Final Report

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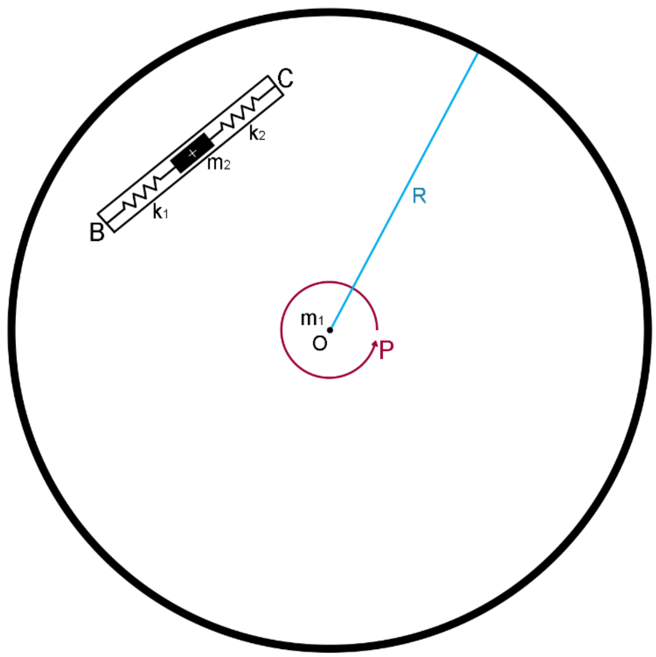
Aim: To find the Lagrangian, a scalar quantity, of a dynamical, mechanical system, model the system's dynamics as equations of motion and simulate the dynamics under different conditions.

System introduction:

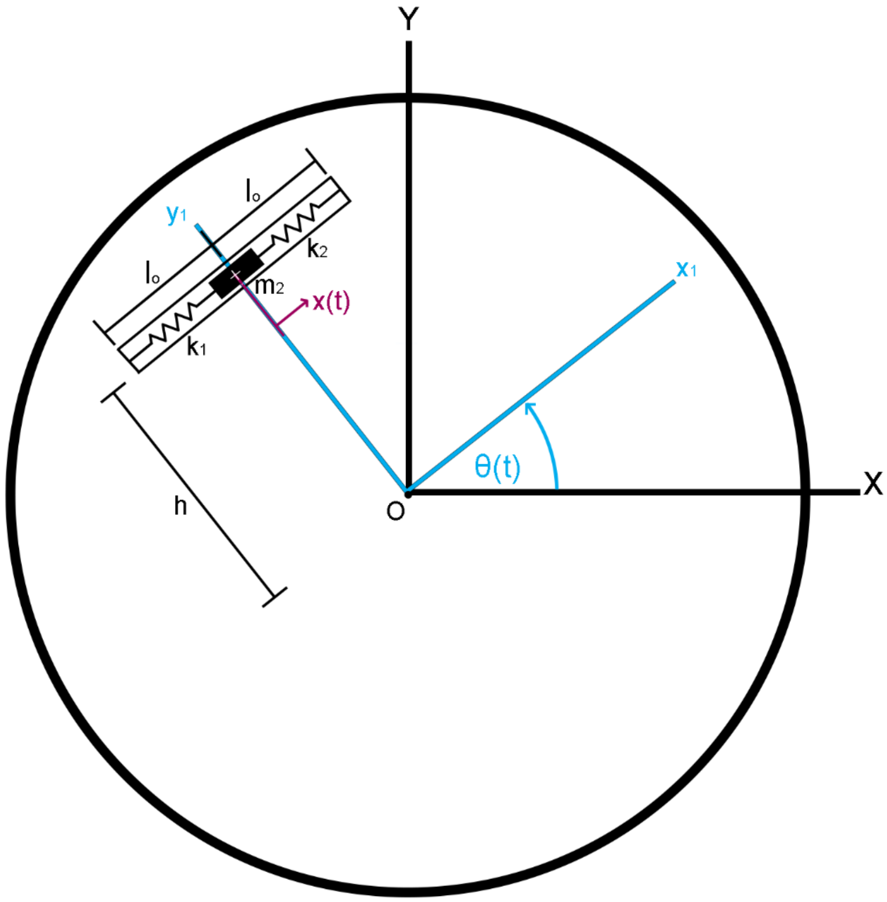
A disk platform rotates in the horizontal plane about a bearing at point O due to an applied torque, P. The mass of the platform is m_1 and its radius is R. A shuttle moves up and down a channel, BC, under the restraint of two springs with spring constants k. The mass of the shuttle is m_2.



To model the dynamics of this system we will:

1. Define coordinate systems for the mechanical system.
2. Develop constraint equations or system dynamics that incorporate the constraints.
3. Identify the degrees of freedom of the system.
4. Find the kinetic energy of the system.
5. Find the potential energy of the system.
6. Find the Lagrangian of the system.
7. Find virtual work.
8. Develop the equations of motion.
9. Model the collisions.
10. Simulate motion of the system

01 COORDINATE FRAMES



We will attached frame {1} to the disk, as such:

$$ {}^{0}_{1}R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

02 CONSTRAINT EQUATIONS

We will treat the shuttle as a point mass, located at G_2.

With the constraint equations expressed in {1}:

$$ y = h \quad where \quad h = constant $$

$$ and $$

$$ -l_o \leq x \leq l_o $$

The shuttle is constrained to move along the x-axis of {1} only, giving:

$$^{1}\boldsymbol{r}_{OG_2} = \begin{bmatrix} x \\ h \\ 0 \end{bmatrix}$$

$$\therefore\ ^{0}\boldsymbol{r}_{OG_2} = {}^{0}_{1}R\ ^{1}\boldsymbol{r}_{OG_2} = \begin{bmatrix} xcos(\theta) - hsin(\theta) \\ xsin(\theta) + h\cos(\theta) \\ 0 \end{bmatrix}$$

03 DEGREES OF FREEDOM

The system requires two independent variables, as such we have 2 DOF

$$x = f_1(t) \quad and \quad \theta = f_2(t)$$

04 KINETIC ENERGY

Since our problem contains two bodies (body 1 is the disk and body 2 is the shuttle) we can analyse the kinetic energy for each body separately. Our total kinetic energy is therefore:

$$T_{total} = T_{body\ 1} + T_{body\ 2} = \frac{1}{2}\ {}^{1}\boldsymbol{\omega}_1{}^{T} I_{G_1}\ {}^{1}\boldsymbol{\omega}_1 + \frac{1}{2} m_2\ {}^{1}\dot{\boldsymbol{r}}_{OG_2}{}^{T}\ {}^{1}\dot{\boldsymbol{r}}_{OG_2}$$

Body 1

Body 1 is undergoing purely rotational motion as such:

$$T_{body\ 1} = \frac{1}{2}\ {}^{1}\boldsymbol{\omega}_1{}^{T} I_{G_1}\ {}^{1}\boldsymbol{\omega}_1$$

The inertial tensor for a disk and angular velocity in {1} will be:

$$I_{G_1} = \begin{bmatrix} \frac{m_1 R^2}{4} & 0 & 0 \\ 0 & \frac{m_1 R^2}{4} & 0 \\ 0 & 0 & \frac{m_1 R^2}{2} \end{bmatrix} \quad and \quad {}^{1}\boldsymbol{\omega}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$$

$$\therefore\ T_{body\ 1} = \frac{m_1 \dot{\theta}^2 R^2}{4}$$

Body 2

Body 2 is defined as a point mass. Therefore it has no internal rotational kinetic energy about G_2 .

Its contribution to the kinetic energy will be:

$$T_{body\ 2} = \frac{1}{2} m_2\ {}^{1}\dot{\boldsymbol{r}}_{OG_2}{}^{T}\ {}^{1}\dot{\boldsymbol{r}}_{OG_2}$$

The velocity of the shuttle in {1} is:

$${}^{1}\dot{\boldsymbol{r}}_{OG_2} = {}^{1}\boldsymbol{r}'_{OG_2} + {}^{1}\boldsymbol{\omega}_1 \times {}^{1}\boldsymbol{r}_{OG_2} = \begin{bmatrix} \dot{x} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} \times \begin{bmatrix} x \\ h \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \dot{x} - \dot{\theta} h \\ \dot{\theta} x \\ 0 \end{bmatrix}$$

$$\therefore\ T_{body\ 2} = \frac{1}{2} m_2 \begin{bmatrix} \dot{x} - \dot{\theta} h \\ \dot{\theta} x \\ 0 \end{bmatrix}^{T} \begin{bmatrix} \dot{x} - \dot{\theta} h \\ \dot{\theta} x \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} m_2 \left(\left(\dot{x} - \dot{\theta} h\right)^2 + \left(\dot{\theta} x\right)^2 \right)$$

$$T_{total} = \frac{m_1 \dot{\theta}^2 R^2}{4} + \frac{1}{2} m_2 \left(\left(\dot{x} - \dot{\theta} h\right)^2 + \left(\dot{\theta} x\right)^2 \right)$$

05 POTENTIAL ENERGY

The potential energy of the system is supplied by two springs attached to m_2, and the potential displacement from an equilibrium position we will define as $x_{eq} = 0$.

As such we have:

$$V_{total} = \tfrac{1}{2} k_1 x^2 + \tfrac{1}{2} k_2 (-x)^2$$

$$= \frac{1}{2} (k_1 + k_2) x^2$$

06 LAGRANGIAN

$$L = T - V$$

Given $k = k_1 = k_2$.

$$V_{total} = kx^2$$

And the Lagrangian is:

$$\therefore\ L = \frac{m_1 \dot{\theta}^2 R^2}{4} + \frac{1}{2} m_2 \left(\left(\dot{x} - \dot{\theta} h\right)^2 + \left(\dot{\theta} x\right)^2 \right) - kx^2$$

07 VIRTUAL WORK

We will find an expression for the virtual work in terms of our generalized coordinates. This will be used to obtain the generalized impressed forces which will be used in the Lagrange equations.

$$\delta W = \sum \vec{F_i} \cdot \delta \vec{r_i} + \sum \vec{M_i} \cdot \delta \theta_i = P \cdot \delta \theta$$

$$\delta W = P \cdot \delta \theta = \sum_{k=1}^{n} Q_k \cdot \delta q_k$$

$$Q_x = 0 \quad and \quad Q_\theta = P$$

08 EQUATIONS OF MOTION:

We will develop the equations of motion for the dynamical system using the Lagrangian and Lagrange's equation.

For x:

$$\frac{\partial L}{\partial \dot{x}} = m_2(\dot{x} - h\dot{\alpha})$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m_2(\ddot{x} - h\ddot{\alpha})$$

$$\frac{\partial L}{\partial x} = m_2 x \dot{\alpha}^2 - 2kx$$

For θ:

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{m_1 r^2}{2}\dot{\theta} - hm_2(\dot{x} - h\dot{\alpha}) + m_2 x^2 \dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{m_1 r^2}{2}\ddot{\theta} - hm_2\left(\ddot{x} - h\ddot{\theta}\right) + m_2 x^2 \ddot{\theta} + 2m_2 x\dot{\theta}\dot{x}$$

$$\frac{\partial L}{\partial \theta} = 0$$

Equation of Motion for x

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = Q_x$$

$$m_2\left(\ddot{x} - h\ddot{\theta}\right) - m_2 x\dot{\theta}^2 + 2kx = 0$$

$$\therefore \;\; \ddot{x} - h\ddot{\theta} = x\dot{\theta}^2 + \frac{2kx}{m_2} \ldots (i)$$

Equation of Motion for θ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = Q_\theta$$

$$\frac{m_1 r^2}{2}\ddot{\theta} - hm_2\left(\ddot{x} - h\ddot{\theta}\right) + m_2 x^2 \ddot{\theta} + 2m_2 x\dot{\theta}\dot{x} = P$$

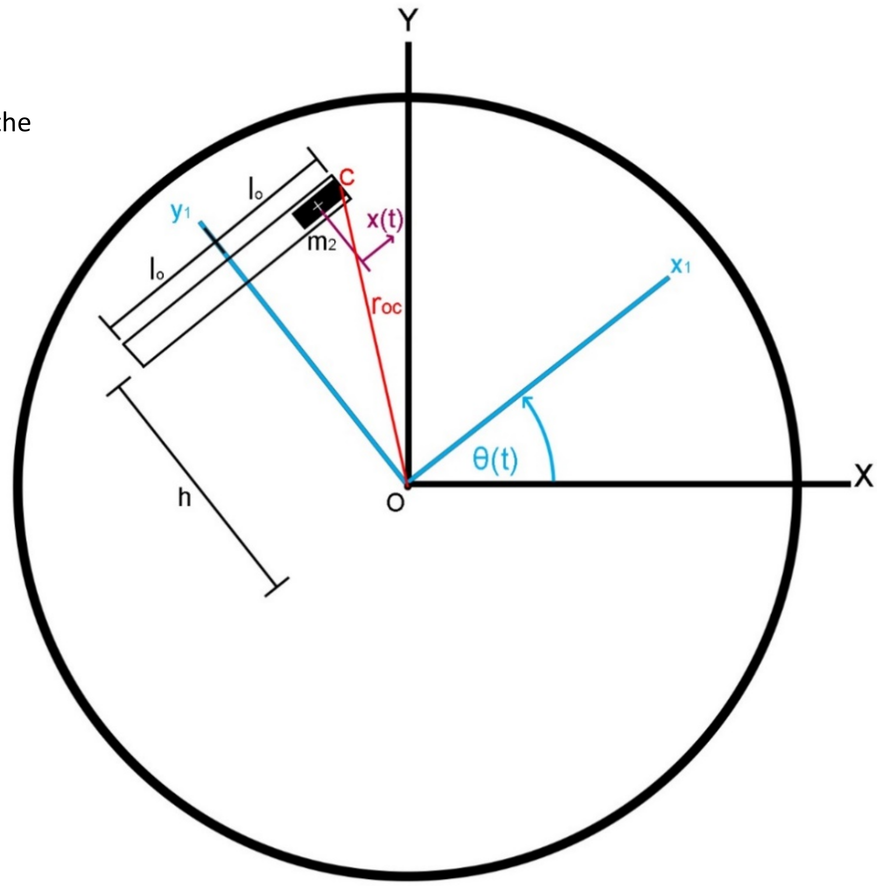
$$\therefore \;\; -hm_2\ddot{x} + \ddot{\theta}\cdot\left(\frac{m_1 r^2}{2} + hm_2 h + m_2 x^2\right) = P - 2m_2 x\dot{\theta}\dot{x} \ldots (ii)$$

We can express the equations of motion as the following system of equations:

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 1 & -h \\ -hm_2 & \left(\frac{m_1 r^2}{2} + hm_2 h + m_2 x^2\right) \end{bmatrix}^{-1} \cdot \begin{bmatrix} x\dot{\theta}^2 + \frac{2kx}{m_2} \\ P - 2m_2 x\dot{\theta}\dot{x} \end{bmatrix}$$

09 COLLISION

When the shuttle reaches the end of the channel it collides with the disk.



The analysis will be performed in {1}.

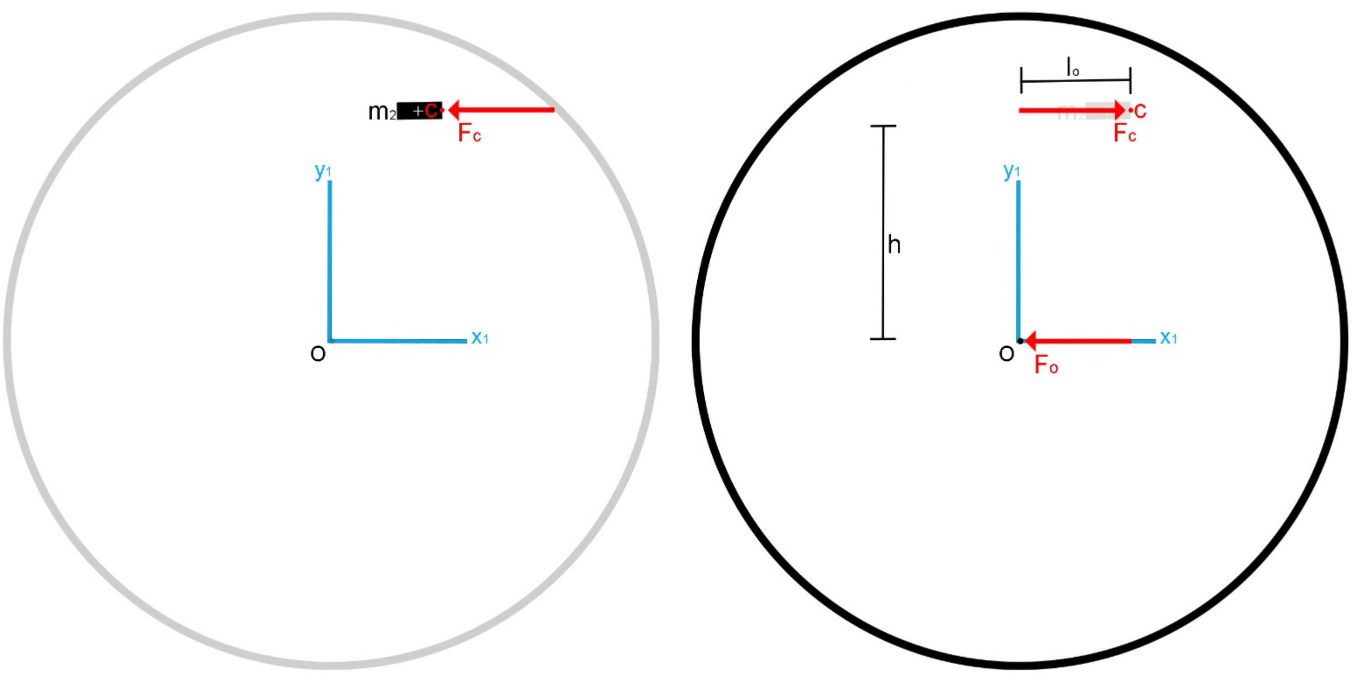


Figure: The above left diagram shows the impulsive forces acting on the shuttle. The above right diagram shows the impulsive forces acting on the disk.

The forces acting on the disk are:

$$ {}^{1}F_c = \begin{bmatrix} F_C \\ 0 \\ 0 \end{bmatrix} \quad {}^{1}F_O = \begin{bmatrix} -F_O \\ 0 \\ 0 \end{bmatrix} $$

Collisions only occur when the shuttle is at the end of the channel.

$$ x(t_{collision}) = \mp l_o $$

$$ left\ side\colon\ {}^{1}r_{oc} = \begin{bmatrix} -l_o \\ h \\ 0 \end{bmatrix},\ right\ side:\ {}^{1}r_{oc} = \begin{bmatrix} l_o \\ h \\ 0 \end{bmatrix} $$

The velocity of the collision point C_2 on the shuttle is given by:

$$ {}^{1}\dot{r}_{OC_2} = \begin{bmatrix} \dot{x} - \dot{\theta}h \\ \dot{\theta}x \\ 0 \end{bmatrix} $$

On the disk, C_1:

$$ {}^{1}\dot{r}_{OC_1} = {}^{1}r'_{OC_1} + {}^{1}w_1 \times {}^{1}r_{OC_1} $$

$$ = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} \times \begin{bmatrix} l_o \\ h \\ 0 \end{bmatrix} $$

$$ = \begin{bmatrix} -h\dot{\theta} \\ l_o\dot{\theta} \\ 0 \end{bmatrix} $$

The virtual work performed by these impulsive forces is given by:

$$ \partial W = \sum \hat{F}_i \cdot \partial \vec{r}_i = \sum \hat{Q}_k \cdot \partial q_k $$

Only impulsive forces are considered. We have two generalized cords, $x(t), \theta(t)$.

So:

$$ {}^{1}\dot{r}_{OC_1} = \begin{bmatrix} -h\dot{\theta} \\ l_o\dot{\theta} \\ 0 \end{bmatrix} \xrightarrow{yields} {}^{1}\partial r_{OC_1} = \begin{bmatrix} -h \cdot \partial\theta \\ l_o \cdot \partial\theta \\ 0 \end{bmatrix} $$

$$ {}^{1}\dot{r}_{OC_2} = \begin{bmatrix} \dot{x} - \dot{\theta}h \\ \dot{\theta}x \\ 0 \end{bmatrix} \xrightarrow{yields} {}^{1}\partial r_{OC_2} = \begin{bmatrix} \partial x - h \cdot \partial\theta \\ x \cdot \partial\theta \\ 0 \end{bmatrix} $$

$$\therefore\ \partial W = \begin{bmatrix} -F_C \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \partial x - h \cdot \partial\theta \\ x \cdot \partial\theta \\ 0 \end{bmatrix} + \begin{bmatrix} F_C \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -h \cdot \partial\theta \\ l_o \cdot \partial\theta \\ 0 \end{bmatrix} + \begin{bmatrix} F_O \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= -F_C \cdot (\partial x - h \cdot \partial\theta)\ +\ F_C \cdot (-h \cdot \partial\theta)$$

$$= -F_C \cdot \partial x + F_C \cdot h \cdot \partial\theta\ - F_C \cdot h \cdot \partial\theta$$

$$= -F_C \cdot \partial x + 0 \cdot \partial\theta$$

We will also find the generalized momentum.

$$P_k(t) = \frac{\partial L}{\partial \dot{q_k}} = \frac{\partial T}{\partial \dot{q_k}}$$

$$\frac{\partial L}{\partial \dot{x}} = m_2(\dot{x} - h\dot{\theta})$$

$$and$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{m_1 r^2}{2}\dot{\theta} - hm_2(\dot{x} - h\dot{\theta}) + m_2 x^2 \dot{\theta}$$

We will use the generalized impulsive force momentum equation.

$$P_x(t^+) -\ P_x(t^-) = -F_c$$

$$P_\theta(t^+) -\ P_\theta(t^-) = 0$$

Which gives:

$$m_2(\dot{x}^+ - h\dot{\theta}^+)\ =\ -F_c + m_2(\dot{x}^- - h\dot{\theta}^-) \ldots (i)$$

$$\frac{m_1 r^2}{2}\dot{\theta}^+ - hm_2(\dot{x}^+ - h\dot{\theta}^+) + m_2 x^2 \dot{\theta}^+ = \frac{m_1 r^2}{2}\dot{\theta}^- - hm_2(\dot{x}^- - h\dot{\theta}^-) + m_2 x^2 \dot{\theta}^- \ldots (ii)$$

We have, 3 unknowns: $\dot{\theta}^+$, $\dot{x}^+$ and F_c , so a third equation is required. The conservation equation gives

$$e =\ -\frac{(\dot{r}^+_{OC_2} -\ \dot{r}^+_{OC_1}) \cdot {}^1\hat{x}}{(\dot{r}^-_{OC_2} -\ \dot{r}^-_{OC_1}) \cdot\ {}^1\hat{x}}$$

$$\therefore\ e =\ -\frac{\left(\begin{bmatrix} \dot{x}^+ -\ \dot{\theta}^+ h \\ \dot{\theta}^+ x^+ \\ 0 \end{bmatrix} - \begin{bmatrix} -h\dot{\theta}^+ \\ l_o \dot{\theta}^+ \\ 0 \end{bmatrix}\right) \cdot {}^1\hat{x}}{\left(\begin{bmatrix} \dot{x}^- -\ \dot{\theta}^- h \\ \dot{\theta}^- x^- \\ 0 \end{bmatrix} - \begin{bmatrix} -h\dot{\theta}^- \\ l_o \dot{\theta}^- \\ 0 \end{bmatrix}\right) \cdot\ {}^1\hat{x}} = -\frac{(\dot{x}^+ -\ \dot{\theta}^+ h + h\dot{\theta}^+\ )}{(\dot{x}^- -\ \dot{\theta}^- h + h\dot{\theta}^-\ )} = -\frac{\dot{x}^+}{\dot{x}^-}$$

$$\therefore \dot{x}^+ =\ -e\dot{x}^- \ldots (iii)$$

$(iii) \rightarrow (ii)$ gives:

$$\dot{\theta}^{+} = \dot{\theta}^{-} - \frac{hm_2(e+1)}{\left(\frac{m_1 r^2}{2} + hm_2 h + m_2 l_o^2\right)} \cdot \dot{x}^{-}$$

Now (iii) and $(ii) \rightarrow (i)$ gives:

$$F_c = m_2(1+e)\left(1 - \frac{m_2 h^2}{\left(\frac{m_1 r^2}{2} + hm_2 h + m_2 l_o^2\right)}\right) \cdot \dot{x}^{-}$$

For simulation our Collision model becomes:

$$X(t^{+}) = \begin{bmatrix} x(t^{+}) \\ \theta(t^{+}) \\ \dot{x}(t^{+}) \\ \dot{\theta}(t^{+}) \end{bmatrix} = \begin{bmatrix} x(t^{-}) \\ \theta(t^{-}) \\ -e\dot{x}(t^{-}) \\ \dot{\theta}(t^{-}) - \frac{hm_2(e+1)}{\left(\frac{m_1 r^2}{2} + hm_2 h + m_2 l_o^2\right)} \cdot \dot{x}(t^{-}) \end{bmatrix}$$

10 SIMULATION

We define a vector to describe the state of the system at a given time as follows:

$$X(t) = \begin{bmatrix} x(t) \\ \theta(t) \\ \dot{x}(t) \\ \dot{\theta}(t) \end{bmatrix} \quad and \quad \dot{X}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{\theta}(t) \\ \ddot{x}(t) \\ \ddot{\theta}(t) \end{bmatrix}$$

With our equations of motion developed in section 08, we have explicit expressions for:

$$\ddot{x}(t) = f_1\big(\, x(t),\ \theta(t),\ \dot{x}(t),\ \dot{\theta}(t),\ P(t)\,\big) = f_1(\,X(t)\,)$$

$$\ddot{\theta}(t) = f_2\big(\, x(t),\ \theta(t),\ \dot{x}(t),\ \dot{\theta}(t)\,,\ P(t)\big) = f_2(\,X(t)\,)$$

A function to $P(t)$ must be defined when $\dot{X}(t)$ is evaluated.

The system can be simulated by wrapping this in a time loop such that each consecutive state is evaluated using:

$$X(t+dt) = \begin{bmatrix} x(t) \\ \theta(t) \\ \dot{x}(t) \\ \dot{\theta}(t) \end{bmatrix} + \begin{bmatrix} \dot{x}(t) \\ \dot{\theta}(t) \\ \ddot{x}(t) \\ \ddot{\theta}(t) \end{bmatrix} \cdot dt$$

To simulate this system, we then require that an initial state, time limits and a step size be defined. That is:

$$X_O = \begin{bmatrix} x(t_O) \\ \theta(t_O) \\ \dot{x}(t_O) \\ \dot{\theta}(t_O) \end{bmatrix}, \qquad t_O < t_f \qquad and \qquad dt \neq 0$$

Note, setting $dt < 0$ and $t_f < t_O$ runs the simulation backwards.

In order to accommodate the inequality constraint:

$$-l_o \leq x(t) \leq l_o$$

a simple trigger is set up to break the simulation loop if a collision between the shuttle and either end of the rail is detected:

$$|x(t)| - l_o \geq 0$$

When this occurs, the simulation is stopped and a new state is evaluated based on the collision model developed in part 09:

$$X(t^+) = \begin{bmatrix} x(t^+) \\ \theta(t^+) \\ \dot{x}(t^+) \\ \dot{\theta}(t^+) \end{bmatrix} = \begin{bmatrix} x(t^-) \\ \theta(t^-) \\ -e\dot{x}(t^-) \\ \dot{\theta}(t^-) - \dfrac{hm_2(e+1)}{\left(\dfrac{m_1 r^2}{2} + hm_2 h + m_2 l_o^2\right)} \cdot \dot{x}(t^-) \end{bmatrix}$$

The simulation is then re-initiated with $X(t^+)$ as the state.

Two simulations and the results are described below. For all simulations the model constants are set as:

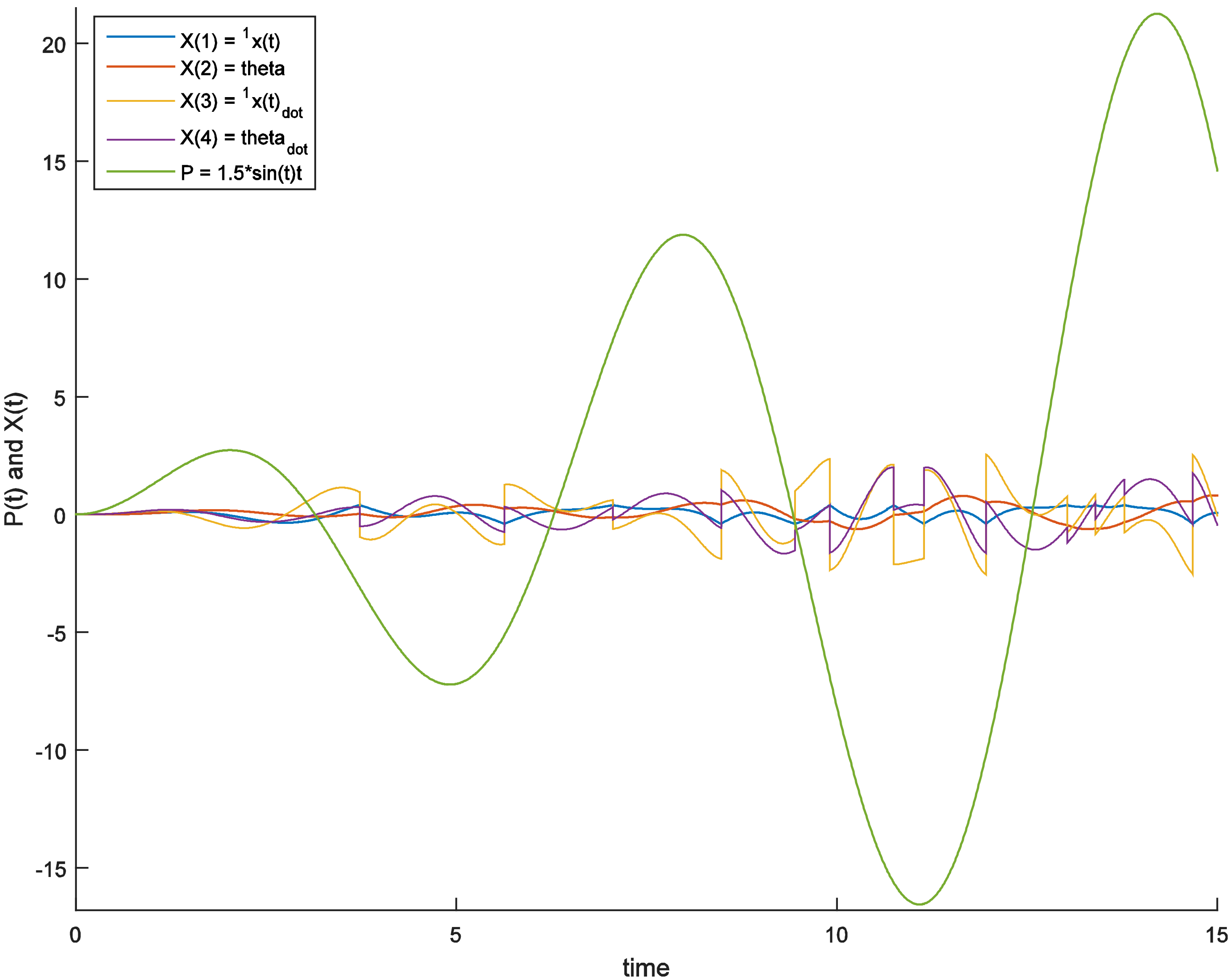
$$e = 1 \quad k = 2 \quad m_1 = m_2 = 1 \quad r = 1.5 \quad h = 1.2 \quad l_O = \; 0.5$$

Simulation 01:

$$X_{O_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad t_{O_1} = 0 \quad t_{f_1} = 15 \quad dt_1 = 0.01$$

$$and$$

$$P_1 = 0.6 \cdot \sin(2t) \cdot t$$



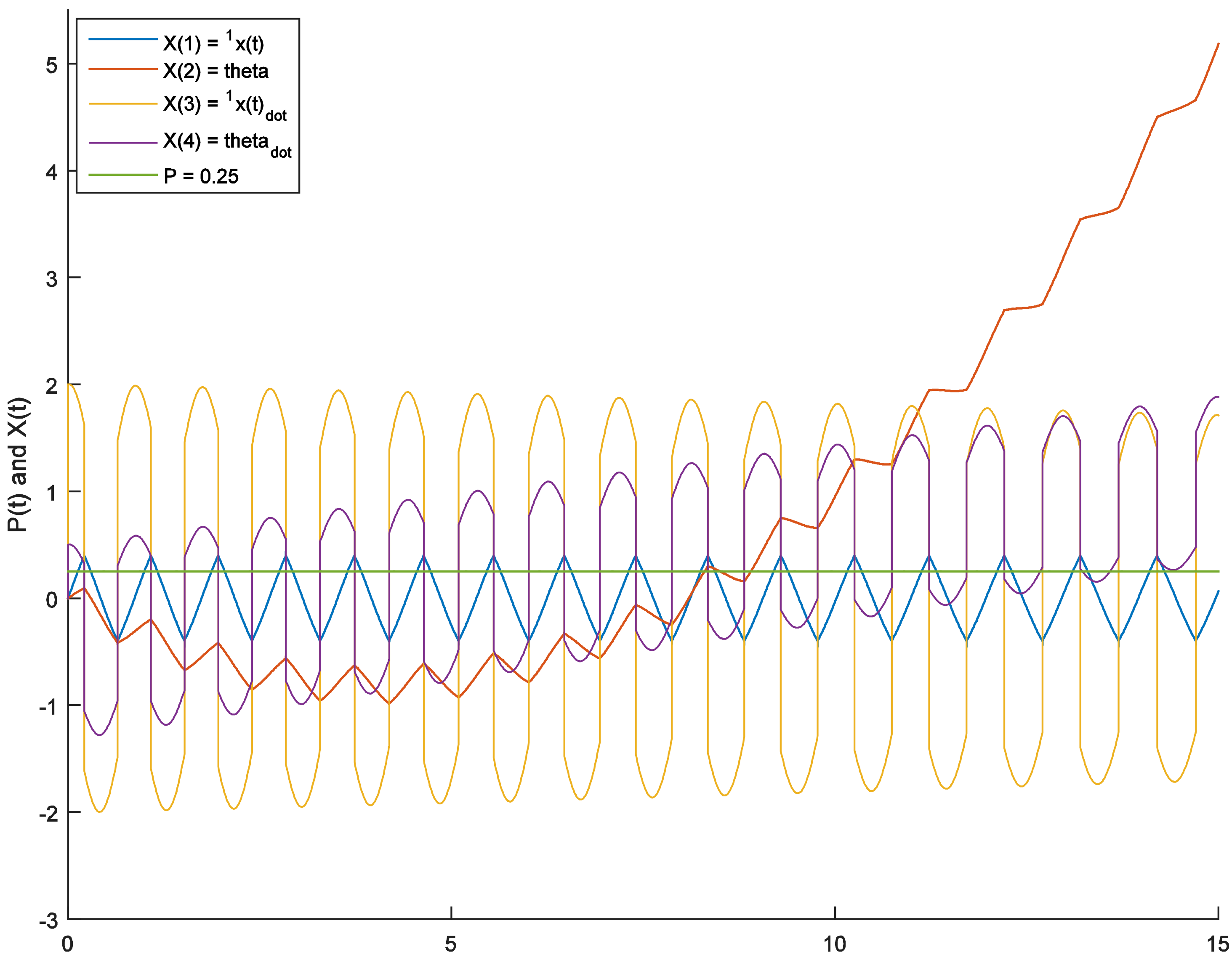
Run: simulation_01.mp4

Simulation 02:

$$X_{O_2} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0.5 \end{bmatrix} \qquad t_{O_2} = 0 \quad t_{f_2} = 15 \quad dt_2 = 0.01$$

$$and$$

$$P_2 = 0.25$$



Run: simulation_02.mp4

The simulation exhibits the expected dynamics, momentum is transferred from the shuttle to the disk and vise versa during a collision. Observe below that across a collision $\dot{\theta}(t)$ and $\dot{x}(t)$ experience a sudden change while θ(t) and $x(t)$ are continuous.

Conditions for simulation without collision:

$$X_O = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad t_O = 0 \quad t_f = 5 \quad dt = 0.01$$

$$and$$

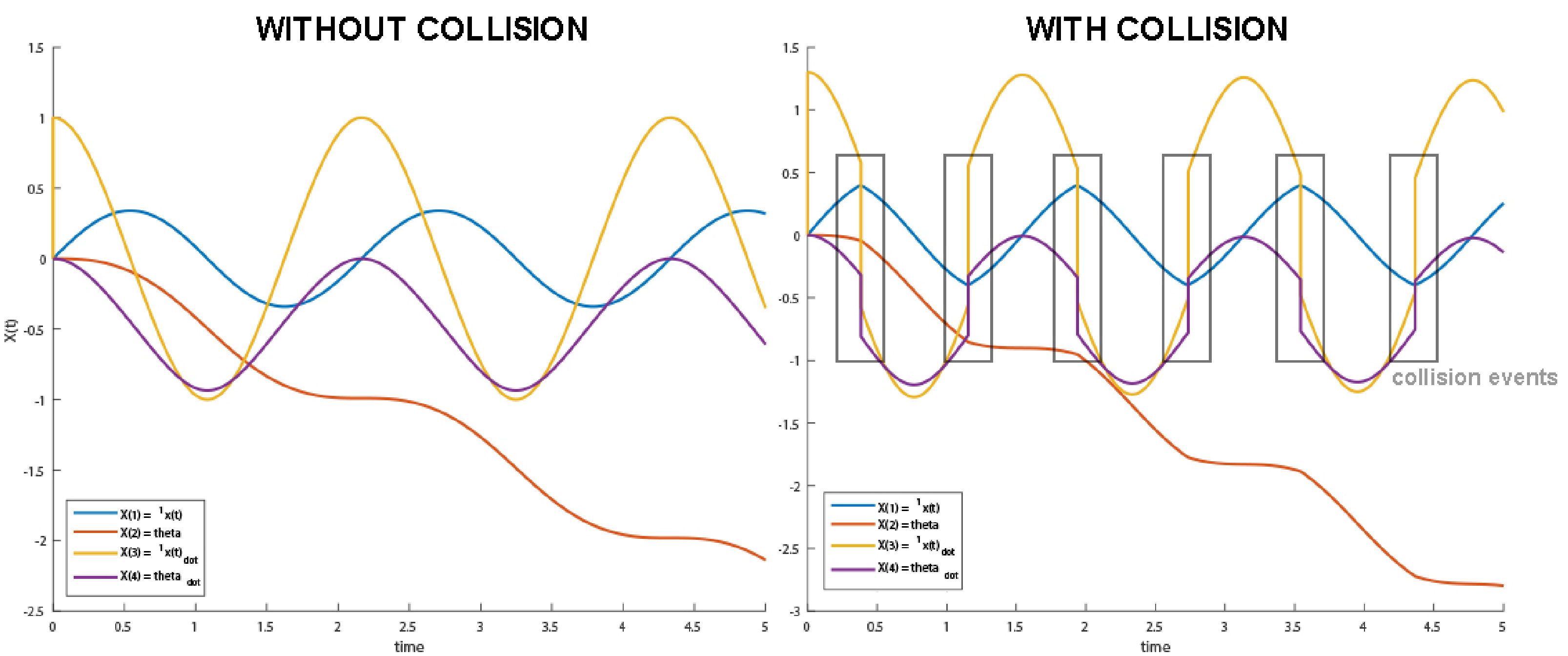
$$P = 0$$

Conditions for simulation with collisions:

$$X_O = \begin{bmatrix} 0 \\ 0 \\ 1.3 \\ 0 \end{bmatrix} \qquad t_O = 0 \quad t_f = 5 \quad dt = 0.01$$

$$and$$

$$P = 0$$



Run: shuttle_velocity_no_collision.mp4 and shuttle_velocity_with_collision.mp4

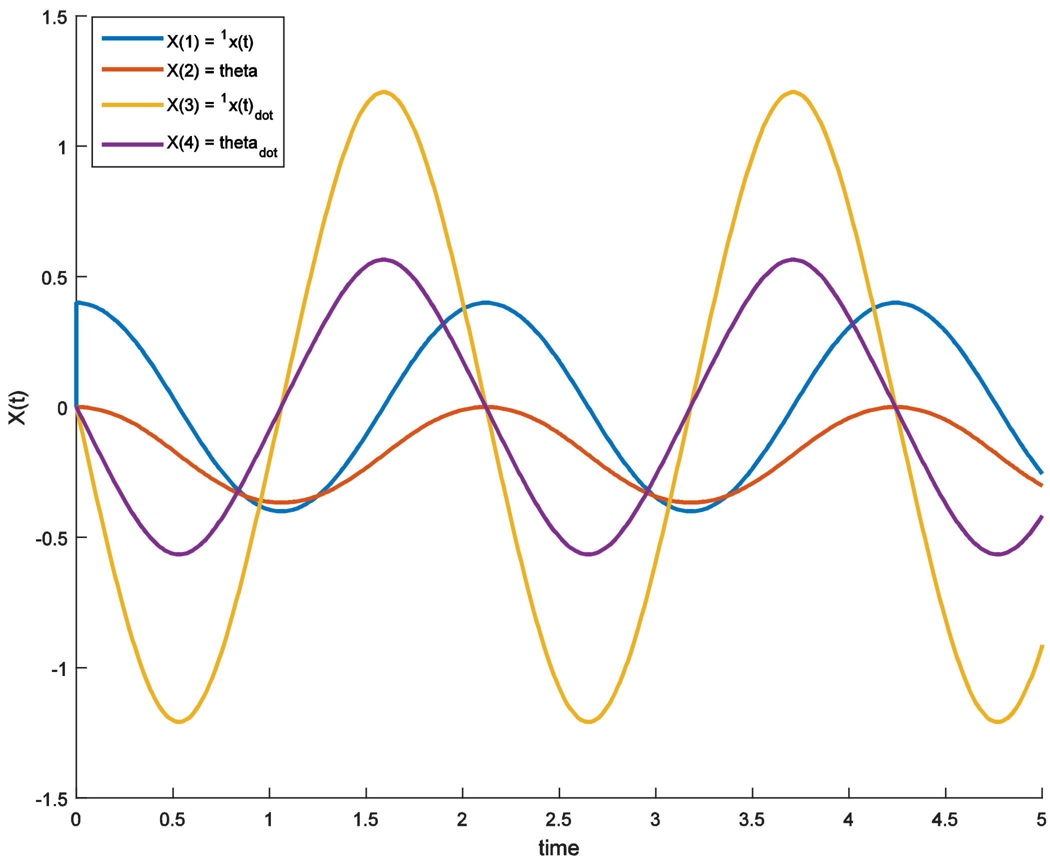
Friction however was not modeled, as such it can be observed below that the system will not lose energy over time.

Conditions for simulation:

$$X_O = \begin{bmatrix} 0.4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad t_O = 0 \quad t_f = 5 \quad dt = 0.01$$

$$and$$

$$P = 0$$



Run: energy_concervation.mp4