Esercizi di ripasso

1. Sia $U \subset \mathbb{R}^4$ l'insieme dato dalle equazioni $x + y - 2z = w - 3y + x = 0$. Si consideri l'operatore lineare $T : \mathbb{R}^4 \to \mathbb{R}^4$ tale che $T(x, y, z, w) = (x + y, x + z, y + w, z - w)$.
i) Determinare una base di $U$ e la sua dimensione.
ii) Dire se $T$ è biunivoco.
iii) Determinare una base di $T(U)$.

• $U$ e' dato dalle equazioni $\begin{array}{l} x = -y + 2z \\ x = 3y - w \end{array}$ ; $x = -y + 2z + 3y - w$

$$f(x, y, z, w)(-y + 2z + 3y - w, y, z, w)$$

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\0\\1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\0\\0\\1\end{pmatrix}$$

$$\begin{matrix}0 & 2 & 2 & -1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{matrix} \qquad \begin{matrix}2 & 2 & -1\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix} \qquad 2R_2 - R_1 \begin{matrix}2 & 2 & -1\\ 0 & -2 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}$$

$$2R_3 + R_2 \begin{vmatrix}2 & 2 & -1\\ 0 & -2 & 1\\ 0 & 0 & 1\\ 0 & 0 & 1\end{vmatrix}$$

sono 3 vettori linearmente indipendenti dunque costituiscono già una base per $U$

$\dim U (3)$

$$\begin{array}{c|ccc} x & 2 & 2 & -1\\ y & 0 & -2 & 1\\ z & 0 & 0 & 1\\ w & 0 & 0 & 1\end{array} \qquad \begin{cases} 2x = 0\\ 2x - 2y = 0\\ -x + y + z + w = 0\end{cases}$$

$x = 0$
$y = 0$
$z = -w$

$w \begin{vmatrix}0\\0\\-1\\1\end{vmatrix}$ cost. una base

• $T$ e' già un'applicazione lineare

$M_T$ rispetto a $C$

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \quad \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$$

$$M_T = \begin{vmatrix}1 & 1 & 0 & 0\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & -1\end{vmatrix}$$

$$
\begin{array}{l}
R_2 - R_1 \\
R_3 + R_2 \\
R_4 - R_3
\end{array}
\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 \end{vmatrix}
$$

$\dim \operatorname{Im}(M_T) = 4$

$\dim \operatorname{Ker}(M_T) = 4 - 4 = 0$

$T$ è biunivoco

$$
T(v) \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix}
\begin{vmatrix} 2 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}
$$

$$
\begin{vmatrix}
(1\cdot 2)+(1\cdot 1)+0+0 & (1\cdot 2)+0+0+0 & (1)\cdot(-1)+0+0+0 \\
(1\cdot 2)+0+0+0 & (1\cdot 2)+0+(1\cdot 1)+0 & (1)\cdot(-1)+0+0+0 \\
0+(1\cdot 2)+0+0 & 0+0+0+0 & 0+0+0+1 \\
0+0+0+0 & 0+0+(1\cdot 1)+0 & 0+0+0+(-1)(1)
\end{vmatrix}
$$

$$
\begin{vmatrix} 3 & 2 & -1 \\ 2 & 3 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}
\quad \frac{3}{2} R_2 - R_1 \quad
\begin{vmatrix} 3 & 2 & -1 \\ 0 & 5/2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}
$$

$$
\frac{3}{2} R_3 - R_1 \quad
\begin{matrix} 3 & 2 & -1 \\ 0 & 5/2 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -1 \end{matrix}
\qquad R_2 + 2R_4 \quad
\begin{vmatrix} 3 & 2 & -1 \\ 0 & 5/2 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}
$$

$$\frac{5}{2}R_4 - R_2 \quad \begin{vmatrix} 3 & 2 & -1 \\ 0 & 5/2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 7/2 \end{vmatrix}$$

sono 3 vettori linearmente indipendenti quindi costituiscono una base

$$\begin{vmatrix} x \\ y \\ z \\ w \end{vmatrix} \begin{vmatrix} 3 & 2 & -1 \\ 0 & 5/2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 7/2 \end{vmatrix} \qquad \begin{cases} 3x = 0 \\ 2x + \frac{5}{2}y = 0 \\ -x - z + \frac{7}{2}w = 0 \end{cases}$$

$$\begin{cases} x = 0 \\ y = 0 \\ z = \frac{7}{2}w \end{cases} \qquad \left\{ \begin{pmatrix} 0 \\ 0 \\ 7/2\, w \\ w \end{pmatrix} : w \in \mathbb{R} \right\}$$

$$w \begin{vmatrix} 0 \\ 0 \\ 7/2 \\ 1 \end{vmatrix}$$

costituisce una base per $T(U)$