UNIT I

FINITE IMPULSE RESPONSE (FIR) FILTER

Digital transfer fn (time domain classification)
based on length ⟹ FIR & IIR.

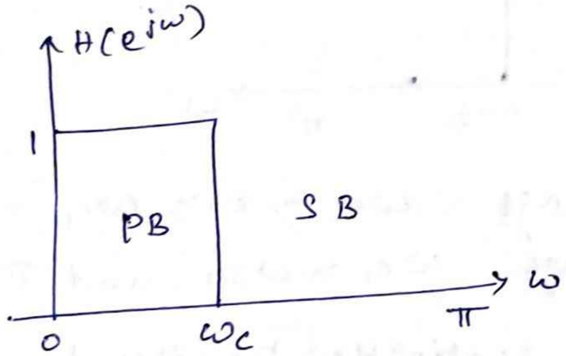
based on digital transfer fn - four types of filters.

Pass band (PB) - the range of frequencies where the freq response of the signal takes the value

stop band (SB) - the range of frequencies where the freq response of the signal is zero.

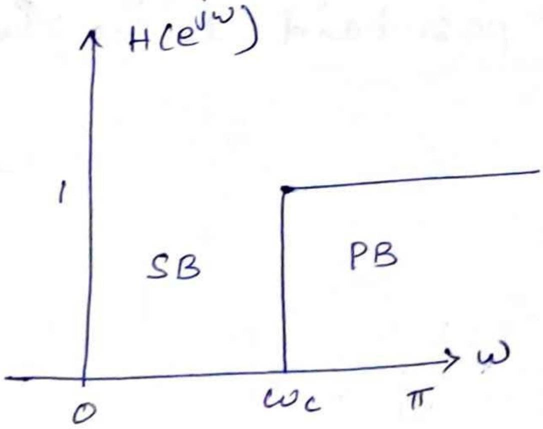
(1) Low pass filter (LPF)
(2) High pass filter (HPF)
(3) Band pass filter (BPF)
(4) band stop filter (BSF).

LPF



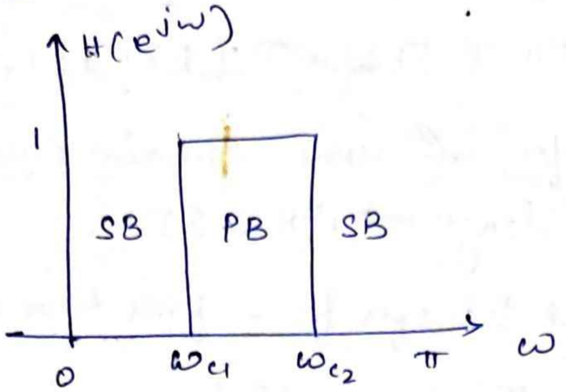
passes low freq $\omega_c \geq \omega \geq 0$
blocks high freq $\pi \geq \omega \geq \omega_c$

HPF



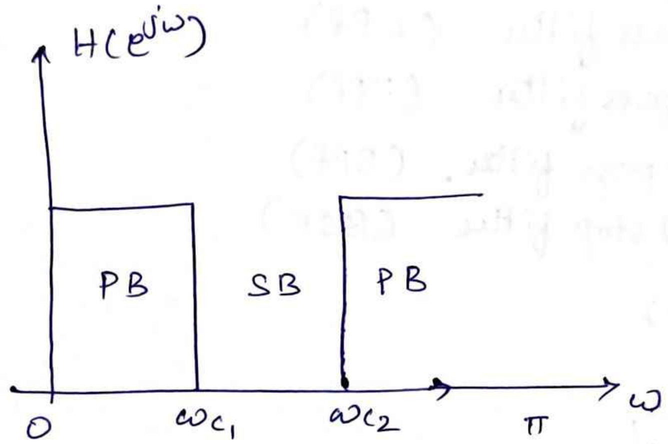
passes high freq $\pi \geq \omega \geq \omega_c$
blocks low freq : $\omega_c \geq \omega \geq 0$

Band pass filter



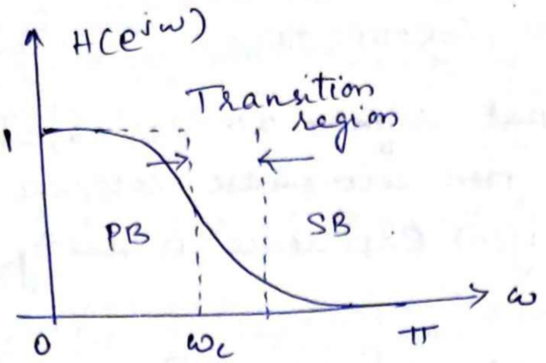
passes band of freq in the range $\omega_{c_2} \geq \omega \geq \omega_{c_1}$
blocks freq in the range $\omega_{c_1} \geq \omega \geq 0$ & $\pi \geq \omega \geq \omega_{c_2}$

Band stop filter

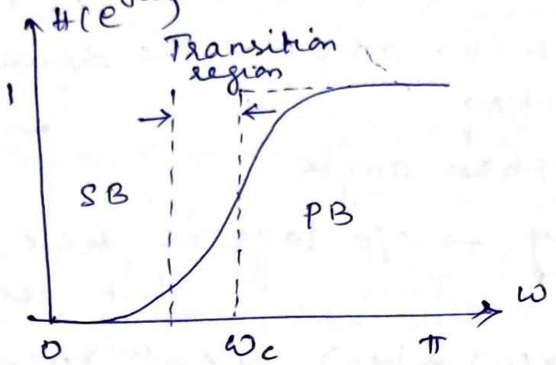


Stops freq in the range $\omega_{c_2} \geq \omega \geq \omega_{c_1}$
pass freq in the range $\omega_{c_1} \geq \omega \geq 0$ and $\pi \geq \omega \geq \omega_{c_2}$.

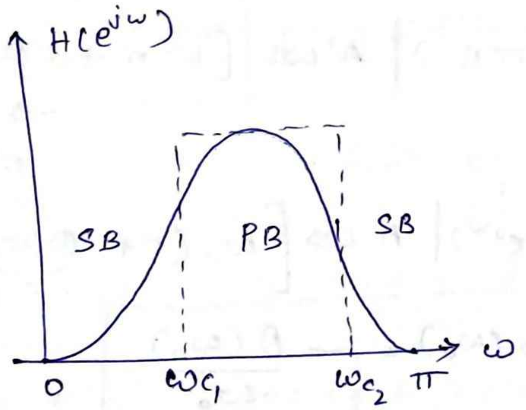
To develop stable and realizable transfer fn, the ideal freq response of the filters are relaxed, by introducing a transition band between the pass band and the stop band to permit the mag response to decay more gradually from its maximum value in the pass band to the zero value in the stop band.



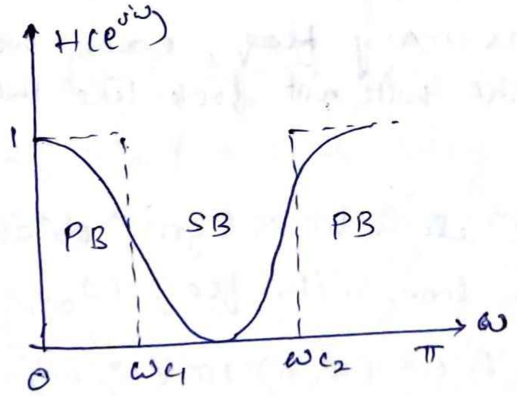
LPF



HPF



BPF



BSF

PHASE DELAY AND GROUP DELAY

When the input signal $x(n)$ is applied to a system which has non-zero phase response $\theta(\omega) = \arg[H(e^{j\omega})]$, the output signal $y(n)$ experience a delay w.r.t the input signal.

Let $x(n) = A\cos[\omega_0 n + \phi]$, $+\infty > n > -\infty$

$A \rightarrow$ max. amp. of the signal

$\omega_0 \rightarrow$ freq

$\phi \rightarrow$ phase angle

due to the delay $\rightarrow$ o/p lags in phase $\theta(\omega_0)$ but freq remains same.

$$y(n) = x(n) * h(n) = X(e^{j\omega})H(e^{j\omega})$$

$$y(n) = |H(e^{j\omega})|\, A\cos[\omega_0 n + \theta(\omega_0) + \phi]$$

$\rightarrow$ time delayed

Phase delay at $\omega = \omega_0$

$$y(n) = |H(e^{j\omega})|\, A\cos\left[\omega_0\left(n + \frac{\theta(\omega_0)}{\omega_0}\right) + \phi\right]$$

$$\boxed{\text{Phase delay} \quad \tau_p(\omega_0) = -\frac{\theta(\omega_0)}{\omega_0}}$$

When i/p signal has many freq, each with diff phase delay, then o/p signal will not look like the i/p signal.

- Group delay

Consider ex. a discrete-time signal obtained by DSB-SC with carrier freq ω_c & sine with freq ω_0.

$$x(n) = A\cos(\omega_0 n)\cos(\omega_c n)$$

$$x(n) = \frac{A}{2}\cos(\omega_c - \omega_0)n + \frac{A}{2}\cos(\omega_c + \omega_0)n.$$

let $\omega_c - \omega_0 = \omega_l$

$\omega_c + \omega_0 = \omega_u$

$$x(n) = \frac{A}{2}\cos\omega_l n + \frac{A}{2}\cos\omega_u n.$$

If the i/p signal is passed thru LTI system whose response is $H(e^{j\omega})$, Then o/p is

$$y(n) = \frac{A}{2}|H(e^{j\omega})| \cos[\omega_l n + \theta(\omega_l)] + \frac{A}{2}|H(e^{j\omega})| \cos[\omega_u(n) + \theta(\omega_u)]$$

$$y(n) = A|H(e^{j\omega})| \cos\left[\left(\frac{\omega_u + \omega_l}{2}\right)n + \frac{\theta(\omega_u) + \theta(\omega_l)}{2}\right] \cos\left[\left(\frac{\omega_u - \omega_l}{2}\right)n + \frac{\theta(\omega_u) - \theta(\omega_l)}{2}\right]$$

$$y(n) = A\cos\left[\omega_c n + \frac{\theta(\omega_u) + \theta(\omega_l)}{2}\right] \cos\left[\omega_0 n + \frac{\theta(\omega_u) - \theta(\omega_l)}{2}\right] |H(e^{j\omega})|$$

$\omega_c - \omega_0 = \omega_l$

$\omega_c + \omega_0 = \omega_u$

$2\omega_c = \omega_l + \omega_u$

$\omega_c = \frac{\omega_l + \omega_u}{2}$

$\omega_c - \omega_0 = \omega_l$

$-\omega_c \mp \omega_0 = \omega_u$

$-2\omega_0 = \omega_l - \omega_u$

$\frac{\omega_u - \omega_l}{2} = \omega_0$

assume $|H(e^{j\omega})| \simeq 1$ in the range $\omega_u \geq \omega \geq \omega_l$

$$y(n) = A\cos\left[\omega_c n + \frac{\theta(\omega_u) + \theta(\omega_l)}{2}\right] \cos\left[\omega_0 n + \frac{\theta(\omega_u) - \theta(\omega_l)}{2}\right]$$

→ eqn shows signal has diff phase delay

Let us Redefine the eqn $x(n) = \frac{A}{2}\cos\omega_l n + \frac{A}{2}\cos\omega_u n$.

For a narrow band signal wherein ω_l and ω_u are so close to the carrier freq ω_c (ω_0 is very small), then the phase delay due to carrier can be expanded using Taylor's series as

$$\theta_c(\omega) \simeq \theta_c(\omega_c) + \left.\frac{d\theta_c(\omega)}{d\omega}\right|_{\omega=\omega_c} (\omega - \omega_c) + \cdots$$

// Taylor series

real fn $f(x)$ at a point $x = a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

3½ 13, 7, 15, 33, 35, 04, 303, 305

Let us consider first term in the eqn

$\frac{\omega_u+\omega_l}{2}=\omega_c$

$\frac{\omega_u-\omega_l}{2}=\omega_o$

$$-\left[\frac{\theta(\omega_u)+\theta(\omega_l)}{2\omega_c}\right] \simeq -\frac{\theta_c(\omega_c)}{\omega_c}$$

which is same as the phase delay

-ve sign indicates phase lag.

The second term can be written as

$-[\theta(\omega_u)-\theta(\omega_l)]$

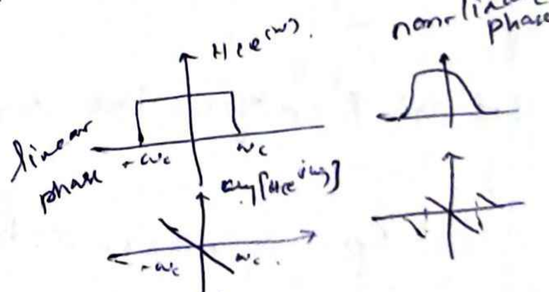
$$-\frac{\theta(\omega_u)+\theta(\omega_l)}{2\omega_o} = -\frac{\theta(\omega_u)+\theta(\omega_l)}{2(\omega_u-\omega_l)} \simeq -\frac{d\theta_c(\omega_c)}{d\omega}\Bigg|_{\omega=\omega_c}$$

$$\boxed{\tau_g(\omega_c) = -\frac{d\theta_c(\omega)}{d\omega}\Bigg|_{\omega=\omega_c} \quad \text{Group delay}}$$

Group delay is differential of phase delay.

The o/p will be distorted if group delay is not constant in LTI system.

delay equalizer is used to obtain distortionless overall o/p response.



LINEAR PHASE TRANSFER FUNCTION.

Filters with linear or zero phase is required for signal processing applications.

but zero phase are not causal. (made arbitrarily causal)

possible to design FIR filter but impossible to design IIR filter.

The transfer fn has linear phase if its impulse response h[n] is either symmetric or antisymmetric.

Let us analyze a causal transfer fn of a FIR filter.

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

F.T of impulse fn $h(n)$ is

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n}.$$

Polar representation

$$H(e^{j\omega}) = \pm |H(e^{j\omega})| e^{j\theta(\omega)}.$$

where $H(e^{j\omega})$ → mag response

$e^{j\theta(\omega)}$ → phase response.

Let us assume the phase of FIR filter transfer fn is linear

i) $\theta(\omega) = -\alpha\omega$, $\pi \geq \omega \geq -\pi$.

α → constant delay in phase.

$$\tau_p = -\frac{\theta(\omega)}{\omega} = \frac{+\alpha\omega}{\omega} = \alpha$$

$$\tau_g = -\frac{d\theta(\omega)}{d\omega} = \frac{-d(-\alpha\omega)}{d\omega} = \alpha.$$

Both phase delay and group delay are constant and independent of freq.

The impulse response of FIR filter has linear phase if its impulse response satisfies the following condition

(i) for symmetric impulse response $h(n)$

$$h(n) = h(N-1-n) \quad , \quad (N-1) \geq n \geq 0$$

(ii) For antisymmetric impulse response $h(n)$

$$h(n) = -h(N-1-n) \quad , \quad (N-1) \geq n \geq 0.$$

The length of the impulse response h(n) can be either even or odd. Based on this, FIR is divided into

(i) Symmetric impulse response with odd length
(ii) Symmetric impulse response with even length
(iii) Antisymmetric impulse response with odd length
(iv) Antisymmetric impulse response with even length.

(i) Symmetric impulse response with odd length.

Let $N = 9$.

$$H(z) = \sum_{n=0}^{8} h(n) z^{-n}.$$

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7} + h(8)z^{-8}.$$

using symmetric property.

$$h(n) = h(N-1-n)$$

$$h(0) = h(9-1-0) = h(8)$$

$$h(1) = h(7)$$

$$h(2) = h(6)$$

$$h(3) = h(5)$$

$$h(4) = h(4).$$

$$H(z) = h(0)(1+z^{-8}) + h(1)(z^{-1}+z^{-7}) + h(2)(z^{-2}+z^{-6}) + h(3)(z^{-3}+z^{-5}) + h(4)z^{-4}.$$

$$H(z) = z^{-4}\left[h(0)(z^{4}+z^{-4}) + h(1)(z^{3}+z^{-3}) + h(2)(z^{2}+z^{-2}) + h(3)(z^{1}+z^{-1}) + h(4)\right]$$

Freq response is obtained by taking F.T $z = e^{j\omega}$

⑤

$$H(e^{j\omega}) = e^{-j4\omega}\Big[h(0)\left(e^{j4\omega}+e^{-j4\omega}\right) + h(1)\left(e^{j3\omega}+e^{-j3\omega}\right) + h(2)\left(e^{j2\omega}+e^{-j2\omega}\right) + h(3)\left(e^{j\omega}+e^{-j\omega}\right) + h(4)\Big]$$

$$H(e^{j\omega}) = e^{-j4\omega}\Big[2h(0)\cos 4\omega + 2h(1)\cos 3\omega + 2h(2)\cos 2\omega + 2h(3)\cos\omega + h(4)\Big]$$

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1st hr.
3, 4, 9, 17, 24, 32, 36, 37, 38, 45, 50, 53, 60, 63, 302,

$$\theta(\omega) = -4\omega + \beta$$

$\beta \rightarrow 0$ or 2π

phase delay $\tau_p = 4$

group delay $\tau_g = 4$

Since both are constant $H(e^{j\omega})$ is a linear fn of ω.

In general case.

$$H(e^{j\omega}) = \sum_{n=0}^{8} h(n)\,e^{-j\omega n}$$

$$H(e^{j\omega}) = \sum_{n=0}^{3} h(n)\,e^{-j\omega n} + h(4)\,e^{-j4\omega} + \sum_{n=5}^{8} h(n)\,e^{-j\omega n}$$

In general

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)\,e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=\left(\frac{N+1}{2}\right)}^{N-1} h(n)\,e^{-j\omega n}$$

Let $n = N-1-m$ be the third term — $m=0$, $m=\frac{N-3}{2}$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)\,e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{m=0}^{\left(\frac{N-3}{2}\right)} h(N-1-m)\,e^{-j\omega(N-1-m)}$$

using symmetry property

$$h(n) = h(N-1-n)$$

$$\therefore H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)\,e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n)\,e^{-j\omega(N-1-n)}$$

$$H(e^{j\omega}) = h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n)\left[e^{-j\omega n} + e^{-j\omega(N-1-n)}\right]$$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n)\left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right]\right\}$$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} 2h(n)\cos\omega\left(\frac{N-1}{2}-n\right)\right]$$

Let $\frac{N-1}{2} - n = m$, Then

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + 2\sum_{m=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2}-m\right)\cos\omega m\right]$$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + 2\sum_{n=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2}-n\right)\cos\omega n\right]$$

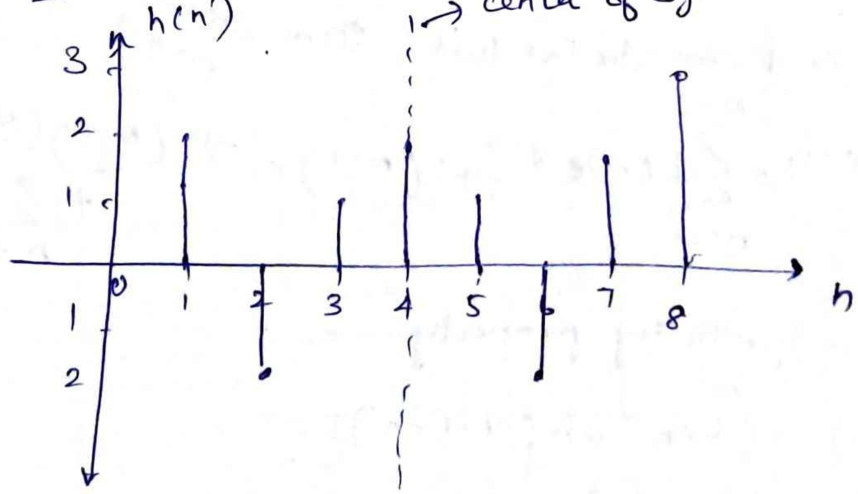
$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\bar{H}(e^{j\omega})$$

$$\bar{H}(e^{j\omega}) = h\left(\frac{N-1}{2}\right) + 2\sum_{n=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2}-n\right)\cos\omega n$$

$$\theta(\omega) = -\alpha\omega = \left(\frac{N-1}{2}\right)\omega.$$

$$\therefore \alpha = \frac{N-1}{2}.$$

→ center of symmetric $\left(\frac{N-1}{2}\right)$.



Zero phase filter.

To avoid phase distortion the freq response of the filter should not delay the spectral components. Such transfer fn is said to have zero-phase characteristic.

A zero-phase TF has no phase component i.e) spectrum is purely real - no img component & non-negative.

But it is not possible to design a causal digital filter with a zero phase.

causal real-coefficient filter is used.

All freq are delayed by α sec, ∴ there is no distortion.

If the filter does not have linear phase, then diff freq components are delayed by diff amounts, causing significant distortion.

Zero phase - filtering:

(1) process i/p data (finite length) with a causal real-coefficient filter $H(z)$

(2) Time reverse the o/p of the filter and process by the same filter

(3) Time reverse once again the o/p of the second filter.

$$x(n) \longrightarrow \boxed{H(z)} \longrightarrow v(n) \qquad u(n) \longrightarrow \boxed{H(z)} \longrightarrow w[n]$$

$u(n) = v[-n]$ $\qquad$ $y[n] = w[-n]$.

$V(\omega) = H(\omega) X(\omega)$ $\qquad$ $W(\omega) = H(\omega) U(\omega)$

$U(\omega) = V^*(\omega)$ $\qquad$ $Y(\omega) = W^*(\omega) = H^*(\omega) U^*(\omega)$

$Y(\omega) = H^*(\omega) V(\omega) = H^*(\omega) \cdot H(\omega) X(\omega) = |H(\omega)|^2 X(\omega)$

2. Symmetric Impulse response with Even length.

Consider length of impulse response is even

$$H(z) = \sum_{n=0}^{7} h(n) z^{-n}$$

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7}.$$

using symmetric property

$$h(n) = h(N-1-n)$$

$h(0) = h(7)$
$h(1) = h(6)$
$h(2) = h(5)$
$h(3) = h(4)$.

$$H(z) = h(0)(1+z^{-7}) + h(1)(z^{-1}+z^{-6}) + h(2)(z^{-2}+z^{-5}) + h(3)(z^{-3}+z^{-4})$$

$$H(z) = z^{-7/2}\Big[h(0)(z^{7/2}+z^{-7/2}) + h(1)(z^{5/2}+z^{-5/2}) + h(2)(z^{3/2}+z^{-3/2}) + h(3)(z^{1/2}+z^{-1/2})\Big]$$

Freq response is obtained by taking F.T ie) $z = e^{j\omega}$

$$H(e^{j\omega}) = e^{-j\frac{7}{2}\omega}\Big[h(0)\big[e^{j\frac{7}{2}\omega} + e^{-j\frac{7}{2}\omega}\big] + h(1)\big[e^{j\frac{5}{2}\omega} + e^{-j\frac{5}{2}\omega}\big] + h(2)\big[e^{j\frac{3}{2}\omega} + e^{-j\frac{3}{2}\omega}\big] + h(3)\big[e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}\big]\Big]$$

$$H(e^{j\omega}) = e^{-j\frac{7}{2}\omega}\Big[2h(0)\cos\frac{7}{2}\omega + 2h(1)\cos\frac{5}{2}\omega + 2h(2)\cos\frac{3}{2}\omega + 2h(3)\cos\frac{\omega}{2}\Big]$$

Comparing this expression with $H(e^{j\omega}) = \pm|H(e^{j\omega})|e^{j\theta(\omega)}$

$$\theta(\omega) = -\alpha\omega = -\frac{7}{2}\omega + \beta$$

β is either 0 or 2π

phase delay $\tau_p = -\alpha = \frac{7}{2}$

Group delay $\tau_g = -\alpha = \frac{7}{2}$

Since, both delay are constant, it is a linear fn of ω.

In general case.

$$H(e^{j\omega}) = \sum_{n=0}^{7} h(n) e^{-j\omega n}$$

$$H(e^{j\omega}) = \sum_{n=0}^{3} h(n) e^{-j\omega n} + \sum_{n=4}^{7} h(n) e^{-j\omega n}$$

in general

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n) e^{-j\omega n}$$

Let $n = (N-1-m)$ in the second term

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega n} + \sum_{m=0}^{\frac{N}{2}-1} h(N-1-m) e^{-j\omega(N-1-m)}$$

$m=n$

using symmetry $= \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(N-1-n) e^{-j\omega(N-1-n)}$

$h(n) = h(N-1-n)$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega(N-1-n)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n) \left[e^{-j\omega n} + e^{-j\omega(N-1-n)}\right]$$

$\frac{N-1}{2} - n - \frac{N-1}{2}$

$\frac{N-1}{2} - n + \frac{N-1}{2}$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left\{\sum_{n=0}^{\frac{N}{2}-1} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right]\right\}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\frac{N}{2}-1} 2h(n) \cos\omega\left(\frac{N-1}{2}-n\right)$$

let $\frac{N}{2} - n = m$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{m=\phi}^{\frac{N}{2}} 2h\left(\frac{N}{2} - m\right) \cos\omega\left(m - \frac{1}{2}\right)\omega \right]$$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{n=\phi}^{\frac{N}{2}} 2h\left(\frac{N}{2} - n\right) \cos\left(n - \frac{1}{2}\right)\omega \right]$$

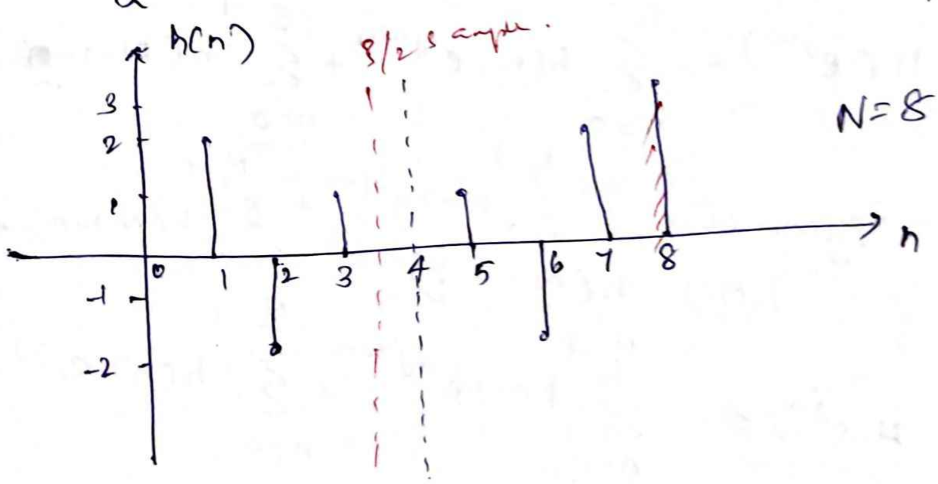
$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \bar{H}(e^{j\omega})$$

$$\bar{H}(e^{j\omega}) = \sum_{n=\phi}^{\frac{N}{2}} 2h\left(\frac{N}{2} - n\right) \cos\left(n - \frac{1}{2}\right)\omega$$

on comparing $\theta(\omega) = -\alpha\omega = \left(\frac{N-1}{2}\right)\omega$

$\alpha = \frac{N-1}{2}$ → center of symmetric.

0, 2, −2, 1, 1, −2, 2, 0



(3) Antisymmetric Impulse response with odd length.

Let length $N = 9$

$$H(z) = \sum_{n=0}^{8} h(n) z^{-n}$$

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + \cancel{h(4)}z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7} + h(8)z^{-8}$$

using antisymm property

$h(n) = -h(N-1-n)$

$h(0) = -h(8)$ $\quad$ $h(3) = -h(5)$

$h(1) = -h(7)$ $\quad$ $h(4)$ ~~$= -h(4)$~~ does not have center value $h(4) = 0$. $h\left(\frac{N-1}{2}\right) = 0$

$h(2) = -h(6)$

$$H(z) = h(0)(1 - z^{-8}) + h(1)(z^{-1} - z^{-7}) + h(2)(z^{-2} - z^{-6}) + h(3)(z^{-3} - z^{-5})$$

Taking F.T i.e) $z = e^{j\omega}$

$$H(e^{j\omega}) = h(0)(1 - e^{-j8\omega}) + h(1)(e^{-j\omega} - e^{-j7\omega}) + h(2)(e^{-j2\omega} - e^{-j6\omega}) + h(3)(e^{-j3\omega} - e^{-j5\omega}).$$

$$H(e^{j\omega}) = e^{-j4\omega}\Big[h(0)(e^{j4\omega} - e^{-j4\omega}) + h(1)(e^{j3\omega} - e^{-j3\omega}) + h(2)(e^{j2\omega} - e^{-j2\omega}) + h(3)(e^{j\omega} - e^{-j\omega})\Big]$$

$$= e^{-j4\omega}(2j)\Big[h(0)\sin 4\omega + h(1)\sin 3\omega + h(2)\sin 2\omega + h(3)\sin\omega\Big]$$

$$= e^{-j4\omega} \cdot e^{j\pi/2} \cdot 2 \cdot \Big[h(0)\sin 4\omega + h(1)\sin 3\omega + h(2)\sin 2\omega + h(3)\sin\omega\Big]$$

$$= e^{-j(4\omega - \pi/2)} \, 2 \cdot \Big[h(0)\sin 4\omega + h(1)\sin 3\omega + h(2)\sin 2\omega + h(3)\sin\omega\Big]$$

$$\theta(\omega) = -(\alpha\omega + \beta).$$

$$\theta(\omega) = -\left(4\omega + \frac{\pi}{2}\right) = -4\omega + \frac{\pi}{2}$$

Phase delay $\tau_p = -\alpha = 4$ $\quad \beta = -\frac{\pi}{2}$

Group delay $\tau_g = -\alpha = 4.$

Since both delay are constant, it is a linear fn of ω.

In general case.

$$H(e^{j\omega}) = \sum_{n=0}^{8} h(n) e^{-j\omega n}$$

$$H(e^{j\omega}) = \sum_{n=0}^{3} h(n) e^{-j\omega n} + \sum_{n=5}^{8} h(n) e^{-j\omega n}$$

$h(4) = 0$ for antisymmetric impulse response sequence of odd length.

In general,

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n) e^{-j\omega n}$$

For antisymmetric with odd length $h\left(\frac{N-1}{2}\right) = 0$

$n = N-1-m$ $\quad H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n) e^{-j\omega(N-1-n)}$

using antisymm property

$$h(n) = -h(N-1-n)$$

$$\therefore H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} - \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega(N-1-n)}$$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n) \left[e^{-j\omega n} - e^{-j\omega(N-1-n)} \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left\{ \sum_{n=0}^{\frac{N-3}{2}} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} - e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right] \right.$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} (2j) \left\{ \sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot \sin \omega\left(\frac{N-1}{2}-n\right) \right\}$$

let $\frac{N-1}{2} - n = m$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot e^{j\pi/2} \left[\sum_{m=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-m\right) \sin \omega m \right]$$

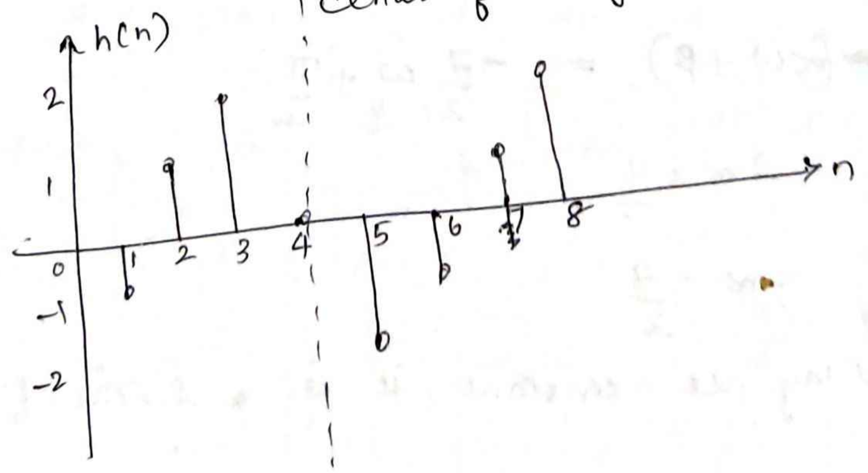
$$H(e^{j\omega}) = e^{-j\left[\omega\left(\frac{N-1}{2}\right) - \pi/2\right]} \overline{H}(e^{j\omega})$$

$$\overline{H}(e^{j\omega}) = \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-n\right) \sin \omega n.$$

$$\theta(\omega) = -\alpha\omega + \beta = -\left(\frac{N-1}{2}\right)\omega + \frac{\pi}{2}.$$

$$\alpha = \frac{N-1}{2}, \quad \beta = -\frac{\pi}{2}.$$

center of antisymmetric $\left(\frac{N-1}{2}\right)$.



(A) Antisymmetric impulse response with even length.

Let $N=8$

$$H(z)=\sum_{n=0}^{7} h(n)z^{-n}$$

$$H(z)=h(0)+h(1)z^{-1}+h(2)z^{-2}+h(3)z^{-3}+h(4)z^{-4}+h(5)z^{-5}$$
$$+h(6)z^{-6}+h(7)z^{-7}.$$

using antisymm property

$$h(n)=-h(N-1-n).$$

$h(0)=-h(7)$ $\quad h(2)=-h(5)$

$h(1)=-h(6)$ $\quad h(3)=-h(4)$

$$H(z)=h(0)(1-z^{-7})+h(1)(z^{-1}-z^{-6})+h(2)(z^{-2}-z^{-5})+h(3)(z^{-3}-z^{-4})$$

$$=z^{-7/2}\left[h(0)(z^{7/2}-z^{-7/2})+h(1)(z^{5/2}-z^{-5/2})+h(2)(z^{3/2}-z^{-3/2})\right.$$
$$\left.+h(3)(z^{1/2}-z^{-1/2})\right]$$

Taking f.T $z=e^{j\omega}$

$$H(e^{j\omega})=e^{-j7/2\omega}(2j)\left[h(0)\sin\frac{7}{2}\omega+h(1)\sin\frac{5}{2}\omega+h(2)\sin\frac{3}{2}\omega\right.$$
$$\left.+h(3)\sin\frac{1}{2}\omega\right]$$

$$=e^{-j7/2\omega}\cdot e^{j\pi/2}\left\{2\left[h(0)\sin\frac{7}{2}\omega+h(1)\sin\frac{5}{2}\omega+h(2)\sin\frac{3}{2}\omega\right.\right.$$
$$\left.\left.+h(3)\sin\frac{1}{2}\omega\right]\right\}$$

$$\theta(\omega)=(\alpha\omega+\beta)=-\frac{7}{2}\omega+\frac{\pi}{2}$$

$$\tau_p=-\alpha=\frac{7}{2}$$

$$\tau_g=-\alpha=\frac{7}{2}$$

since both delay are constant, it is a linear fn of ω.

In general

$$H(e^{j\omega}) = \sum_{n=0}^{7} h(n) e^{-j\omega n}$$

$$H(e^{j\omega}) = \sum_{n=0}^{3} h(n) e^{-j\omega n} + \sum_{n=4}^{7} h(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(N-1-n) e^{-j\omega (N-1-n)}$$

using antisymm

$$h(n) = -h(N-1-n)$$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega n} - \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega (N-1-n)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n) \left[e^{-j\omega n} - e^{-j\omega (N-1-n)} \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\frac{N}{2}-1} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} - e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot 2j \cdot \sum_{n=0}^{\frac{N}{2}-1} h(n) \sin\omega\left(\frac{N-1}{2}-n\right)$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot e^{j\pi/2} \, 2 \sum_{n=0}^{\frac{N}{2}-1} h(n) \sin\omega\left(\frac{N-1}{2}-n\right)$$

let $\frac{N}{2} - n = m$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} e^{j\pi/2} \left[\sum_{m=1}^{\frac{N}{2}} 2h\left(\frac{N}{2} - m\right) \sin\omega\left(m - \frac{1}{2}\right) \right]$$

$$= e^{-j\left[\left(\frac{N-1}{2}\right)\omega - \frac{\pi}{2}\right]} \underbrace{\left[\sum_{n=1}^{\frac{N}{2}} 2h\left(\frac{N}{2} - n\right) \sin\omega\left(n - \frac{1}{2}\right) \right]}_{\bar{H}(e^{j\omega})}$$

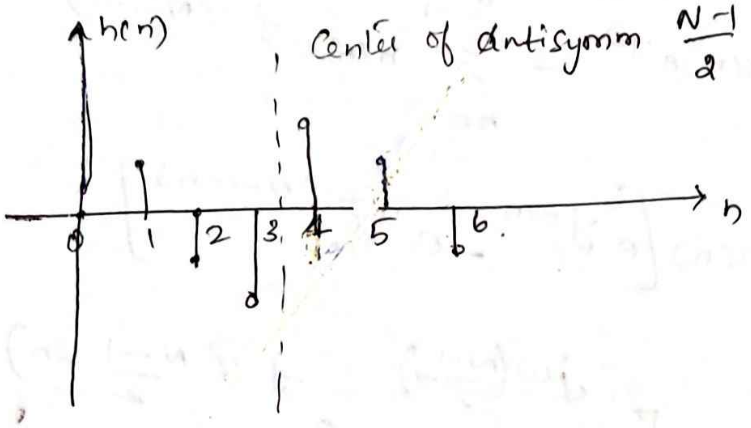
$$\theta(\omega) = -\left(\frac{N-1}{2}\omega + \frac{\pi}{2}\right)$$

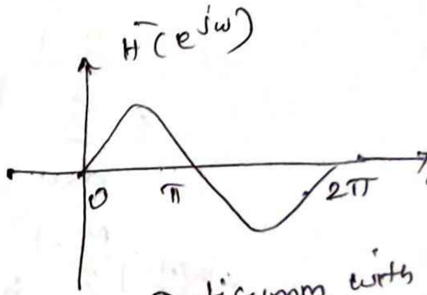
$$= -(\alpha\omega + \beta)$$

$$= -\left(\frac{N-1}{2}\right)\omega + \frac{\pi}{2}$$

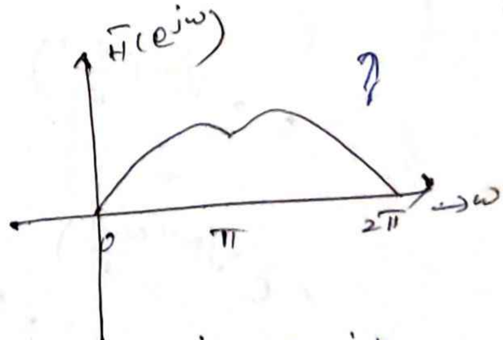
$\tau_p = -\alpha = \frac{N-1}{2}$ $\quad \beta = -\frac{\pi}{2}$.

$\tau_g = -\alpha = \frac{N-1}{2}$.





antisymm with N odd.



antisymm with N even.

18, 58, 200

DESIGN OF FIR FILTER - FOURIER METHOD

The desired frequency response of a FIR filter can be represented as

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n)\, e^{-j\omega n} \quad \text{--- (1)}$$

The freq response $H(e^{j\omega})$ is periodic with 2π.

The desired impulse response of the filter is given by

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega \quad \text{--- (2)}$$

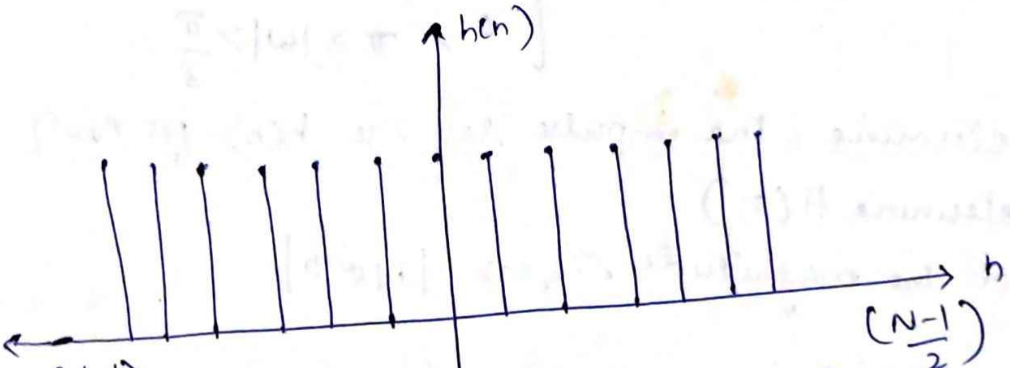
Z.T

$$H_d(z) = \sum_{n=-\infty}^{\infty} h_d(n)\, z^{-n} \quad \text{--- (3)}$$

→ this is a infinite non-causal system. Not realizable.

In order to realize a finite system, the impulse response has to be finitized. This is done by truncating the long sequence as given below

$$h(n) = \begin{cases} h_d(n), & |n| \le \left(\frac{N-1}{2}\right) \\ 0, & \text{elsewhere} \end{cases} \quad \text{--- (4)}$$



Impulse response of data sequence

Sub ④ in ③

$$H(z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n) z^{-n} = H(0) + \sum_{n=1}^{\frac{N-1}{2}} \left[h(-n) z^{n} + h(n) z^{-n} \right]$$

$h(n) = h(-n)$

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) \left(z^{n} + z^{-n} \right)$$

Since impulse response is symmetric at center

$$h(n) = h(-n)$$

$$\therefore \ H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) \left[z^{n} + z^{-n} \right] . \rightarrow \text{non-causal}$$

not realizable.

delay unit can be realized using flip flop but advance unit is not realizable.

It can be converted to realizable filter by multiplying entire system with $z^{-\left(\frac{N-1}{2}\right)}$

$$\overline{H}(z) = \left\{ h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) \left[z^{n} + z^{-n} \right] \right\} z^{-\left(\frac{N-1}{2}\right)}$$

$$\overline{H}(z) = z^{-\left(\frac{N-1}{2}\right)} H(z)$$

Problems

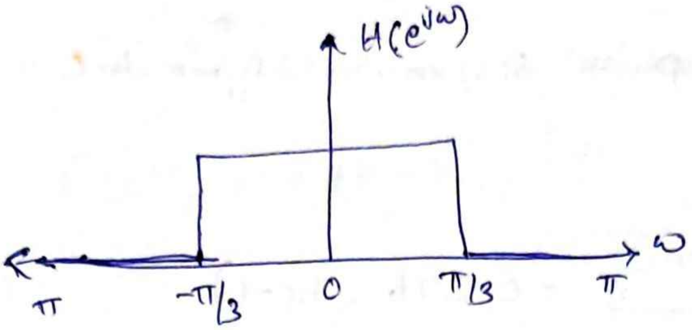
(1) Design an ideal low pass filter whose desired freq response is

$$H_d(e^{j\omega}) = \begin{cases} 1, & \frac{\pi}{3} \geq \omega \geq -\frac{\pi}{3} \\ 0, & \pi \geq |\omega| > \frac{\pi}{3} \end{cases}$$

(i) Determine the impulse response $h(n)$ for $N=9$

(ii) Determine $H(z)$

(iii) plot the magnitude response $|H(e^{j\omega})|$



desired impulse response

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}\,d\omega$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi/3}^{\pi/3} 1\cdot e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi/3}^{\pi/3}$$

$$= \frac{1}{2\pi jn}\left[e^{j\pi/3\,\omega} - e^{-j\pi/3\,\omega}\right]$$

$$= \frac{1}{\pi n}\sin\frac{\pi}{3}n, \quad \infty \geq n \geq -\infty$$

To obtain desired impulse response truncate the infinite sequence of hd(n) to 9 samples

$$h(n) = \begin{cases} h_d(n) & |n| \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

$$h_d(n) = \frac{\sin\frac{\pi}{3}n}{\pi n} \qquad n = -4, -3, -2, -1, 0, 1, 2, 3, 4$$

For $n=0$

$$h(0) = \underset{n\to 0}{Lt}\ \frac{\sin\frac{\pi}{3}n}{\pi n}$$

$$\underset{n\to 0}{Lt}\left(\frac{1}{3}\right)\frac{\sin\frac{\pi}{3}n}{\frac{\pi}{3}n} = \frac{1}{3}\cdot n = \frac{1}{3}$$

$$= \underset{n\to 0}{Lt}\left(\frac{1}{3}\right)\frac{\sin\frac{\pi}{3}n}{\frac{\pi}{3}n} = \frac{1}{3}$$

$$\underset{\theta\to 0}{Lt}\ \frac{\sin\theta}{\theta} = n.$$

Since impulse response is symmetric

For $4 \geq n \geq 1$

$$h(1) = \frac{\sin\frac{\pi}{3}}{\pi} = 0.276 = h(-1)$$

$$h(2) = \frac{\sin\frac{2\pi}{3}}{2\pi} = 0.138 = h(-2)$$

$$h(3) = \frac{\sin\pi}{3\pi} = 0 = h(-3)$$

$$h(4) = \frac{\sin\frac{4\pi}{3}}{4\pi} = -0.069 = h(-4)$$

(ii) $H(z)$

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\left(z^{n} + z^{-n}\right)$$

$$= \frac{1}{3} + h(1)(z^{1}+z^{-1}) + h(2)(z^{2}+z^{-2}) + h(3)(z^{3}+z^{-3}) + h(4)(z^{4}+z^{-4})$$

$$= \frac{1}{3} + 0.276(z+z^{-1}) + 0.138(z^{2}+z^{-2}) - 0.069(z^{4}+z^{-4})$$

To obtain realizable filter multiply by $z^{-\left(\frac{N-1}{2}\right)} = z^{-4}$

$$\bar{H}(z) = \frac{1}{3}z^{-4} + 0.276\left(z^{-3}+z^{-5}\right) + 0.138\left(z^{-2}+z^{-6}\right) - 0.069\left(\cancel{1} + z^{-8}\right)$$

$= \cancel{0.33}$

$$= -0.069 + 0.138z^{-2} + 0.276z^{-3} + 0.33z^{-4} + 0.276z^{-5} + 0.138z^{-6} - 0.069z^{-8}.$$

The impulse response is symm with odd length. (13)

$$\bar{H}(e^{j\omega}) = h\left(\frac{N-1}{2}\right) + 2\sum_{n=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2}-n\right)\cos\omega n.$$

$N=9$

$$\bar{H}(e^{j\omega}) = h(4) + 2\sum_{n=1}^{4} h(4-n)\cos\omega n.$$

$\therefore$ it satisfies the condition

$$h(n) = h(N-1-n)$$

The filter coefficients of causal filter are

$h(0) = h(8) = -0.069$

$h(1) = h(7) = 0$

$h(2) = h(6) = 0.138$

$h(3) = h(5) = 0.276$

$h(4) = 0.333$

The freq response is

$$\bar{H}(e^{j\omega}) = 0.333 + 2h(3)\cos\omega + 2h(2)\cos 2\omega + 2h(1)\cos 3\omega + 2h(0)\cos 4\omega$$

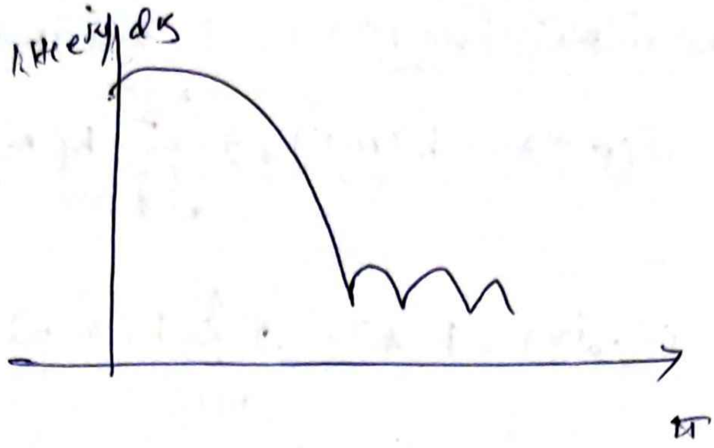
$$\bar{H}(e^{j\omega}) = 0.333 + 2\times 0.276\cos\omega + 2\times 0.138\cos 2\omega + 2\cdot(0)\cos 3\omega + 2(-0.069)\cos 4\omega$$

$$= 0.333 + 0.552\cos\omega + 0.276\cos 2\omega - 0.138\cos 4\omega.$$

mag $|H(e^{j\omega})|_{dB} = 20\lg|\bar{H}(e^{j\omega})|$

ω (in deg)	0	10	20	30	40	50	60	70	80
$\lvert H(e^{j\omega})\rvert_{dB}$									

90	100	110	120	130	140	150	160	170	180



Freq response of LPF

FIR Filter design

(1) window method — Rectangular, Hanning, Hamming and Kaiser.

Generalized cosine windows.

(triangular or Bartlett window raised cosine).

(2) frequency sampling technique

(3) optimal filter design methods.

34, 25, 27, 49, 51

DESIGN OF FIR FILTER – WINDOWING TECHNIQUES

The frequency response of desired sequence $H_d(e^{j\omega})$ is given by

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega n}$$

For causal system

$$H_d(e^{j\omega}) = \sum_{n=0}^{\infty} h_d(n) e^{-j\omega n}$$

The $h_d(n)$ is inverse fourier transform of $H_d(e^{j\omega})$, ie,

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} \, d\omega .$$

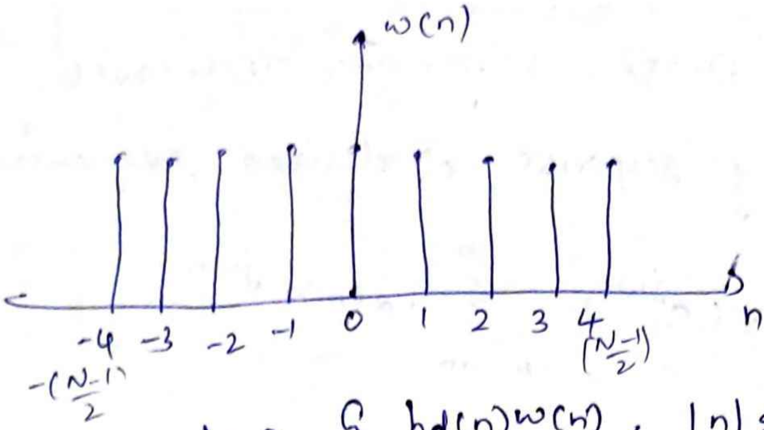
In Fourier method, inorder to obtain filter of length N, the infinite length of desired impulse response $h_d(n)$ is truncated at $n = \pm\left(\frac{N-1}{2}\right)$.

Instead of truncating the desired impulse response $h_d(n)$, the same can be obtained by using a 'rectangular window'.

The rectangular window can be defined as

$$w(n) = \begin{cases} 1, & |n| \leq \left(\frac{N-1}{2}\right) \\ 0, & \text{otherwise} \end{cases}$$

The finite duration impulse response is obtained when the desired impulse response (infinite duration) is multiplied with rectangular window fn $(w(n))$.



$N=9$

$$h(n) = \begin{cases} h_d(n)\,w(n), & |n| \le \left(\frac{N-1}{2}\right) \\ 0, & \text{otherwise.} \end{cases}$$

$$h_d(n) \times w(n) \xleftrightarrow{F.T} H_d(e^{j\omega}) * W(e^{j\omega}).$$

$\therefore$ Frequency of finite data is given by

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})\, W(e^{j(\omega-\theta)})\, d\theta.$$

F.T of Rectangular window is

$$W(e^{j\omega}) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} 1\cdot e^{-j\omega n}$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)} + \dots e^{j\omega} + 1 + e^{-j\omega} + \dots + e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)}\left[1 + e^{-j\omega} + \dots + e^{-j\omega(N-1)}\right]$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)}\left[\frac{1-e^{-j\omega N}}{1-e^{-j\omega}}\right]$$

$$1 + a + a^2 \dots a^{N-1} = \frac{1-a^N}{1-a}$$

$$\frac{1-e^{-j\omega(N)}}{1-e^{-j\omega}}$$

$$= \sum_{n=0}^{N-1} e^{-j\omega n} = e^{j\omega\left(\frac{N-1}{2}\right)}\left[\frac{1-e^{-j\omega N}}{1-e^{j\omega}}\right]$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)}\left[\frac{1-e^{-j\omega N}}{1-e^{-j\omega}}\right]$$

$$= \frac{e^{j\frac{\omega N}{2}}}{e^{j\omega/2}}\left[\frac{1-e^{-j\omega N}}{1-e^{-j\omega}}\right]$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)} * \frac{e^{-j\omega\left(\frac{N}{2}\right)}}{e^{-j\omega/2}}\left[\frac{e^{j\omega N/2} - e^{-j\omega N/2}}{e^{j\omega/2} - e^{-j\omega/2}}\right]$$

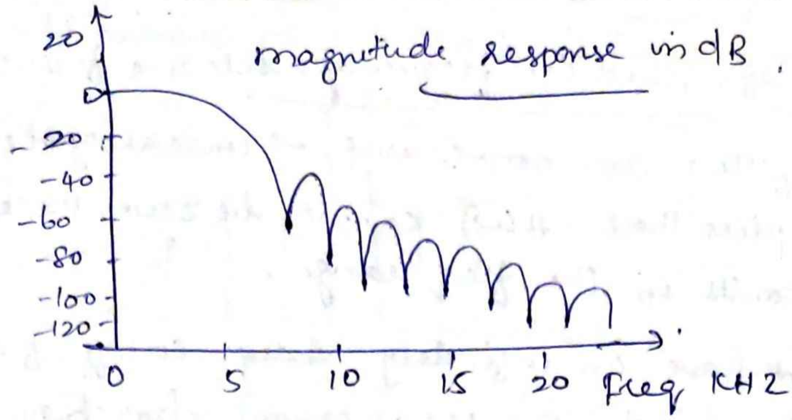
$$= e^{j\omega\left(\frac{N-1}{2}\right)} \frac{e^{-j\frac{\omega N}{2}}\left[e^{j\frac{\omega N}{2}} - e^{-j\frac{\omega N}{2}}\right]}{e^{-j\frac{\omega}{2}}\left[e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right]}$$

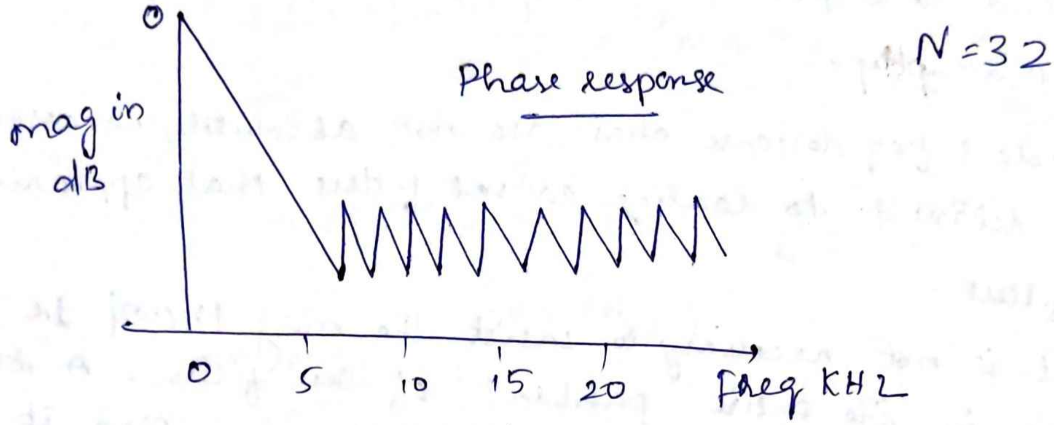
$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \frac{\sin \omega N/2}{\sin \omega/2}$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)} e^{-j\omega\left(\frac{N-1}{2}\right)} \frac{\sin \omega \frac{N}{2}}{\sin \frac{\omega}{2}}$$

magnitude response

$$|W(\omega)| = \frac{|\sin(\omega N/2)|}{|\sin(\omega/2)|} \qquad -\pi \le \omega \le \pi, \qquad \boxed{W(e^{j\omega}) = \frac{\sin \omega\frac{N}{2}}{\sin\frac{\omega}{2}}}$$





The freq response of rect. window consists of main lobe and side lobes. The height and width of the main lobe is large compared to side lobes.

Main lobe – central part has energy fn. width $\frac{4\pi}{N}$ $\frac{-2\pi}{N}$ to $\frac{2\pi}{N}$

Side lobes – ~~also~~ spectral leakage (should be (Ringing effect) negligible)

convolution of $H_d(\omega)$ with $W(\omega)$ has smoothing effect.

as N ↑ses, main lobe becomes narrower and side lobe width also decreases. But height ↑ses and effective area remains same. Smoothing effect reduces. but reduces spectral leakage.

Characteristics of Practical frequency-selective filters.

ideal filters are non-causal → unrealizable

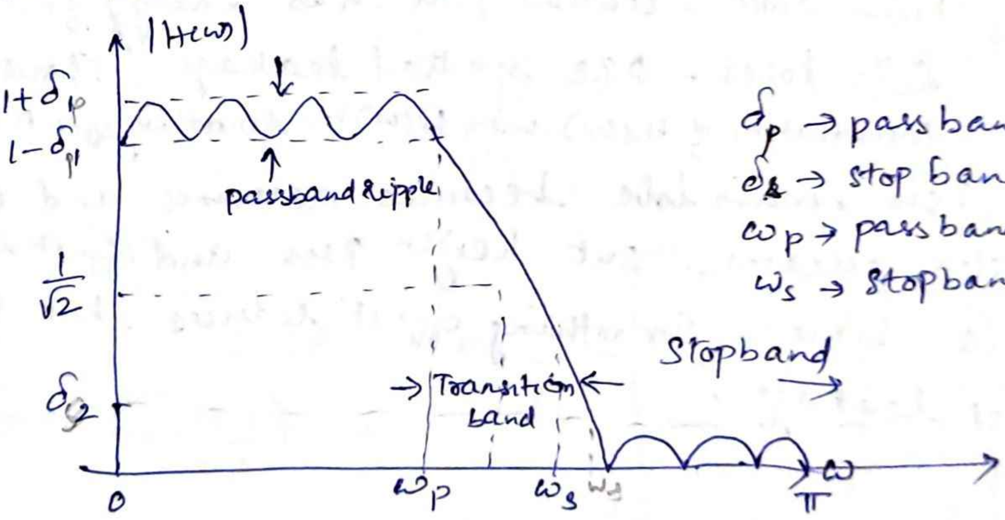
causality implies That $H(\omega)$ cannot be zero, except at a finite set of points in The freq range.

$H(\omega)$ cannot have an infinitely sharp cutoff from passband to stop band ie) $H(\omega)$ cannot drop from unity to zero abruptly.

ideal freq response char are not absolutely necessary. It is relaxed to realize causal filters that approximate ideal filters.

It is not necessary to insist the mag $|H(\omega)|$ be constant in the entire passband of the filter. A small amount of ripple in the passband is tolerable. Also, it is not necessary that $|H(\omega)|$ to be zero in the stopband.

transition of freq response from passband to stopband defines the transition band.



δ_p → passband ripple
δ_s → stop band ripple
ω_p → passband edge ripple
ω_s → stopband edge ripple.

ω_c (3dB freq or half power point).

The approximation of $H(\omega)$ depends on the filter co-efficients as well The length of samples.

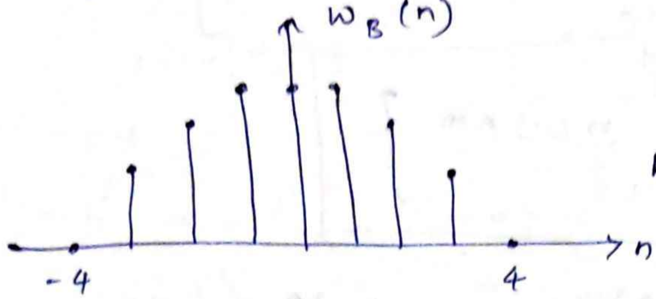
Passband $0 \le \omega \le \omega_p$

$$1-\delta_p \le H(e^{j\omega}) \le 1+\delta_p \quad |\omega| \le \omega_p$$

Stopband $\omega_s \le \omega < \infty$

$$H(e^{j\omega}) \le \delta_s \quad \omega_s \le |\omega| < \infty$$

(2) Triangular window (Barlett window).



$N = 9$

Gibbs phenomenon is reduced by choosing a window which has smooth transition in the filter.

$$w_B(n) = 1 - \frac{2|n|}{N-1}, \quad \left(\frac{N-1}{2}\right) \geq n \geq -\left(\frac{N-1}{2}\right)$$

F.T

$$W_B(e^{j\omega}) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} \left(1 - \frac{2|n|}{N-1}\right) e^{-j\omega n}.$$

$$= \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} e^{-j\omega n} - \frac{2}{N-1} \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} |n| e^{-j\omega n}$$

$e^{j\omega\left(\frac{N-1}{2}\right)}[\ldots e^{j\omega} + 1 + e^{-j\omega} \ldots + e^{-j\omega\left(\frac{N-1}{2}\right)}$

$e^{j\omega\left(\frac{N-1}{2}\right)}[1 + \ldots e^{-j\omega\left(\frac{N-1}{2}\right)} + \ldots e^{-j\omega(N-1)}]$

$= e^{j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{N-1} e^{-j\omega n}$

$= e^{j\omega\left(\frac{N-1}{2}\right)} \frac{(1 - e^{-j\omega N})}{1 - e^{-j\omega}}$

First term

$$W_B(e^{j\omega}) = e^{j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{\sin \frac{\omega N}{2}}{\sin \frac{\omega}{2}}$$

second term

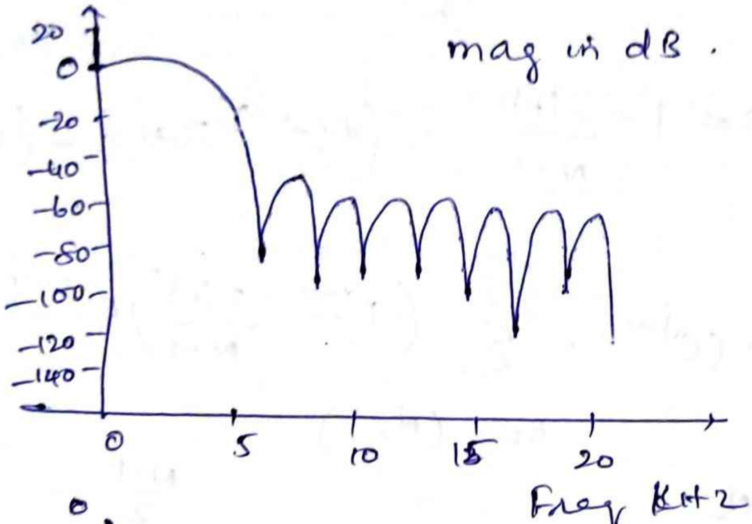
$$\frac{2}{N-1} \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} |n| e^{-j\omega n} = \frac{2}{N-1}\left[\sum_{n=-\left(\frac{N-1}{2}\right)}^{-1} -n e^{-j\omega n} + \sum_{n=0}^{\frac{N-1}{2}} n e^{-j\omega n}\right]$$

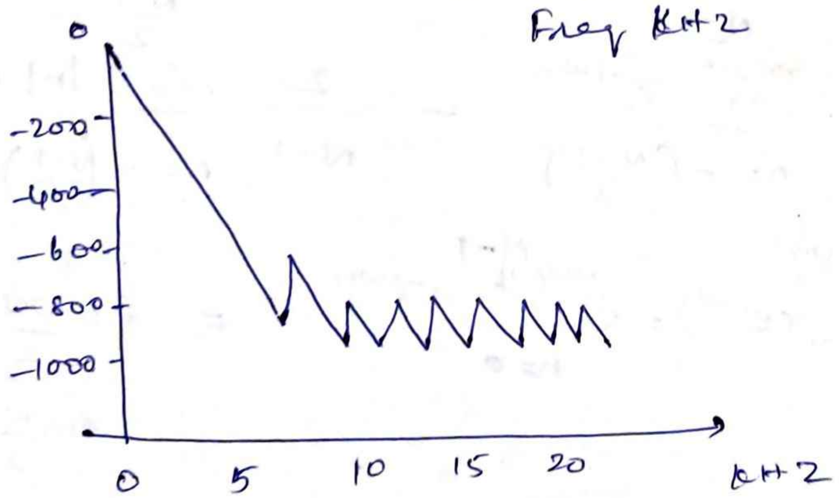
$$= \frac{2}{N-1}\left[\sum_{n=1}^{\frac{N-1}{2}} n e^{j\omega n} + \sum_{n=0}^{\frac{N-1}{2}} n e^{-j\omega n}\right]$$

$$= \frac{2}{N-1}\left[\sum_{n=0}^{\frac{N-1}{2}} n e^{j\omega n} + \sum_{n=0}^{\frac{N-1}{2}} n e^{-j\omega n}\right]$$

$$= \frac{2}{N-1}\left[\sum_{n=0}^{\frac{N-1}{2}} n\left(e^{j\omega n}+e^{-j\omega n}\right)\right]$$

$$\boxed{W_B(e^{j\omega}) = \frac{4}{N-1}\left[\sum_{n=0}^{\frac{N-1}{2}} n\cos\omega n\right]}$$





Raised cosine window.

$$W_\alpha(n) = \alpha + (1-\alpha)\cos\left(\frac{2\pi n}{N-1}\right), \quad \frac{N-1}{2} \geq n \geq -\left(\frac{N-1}{2}\right)$$

Taking F.T

$$W_\alpha(e^{j\omega}) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} \left[\alpha + (1-\alpha)\cos\left(\frac{2\pi n}{N-1}\right)\right]e^{-j\omega n}$$

$$W_\alpha(e^{j\omega}) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} \alpha e^{-j\omega n} + (1-\alpha)\sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} \left[\frac{e^{j\frac{2\pi n}{N-1}} + e^{-j\frac{2\pi n}{N-1}}}{2}\right] e^{-j\omega n}$$

$$= \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} \alpha e^{-j\omega n} + \left(\frac{1-\alpha}{2}\right)\sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} e^{+j\left(\omega-\frac{2\pi}{N-1}\right)n} + \left(\frac{1-\alpha}{2}\right)\sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} e^{-j\left(\omega+\frac{2\pi}{N-1}\right)n}$$

First term

$$\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} \alpha e^{-j\omega n} = e^{j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{N-1} \alpha e^{-j\omega n}$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)} \alpha \left[\frac{1-e^{-j\omega N}}{1-e^{-j\omega}}\right]$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)} \alpha \frac{e^{-j\omega N/2}}{e^{-j\omega/2}} \frac{\left[e^{j\omega N/2} - e^{-j\omega N/2}\right]}{\left[e^{j\omega/2} - e^{-j\omega/2}\right]}$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)} \cdot e^{-j\omega\left(\frac{N-1}{2}\right)} \alpha \frac{\sin \omega N/2}{\sin \omega/2}$$

$$= \alpha \frac{\sin \omega N/2}{\sin \omega/2}$$

Second term

$$\frac{1-\alpha}{2} \sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} e^{+j\left(\omega-\frac{2\pi}{N-1}\right)n} = \left(\frac{1-\alpha}{2}\right) \cdot e^{j\left(\omega-\frac{2\pi}{N-1}\right)\left(\frac{N-1}{2}\right)} \cdot \sum_{n=0}^{N-1} e^{-j\omega\left(\omega-\frac{2\pi}{N-1}\right)}$$

$$= e^{j(\omega - \frac{2\pi}{N-1})(\frac{N-1}{2})}\left(\frac{1-\alpha}{2}\right)\left[\frac{1 - e^{-j(\omega - \frac{2\pi}{N-1})N}}{1 - e^{-j(\omega - \frac{2\pi}{N-1})}}\right]$$

$$= e^{j(\omega - \frac{2\pi}{N-1})(\frac{N-1}{2})}\left(\frac{1-\alpha}{2}\right)\left[\frac{e^{-j(\omega - \frac{2\pi}{N-1})N/2}}{e^{-j(\omega - \frac{2\pi}{N-1})\frac{1}{2}}}\frac{\left(e^{j(\omega - \frac{2\pi}{N-1})N/2} - e^{-j(\omega - \frac{2\pi}{N-1})\frac{N}{2}}\right)}{\left(e^{j(\omega + \frac{2\pi}{N-1})\frac{1}{2}} - e^{-j(\omega - \frac{2\pi}{N-1})\frac{1}{2}}\right)}\right]$$

$$= \frac{1-\alpha}{2}\;\frac{\sin(\omega - \frac{2\pi}{N-1})\frac{N}{2}}{\sin(\omega - \frac{2\pi}{N-1})\frac{1}{2}}$$

Third term

$$\left(\frac{1-\alpha}{2}\right)\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} e^{-j(\omega + \frac{2\pi}{N-1})n} = \frac{1-\alpha}{2}\left[\frac{\sin(\omega + \frac{2\pi}{N-1})\frac{N}{2}}{\sin(\omega + \frac{2\pi}{N-1})\frac{1}{2}}\right]$$

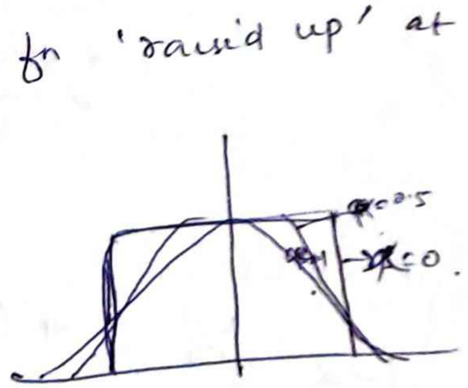
$$W_\alpha(e^{j\omega}) = \alpha\frac{\sin\omega N/2}{\sin\omega/2} + \left(\frac{1-\alpha}{2}\right)\left\{\frac{\sin(\omega - \frac{2\pi}{N-1})\frac{N}{2}}{\sin(\omega - \frac{2\pi}{N-1})\frac{1}{2}} + \frac{\sin(\omega + \frac{2\pi}{N-1})\frac{N}{2}}{\sin(\omega + \frac{2\pi}{N-1})\frac{1}{2}}\right\}$$

Raised cosine filter has a cosine fn 'raised up' at non-zero portion of the freq spectrum.

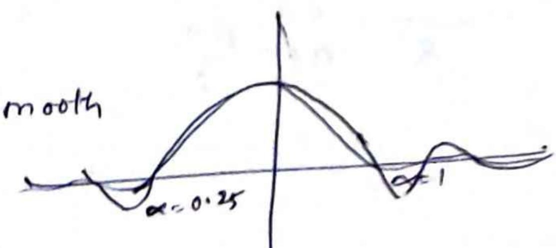
$\alpha \to$ roll off factor

measure of excess Bandwidth.

ripple increases as α decreases.



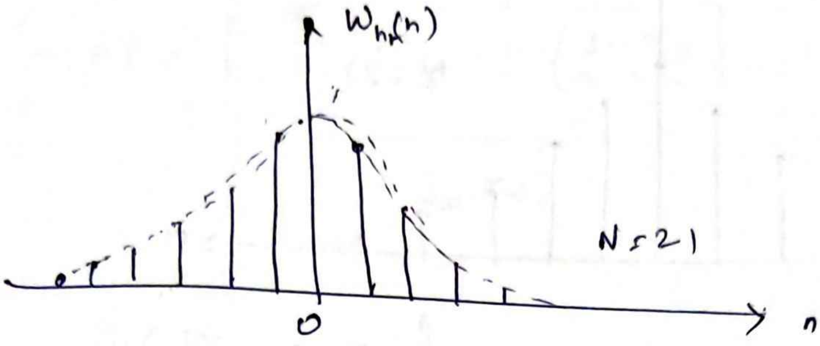
avoids spectral leakage and has smooth envelope.



Hanning window.

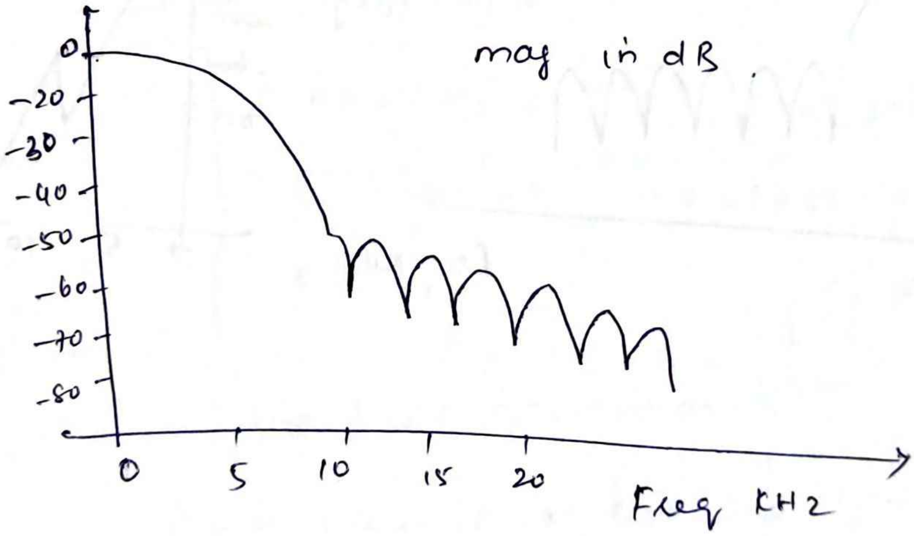
$\alpha = 0.5$ in raised cosine window.

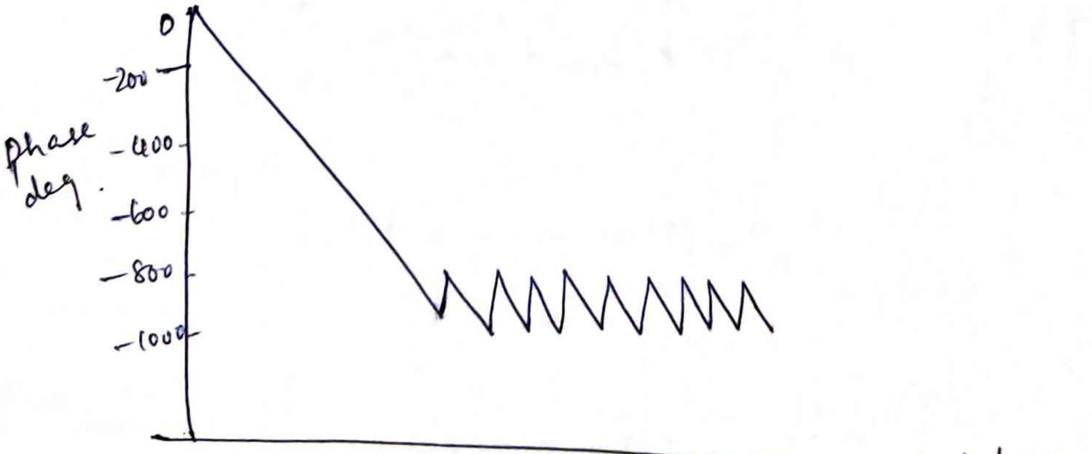
$$W_{hn} = 0.5 + 0.5 \cos\left(\frac{2\pi n}{N-1}\right), \quad \frac{N-1}{2} \geq n \geq -\frac{N-1}{2}.$$



$$W_{hn}(e^{jw}) = 0.5 \frac{\sin wN/2}{\sin w/2} + 0.25 \left\{ \frac{\sin\left(w - \frac{2\pi}{N-1}\right)\frac{N}{2}}{\sin\left(w - \frac{2\pi}{N-1}\right)\frac{1}{2}} + \frac{\sin\left(w + \frac{2\pi}{N-1}\right)\frac{N}{2}}{\sin\left(w + \frac{2\pi}{N-1}\right)\frac{1}{2}} \right.$$

mag response.

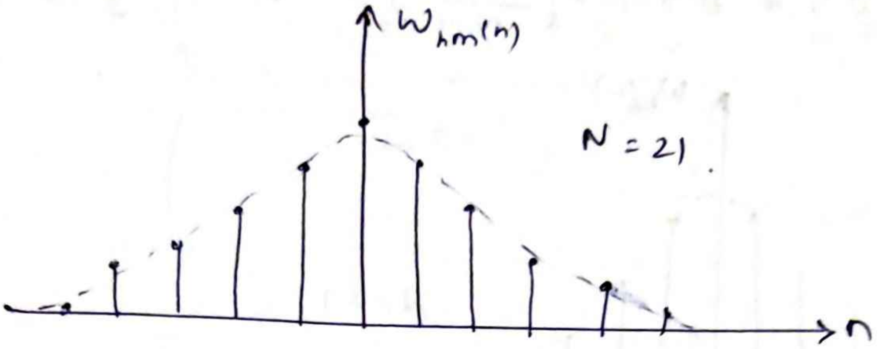




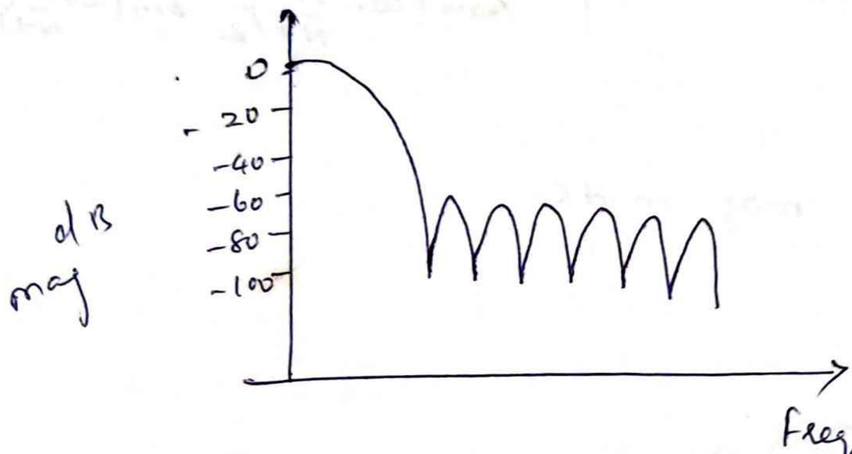
Hamming window.

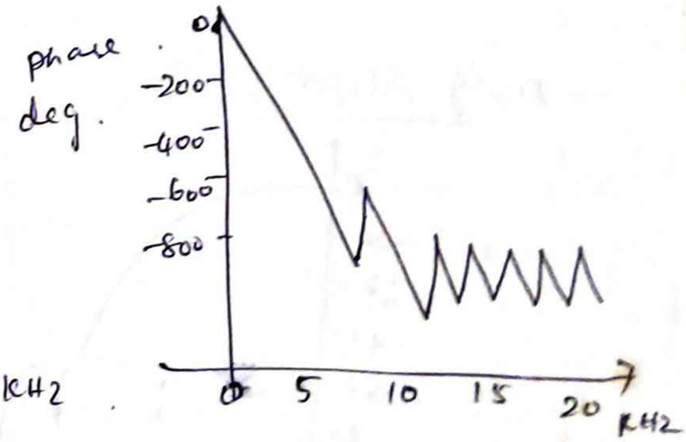
$\alpha = 0.54$

$$W_{hm}(n) = 0.54 + 0.46 \cos\left(\frac{2\pi n}{N-1}\right) \qquad \left(\frac{N-1}{2}\right) \geq N \geq -\left(\frac{N-1}{2}\right)$$



$$W_{hm}(e^{j\omega}) = 0.54 \frac{\sin \omega N/2}{\sin \omega/2} + 0.23 \left\{ \frac{\sin\left(\omega - \frac{2\pi}{N-1}\right)\frac{N}{2}}{\sin\left(\omega - \frac{2\pi}{N-1}\right)\frac{1}{2}} + \frac{\sin\left(\omega + \frac{2\pi}{N-1}\right)\frac{N}{2}}{\sin\left(\omega + \frac{2\pi}{N-1}\right)\frac{1}{2}} \right.$$





KAISER WINDOW

It has control over ripples by means of an additional parameter characterizing the window.

$$W_K(\alpha, n) = \begin{cases} \dfrac{I_0\left[\alpha\sqrt{1-\left(\frac{2n}{N-1}\right)^2}\right]}{I_0(\alpha)}, & \frac{N-1}{2} \geq n \geq -\frac{N-1}{2} \\ 0, & \text{otherwise.} \end{cases}$$

where $$I_0(\alpha) = \sum_{k=0}^{\infty}\left[\frac{(\alpha/2)^k}{k!}\right]^2$$

$I_0(\alpha)$ → modified Bessel function of its first kind of order zero.

α → adjustable parameter.

$$\alpha = \begin{cases} 0.1102(\alpha_s' - 8.7) & \alpha_s' > 50 \\ 0.5842(\alpha_s' - 21)^{0.4} + 0.07886(\alpha_s' - 21), & 50 \geq \alpha_s' \geq 21 \\ 0 & \alpha_s' < 21. \end{cases}$$

α_s' → stop band attenuation factor. $\alpha_s' = -20\log_{10}\delta$

$\delta = \min(\delta_s, \delta_p)$

δ_s → minimal of stop band ripple

δ_p → minimal pass band ripple

δ → ripple factor

stop band ripple

$$\alpha_s = -20\log_{10}\delta_s$$

$$\delta_s = 10^{-0.05\alpha_s}$$

pass band ripple

$$\alpha_p = -20\log_{10}\left[\frac{1+\delta_p}{1-\delta_p}\right]$$

$$\delta_p = \left[\frac{10^{0.05\alpha_p}-1}{10^{0.05\alpha_p}+1}\right]$$

$\delta_s \geq \delta_p$

Order of Kaiser

$$N = \begin{cases} \dfrac{\alpha_s' - 7.95}{14.36(\Delta f)} & , \alpha_s' > 21 \\ \dfrac{0.4611}{\Delta f} & , \alpha_s' < 21 \end{cases}$$

$\Delta f \rightarrow$ desired transition width.

Frequency response of kaiser window.

Taking F.T of $W_K(\alpha, n)$

$$W_K(e^{j\omega}) = FT\left[w_K(n)\right]$$

$$W_K(e^{j\omega}) = \frac{2 \sin\left\{\left(\frac{N-1}{2}\right)\left[\omega^2 - \left(\frac{2\alpha}{N-1}\right)^2\right]^{1/2}\right\}}{I_0(\alpha)\left[\omega^2 - \left(\frac{2\alpha}{N-1}\right)^2\right]^{1/2}}$$

Design a linear phase FIR filter using Hamming and Hanning windows for the following desired freq response

$$H_d(\omega) = \begin{cases} e^{-j3\omega} & , \frac{\pi}{4} \leq |\omega| \leq \pi \\ 0 & , \text{otherwise} \end{cases}$$

$$\omega_c = \frac{\pi}{4}$$

$$\frac{M-1}{2} = 3 \quad \therefore \text{filter length} = 7.$$

$$\boxed{h(n) = h_d(n) \cdot W_{ham}(n)}$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{-j3\omega}\cdot e^{j\omega n}\,d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{-j3\omega}\cdot e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j(n-3)\omega}}{j(n-3)}\right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi}\left[\frac{e^{j(n-3)\omega}}{j(n-3)}\right]_{\omega_c}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j(n-3)\omega_c}}{j(n-3)} - \frac{e^{-j(n-3)\pi}}{j(n-3}\right] + \frac{1}{2\pi}\left[\frac{e^{j(n-3)\pi}}{j(n-3)} - \frac{e^{j(n-3)\omega_c}}{j(n-3)}\right]$$

$$= \frac{1}{2\pi j(n-3)}\left[e^{-j(n-3)\omega_c} - e^{j(n-3)\omega_c} + e^{j(n-3)\pi} - e^{-j(n-3)\pi}\right]$$

$$= -\frac{\sin(n-3)\omega_c}{(n-3)\pi} \qquad n \neq 3.$$

$$h_d(3) = \frac{1}{2\pi}\left[\int_{-\pi}^{-\omega_c} d\omega + \int_{\omega_c}^{\pi} d\omega\right]$$

$$= \frac{\pi-\omega_c}{\pi}.$$

$\omega_c = \frac{\pi}{4}$

$$h_d(n) = \frac{\sin\frac{\pi}{4}(n-3)}{\pi(n-3)} \qquad n \neq 3$$

$$h_d(3) = \frac{\pi-\frac{\pi}{4}}{\pi} = \frac{4\pi-\pi}{4\pi} = \frac{3}{4} = 0.75$$

$h_d(0) = h_d(6) = -0.07506$

$h_d(1) = h_d(5) = -0.15923$

$h_d(2) = h_d(4) = -0.22586$

Hamming window

$$W_{ham}(n) = 0.54 - 0.46 \cos\left(\frac{2\pi n}{6}\right) \qquad 0 \le n \le N-1$$

$$\boxed{W_{ham}(n) = 0.54 + 0.46 \cos\left(\frac{2\pi n}{6}\right) \quad -\left(\frac{N-1}{2}\right) \le n \le -\frac{N-1}{2}}$$

$W_{ham}(0) = W_{ham}(6) = 0.08$

$W_{ham}(1) = W_{ham}(5) = 0.31$

$W_{ham}(2) = W_{ham}(4) = 0.77$

$W_{ham}(3) = 1$

$h(n) = h_d(n) \cdot W_{ham}(n)$

$= \{-0.006, -0.04936, -0.17391, 0.75, -0.17396, -0.04936, -0.006\}$
$\uparrow$

Hanning window

$$W_{han}(n) = 0.5 - 0.5 \cos \frac{2\pi n}{6}$$

$W_{han}(0) = W_{han}(6) = 0$

$W_{han}(1) = W_{han}(5) = 0.25$

$W_{han}(2) = W_{han}(4) = 0.75$

$W_{han}(3) = 1$

$h(n) = h_d(n) W_{han}(n) = \{0, -0.0398, -0.16939, 0.75, -0.16939, -0.0398, 0\}$

when $n - \frac{N-1}{2}$.
use $H(z)$ for symmetric odd
$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)(z^n + z^{-n})$
and multiply with $z^{-\left(\frac{N-1}{2}\right)}$
$\to H(z)$

$h(n) = h_d(n) \cdot W_{han}(n)$

$= \{0, -0.0398, -0.16939, 0.75, -0.16939, -0.0398, 0\}$

$H(z) = 0.0z^0 - 0.0398z^{-1} - 0.16939z^{-2} + 0.75z^{-3} - 0.16939z^{-4} - 0.0398z^{-5} + 0.0z^{-6}$

STRUCTURE REALIZATION OF FIR SYSTEM

In general, the FIR system is a non-recursive system, described by the difference equation

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k)$$

Taking z-transform

$$Y(z) = \sum_{k=0}^{M-1} b_k z^{-k} X(z)$$

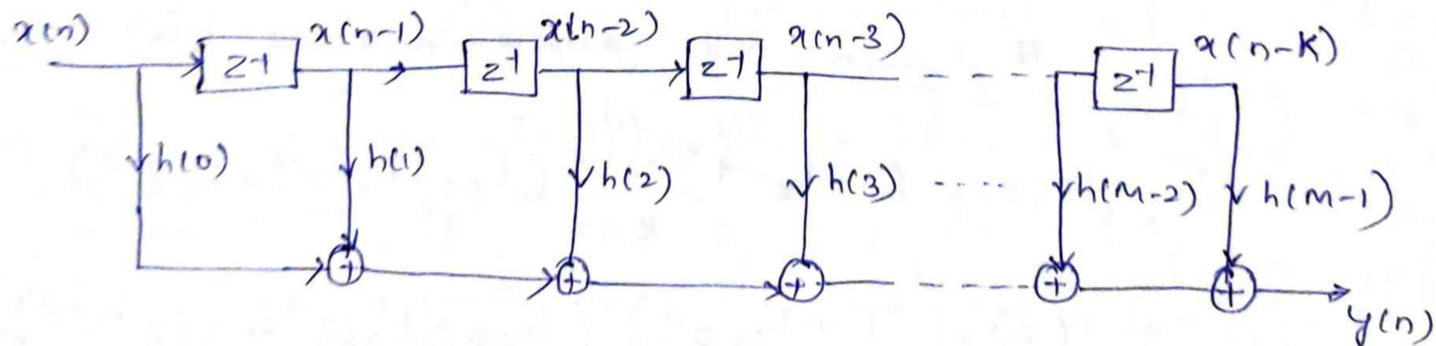
$$\frac{Y(z)}{X(z)} = H(z) = \sum_{k=0}^{M-1} b_k z^{-k}.$$

where M = length of the FIR filter.

$$h(n) = \begin{cases} b_k, & 0 \le n \le M-1 \\ 0 & \text{otherwise}. \end{cases}$$

Direct form (or) Transversal structure

$$H(z) = \sum_{n=0}^{M-1} h(n) z^{-n}$$



The structure has M-1 memory locations

M → complex multiplications

M-1 → additions.

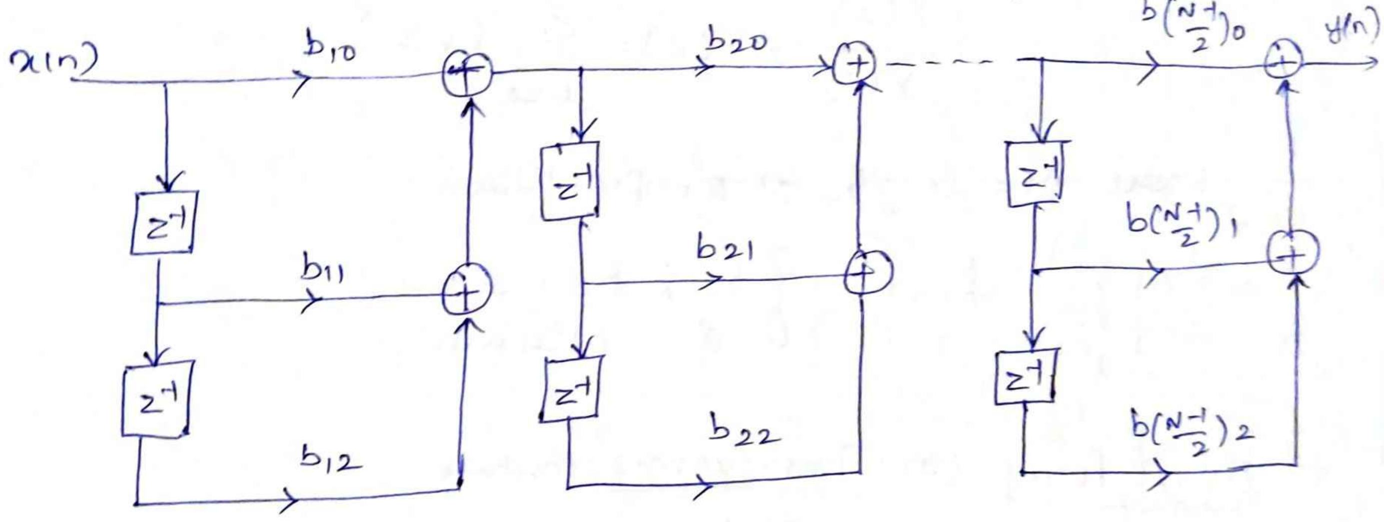
Cascade structure realization.

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

(1) when N = odd.

$$H(z) = \prod_{k=1}^{\left(\frac{N-1}{2}\right)} \left(b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2}\right)$$

$$H(z) = (b_{10} + b_{11} z^{-1} + b_{12} z^{-2})\,(b_{20} + b_{21} z^{-1} + b_{22} z^{-2}) \cdots$$
$$\left(b_{\left(\frac{N-1}{2}\right)0} + b_{\left(\frac{N-1}{2}\right)1} z^{-1} + b_{\left(\frac{N-1}{2}\right)2} z^{-2}\right)$$



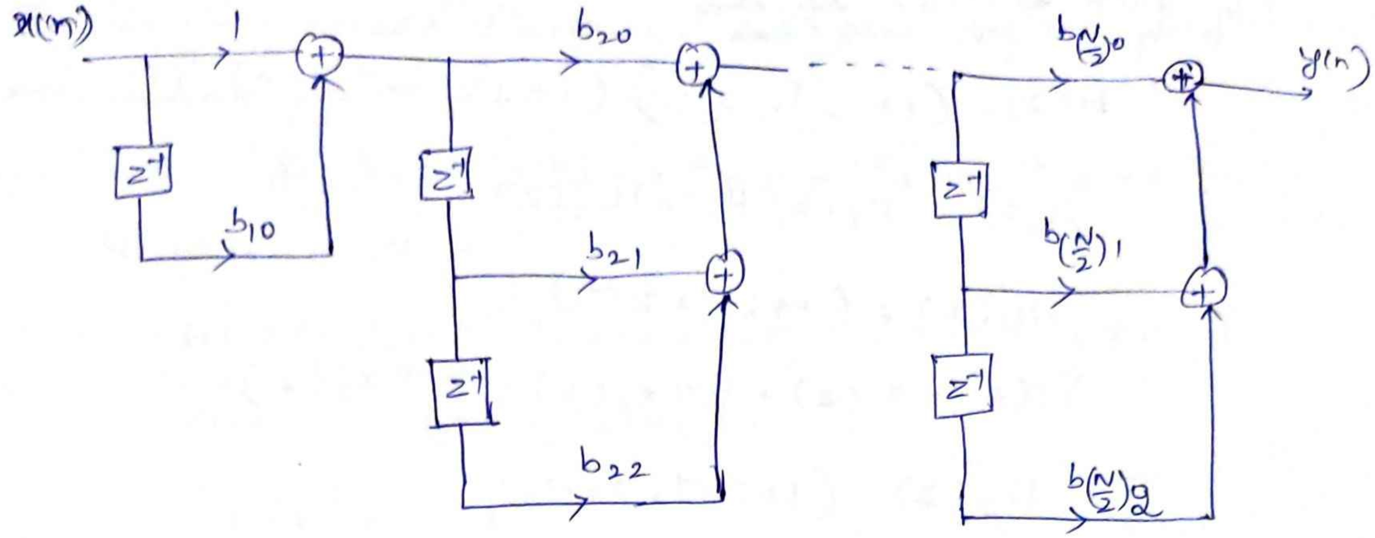
$\frac{N-1}{2} \rightarrow$ second order factors

(2) when N = even.

$\frac{N-1}{2}$ second order factors and one first order factor

$$H(z) = (1 + b_{10} z^{-1}) \prod_{K=2}^{N/2} \left(b_{k0} + b_{k1} z^{-1} + b_{k} z^{-2}\right)$$

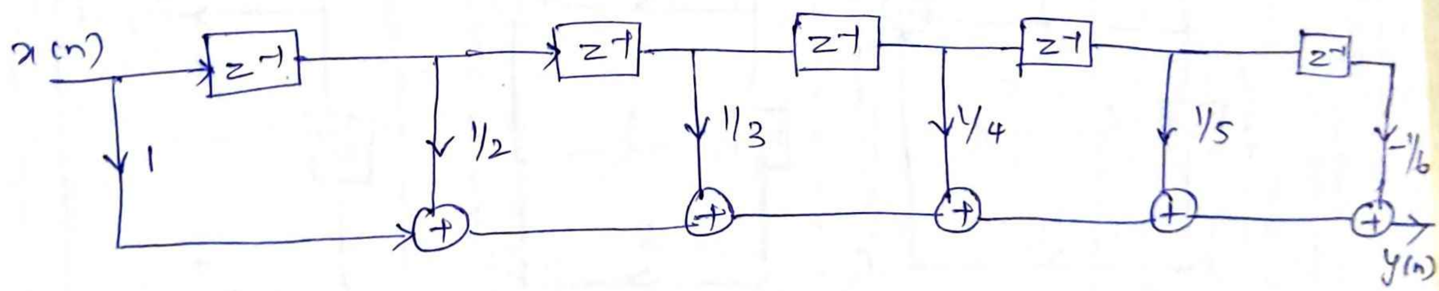
$$H(z) = (1 + b_{10} z^{-1}) \cdot (b_{20} + b_{21} z^{-1} + b_{22} z^{-2}) +$$
$$(b_{30} + b_{31} z^{-1} + b_{32} z^{-2}) \cdots$$
$$\left(b_{\left(\frac{N}{2}\right)0} + b_{\left(\frac{N}{2}\right)1} z^{-1} + b_{\left(\frac{N}{2}\right)2} z^{-2}\right)$$



(1) Obtain the direct form structure realization of the non recursive system.

$$H(z) = 1 + \frac{1}{2} z^{-1} + \frac{1}{3} z^{-2} + \frac{1}{4} z^{-3} + \frac{1}{5} z^{-4} - \frac{1}{6} z^{-5}$$

$$Y(z) = X(z) + \frac{1}{2} z^{-1} X(z) + \frac{1}{3} z^{-2} X(z) + \frac{1}{4} z^{-3} X(z) + \frac{1}{5} z^{-4} X(z) - \frac{1}{6} z^{-5} X(z)$$



(2) obtain the cascade structure realization the given non-recursive system.

$$H(z) = 1 + 6z^{-1} + 19z^{-2} + 35z^{-3} + 42z^{-4} + 29z^{-5} + 12z^{-6}$$

$N - 1 = 6 \quad N = 7$

odd.

① On factorizing the equation

$$H(z) = (1+z^{-1}+z^{-2})(1+2z^{-1}+3z^{-2})(1+3z^{-1}+4z^{-2})$$

$$H(z) = H_1(z)\,H_2(z)\,H_3(z)$$

$$H_1(z) = (1+z^{-1}+z^{-2})$$

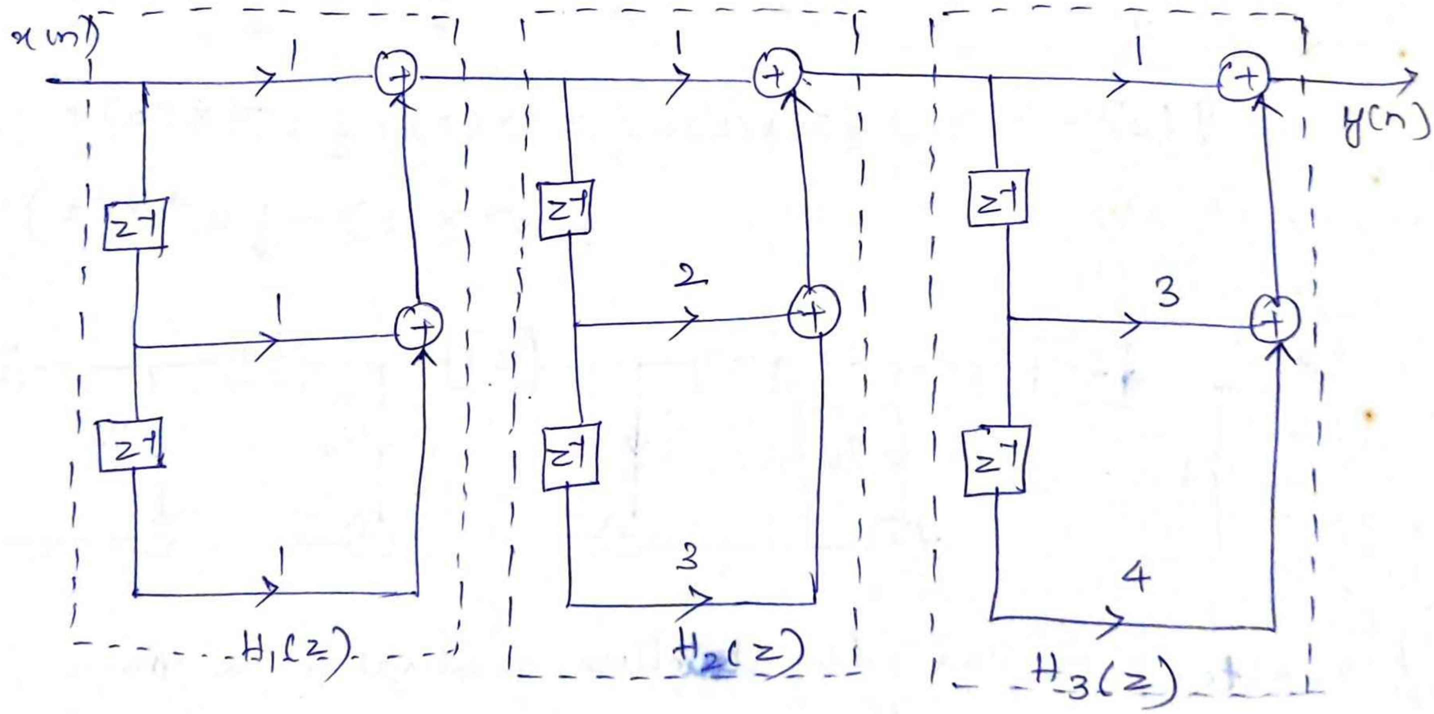
$$Y_1(z) = X_1(z) + z^{-1}X_1(z) + z^{-2}X_2(z)$$

$$H_2(z) = (1+2z^{-1}+3z^{-2})$$

$$Y_2(z) = X_2(z) + 2z^{-1}X_2(z) + 3z^{-2}X_2(z)$$

$$H_3(z) = 1+3z^{-1}+4z^{-2}$$

$$Y_3(z) = X_3(z) + 3z^{-1}X_3(z) + 4z^{-2}X_3(z).$$



Obtain the cascade structure realization of the given non recursive system

$$H(z) = 1 + 8z^{-1} + 21z^{-2} + 35z^{-3} + 28z^{-4} + 15z^{-5}.$$

$N-1 = 5 \quad \therefore N = 6.$

$$H(z) = (1+5z^{-1})(1+z^{-1}+z^{-2})(1+2z^{-1}+3z^{-2})$$

$$H(z) = H_1(z)\, H_2(z)\, H_3(z)$$

$$H_1(z) = 1+5z^{-1}$$

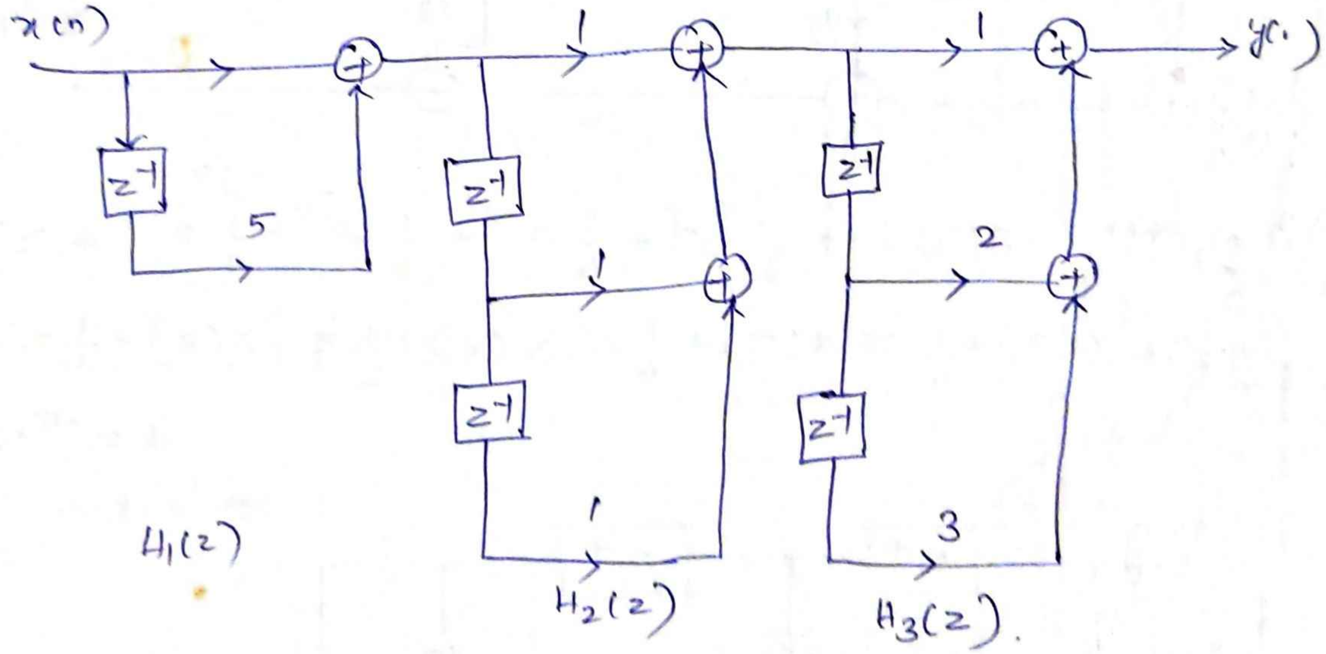
$$Y_1(z) = X_1(z) + 5z^{-1}X_1(z)$$

$$H_2(z) = 1 + z^{-1} + z^{-2}$$

$$Y_2(z) = X_2(z) + z^{-1}X_2(z) + z^{-2}X_2(z)$$

$$H_3(z) = 1 + 2z^{-1} + 3z^{-2}$$

$$Y_3(z) = X_3(z) + 2\cdot z^{-1}X_3(z) + 3\cdot z^{-2}X_3(z)$$



LINEAR PHASE FIR SYSTEM

$$h(n) = h(N-1-n)$$

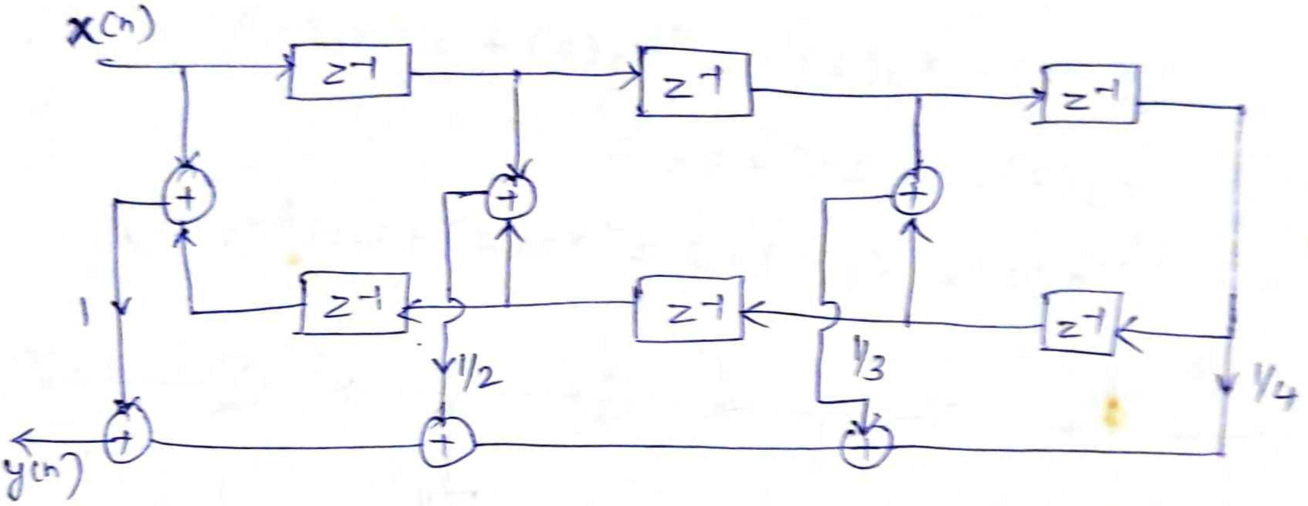
(1) Realize a direct form structure for a linear phase FIR system.

$$H(z) = 1 + \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2} + \frac{1}{4}z^{-3} + \frac{1}{3}z^{-4} + \frac{1}{2}z^{-5} + z^{-6}$$

$$Y(z) = X(z) + \frac{1}{2}z^{-1}X(z) + \frac{1}{3}z^{-2}X(z) + \frac{1}{4}z^{-3}X(z) + \frac{1}{3}z^{-4}X(z) + \frac{1}{2}z^{-5}X(z) + z^{-6}X(z)$$

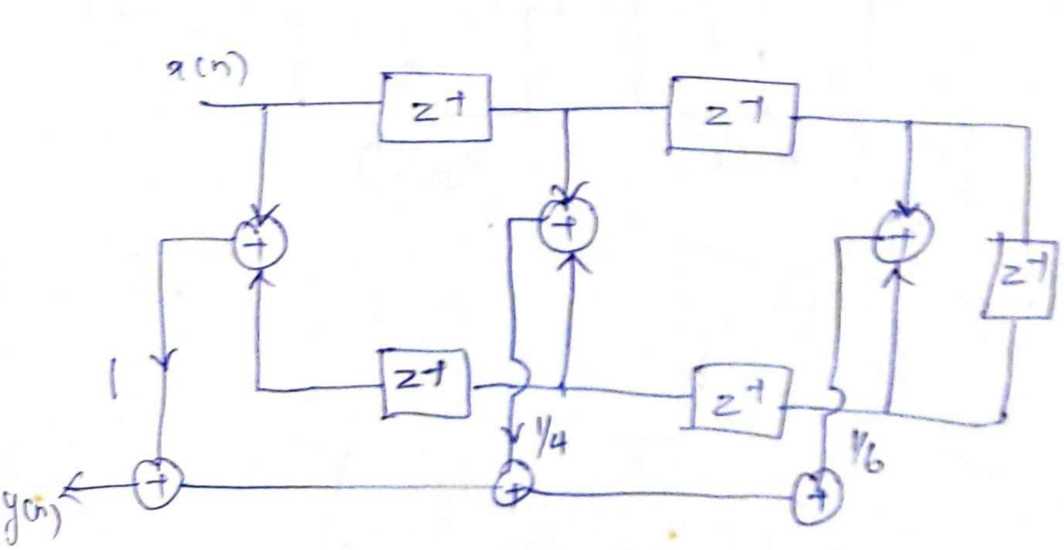
N odd

$N-1 = 6 \quad N = 7$



(2) N even $\quad H(z) = 1 + \frac{1}{4}z^{-1} + \frac{1}{6}z^{-2} + \frac{1}{6}z^{-3} + \frac{1}{4}z^{-4} + z^{-5}$

$$Y(z) = X(z) + \frac{1}{4}z^{-1}X(z) + \frac{1}{6}z^{-2}X(z) + \frac{1}{6}z^{-3}X(z) + \frac{1}{4}z^{-4}X(z) + z^{-5}X(z)$$



Linear phase realization

$h(n) = h(N-1-n)$

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}.$$

N even

$$H(z) = \sum_{n=0}^{\frac{N-2}{2}} h(n) z^{-n} + \sum_{n=N/2}^{N-1} h(n) z^{-n}$$

$$= \sum_{n=0}^{\frac{N-2}{2}} h(n) z^{-n} + \sum_{n=0}^{\frac{N-2}{2}} h(N-1-n) z^{-(N-1-n)}$$

$$= \sum_{n=0}^{\frac{N-2}{2}} h(n) \left[z^{-n} + z^{-(N-1-n)} \right].$$

$N/2$ multipliers required.

N odd

$$H(z) = \sum_{n=0}^{\frac{N-3}{2}} h(n) z^{-n} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n) z^{-n} + h\left(\frac{N-1}{2}\right) z^{-(N-1)/2}$$

$$= h\left(\frac{N-1}{2}\right) z^{-(N-1)/2} + \sum_{n=0}^{\frac{N-3}{2}} h(n) \left[z^{-n} + z^{-(N-1-n)} \right]$$

$\frac{N+1}{2}$ multipliers required.

h