Exam 01 Solutions

1. Using Induction

   (base case) Letting $n = 1$ yields

$$\begin{aligned}\sum_{i=1}^{1} 4i - 3 &= 1(2(1) - 1)\\ 4(1) - 3 &= 1(2 - 1)\\ 4 - 3 &= 1(1)\\ 1 &= 1\end{aligned}$$

   as required.

   (induction hypothesis) Thus, assume for $n = k$ that

$$\sum_{i=1}^{k} 4i - 3 = k(2k - 1)$$

   is true.

   Now, $n = k + 1$ yields

$$\begin{aligned}\sum_{i=1}^{k+1} 4i - 3 &= \sum_{i=1}^{k} 4i - 3 + 4(k + 1) - 3\\ &= k(2k - 1) + 4(k + 1) - 3 \quad \text{by the induction hypothesis}\\ &= 2k^2 - k + 4k + 4 - 3\\ &= 2k^2 + 3k + 1\\ &= 2k^2 + 2k + k + 1\\ &= 2k(k + 1) + 1(k + 1)\\ &= (k + 1)(2k + 1)\\ &= (k + 1)(2k + 2 - 1)\\ &= (k + 1)[2(k + 1) - 1]\end{aligned}$$

   as required.

2. Direct Proof

$$\begin{aligned}\sum_{i=1}^{n} i(i + 2) &= \sum_{i=1}^{n} i^2 + 2i\\ &= \sum_{i=1}^{n} i^2 + 2\sum_{i=1}^{n} i\\ &= \frac{n(n + 1)(2n + 1)}{6} + \frac{2n(n + 1)}{2}\\ &= \frac{n(n + 1)(2n + 1) + 6n(n + 1)}{6}\\ &= \frac{n(n + 1)[2n + 1 + 6]}{6}\\ &= \frac{n(n + 1)(2n + 7)}{6}\end{aligned}$$

   as required.

3. The language $L = \{\texttt{001}, \texttt{010}, \texttt{100}\}$; hence, the states of the DFA need to keep track of the counts of 0s and 1s seen. Therefore, a DFA that recognizes the language is as follows

   - $Q = \{q_0, q1, q_2, q_3, q_4, q_5, q_6\}$
   - $\Sigma = \{\texttt{0}, \texttt{1}\}$
   - $\delta : Q \times \Sigma \longrightarrow Q$

	0	1
q_0	q_1	q_2
q_1	q_3	q_4
q_2	q_4	q_6
q_3	q_6	q_5
q_4	q_5	q_6
q_5	q_6	q_6
q_6	q_6	q_6

   - $q_0 = q_0$
   - $F = \{q_5\}$

   as required.

4. The proofs are

   a) $(q_0, q_1, q_3, q_0, q_2)$ rejected
   b) $(q_0, q_2, q_3, q_0, q_2, q_5, q_5)$ accepted
   c) $(q_0, q_2, q_5, q_5, q_5)$ accepted
   d) $(q_0, q_2, q_5, q_5, q_5, q_5, q_5)$ accepted
   e) $(q_0, q_2, q_3, q_1, q_0, q_1)$ rejected
   f) $(q_0, q_2, q_3, q_1, q_3, q_0, q_2)$ rejected
   g) $(q_0, q_1, q_3, q_1, q_3, q_0)$ accepted
   h) $(q_0, q_1, q_3, q_0, q_1, q_3, q_0)$ accepted

5. The DFA that recognizes the union of the languages will consist of states that have a purpose for both languages and a purpose for exactly one language. Additionally, a single-purpose state cannot transition to a dual-purpose state. Thus, a DFA that recognizes the union of the languages is as follows

   - $Q = \{q_0, q_1, q_2, q_3, q_4, q_5\}$
   - $\Sigma = \{\texttt{0}, \texttt{1}\}$
   - $\delta : Q \times \Sigma \longrightarrow Q$

	0	1
q_0	q_1	q_2
q_1	q_3	q_3
q_2	q_5	q_4
q_3	q_1	q_4
q_4	q_5	q_4
q_5	q_5	q_4

   - $q_0 = q_0$
   - $F = \{q_1, q_3, q_4\}$

   as required.