FAST FOURIER TRANSFORM

The computation of the DFT involves the multiplication of a matrix by a vector, where as fast fourier algorithm employ a divide and conquer paradigm.

Divide and conquer paradigm

1. Divide the data set into two or more sub-data sets
2. Solve each sub-data set recursively and terminate the recursion when the data-set length is small.
3. Obtain the solution to the original data-set by combining the solutions to each sub data-sets.

Radix-2. FFT

A recursion algorithm is obtained by dividing the original data set into two sub-data sets of half the size. The sub-division is proceeded until their size is one.

The solution to the given data set can be obtained in time-domain or frequency domain.

(1) Decimation-in-time FFT (time domain analysis)
(2) Decimation-in-frequency FFT (frequency domain analysis)

Adv of FFT

The direct evaluation of DFT requires N^2 complex multiplications and $N(N-1)$ complex additions. By using the properties of twiddle factor, the complexity is reduced.

Symmetry property : $W_N^{k+\frac{N}{2}} = -W_N^k$

periodicity property : $W_N^{k+N} = W_N^k$.

$W_N^{kn} = W_N^{k(n+N)} = W_N^{(k+N)n}$.

The FFT reduces the number of complex multiplications required to perform DFT from N^2 to $\frac{N}{2} \log_2 N$.

eg DFT $N = 8$ $\quad N^2 = 64$.

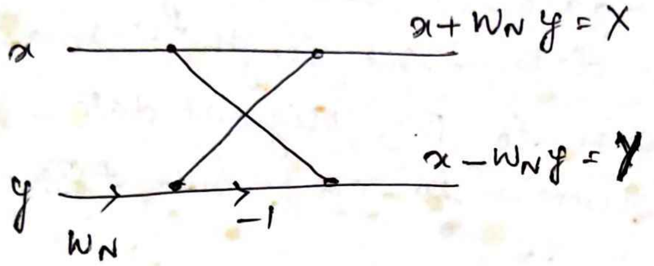
FFT $\frac{N}{2} \log_2 (N) = \frac{8}{2} \log_2 (8) = 4 \times 3 = 12$

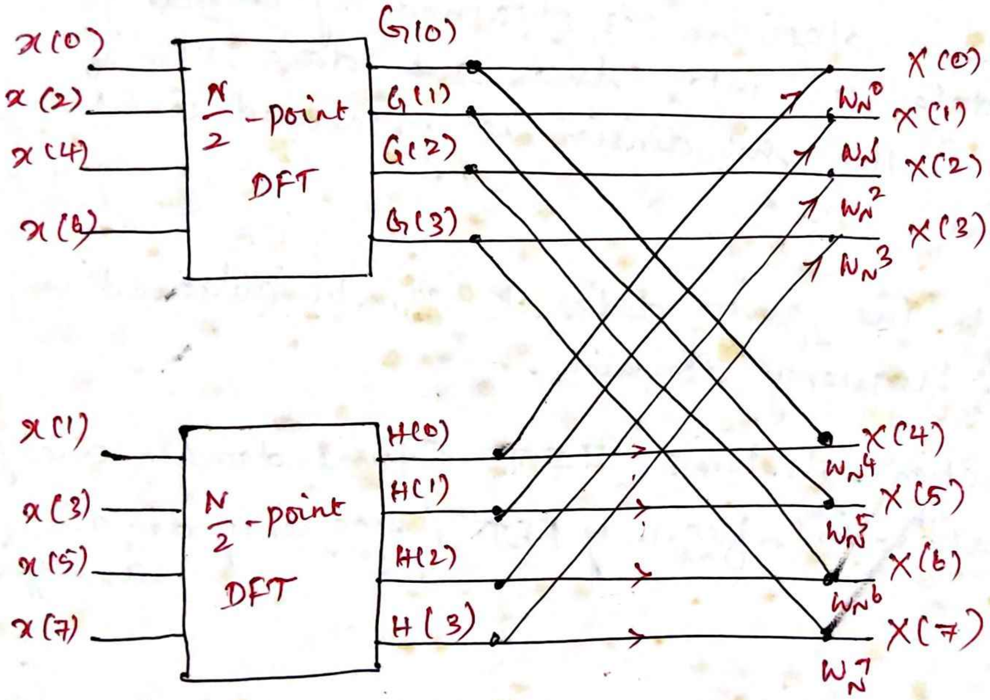
$(2^3 = 8)$.

$N \log_2 N$ additions

24.

Butterfly structure.





FIRST LEVEL OF D-I-T of 8-pt DFT

$x(n) = \{0, 1, 2, 3, 4, 5, 6, 7\}$

I level decimation ⇒

x(n) even $\{0, 2, 4, 6\}$ $\quad$ x(n) odd $\{1, 3, 5, 7\}$

Data-sequence decimation by Radix-2 method.

II level decimation ⇒ $\{0, 4\}$ $\quad \{2, 6\}$ $\quad \{1, 5\}$ $\quad \{3, 7\}$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}, \quad k = 0,1,2 \dots N-1$$ (2)

odd & even

$$X(k) = \sum_{n=\text{even}} x(n) W_N^{nk} + \sum_{n=\text{odd}} x(n) W_N^{nk} \quad k = 0,1 \dots N-1$$

even pts $n = 2r$

odd pts $n = 2r+1$

$$X(k) = \sum_{r=0}^{\frac{N}{2}-1} x(2r) W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} x(2r+1) W_N^{(2r+1)k}$$

$$W_N^{2rk} = (W_N^2)^{rk}$$

$$W_N^2 = e^{-j2\pi(2)/N} = e^{-j2\pi/(N/2)} = W_{\frac{N}{2}}.$$

$$W_N^{2rk} = W_{\frac{N}{2}}^{rk}.$$

$$X(k) = \sum_{r=0}^{\frac{N}{2}-1} x(2r) W_{\frac{N}{2}}^{rk} + W_N^k \sum_{r=0}^{\frac{N}{2}-1} x(2r+1) W_{\frac{N}{2}}^{rk}$$

$$X(k) = G(k) + W_N^k H(k) \quad k = 0,1, \dots N-1$$

where
$$G(k) = \sum_{r=0}^{\frac{N}{2}-1} x(2r) W_{\frac{N}{2}}^{rk}$$

$$H(k) = \sum_{r=0}^{\frac{N}{2}-1} x(2r+1) W_{\frac{N}{2}}^{rk}.$$

using periodic & symmetry property.

$$\left.\begin{aligned} G(k+\tfrac{N}{2}) &= G(k) \\ H(k+\tfrac{N}{2}) &= H(k) \end{aligned}\right\} \text{periodic property}$$

$$W_N^{(k+\frac{N}{2})} = -W_N^k \;\} \text{ symmetric property}$$

$$X(k) = G(k) + W_N^k H(k), \quad k = 0,1, \dots \tfrac{N}{2}-1$$

$$X(k+\tfrac{N}{2}) = G(k) - W_N^k H(k) \quad k = 0,1, \dots \tfrac{N}{2}-1$$

k ranges b/w 0 to $\frac{N}{2}-1$

$$X(k) = G(k) + W_N^{kn} H(k) \qquad k = 0, 1, \ldots N-1$$

when

$k=0 \qquad X(0) = G(0) + W_N^0 H(0)$

$k=1 \qquad X(1) = G(1) + W_N H(1)$

$k=2 \qquad X(2) = G(2) + W_N^2 H(2)$

$k=3 \qquad X(3) = G(3) + W_N^3 H(3)$

$$\left.\begin{aligned}
k=4 \qquad X(4) &= G(4) + W_N^4 H(4) = G(0) + W_N^4 H(0) \\
k=5 \qquad X(5) &= G(5) + W_N^5 H(5) = G(1) + W_N^5 H(1) \\
k=6 \qquad X(6) &= G(6) + W_N^6 H(6) = G(2) + W_N^6 H(2) \\
k=7 \qquad X(7) &= G(7) + W_N^7 H(7) = G(3) + W_N^7 H(3)
\end{aligned}\right\} \text{By periodic property}$$

By Symmetric property

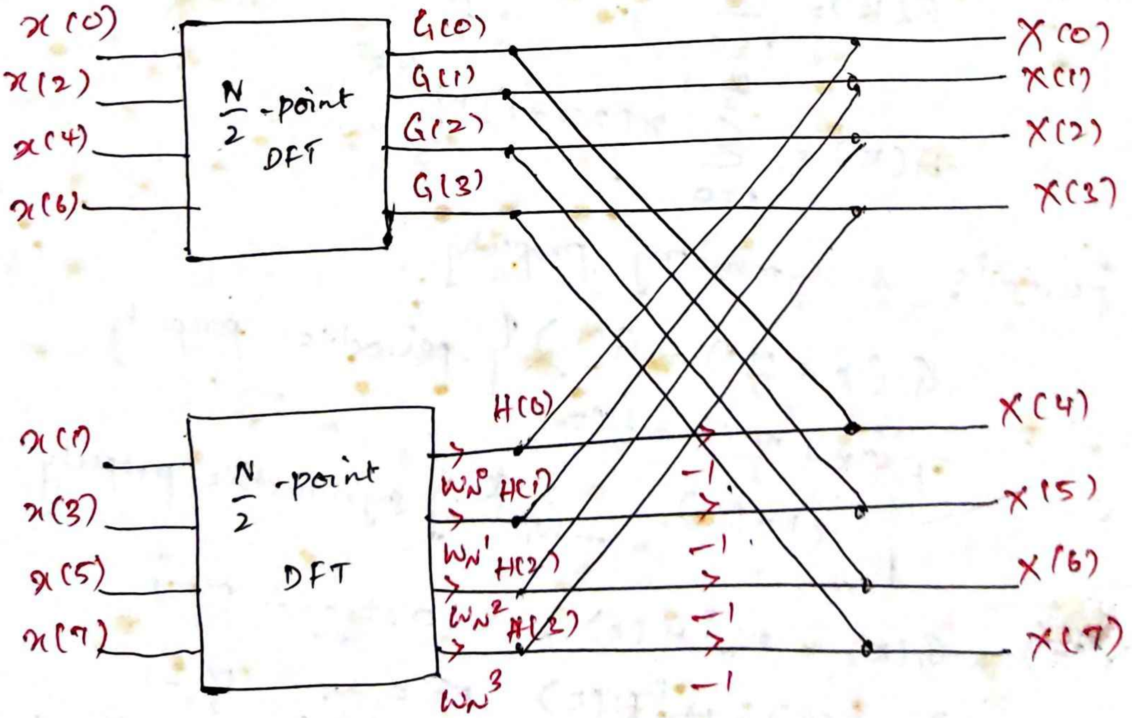
$$W_N^{k+\frac{N}{2}} = -W_N^k$$

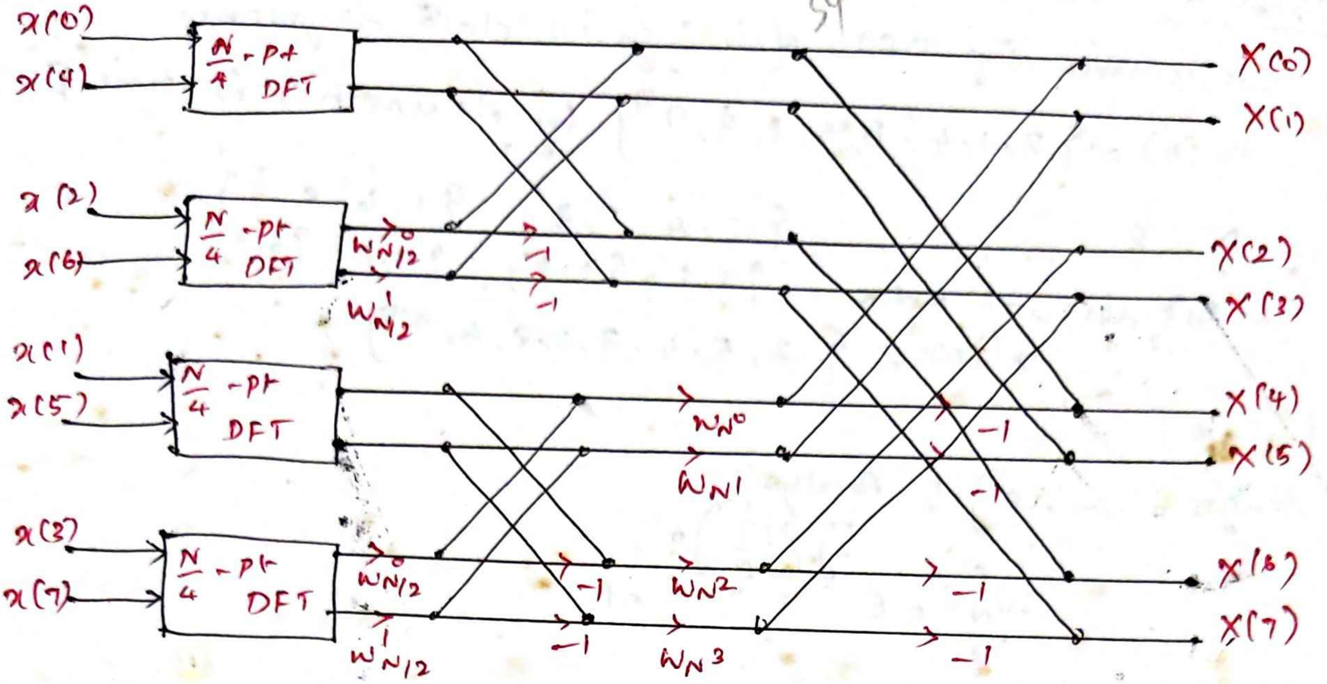
$$X(4) = G(0) - W_N^0 H(0)$$

$$X(5) = G(1) - W_N^1 H(1)$$

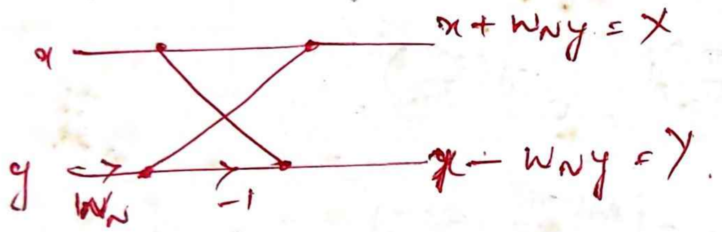
$$X(6) = G(2) - W_N^2 H(2)$$

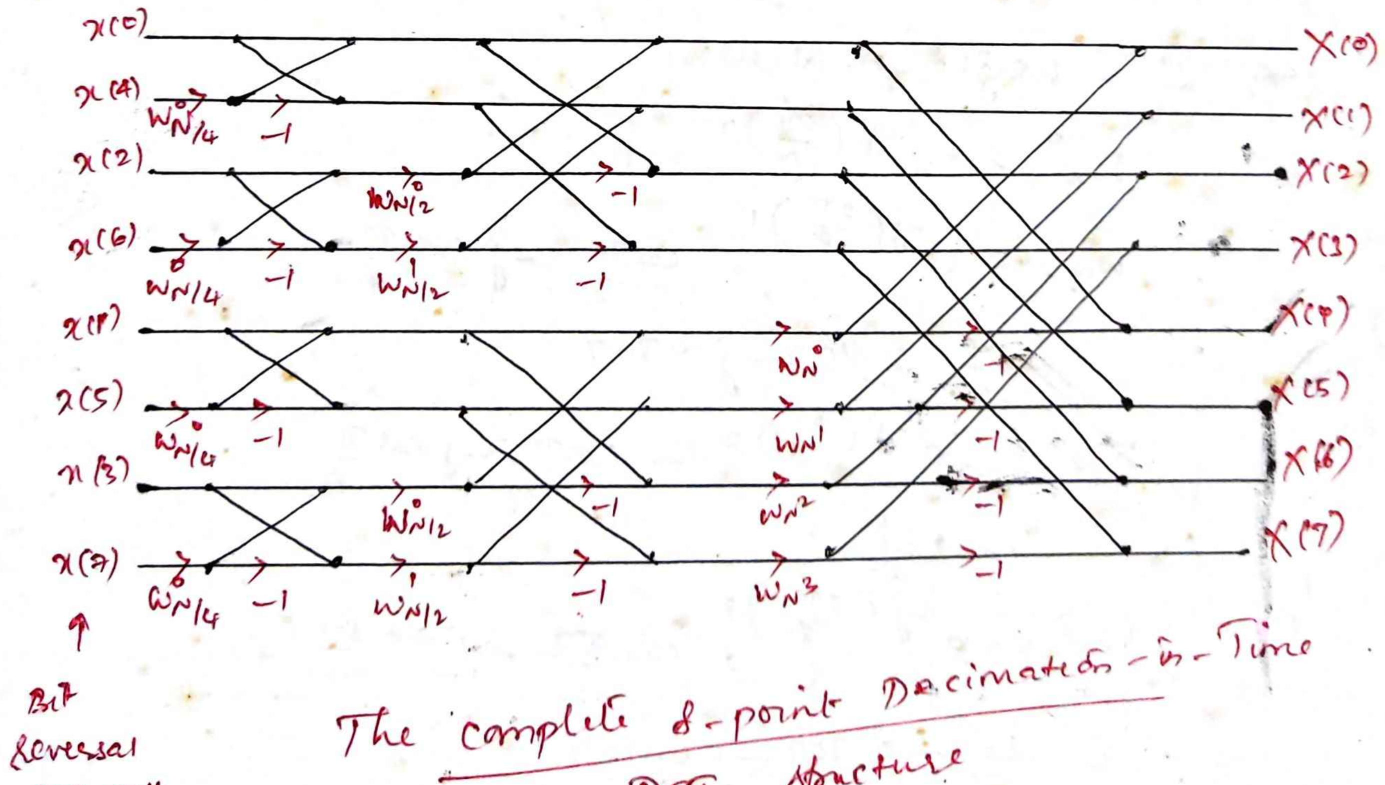
$$X(7) = G(3) - W_N^3 H(3)$$





II - level decimation - in - time FFT Algorithm.





The complete 8-point Decimation - in - Time DFT structure

Problem

1. Determine the DFT of the given data sequence
$x(n) = \{2,1,4,6,5,8,3,9\}$ by decimation in-time FFT

$N = 8$.

$\{2,4,5,3\}$ $\{1,6,8,9\}$

Bit reversal order $\{2,5\}\{4,3\}$ $\{1,8\}\{6,9\}$

$x'(n) = \{2,5,4,3,1,8,6,9\}$

Stage I - Weight Analysis

$$W_{\frac{N}{4}}^{0} = e^{-j\left(\frac{2\pi}{\frac{N}{4}}\right)0} = 1$$

Stage II - weight Analysis

$$W_{\frac{N}{2}}^{0} = e^{-j\left(\frac{2\pi}{\frac{N}{2}}\right)0} = 1$$

$$W_{\frac{N}{2}}^{1} = e^{-j\left(\frac{2\pi}{N/2}\right)1} = e^{-j\left(\frac{2\pi}{4}\right)}$$

$$= \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

Stage III - weight Analysis

$$W_N^{0} = e^{-j\left(\frac{2\pi}{N}\right)0} = 1$$

$$W_N^{1} = e^{-j\left(\frac{2\pi}{8}\right)1} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4}$$

$$= 0.707 - j0.707$$

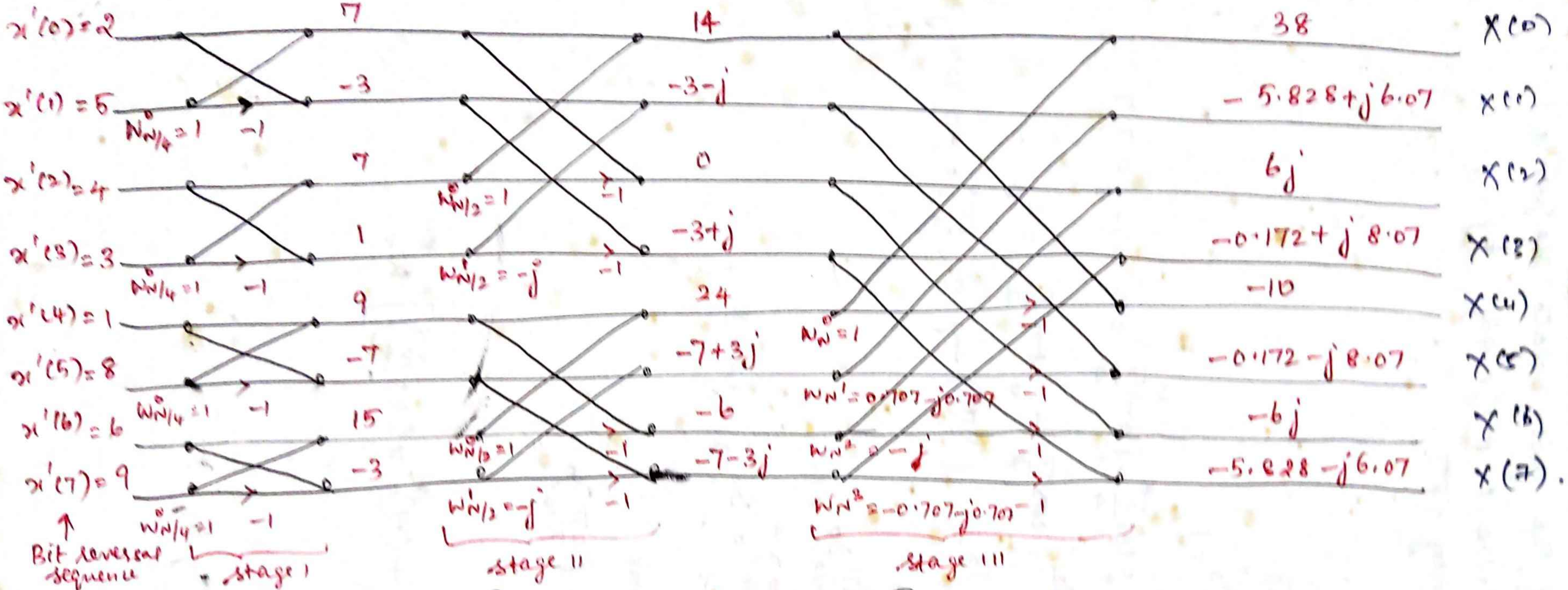
$$W_N^{2} = e^{-j\left(\frac{2\pi}{8}\right)2} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2}$$

$$= -j$$

$$W_N^{3} = e^{-j\left(\frac{2\pi}{8}\right)3} = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4}$$

$$= -0.707 - j0.707$$

$X(K) = \{38, -5.828+j6.07, j6, -0.172+j8.07, -10, -0.172-j8.07, -j6, -5.828-j6.07\}$.



Input $x'(n)$	Stage I computation	Stage II computation	Stage III computation
2	$2+5 = 7$	$7+7 = 14$	$14+24 = 38$
5	$2-5 = -3$	$-3+[(-j)1] = -3-j$	$-3-j+[(-7+3j)(0.707-j0.707)] = -5.828+j6.07$
4	$4+3 = 7$	$7-7 = 0$	$0+[-6(-j)] = 6j$
3	$4-3 = 1$	$-3-[(-j).1] = -3+j$	$-3+j+[(-7-3j)(-0.707-j0.707)] = -0.172+j8.07$
1	$1+8 = 9$	$9+15 = 24$	$14-[24\times1] = -10$
8	$1-8 = -7$	$-7+[(-j)(-3)] = -7+3j$	$-3-j-[-7+3j(0.707-j0.707)] = -0.172-j8.07$
6	$6+9 = 15$	$9-[(15).1] = -6$	$0-[-6(-j)] = -6j$
9	$6-9 = -3$	$-7-[(-j)(-3)] = -7-3j$	$-3+j-[-7-3j(-0.707-j0.707)] = -5.828-j6.07$

Decimation-In-Frequency FFT Algorithm

1. Determine the DFT of the given data sequence $x(n) = \{2,1,4,6,5,8,3,9\}$ by decimation in frequency FFT.

$N=8$.

Stage 1 : Weight Analysis

$$W_N^0 = e^{-j\left(\frac{2\pi}{N}\right)0} = 1$$

$$W_N^1 = e^{-j\left(\frac{2\pi}{8}\right)} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = 0.707 - j0.707$$

$$W_N^2 = e^{-j\left(\frac{2\pi}{8}\right)2} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$W_N^3 = e^{-j\left(\frac{2\pi}{8}\right)3} = \cos\left(\frac{3\pi}{4}\right) - j\sin\left(\frac{3\pi}{4}\right) = -0.707 - j0.707$$

Stage 2 weight Analysis

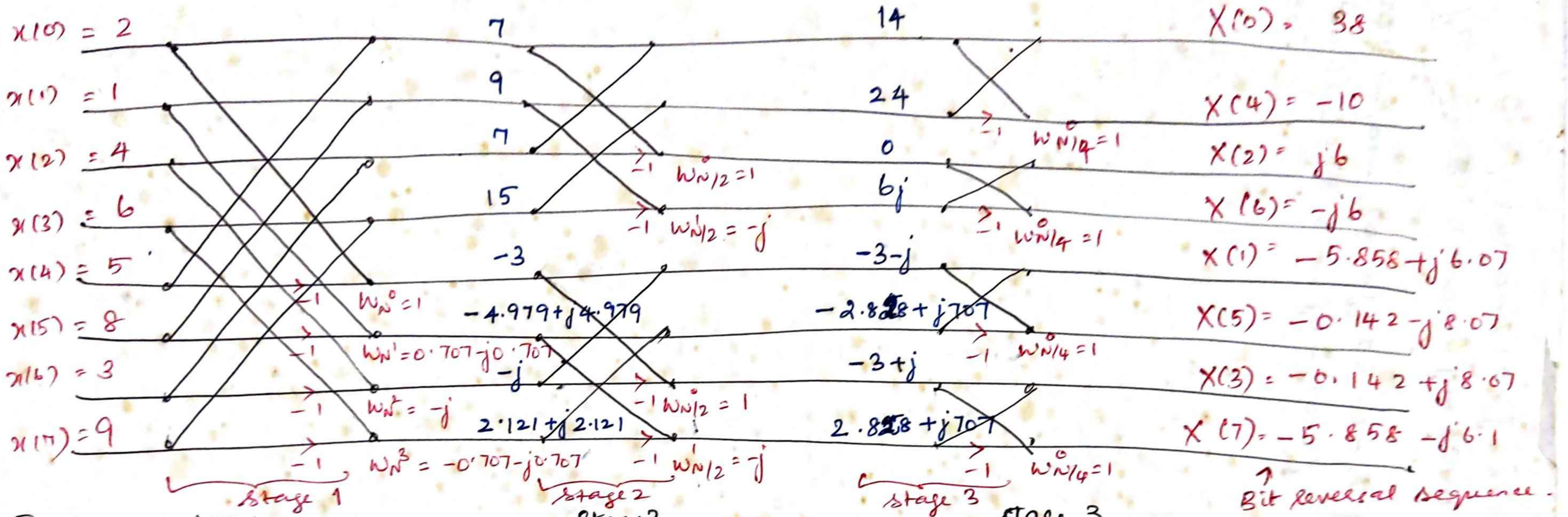
$$W_{N/2}^0 = e^{-j\left(\frac{2\pi}{N/2}\right)0} = 1$$

$$W_{N/2}^1 = e^{-j\left(\frac{2\pi}{4}\right)} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

stage 3 Weight Analysis

$$W_{N/4}^0 = e^{-j\left(\frac{2\pi}{N/4}\right)0} = 1$$

$$X(k) = \sum_{n=0}^{N/2-1}\left[x(n) + (-1)^k x\left(n+\frac{N}{2}\right)\right] W_N^{kn}.$$



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Input $x(n)$	Stage 1 computation	Stage 2 computation	Stage 3 computation
2	$2+5=7$	$7+7=14.$	$14+24=38$
1	$1+8=9$	$9+15=24$	$(14-24)\cdot 1=-10$
4	$4+3=7$	$(7-7)1=0$	$(0+6j)\cdot 1=6j$
6	$6+9=15$	$(9-15)(-j)=6j$	$(0-6j)\cdot 1=-6j$
5	$2-5=-3$	$-3-j=-3-j$	$(-3-j)+(-2.828+j707)\cdot 1=-5.828+6.07j$
8	$(1-8)(0.707-j0.707)$ $=-4.979+j4.979$	$-4.979+j4.979+2.121+j2.121$ $=-2.828+j7.07$	$(-3-j)-(-2.828+j707)\cdot 1=-0.142-j8.07$
3	$(4-3)(-j)=-j$	$(-3-(-j))\cdot 1=-3+j$	$(-3+j)+(2.828+j7.07)=-0.142+j8.07$
9	$(6-9)(-0.707-j0.707)$ $=2.121+j2.121$	$[-4.979+j4.979-(2.121+j2.121)](-j)=$ $=2.828+7.07j$	$(-3+j)-(2.828+j7.07)\cdot 1=-5.828-j6.07$

$$X'(k) = \{38, -10, j6, -j6, -5.828 + j6.07, -0.142 - j8.07, -0.142 + j8.07, -5.828 - j6.07\}$$

$$X(k) = \{38, -5.828 + j6.07, j6, -0.142 + j8.07, -10, -0.142 - j8.07, -6j, -5.828 - j6.07\}$$

$$x(n) = \{-1, 2, -3, 4, 9, -20, 12, 6\}$$