after 76 iterations (with the same bounds on the entries and the same constraints) is described by

$$A_{c1} = \begin{bmatrix} -50.18 & -13.65 & 13.80 \\ -13.88 & -49.03 & -14.78 \\ 14.06 & -14.82 & -48.80 \end{bmatrix},$$

$$K_1 = [-3.6582 \quad 3.6007 \quad -3.5741],$$

$$A_{c2} = A_{c3} = A_{c1}, K_2 = K_3 = K_1, b_{c2} = 0,$$

$$b_{c1} = \begin{bmatrix} -6.8647 \\ 6.8622 \\ -6.8604 \end{bmatrix}, b_{c3} = \begin{bmatrix} -61.7827 \\ 61.7598 \\ -61.7436 \end{bmatrix},$$

$$m_1 = 0.925, \quad m_2 = 0.9, \quad m_3 = 0, \quad \tau = 20.5417,$$

$$L_1 = \begin{bmatrix} 27.4590 \\ -27.4488 \\ 27.4416 \end{bmatrix}, L_2 = \begin{bmatrix} -6.8647 \\ 6.8622 \\ -6.8604 \end{bmatrix},$$

$$L_3 = \begin{bmatrix} 96.1065 \\ -96.0709 \\ 96.0456 \end{bmatrix}, \min_i \alpha_i = 0.02757,$$

and gives the results shown in Figures 4 and 5. It is clearly seen from Figure 5 that the controller obtained after the V-K iteration is much faster and has much better performance than the initial controller.

Example 2: Assume now that we only have access to the current, i.e. x_1 , which is not the variable that drives the switching. We are now going to design a regulator/estimator controller. As before, the initial controller is designed using LQG and the technique described in Section 6. The initial filter bandwidth was $\tau = 500$. The state/input weighting matrix \tilde{Q} and the covariance matrix W were the same as before. The controller after 69

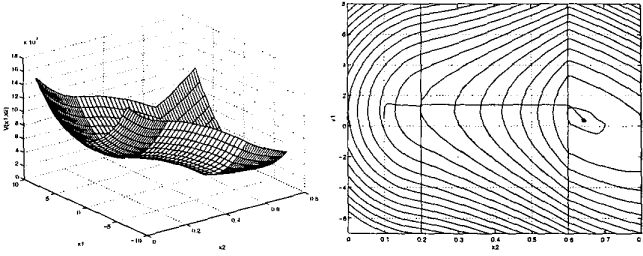


Figure 6: 1x1 Block of Lyapunov Function for Last Controller (Example 2)

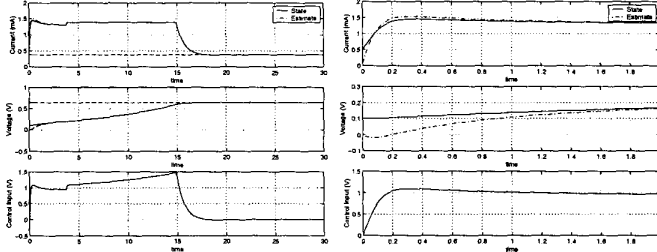


Figure 7: Time Response for Regulator/Estimator with $x_{10} = 0.5[mA]$, $x_{20} = 0.1[V]$

iterations (subjected to $l_0 = 0.006$ and the same other bounds as in Example 1) is described by

$$K_1 = [0.0082 \quad -0.0066 \quad 0.9624],$$

$$K_2 = [0.0072 \quad 0.0509 \quad 0.9469],$$

$$K_3 = [0.0137 \quad -0.0342 \quad 0.9872],$$

$$m_1 = 0.925, \quad m_2 = 0.9, \quad m_3 = 0, \quad \tau = 635.1348,$$

$$L_1 = \begin{bmatrix} -7.1722 \\ -1.3419 \\ -19.722 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -5.7785 \\ -0.5796 \\ -25.466 \end{bmatrix},$$

$$L_3 = \begin{bmatrix} 13.7737 \\ -2.2239 \\ 51.657 \end{bmatrix}, \quad \min_t \alpha_i = 0.00793,$$

and gives the results shown in Figures 6 and 7. It is clearly seen in Figure 7 that the estimated states converge very fast to the actual states. Finally, if Gaussian noise with zero mean and unit variance is added to the output and if the initial error of the estimator is increased (which is certainly a worst case scenario) we get the results shown in Figure 8. Because of the noise, the switching was implemented with an hysteresis with width equal to 0.01. The results of Figure 8 show that this design is robust to noise in the measurements.

8 Conclusions

This work has shown how to extend a technique that searches for piecewise-quadratic Lyapunov functions for piecewise-affine systems to the synthesis of output feedback controllers. More precisely, dynamic output feedback controllers were designed by iteratively solving a BMI. A key point in this design technique is that many different desired features can be included such as controllers with different structures, boundedness of the

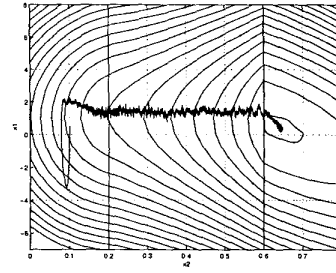


Figure 8: State Trajectory with Gaussian Noise

control signals, continuity of the control signals at the boundaries of the different regions of the state space, continuity of the Lyapunov function at the boundaries and actuator dynamics.

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