

$$Q1. (a) \omega_0 = \dot{\beta} \hat{z}_0 = \dot{\beta} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\omega_1 = \dot{\beta} \hat{z}_0 + \dot{\theta} \cdot \hat{y}_1 \quad \therefore \hat{y}_1 = \text{Rot}(\hat{z}_0, \beta) \text{Rot}(\hat{x}_0, \gamma) \cdot \hat{y}_0$$

$$\omega_1 = \dot{\beta} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \dot{\theta} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.766 & -0.643 \\ 0 & 0.643 & 0.766 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\therefore \dot{\theta} = 60 \text{ rev/min} = 6.28 \text{ rad s}^{-1}$$

$$= 3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 6.28 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.766 & -0.643 \\ 0 & 0.643 & 0.766 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= 3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 6.28 \begin{pmatrix} 0 \\ 0.766 \\ -0.643 \end{pmatrix} = \begin{pmatrix} 0 \\ 4.81 \\ 7.03 \end{pmatrix} \text{ rad s}^{-1}$$

$$\therefore \|\hat{\omega}_1\| = \sqrt{4.81^2 + 7.03^2} = 8.125 \text{ rad s}^{-1}$$

$$\therefore \alpha_0 = \ddot{\beta} \hat{z}_0 = 0$$

$$\alpha_1 = \ddot{\beta} \hat{z}_0 + \ddot{\theta} \hat{y}_1 + \vec{\beta}_0 \times \vec{\theta}_1$$

$$= \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 4.81 \\ 7.03 \end{pmatrix} = \begin{pmatrix} -14.43 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \|\vec{\alpha}_1\| = 14.43 \text{ rad s}^{-2}$$

$$(b) P_{0, \text{DirR}} = R_{01} P_1 + R_{01} R_{12} P_{2, \text{DirR}}$$

$$\therefore R_{01} = \text{rot}(\hat{\mathbf{z}}, \beta) = \begin{bmatrix} c_\beta & -s_\beta & 0 \\ s_\beta & c_\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{12} = \text{rot}(\hat{\mathbf{x}}, 40^\circ) \cdot \text{rot}(\hat{\mathbf{y}}, \theta)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{40} & -s_{40} \\ 0 & s_{40} & c_{40} \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix}$$

$$= \begin{bmatrix} c_\theta & 0 & s_\theta \\ s_{40}s_\theta & c_{40} & -s_{40}c_\theta \\ -c_{40}s_\theta & s_{40} & c_{40}c_\theta \end{bmatrix}$$

$$\therefore R_{02} = \begin{bmatrix} c_\beta & -s_\beta & 0 \\ s_\beta & c_\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ s_{40}s_\theta & c_{40} & -s_{40}c_\theta \\ -c_{40}s_\theta & s_{40} & c_{40}c_\theta \end{bmatrix}$$

$$= \begin{bmatrix} c_\theta c_\beta - s_{40} s_\theta s_\beta & -c_{40} s_\beta & c_\beta s_\theta + s_{40} s_\beta c_\theta \\ s_\beta c_\theta + s_{40} c_\beta s_\theta & c_{40} c_\beta & s_\beta s_\theta - s_{40} c_\beta c_\theta \\ -c_{40} s_\theta & s_{40} & c_{40} c_\theta \end{bmatrix}$$

$$\dot{R}_{02} = \begin{bmatrix} -\dot{\beta} S_B C_\theta - \dot{\theta} C_B S_\theta - S_{40} [\dot{\beta} C_B S_\theta + \dot{\theta} C_\theta S_B] & -\dot{\beta} C_{40} C_\beta \\ -\dot{\beta} (S_{40} S_B S_\theta - C_B C_\theta) + \dot{\theta} (S_{40} C_B C_\theta - S_B S_\theta) & -\dot{\beta} C_{40} S_B \\ -\dot{\theta} C_{40} C_\theta & 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\beta} (S_{40} C_B C_\theta - S_B S_\theta) - \dot{\theta} (S_{40} S_B S_\theta - C_B C_\theta) \\ \dot{\beta} (S_{40} S_B C_\theta - S_\theta C_B) + \dot{\theta} (S_{40} S_\theta C_B + S_B C_\theta) \\ -\dot{\theta} C_{40} S_\theta \end{bmatrix}$$

$$\therefore R_{01} = \begin{bmatrix} -\dot{\beta} S_B & -\dot{\beta} C_B & 0 \\ \dot{\beta} C_B & -\dot{\beta} S_\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore V_{0,A} = \dot{R}_{01} P_1 + \dot{R}_{02} P_{2,A}$$

$$\theta = 0, \beta = 0, \dot{\theta} = 6.28 \text{ rad/s}, \dot{\beta} = 3.0 \text{ rad/s}^2$$

$$= \dot{R}_{01} \begin{pmatrix} 0 \\ 3j0 \\ 0 \end{pmatrix} + \dot{R}_{02} \begin{pmatrix} 0 \\ 300 \\ 120 \end{pmatrix}$$

$$= \begin{bmatrix} 0 & -3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 2j0 \\ 1j0 \end{pmatrix} + \begin{bmatrix} 0 & -2.298 & 8.208 \\ 7.037 & 0 & 0 \\ -4.811 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 300 \\ 120 \end{pmatrix}$$

$$= \begin{pmatrix} -10\text{J}_0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 295.6 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -754.4 \\ 0 \\ 0 \end{pmatrix} \text{ mm s}^{-1}$$

$$\therefore \overrightarrow{|V_{gA}|} = 754.5 \text{ mm s}^{-1}$$

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

From Lie's algebra,  $\dot{R} = [\hat{\omega}] R$

$$\ddot{R} = [\hat{\omega}] \dot{R} + [\hat{\omega}] \dot{R}$$

$$[\omega_0]^2 \cdot R_{01} = [\hat{\omega}]^2 R \quad \therefore \frac{d}{dt} [\hat{\omega}] = 0$$

$$\therefore \ddot{R}_{01} = [\hat{\omega}] \dot{R} + [\hat{\omega}] \dot{R}$$

$$= \begin{bmatrix} -9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 350 \\ 150 \end{pmatrix} + \begin{bmatrix} -72.66 & 0 & 0 \\ 0 & -6.89 & 68.81 \\ 0 & 0 & -30.21 \end{bmatrix} \begin{pmatrix} 0 \\ 300 \\ 120 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 3039.5 \\ -3625.4 \end{pmatrix} \text{ mm s}^{-2}$$

$$\therefore \overrightarrow{|a_{0,A}|} = \sqrt{6189.5^2 + 3625.4^2} = 7173.1 \text{ mm s}^{-2}$$

(c) Similarly, we sub  $\Theta = 90^\circ$  while keep other variables

unchanged: (or sub  $P_2 = \begin{pmatrix} 120 \\ 300 \\ 0 \end{pmatrix}$  is the same)

$$v_{0,A} = \begin{bmatrix} 0 & -3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 350 \\ 150 \end{pmatrix} + \begin{bmatrix} -8.21 & -2.298 & 0 \\ 0 & 0 & 7.037 \\ 0 & 0 & -4.81 \end{bmatrix} \begin{pmatrix} 0 \\ 300 \\ 120 \end{pmatrix}$$

$$= \begin{pmatrix} -1050 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -689.4 \\ 844.4 \\ -577.3 \end{pmatrix} = \begin{pmatrix} -1739.4 \\ 844.4 \\ -577.3 \end{pmatrix}$$

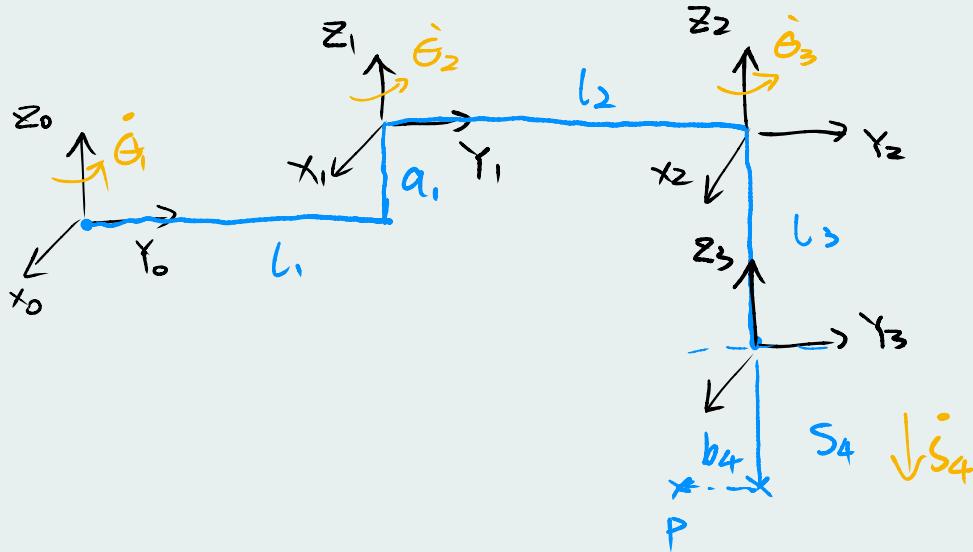
$$\therefore \vec{v}_{0,A} = 2017.9 \text{ mm s}^{-1}$$

$$\therefore a_{0,A} = \begin{bmatrix} -9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 350 \\ 150 \end{pmatrix} + \begin{bmatrix} 0 & 0 & -72.7 \\ -68.8 & -6.89 & 0 \\ 30.2 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 300 \\ 120 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -3150 \\ 0 \end{pmatrix} + \begin{pmatrix} -8719.04 \\ -2068.3 \\ 0 \end{pmatrix} = \begin{pmatrix} -8719.04 \\ -5218.3 \\ 0 \end{pmatrix}$$

$$\vec{a}_{0,A} = 10161.3 \text{ mm s}^{-2}$$

Q2. (i)



$$P_0 = R_{01} P_1 + R_{01}R_{12} P_2 + R_{01}R_{12}R_{23} P_3$$

$$R_{01}R_{12} = \text{Rot}(z_0, \theta_1) \cdot \text{Rot}(z_1, \theta_2) = \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 \\ \sin \theta_1 & \cos \theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 & 0 \\ \sin \theta_2 & \cos \theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta_1 + \cos \theta_2 & -\sin \theta_1 - \sin \theta_2 & 0 \\ \sin \theta_1 + \sin \theta_2 & \cos \theta_1 + \cos \theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly:

$$R_{01}R_{12}R_{23} = \begin{bmatrix} \cos \theta_1 + \cos \theta_2 + \cos \theta_3 & -\sin \theta_1 - \sin \theta_2 - \sin \theta_3 & 0 \\ \sin \theta_1 + \sin \theta_2 + \sin \theta_3 & \cos \theta_1 + \cos \theta_2 + \cos \theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_0 = \begin{bmatrix} C_{\theta_1} & -S_{\theta_1} & 0 \\ S_{\theta_1} & C_{\theta_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ l_1 \\ a_1 \end{pmatrix} + \begin{bmatrix} C_{\theta_1+\theta_2} & -S_{\theta_1+\theta_2} & 0 \\ S_{\theta_1+\theta_2} & C_{\theta_1+\theta_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ l_1 \\ 0 \end{pmatrix}$$

$$+ \begin{bmatrix} C_{\theta_1+\theta_2+\theta_3} & -S_{\theta_1+\theta_2+\theta_3} & 0 \\ S_{\theta_1+\theta_2+\theta_3} & C_{\theta_1+\theta_2+\theta_3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ -b_4 \\ -l_3 - s_4 \end{pmatrix}$$

$$\therefore P = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} -l_1 \sin \theta_1 - l_2 \sin(\theta_1 + \theta_2) + b_4 \sin(\theta_1 + \theta_2 + \theta_3) \\ l_2 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) - b_4 \cos(\theta_1 + \theta_2 + \theta_3) \\ -l_3 - s_4 + a_1 \end{pmatrix}$$

$$\text{i)} \frac{dp}{dt} = J \dot{\theta}$$

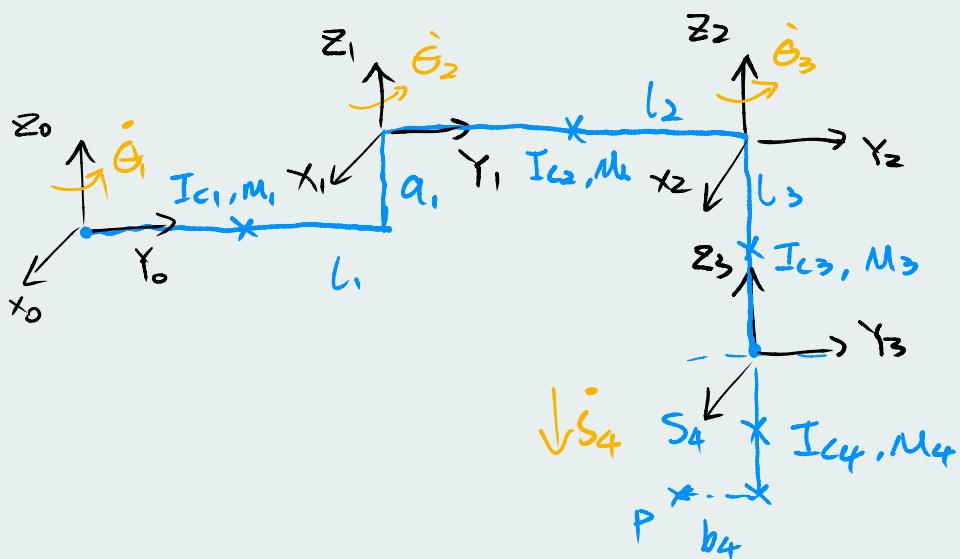
$$\theta_{12} = \theta_1 + \theta_2$$



$$\frac{dp}{dt} = \begin{bmatrix} -l_1 C_{\theta_1} - l_2 C_{\theta_{12}} + b_4 C_{\theta_{123}} & -l_2 C_{\theta_{12}} + b_4 C_{\theta_{123}} & b_4 C_{\theta_{123}} & 0 \\ -l_1 S_{\theta_1} - l_2 S_{\theta_{12}} + b_4 S_{\theta_{123}} & -l_2 S_{\theta_{12}} + b_4 S_{\theta_{123}} & b_4 S_{\theta_{123}} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$\left. \begin{array}{c} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{s}_4 \end{array} \right\}$

(iii)  $L = K - P$ , Assuming centroid at half of Link Length.



By treating frame 0 as Datum:

$$P = M_2 g a_1 + M_3 g (a_1 - l_3/2) + M_4 g (a_1 - l_3/2 - s_4/2)$$

To find Kinetic Energy, we first find velocity of individual centroids:

$$\begin{aligned} \dot{r}_{c1} &= \dot{\theta}_0 r_{c1} \\ &= \begin{bmatrix} -\dot{\theta}_0 \sin \theta_0, & -\dot{\theta}_0 \cos \theta_0, & 0 \\ \dot{\theta}_0 \cos \theta_0, & -\dot{\theta}_0 \sin \theta_0, & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ l_1/2 \\ 0 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} -\dot{\theta}_0 \frac{l_1}{2} \cos \theta_0, \\ -\dot{\theta}_0 \frac{l_1}{2} \sin \theta_0, \\ 0 \end{pmatrix} = \dot{\theta}_0 \frac{l_1}{2}$$

$$\dot{P}_{C2} = \dot{R}_{01} P_1 + \dot{R}_{02} P_{C2}$$

$\Theta_{12} = \Theta_1 + \Theta_2$

$$= \begin{pmatrix} -\dot{\theta}_1 L_1 \cos \theta_1 \\ -\dot{\theta}_1 L_1 \sin \theta_1 \\ 0 \end{pmatrix} + \begin{bmatrix} -\Theta_{12} \sin \Theta_{12} & -\Theta_{12} \cos \Theta_{12} & 0 \\ \Theta_{12} \cos \Theta_{12} & -\Theta_{12} \sin \Theta_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ L_2/2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\dot{\theta}_1 L_1 \cos \theta_1 & -\frac{L_2}{2} \Theta_{12} \cos \Theta_{12} \\ -\dot{\theta}_1 L_1 \sin \theta_1 & -\frac{L_2}{2} \Theta_{12} \sin \Theta_{12} \\ 0 \end{pmatrix}$$

$$\dot{P}_{C3} = \dot{R}_{01} P_1 + \dot{R}_{02} P_2 + \dot{R}_{03} P_{C3}$$

$$= \begin{pmatrix} -\dot{\theta}_1 L_1 \cos \theta_1 & -\frac{L_2}{2} \Theta_{12} \cos \Theta_{12} \\ -\dot{\theta}_1 L_1 \sin \theta_1 & -\frac{L_2}{2} \Theta_{12} \sin \Theta_{12} \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} -\Theta_{123} S_{\Theta_{123}} & -\Theta_{123} C_{\Theta_{123}} & 0 \\ \Theta_{123} C_{\Theta_{123}} & -\Theta_{123} S_{\Theta_{123}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ -\frac{L_3}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -\dot{\theta}_1 L_1 \cos \theta_1 & -\frac{L_2}{2} \Theta_{12} \cos \Theta_{12} \\ -\dot{\theta}_1 L_1 \sin \theta_1 & -\frac{L_2}{2} \Theta_{12} \sin \Theta_{12} \\ 0 \end{pmatrix} = \dot{P}_{C2}$$

$$\dot{P}_{c4} = \dot{R}_{01} P_1 + \dot{R}_{02} P_2 + \dot{R}_{03} P_{c4} + \dot{R}_{03} \dot{S}_4$$

$$= \begin{pmatrix} -\dot{\theta}_1 L_1 \cos\theta_1 & -\frac{L_2}{2} \dot{\theta}_{12} \cos\theta_{12} \\ -\dot{\theta}_1 L_1 \sin\theta_1 & -\frac{L_2}{2} \dot{\theta}_{12} \sin\theta_{12} \\ \dot{S}_4 \end{pmatrix}$$

Therefore, we are able to find the KE of the system:

$$\begin{aligned} K = & \frac{1}{2} m_1 \|\dot{P}_{c1}\|^2 + \frac{1}{2} I_{d1} \dot{\theta}_1^2 + \frac{1}{2} m_2 \|\dot{P}_{c2}\|^2 + \frac{1}{2} I_{d2} (\dot{\theta}_1 + \dot{\theta}_2)^2 \\ & + \frac{1}{2} m_3 \|\dot{P}_{c3}\|^2 + \frac{1}{2} I_{d3} (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2 + \frac{1}{2} m_4 \|\dot{P}_{c4}\|^2 \\ & + \frac{1}{2} I_{d4} (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2 \end{aligned}$$

sub P, K into the  $\boxed{L = K - P}$  we get the Lagrangian

