We can modify the Lemma D.2 as follows: Under Assumption 4.1, $\forall r > 0$, we define $J_r := \sup_{\theta \in B_{\theta,r}(\tilde{\theta})} \mathcal{J}(\theta)$. Then $\forall r \in [0, R_0]$ and $q > 0$ such that $(q + J_r - \underline{J})^\mu \leq \delta_J$, we have

$$\|\mathcal{M}_\beta[\mu] - \tilde{\theta}\| \leq \frac{(q + J_r - \underline{J})^\mu}{\eta} + \frac{\exp(-\beta q)}{\rho(B\_\theta, r(\tilde{\theta}))} \int \|\theta - \tilde{\theta}\| \mathrm{d}\rho(\theta, \omega).$$

Proof of Lemma D.2 Let $\tilde{r} = \frac{(q+J_r-\underline{J})^\mu}{\eta} \geq \frac{(J_r-\underline{J})^\mu}{\eta} \geq r$, we have

$$\begin{aligned}
\|\mathcal{M}[\mu] - \tilde{\theta}\| \leq & \int_{B_{\theta,\tilde{r}}(\tilde{\theta})} \|\theta - \tilde{\theta}\| \frac{w_\beta(\theta)}{\|w_\beta(\theta)\|_{L^1(\rho)}} \mathrm{d}\rho + \int_{B^c_{\theta,\tilde{r}}(\tilde{\theta})} \|\theta - \tilde{\theta}\| \frac{w_\beta(\theta)}{\|w_\beta(\theta)\|_{L^1(\rho)}} \mathrm{d}\rho \\
\leq & \tilde{r} + \int_{B^c_{\theta,\tilde{r}}(\tilde{\theta})} \|\theta - \tilde{\theta}\| \frac{w_\beta(\theta)}{\|w_\beta(\theta)\|_{L^1(\rho)}} \mathrm{d}\rho.
\end{aligned}$$

For the second term, we still obtain the original inequality

$$\int_{B^c_{\theta,\tilde{r}}(\tilde{\theta})} \|\theta - \tilde{\theta}\| \frac{w_\beta(\theta)}{\|w_\beta(\theta)\|_{L^1(\rho)}} \mathrm{d}\rho \leq \frac{\exp(-\beta(\inf_{B^c_{\theta,\tilde{r}}(\tilde{\theta})} J(\theta) - J_r))}{\rho(B_{\theta,r}(\tilde{\theta}))} \int \|\theta - \tilde{\theta}\| \mathrm{d}\rho.$$

We also notice

$$\inf_{B^c_{\theta,\tilde{r}}(\tilde{\theta})} J(\theta) - J_r \geq \min\{\delta_J + \underline{J}, (\eta\tilde{r})^{1/\mu} + \underline{J}\} - J_r \geq (\eta\tilde{r})^{1/\mu} - J_r + \underline{J} = q,$$

Combining the above inequality and the definition of $\tilde{r}$, we have

$$\begin{aligned}
\|\mathcal{M}[\mu] - \tilde{\theta}\| \leq & \frac{(q + J_r - \underline{J})^\mu}{\eta} + \frac{\exp(-\alpha(\inf_{B^c_{\theta,\tilde{r}}(\tilde{\theta})} J(\theta) - J_r))}{\rho(B_{\theta,r}(\tilde{\theta}))} \int \|\theta - \tilde{\theta}\| \mathrm{d}\rho \\
\leq & \frac{(q + J_r - \underline{J})^\mu}{\eta} + \frac{\exp(-\beta q)}{\rho(B_{\theta,r}(\tilde{\theta}))} \int \|\theta - \tilde{\theta}\| \mathrm{d}\rho.
\end{aligned}$$

Then in the proof of Theorem 4.5, we can replace the definition of q_ϵ with

$$q_\epsilon = \frac{1}{2} \min\left\{ \left(\frac{c(\tau, \lambda)\sqrt{\epsilon}\eta}{2} \right)^{\frac{1}{\mu}}, \delta_J \right\}$$

and then

$$\begin{aligned}
\|\mathcal{M}_\beta[\mu_0] - \tilde{\theta}\| \leq & \frac{(q_\epsilon + J_{r_\epsilon} - \underline{J})^\mu}{\eta} + \frac{\exp(-\beta q_\epsilon)}{\rho(B_{\theta,r_\epsilon}(\tilde{\theta}))} \int \|\theta - \tilde{\theta}\| \mathrm{d}\rho_0(\theta, \omega) \\
\leq & \frac{(q_\epsilon + J_{r_\epsilon} - \underline{J})^\mu}{\eta} + \frac{\exp(-\beta q_\epsilon)}{\rho(B_{r_\epsilon}(\tilde{\theta}, 0))} \int \|\theta - \tilde{\theta}\| \mathrm{d}\rho_0(\theta, \omega) \\
\leq & \frac{\mathrm{c}(\tau, \lambda)\sqrt{\epsilon}}{2} + \frac{\exp(-\beta q_\epsilon)}{\rho(B_{r_\epsilon}(\tilde{\theta}, 0))} \sqrt{2E[\rho_0]} \\
\leq & C(0),
\end{aligned}$$

and for **Case $T_\beta < T^*$ and $E[\rho_{T_\beta}] > \epsilon$:** We shall replace the current proof with the following one:

$$\begin{aligned}\|\mathcal{M}_\beta[\mu_{T_\beta}] - \tilde{\theta}\| &\leq \frac{(q_\epsilon + J_{r_\epsilon} - \underline{J})^\mu}{\eta} + \frac{\exp(-\beta q_\epsilon)}{\rho(B_{\theta,r_\epsilon}(\tilde{\theta}))} \int \|\theta - \tilde{\theta}\| \mathrm{d}\rho_{T_\beta}(\theta, \omega) \\ &< \frac{\mathrm{c}(\tau,\lambda)\sqrt{E[\rho_{T_\beta}]}}{2} + \frac{\exp(-\beta q_\epsilon)}{\rho(B_{\theta,r_\epsilon}(\tilde{\theta}))} \sqrt{E[\mu_{T_\beta}]}.\end{aligned}$$