Q1. (a) $\omega_0 = \dot{\beta}\,\hat{z}_0$

$$w_1 = \dot{\beta}\,\hat{z}_0 + \dot{\theta}\cdot\hat{y}_1 \qquad \because\ \hat{y}_1 = Rot(\hat{x}_0, \gamma)\cdot\hat{y}_0$$

$$w_1 = \dot{\beta}\begin{pmatrix}0\\0\\1\end{pmatrix} + \dot{\theta}\cdot\begin{bmatrix}1 & 0 & 0\\0 & 0.766 & -0.643\\0 & 0.643 & 0.766\end{bmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$\because\ \dot{\theta} = 60$ rev/min $= 6.28$ rad s^{-1}

$$= 3\begin{pmatrix}0\\0\\1\end{pmatrix} + 6.28\begin{bmatrix}1 & 0 & 0\\0 & 0.766 & -0.643\\0 & 0.643 & 0.766\end{bmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$= 3\begin{pmatrix}0\\0\\1\end{pmatrix} + 6.28\begin{pmatrix}0\\0.766\\0.643\end{pmatrix} = \begin{pmatrix}0\\4.81\\7.03\end{pmatrix} \text{ rad } s^{-1}$$

$$\therefore \|\hat{w}_1\| = \sqrt{4.81^2 + 7.03^2} = 8.525 \text{ rad } s^{-1}$$

$$\because\ \alpha_0 = \ddot{\beta}\,\hat{z}_0 = 0$$

$$\alpha_1 = \ddot{\beta}\,\hat{z}_0 + \ddot{\theta}\,\hat{y}_1 + \vec{\beta}_0 \times \vec{\theta}_1$$

$$= \begin{pmatrix}0\\0\\3\end{pmatrix} \times \begin{pmatrix}0\\4.81\\4.03\end{pmatrix} = \begin{pmatrix}-14.43\\0\\0\end{pmatrix}$$

$$\therefore \|\vec{\alpha}_1\| = 14.43 \text{ rad } s^{-2}$$

(b) $P_{0,\text{DiskR}} = R_{01} P_1 + R_{01} R_{12} P_{2,\text{DiskR}}$

$$\because R_{01} = \text{rot}(\hat{z}, \beta) = \begin{bmatrix} C_\beta & -S_\beta & 0 \\ S_\beta & C_\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{12} = \text{rot}(\hat{x}, 40^\circ) \cdot \text{rot}(\hat{y}, \theta)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_{40} & -S_{40} \\ 0 & S_{40} & C_{40} \end{bmatrix} \begin{bmatrix} C_\theta & 0 & S_\theta \\ 0 & 1 & 0 \\ -S_\theta & 0 & C_\theta \end{bmatrix}$$

$$= \begin{bmatrix} C_\theta & 0 & S_\theta \\ S_{40} S_\theta & C_{40} & -S_{40} C_\theta \\ -C_{40} S_\theta & S_{40} & C_{40} C_\theta \end{bmatrix}$$

$$\therefore R_{02} = \begin{bmatrix} C_\beta & -S_\beta & 0 \\ S_\beta & C_\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_\theta & 0 & S_\theta \\ S_{40} S_\theta & C_{40} & -S_{40} C_\theta \\ -C_{40} S_\theta & S_{40} & C_{40} C_\theta \end{bmatrix}$$

$$= \begin{bmatrix} C_\theta C_\beta - S_{40} S_\theta S_\beta & -C_{40} S_\beta & C_\beta S_\theta + S_{40} S_\beta C_\theta \\ S_\beta C_\theta + S_{40} C_\beta S_\theta & C_{40} C_\beta & S_\beta S_\theta - S_{40} C_\beta C_\theta \\ -C_{40} S_\theta & S_{40} & C_{40} C_\theta \end{bmatrix}$$

$$\dot{R}_{02} = \begin{bmatrix} -\dot{\beta} S_\beta C_\theta - \dot{\theta} C_\beta S_\theta - S_{40}\left[\dot{\beta} C_\beta S_\theta + \dot{\theta} C_\theta S_\beta\right] & -\dot{\beta} C_{40} C_\beta & \dot{\beta}(S_{40} C_\beta C_\theta - S_\beta S_\theta) - \dot{\theta}(S_{40} S_\beta S_\theta - C_\beta C_\theta) \\ -\dot{\beta}(S_{40} S_\beta S_\theta - C_\beta C_\theta) + \dot{\theta}(S_{40} C_\beta C_\theta - S_\beta S_\theta) & -\dot{\beta} C_{40} S_\beta & \dot{\beta}(S_{40} S_\beta C_\theta - S_\theta C_\beta) + \dot{\theta}(S_{40} S_\theta C_\beta + S_\beta C_\theta) \\ -\dot{\theta} C_{40} C_\theta & 0 & -\dot{\theta} C_{40} S_\theta \end{bmatrix}$$

$$\therefore \dot{R}_{01} = \begin{bmatrix} -\dot{\beta} S_\beta & -\dot{\beta} C_\beta & 0 \\ \dot{\beta} C_\beta & -\dot{\beta} S_\beta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore V_{0,A} = \dot{R}_{01} P_1 + \dot{R}_{02} P_{2,A}$$

$$\theta = 0, \quad \beta = 0, \quad \dot{\theta} = 6.28 \text{ rad s}^{-1}, \quad \dot{\beta} = 3.0 \text{ rad s}^{-1}$$

$$= \dot{R}_{01} \begin{pmatrix} 0 \\ 310 \\ 0 \end{pmatrix} + \dot{R}_{02} \begin{pmatrix} 0 \\ 300 \\ 120 \end{pmatrix}$$

$$= \begin{bmatrix} 0 & -3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 310 \\ 150 \end{pmatrix} + \begin{bmatrix} 0 & -2.298 & 8.208 \\ 7.037 & 0 & 0 \\ -4.811 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 300 \\ 120 \end{pmatrix}$$

$$= \begin{pmatrix} -1050 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 295.6 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -754.4 \\ 0 \\ 0 \end{pmatrix} \text{ mm s}^{-1}$$

$$\therefore \|\vec{V}_{oA}\| = 754.5 \text{ mm s}^{-1}$$

$$\hat{\omega} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}$$

From Lie's algebra, $\dot{R} = [\hat{\omega}] R$

$$\ddot{R} = [\dot{\hat{\omega}}] R + [\hat{\omega}] \dot{R}$$

$$= [\hat{\omega}]^2 R \qquad \because \frac{d}{dt}[\hat{\omega}] = 0$$

$$\therefore a_{o,\text{Disk}} = \underbrace{\ddot{R}_{01}}_{= [w_0]^2 \cdot R_{01}} p_1 + \underbrace{\ddot{R}_{02}}_{= [w_1]^2 R_{02}} p_{2,\text{Disk}}$$

$$= \begin{bmatrix} -9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 350 \\ 150 \end{pmatrix} + \begin{bmatrix} -72.66 & 0 & 0 \\ 0 & -6.89 & 68.81 \\ 0 & 0 & -30.21 \end{bmatrix} \begin{pmatrix} 0 \\ 300 \\ 120 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 6189.5 \\ -3625.4 \end{pmatrix} \text{ mm s}^{-2}$$

$$\therefore \|\vec{a}_{o,A}\| = \sqrt{6189.5^2 + 3625.4^2} = 7173.1 \text{ mm s}^{-2}$$

(c) Similarly, we sub $\theta = 90^\circ$ while keep other variables unchanged : (OR sub $p_2 = \begin{pmatrix} 120 \\ 300 \\ 0 \end{pmatrix}$ is the same)

$$v_{0,A} = \begin{bmatrix} 0 & -3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 350 \\ 150 \end{pmatrix} + \begin{bmatrix} -8.21 & -2.298 & 0 \\ 0 & 0 & 7.037 \\ 0 & 0 & -4.81 \end{bmatrix} \begin{pmatrix} 0 \\ 300 \\ 120 \end{pmatrix}$$

$$= \begin{pmatrix} -1050 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -689.4 \\ 844.4 \\ -577.3 \end{pmatrix} = \begin{pmatrix} -1739.4 \\ 844.4 \\ -577.3 \end{pmatrix}$$

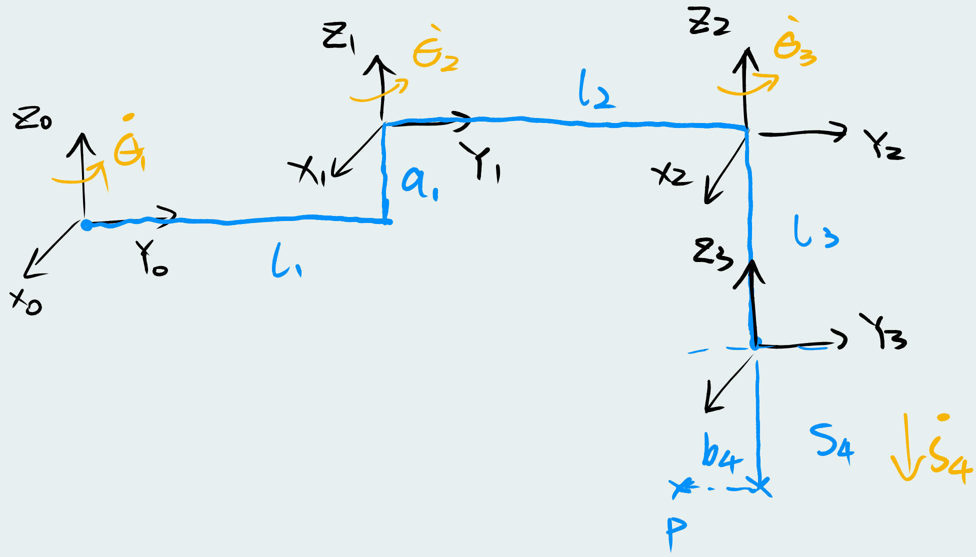
$$\therefore \|\vec{v}_{0,A}\| = 2017.9 \text{ mm s}^{-1}$$

$$\therefore a_{0,A} = \begin{bmatrix} -9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 350 \\ 150 \end{pmatrix} + \begin{bmatrix} 0 & 0 & -72.7 \\ -68.8 & -6.89 & 0 \\ 30.2 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 300 \\ 120 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -3150 \\ 0 \end{pmatrix} + \begin{pmatrix} -8719.04 \\ -2068.3 \\ 0 \end{pmatrix} = \begin{pmatrix} -8719.04 \\ -5218.3 \\ 0 \end{pmatrix}$$

$$\|\vec{a}_{0,A}\| = 10161.3 \text{ mm s}^{-2} \quad \#$$

Q2. (i)



$$P_0 = R_{01} P_1 + R_{01} R_{12} P_2 + R_{01} R_{12} R_{23} P_3$$

$$R_{01} R_{12} = rot(\hat{z}_0, \theta_1) \cdot rot(\hat{z}_1, \theta_2) = \begin{bmatrix} c_{\theta_1} & -s_{\theta_1} & 0 \\ s_{\theta_1} & c_{\theta_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_{\theta_2} & -s_{\theta_2} & 0 \\ s_{\theta_2} & c_{\theta_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_{\theta_1+\theta_2} & -s_{\theta_1+\theta_2} & 0 \\ s_{\theta_1+\theta_2} & c_{\theta_1+\theta_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

similarly:

$$R_{01} R_{12} R_{23} = \begin{bmatrix} c_{\theta_1+\theta_2+\theta_3} & -s_{\theta_1+\theta_2+\theta_3} & 0 \\ s_{\theta_1+\theta_2+\theta_3} & s_{\theta_1+\theta_2+\theta_3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_0 = \begin{bmatrix} C_{\theta_1} & -S_{\theta_1} & 0 \\ S_{\theta_1} & C_{\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ L_1 \\ a_1 \end{pmatrix} + \begin{bmatrix} C_{\theta_1+\theta_2} & -S_{\theta_1+\theta_2} & 0 \\ S_{\theta_1+\theta_2} & C_{\theta_1+\theta_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ L_2 \\ 0 \end{pmatrix}$$

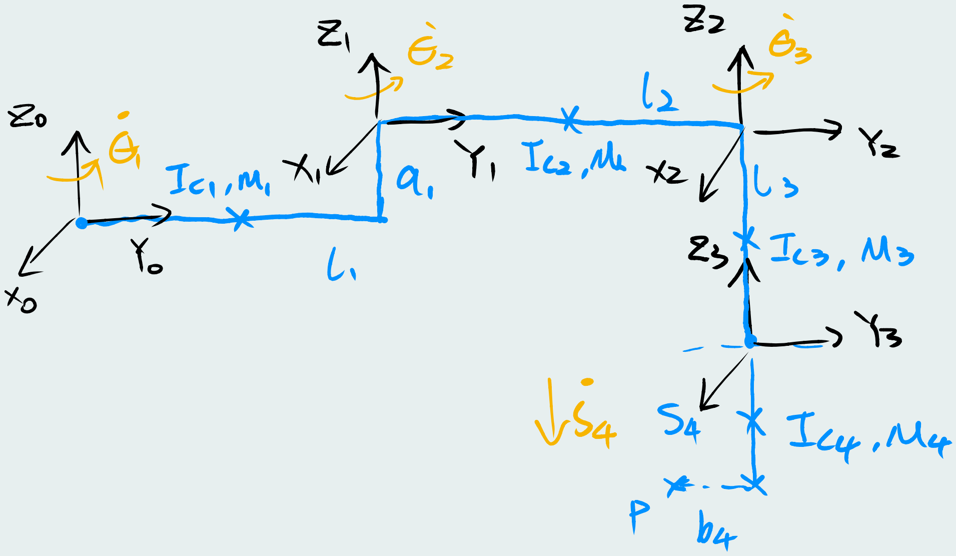
$$+ \begin{bmatrix} C_{\theta_1+\theta_2+\theta_3} & -S_{\theta_1+\theta_2+\theta_3} & 0 \\ S_{\theta_1+\theta_2+\theta_3} & S_{\theta_1+\theta_2+\theta_3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ -b_4 \\ -L_3-S_4 \end{pmatrix}$$

$$\therefore P = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1+\theta_2) + b_4 \sin(\theta_1+\theta_2+\theta_3) \\ L_2 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2) - b_4 \cos(\theta_1+\theta_2+\theta_3) \\ -L_3 - S_4 + a_1 \end{pmatrix}$$

ii) $\frac{dp}{dt} = J\dot{\theta}$ $\qquad \theta_{12} = \theta_1 + \theta_2$

$$\frac{dp}{dt} = \begin{bmatrix} -L_1 C_{\theta_1} - L_2 C_{\theta_{12}} + b_4 C_{\theta_{123}} & -L_2 C_{\theta_{12}} + b_4 C_{\theta_{123}} & b_4 C_{\theta_{123}} & 0 \\ -L_1 S_{\theta_1} - L_2 S_{\theta_{12}} + b_4 S_{\theta_{123}} & -L_2 S_{\theta_{12}} + b_4 S_{\theta_{123}} & b_4 S_{\theta_{123}} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{S}_4 \end{Bmatrix}$$

(iii) $L = K - P$, Assuming centroid at half of Link Length.



By Treating Frame O as Datum:

$$P = M_2 g a_1 + M_3 g (a_1 - l_3/2) + M_4 g (a_1 - l_3/2 - S_4/2)$$

To Find Kinetic Energy, we first find velocity of individual centroids:

$$\dot{p}_{c1} = \dot{R}_{01} p_{c1}$$

$$= \begin{bmatrix} -\dot{\theta}_1 \sin\theta_1 & -\dot{\theta}_1 \cos\theta_1 & 0 \\ \dot{\theta}_1 \cos\theta_1 & -\dot{\theta}_1 \sin\theta_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ L_1/2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\dot{\theta}_1 \frac{L_1}{2} \cos\theta_1 \\ -\dot{\theta}_1 \frac{L_1}{2} \sin\theta_1 \\ 0 \end{pmatrix} = \dot{\theta}_1 \frac{L_1}{2}$$

$$\dot{p}_{c2} = \dot{R}_{01}\, p_1 + \dot{R}_{02}\, p_{c2}$$

$\theta_{12} = \theta_1 + \theta_2$

$$= \begin{pmatrix} -\dot{\theta}_1 L_1 \cos\theta_1 \\ -\dot{\theta}_1 L_1 \sin\theta_1 \\ 0 \end{pmatrix} + \begin{bmatrix} -\theta_{12}\sin\theta_{12} & -\theta_{12}\cos\theta_{12} & 0 \\ \theta_{12}\cos\theta_{12} & -\theta_{12}\sin\theta_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ L_2/2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\dot{\theta}_1 L_1 \cos\theta_1 - \frac{L_2}{2}\theta_{12}\cos\theta_2 \\ -\dot{\theta}_1 L_1 \sin\theta_1 - \frac{L_2}{2}\theta_{12}\sin\theta_2 \\ 0 \end{pmatrix}$$

$$\dot{p}_{c3} = \dot{R}_{01}\, p_1 + \dot{R}_{02}\, p_2 + \dot{R}_{03}\, p_{c3}$$

$$= \begin{pmatrix} -\dot{\theta}_1 L_1 \cos\theta_1 - \frac{L_2}{2}\theta_{12}\cos\theta_2 \\ -\dot{\theta}_1 L_1 \sin\theta_1 - \frac{L_2}{2}\theta_{12}\sin\theta_2 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} -\theta_{123} S_{\theta 123} & -\theta_{123} C_{\theta 123} & 0 \\ \theta_{123} C_{\theta 123} & -\theta_{123} S_{\theta 123} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ -\frac{L_3}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -\dot{\theta}_1 L_1 \cos\theta_1 - \frac{L_2}{2}\theta_{12}\cos\theta_2 \\ -\dot{\theta}_1 L_1 \sin\theta_1 - \frac{L_2}{2}\theta_{12}\sin\theta_2 \\ 0 \end{pmatrix} = \dot{p}_{c2}$$

$$\dot{p}_{c4} = \dot{R}_{01} p_1 + \dot{R}_{02} p_2 + \dot{R}_{03} p_{c4} + R_{03} \dot{s}_4$$

$$= \begin{pmatrix} -\dot{\theta}_1 L_1 \cos\theta_1 - \frac{L_2}{2}\theta_{12} \cos\theta_{12} \\ -\dot{\theta}_1 L_1 \sin\theta_1 - \frac{L_2}{2}\theta_{12} \sin\theta_2 \\ \dot{s}_4 \end{pmatrix}$$

Therefore, we are able to find the K.E of the system:

$$K = \frac{1}{2}m_1 \|\dot{p}_{c1}\|^2 + \frac{1}{2}I_{c1}\dot{\theta}_1^2 + \frac{1}{2}m_2\|\dot{p}_{c2}\|^2 + \frac{1}{2}I_{c2}(\dot{\theta}_1+\dot{\theta}_2)^2$$

$$+ \frac{1}{2}m_3\|\dot{p}_{c3}\|^2 + \frac{1}{2}I_{c3}(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)^2 + \frac{1}{2}m_4\|p_{c4}\|^2 \quad \#$$

sub P, K into the $\boxed{\mathcal{L} = K - P}$ we get the Lagrangian