Assumption 4.1. The following assumptions are imposed on the cost function $\mathcal{J}$

1. There exist $\tilde{\theta}$ such that $\mathcal{J}(\tilde{\theta}) = \inf_{\theta} \mathcal{J}(\theta) =: \underline{J}$. Also, it is bounded from above by $\sup \mathcal{J} \leq \overline{J}$.

2. The cost function $\mathcal{J}$ is locally Lipschitz continuous $\|\mathcal{J}[\theta_1] - \mathcal{J}[\theta_2]\| \leq L_J(\|\theta_1\| + \|\theta_2\|)\|\theta_1 - \theta_2\|$.

3. There exists a constant $c_{\mathcal{J}} > 0$ such that $\mathcal{J}(\theta) - \underline{J} \leq c_{\mathcal{J}}(1 + \|\theta\|^2)$.

4. There exist $\delta_J, R_0, \eta, \mu > 0$ such that $\|\theta - \tilde{\theta}\| \leq \frac{(\mathcal{J} - \underline{J})^{\mu}}{\eta}$, for all $\theta \in B_{\theta,R_0}(\tilde{\theta}) = \{\theta : \|\theta - \tilde{\theta}\| \leq R_0\}$, and $\mathcal{J}(\theta) - \underline{J} > \delta_J$ for all $\theta \in \left(B_{\theta,R_0}(\tilde{\theta})\right)^c$.

5. The parameters we choose $\sigma(t)$ has upper and lower bound $\underline{\sigma} \leq \sigma(t) \leq \overline{\sigma}$.

Lemma D.2 Under Assumption 4.1, $\forall r > 0$, we define $J_r := \sup_{\theta \in B_{\theta,r}(\tilde{\theta})} \mathcal{J}(\theta)$. Then $\forall r \in [0, R_0]$ and $q > 0$ such that $(q + J_r - \underline{J})^{\mu} \leq \delta_J$, we have

$$\|\mathcal{M}_{\beta}[\mu] - \tilde{\theta}\| \leq \frac{(q + J_r - \underline{J})^{\mu}}{\eta} + \frac{\exp(-\beta q)}{\rho(B_{\theta,r}(\tilde{\theta}))} \int \|\theta - \tilde{\theta}\| \mathrm{d}\rho(\theta, \omega).$$

Proof. Let $\tilde{r} = \frac{(q+J_r-\underline{J})^{\mu}}{\eta} \geq \frac{(J_r-\underline{J})^{\mu}}{\eta} \geq r$, we have

$$\begin{aligned}
\|\mathcal{M}_{\beta}[\mu] - \tilde{\theta}\| &\leq \int_{B_{\theta,\tilde{r}}(\tilde{\theta})} \|\theta - \tilde{\theta}\| \frac{w_{\beta}(\theta)}{\|w_{\beta}(\theta)\|_{L^1(\rho)}} \mathrm{d}\rho + \int_{B^c_{\theta,\tilde{r}}(\tilde{\theta})} \|\theta - \tilde{\theta}\| \frac{w_{\beta}(\theta)}{\|w_{\beta}(\theta)\|_{L^1(\rho)}} \mathrm{d}\rho \\
&\leq \tilde{r} + \int_{B^c_{\theta,\tilde{r}}(\tilde{\theta})} \|\theta - \tilde{\theta}\| \frac{w_{\beta}(\theta)}{\|w_{\beta}(\theta)\|_{L^1(\rho)}} \mathrm{d}\rho.
\end{aligned} \tag{1}$$

By Markov's inequality, we have $\|w_{\beta}\|_{L^1(\rho)} \geq a\rho(\{(\theta, \omega) : \exp(-\beta\mathcal{J}(\theta) \geq a)\})$. By choosing $a = \exp(-\beta J_r)$, we have

$$\begin{aligned}
\|w_{\beta}\|_{L^1(\rho)} &\geq \exp(-\beta J_r)\rho\left(\{(\theta, \omega) : \exp(-\beta\mathcal{J}(\theta) \geq \exp(-\beta J_r))\}\right) \\
&= \exp(-\beta J_r)\rho\left(\{(\theta, \omega) : J(\theta) \leq J_r)\}\right) \\
&\geq \exp(-\beta J_r)\rho(B_{\theta,r}(\tilde{\theta})),
\end{aligned}$$

where the second inequality comes from the definition of J_r. Thus for the second term in (1), we obtain

$$\begin{aligned}
\int_{B^c_{\theta,\tilde{r}}(\tilde{\theta})} \|\theta - \tilde{\theta}\| \frac{w_{\beta}(\theta)}{\|w_{\beta}(\theta)\|_{L^1(\rho)}} \mathrm{d}\rho &\leq \frac{1}{\exp(-\beta J_r)\rho(B_{\theta,r}(\tilde{\theta}))} \int_{B^c_{\theta,\tilde{r}}(\tilde{\theta})} \|\theta - \tilde{\theta}\| w_{\beta}(\theta) \mathrm{d}\rho \\
&\leq \frac{\exp(-\beta(\inf_{B^c_{\theta,\tilde{r}}(\tilde{\theta})} J(\theta) - J_r))}{\rho(B_{\theta,r}(\tilde{\theta}))} \int_{B^c_{\theta,\tilde{r}}(\tilde{\theta})} \|\theta - \tilde{\theta}\| \mathrm{d}\rho \\
&\leq \frac{\exp(-\beta(\inf_{B^c_{\theta,\tilde{r}}(\tilde{\theta})} J(\theta) - J_r))}{\rho(B_{\theta,r}(\tilde{\theta}))} \int \|\theta - \tilde{\theta}\| \mathrm{d}\rho.
\end{aligned}$$

We also notice

$$\inf_{B^c_{\theta,\tilde r}(\tilde\theta)} J(\theta) - J_r \geq \min\{\delta_J + \underline{J}, (\eta\tilde r)^{1/\mu} + \underline{J}\} - J_r \geq (\eta\tilde r)^{1/\mu} - J_r + \underline{J} = q,$$

where the first inequality comes from Assumption 4.1 and the second inequality comes from the definition of $\tilde r$ and q, $\tilde r = \frac{(q+J_r-\underline{J})^\mu}{\eta} \leq \frac{\delta_J}{\eta}$. Combining the above inequality and the definition of $\tilde r$, we have

$$\begin{aligned}\|\mathcal{M}_\beta[\mu] - \tilde\theta\| &\leq \frac{(q+J_r-\underline{J})^\mu}{\eta} + \frac{\exp(-\alpha(\inf_{B^c_{\theta,\tilde r}(\tilde\theta)} J(\theta) - J_r))}{\rho(B_{\theta,r}(\tilde\theta))}\int \|\theta-\tilde\theta\|\mathrm{d}\rho \\ &\leq \frac{(q+J_r-\underline{J})^\mu}{\eta} + \frac{\exp(-\beta q)}{\rho(B_{\theta,r}(\tilde\theta))}\int \|\theta-\tilde\theta\|\mathrm{d}\rho.\end{aligned}$$

□

Theorem 4.5 Let $\mathcal{J}$ satisfy the Assumption 4.1. Moreover, let $\rho_0 \in \mathcal{P}_4(\mathbb{R}^{2D})$ and $(\tilde\theta, 0) \in supp(\rho_0)$. By choosing parameters $\sigma(t)$ is exponentially decaying as $\sigma(t) = \sigma_1\exp(-\sigma_2 t)$ with $\sigma_1 > 0$ and $\sigma_2 > 1$ and $\lambda = \max\{m, \gamma_1\} \geq 2\sigma_2$ and $\gamma = \min\{\gamma_1,\gamma_2\} > 0$. Fix any $\epsilon \in (0, E[\rho_0])$ and $\tau \in (0, 1-\frac{2\sigma_2}{\lambda})$, and define the time horizon

$$T^* := \frac{1}{(1-\tau)\lambda}\log\left(\frac{E[\rho_{T_0}])}{\epsilon}\right) \tag{2}$$

Then there exists $\beta > 0$ such that for all $\beta > \beta_0$, if $\rho \in \mathcal{C}([0,T^*], \mathcal{P}_4(\mathbb{R}^{2D}))$ is a weak solution to the Fokker-Planck equation in the time interval $[0,T^*]$ with initial condition ρ_0, we have

$$\min_{t\in[0,T^*]} E[\rho_t] \leq \epsilon.$$

Furthermore, until $E[\rho_t]$ reaches the prescribed accuracy ϵ, we have the exponential decay

$$E[\rho_t] \leq E[\rho_0]\exp(-(1-\tau)\lambda t) \tag{3}$$

and, up to a constant, the same behavior for $W_2^2(\rho_t, \delta_{(\tilde\theta,0)})$.

Proof of Theorem 4.5. We choose parameters β such that

$$\beta > \beta_0 := \frac{1}{q_\epsilon}\left(\log\left(\frac{4\sqrt{2E[\rho_0]}}{\mathrm{c}(\tau,\lambda)\sqrt{\epsilon}} + \frac{p}{(1-\tau)\lambda}\log\left(\frac{E[\rho_0]}{\epsilon}\right) - \log\rho_0(B_{\frac{r_\epsilon}{2}}(\tilde\theta, 0))\right)\right),$$

where we introduce

$$\mathrm{c}(\tau,\lambda) = \frac{\tau\gamma}{\lambda},\quad q_\epsilon = \frac{1}{2}\min\left\{\left(\frac{c(\tau,\lambda)\sqrt{\epsilon}\eta}{2}\right)^{1/\mu}, \delta_J\right\},\ \text{and } r_\epsilon = \max_{x\in[0,R_0]}\{\max_{(\theta,\omega)\in B_s(\tilde\theta,0)} J(\theta) \leq q_\epsilon + \underline{J}\},$$

and define the time horizon $T_\beta \geq 0$, which may depend on β, by

$$T_\beta = \sup\{t \geq 0 : E[\mu_{t'}] > \epsilon \text{ and } \|\mathcal{M}_\beta[\mu_{t'}] - \tilde\theta\| < C(t') \text{ for all } t' \in [0,t]\}$$

with $C(t) = \mathrm{c}(\tau,\lambda)\sqrt{E(\rho_t)}$. First we want to prove $T_\beta > 0$, which follows from the continuity of the mappings $t \to E[\rho_t]$ and $t \to \|\mathcal{M}_\beta[\mu_t]-\tilde{\theta}\|$ since $E[\rho_0] > 0$ and $\|\mathcal{M}_\beta[\mu_0]-\tilde{\theta}\| < C(0)$. While the former holds by assumption, the latter follows by

$$
\begin{aligned}
\|\mathcal{M}_\beta[\mu_0]-\tilde{\theta}\| &\leq \frac{(q_\epsilon + J_{r_\epsilon} - \underline{J})^\mu}{\eta} + \frac{\exp(-\beta q_\epsilon)}{\rho(B_{\theta,r_\epsilon}(\tilde{\theta}))}\int \|\theta-\tilde{\theta}\| \mathrm{d}\rho_0(\theta,\omega) \\
&\leq \frac{(q_\epsilon + J_{r_\epsilon} - \underline{J})^\mu}{\eta} + \frac{\exp(-\beta q_\epsilon)}{\rho(B_{r_\epsilon}(\tilde{\theta},0))}\int \|\theta-\tilde{\theta}\| \mathrm{d}\rho_0(\theta,\omega) \\
&\leq \frac{\mathrm{c}(\tau,\lambda)\sqrt{\epsilon}}{2} + \frac{\exp(-\beta q_\epsilon)}{\rho(B_{r_\epsilon}(\tilde{\theta},0))}\sqrt{2E[\rho_0]} \\
&\leq \mathrm{c}(\tau,\lambda)\sqrt{\epsilon} \leq \mathrm{c}(\tau,\lambda)\sqrt{E[\rho_0]} = C(0),
\end{aligned}
$$

where we use the definition of β in the first inequality of the last line. Recall the Lemma **??**, up to time T_β

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E[\rho_t] &\leq -\gamma E[\rho_t] + \lambda\sqrt{E[\rho_t]}\|\mathcal{M}_\beta[\mu_t]-\tilde{\theta}\| + \frac{\sigma^2(t)D(m+1)}{2} \\
&\leq -\left(1-\tau\right)\gamma E[\rho_t] + \frac{\sigma^2(t)D(m+1)}{2}.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left((1-\tau)\gamma t\right)E[\rho_t]\right) =& (1-\tau)\gamma\left(\exp\left((1-\tau)\gamma t\right)E[\rho_t]\right) + \exp\left((1-\tau)\gamma t\right)\frac{\mathrm{d}}{\mathrm{d}t}E[\rho_t] \\
\leq& \exp\left((1-\tau)\gamma t\right)\frac{\sigma^2(t)D(m+1)}{2}.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
\left(\exp\left((1-\tau)\gamma t\right)E[\rho_t]\right) - E[\rho_0] \leq& \int_0^t \exp\left((1-\tau)\lambda s\right)\sigma^2(s)\mathrm{d}s \\
=& \frac{\sigma_1^2(1-\exp\left((-2\sigma_2+\lambda(1-\tau))t\right)}{2\sigma_2-\lambda(1-\tau)}.
\end{aligned}
$$

We can get the boundedness for $E[\rho_t]$, for $2\sigma_2 - \lambda(1-\tau) < 0$ by the chosen of τ and λ, then we have

$$
E[\rho_t] \leq \exp\left(-(1-\tau)t\lambda\right)E[\rho_0].
$$

Accordingly, we note that $E(\rho_t)$ is decreasing in t, which implies the decay of the function $C(t)$ as well. Hence, recalling the definition of T_β, we may bound $\max_{t\in[0,T_\beta]}\|\mathcal{M}_\beta[\rho_{t'}]-\tilde{\theta}\| \leq \max_{t\in[0,T_\beta]} C(t) \leq C(0)$. We now conclude by showing $\min_{t\in[0,T_\beta]} E(\rho_t) \leq \epsilon$ with $T_\beta \leq T^*$. For this, we distinguish the following three cases.

Case $T_\beta \geq T^*$: If $T_\beta \geq T^*$, we can use the definition of $T^* = \frac{1}{(1-\tau)\lambda}\log(\frac{E[\rho_0]}{\epsilon})$

and the time evolution bound of $E[\rho_t]$ to conclude that $E[\rho_{T^*}] \leq \epsilon$. Hence, by definition of T_β, we find $E[\rho_{T_\beta}] \leq \epsilon$ and $T_\beta = T^*$.
Case $T_\beta < T^*$ **and** $E[\rho_{T_\beta}] \leq \epsilon$**:** Nothing need to discussed in this case.
Case $T_\beta < T^*$ **and**$E[\rho_{T_\beta}] > \epsilon$**:** We shall prove that this case will never occur.

$$\begin{aligned}\|\mathcal{M}_\beta[\mu_{T_\beta}] - \tilde{\theta}\| &\leq \frac{(q_\epsilon + J_{r_\epsilon} - \underline{J})^\mu}{\eta} + \frac{\exp(-\beta q_\epsilon)}{\rho(B_{\theta,r_\epsilon}(\tilde{\theta}))} \int \|\theta - \tilde{\theta}\| \mathrm{d}\rho_{T_\beta}(\theta,\omega) \\ &< \frac{\mathrm{c}(\tau,\lambda)\sqrt{E[\rho_{T_\beta}]}}{2} + \frac{\exp(-\beta q_\epsilon)}{\rho(B_{\theta,r_\epsilon}(\tilde{\theta}))}\sqrt{E[\mu_{T_\beta}]}.\end{aligned}$$

Since, we have $\max_{t\in[0,T_\beta]} \|\mathcal{M}_\beta[\mu_{t'}] - \tilde{\theta}\| = B = C(0)$ guarantees that there exist a $p > 0$ with

$$\rho_{T_\beta}(B_{\theta,r_\epsilon}(\tilde{\theta})) \geq \left(\int \phi_{r_\epsilon}(\theta,\omega)\mathrm{d}\rho_0(\theta,\omega)\right)\exp(-pT_\beta) \geq \frac{1}{2}\rho_0\left(B_{\frac{r_\epsilon}{2}}(\tilde{\theta},0))\right)\exp(-pT^*), \tag{4}$$

where we used $(\tilde{\theta},0) \in supp(\rho_0)$ for bounding the initial mass ρ_0 and the fact that ϕ_r is bounded from below on $B_{\frac{r_\epsilon}{2}}(\tilde{\theta},0))$ by $1/2$. With this, we can conclude that

$$\begin{aligned}\|\mathcal{M}_\beta[\mu_{T_\beta}] - \tilde{\theta}\| &< \frac{\mathrm{c}(\tau,\lambda)\sqrt{E[\rho_{T_\beta}]}}{2} + \frac{2\exp(-\beta q_\epsilon)}{\rho(B_{\frac{r_\epsilon}{2}}(\tilde{\theta},0))\exp(-pT^*)}\sqrt{E[\rho_{T_\beta}]} \\ &\leq c(\tau,\lambda)\sqrt{E[\rho_{T_\beta}]} = C(T_\beta),\end{aligned}$$

where the first inequality in the last line holds by the choice of β. This establishes the desired contradiction, against the consequence of the continuity of the mappings $t \to E[\rho_t]$ and $t \to \|\mathcal{M}_\beta[\mu_t] - \tilde{\theta}\|$. □