ASEN6023 Notes

University of Colorado Boulder

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There are 3 exams – 30% each, and all take home exams. The Homework is 10%. OH is on Tuesdays 3-4pm

We can think of a system as a simple relationship between two signals, the input u and the output y. Linearity assumes the satisfaction of the superposition principle. Therefore the function $y = F(u, x_0)$, this means that $f(u_1 + u_2, x_0) = f(u_1) + f(u_2) \quad \forall u_1 u_2$. We can similarly write this as the summation of two initial conditions $\forall x_1(0) x_2(0)$. When you think about it, most systems are not truly linearly. But, these things can be approximately true over a wide variety of signals. These are linear regions. When we design a controller, we usually take linear approach first. There exist many powerful design techniques, which work 80% the time. This leads you to understanding what nonlinearities give you trouble, to motivate your nonlinear techniques. Don't waste time by jumping into a nonlinear approach off the bat.

Let's focus on the state system point of view now, for $\dot{x} = f(x, u) = f(x) + g(u)$. Which we know can be $\dot{x} = Ax + Bu$ as a standard linear system, from a state space point of view.

Where do nonlinearities come from? Well to pick a proper control, changing indirectly the state of the system, we must understand the body dynamics. The plant itself may be the nonlinear factor. It could also be in the control: saturation in poeration current of actuators, mechanism motion limits, ADC ranges, powere amplifier slew rate limits, trigonometric non-lienarities, etc. They can also come from non-ideologies; assuming ideal systems. Sensors can also be nonlinear. Of course, many times we want to introduce nonlinearities to get a specific behaviour.

Example: shape memory alloy. They have a hysteresis in the relationship between stress and strain. But these are all smooth differentiable curves. But there are plenty of non-diff systems: like friction. Example: stiction. There are discrete friction transitions from static to dynamic friction. "Coulombic" friction an example of one. Example: quantization error. The ADC will be noncontinuous and chops up analog signals. This is the same way with DACs on the output.

Robotic arm dynamics are nonlinear due to trigonometric linkage setups and the moment of inertia changes over time. These are dynamic linearities. For $\Theta = [\theta_1 \theta_2 \theta_3]^T$, dynamics is in the form $M(\theta)\ddot{\theta} + S(\theta, \dot{\theta}) + G(\theta) = [\tau_1 \tau_2 \tau_3]^T$. The M and S variable are the nonlinear components of the dynamics. G is often trigonometric. Mode switching logic gives you hybrid control systems. On-off switching amplifiers, like those driven by PWM, are nonlinear (NL) systems.

Let's say that the NL are bad: assume they are not there, get rid of them, or dominate them. The second means canceling the NL dynamics with a NL controller. High gains to dominate NLs; but these might not be feasible. If NL are good: design to produce a desired NL behaviour. If they are indifferent, then just design to produce good behaviour.

Homework: read Khalil Ch1,2 and Hw 1 assigned next week.

January 17th, 2018

Characterization of Linear systems: spoke about superposition wrt inputs and states. Recall that we can write the solution of a linear system $\dot{x}(t) = Ax(t) + Bu(t)$ with solution $x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$ where the latter is teh convolution integral of the input function with the impulse response. These are the forced response to the inputs. And let us recall that the matrix exponential is an infinite Peano Baker power series, and the natural response of the system.

Checking the superposition property with two inputs should be a simple sum of the two. Therefore $\int_{t_0}^{t} e^{A(t-\tau)}B(u_1 + u_2)d\tau = \int_{t_0}^{t} e^{A(t-\tau)}Bu_1(\tau)d\tau + \int_{t_0}^{t} e^{A(t-\tau)}Bu_2(\tau)d\tau$.

Natural responses are exponential for linear systems. So $x(t) = e^{A(t-t_0)}x(t_0)$. And the scalar form e^{at} has a similar infinite series description. We can break the matrix form down into modal componenets. If A is semi-simple $T^{-1}AT = diag\{\lambda_i(A)\}$. The property of semi-simple are robust to perturbation so that $A + \delta A$ acts similar to A.

$\dot{x} = Ax$ for $x = Tv$ then we see that $T\dot{v} = ATv$ therefore $\dot{v} = T^{-1}ATv$ which decouples the system completely where $T^{-1}AT$ acts as a diagonal eigenvalue matrix. Therefore $\dot{v}_i = \lambda_i v_i$ has the solution $v_i(t) = e^{\lambda_i t}v_i(0)$. The columns of T, T_i are eigenvectors. So for $\dot{x} = Ax$ we know that $AT_i = \lambda_i T_i$ therefore T_i is an invariant basis. So if you start with $x = T_i$, by operatitng with A, you will stay in the same direction, it evolves with the behavior of $e^{\lambda_1 t}$. Or T_i is an invariant set of the operator A.

We see that positive λ values blow up and negative λ values decay to zero. Complex λ values, $e^{\lambda_i} = e^{\sigma_i t}e^{j\omega_i t}$ which encode oscillatory behaviour, and so the eigenvector is complex. Of coursee the complex values come in pairs. The combination of the two eigensolutions create a real solution together. Conjugate twins satisfy symmetry of $AT_i = \lambda_i T_i$ has pair $AT_i^* = \lambda_i^* T_i^*$. Seeing them in a complex plane shows that for $\sigma = 0$ we get closed orbits. The origin in this situation is neutrally stable. For $\sigma > 0$ you will spiral out, and for $\sigma < 0$ you spiral to the origin. Other conjugate pairs (so 4 λs) will create another complex subspace having its own behaviour.One real λ with a conjugate pair λ will create a line and a complex plane.

The frequency domain overview is also important. Linear systems do not skew the input frequency, this is not true for NL systems.

Limit sets in linear systems are either equilibria or periodic orbits. Example, $\dot{x} = Ax = f(x) = 0$ for equilibria. All linear systems have an equilibrium at the origin. The nullspaces are also the equilibria, this could be a whole subspace. Therefore you cannot have an individual equilibrium, it must be connected to the origin with a subspace. This is true for λ values equal to zero. Otherwise this is not true, only equilibrium is the origin. The ω limit points p: $x(t_i)$ converges to point p where t_i is an infinite sequence. So if a state trajectory converges to this point, it is an ω limit point. So an ω limit set is the set of all such points p where this is true. Therefore everything on a periodic orbit has an infinite number of points, or solutions. "Points that you visit infinitely often!" The book calls these "positive" limit points/sets. α limit sets are considered negative, so in backwards recursed time. For unstable systems, in backwards time it converges to the origin. Hyperbolic equilibrium for linear systems: the real part of the eigenvalues of A are negative: $Re(\lambda_i(A)) < 0 \quad \forall i$. In this situation, for no zero values of λ, $p = 0$ is the only ω limit point. For $Re(\lambda_i(A)) > 0 \quad \forall i$, $p = 0$ is the only α

limit point. For a mix of the two, without any $\lambda = 0$ we will have the same equilibrium point forward or backward in time, so therefore both an α and ω limit point. Linearization does not destroy behaviour of the hyperbolic from NL systems.

We also have a "center-equilibrium" where $Re(\lambda_i(A)) = 0$ for some i, the system has a center equilibrium or center manifold. Therefore it can have entire subspaces of equilibria and periodic orbits. Linearization may destroy this behaviour.

January 22nd

Previously we spoke about phase portraits and the invariant spaces that eigenspaces created on those portraits. He defines the eigenspace as invariant under the dynamics. We can also call these sub-manifolds of the state space. These invariant sets are "flat," or scale invariant. Anything in the nullspace of A is in equilibrium, which only happens if $\lambda = 0$.

Stable manifold is a set of states that go to the equilibrium as $t \to \inf$. Unstable manifolds are a set of states that go to equilibrium as $t \to -\inf$. Therefore for the phase portrait (phase planes are when the axes are derivatives of each other) lines that describe $\lambda < 0$, this line is a stable manifold. The origin, in the case for a single positive and negative eigenvalue, is in both the stable and unstable manifold sets. For complex eigenvalues with zero real value, you orbit the equilibrium.

For looking at stability, we must ask "what happens in the neighborhood of the equilibrium points?" Lyapunov stability, coming later, is described in a more concrete manner. sStability of an equilibrium can be "transferred" to studying the stability of the whole system (or whole state space). In linear systems, equilibrium stability is a global property. This is not true for nonlinear systems.

For systems with large number of eigenvalues, we cannot just look at portraits. So lets look at all of the eigenvectors. We group conjugate terms; either there are first order or second order sub-eigenspaces which describe simple behaviour.

For a mass spring damper system with an external force: $f_{net} = f_{ext} - k(d - l_0) - B\dot{d}$. Let us define that $\delta d = d - l_0$, and $\dot{\delta} = \dot{d}$, $\ddot{\delta} = \ddot{d}$. We can then write that $f_{ext} = m\delta\ddot{d} + B\delta\dot{d} + K\delta d$. We can write in state space such that $x = [\delta d \quad \delta\dot{d}]^T = [x_1 \quad x_2]^T$, and then $\dot{x} = Ax + [0 \quad \frac{1}{m}]^T f_{ext}$ for

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & \frac{-B}{m} \end{bmatrix} \tag{1}$$

And of course it must be the case that $A\bar{v} = \lambda\bar{v}$ so then $(A - \lambda I)\bar{v} = \bar{0}$ therefore, $det(A - \lambda I) = 0$. For the MSD system, the determinant yields a charateristic polynomial $\lambda^2 + \frac{B}{m}\lambda + \frac{K}{m} = 0$ which has its roots as the λs of the system. We can then symbolically solve for each eigenvector.

Sometimes with repeated roots you get non-semisimple cases. He then shows MATLAB simulations.

He lists the single equilibrium: node (real eigenvalues, same sign), saddle (real eigenvalues opposite sign), focus (complex eigenvalues), center (nonzero imaginary eigenvalues).

He lists distributed equilibria: center (zero eigenvalues).

January 24th

Again we recall our linear system of form $\dot{x} = Ax + Bu = f(x, u)$ with solution $x(t) = e^{\lambda t} x_0$. Let us say that we have some initial, nominal solution such that $\dot{x}_0 = Ax_0 + Bu_0$; we want to perturb the dynamics about this nominal condition by a small amount. Let's expand $\dot{x} = f(x, u)$ about the nominal condition with a Taylor series: $\dot{x} = f(x_0, u_0) + Df_x|_{x_0, u_0} \delta x + Df_u|_{x_0, u_0} \delta u + HOT = \dot{x}_0 + \delta \dot{x}$ but we know that the derivative of the initial condition is zero, so then it is just $\delta \dot{x}$. We know that the partial matrices are just the Jacobian matrices A and B for each. So we say that $Df_x = A$ and $Df_u = B$. Therefore, the deviations have the same form as the original: $\dot{\delta x} = A \delta x + B \delta u$.

Holmes Equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \zeta x_1 - \delta x_2 + x_1 x_2 \\ \delta x_1 - \zeta x_2 + \frac{1}{2}(x_1^2 + x_2^2) \end{bmatrix} \tag{2}$$

Recall that the equilibrium are the origin and the nullspace of A. Here we don't have A, so don't have that to work off of. We do have an equilibrium at the origin obviously. We have that $0 = \zeta x_1 - \delta x_2 + x_1 x_2$ and that $0 = \delta x_1 - \zeta x_2 + \frac{1}{2}(x_1^2 + x_2^2)$. Let us take that $\zeta = 0$ for now (as a special case). In this situation we have that $x_2(x_1 - \delta)$ and $0 = \delta x_1 + \frac{1}{2}(x_1^2 + x_2^2)$. In this situation it must be that $x_2 = 0$ and $x_1 = 0$ $x_1 = -2\delta$. Lets say that $x_1 = \delta$, then we have that $\frac{3}{2}\delta^2 = \frac{1}{2}x_2^2$ so then $x_2 = \pm\sqrt{3}\delta$.

Let's look at the equilibrium point that $x_1^\star = x_2^\star = 0$. So then we can look at the local deviation defined by $y = x - x^\star$. This system is an autonomous so we only have $\dot{x} = f(x) = f(x^\star) + Df_x|_{x^\star} y + HOT$. We can then show that $\dot{y} = \dot{x} - \dot{x^\star} = \dot{x} = Df_x|_{x^\star} y$. Our Jacobian is the following

$$Df_x = \begin{bmatrix} -\zeta + x_2 & -\delta + x_1 \\ \delta + x_1 & -\zeta - x_2 \end{bmatrix}\Bigg|_{x^\star} = \begin{bmatrix} 0 & -\delta \\ \delta & 0 \end{bmatrix} \tag{3}$$

Therefore we can show that the linear system is as such

$$\dot{y} = \begin{bmatrix} 0 & -\delta \\ \delta & 0 \end{bmatrix} y \tag{4}$$

The eigenvalues are such that $\lambda^2 + \delta^2 = 0$. These are complex, therefore the origin is called a "center" as near to the origin, the system will orbit.

In the other case where $x_1 = -2\delta$ we have that:

$$Df_x = \begin{bmatrix} 0 & -3\delta \\ -\delta & 0 \end{bmatrix} \tag{5}$$

Which yields the characteristic polynomial $\lambda^2 - 3\delta^2$ therefore $\lambda = \pm\sqrt{3}\delta$. This is a saddle equilibrium. We want to look at the eigenvectors now to see the stable and unstable manifolds for this saddle.

$$\begin{bmatrix} 0 & -3\delta \\ -\delta & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \pm\sqrt{3}\delta \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{6}$$

We can trivially solve this for the positive eigenvalue and see that $-3\delta v_2 = \sqrt{3}\delta v_1$ and $\delta v_1 = \sqrt{3}\delta v_2$ therefore $v_1 = \sqrt{3}$ $v_2 = -1$. This makes an unstable manifold. For the negative eigenversion, we get the opposite eigenvector, which is stable. We can use these vectors to look at the stability in the locale of the equilibrium point.

Doing this similarly, we find the other equilibria to have saddle point dynamics with a stable and unstable mode. As you stray farther from these equilibrium points, the linearization behaviour cannot tell you the stability flows.

The orbit center in this system is a heteroclinic orbit.c

We picked ζ here. As you change this parameter the dynamics will change, and the stability properties will change. Sometimes this will cause the system to snap into a different behaviour, this is called a bifurcation.

For $\zeta = -0.03$, the EQ points have changed slightly. But here we have an unstable focus where the real part of the eigenvalues is nonzero at the origin.

For $\zeta = 0.174$, one of the EQ points disappear. There are now two close at the origin, making it a stable focus. There are still two saddles.

January 29, 2019

Recall the holmes problem with a center, and nonlinear behaviour. Other types of equilibria: for a second order thing described by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 - ax_2x_2 - bx_1^2 \\ -x_1x_2 - \frac{b}{2}x_2^2 \end{bmatrix} \tag{7}$$

We have the points such that $x_2^\star = x_2^\star = 0$ as equilibria, we get that $x_1^\star = \frac{-b}{2}x_2^\star$ and $x_2^\star = \frac{-1}{\frac{ab}{2} - \frac{b^3}{9}}$. In the case, we can write an A matrix and find that it has two 0 eigenvalues with a single eigenvector. This is a degenerate special case. Since the eigenvalue is zero, anywhere that is spanned by the eigenvector is an equilibrium.

For when $a = \frac{1}{2}$ $b = 3$, we have that $x_1^\star = -0.25$ $x_2^\star = 0.166$ we have that $\lambda_1 = 1.29$ $\lambda_2 = -0.129$. This is a saddle point.

Talks about semi-stable equilibrium, where approaching the point from one side is "stable" but from the other is "unstable." In this prior example, there is an unstable manifold between two points leading back into a stable manifold. Linearization does not preserve the nonlinear behaviour. This happens when eigenvalues are zero! Homoclinic orbit because it revisits the same equilibrium point.

For the case that:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ \epsilon x_1^2 x_2 \end{bmatrix} \tag{8}$$

Here we only have one equilibrium at the origin. We have complex conjugate eigenvalues $\pm j$. This acts as a "center" which is a closed orbit. This is neutrally stable for a linearization. But his plots show that this does not properly convey the nonlinear behaviour as the orbit decays towards the center.

Van der Pol oscillator: a famous example.

$$m\ddot{z} + zc(z^2 - 1)\dot{z} + kz = 0 \tag{9}$$

This is effectively a nonlinear damper. We see that there appears to be only one equilibrium at the origin. Let's linearize about the origin. We get that

$$A = \begin{bmatrix} 0 & 1 \\ \frac{-k}{m} & \frac{2c}{m} \end{bmatrix} \tag{10}$$

The eigenvalues evaluate to: $\lambda_{1,2} = \frac{c}{m} \pm j\sqrt{\frac{k}{m}}$. This has a real part. So about the equilibrium origin, it spirals out in an unstable fashion, according to our linear analysis. We see the proper nonlinear behaviour has a limit cycle. There exists a closed orbit at a specific orbit from the origin. This orbit is stable. This is a limit cycle. It is attractive to initial conditions in its neighborhood. This is not a center behaviour because it is not neutrally stable.

You want this behaviour if you are engineering an oscillator. Additionally, this is important for engineering locomotion. A limit cycle is basically an isolated periodic orbit. There is only one, while a center will have many in neighborhood.

Rossler's Band: another example. This is a third order system that follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & -c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b + x_1 x_3 \end{bmatrix} \tag{11}$$

For values $c = 4 \quad b = 2 \quad a = 0.3891$ we have that there is an equilibrium at $x_2^\star = -x_3^\star \quad x_1^\star = ax_3^\star \quad x_3^\star = \frac{c}{2a} \pm \frac{1}{2a}\sqrt{c^2 - 4ab}$. These take values of $x_1^\star = 3.79 \quad x_2^\star = -0.975 \quad x_3^\star$ for eigen structure that is $\lambda_1 = 0.335 \quad \lambda_{2,3} = -0.07 \pm j3.27$. If you are on the complex plane you are stable; but this is knife edge. Any overshoot off the edge and you shoot off, so this equilibrium is unstable.

The second equilibrium is at $x_1^\star = 0.21 \quad x_2^\star = -0.53 \quad x_3^\star = 0.53$ with eigenvalues $\lambda_1 = -0.366 \quad \lambda_{2,3} = 0.127 \pm j0.98$. This is an unstable spiral, with a stable manifold that approaches it.

There is no equilibrium at the origin in this case. Plotting this in 3D shows a potato chip structure that represents a band. This is a chaotic orbit. There are no closed orbits. There are some banded orbits or groupings, but no repetition. Turns out that you need at least three states to create chaos.

January 31st

Properties of NL ODEs: (1) solutions may not exist starting from some initial conditions. Example given: $\dot{x} = -x^{\frac{1}{2}}$, which is not defined on the vector field function plot for x values below 0 because it is complex. Another is $\dot{x} = -sgn(x)$. The phase plane plot is discontinuous at zero, and therefore not defined at zero. This function is not a continuously differentiable C^1 solution. It is however a C^0 solution.

Peano's theorem: if $f(x)$ is continuous at some x_0, then there exists at least one C^1 solution $x(t)$ to $\dot{x} = f(x)$ passing through x_0. There also might be other solutions starting from that point.

Properties of NL ODEs: (2) solutions starting from some x_0 may not be unique. Example given: $\dot{x} = \mid x \mid^{\frac{1}{2}}$; solutions here are that $x(t) = \frac{1}{4}t^2, \quad 0$. This function is not continuously differentiable. THM: if f(x) satisfies the Lipschitz condition $\mid f(x_1) - f(x_2) \mid < k \mid x_1 - x_2 \mid$ $\forall x_1, x_2$ in the neighborhood of x_0, then $\dot{x} = f(x), \quad x(t_0) = x_0$ has a unique solution $x(t)$ passing through x_0 for t defined on a neighborhood of t_0. Basically around these initial state and time conditions, the system can go forward and backward smoothly. This is a stronger form of continuity. However, this allows some strange functions to be considered continuous that can be nasty. Example given $\sin(\frac{1}{x})$ which oscillates faster and faster towards the zero value. This is not uniformly continuously differentiable, but it is Lipschitz because it abides by the gain boundary.

What if the RHS is not Lipschitz continuous? What can we say about the solutions? You can still have unique solutions but is not Lipschitz continuous. It only goes one way.

Properties of NL ODEs: (3) solutions may not exist for all time. Example given: $\dot{x} = 1 + x^2$. Here the k values for the neighborhood of the chosen x_0 will be different over the function. But this is still Lipschitz continuous. NOT uniformly Lipschitz continuous. A solution to this is $x(t) = \tan(t)$, which escapes in finite time.

Theorem 3.2 in Khalil: if f(x) is uniformly Lipschitz continuous, the solution exists $\forall t$. This uniformly assumption means the same k holds $\forall x \in \mathcal{R}^n$. This prevents the finite time escape issues, if that is actually an issue.

$f(x) \quad C^0 \Leftarrow$ Locally Lipschitz continuous $\Leftarrow$ uniformly Lipschitz is stronger. C^1 is stronger than Locally Lipschitz and implies locally Lipschitz. So $C^1 \Leftarrow$ uniformly C^1. A convex function has an unbounded slope, so is not technically C^1. See hand written notes for this to be clearer.

Theorem (Hartman): If $Df|_{x_0}$ has eigenvalues with nonzero real parts, then there exists a homeomorphism h defined on some neighborhood U of x_0 mapping the nonlinear flow of $\dot{x} = f(x)$ to the linear flow $\dot{\delta x} = Df|_{x_0}\delta x$. This homeomorphism h is continuous and an invertible process. Furthermore, the stable and unstable manifolds of $\dot{x} = f(x)$ about x_0 are tangent to the stable and unstable manifolds of $\dot{\delta x} = Df|_{x_0}\delta x$.

Note: if $Df|_{x_0}$ has eigenvalues with nonzero real parts, then x_0 is "hyperbolic."

Limit cycles: a limit cycle is an isolated closed orbit. Isolated here means that there is some neighborhood of the orbit that has no other closed orbits in it.

Let us take:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} \tag{12}$$

This is linear, and turns out that it is a center. This does have an orbiting type of vector field, but is it a limit cycle? We must look at the neighborhood. There are an infinite amount of options that can create a closed orbit. Every neighborhood has other closed orbits in it. This means that there is an implied convergence/divergence when talking about limit cycles. We can study this with the following:

$\hat{n} \cdot f(x) > 0 \rightarrow$ orbit radius increasing

$\hat{n} \cdot f(x) < 0 \rightarrow$ orbit radius decreasing

$\hat{n} \cdot f(x) = 0 \rightarrow$ closed orbit

February 5th

More on limit cycles: Let's look at this system:

$$\begin{bmatrix} \dot{z} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} z \\ \dot{z} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f_c \tag{13}$$

This produces simple harmonic motion, but with friction, the circular motion will head to the origin. It is also sensitive to the initial conditions, and is NOT a limit cycle. We can make it one with feedback control, however. For $f_c = m(r^2 - z^2 - \dot{z}^2)\dot{z}$, which looks a little like a damper. The r value encodes the orbit height we would like in the phase plane. If we are on the circle, this will go to zero. If we are inside the circle, the force will be positive and push the system out. The converse is true for being outside the circle. This simple feedback law would create a limit cycle in the system dynamics.

Let's take another circle with radius R, the outward normal inline with radius $n_c = [z \quad \dot{z}]^T$. We can write the dynamics as

$$\begin{bmatrix} \dot{z} \\ \ddot{z} \end{bmatrix} = f(z, \dot{z}) = \begin{bmatrix} f_1(z, \dot{z}) \\ f_2(z, \ddot{z}) \end{bmatrix} \tag{14}$$

And $f \cdot n_c = f_1 n_1 + f_2 n_2 = z\dot{z} + (-z + (r^2 - z^2 - \dot{z}^2)\dot{z})\dot{z} = \dot{z}^2(r^2 - z^2 - \dot{z}^2)$. This value is positive when $R^2 < r^2$, negative when $R^2 > r^2$, and 0 when you are on the orbit so $R^2 = r^2$.

This creates an annulus about the stable orbit that pushes everything towards that closed orbit. This is called a "positively invariant set" .. also closed, bounded, compact. The flows cannot cross each other in this region, (non-unique), they must swirl around. This implies that the region, bounded by $R_1 < R < R_2$ contains a closed orbit or an equilibrium. If there is not an equilbrium point, this implies that there exists a closed orbit. We can look at the Poincare Bendixson theorem.

REST OF NOTES ON PAPER! COMPUTER DIED

February 7th

For the weak non-linearity

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ \epsilon x_1^2 x_2 \end{bmatrix} \tag{15}$$

We see, like last time, that the phase diagram shows a spiral decay towards an equilibrium at the origin. With the Poincare Bendixson we can see if it truly converges.

Consider the energy function $V(x) : \mathcal{R}^2 \to \mathcal{R}$ for the function $V(x) = x_1^2 + x_2^2$ which measures energy in the state about $(0,0)$. Let's look at $\frac{d}{dt} V(x) = \frac{\partial V}{\partial x_1} \frac{d}{dt} x_1 + \frac{\partial}{\partial x_2} \frac{d}{dt} x_2 + \frac{\partial V}{\partial t} = 2x_1(x_2) + 2x_2(-x_1 - \epsilon x_1^2 x_2) = -2\epsilon x_1^2 x_2$. Over time, we see that this is monotonically decreasing. Looking at V we see that it can either converge to a nonzero value or converge to zero; but regardless it is a monotonically decreasing function. This nature of this energy function can geometrically draw out a circle and still be constant. Looking at $\dot{V}$ we see that it is zero on the x_1, x_2 axes as well as at the origin. $\dot{V} = 0$ is not an invariant set for the flow, which implies that $V \to 0$ which means that x_1 and x_2 go to zero.

Another example:
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{1}{2} x_1 - x_2^2 \end{bmatrix} \tag{16}$$

Perhaps our first guess for constructing V is $V = x_1^2 + ax_2^2$ and therefore the solution for $\dot{V} = 2x_1 x_2 + 2ax_2(-\frac{1}{2} x_2 - x_2^2)$ choose that $a = 2$ such that the non-definite term $2x_1 x_2$ disapears (recall that we want sign definite terms in the derivative of V). This leaves us with $\dot{V} = -4x_2^3$ which is not sign definite. So, we go back to playing around with the Lyapunov function V again.

Let us look at $V = x_1^2 x_2 + \frac{4}{3} x_2^3$ which gives $\dot{V} = 2x_1 x_2^2 + (x_1^2 + 4x_2^2) * (\frac{-1}{2} x_2 - x_2^2) = \frac{-1}{2} x_1^3 - x_1^2 x_2^2 - 4x_2^4$. We see that we still have an bad term (the cubed) as it is not sign definite. Sometimes you can argue around this mathematically, but here it does not work out. There will be sections of the phase plane where $\dot{V}$ is positive, and portions that are negative. If this is your function, you must walk about in the negative section. But try to tweak your Lyapunov function. We need both orders to satisfy different conditions. These are all sufficient conditions for convergence. Also if you cannot show convergence, that does not mean it is unstable.

Now with the following with input control:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{1}{2}x_1 - x_2^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{17}$$

We can look again at the bad function $V = x_1^2 + 2x_2^2$. We can use this Lyapunov analysis to choose a u to converge. We now have $\dot{V} = -4x_2^3 + 4x_2 u$. Let us choose that $u = x_2^2 - x_2$ and we get that $\dot{V} = -4x_2^2$ we have canceled the sign indefinite portion and the derivative is monotonically decreasing. But does the system converge to zero? It is not clear that x_1 will go to zero. This control makes the system linear! This is also a sort of feedback linearization technique. j

What if we choose $V = x_2^2$? We see that $\dot{V} = 2x_2(-x_2^2 - \frac{1}{2}x_1 + u)$. We can have $u = \frac{1}{2}x_1 + x_2^2 - x_2$ so then it is such that $\dot{V} = -2x_2^2$. We have a monotone decreasing definite function, and it looks a lot like $-2V$. This ODE can be solved directly. Therefore $V = e^{-2t}V(0)$ which converges exponentially. This tells you nothing about the convergence properties wrt x_1. Looking at the state space, clearly we see that x_1 converges to some nonzero value as the rate of change depends on x_2. x_1 can go to somewhere nonzero and stop. Essentially V itself has bad level sets: lines along the x_1 axis.

From this, we see we will need: $V(x)$ to be lower-bounded ($\dot{V} \leq 0 \rightarrow V$ converges), we want that a bounded V implies a bounded x state (we want the level sets of V to bound x), we want that for when V goes to zero it implies that x goes to zero as well, we want that $\dot{V} \leq \forall x$ in neighborhood of the equilibrium.

Main theorem: let $\bar{x}$ as an equilibrium for differential equation $\dot{x} = f(x)$ and $D \subset \mathcal{R}^n$ is a domain containing $\bar{x}$. If $V : D \rightarrow \mathcal{R}$ is C^1 (continuously differentiable) and positive definite on D, $\dot{V}$ is negative semi-definite on D, then the equilibrium point $\bar{x}$ is stable i.s.l (in the sense of Lyapunov). If in addition, $\dot{V}(x)$ is negative definite on D, then $\bar{x}$ is asymptotically stable i.s.l.

Lyapunov Stability

This is a property of an equilibrium! How do flows behave in the neighborhood of an equilibrium? Definition: An equilibrium $\bar{x}$ of $\dot{x} = f(x)$ is stable isl if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\|x(0) - \bar{x}\| < \delta$ then $\|x(t) - \bar{x}\| < \epsilon, \quad \forall t \geq 0$.

February 12th

Recall the main Lyapunov theorem from last class. If $V : D \rightarrow R$ is C_1 and positive definite on D AND $\dot{V}$ is negative semi-definite on D then $\bar{x}$ is stable. If $\dot{V}$ is negative definite on D then $\bar{x}$ is asymptotically stable.

Sign-definite functions: A functional $f : \mathcal{R}^n \rightarrow R$ is PSD on D if $f(x) \geq 0, \quad \forall x \in D$. It is PD if in addition, $f(x) = 0 \rightarrow x = \bar{0}$. Let us look at the contrapositive of this statement: where $x \neq 0 \rightarrow f(x) \neq 0$. (here we say that $\rightarrow$ is basically "only if").

Similarly considering NSD functions, it is such that $f(x) \leq 0, \quad \forall x \in D$ as well as ND including the statement that $f(x) = 0 \rightarrow x = \bar{0}$.

Prototype definite functions: $f(x) = x^T P x$ has quadratic form for P as a PSD (PD) matrix. And for $x^T P x > 0$ the matrix P must have nonnegative eigenvalues and they must be strictly positive. Or they must be ≥ 0 for the PD case.

Now let us talk about ISL: in the sense of Lyapunov. Lyapunov stability is a property of an equilibrium. Def: an equilibrium $\bar{x}$ of $\dot{x} = f(x)$ is stable ISL if $\forall \epsilon > 0 \quad \exists \delta > 0$ such that $\|x(0) - \bar{x}\| < \delta$ then $\|x(t) - \bar{x}\| < \epsilon, \quad \forall t \geq 0$. We can think of this as a ball $B_\delta : x \in \mathcal{R}^n$ for $\|x\| \leq \delta$. These balls are general: can be spherical, circular, ellipsoidal, etc. Some closed smooth surface generally.

We show on a plot two enclosed surfaces: the epsilon and delta ball. To show stability, we must show that $\forall \epsilon$ the system stays within.

EG: Suppose we have a limit cycle, where things from inside converge, and outside systems converge in. For B_ϵ that is smaller than the cycle, we must pick a B_δ that is exactly the origin. Nothing else qualifies as stable.

If $\bar{x}$ is stable ISL, $\lim_{t \to \inf} \|x(t) - \bar{x}\| = 0$ starging from some D containing $\bar{x}$ then this $\bar{x}$ is asymptotically stable.

More on norms on $\mathcal{R}^n$. We know that $\|x\|_{infty} = \max_i |x_i|$. A norm must satisfy: (1) $\|x\| \geq 0$ $\forall x \in \mathcal{R}^n$, (2) if the $\|x\| = 0 \rightarrow x = \bar{0}$ which means it is a PD function, (3) $\|ax\| = |a| \|x\|$ $\forall x \in \mathcal{R}^n \quad \forall a$ scalars, (4) $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$ or the triangle inequality. All norms on $\mathcal{R}^n$ are equivalent in the sense that $\|x\|_\alpha, \|x\|_\beta$ can be related by $c_1 \|x\|_\alpha \leq \|x\|_\beta \leq c_2 \|x\|_\alpha$. Therefore you can just use whatever norms you want in Lyapunov candidate functions.

Lyapunov instability: $\bar{x}$ is unstable ISL if there exists an $\epsilon > 0$ such that $\forall \delta > 0$ there exists a $x(0)$ initial condition such that $\|x(0) - \bar{x}\| < \delta \rightarrow \|x(t) - \bar{x}\| > \epsilon$ some $t \geq 0$.

Lyapunov methods can provide: (1) rigorous analysis of stability (which you do not often get with linearization); (1a) useful for local equilibrium that are not hyperbolic; (1b) global stability properties, (2) can estimates of the region of attraction of an asymptotically stable equilibrium; (3) estimates on the speed of convergence; (4) control design methods to make V decrease; (5) guidelines for constructing vector fields that have inherent stability properties.

February 14th, 2019

For autonomous vector field $\dot{x} = f(x)$ for $V : \mathcal{R}^n \to \mathcal{R}$. And we viewed $\dot{V}$ as a set of partial derivatives, which ends up looking like

$$\dot{V} = \begin{bmatrix} \nabla V \end{bmatrix} \begin{bmatrix} f_1(x) \\ f_2(x) \\ \dots \end{bmatrix} \tag{18}$$

This gradient is orthogonal to the level set of V for $V(x) = c$. Therefore $\dot{V}$ is the inner product. We can look at the sign to see it entering smaller level sets (we want V to decrease!). But some $\dot{V}$ flows behave properly but seem to diverge under poorly constructed level sets. So make your level sets with an understanding of the flows.

So far we have said to choose a V then choose $\dot{V}$. But we know this is suboptimal. Try choosing ∇V first, then find V to see if it is PD. Generally speaking it seems to be perfect to choose $\nabla V = -f(x)$ such that $\dot{V} = f(x)^T f(x)$. Now what if we do $\dot{V} = f(x)^T P(x) f(x) = G$? This gives us more flexibility.

The book suggests $\dot{V} = g(x)f(x)$ such that it is negative definite. Then you can integrate this and make sure that it is positive definite. Once this is satisfied, the Lyapunov function satisfies ISL requirements.

For $\dot{V} = \frac{d}{dx}V(x)f(x)$, and there is a lot of flexibility for choosing $g(x)$. Additionally $V(x) = \int_0^x g(\sigma)d\sigma$.

And for $\dot{x} = ax^n = f(x)$ we know that $\dot{V} = g(x)f(x)$, and for $g(x) = -ax^n$ we have that $\dot{V} = -a^2 x^{2n}$ therefore

$$V(x) = \int_0^x -a\sigma^n d\sigma = \frac{-a\sigma^{n+1}}{n+1}\Big|_0^x = \frac{-ax^{n+1}}{n+1} \tag{19}$$

Where the exponent is even. We can understand the stability result without explicitly knowing the solution.

Another given vector field case example: $\dot{V} = g(x)f(x)$ for $g(x) = \nabla V$. We can look at the line integral (plots).

Some Lyapunov functions take years to find. Prof. Lawrence describes a satellite ADCS system with a single RWA that took him a year.

Domain of attraction: another thing that linearization cannot tell you! Let's say $V(x)$ is PD on domain D and $\dot{V}(x)$ is ND on domain (both "about equilibrium"). Obviously remember this equilibrium does not have to be at the origin.

We draw large level sets $V(x)$ in D that are bound by D. These are also domains of attraction. Anything outside D is not necessarily a domain of attraction because $V(x)$ is only qualified to be PD on the domain. The level sets that qualify as domains of attraction are bounded.

EG: $V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$. For global asymptotic stability, the level sets of $V(x)$ must be globally bounded.

February 19th

On global asymptotic stability (GAS): last time we stated that we can look at the Lyapunov level sets. If we can show that the derivative of V is ND, then it is definitely entering the set, and being attracted to the equilibrium. For local asymptotic stability, $V(x)$ is PD on D and $\dot{V}(x)$ is ND on D.

From diagrams: essentially we want to make sure that our Lyapunov function is stricly increasing and has no minima or maxima. If $\dot{V}(x)$ is ND $\in \mathcal{R}$, then $V(x) \to 0$. This also implies that $x \to 0$ as the PD function $V(x)$ only has a zero point at the origin. This is GAS.

EG: for a function $g : [0, a) \to [0, \infty)$ where a could be finite or infinite, with $g(0) = 0$ and strictly increasing on $[0, a)$, then function g is said to be "class K."

If $\dot{x} = f(x)$ for $f : \mathcal{R} \to \mathcal{R}$ and $V(|x|)$ is class K with $a = \infty$ and $\dot{V}(x)$ is ND on $\mathcal{R}$. This means that $x = 0$ is GAS. This is a scalar vector field; this is not rigorous enough for a higher dimension vector field.

For $f : \mathcal{R}^@ \to \mathcal{R}^2$, with $V(x) : \mathcal{R}^2 \to \mathcal{R}$ and we have that $V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$ (as seen in last class). We remember that the level sets about the origin are circular and become ovular as they go out, and asymptoticall become unbounded. Is this class K? It turns out that $V(\|x\|)$ is class K with $a = \infty$. In general it does not work for the multivariate case. The essential problem with this is that some of the level sets are not bounded or closed.

EG: suppose we have $g : [0, a) \to [0, \infty)$ where $a = \infty$ and $r \to \infty \Rightarrow g(r) \to \infty$, then g is called "class K_∞." So if $g(r) = V(\|x\|)$. Our previous example $V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$ is class K but NOT class K_∞.

Instead, g (class K_∞) is called radially unbounded more frequently.

THM 4.2: Let $\bar{x}$ be an equilibrium of $\dot{x} = f(x)$ if $V : \mathcal{R}^2 \to \mathcal{R}$ is C^1 and PD (wrt x) and radially unbounded and if $\dot{V}$ is ND then $\bar{x}$ is GAS.

Here with diagram we also discuss unstable ISL equilibria and how they help define regions of attraction.

Lyapunov Instability: Def: $\exists$ $\epsilon > 0$ and (upside down f??) $\forall \delta > 0$ for $\|x(0) - \bar{x}\| < \delta \Rightarrow \|x(t) - \bar{x}\| > \epsilon$ for some t.

If V is PD on D (containing $\bar{x}$) and $\dot{V}$ is PD on D $\Rightarrow \bar{x}$ is unstable ISL. The book talks about Chetaev's theorem (4.3) is strronger as it has a weaker condition on $\dot{V}$.

Let us look at a linear system such that $\dot{x} = f(x) = Ax$ with $V(x) = \|x\|_p = x^T P x$ for P a PD symmetric matrix. And then $\dot{V} = x^T P \dot{x} + \dot{x}^T P x = x^T P A x + x^T A^T P x = -x^T Q x$ where the last portion is just a quadratic form where Q is a PD matrix.

THM 4.6: For $Re(\lambda_i(A)) < 0, \quad \forall i \Leftrightarrow$ for any PD Q matrix there exists a PD P matrix such that $A^T P + P A = -Q$. This is the Lyapunov Equation from linear systems theory. This can be a way to pick a V function that works with the linearized dynamics about a hyperbolic equilibrium.

1 February 21st

Today we are going to talk about performance (convergence rate). EG: $\dot{x} = f(x)$ with $V(x)$ PD and $\dot{V}(x) = -\alpha V(x) \Rightarrow V(x) = e^{-\alpha t} V(x_0)$. Suppose our $V(x) \geq b \|x\|^2$ as an underbound and impose an upper bound on the derivative as such $\dot{V}(x) \leq -a \|x\|^2 \leq \frac{-a}{b} V(x)$. This is

similar to what we said at the beginning but an innequality and it is a differential "in-equation" if you will. We can look at a system defined by $\dot{\omega}(t) = \frac{-a}{b}\omega(t)$ which has solution $\omega = e^{\frac{-a}{b}t}\omega_0$. Now if $\omega_0 = V(x_0) \Rightarrow V(t) \leq \omega(t) \quad \forall t \geq t_0$. This means that $V(t) \leq e^{\frac{-a}{b}t}V(x_0) \quad \forall t$. Then it is such that $\|x\|^2 \leq \frac{1}{b}e^{\frac{-a}{b}t}V(x_0), \quad \forall t$ We also have $V(x) \leq c\|x\|^2, \quad \forall x$ then we have that $\|x\|^2 \leq \frac{c}{b}e^{frac-abt}\|x_0\|^2$. We now have an expression that bounds the system by the initial condition. We do similar things in linear systems. This can give you a way to estimate the exponential convergence rate as a time constant.

New topic: LaSalle invariance. If V is C^1 and PD about some $\bar{x}$ equilibrium point of interest on domain D. We will call the set of x that is in domain D, $\Omega_b : \{x|V(x) \leq b\}$ is bounded (for | meaning such that). Basically b is an upper bound on the $V(x)$ function from above that caps it off. if $\dot{V}(x) \leq 0$ on D, what can we say about x? Well if $x_0 \in \Omega_b, \quad V(x(t)) \leq b \quad \forall t$. Therefore Ω_b is a positively invariant set.

We showed some diagrams of this invariance where the state vector can converge to a level set Ω_a smaller than the initial level set. Sometimes there are flat spots in the $\dot{V}(x)$ function that cause $V(x)$ to stop which create these stopping level sets where a system will stabilize about.

EQ: $\dot{x_1} = x_2, \quad \dot{x_2} = -x_1 - x_2^3$ with our Lyapunov candidate function $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ with $\dot{V}(x) = (x_1\dot{x_1} + x_2\dot{x_2}) = -x_2^4$ after plugging in our dynamics, which is NOT ND about (0,0). Let us define the set $E = \{x|\dot{V}(x) = 0\}$. We see that we will converge to this set E along the x_1 line. But as we look at the flows, we see that starting in E does not mean we will stay in E. In this case, the largest invariant set in E is (0,0). Therefore $x(t) \to 0$ as $t \to \infty$.

THM LaSalle: Let $\Omega \in D$ be compact and positively invariant (forward in time). If $V(x)$ is C^1 and $\dot{V} \leq 0$ in Ω, then x converges to the largest invariant set $\mathcal{M}$ in $\mathcal{E} : \{x|\dot{V}(x) = 0\}$. In the previous example $\mathcal{M} : \{(0, 0)\}$. In this theorem, you do not necessarily need positively definiteness.

Typically V is positive definite, and Ω = level set of V.

Corollary 1 (4.1): For and equilibrium $\bar{x}$, if V is C^1 and PD about $\bar{x}$, $\dot{V} \leq 0$ on D and the only invariant set in $\mathcal{E}$ is $\bar{x}$. Then $\bar{x}$ is asymptotically stable.

Corollary 2 (4.2): If V above is radially unbounded and $\dot{V} \leq 0$ on $\mathcal{R}^n$ then $\bar{x}$ is GAS.

LaSalle is USEFUL.

2 February 26

More general invariant sets: Let us look at the hopper robot.

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-k}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f_c \tag{20}$$

Now if we want it to follow a stable limit cycle behaviour, we will want to develop a nonlinear control method. Let us use

$$f_c = (r_0^2 - x_1^2 - x_2^2)x_2 \tag{21}$$

The damping will go to zero on the circle defined by r_0. And the force will oppose x_2 as the internal term will be negative. We are using the state space to geometrically define a circle with offset r_0. Inside the state space circle, we generate energy with the input force, and outside this we dissipate energy.

Let us construct a Lyapunov approach.

$$V(x) = (r_0^2 - x_1^2 - x_2^2)^2 \tag{22}$$

We see that this function has a zero at r_0 and is positive definite about this level set. This is not an equilibrium point but a level set.

Now let us look at

$$\dot{V}(x) = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = 2(r_0^2 - x_1^2 - x_2^2)\{-2x_1x_2 - 2x_2(\frac{-k}{m}x_1 + \frac{1}{m}(r_0^2 - x_1^2 - x_2^2)x_2)\} \tag{23}$$

If we are stuck with the spring constant, we can change the input function by forcing the x_1 and x_2 dependence. This would effectively be elongating the circle in the phase portrait domain. And for $\frac{k}{m} = 1$ we will have that $\dot{V}(x) = \frac{-4}{m}x_2^2(r_0^2 - x_1^2 - x_2^2)^2$. We see that this is negative semi definite. Is it ND relative to the circle? No! It has an x_2 dependence that can mess things up.

But via LaSalle, x converges to the larges invariant set $\mathcal{M} \subset \mathcal{E} = \{x|\dot{V}(x) = 0\}$. The set $\mathcal{M}$ is the equilibrium of the dynamics, and is only the origin. To be on the circle:

$$r = r_0 \tag{24}$$

$$\frac{dr^2}{dt} = 0 = 2x_1x_2 - 2x_2x_1 \tag{25}$$

This is an invariant set on the flow. So the total invariant set $\mathcal{M}$ just contains the circle and the point. LaSalle says that we converge to the largest set. If you start at the origin, you stay there. If you start right off center, it isn't clear whether you would converge to the center. Lawrence says he hasn't looked at the flow analysis. But there is enough using the flows to show essential GAS (essential meaning basically everything except origin). There is a really small set of points that do not end up at the invariant set.

Let us now talk about Lyapunov vector fields that provide nice arguments for stability. For $\dot{x} = f(x)$. He mentions a guidance vector field for UAVs. Basically where you should be moving in the state space vector field. Let us say that you want to fly about a limit cycle. Let's say you want to fly about some radius in space ρ about the origin point. We shall construct the Lyapunov function to begin with $V(x) = \frac{1}{2}(x_1^2 + x_2^2 - \rho^2)^2$, we then have that $\dot{V}(x) = 2(x_1^2 + x_2^2 - \rho^2)[x_1x_2][\dot{x}_1\dot{x}_2]^T$. We see the second part is the $f(x)$ and the first part is jujst the gradient of V. So what would you pick $f(x)$ to be? We would like $f(x) = -\nabla V^T$. Therefore we have that $\dot{V}(x) = -\nabla V(\nabla V)^T = -[\frac{\partial V}{\partial x}][\frac{\partial V}{\partial x}]^T$. But we want to be flying

at a constant velocity, so we add a new portion such that $\dot{V}(x) = -\nabla V \nabla V^T + \nabla V S(x)$ for $f(x) = -\nabla V^T + S(x)$. We choose that $[\frac{\partial V}{\partial x}]S(x) = 0$. So we structure it such that $S(x) = c[x_1 - x_2]^T$.

3 March 5th: Starting I/O System Approaches

The big two nonlinear approaches: Lysapunov state space approaches and input output systems. Next major phase of the course: IO systems and stability.

SEE WRITTEN NOTES: PREDOMINANTLY FIGURES

4 March 7th: I/O Stability Examples

EG: $y = H(u) = u^2$. Using an $\mathcal{L}_\infty$ norm to talk about $\mathcal{L}$ stability. If we take $\|y_\tau\|_{\mathcal{L}_\infty} \leq (?)\alpha(\|u_\tau\|_\infty) + \beta$, for τ indicating truncation. It is that $\forall \tau \geq 0, \quad \forall u \in \mathcal{L}_{\infty_e}$. Here he states that α is an increasing function that is not necessarily linear. We find a γ instead of α that makes the right side of the inequality describe an affine function (linear over domain). So, $\|y_\tau\|_{\mathcal{L}_\infty} = \sup_{\tau \geq t \geq 0} |y_\tau(t)| \leq \alpha(\sup_{\tau \geq t \geq 0} |u_\tau(t)|) + \beta$. Can we show this inequality $\forall \tau, \forall u$.

In our scenario, $\|y_\tau\|_{\mathcal{L}_\infty} = \sup_{\tau \geq t \geq 0} |u_\tau^2(t)| = (\sup_{\tau \geq t \geq 0} |u_\tau(t)|)^2$; we just substituted and brought the supremum out. We know this to be $\|y_\tau\|_{\mathcal{L}_\infty} = \alpha(\|u_\tau\|_{\mathcal{L}_\infty})$. He states that this demonstrates H being $\mathcal{L}_\infty$ stable. Can this be true? : $\sup_{\tau \geq t \geq 0} |u_\tau^2(t)| \leq \gamma \sup_{\tau \geq t \geq 0} |u_\tau(t)| + \beta$. This is NOT true. So in this scenario, H is not $\mathcal{L}_\infty$ stable.

This is one of the simplest examples we can think of. In combinations of systems, you can use the small gain theorem to show linear stability. In the combination of linear and nonlinear components, we find the IO stability of the linear system.

In the case of $\dot{x} = Ax + Bu, \quad y = Cx$, we can write down the solution: $y(t) = Ce^{At}x_0 + C \int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma$. Looking at $\mathcal{L}_\infty$ stability, we see (where bars are norm on $\mathcal{R}^p$):

$$\sup_{\tau \geq t \geq 0} |y_\tau(t)|_p \leq \alpha(\sup_{\tau \geq t \geq 0} |u_\tau(t)|) + \beta, \quad \forall \tau \geq 0, \forall u \in \mathcal{L}_\infty \tag{26}$$

We can plug in our dynamics and we must perform, and with triangle inequality :

$$|Ce^{At}x_0 + C \int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma|_p \leq |Ce^{At}x_0| + |C \int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma| \tag{27}$$

This gets into the topic of reduced norms. For example, $w = Mz, \quad M : \mathcal{R}^n \to \mathcal{R}^m$. Then if we take norms: $\|w\|_m = \|Mz\| \leq (\ )\|z\|, \forall z \in \mathcal{R}^n$. We want the worst case for scaling z. Let us define:

$$\|M\|_{ind} = \max_{normz=1} \frac{\|w\|_m}{\|z\|_n} \tag{28}$$

So this questions asks about the worse case direction of z for this matrix. This ratio is not a function of scale, but more of direction. So we state $\|w\| \leq \|M\| \|z\|, \; \forall z \in \mathcal{R}^n$

$$\left\|Ce^{At}x_0\right\|_{ind} \leq \left\|Ce^{At}\right\|_{ind} \|x_0\|_{ind} \tag{29}$$
$$\leq \|C\|_{ind} \left\|e^{At}x_0\right\|_{ind} \tag{30}$$
$$\leq \|C\|_{ind} \left\|e^{At}\right\|_{ind} \|x_0\|_{ind} \tag{31}$$

But the dynamics are hard things to bound; especially if they hold for all time. The initial condition is finite and C can be bound. The eigenstructure of the matrix will give us insight into bounding this term.

Let us use the Euclidean norm on $\mathcal{R}^n$, the induced norm, using the Hermitian operator:

$$\|M\|_{ind} = \sigma_{max}(M) = (\lambda_{max}(M^H M))^{1/2} \tag{32}$$

And in our case from previous, knowing the A matrix to be real:

$$\left\|e^{At}\right\|_{ind} = \sigma_{max}(e^{At}) = (\lambda_{max}(e^{A^T t}e^{At}))^{1/2} \tag{33}$$

Now you may be tempted to say that $e^M e^N = e^M N$ but this is almost never the case. This is only true if and only if M and N commute. Let us use our eigenvalue perspective. If A is semi-simple, then there exists a nonsingular T such that $A = TDT^{-1}$ as a similarity transformation, where D is $diag\{\lambda_i(A)\}$. There are also ways to make A semi-simple with small perturbations. This can be a knife-edge problem.

Looking at the term e^{At}, substituting the similarity transform into the power series, we see that it evaluates to :

$$e^{At} = T(I + Dt + \frac{D^2 t^2}{2!} + ...)T^{-1} \tag{34}$$

Plugging this into our induced norm description, we have that

$$\left\|e^{At}\right\|_{ind} = (\lambda_{max}((T^{-1})^T (e^{Dt})^T T^T T e^{Dt} T^{-1}))^{1/2} = \sigma_{max}(Te^{Dt}T^{-1}) \tag{35}$$
$$\leq \sigma_{max}(T)\sigma_{max}(e^{DT})\sigma_{max}(T^{-1}) \tag{36}$$

The inner maximum singular value portion evaluations to $(\lambda_{max}(e^{D^H t}e^{Dt}))^{1/2} = (\lambda_{max}(e^{(D^H + D)t}))^{1/2}$ and we can say that $D^H + D = diag\{\lambda_i(A)^H + \lambda_i(A)\} = diag\{2 * Re\{\lambda_i(A)\}\}$.

Of course we want that $Re\{\lambda_i(A)\} < 0$, but what if they are zero? Well then the diagonal element of the e^{Dt} matrix will be one and are inherently bounded. Therefore the overbound of the eigen value matrix is one. They all start at one, and dip below for negative eigenvalues.

SO back to $\left\|Ce^{At}x_0\right\|_{ind} \leq \sigma_{max}(C)\sigma_{max}(T)\sigma_{max}(T^{-1}) \|x_0\|_{ind}$. We can lump this into : $\beta = \sigma_{max}(C)\sigma_{max}(T)\sigma_{max}(T^{-1})$.

5 March 12: I/O Stability of Linear Dynamical Systems

Recall our standard input-output block defined by the linear system $\dot{x} = Ax + Bu$, $y = Cx$. We wrote the solution to this being the convolution integral, and worked on only the initial condition response. We found that we could bound this in terms of a constant bound under the condition that $Re\{\lambda_i(A)\} \leq 0$, $\forall i$. Now can we bound the convolution portion such that it too is $\leq \gamma \|u\|$. So:

$$\sup_{\tau \geq t \geq 0} \left| C \int_0^\tau e^{A(t-\sigma)} Bu(\sigma) d\sigma \right|_p \leq \bar{\sigma}(C) \left| \int_0^\tau e^{A(t-\sigma)} Bu(\sigma) d\sigma \right|_p \tag{37}$$

Where the vertical bars denote a norm on $\mathcal{R}^p$, we will use the Euclidean norm because this allows us to use singular values. Recall that we can assume, for A semi-simple, there exists a similarity transform $A = TDT^{-1}$. But this into the convolution, and factor out such that we get $Te^{D(t-\sigma)}T^{-1}$. Then we can write this as such:

$$\sup_{\tau \geq t \geq 0} |C \int_0^t e^{A(t-\sigma)} Bu(\sigma) d\sigma|_p \leq \bar{\sigma}(C)\bar{\sigma}(T)\bar{\sigma}(T^{-1}) \left| \int_0^\tau e^{D(t-\sigma)} Bu(\sigma) d\sigma \right|_p \tag{38}$$

Utilizing the triangle inequality, we can split up the summation such that

$$\bar{\sigma}(C)\bar{\sigma}(T)\bar{\sigma}(T^{-1}) \left| \int_0^\tau e^{D(t-\sigma)} Bu(\sigma) d\sigma \right|_p \leq \int_0^\tau \left| e^{D(t-\sigma)} Bu(\sigma) \right| d\sigma \, \bar{\sigma}(C)\bar{\sigma}(T)\bar{\sigma}(T^{-1}) \tag{39}$$

This is also called Minkowskis Inequality. With this, we can also show that :

$$\left| e^{D(t-\sigma)} Bu(\sigma) \right| \leq \left| e^{D(t\sigma)} B \right| \sup_{\tau \geq t \geq 0} \left| u(\sigma) \right| \tag{40}$$

Where clearly we have now

$$\sup_{\tau \geq t \geq 0} \left| u(\sigma) \right| = \|u\|_{\mathcal{L}_{\infty e}} \tag{41}$$

Putting this back together, we can show that

$$\int_0^\tau \left| e^{D(t-\sigma)} Bu(\sigma) \right| d\sigma \leq \int_o^\tau \left| e^{D(t-\sigma)} \right| d\sigma \, \bar{\sigma}(B) \|u\|_{\mathcal{L}_{\infty e}} \tag{42}$$

So the last thing to be bounded is the integration of matrix exponential term. Using our truncation we may as well let the t in the integral be a τ.

You can show that

$$\int_o^\tau e^{d_i(t-\sigma)} d\sigma = e^{d_i t} \int_o^\tau e^{-d_i \sigma} d\sigma = \frac{e^{d_i \tau}}{-d_i} \int_o^\tau e^{-d_i \sigma}(-d_i) d\sigma = \frac{e^{d_i \tau}}{-d_i}(e^{-d_i \tau} - e^0) = \frac{-1}{d_i} + \frac{1}{d_i} e^{d_i \tau} \tag{43}$$

So this thing converges to the constant $\frac{-1}{d_i}$; it remains finite. The area under a decaying exponential is bounded.

Therefore, that integral portion $\leq \gamma \iff$ the eigenvalues are negative. Alll together we have that

$$\sup_{\tau \geq t \geq 0} \left| \int_0^\tau e^{A(t-\sigma)} Bu(\sigma) d\sigma \right| \leq \bar{\sigma}(C)\bar{\sigma}(T)\bar{\sigma}(T^{-1})\bar{\sigma}(B)\gamma \sup_{\tau \geq t \geq 1} |u(t)|, \quad \forall u \in \mathcal{L}_{\infty e} \tag{44}$$

And so back to the original, we have that

$$\|y\|_{\mathcal{L}_{\infty e}} \leq \beta + \bar{\gamma} \|u\|_{\mathcal{L}_{\infty e}} \tag{45}$$

For $\bar{\gamma} = \bar{\sigma}(C)\bar{\sigma}(T)\bar{\sigma}(T^{-1})\bar{\sigma}(B)\gamma$. Which means that this system is finite gain $\mathcal{L}_\infty$ stable. This was the time invariant solution. The $\mathcal{L}_1$ convolution theorem accounts for linear time varying systems.

$\mathcal{L}_1$ **Convolution Theorem:** The system $y = H(u)$ where (when initial conditions are zero) $y(t) = \int_0^t h(t-\sigma)u(\sigma)d\sigma$ where small h is the impulse response of H. When $h \in \mathcal{L}_1$, i.e:

$$\left(\int |h(\sigma)|^1 d\sigma \right)^{\frac{1}{1}} < \infty \tag{46}$$

Where the internal induced norm is from the norm on $\mathcal{R}^m$ and norm on $\mathcal{R}^p$. Then there exists $\gamma < \infty$ such that

$$\|y\|_{\mathcal{L}_{pe}} \leq \gamma \|u\|_{\mathcal{L}_{pe}}, \quad \forall u \in \mathcal{L}_{pe}, \quad 1 \leq p \leq \infty \tag{47}$$

Interconnected Systems $\to$ Small Gain Theorem: For two systems S_1 and S_2

The rest of the stuff is on the paper.

March 14: Small Gain Theorem

From the paper we know that, dropping the $\mathcal{L}_{pe}$s:

$$\|y_1\| \leq \gamma_1(\|u_1\| + \|u_2\|) + \beta_1 \tag{48}$$
$$\leq \gamma_1 \|u_1\| + \gamma_1\gamma_2 \|u_2\| + \gamma_1\gamma_2 \|y_1\| + \gamma_1\beta_2 + \beta_1 \tag{49}$$

and similarly:

$$\|y_2\| \leq \gamma_2(\|u_2\| + \|u_1\|) + \beta_2 \tag{50}$$
$$\leq \gamma_2 \|u_2\| + \gamma_1\gamma_2 \|u_1\| + \gamma_1\gamma_2 \|y_2\| + \gamma_2\beta_1 + \beta_2 \tag{51}$$

Therefore, from what we said previously we can bound y as such:

$$\|y\| \leq \left\|[y_1\ 0]^T\right\| + \left\|[0\ y_2]^T\right\| \tag{52}$$

Which, for Euclidean norm, collapses to

$$\|y\| \leq \|y_1\| + \|y_2\| \tag{53}$$

We can simplify what we had before with the following:

$$\|y_1\| \leq \frac{\gamma_1}{1 - \gamma_1\gamma_2}\|u_1\| + \frac{\gamma_1\gamma_2}{1 - \gamma_1\gamma_2}\|u_2\| + \frac{\gamma_1\beta_2 + \beta_1}{1 - \gamma_1\gamma_2} \tag{54}$$

$$\|y_2\| \leq \frac{\gamma_2}{1 - \gamma_1\gamma_2}\|u_2\| + \frac{\gamma_1\gamma_2}{1 - \gamma_1\gamma_2}\|u_1\| + \frac{\gamma_2\beta_1 + \beta_2}{1 - \gamma_1\gamma_2} \tag{55}$$

Therefore in total we have that:

$$\|y\| \leq \frac{\gamma_1}{1 - \gamma_1\gamma_2}(\|u_1\| + \gamma_2\|u_1\|) + \frac{\gamma_2}{1 - \gamma_1\gamma_2}(\gamma_1\|u_2\| + \|u_2\|) + (...) + (...) \tag{56}$$

The (...) terms are β terms. However we want it in form $\leq \gamma\|u\| + \beta$. Therefore we must continue to work on this.

We shall pull the same trick with the norm of the input vector by using the triangle inequality and splitting up the norm into elements. Also use the trick that $\|u\| \geq \|u_1\|$ and $\|u\| \geq \|u_2\|$ So, we have that then:

$$\|y\| \leq (...) + (...) + \frac{\gamma_1 + \gamma_1\gamma_2}{1 - \gamma_1\gamma_2}\|u\| + \frac{\gamma_2 + \gamma_1\gamma_2}{1 - \gamma_1\gamma_2}\|u\| \tag{57}$$

$$\leq \gamma\|u\| + \beta \tag{58}$$

As a result it must be that $\gamma_1\gamma_2 < 1$.

March 19th: Passivity

This method of passivity explicitly allows large gains while focusing on the phase.

We look at the passivity block diagram which has a through variable and an across variable. For a mechanical system the through variable is a force or torque. The across variable is typically velocity or displacement. Power supplied to the system from the outside world is :

$$P(t) = T(t)A(t) \tag{59}$$

The energy supplied is the time integral of this function.

$$E(t) = \int_0^t T(\tau)A(\tau)d\tau \tag{60}$$

The dynamics dissipate energy. Overall the integral will be increasing and we consider it to be dissipative. Lossless energy transfer is if there is no dissipation whatsoever; manifests itself in a sinusoid.

If we substitude T with u and A with y in the block diagram form we get that the energy supplied to the system is as such:

$$E(t) = \int_0^t u(\tau)y(\tau)d\tau \tag{61}$$

This is passive in a sense if this integral $E(t) \geq \beta$, $\forall u$, $\forall y$, $\forall t$. We cann extend this to multivariable cases. This integrand depends on purely $u(t)$. So it seems that the two important parts are $u(t)$ and the system itself. But it must work $\forall u$. So passivity is purely a system property and how the systemr responds to these inputs.

Basically the idea is that there is a lower finite bound β that shows that the system cannot generate more energy than this. This energy is actually dependant upon input/output phase. If you're entirely out of phase, then you're technically generating energy and the integral will go negative. So this is a figure of phase essentially.

But make sure the thing you're integrating is actually energy, sometimes it can be abstract and lead you down the wrong path.

For multivariate systems we have that:

$$E(t) = \int_0^t u(\tau)^T y(\tau)d\tau \geq \beta, \ \forall u, \ \forall t \tag{62}$$

We see that in the vector sense, u and y can be at most 90 degrees away. It appears to be an angled characterization now between input and output. What if the output is a different size? Well, you're out of luck. Passivity approaches generally don't work. You can be clever though in some cases.

Passivity Examples

Mechanical damper: for damping coefficient B

$$E(t) = \int_0^t fv d\tau = B \int_0^t v^2 d\tau \geq 0, \forall t \tag{63}$$

So clearly this is just a dissipative element and never produces energy. check paper.

Passivity Maps

For a nondynamic memoryless block. check paper.

The book Uses the differential formulation for the passivity argument:

$$u^T(t)y(t) \geq \dot{V} \tag{64}$$

This V function is called a storage function, and intuitively is energy in most situations.

For passivity maps it is is such that $u^T(t)y(t)$, $\forall u$, $\forall t$. This agrees with our argument that you should stay out of the 2nd and 4th quadrants.

Why are we doing this crap?

It is used in conjunction with small gain theorem. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are passive. Then $e_1^T y_1 \geq \dot{v}_1$, $\forall e_1, t$. Simmilarly $e_2^T y_2 \geq \dot{v}_2$, $\forall e_2, t$. If we want to say the close loop is passive then we must use the multivariate argument $u = [u_1 \quad u_2]^T$ and $y = [y_1 \quad y_2]^T$ so closed loop must ask the question if $u^T y \geq \dot{v}$, $\forall u, t$.

$$u^T y = e_1^T y_1 + e_2^T y_2 \tag{65}$$

Knowing that $e_1 = u_1 - y_2$ and that $e_2 = u_2 + y_1$. And this summation is $\geq \dot{v}_1 + \dot{v}_2$. Therefore passivity is preserved in feedback systems.

March 21st: Passivity variations

Recall from las class where we defined regions of passivity being in the first and third quadrant. The differential formulation's integrand had form $u^T y \geq 0$.

The question we are going to look at is expanding this to treat subsystems that are not passive. So we will look at sharing passivity, as we got into at the end of last lecture.

Suppose we had a charaterization where $u^T y - \epsilon u^T u \geq 0$. For the case where $\epsilon > 0$ then the left hand must be positive and the system has to be passive.

Let us rewrite to understand as such $u^T y \geq \epsilon u^T u$. This creates a much more restricted system where you must not exist in a larger space then the second and fourth quadrant.

For the case of $\epsilon < 0$ then the reachable area is expanded into the second and fourth quadrant. This is not passive but almost passive. Here we have the ability to change the passivity to make it agree.

Suppose for the system we feedforward an ϵ according to the block diagram figure. We know have $u^T \bar{y} = u^T(y - \epsilon u) = u^T y - \epsilon u^T u$. This makes the resulting system with interconnection passive. We could also call teh ϵ the excess passivity of a system that is already passive. This

system would be called input-feedforward passive. And if $\epsilon > 0$ then the system is said to be input-strictly passive.

Before we said that $u^T y$ is the power supplied to the system, and therefore $\epsilon u^T u$ is the power lost/dissipated $\epsilon > 0$ or power generated by the system $\epsilon < 0$.

Now suppose our system looks like $u^T y - \delta y^T y \geq 0$. The second term is the power lost or generated. Therefore $u \geq \delta y$. See the figure for the areas you cannot exist in. For $\delta > 0$ the system will have excess passivity. Now for $\delta < 0$, we have a system that is almost passive just like what we showed above.

We can do a positive feedback network and show that $u = \bar{u} + \delta y$ and therefore we have that $\bar{u}^T y = u^T y - \delta y^T y \geq \delta y^T y - \delta y^T y = 0$.

The system defined by $u^T y - \delta y^T y \geq 0$ is called output feedback passive. For positive δ it is called output strictly passive.

Let us return to the interconnected system block diagram figure, on the paper. Suppose $\mathbb{S}_1$ is output strictly passive (OSP) ($\delta > 0$) but suppose that $\mathbb{S}_2$ is only input feedforward passive (IFP) ($\epsilon < 0$). We can take out some constants in the loop, as seen by the figure on the paper. We pull out constant g and call the new blocks $\mathbb{S}'_1$ and $\mathbb{S}'_2$. We have denoted the notation on the paper. The new power function is $v^T y_1 = (e_1 - g y_1)^T y_1 = e_1^T y_1 - g y_1^T y_1$. The first term in this subtraction is positive because it is OSP. This term $e_1^T y_1 \geq \delta y_1^T y_1$ and the whole term $e_1^T y_1 - g y_1^T y_1 \geq (\delta - g) y_1^T y_1$ and is passive when $g \leq \delta$.

Now let us look at $\mathbb{S}'_2$. The function power, knowing that $w = x + g u_2$ is $u_2^T w = (x + g u_2)^T u_2 = x^T u_2 + g u_2^T u_2$. The first term $x^T u_2 \geq u_2^T u_2$ and the whole thing $u_2^T w \geq (\epsilon + g) u_2^T u_2$ therefore it must be that $g \geq -\epsilon$ for passivity.

Finally we have that $-\epsilon \leq g \leq \delta$. This only happens if $\delta \geq -\epsilon$.

Again the driving equations are the following:

$$u^T y - \epsilon u^T u \geq 0 \tag{66}$$

$$u^T y - \delta y^T y \geq 0 \tag{67}$$

These are very general and can be used to describe most systems. Truly acquiring the ϵ and δ values are the nontrivial jobs.

Looking at his drawing of sectorized passivity: to be passive means $\Rightarrow sector[0, \infty]$, OSP $\Rightarrow sector[0, k_2]$ for $k_2 > 0$, ISP $\Rightarrow [k_1, \infty]$ for $k_1 > 0$.

April 2nd: More on Passivity

After passivity, towards the end of the course, we will cover feedback linearization as well as describing functions.

Recall that wee spoke about total passivity as well as subsystem passivity sharing. We also spoke about sector bounds for describing passive systems.

Passivity of Dynamical Systems: Previously we studied the passivity of a spring system with forces on either end. Now let us add a damper to the system according to our diagram. Therefore the dynamics are described by $f = f_k + f_b = l(l - l_0) + B\dot{l}$.

We can describe this as a block diagram with input $\dot{l}$ and output f therefore we have that $x = l - l_0$, $\dot{x} = \dot{l} = u$ and $y = f = kx + Bu$. The energy integral is as follows:

$$E(t) = \int_0^t u(\tau)y(\tau)d\tau \tag{68}$$

$$= \int_0^t u(\tau)(kx + Bu)d\tau \tag{69}$$

$$= \int_0^t \dot{x}(\tau)kx(\tau)d\tau + B\int_0^t u^2(\tau)d\tau \tag{70}$$

We can see that the antiderivative of the integrand follows such that

$$E(t) = \frac{k}{2}\int_0^t \frac{d}{d\tau}x^2(\tau)d\tau + B\int_0^t u^2(\tau)d\tau \tag{71}$$

$$= \frac{k}{2}(x^2(t) - x^2(0)) + B\int_0^t u^2(\tau)d\tau \tag{72}$$

We see that there exists a lower bound and can write that

$$\frac{k}{2}(x^2(t) - x^2(0)) + B\int_0^t u^2(\tau)d\tau \geq \frac{k}{2}(x^2(t) - x^2(0)) \tag{73}$$

We see that the second term adds energy and is a dissipative term. Now energy can be stored but, overall it is still passive.

Now let $V(x) = \frac{k}{2}x^2$, we can have that $E(t) = V(x(t)) - V(x_0) + B\int_0^t u^2(\tau)d\tau$. Differentiating with respect to time we have that $\dot{E}(t) = u(t)y(t) = \dot{V} + Bu^2(t) \geq \dot{V}$. Again this is more than passive. We see that $uy - Bu^2 \geq \dot{V}$ where the second term is considered excess passivity. V is the storage function.

Previously we discussed a system with no storage and excess passivity called input strictly passive: $uy - \epsilon u^2 \geq 0$. Therefore our system is considered input strictly passive. So damping gives us passivity and dissipation!

Now what if we have something more complicated? Can we do this in a more general way? In the case of a linear system, let us try a transfer function approach with impulse response h(t). Let us write the convolution that that

$$y(t) = \int_0^t h(t - \tau)u(\tau)d\tau + ICs(\textbf{assumed to be zero}) \tag{74}$$

Let us write an energy expression in the frequency domain, with truncation function τ:

$$E(t) = \int_o^t u(\tau)y(\tau)d\tau \tag{75}$$

$$= \int_{-\infty}^{\infty} u_\tau(\tau)y_\tau(\tau)d\tau \tag{76}$$

We do this to get it into a form we are interested in.. he calls this a "mathematical slight of hand." Let us now write this as:

$$E(t) = \int_{-\infty}^{\infty} \hat{u}_\tau^*(j\omega)\hat{y}_\tau(j\omega)d\omega \tag{77}$$

This is not obvious at all. By the famous Plancherel (Parseval) theorem, this is known to be true.

Let us recall the fourier transform:

$$\hat{u}_\tau(j\omega) = \int_{-\infty}^{\infty} u_\tau(\tau)e^{-j\omega t}d\omega \tag{78}$$

and of course we know from the definition of a transfer function that

$$\hat{y}_\tau(j\omega) = \hat{h}(j\omega)\hat{u}_\tau(j\omega) \tag{79}$$

Back to the integral, we can then write it as:

$$E(t) = \int_{-\infty}^{\infty} \hat{u}_\tau^*(j\omega)\hat{h}(j\omega)\hat{u}_\tau(j\omega)d\omega \tag{80}$$

$$= \int_{-\infty}^{\infty} \hat{h} \left\| \hat{u}_\tau \right\|^2 d\omega \tag{81}$$

So now the sign entirely depends on the transfer function. We want it to be positive so that we can put a lower bound on the energy.

Represented in by the Laplace transform we have that $\hat{h}(s)$, and require this to be positive in order to input a bounding inequality in our study of the energy function.

We know that the transfer function in the complex plane has conjugate symmetry due to real ODE coefficients. Know that $\hat{h} = (\hat{h}_R + j\hat{h}_I, \ \hat{h}_R - j\hat{h}_I)$. Knowing this, then the integral on the entire domain should cancel all of the imaginary components and yield only the real components. Therefore we can reformulate as such:

$$E(t) = 2\int_0^{\infty} \mathbf{Re}(\hat{h}(j\omega)) \left\| \hat{u}_\tau(j\omega) \right\|^2 d\omega \geq 0 \quad \Longleftrightarrow \quad \mathbf{Re}(\hat{h}(j\omega)) \geq 0 \ , \forall \omega \tag{82}$$

Therefore $\mathbf{Re}(\hat{h}(j\omega))$ must be in the right half of the complex plane! The term that people use is **positive real transfer function**. If the transfer function is positive real, then it is passive. This is considered a synonym for passivity. So we can immediately state the passivity by looking at the frequency response.

For example, let us state that $h(s) = \frac{a}{s+b} \rightarrow h(j\omega) = \frac{a}{j\omega+b}$. We know the angular displacement on the complex plane to a specific frequency is the phase, and the length to that point is the magnitude. We showed that this low pass filter is a passive system as the real components are always positive.

Now let us consider thee system $h(s) = \frac{a}{s}$. We see that integrator remains on the imaginary axis and is still passive.

Now we consider the system defined by $h(s) = \frac{s+a}{s+b}$. We know that the phase is the most important decider for passivity arguments because we must stay on the left hand side. We know that the total phase can never exceed 90 degrees for these first order systems. Since the real gain goes to unity as $\omega \rightarrow \infty$, it is stronger than passive and is considered strictly positive real (SPR). We can lower bound the real magnitude. This is passive plus dissipation.

So we can then write

$$E(t) = 2 \int_0^\infty \mathbf{Re}(\hat{h}(j\omega)) \left\| \hat{u}_\tau(j\omega) \right\|^2 d\omega \tag{83}$$

$$\geq 2\sigma \int_0^\infty \left\| \hat{u}_\tau(j\omega) \right\|^2 d\omega \tag{84}$$

$$= \sigma \int_0^t u^2(\tau) d\tau \tag{85}$$

For $\sigma = \mathbf{Re}(\hat{h}(j\omega))$. This is input strictly passive. It has a positive lower bound, so "has more to give and still be passive." This is true for SPR systems.

For second order systems, showing passivity is harder as you must look at the Nyquist plot.

Missed a Class

April 9th: Absolute stability

April 11th: More on Absolute stability

We are looking at a loop with an LTI system inside of it. Let us talk about the circle criterion (Thm 7.2 in book).

Let us assume that the LTI portion of the feedback loop is defined by $\dot{x} = Ax + Bu$, $y = Cx + Du$, (with Laplace transform of $G(s)$) and we assume that it is a minimal system

meaning that it is fully controllable and observable. Our feedback element Ψ is in a sector bound defined by $[\alpha, \ \beta]$. If all of this is true, then the closed loop is "absolutely stable" if we pass other conditions:

- $0 < \alpha < \beta$ and the Nyquist plot of $G(j\omega)$ does not enter the disk $D(\alpha, \beta)$ and encircles it M times counterclockwise when $G(s)$ has M poles in the right half plane. This circle has left intercept of $\frac{-1}{\alpha}$ and right intercept of $\frac{-1}{\beta}$. We cannot enter this circle and have to encircle it the same number of times as we have poles.

- $0 = \alpha < \beta$ which sector bounds our Ψ such that it remains in the first and third quadrant, and G is Hurwitz and the Nyquist plot of $G(j\omega)$ lies to the right of $Re(s) = \frac{-1}{\beta}$. With the left bound being zero, it effectively closes off the entire left half plane beyond the right most bound.

- $\alpha < 0 < \beta$ gives us a sector bound which allows for some space in all quadrants. If G is Hurwitz and the Nyquist plot of $G(j\omega)$ lies inside the disk $D(\alpha, \beta)$. Therefore you have the disk now moving into the right half plane. The Nyquist plot must be entirely contained by the disk. In this case it turns out the the system can still be stable under positive feedback coefficients, showing that the LTI system is highly robust.

Describing Functions

I have drawn a loop in my notes. We break a loop and insert a sinusoidal function $A\sin\omega t$. We can look at the output of this separation and look the steady state response. In the case where the output $B\sin\sin\omega t + \phi$ has $B = A$ and $\phi = 0$ we see that the system is marginally or neutrally stable. You can think of this as if you instantly reconnected the system.

The loop gain $L_g = -GH$ or can view it as $-L_g = GH$ for our Nyquist analysis. We end up starting at -1 on the Nyquist plot, or 180 degree shift. Now if we change this to $-L_g = kGH$? We could scale the system such that it does not go through the critical point. For k constant greater than one, it goes unstable, because it exponentially increases.

The closed loop transfer function described by $\frac{y}{r} = \frac{kG}{1+kGH}$. So when does $GH = \frac{-1}{k}$? Because this is the point of instability. This is our new critical point.

With nonlinear systems with weak nonlinearities, we can attribute describing functions to specific frequencies. But again with leakage into other frequencies, it is not necessarily a rigorous approach.

April 22nd: Finding the Describing Function

Let us recall that we would insert a sinusoidal function into a nonlinearity. We see the spectrum of the output, and can selectively generate an approximate function that describes the dominant frequency.

How do we find this exactly? We must also remember there exists a transient. This decays, leaving the steady state, given the function is stable.

We can describe the steady state output as such

$$y_{ss} = \hat{C}_0 + \sum_{k=-\infty}^{\infty} \hat{C}_k e^{j(\omega k)t} \tag{86}$$

Where ω is the fundamental while ωk is the harmonic frequency. The spectral distribution amplitude is denoted with $\hat{C}_k$. It is defined by

$$\hat{C}_k = \frac{1}{T} \int_0^T y_{ss}(t) e^{-j(\omega k)t} dt \tag{87}$$

Now we must substitute $y_{ss}(t)$ can carry out the integration. We want to find the ratio of the input and output spectral amplitudes. We can describe the input as such:

$$u(t) = a\sin(\omega t) = \frac{a(e^{j\omega t} - e^{-j\omega t})}{2j} \tag{88}$$

$$= \hat{a}_0 + \sum_{k=-\infty}^{\infty} \hat{a}_k e^{j(\omega k)t} \tag{89}$$

In the case where $\hat{a}_0 = 0$ we see that $\hat{a}_1 = \frac{a}{2j}$. In our example suppose the nonlinearity is $\psi(u) = y_{ss}$. Lets take the case of a saturation nonlinearity (drawn on paper). The saturation points on the output vs input plot are at $+\delta$, $-\delta$. For $|u| < \delta \rightarrow y = ku$ and we are ok, and the output stays sinusoidal. When $|u| > \delta$ so we get clipping on the output waveform. The amplitude of the first harmonic can be approximated by a slightly smaller sinusoidal function.

We can state that $|\hat{C}_1| < k$. The first harmonic will not change as the input grows. Our describing function is $\Psi = \frac{\hat{C}_1}{\hat{a}_1}$ which is the ratio of amplitudes as a gain. Linear plot of the relationship on paper. Typically these describing functions are precomputed and are in tables in the older texts.

The ratio:

$$\Psi = \frac{\hat{C}_1}{\hat{a}_1} = \frac{2j}{a} \frac{\omega}{2\pi} \int_0^T \psi(a\sin(\omega t)) e^{-j(k\omega)t} dt \tag{90}$$

$$= \frac{2j}{a} \frac{\omega}{2\pi} \int_0^T \psi(a\sin(\omega t))(cos(\omega t) - j\sin(\omega t)) dt \tag{91}$$

The cosine portion averages to zero of the integral but the $\sin^2$ term does not leaving us with the following:

$$\Psi = \frac{-2j^2}{a} \frac{\omega}{2\pi} \int_0^T \psi(a\sin(\omega t))\sin(\omega t)dt \tag{92}$$

$$= \frac{1}{a\pi} \int_0^{2\pi} \psi(a\sin(\theta))\sin(\theta)d\theta \tag{93}$$

$$= \frac{2}{a\pi} \int_0^{\pi} \psi(a\sin(\theta))\sin(\theta)d\theta \tag{94}$$

We see this is not dependent upon time.

NOTE: read book for more information on ψ. Lowercase is the nonlinearity, uppercase is the describing function.

April 25th, 2019: Feedback Linearization

Robot manipulator example, with joint deflections $q \in \mathbb{R}^n$, where n is usually 6 for three positional and attitude DOF. We can get them into this form:

$$H(q)\ddot{q} + S(q,\dot{q})\dot{q} + g(q) = \tau \tag{95}$$

All of the elements in this equation are torques, with τ representing joint torques. We see immediately that $H(q)$ is the inertial torque, and that $S(q,\dot{q})$ is a damping kind of torque and other stuff. This second portion includes Coriolis gyroscope, damping, and centripetal type torques.

How do we generate a control torque which linearizes the dynamics? First we must find a way to linearize the dynamics and then make sure it tracks our control references.

Suppose $\tau = H(q)v + S(q,\dot{q})\dot{q} + g(q)$. Plugging in we have that, assuming that $H(q)^{-1}$ exists $\forall \ q$:

$$H(q)(\ddot{q} - v) = 0 \tag{96}$$

In this case we have that $\ddot{q} = v$ in the nontrivial case. We can see the system as v being double integrated to get q. This double integrator is a linear, decoupled, second-order system. Given that it is decoupled, it is a stackup of individual differential equations. There are no zeros, as it looks like $\frac{1}{s^2}$. So now let us do some control design to make this thing track a reference q_r.

This could be $v = K_p(q_r - q) - K_d\dot{q}$. However, the dynamics are very complicated and it is very hard to implement. Most people would not implement this, as the cancellation terms are very difficult to calculate with the complicated dynamics. In most scenarios the middle term has small Coriolis and gyroscopic forces. Additionally, with large gear ratios, the gravity backdrive ends up being very small. The inertia terms dominate by far. The H matrix ends up being almost diagonal. Most people ended up only implementing a PD controller at each joint with high gains.

If we crank out the linearization it ends up looking roughly like:

$$M\ddot{q} + B\dot{q} + Kq = \tau \tag{97}$$

For matrices M, B, and K. The decoupled case ends up being approximately:

$$m_i\ddot{q}_i + b_i\dot{q}_i + k_iq = \tau_i \tag{98}$$

$$m_i\ddot{q}_i = \tau_i - b_i\dot{q}_i - k_iq \tag{99}$$

With the transfer function being:

$$\frac{q_i(s)}{\tau_i(s)} = \frac{1}{m_is^2 + b_is + k_i} \tag{100}$$

The block diagram is written on paper. We see that q_i is related to τ_i through a double integrator process. He says that q_i is "two integrators away" from τ_i, independent of the feedback.

Now, as shown on paper, the feedback linearization function effectively becomes the new dynamics function. We add another feedback loop to add more zeros to the system. We use our own feedback to cancel the internal feedback! Interesting to note what is written on paper.

Now let us try to add zeros to the reduced chain of integrators. We have the standard $y(s) = H(s)u(s) = \frac{N(s)}{D(s)}u(s)$ with denominator of order m and denominator of order n. It must be proper to be realizable, therefore $m \leq n$. We also have the idea of relative degree where $p = n - m$. Proper for $p = 0$ and strictly proper for $p > 0$. Obviously the relative degree of the chain of integrators was 2.

Ideally we add feedback such that we reduce it to chain of integrators.

Now we describe the system with $D(s)y(s) = N(s)u(s)$. So if we could divide N into D, we get a quotient and a remainder. Therefore $D(s) = N(s)Q(s) + R(s)$ with $deg(Q(s)) = p$ and with $deg(R(s)) < deg(N)$.

We now have that $(NQ + R)y = Nu$ which is manipulated to

$$NQy = Nu - Ry \tag{101}$$

$$Qy = u - \frac{R}{N}y \tag{102}$$

This block diagram, on paper, is now our $H(s)$ which is an integrator with internal feedback. We can cancel this internal feedback. It becomes a reduced order system as we have lost some of the poles. If we look at the new transfer function we have that

$$\frac{y}{r} = \frac{\frac{N}{D}}{1 - (\frac{R}{N}\frac{N}{D})} \tag{103}$$

$$= \frac{N}{D - R} \tag{104}$$

$$= \frac{N}{(NQ + R) - R} \tag{105}$$

$$= \frac{1}{Q} \tag{106}$$

However, it is hard to cancel N exactly if you do not have the perfect model. So if there is an unstable zero, it could not cancel perfectly. We must show that the $\mathbb{R}(\lambda_i(N(s))) < 0$. Another way to say this is that $H(s)$ is minimum phase (zeros are strictly left hand plane) or "zero dynamics of H are stable."

April 30th, 2019: More on Feedback Linearization

Recall that we were left with a chain of integrators with our new defined input as z. Our system is described by, using the polynomial division version of the block diagram:

$$e(t) = q_p\frac{d^p y(t)}{dt^p} + ... + q_0 y(t) \tag{107}$$

We would like to make a state space model (book usees ξ). We have that $\xi_1 = y$, $\xi_2 = \dot{\xi}_1$, and they continue to successive derivatives such that $\xi_p = \dot{\xi}_{p-1} = \frac{d^{p-1}y(t)}{dt^{p-1}}$ and therefore $\dot{\xi}_p = \frac{d^p y(t)}{dt^p} = \frac{1}{q_p}e(t) - \frac{q_{p-1}}{q_p}\xi_p - ... - \frac{q_0}{q_p}\xi_{p-1}$.

So we can describe it as such: $\dot{\xi} = A\xi + Be$ with output equation $y = C\xi$. We see that, with our definition, A is a companion matrix form which has mostly entries of 1 that are off diagonal. The bottom row is all of the constants q that we described in the previous paragraph. If, with feedback, we cancel all of the elements on the bottom row and make this a chain of integrators, then all eigenvalues are at the origin: $A_c + B_c\lambda^T$ for $\lambda^T = [\frac{-q_o}{q_p} ... \frac{-q_{p-1}}{q_p}]$.

We can then describe a minimal realization of the bottom portion of the block diagram $\frac{R}{N}$.

$$\dot{\eta} = A_0\eta + B_0 y \tag{108}$$
$$w = C_0 y \tag{109}$$

This is the state space representation of this lower block system. We can stack this up like the following (Normal form):

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A_c + B_c \lambda^T & \frac{B_c C_0}{q_p} \\ B_0 C_c & A_0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} \frac{1}{q_p} u \tag{110}$$

$$y = \begin{bmatrix} C_c & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \tag{111}$$

We then have that

$$\dot{\xi} = (A_c + B_c \lambda^T)\xi + \frac{B_c C_0}{q_0}\eta + B_c \frac{1}{q_p} u \tag{112}$$

We can choose that

$$u = q_p(-\lambda^T \xi + \frac{1}{q_p} C_0 \eta + v) \tag{113}$$

We then have that

$$\dot{\xi} = A_c \xi + B_c v \tag{114}$$
$$y = C\xi \tag{115}$$
$$\dot{\eta} = B_0 C_c \eta + A_0 \eta \tag{116}$$

Which is a chain of p integrators. This feedback results in a system that appears as:

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A_c & 0 \\ B_0 C_c & A_0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} v \tag{117}$$

$$y = \begin{bmatrix} C_c & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \tag{118}$$

Which is upper triangular and there is a one way dependence. This matrix is companion form and this is a chain of integrators. Ideally we would like to take a nonlinear system and put it into normal form. If we can, then we can do state variable feedback and cancel elements to make a chain of integrators to which we could add linear control design.

So what does it take to get a nonlinear system into a form like this? Suppose, as a special case, we have the system in the form $\dot{x} = f(x) + g(x)u$ called the separable form. Now usually it is like $\dot{x} = f(x, u)$, so this is special. Of course we just say that $y = h(x)$. We assume that $f(x), g(x), h(x)$ are smooth: smooth enough to take the number of derivatives required for the theory. We perform a change of variables, as a general transformation:

$$z = T(x) = \begin{bmatrix} \eta \\ \xi \end{bmatrix} = \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_{n-p}(x) \\ - - - \\ \vdots \end{bmatrix} \tag{119}$$

This needs to be a smooth homeomorphism (diffeomorphism): an invertible mapping that is smooth. Basically we aim to make the η states as unobservable on the output. Therefore we need that

$$\dot{\eta} = \frac{d}{dt}\phi(x) = \frac{\partial \phi}{\partial x}\dot{x} = \frac{\partial \phi}{\partial x}(f(x) + g(x)u) \tag{120}$$

We need to choose $\phi(x)$ such that the projection: $\frac{\partial \phi}{\partial x} \cdot g(x) = \bar{0}$. This is not necessarily easy to find this function.

Now we can construct the ξ portion:

$$z = T(x) = \begin{bmatrix} \eta \\ \xi \end{bmatrix} = \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_{n-p}(x) \\ y = h(x) \\ \dot{\xi}_1 = \frac{d}{dt}(h(x)) = \frac{\partial h}{\partial x}(f(x) + g(x)u) \\ \dot{\xi}_2 = \frac{d}{dt}(\frac{\partial h}{\partial x}f(x)) = \frac{\partial}{\partial x}(\frac{\partial h}{\partial x}f(x)) \\ \vdots \\ \dot{\xi}_p = -\delta(x) + \gamma(x)u \end{bmatrix} \tag{121}$$

To get u to appear here, the derivative must exceed the relative degree of the system. Therefore the dimensionality of the bottom half of this vector is p. Notice how I drop the input multiplication in subsequent steps. The input only shows up on the last step. The bottom row is kind of like the companion form chain of integrators. Then the idea is to use u to get rid of the δ term.

We can write:

$$\dot{\xi}_p = -\delta(x) + \gamma(x)u = \gamma(x)\left(u - \frac{\delta(x)}{\gamma(x)}\right) = \gamma(x)(u - \alpha x) \tag{122}$$

We choose that $u = \alpha(x) + \frac{1}{\gamma(x)}$ such that $\dot{\xi}_p = v$. This is a linear chain of integrators. THM 13.1 in the book shows why this is true.

By THM 13.1, such a $T(x)$ always exists with the assumptions of smoothness on some domain D (also must be that $\gamma(x)$ does not go singular) about an x of interest.

Also, he missed the definition of normal form:

$$\dot{\eta} = f(\eta, \xi) \tag{123}$$

$$\dot{\xi} = A_x \xi + B_c \gamma(x)(u - \alpha(x)) \tag{124}$$

$$y = C_0 \xi \tag{125}$$

And now we see that η is internal and do not appear in the input output map.

Bad things will happen if some of your internal states are blowing up - basically you must know that your internal states are stable. This was an initial assumption. So we must have that $\dot{\eta} = f(\eta, 0)$ is asymptotically stable or minimum phase "zero dynamics are stable."