Mathematical Modelling & Graph Theory

Aalekhya Mukhopadhyay[1,a], **Angshika Dey**[1,b], and **Pramit Majumder**[1,c]

[1]Department of Mathematics, Presidency University, 86/1 College Street, Kolkata 700073
[a]mukhopadhyayaalekhya@gmail.com
[b]angshika369@gmail.com
[c]majumderpramit@gmail.com

ABSTRACT

Food chains help us better understand the intricate relationships in ecosystems. Representing the species whose relations we are to study by nodes, we obtain an interconnected structure which can be represented by a graph.
We delve into the study of the coexistence of a three species closed ecosystem consisting of grass, grasshoppers and birds. We construct a system of equations that show their interdependence making a few assumptions about their behaviour so as not to overly complicate the system. We then proceed to portray them in a graph from where we derive the adjacency matrix and study the stability of the system. We once more check the stability for the same system of differential equations by calculating the Jacobian matrix and further examining the eigenvalues. We compare the two results and attempt to establish a connection between the two.

INTRODUCTION

In an ecosystem, all living creatures rely on each other to live. Plants rely on soil, water and the sun, while animals rely on plants and other animals. This is called a food chain and it is an important example of the balance and interactions within living things and nature. In a food chain there are many different links, every link is important to the chain and without one none of the others could survive. There are three major parts to a food chain – producers, consumers and decomposers. Plants are producers because they provide energy for the ecosystem by absorbing energy from the sunlight. Plants are the only place where new energy can be made. All animals are consumers, and they absorb energy from producers. Animals that eat plants are called herbivores, and they are considered primary consumers. Grasshoppers are an example of herbivores. Animals that eat other animals are called carnivores, and they are considered secondary consumers. Frogs and birds are good examples of carnivores. Sometimes animals can be both primary and secondary consumers, these kinds of animals are called omnivores. Pigs are an example of omnivores. Food chains allow for the different parts of the ecosystem to rely on each other, also food chains prevent over population of species, which is very important.

To help us better understand our world, we often describe a particular phenomenon mathematically. Such a mathematical model is an idealization of the real-world phenomenon and never a completely accurate representation. Although any model has its limitations, a good one can provide valuable results and conclusions. In modelling our world, we are often interested in predicting the value of a variable at some time in the future. Perhaps it is a population, a real estate value, or the number of people with a communicative disease. Often a mathematical model can help us understand a behaviour better or aid us in planning for the future. Let's think of a mathematical model as a mathematical construct designed to study a particular real-world system or behaviour of interest. The model allows us to reach mathematical conclusions about the behaviour, as illustrated in following figure. These conclusions can be interpreted to help a decision maker plan for the future.

Definitions of graph theory vary.
One restricted but very common sense of the term, a graph is an ordered pair **G=(V, E)** where,

- **V** set of vertices
- **E** set of edges

The term "graph" was introduced by Sylvester in a paper published in 1878 in Nature, where he draws an analogy between "quantic invariants" and "co-variants" of algebra and molecular diagrams.

Graph theory is useful in biology and conservation efforts where a vertex can represent regions where certain species exist (or inhabit) and the edges represent migration paths or movement between the regions. This information is important when

The expressions for the characteristic polynomials and eigenvalues were obtained using Maple 2021

looking at breeding patterns or tracking the spread of disease, parasites or how changes to the movement can affect other species.

Graphs are also commonly used in molecular biology and genomics to model and analyse datasets with complex relationships.

Graph theory is also used in connectomics; nervous systems can be seen as a graph, where the nodes are neurons and the edges are the connections between them.

LOTKA – VOLTERRA SYSTEMS

$N(t)$ is the prey population at time t
$P(t)$ is the predator population at time t

$$\frac{dN}{dt} = N(a - bP) \tag{1}$$

$$\frac{dP}{dt} = P(cN - d) \tag{2}$$

Where a,b,c,d are positive constants.

The assumptions in the model are:

- The prey in the absence of any predation grows unboundedly in a Malthusian way; this is the aN term in (1).
- The effect of the predation is to reduce the prey's per capita growth rate by a term proportional to the prey and predator populations; this is the -bNP term.
- In the absence of any prey for sustenance the predator's death rate results in exponential decay, that is, the -dP term in (2).
- The prey's contribution to the predators' growth rate is cNP; that is, it is proportional to the available prey as well as to the size of the predator population.

As a first step in analysing the Lotka–Volterra model we nondimensionalise the system by writing:-

$$u(\tau) = \frac{cN(t)}{d} \tag{3}$$

$$v(\tau) = \frac{bP(t)}{a} \tag{4}$$

$$\tau = at \tag{5}$$

$$\alpha = \frac{d}{a} \tag{6}$$

and it becomes

$$\frac{du}{d\tau} = u(1 - v) \tag{7}$$

$$\frac{dv}{d\tau} = \alpha v(u - 1) \tag{8}$$

In the u,v phase plane these give

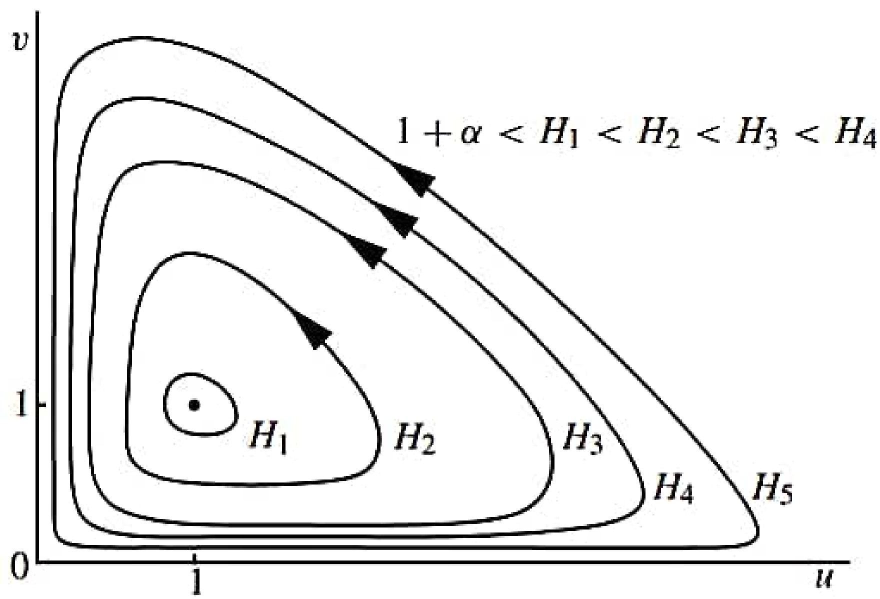
$$\frac{dv}{du} = \alpha \frac{v(u-1)}{u(1-V)} \tag{9}$$

which has singular points at u=v=0 and u=v=1.

We can integrate (9) exactly to get the phase trajectories

$$\alpha u + v - \ln u^{\alpha} v = H \tag{10}$$

where H> H_m is a constant: $H_m = 1 + \alpha$
is the minimum of H over all (u,v) and it occurs at u=v=1. We first consider the steady state (u,v)=(0,0).



Let x and y be small perturbations about (0,0).
If we keep only linear terms, (7,8) becomes

$$\begin{pmatrix} \frac{dx}{d\tau} \\ \frac{dy}{d\tau} \end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \tag{11}$$

The solution is of the form

$$\begin{pmatrix} x(\tau) \\ y(\tau) \end{pmatrix} = Be^{\lambda \tau} \tag{12}$$

where B is an arbitrary constant column vector and the eigenvalues λ are given by the characteristic polynomial of the matrix A and thus are solutions of

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ 0 & -\alpha - \lambda \end{vmatrix} = 0 \Rightarrow \lambda_1 = 1$$

$$\lambda_2 = \alpha$$

Since at least one eigenvalue, $\lambda_1 > 0, x(\tau)$ and $y(\tau)$ grow exponentially and so u=0=v is linearly unstable.
Since $\lambda_1 > 0$ and $\lambda_2 < 0$ this is a saddle point singularity. Linearising about the steady state u=v=1 by setting
u=1+x
v=1+y
with $|x|$ and $|y|$ small, (7,8) becomes

$$\begin{pmatrix} \frac{dx}{d\tau} \\ \frac{dy}{d\tau} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \tag{13}$$

where,

$$A = \begin{pmatrix} 0 & -1 \\ \alpha & 0 \end{pmatrix}$$

with eigenvalues λ given by

$$\begin{vmatrix} -\lambda & -1 \\ \alpha & -\lambda \end{vmatrix} = 0 \tag{14}$$

$$\Rightarrow \lambda_1, \lambda_2 = \pm \mathrm{i}\sqrt{\alpha}$$

Thus u=v=1 is a centre singularity since the eigenvalues are purely imaginary. Since Re λ= 0 the steady state is neutrally stable. The solution of (13) is of the form

$$\begin{pmatrix} x(\tau) \\ y(\tau) \end{pmatrix} = Ie^{\mathrm{i}\sqrt{\alpha}\tau} + me^{-\mathrm{i}\sqrt{\alpha}\tau} \tag{15}$$

where l and m are eigenvectors.

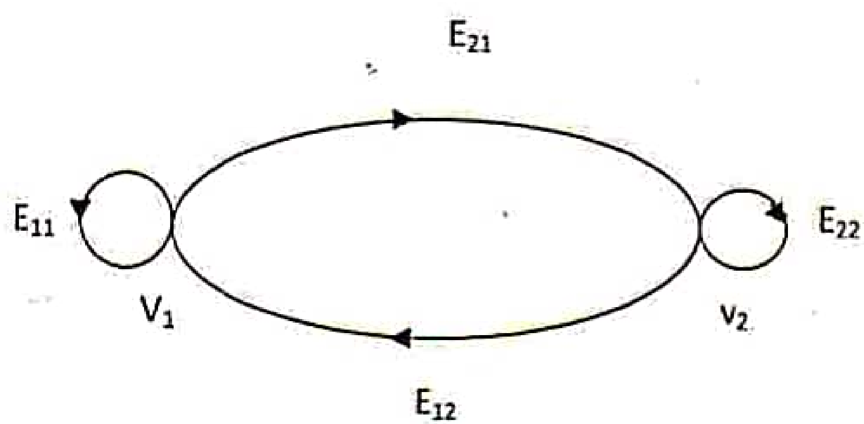
BI–VARIABLE PREY PREDATOR MODEL

In this section, we study matrix differential equation for a prey predator model involving a prey and a predator. Let x denote the prey population and y denote the predator population. Then the rate of change of prey and that of predator population gives rise to a system of nonlinear differential equations given by

$$\frac{dx}{dt} = ax + bxy, a > 0, b < 0 \tag{16}$$

$$\frac{dy}{dt} = cyx = dy, c > 0, d < 0 \tag{17}$$

We now express the above system as a graph differential equation and consider the corresponding matrix differential equation. We will see how the nonlinearity is preserved in this set up. Let the vertex v_1 denote the prey and v_2 denote the predator. Set e_{11} = x as population of the prey and e_{22} = y as the population of the predator. It can be seen that e_{12} is the edge going outward from v_2 and is incident on v_1. The edge e_{12} gives the status of predators finding the prey. Similarly the edge e_{22} going outward from v_1 and incident on v_2, indicates the status of prey that fall prey to predators. Now the graph of the prey predator model is of the form



and its adjacency matrix is given by

$$\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \tag{18}$$

The equations (16) and (17) reduce to the form

$$e'_{11} = ae_{11} + be_{21} \tag{19}$$

$$e'_{22} = ae_{12} + be_{22} \tag{20}$$

The aim is to obtain a matrix differential equation of the form

$$\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}' = A \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \tag{21}$$

Where $A_{2\times 2}$ is the coefficient matrix.
The coefficient matrix is chosen as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{22}$$

And obtain matrix differential equation of the form

$$\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \tag{23}$$

The system (23) yields the equations (19), (20) and the following two differential equations given by

$$e'_{12} = ae_{12} + be_{22} \tag{24}$$
$$e'_{21} = ce_{11} + de_{21} \tag{25}$$

The equation (24) describe the rate of change of predator finding prey and it is positively proportional to the predator finding prey and negatively proportional to the predator population.
The equation (25) gives the rate of change of prey falling prey to predator and this is positively proportional to prey available and negatively proportional to prey falling to predator.
Hence it can be seen that all the four equations given by (19), (20), (24) and (25) are consistent with the standard prey predator problem. The beauty in this set up is that the nonlinearity is preserved and effectively used.

OUR MODEL

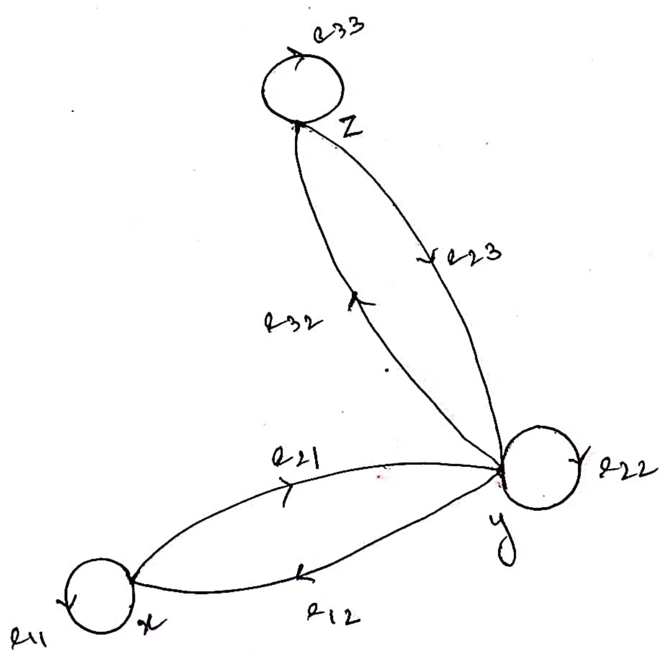
In this section, we formulate a prey predator model where we describe a closed environment with three prevalent populations: grass, grasshoppers and birds. We describe the system such that the only source of food for the grasshoppers is grass, and that of the birds are grasshoppers. The grass grows in the area by photosynthesis with the help of sunlight, water, nutrients available in the soil and carbon dioxide emitted by the grasshoppers and birds. We assume that the only way the grasshoppers survive is by consuming the grass, and the only way the birds survive is by consuming the grasshoppers. We also assume the birds to be the highest in the food chain, meaning that they have no predators and die only due to natural causes.

$$\text{GRASS} \longrightarrow \text{GRASSHOPPER} \longrightarrow \text{BIRD}$$

Let x denote the grass population, y denote the grasshopper population and z denote the bird population. Then the rate of change of grass, grasshopper and bird gives rise to a system of nonlinear differential equations given by

$$\frac{dx}{dt} = ax - (by)x - cx = (a-c)x - (by)x \tag{26}$$
$$\frac{dy}{dt} = (qx)y - (pz)y - fy = (c_1bx) - fy - (pz)y \tag{27}$$
$$\frac{dz}{dt} = (hy)z - jz = (c_2py)z - jz \tag{28}$$

We now express the above system as a graph differential equation and consider the corresponding matrix differential equation. Let x denote the grass, y denote the grasshoppers, and z denote the birds. Lets set e_{11} as the population of grass, e_{22} as the population of grasshoppers and e_{33} as the population of birds. It can be seen that e_{12} goes outwards from y and is incident on x, e_{12} gives the status of grasshoppers finding grass. Similarly, e_{21} gives the status of grass falling prey to grasshoppers. Also, e_{23} gives the status of birds finding grasshoppers. Finally, e_{32} gives the status of grasshoppers falling prey to birds.
Now, the graph of our prey predator model is of the form



and its adjacency matrix is given by

$$\begin{bmatrix} e_{11} & e_{12} & 0 \\ e_{21} & e_{22} & e_{23} \\ 0 & e_{32} & e_{33} \end{bmatrix} \tag{29}$$

The equations (26), (27) and (28) reduce to the form

$$e'_{11} = (a-c)e_{11} - be_{21} \tag{30}$$
$$e'_{22} = (c_1 b)e_{12} - fe_{22} - pe_{32} \tag{31}$$
$$e'_{33} = (c_2 p)e_{23} - je_{33} \tag{32}$$

Our aim is to obtain a matrix differential equation of the form

$$\begin{bmatrix} e_{11} & e_{12} & 0 \\ e_{21} & e_{22} & e_{23} \\ 0 & e_{32} & e_{33} \end{bmatrix}' = A \begin{bmatrix} e_{11} & e_{12} & 0 \\ e_{21} & e_{22} & e_{23} \\ 0 & e_{32} & e_{33} \end{bmatrix} \tag{33}$$

where $A_{3\times 3}$ is the coefficient matrix.

We propose to choose

$$\text{A}=\begin{bmatrix} (a-c) & -b & 0 \\ c_1 b & -f & -p \\ 0 & c_2 p & -j \end{bmatrix}$$

and obtain matrix differential equation of the form

$$\begin{bmatrix} e_{11} & e_{12} & 0 \\ e_{21} & e_{22} & e_{23} \\ 0 & e_{32} & e_{33} \end{bmatrix}' = \begin{bmatrix} (a-c) & -b & 0 \\ c_1 b & -f & -p \\ 0 & c_2 p & -j \end{bmatrix} \begin{bmatrix} e_{11} & e_{12} & 0 \\ e_{21} & e_{22} & e_{23} \\ 0 & e_{32} & e_{33} \end{bmatrix} \tag{34}$$

The system (34) yields equations (30), (31), (32) and the following four differential equations given by

$$e'_{12} = (a-c)e_{12} - be_{22} \tag{35}$$
$$e'_{21} = (c_1 b)e_{11} - fe_{21} \tag{36}$$
$$e'_{23} = -fe_{23} - pe_{33} \tag{37}$$
$$e'_{32} = (c_2 p)e_{22} - je_{32} \tag{38}$$

Equation (35) describes the rate of change of grasshoppers finding grass and it is positively proportional to the grasshoppers finding grass and negatively proportional to the grasshopper population.
Equation (36) gives the rate of change of grass falling prey to grasshoppers and it is positively proportional to the grass population and negatively proportional to grass falling prey to grasshoppers.
Equation (37) provides the rate of change of birds finding grasshoppers and it is negatively proportional to birds finding grasshoppers and negatively proportional to the bird population. This statement is not consistent with a stable system.
Equation (38) states the rate of change of grasshoppers falling prey to birds and it is positively proportional to grasshopper population and negatively proportional to grasshoppers falling prey to birds.
It is evident from equation (37) that the prey predator system we have formed is not stable. Next we will study the inconsistency in this system and try to formulate some corrections.

Inconsistency

Calculating Equilibrium Points

We calculate the equilibrium points by equating $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ to 0.
From eqn (26), we get $G = \frac{a-c}{b}$

From eqn (28), we get $G = \frac{j}{c_2 p}$

This is inconsistent.

MODEL 2.0

We introduce the concept of **Carrying Capacity**.
We are redefining the model 1.0 with a few change of variables for convenience purposes.
We describe a closed environment with three prevalent populations: grass, grasshoppers and birds. We describe the system such that the only source of food for the grasshoppers is grass, and that of the birds are grasshoppers. The grass grows in the area by photosynthesis with the help of sunlight, water, nutrients available in the soil and carbon dioxide emitted by the grasshoppers and birds. We assume that the only way the grasshoppers survive is by consuming the grass, and the only way the birds survive is by consuming the grasshoppers. We also assume the birds to be the highest in the food chain, meaning that they have no predators and die only due to natural causes.
Here, we try to construct a mathematical modelling about the coexistence of these populations in a closed environment by assuming a few factors as listed below:

x	population of grass
a	growth of grass due to photosynthesis
b	death of grass by grasshoppers
c	death of grass by natural causes
y	population of grasshoppers
d	rate at which the grasshoppers consume the grass
q	death rate of grasshoppers due to birds
f	death rate of grasshoppers due to natural causes
z	population of birds
h	rate at which birds consume grasshoppers
j	death rate of birds due to natural causes
t	time

$\frac{dx}{dt}$ denotes rate of growth of grass population

$\frac{dy}{dt}$ denotes rate of growth of grasshopper population

$\frac{dz}{dt}$ rate of growth of bird population

We define

$$\frac{dx}{dt} = ax - (by)x - cx^2 = ax(1 - \frac{x}{s}) - (by)x = u_1 \tag{39}$$

where, s denotes the carrying capacity for grass population such that $s = \frac{1}{c}$ and $a > c$

$$\frac{dy}{dt} = (dx)y - (qz)y - fy = (pbx)y - (qz)y - fy = (pbx - qz - f)y = u_2 \tag{40}$$

where, d=pb and p>0

$$\frac{dx}{dt} = (hy)z - jz = (kqy)z - jz = u_3 \tag{41}$$

where, h=kq and $q < 1$

ANALYZING

We calculate the Jacobian Matrix of the equations (39), (40) and (41) using the formula given below:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial z} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} & u\frac{\partial u_2}{\partial z} \\ \frac{\partial u_3}{\partial x} & \frac{\partial u_3}{\partial y} & \frac{\partial u_3}{\partial z} \end{bmatrix}$$

$$\implies \mathbf{J} = \begin{bmatrix} a - \frac{2ax}{s} - by & pby & 0 \\ -bx & pbx - qz - f & kqz \\ 0 & -qy & kqy - j \end{bmatrix} \tag{42}$$

We then proceed to calculate the characteristic polynomial from the Jacobian obtained in equation (44) assuming the eigenvalue to be λ

$$|\text{J-}\lambda I| = 0$$

$$\left(\lambda^3 + (by - a + f + j - kqy - pbx + qz)\lambda^2 \right.$$
$$+((-by + a - f + pbx - qz)\,kqy + (by - a + f - pbx + qz)\,j + (pbx - qz - f)\,a + y\,(-pbx + qz + f)\,b + kqzqy + pbybx)\,\lambda +$$
$$\left((-pbx + qz + f)\,a - y\left(-pbx + qz + \ddot{f}\right)b - pbybx\right)kqy + ((pbx - qz - f)\,a + y\,(-pbx + qz + f)\,b + pbybx)\,j - kqzqy\,(-by + a)\Big)\,s -$$
$$2\left(-\lambda^2 + (-qz + kqy + pbx - f - j)\,\lambda + (-pbx + qz + f)\,kqy + (pbx - qz - f)\,j - kqzqy\right)ax)/s = 0$$

POSSIBILITIES

Assuming that x, y and z represent population of grass, grasshoppers and birds, we obtain 8 different possibilities about the outcome of our environment:

Cases	x	y	z
A	0	0	0
B	0	0	1
C	0	1	0
D	0	1	1
E	1	0	0
F	1	0	1
G	1	1	0
H	1	1	1

We now delve into each of these cases in detail to determine their whether or not they are feasible.

Case A:

x	y	z
0	0	0

Replacing the values of x, y and z by 0 in equations (39), (40) and (41)
we get

$$J = \begin{bmatrix} a & 0 & 0 \\ 0 & -f & 0 \\ 0 & 0 & -j \end{bmatrix} \quad (43)$$

This gives the following characteristic polynomial:

$$(\lambda - a)(\lambda + f)(\lambda + j) = 0 \quad (44)$$

$\implies \lambda = a, -f, -j$

Case B:

x	y	z
0	0	1

Replacing the values of x and y by 0 in equations (39), (40) and (41), we get the following:
From equation (41),
(kqy - j) z = 0
We know, $z \neq 0$
$\implies kqy - j = 0$
$\implies j = kqy$ but y=0 (given)
$\implies j = 0$ which is a contradiction.
From a logical standpoint, we see that only the bird population is thriving while the grass and grasshopper populations are extinct. This is not possible as the only source of nutrition for the birds are grasshoppers and they can't exist without them. Hence, this case is impossible.

Case C:

x	y	z
0	1	0

Replacing the values of x and z by 0 in equations (39), (40) and (41), we get the following:
From equation (40),
(pbx-qz-f)y=0
We know, y $\neq 0$
$\implies pbx - qz - f = 0$
$\implies f = pbx - qz$
$\implies f = 0$ (which is a contradiction.)
From a logical standpoint, this case implies that only the grasshopper population is surviving whereas the grass and bird population is extinct. This is not possible as without the grass, the grasshoppers have no other source of nutrition. This case is, therefore, not possible.

Case D:

x	y	z
0	1	1

Replacing the value of x by 0 in equations
(39), (40) and (41), we get the following:
From equation (40),
(pbx-qz-f)y=0
We know, y $\neq 0$
$\implies pbx - qz - f = 0$

The expressions for the characteristic polynomials and eigenvalues were obtained using Maple 2021

$\implies z = -\frac{f}{q}$
We have assumed f>0, q>0 which implies z<0, which is not possible since a population cannot be negative. This, we get a contradiction.
From a logical standpoint, in this system, we get the population of grass to be 0, whereas the grasshopper and bird populations are thriving. This is not possible as grass is the producer and an ecosystem cannot survive without one. This case is unfeasible too.

Case E:

x	y	z
1	0	0

Replacing the values of y and z by 0 in equations (39), (40) and (41), we get the following:
From equation (39),
ax(1- $\frac{x}{s}) - (by)x = 0$

We know, x $\neq 0$
$\implies a(1-\frac{x}{s}) - by = 0$
$\implies a(1-\frac{x}{s}) = 0$
$\implies x = s$

$$J = \begin{bmatrix} -a & 0 & 0 \\ -bs & pbs-f & 0 \\ 0 & 0 & -j \end{bmatrix} \tag{45}$$

The characteristic polynomial is given by:
$(\lambda + a)\,(\lambda - pbs + f)\,(\lambda + j) = 0$
$\implies \lambda = -a, \quad pbs - f, \quad -j$

Case F:

x	y	z
1	0	1

Replacing the value of y by 0 in equations (39), (40) and (41), we get the following:
From equation (41),
(kqy-j)z=0
We know, z not equals to 0
$\implies kqy - j = 0$
$\implies$j=kqy but y=0 (given)
$\implies$ j=0 which is a contradiction.
From a logical standpoint, we see that the bird and grass population exist whereas the grasshopper population is extinct. This is not possible as the only source of nutrition for the birds are grasshoppers and without them, their survival is impossible. This case is not possible.

Case G:

x	y	z
1	1	0

Replacing the value of z by 0 in equations (39), (40) and (41), we get the following:
From equation (40),
(pbx-qz-f)y=0
We know, y $\neq 0$,
$\implies pbx - qz - f = 0$
$\implies$x= $\frac{f}{bp}$
From equation (39),
ax(1- $\frac{x}{s}) - (by)x = 0$

The expressions for the characteristic polynomials and eigenvalues were obtained using Maple 2021

We know, x $\neq 0$
$\Longrightarrow$a(1- $\frac{x}{s}) - by = 0$
$\Longrightarrow y = \frac{a(bps-f)}{b^2ps}$

$$J = \begin{bmatrix} -\frac{af}{bps} & \frac{a(bps-f)}{bs} & 0 \\ -\frac{f}{p} & 0 & 0 \\ 0 & -\frac{aq(bps-f)}{b^2ps} & \frac{akq(bps-f)}{b^2ps} - j \end{bmatrix} \tag{46}$$

The characteristic polynomial is given by:

$$\frac{((bj+b\lambda+akq)\,bps-akqf)\left(fap\left(\frac{bps-f}{bps}\right)bps+(\lambda bps+af)\,\lambda p\right)}{bps^2bp} = 0 \tag{47}$$

Eigenvalues(λ) = $\frac{(-bj-akq)bps+akqf}{bbps}$, $\frac{-afp+\sqrt{-4\left(ap\left(\frac{bps-f}{bps}\right)bps^2f-\frac{af^2p}{4}\right)p}}{2bpsp}$, $\frac{-afp-\sqrt{-4\left(ap\left(\frac{bps-f}{bps}\right)bps^2f-\frac{af^2p}{4}\right)p}}{2bpsp}$

Case H:

x	y	z
1	1	1

In this case, we represent the values of x, y and z by x^*, y^* and z^* respectively.
From equation (41),
$(kqy^* - j)z^* = 0$
We know, z^* not equals to 0
$\Longrightarrow kqy^* - j = 0$
$\Longrightarrow y^* = \frac{j}{kq}$
From equation (39),
ax$^*(1 - \frac{x^*}{s}) - (by^*)x^* = 0$

We know, x$^* \neq 0$
$\Longrightarrow$a(1- $\frac{x^*}{s}) - by^* = 0$
$\Longrightarrow x^* =$(1- $\frac{bj}{akq})s$
From equation (40),
(pbx$^* - qz^* - f)y^* = 0$
We know, y$^* \neq 0$,
$\Longrightarrow pbx^* - qz^* - f = 0$
$\Longrightarrow$z$^* = \left(\frac{1}{q}\left(pbs\left(1-\frac{bj}{akq}\right)-f\right)\right)$

$$\text{J=} \begin{bmatrix} -a+b\left(\frac{j}{kq}\right) & pb\left(\frac{j}{kq}\right) & 0 \\ -b\left(1-\frac{bj}{akq}\right)s & 0 & k\left(pbs\left(1-\frac{bj}{akq}\right)-f\right) \\ 0 & -\frac{j}{k} & 0 \end{bmatrix}$$

The characteristic polynomial is given by:
$$\frac{j\left(\lambda+a-b\left(\frac{j}{kq}\right)\right)k\left(pbs\left(\frac{akq-bj}{akq}\right)-f\right)+\lambda k\left(b\left(\frac{akq-bj}{akq}\right)pb\left(\frac{j}{kq}\right)s+\lambda\left(\lambda+a-b\left(\frac{j}{kq}\right)\right)\right)}{k} = 0$$

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The eigenvalues from this equation were too long to be represented here.

STABILITY

The stability of the system depends on the eigenvalues as follows:

Eigenvalue Type	Stability
All Real and +	Unstable
All Real and -	Stable
Mixed + & - Real	Unstable
+a + b*i*	Unstable
-a + b*i*	Stable
0 + b*i*	Unstable
Repeated values	Depends on orthogonality of eigenvectors

From the above table we infer that the system can be said to be stable if the eigenvalues obtained are:

- all real, distinct and negative
- complex with negative real part
- repeated (in this case it depends on the orthogonality of eigenvectors)

CONCLUSION

The nature of the eigenvalues depend on the initial assumptions. We are yet to run pre-existing data on our model and verify which of the four cases are feasible.
However, from a biological standpoint we see that the model we assumed should be stable in each of the four cases.

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