09/09/2021

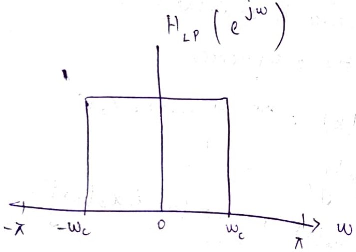
UNIT - 3 $\longrightarrow$ FIR - Filter Design

Filter
- $\rightarrow$ Remove unwanted signals
- $\rightarrow$ FIR - Finite Impulse Response Filter
- $\rightarrow$ IIR - Infinite Impulse Response Filter

FIR Filter Design - Methods
- $\rightarrow$ Fourier series
- $\rightarrow$ Windowing
- $\rightarrow$ Freq. sampling

Design of FIR Filters using Windowing Method:

★ LPF:



α - phase delay

$\boxed{\alpha = \frac{N-1}{2}}$

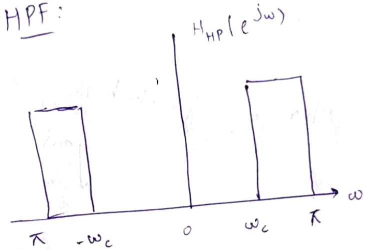
Magnitude response of ideal low pass filter

$$H_d(\omega) = \begin{cases} Ce^{-j\omega\alpha}, & -\omega_c \le \omega \le \omega_c \\ 0, & -\pi < \omega < -\omega_c \;\&\; \omega_c < \omega < \pi \end{cases}$$

Pass band: $-\omega_c$ to ω_c

Remaining region: Stop band

★ HPF:



Magnitude response of ideal high pass filter

ω_c - cut off freq.

Pass band: ω_c to π; $-\omega_c$ to $-\pi$

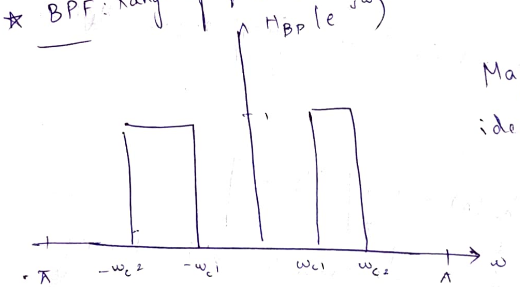
$$H_d(\omega) = \begin{cases} Ce^{-j\omega\alpha}, & -\pi < \omega < -\omega_c \;\&\; \omega_c < \omega < \pi \\ 0, & -\omega_c \le \omega \le \omega_c \end{cases}$$

Desired response

Step ~~1: Identify the design response [$H_d(\omega)$]~~

Range of freq. for which the filter passes the signal.

★ BPF:

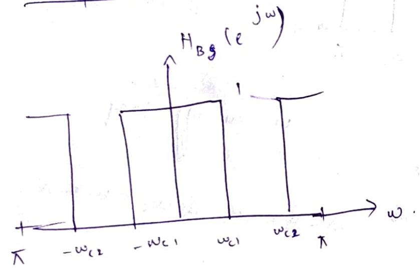


Magnitude response of ideal band pass filter

Pass band: ω_{c1} to ω_{c2}; $-\omega_{c1}$ to $-\omega_{c2}$

$$H_d(\omega) = \begin{cases} Ce^{-j\omega\alpha}, & -\omega_{c2} < \omega < -\omega_{c1} \ \& \ \omega_{c1} < \omega < \omega_{c2} \\ 0, & -\pi \le \omega \le -\omega_{c2} \ \& \ -\omega_{c1} < \omega < \omega_{c1} \ \& \\ & \omega_{c2} < \omega < \pi \end{cases}$$

★ **Band Stop Filter** : Range of ~~filter~~ freq. for which the filter stops the signal.



$$H_d(\omega) = \begin{cases} Ce^{-j\omega\alpha}, & -\pi \le \omega \le -\omega_{c2} \ \& \ -\omega_{c1} < \omega < \omega_{c1} \ \& \ \omega_{c2} < \omega < \pi \\ 0, & -\omega_{c2} < \omega < -\omega_{c1} \ \& \ \omega_{c1} < \omega < \omega_{c2} \end{cases}$$

Step 1: To find out the desired freq. response $H_d(\omega)$

Step 2: To find $h_d(n) = IFT[H_d(\omega)]$

$h_d(n)$ ↓ ~~Desired~~ impulse response in time domain

IFT ↳ Inverse Fourier Transform

$$\boxed{h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega}\, d\omega}$$

Step 3:

Applying windowing Technique

[Convert infinite no. of samples to finite no. of samples]

To find $h(n)$

$$\boxed{h(n) = h_d(n) \cdot w(n)}$$

$w(n)$ - windowing function / window seq.

→ cont. next page

Windowing Techniques:

(i) **Rectangular window:** N-pt. rectangular window

$$W_R(n) = \begin{cases} 1, & -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right) \\ 0, & \text{elsewhere} \end{cases}$$

(or)

$$W_R(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$$

N - order of filter

(ii) Hanning window:

$$W_{HANN}(n) = \begin{cases} 0.5+0.5\cos\frac{2n\pi}{N-1}, & -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right) \\ 0, & \text{elsewhere} \end{cases}$$

(or)

$$W_{HANN}(n) = \begin{cases} 0.5-0.5\,\frac{2n\pi}{N-1}, & 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$$

(iii) Hamming window:

$$W_{HAM}(n) = \begin{cases} 0.54+0.46\cos\frac{2n\pi}{N-1}, & -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right) \\ 0, & \text{elsewhere} \end{cases}$$

(or)

$$W_{HAM}(n) = \begin{cases} 0.54-0.46\cos\frac{2n\pi}{N-1}, & 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$$

Step 4: Transfer function $H(z)$ is obtained by taking z-transform of $h(n)$

Step 5: Using $h(n)$ obtained the eqn for $|H(\omega)|$ for various values of ω in the range $0 \le \omega \le \pi$ & draw the graph b/w $|H(\omega)|$ & ω. It is the freq. response of the filter.

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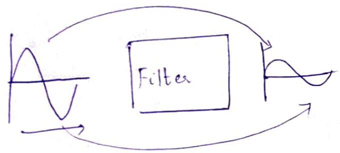
Design of FIR Filter

→ LP

* Linear phase characteristics.



} After filter we are getting the same shape

(i) $\boxed{h(n) = h(N-1-n)}$

h(n) - impulse response of filter
N - order of the filter

(ii) $\boxed{\alpha = \frac{N-1}{2}}$ } FIR filter to become linear phase

α - phase delay.

Eg: N = 5 :

$\alpha = \frac{N-1}{2} = \boxed{2}$

$\boxed{h(n) = h(N-1-n)}$ → Impulse responses are symmetric

$h(0) = h(4)$

$h(1) = h(3)$

Q:

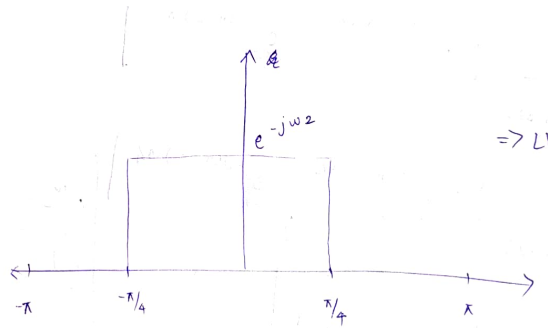
① Design a LPF for the foll. specification :

$$H_d(\omega) = \begin{cases} e^{-j2\omega} , & |\omega| \le \pi/4 \\ 0 , & \pi/4 \le |\omega| \le \pi \end{cases}$$

Use rectangular window & Hamming window for $0 \le n \le 4$
Plot the magnitude response of filter.

→ $$H_d(\omega) = \begin{cases} e^{-j2\omega} , & |\omega| \le \pi/4 \\ 0 , & \pi/4 \le |\omega| \le \pi \end{cases}$$

$H_d(\omega)$ ⇒ desired freq. response filter.



⇒ LPF

$\underbrace{0 \le n \le 4}$

⇒ N = 5

⇒ $\alpha = \frac{5-1}{2} = \cancel{4/2}\ 4/2$

$\boxed{\alpha = 2}$ → $e^{-j\omega\alpha}$ ⇒ $e^{-j2\omega}$

Step 1:

To find $h_d(n)$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(w)\, e^{jwn}\, dw \longrightarrow IFT$$

FT → TD

IFT

i.e. $H_d(w) \rightarrow h_d(n)$

IFT

$$= \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{-jw2}\, e^{jwn}\, dw$$

$$= \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{jw(n-2)}\, dw$$

$$= \frac{1}{2\pi}\left[\frac{e^{jw(n-2)}}{j(n-2)}\right]_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{\pi(n-2)2j}\left[e^{j(n-2)\pi/4} - e^{-j(n-2)\pi/4}\right]$$

$$= \frac{1}{\pi(n-2)}\left[\frac{e^{j(n-2)\pi/4} - e^{-j(n-2)\pi/4}}{2j}\right]$$

$$\frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta$$

$$\boxed{h_d(n) = \frac{1}{\pi(n-2)}\left[\sin(n-2)\cdot\pi/4\right]}$$

N = 5 :

n = 0 :

$$h_d(0) = \frac{1}{\pi(0-2)}\left[\sin(0-2)\cdot\pi/4\right]$$

$$= \frac{-1}{2\pi}\sin\left(-\frac{2\pi}{4}\right)$$

$$= \frac{-1}{2\pi}\cdot -\sin(\pi/2)$$

$$= 1/2\pi\ \sin(\pi/2)$$

$$= \frac{1}{2\pi}\cdot 1 \qquad [\pi = 3.14]$$

$$h_d(0) = 0.159$$

$$\Rightarrow \boxed{h_d(0) = h_d(4) = 0.159}$$

$\sin(-\theta) = -\sin\theta$

$h(n) = h(N-1-n)$

n = 1 :

$$h_d(1) = \frac{1}{\pi(1-2)}\left[\sin(1-2)\cdot\pi/4\right]$$

$$= \frac{-1}{\pi}\sin(-\pi/4) \Rightarrow \frac{\sin \pi/4}{\pi}$$

$$= \frac{0.707}{3.14}$$

$$h_d(1) = 0.225$$

$$\Rightarrow \boxed{h_d(1) = h_d(3) = 0.225}$$

$n = 2$:

$h_d(2) = \frac{1}{\pi(2-2)} (\sin(2-2).\pi/4)$

$h_d(n) = \frac{1}{\pi(n-2)} . \sin(n-2).\pi/4$

0/0

$= 0/0$

L-hospital rule

$h_d(2) = \lim_{n \to 2} \frac{1}{\pi} . \pi/4 \cos(n-2).\pi/4$

$= \frac{1}{\pi} \times \frac{\pi}{4} = 1/4 = 0.25$

$\boxed{h_d(2) = 0.25}$

$h_d(0) = 0.159$	$h_d(3) = 0.159$
$h_d(1) = 0.225$	$h_d(4) = 0.225$
$h_d(2) = 0.25$	

Step 2 : Finding Impulse response $h(n)$

$\boxed{h(n) = h_d(n).w(n)}$

$w(n) \rightarrow$ windowing function

$h(0) = h_d(0). w(0) = h(4)$

$h(1) = h_d(1) . w(1) = h(3)$

$h(2) = h_d(2).w(2)$

Rectangular window:

$$w(n) = \begin{cases} 1 & 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$$

$h(0) = h_d(0) . \underbrace{w(0)}_{1}$

$= 0.159 * 1 = 0.159 = h(4)$

$h(1) = h(3) = 0.225$

$h(2) = 0.25$

Ans.

$h(0) = 0.159$
$h(1) = 0.225$
$h(2) = 0.25$
$h(3) = 0.225$
$h(4) = 0.159$

} Impulse response of FIR filter

$h(n) = \{h(0), h(1), h(2), h(3), h(4)\}$

$h(n) = \{0.159, 0.225, 0.25, 0.225, 0.159\}$

$\Rightarrow \boxed{H(z) = 0.159 + 0.225 z^{-1} + 0.25 z^{-2} + 0.225 z^{-3} + 0.159 z^{-4}}$

→ Magnitude response

$$|H(\omega)'| = \underbrace{e^{j\omega\left(\frac{N-1}{2}\right)}}_{e^{j\omega\alpha}}\left[h\left(\frac{N-1}{2}\right) + 2\sum_{n=0}^{\frac{N-1}{2}} h(n)\cos\left[\left(\frac{N-1}{2} - n\right)\omega\right]\right]$$

$$|H(\omega)'| = e^{j\omega 2}\left[h(2) + 2\sum_{n=0}^{2} h(n)\cos\left[(2-n)\omega\right]\right]$$

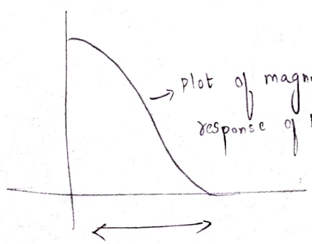
$$= e^{j\omega 2}\left[0.25 + 2\left[h(0)\cos 2\omega + h(1)\cos\omega + h(2)\cos 0\right]\right]$$

$$|H(\omega')| = e^{j\omega 2}\left[0.25 + 2\left[0.159\cos 2\omega + 0.225\cos\omega + 0.25\right]\right]$$

Rough:

$\omega = 0$

H(ω)					
ω	0	25	30	45	90



→ Plot of magnitude response of FIR filter

Steps:

$H_d(\omega) \rightarrow$ desired freq. response filter

$N = ?$

$\alpha = \frac{N-1}{2}$

Step 1:

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega$$

$$\boxed{IFT\left[H_d(\omega)\right] = h_d(n)}$$

$h(n) = h(N-1-n)$

Step 2:

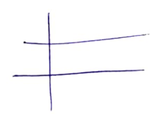
$h(n) = h_d(n)\, w(n)$

Step (3).

$H(z)$

Step (4).

Magnitude response



$|H(\omega)'|$

(2) Same sum but use Hamming window :

$h_d(n) = IFT[H_d(w)]$

$\boxed{h(n) = h(N-1-n)}$

$h(n) = h_d(n) \cdot w(n)$

$h_d(n) = \frac{1}{\pi(n-2)} \sin(n-2)\pi/4$

$$\boxed{\begin{array}{l} h_d(0) = 0.159 = h_d(4) \\ h_d(1) = h_d(3) = 0.225 \\ h_d(2) = 0.25 \end{array}}$$

$w(n) = \begin{cases} 0.54 - 0.46 \cos \frac{2\pi n}{N-1} & 0 \le n < 4 \end{cases}$

$w(0) = 0.54 - 0.46 \cdot 1$

$= 0.08 = w(4)$

$w(1) = w(3) = 0.54 - 0.46 \cos \frac{2\pi}{4}$

$w(1) = 0.54 = w(3)$

$w(2) = 0.54 - 0.46 \cos \frac{4\pi}{2}$

$w(2) = 0.54 + 0.46 \Rightarrow w(2) = 1$

$h(n) = h_d(n) \cdot w(n)$

$h(0) = h_d(0) \cdot w(0)$

$= 0.159 \times 0.08$

$\boxed{h(0) = 0.012 = h(4)}$

$h(1) = h_d(1) \cdot w(1)$

$= 0.225 \times \cancel{0.08}\ 0.54$

$\boxed{h(1) = 0.1215 = h(3)}$

$h(2) = h_d(2) \cdot w(2)$

$= 0.25 * 1$

$\boxed{h(2) = 0.25}$

Q:

(1) Design a linear phase FIR filter using Hamming window for the foll. specifications N = 7.

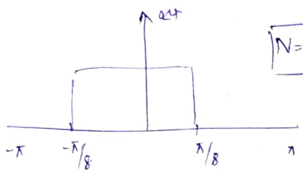
$H_d(w) = \begin{cases} e^{-j3w}, & -\pi/8 \le |w| \le \pi/8 \\ 0, & \pi/8 \le |w| \le \pi \end{cases}$

Plot the magnitude response of the filter.

→ Desired response available from $-\pi/8$ to $\pi/8$ ⟹ Low pass filter.

Step 1.

$H_d(w) = e^{-j3w}$ $\qquad -\pi/8 \le |w| \le \pi/8$



$\boxed{N=7}$ } LPF

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi/8}^{\pi/8} e^{-j3w}\cdot e^{jwn}\, dw$$

$$= \frac{1}{2\pi}\int_{-\pi/8}^{\pi/8} e^{jw(n-3)}\, dw$$

$$= \frac{1}{2\pi}\left[\frac{e^{jw(n-3)}}{j(n-3)}\right]_{-\pi/8}^{\pi/8}$$

$$\boxed{h_d(n) = \frac{1}{\pi(n-3)}\sin(n-3)\pi/8}$$

$h(n) = h(N-1-n)$

$$h_d(0) = h_d(6) = \frac{1}{\pi(-3)}\sin(-3)\pi/8$$

$$= \frac{1}{3\pi}\sin 3\pi/8$$

$$\boxed{h_d(0) = h_d(6) = 0.0980}$$

$$h_d(1) = h_d(5) = \frac{1}{\pi(-2)}\sin(-2)\pi/8$$

$$= \frac{1}{2\pi}\sin\pi/4 = \boxed{0.112}$$

$$\boxed{h_d(2) = h_d(4) = 0.1218}$$

$h_d(3) = \frac{0}{0}$

Applying L hospital rule

$$h_d(3) = \lim_{n\to 3}\frac{1}{\pi}\times\frac{\pi}{8}\left[\cos(n-3).\pi/8\right]$$

$$\boxed{h_d(3) = \frac{1}{8} = 0.125}$$

Step 2: To find the impulse response $h(n)$

$h(n) = h_d(n).w(n)$

Hamming window.

$$w(n) = \{\ 0.54 - 0.46\cos\frac{2\pi n}{N-1}$$

$w(0) = w(6) = 0.54 - 0.46 \times \cos 0$

$= 0.54 - 0.46 \Rightarrow \boxed{0.8}$

$w(1) = w(5) = 0.54 - 0.46\cos\left(\frac{2\pi}{6}\right)$

$= 0.54 - 0.46\cos(\pi/3) \Rightarrow \boxed{0.31}$

$w(2) = w(4) = 0.54 - 0.46\cos\left(\frac{4\pi}{6}\right) = \boxed{0.77}$

$w(3) = 0.54 - 0.46\cos\left(\frac{6\pi}{6}\right) \Rightarrow 0.54 + 0.46 = \boxed{1}$

$h(n) = h_d(n) \times w(n)$

$h(0) = h(6) \Rightarrow 0.0980 \times 0.08 \Rightarrow 0.0018$

$h(1) = h(5) \Rightarrow 0.112 \times 0.31 \Rightarrow 0.034$

$h(2) = h(4) \Rightarrow 0.1218 \times 0.77 \Rightarrow 0.0937$

$h(3) \Rightarrow 0.125 \times 1 = 0.125$

$h(n) = \{0.0018, 0.034, 0.0937, \underset{\uparrow}{0.125}, 0.0937, 0.034, 0.018\}$

$$\boxed{H(z) = 0.0018 + 0.034\,z^{-1} + 0.0937\,z^{-2} + 0.125\,z^{-3} + 0.0937\,z^{-4} + 0.0034\,z^{-5} + 0.018\,z^{-6}}$$

Get formula for magnitude response:

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-1}{2}} a(n) \cos \omega(n)$$

$N = 7$

So limit 0 to 3

Where,

$a(0) = h\left(\frac{N-1}{2}\right)$

$a(n) = 2h\left(\frac{N-1}{2} - n\right)$

$H(e^{j\omega}) = a(0) + a(1)\cos\omega(n) + a(2)\cos 2\omega + a(3)\cos 3\omega$

$a(0) = h(3) = 0.125$

$a(1) = 2h(3-1) = 2h(2) = 2 \times 0.0937 = 0.1874$

$a(2) = 2h(3-2) = 2h(1) = 2 \times 0.034 = 0.068$

$a(3) = 2h(3-3) = 2h(0) = 2 \times 0.0018 = 0.0036$

$\Rightarrow H(e^{j\omega}) = a(0) + a(1)\cos\omega + a(2)\cos 2\omega + a(3)\cos 3\omega$

$$\boxed{H(e^{j\omega}) = 0.125 + 0.1874\cos\omega + 0.068\cos 2\omega + 0.0036\cos 3\omega}$$

ω	0	20	30	40	60	80	90	100
$H(e^{j\omega})$	0.384	0.3548						
$H(e^{j\omega})_{dB}$	-4.1 dB	-4.5 dB						

$\omega = 0$:

$H(e^{j\omega}) = 0.125 + 0.1874 + 0.068 + 0.0036 = \boxed{0.384}$

To convert into dB.

$10 \log(0.384) = \boxed{-4.1\ dB}$

$\omega = 20$:

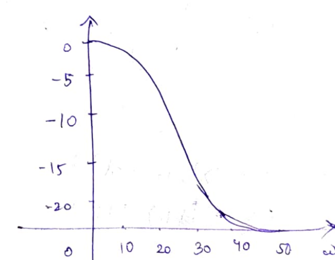
$H(e^{j\omega}) = 0.125 + 0.1874(0.939) + 0.068(0.7660) + 0.036(0.5)$

$= 0.125 + 0.1759 + 0.0520 + 0.018$

$= 0.3548$

dB:

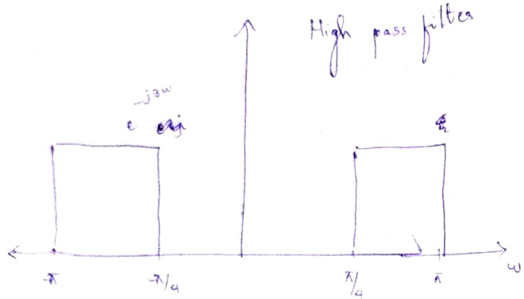
$10 \log(0.3548) = \boxed{-4.5\ dB}$



(2) Design a linear phase FIR filter using Hamming & Hanning window for the foll. specifications

$$H_d(w) = \begin{cases} e^{-j3w}, & \pi/4 \le |w| \le \pi \\ 0, & \text{otherwise} \end{cases}$$

→ High pass filter



$e^{-j3w} \Rightarrow e^{-jw\alpha}$

$\Rightarrow \boxed{\alpha = 3}$

$\alpha = \frac{N-1}{2} \Rightarrow 3 = \frac{N-1}{2}$

$\Rightarrow \boxed{N = 7}$ ← Order

$$h_d(n) = \frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/4} e^{-j3w} e^{jwn}\, dw + \int_{\pi/4}^{\pi} e^{-j3w} e^{jwn}\, dw\right]$$

* $h_d(n) \Rightarrow h_d(0) = h_d(6)$
$h_d(1) = h_d(5)$

* $h(n) = h_d(n) \cdot w(n)$

* $H(z)$

* $H(e^{jw}) = \sum_{n=0}^{\frac{N-1}{2}} a(n) \cos w(n)$

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(3) Design a linear phase FIR filter using Hamming & Hanning window for the foll. specifications. Find the values of $h(n)$ for $N = 11$. Find $H(z)$. Plot the magnitude response

$$H_d(w) = \begin{cases} 1 & \pi/4 \le |w| \le \pi \\ 0 & \text{otherwise} \end{cases}$$

⇒ High pass filter.