Closure Under the Regular Operations

The closure properties of regular languages under the operations of union, concatenation, and Kleene star can be formally proven through automata construction. In essence:

- The union of two regular languages is achieved by constructing an automaton that simultaneously evaluates both component automata and accepts any string recognized by either.
- The concatenation of two regular languages is represented by an automaton that begins evaluating the second automaton immediately after a string is accepted by the first.
- The Kleene star operation is realized by allowing an automaton to accept the empty string and to process strings recognized by the original automaton repeatedly.

Their proofs are:

- **Union Proof**

  Proof. Let the regular languages A and B be recognized by the NFAs $M_1 = (Q_1, \Sigma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma_2, \delta_2, q_2, F_2)$, respectively. Then, the NFA $N = (Q, \Sigma, \delta, q_0, F)$ where

  1. $Q = Q_1 \cup Q_2 \cup \{q_0\}$
  2. $\Sigma = \Sigma_1 \cup \Sigma_2$
  3. $\delta : Q \times \Sigma_\varepsilon \longrightarrow \wp(Q)$

  $$\delta(q,a) = \begin{cases} \delta_1(q,a) & \text{if } q \in Q_1 \wedge a \in \Sigma_{1\varepsilon} \\ \delta_2(q,a) & \text{if } q \in Q_2 \wedge a \in \Sigma_{2\varepsilon} \\ \{q_1, q_2\} & \text{if } q = q_0 \wedge a = \varepsilon \\ \emptyset & \text{if } (q = q_0 \wedge a \neq \varepsilon) \vee (q \in Q_1 \wedge a \notin \Sigma_1) \vee (q \in Q_2 \wedge a \notin \Sigma_2) \end{cases}$$

  4. $q_0 = q_0$
  5. $F = F_1 \cup F_2$

  recognizes $\mathrm{A} \cup \mathrm{B}$ as required. □

- **Concatenation Proof**

  Proof. Let the regular languages A and B be recognized by the NFAs $M_1 = (Q_1, \Sigma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma_2, \delta_2, q_2, F_2)$, respectively. Then, the NFA $N = (Q, \Sigma, \delta, q_0, F)$ where

  1. $Q = Q_1 \cup Q_2$
  2. $\Sigma = \Sigma_1 \cup \Sigma_2$
  3. $\delta : Q \times \Sigma_\varepsilon \longrightarrow \wp(Q)$

  $$\delta(q,a) = \begin{cases} \delta_1(q,a) & \text{if } (q \in Q_1 - F_1 \wedge a \in \Sigma_{1\varepsilon}) \vee (q \in F_1 \wedge a \in \Sigma_1) \\ \delta_1(q,a) \cup \{q_2\} & \text{if } q \in F_1 \wedge a = \varepsilon \\ \delta_2(q,a) & \text{if } q \in Q_2 \wedge a \in \Sigma_{2\varepsilon} \\ \emptyset & \text{if } (q \in Q_1 \wedge a \notin \Sigma_1) \vee (q \in Q_2 \wedge a \notin \Sigma_2) \end{cases}$$

  4. $q_0 = q_1$
  5. $F = F_2$

  recognizes $\mathrm{A} \circ \mathrm{B}$ as required. □

- **Star Proof**

  Proof. Let the regular language A be recognized by the NFA $M_1 = (Q_1, \Sigma_1, \delta_1, q_1, F_1)$. Then the NFA $N = (Q, \Sigma, \delta, q_0, F)$ where

  1. $Q = Q_1 \cup \{q_0\}$
  2. $\Sigma = \Sigma_1$
  3. $\delta : Q \times \Sigma_\varepsilon \longrightarrow \wp(Q)$

  $$\delta(q, a) = \begin{cases} \delta_1(q, a) & \text{if } (q \in Q_1 - F_1) \vee (q \in F_1 \wedge a \neq \varepsilon) \\ \delta_1(q, a) \cup \{q_1\} & \text{if } q \in F_1 \wedge a = \varepsilon \\ \{q_1\} & \text{if } q = q_0 \text{ and } a = \varepsilon \\ \emptyset & \text{if } q = q_0 \text{ and } a \neq \varepsilon \end{cases}$$

  4. $q_0 = q_0$
  5. $F = F_1 \cup \{q_0\}$

  recognizes A* as required. $\square$