Lecture 1 plus 2. ETS models

ETS = Error, Trend, Seasonality

- How to skip current lecture?

Foreasting principles and practice

ETC3550 Slides on ETS

Stationarity, White-Noise

- def. **Stochastic (Random) process** — just the sequence of random variables, (y_t)!

$y_1, y_2, y_3, y_4, \ldots$

- def. **White noise process**, (u_t)

$E(u_t) = 0$ — zero expected value

$Var(u_t) = \sigma_u^2$ — constant variance

$Cov(u_t, u_s) = 0$ for $t \neq s$

- Process (y_t) is **weakly statinary** if

$E(y_t) = \mu_y$ — constant expected value

$Var(y_t) = \sigma_y^2$ — constant variance

$Cov(y_t, y_s) = \gamma_{t-s}$ — covaraince depends only on difference of t and s

Ie that means: $Cov(y_1, y_2) = Cov(y_3, y_4)$.

Q. $Cov(y_1, y_{10})$, $Cov(y_{10}, y_{19})$, $Cov(y_3, y_{10})$ which are equal?

A. $Cov(y_1, y_{10}) = Cov(y_{10}, y_{19})$

Theorem. White noise is a stationary process 😃

Proof. Just check the definition.

- def. **Random walk**, (y_t):

y_0 — constant

$y_t = y_{t-1} + u_t$, where (u_t) — is a white noise process.

Q. Let (y_t) be a random walk

$$\begin{cases} y_0 = 0 \\ y_t = y_{t-1} + u_t \end{cases},$$

(u_t) is a white noise process with variance σ_u^2.

Find $E(y_t), Var(y_t)$?

Is (y_t) weakly stationary?

A. $y_1 = y_0 + u_1 = u_1$

$y_2 = y_1 + u_2 = u_1 + u_2$

$y_3 = y_2 + u_3 = u_1 + u_2 + u_3$

...

$y_t = u_1 + u_2 + \ldots + u_t = \sum_{i=1}^{t} u_i$

$E(y_t) = E(u_1) + E(u_2) + \ldots + E(u_t) = 0 + 0 + \ldots + 0$

$Var(y_1) = Var(u_1) = \sigma_u^2$

$Var(y_2) = Var(u_1 + u_2) = \sigma_u^2 + \sigma_u^2 = 2\sigma_u^2$

$Var(y_t) = Var(u_1 + u_2 + \ldots + u_t) = Var(u_1) + Var(u_t) + \ldots + Var(u_t) = t \cdot \sigma_u^2.$

Random walk is not stationary!

ETS(AAA) model without random noise

(y_t) — quarterly time series

trend + seasonal effects

y_t — observed time series

ℓ_t — time series cleared from seasonal effects

b_t — current growth rate of ℓ_t

s_t — current seasonal effect

Let's write equations!
Current values depend on past values!

ETS(AAA) without noise:

$$\begin{cases} b_t = b_{t-1} \\ s_t = s_{t-4} \\ \ell_t = \ell_{t-1} + b_{t-1} \\ y_t = \ell_{t-1} + b_{t-1} + s_{t-4} \\ s_0 + s_{-1} + s_{-2} + s_{-3} = 0 \end{cases}$$

Initial values:
$b_0 = 2$, $\ell_0 = 100$,

$s_0 = 3$, $s_{-1} = 2$, $s_{-2} = -4$, $s_{-3} = -1$.

Initial values are estimated by maximum likelihood on computer.

- notation: $\Delta \ell_t = \ell_t - \ell_{t-1}$

Q. y_{101}? y_{102}?

A.
$b_1 = b_2 = \ldots = 2$

$s_{101} = \ldots = s_{-3} = -1$

$\ell_{100} = \ell_0 + b_0 + b_1 + \ldots + b_{99} = 100 + 2 \times 100 = 300.$

$\ell_1 = \ell_0 + b_0 = 100 + 2 = 102$

$\ell_2 = \ell_1 + b_1 = 102 + 2 = 104$

$\ell_3 = \ell_2 + b_2 = 104 + 2 = 106 = 100 + 3 \times 2$

$y_{101} = \ell_{100} + b_{100} + s_{97} = 300 + 2 + (-1) = 301$

We need equation $s_0 + s_{-1} + s_{-2} + s_{-3} = 0$ to interpret ℓ_t as version of y_t cleared from seasonal effects.

Q. ETS(AAA) without noise.

How many parameters do we need to estimate for quarterly data?

A. 5 parameters: b_0, ℓ_0, s_0, s_{-1} and s_{-2}.

Q. ETS(AAA) without noise.

How many parameters do we need to estimate for monthly data?

ETS(AAA) model with random noise

Random noise: (u_t)

ETS(AAA) with noise:

$$\begin{cases} u_t \sim N(0; \sigma_u^2), \text{ independent} \\ b_t = b_{t-1} + \beta u_t \\ s_t = s_{t-4} + \gamma u_t \\ \ell_t = \ell_{t-1} + b_{t-1} + \alpha u_t \\ y_t = \ell_{t-1} + b_{t-1} + s_{t-4} + u_t \\ s_0 + s_{-1} + s_{-2} + s_{-3} = 0 \end{cases}$$

Q. ETS(AAA) with noise.

How many parameters do we need to estimate for quarterly data?

A. 5 + alpha + beta + gamma + sigma = 9 parameters

Miracle

Observed values: y_1, y_2, ..., y_{100}.

Maximum Likelihood magick 😃

Imagine that 9 parameters are already estimated:

$\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$, $\hat{\ell}_0$, ...

How to make seasonal decomposition using estimated ETS model?

For simplicity I will omit «hat» above estimated parameters. I will think of true values of parameters.

Observed values: $y_1 = 101$, $y_2 = 104$, $y_3 = 106$, ...

True parameters: $b_0 = 2$, $\ell_0 = 100$, $s_0 = 1$, $s_{-1} = 2$, $s_{-2} = -2$, $\alpha = 0.5$, $\beta = 0.5$, $\gamma = 0.5$, $\sigma = 2$.

Q. Decompose y_1, y_2, y_3 into error, trend and seasonal effect.

Find $\ell_1, \ell_2, \ell_3,$
$s_1, s_2, s_3, b_1, b_2, b_3,$
$u_1, u_2, u_t.$

$$\begin{cases} u_t \sim N(0; \sigma_u^2), \text{ independent} \\ b_t = b_{t-1} + \beta u_t \\ s_t = s_{t-4} + \gamma u_t \\ \ell_t = \ell_{t-1} + b_{t-1} + \alpha u_t \\ y_t = \ell_{t-1} + b_{t-1} + s_{t-4} + u_t \\ s_0 + s_{-1} + s_{-2} + s_{-3} = 0 \end{cases}$$

Sequence:

Step 1. Search for u_1, b_1, s_1, ℓ_1

Look at the equation for y_t:

$y_1 = \ell_0 + b_0 + s_{-3} + u_1$

From $101 = 100 + 2 + (-1) + u_1$ we have $u_1 = 0$.

Look at the equation for b_1:

$b_1 = b_0 + \beta u_1 = 2 + 0.5 \times 0 = 2$

Look at the equation for s_1:

$s_1 = s_{-3} + \gamma u_1 = -1 + 0.5 \times 0 = -1.$

Look at the equation for ℓ_1:

$\ell_1 = \ell_0 + b_0 + \alpha u_1 = ... = 102.$

Step 2. Search for u_2, b_2, s_2, ℓ_2

$u_2 = 2$, $b_2 = 3$, $s_2 = -1$, $\ell_2 = 105$

Step 3. u_3, b_3, s_3, ℓ_3

How to make predictions using estimated ETS model?

True parameters: $b_0 = 2$, $\ell_0 = 100$, $s_0 = 1$, $s_{-1} = 2$, $s_{-2} = -2$, $\alpha = 0.5$, $\beta = 0.5$, $\gamma = 0.5$, $\sigma = 2$.

We have done seasonal decomposition up to y_{100}.

The last step gave me:

$b_{100} = 3$, $\ell_{100} = 400$, $s_{100} = 2$, $s_{99} = 1$, $s_{98} = -3$,
$s_{97} = -2$.

Q. Find point forecast for y_{101}. Find 95% predictive interval for y_{101}.

$y_{101} = \ell_{100} + b_{100} + s_{97} + u_{101} = 400 + 3 + (-2) + u_{101}$

Given my information:
$y_{101} = 401 + u_{101}$.

$\mathcal{F}_{100}$ — all available information up to time 100.

Conditional expected value:
$E(y_{101} \mid \mathcal{F}_{100}) = E_{100}(y_{101}) = 401 + 0$

Future noise u_{101} is independed of my information $\mathcal{F}_{100}$.

$E(u_{101} \mid \mathcal{F}_{100}) = E(u_{101}) = 0$

Point prediction: $E(u_{101} \mid \mathcal{F}_{100}) = E_{100}(u_{101}) = \hat{y}_{101|100} = 401$.

Let's do **two steps** ahead point prediction:

$y_{102} = \ell_{101} + b_{101} + s_{98} + u_{102} = (\ell_{100} + b_{100} + \alpha u_{101}) + (b_{100} + \beta u_{101}) + s_{98} + u_{102}$

$y_{102} = (400 + 3 + 0.5u_{101}) + (3 + 0.5u_{101}) + (-3) + u_{102} = 403 + u_{101} + u_{102}$

Point forecast:

$\hat{y}_{102|100} = E(y_{102} \mid \mathcal{F}_{100}) = 403$

Let's u_{101} is not zero!

We have decomposed y_{101} into two parts: predictable ($\ell_{100} + b_{100} + s_{97}$) and not predictable u_{101}.

My best prediction for u_{101} is zero: $E(u_{101} \mid \mathcal{F}_{100}) = 0$.

Interval forecasts!

Let's calculate conditional variance!

$$Var(y_{101} \mid \mathcal{F}_{100}) = Var(401 + u_{101} \mid \mathcal{F}_{100}) = Var(u_{101} \mid \mathcal{F}_{100}) = Var(u_{101}) = 2^2 = 4$$

Summary:

$$(y_{101} \mid \mathcal{F}_{100}) \sim N(401; 4)$$

95% predictive interval (critical value is 1.96)

$$[401 - 1.96\sqrt{4}; 401 + 1.96\sqrt{4}]$$

Normal Elephant is in $[E(Elephant) - 1.96\sqrt{Var(Elephant)}; E(Elephant) + 1.96\sqrt{Var(Elephant)}]$ with probability 0.95;

Let's move on to two steps ahead forecasts:
$y_{102} = 403 + u_{101} + u_{102}$

$E(y_{102} \mid \mathcal{F}_{100}) = 403$;

$$Var(y_{102} \mid \mathcal{F}_{100}) = Var(403 + u_{101} + u_{102} \mid \mathcal{F}_{100}) = Var(u_{101} + u_{102}) = 4 + 4 = 8$$

Summary:

$$(y_{102} \mid \mathcal{F}_{100}) \sim N(403; 8)$$

95% predictive interval (critical value is 1.96)

$$[403 - 1.96\sqrt{8}; 403 + 1.96\sqrt{8}]$$

Write likelihood functions

Versions of ETS model

ETS(AAA)

Additive Error

Additive Trend

Additive Seasonal

We can switch off some components

ETS(ANA)

Additive Error

No Trend: $b_t = 0$

Additive Seasonal

ETS(ANN)

Additive Error

No Trend: $b_t = 0$

No Seasonal: $s_t = 0$

We can consider Multiplicative components:

ETS(AAA)
$y_t = \ell_{t-1} + b_{t-1} + s_{t-4} + u_t$

ETS(MMM)
$y_t = \ell_{t-1} \cdot b_{t-1} \cdot s_{t-4} \cdot (1 + u_t)$

ETS(MAM)
$y_t = (\ell_{t-1} + b_{t-1}) \cdot s_{t-4} \cdot (1 + u_t)$

ETS(NNN): We never use it 😃
$\ell_t = \ell_{t-1}$
$y_t = \ell_{t-1}$

Train-test split

y_1, y_2, ..., y_{100}

Training sample: y_1, ..., y_{80}

Test sample y_{81}, ..., y_{100}.

Estimate my models (ETS(AAA), ETS(MAM)) using training sample.

Do predictions for test sample.

Metrics

You can compare models using:

Mean absolute error

$$MAE = \frac{1}{20}\sum_{t=81}^{100}|y_t - \hat{y}_t|$$

Mean squared error:

$$MSE = \frac{1}{20}\sum_{t=81}^{100}(y_t - \hat{y}_t)^2$$

Mean absolute percentage error

$$MAPE = \frac{1}{20}\sum_{t=81}^{100}|y_t - \hat{y}_t|/|y_t|$$

Symmetric mean absolute percentage error

$$SMAPE = \frac{1}{20}\sum_{t=81}^{100}|y_t - \hat{y}_t|/s_t$$

where $s_t = 0.5|y_t| + 0.5|\hat{y}_t|$.

Cross-validation