# THE LINEAR MODEL

- Estimable Linear Functions and the Gauss-Markov Theorem

**Definition:** The linear function of the parameters, $\boldsymbol{q}'\boldsymbol{b}$, is estimable if it has an unbiased linear estimator, that is, if a $\boldsymbol{t}'\boldsymbol{y}$ exists such that

$$E(\boldsymbol{t}'\boldsymbol{y}) = \boldsymbol{q}'\boldsymbol{b}.$$

**Theorem:** $\boldsymbol{q}'\boldsymbol{b}$ estimable $\Leftrightarrow \operatorname{rank} \boldsymbol{X} = \operatorname{rank} \begin{pmatrix} \boldsymbol{q}' \\ \boldsymbol{X} \end{pmatrix}$.

The implication is that $\boldsymbol{q}'\boldsymbol{b}$ is estimable if and only if a $\boldsymbol{t}$ exists such that $\boldsymbol{q}' = \boldsymbol{t}'\boldsymbol{X}$.

Proof:

$$\begin{aligned} \boldsymbol{q}'\boldsymbol{b} \text{ estimable } &\Leftrightarrow \exists \;\; \boldsymbol{t} \;\ni\; E(\boldsymbol{t}'\boldsymbol{y}) = \boldsymbol{t}'\boldsymbol{X}\boldsymbol{b} \equiv \boldsymbol{q}'\boldsymbol{b} \quad \forall\, \boldsymbol{b} \\ &\Leftrightarrow \boldsymbol{q}' = \boldsymbol{t}'\boldsymbol{X} \;\text{ for some } \boldsymbol{t}. \end{aligned}$$

If $\operatorname{rank}(\boldsymbol{X}) = p$ all linear functions of the parameters are estimable, because the condition of the Theorem is always met in this case. All the parameters are then also estimable. In this case $\boldsymbol{X}'\boldsymbol{X}$ is non-singular and the solution of the normal equations is unique.

**Theorem (Gauss-Markov):** Consider the estimable function $\boldsymbol{q}'\boldsymbol{b}$.

1. A **unique unbiased linear estimator**, $\boldsymbol{t}_0'\boldsymbol{y}$, of $\boldsymbol{q}'\boldsymbol{b}$ exists such that $\boldsymbol{t_0} \in V_r$. If $\boldsymbol{t}'\boldsymbol{y}$ is any unbiased estimator of $\boldsymbol{q}'\boldsymbol{b}$, then $\boldsymbol{t_0}$ is the projection of $\boldsymbol{t}$ on $V_r$.
2. $\operatorname{var}(\boldsymbol{t}_0'\boldsymbol{y}) \leq \operatorname{var}(\boldsymbol{t}'\boldsymbol{y})$ and the equality holds if and only if $\boldsymbol{t} = \boldsymbol{t_0}$.
3. $\boldsymbol{t}_0'\boldsymbol{y} = \boldsymbol{q}'\widehat{\boldsymbol{b}}$, where $\widehat{\boldsymbol{b}}$ is any set of least squares estimators of the parameters.

**Proof:**

1. Let $\boldsymbol{t}'\boldsymbol{y}$ be any unbiased linear estimator:

$$E(\boldsymbol{t}'\boldsymbol{y}) = \boldsymbol{q}'\boldsymbol{b}.$$

Let $\boldsymbol{t} = \boldsymbol{t_0} + \boldsymbol{e}$, where $\boldsymbol{t_0} \in V_r$ and $\boldsymbol{e} \in V_e$. Then

$$E(\boldsymbol{t}_0'\boldsymbol{y}) = E(\boldsymbol{t}'\boldsymbol{y}) = \boldsymbol{q}'\boldsymbol{b}$$

$E(t^T e) = 0$

since $E(\boldsymbol{e}'\boldsymbol{y}) = \boldsymbol{0}$. Therefore $\boldsymbol{t}_0'\boldsymbol{y}$ is an unbiased estimator of $\boldsymbol{q}'\boldsymbol{b}$ and $\boldsymbol{t}_0 \in V_r$. Suppose that the same holds for $\boldsymbol{t}_1'\boldsymbol{y}$. Then

$(t_0^T - t_1^T)E(y)$

$$q^Tb - q^Tb = E(\boldsymbol{t}_0'\boldsymbol{y}) - E(\boldsymbol{t}_1'\boldsymbol{y}) = (\boldsymbol{t}_0 - \boldsymbol{t}_1)'\boldsymbol{X}\boldsymbol{b} \equiv \boldsymbol{0} \ \forall\, \boldsymbol{b}.$$

Therefore $\boldsymbol{t}_0 - \boldsymbol{t}_1 \in V_e$. But $\boldsymbol{t}_0 \in V_r, \boldsymbol{t}_1 \in V_r$ and $(\boldsymbol{t}_0 - \boldsymbol{t}_1) \in V_r$. Therefore $\boldsymbol{t}_0 - \boldsymbol{t}_1 = \boldsymbol{0}$ and $\boldsymbol{t}_1 = \boldsymbol{t}_0$.

2. $var(\boldsymbol{t}'\boldsymbol{y}) = \boldsymbol{t}'\boldsymbol{t}\sigma^2$

$Var(aZ) = a \, Var Z \, a^T$ ($A$, $A$, $A^T$) $\quad Var(y) = \sigma^2 I$

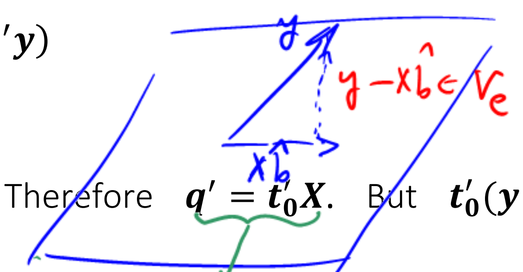
$$= (\boldsymbol{t}_0 + \boldsymbol{e})'(\boldsymbol{t}_0 + \boldsymbol{e})\sigma^2$$

$t_0^T e = e^T t_0 = 0$

$$= \boldsymbol{t}_0'\boldsymbol{t}_0\sigma^2 + \boldsymbol{e}'\boldsymbol{e}\sigma^2 \text{ because } \boldsymbol{t}_0 \perp \boldsymbol{e}$$

$$= \mathrm{var}(\boldsymbol{t}_0'\boldsymbol{y}) + \mathrm{var}(\boldsymbol{e}'\boldsymbol{y})$$

$$\geq \mathrm{var}(\boldsymbol{t}_0'\boldsymbol{y})$$




$V_r \perp V_e$

3. $E(\boldsymbol{t}_0'\boldsymbol{y}) = \boldsymbol{t}_0'\boldsymbol{X}\boldsymbol{b} \equiv \boldsymbol{q}'\boldsymbol{b} \ \ \forall\, \boldsymbol{b}$. Therefore $\boldsymbol{q}' = \boldsymbol{t}_0'\boldsymbol{X}$. But $\boldsymbol{t}_0'(\boldsymbol{y} - \boldsymbol{X}\widehat{\boldsymbol{b}}) = \boldsymbol{0}$ since $\boldsymbol{t}_0 \perp V_e$. Therefore

Best linear Unbiased Estimator

BLUE

$$\boldsymbol{t}_0'\boldsymbol{y} = \boldsymbol{t}_0'\boldsymbol{X}\widehat{\boldsymbol{b}} = \boldsymbol{q}'\widehat{\boldsymbol{b}}.$$

The estimator $\boldsymbol{q}'\widehat{\boldsymbol{b}} = \boldsymbol{t}_0'\boldsymbol{y}$ is called the best estimator of $\boldsymbol{q}'\boldsymbol{b}$ Conversely, for any $\boldsymbol{d} \in V_r \boldsymbol{d}'\boldsymbol{y}$ is the best estimator of the estimable linear function $E(\boldsymbol{d}'\boldsymbol{y}) = \boldsymbol{d}'\boldsymbol{X}\boldsymbol{b}$. Therefore $\boldsymbol{V_r}$ is called the estimation space and the linear set

Regression space

$$L_r = \{\boldsymbol{t}'\boldsymbol{y} \ : \ \boldsymbol{t} \in V_r\}$$

is called the estimation set. The sum of squares of the estimation set is:

$y^Ty = SSR + SSE$

Regression set

$$SSR = \widehat{\boldsymbol{b}}'\boldsymbol{X}'\boldsymbol{X}\widehat{\boldsymbol{b}} = \widehat{\boldsymbol{b}}'\boldsymbol{X}'\boldsymbol{y} = \boldsymbol{y}'\boldsymbol{y} - SSE$$

$rank(V_r) = r$

Suppose that the columns of $\boldsymbol{E}: n \times (n - r)$ generate $V_e$. Then

$$SSE = \boldsymbol{y}'\boldsymbol{E}(\boldsymbol{E}'\boldsymbol{E})^{-1}\boldsymbol{E}'\boldsymbol{y}.$$

It also follows that

$$E(SSE) = (n - r)\sigma^2$$

$\frac{1}{n-r} SSE$ unbiased estimator for $\sigma^2$

and

$$MSE = SSE/(n - r)$$

is an unbiased estimator of $\sigma^2$, based on all the degrees of freedom for error.

Theorem: Suppose that the elements of $\boldsymbol{Q}'\boldsymbol{b}: s \times 1$ are estimable with best estimators the elements of $\boldsymbol{T}'\boldsymbol{y}$. Then

- $\boldsymbol{h}'\boldsymbol{Q}'\boldsymbol{b}$ is estimable with best estimator $\boldsymbol{h}'\boldsymbol{T}'\boldsymbol{y}$ and
- $\text{rank}\ (\boldsymbol{Q}) = \text{rank}\ (\boldsymbol{T})$.

Proof: Since a linear function is estimable if has an unbiased linear estimator, we have

- $E(\boldsymbol{h}'\boldsymbol{T}'\boldsymbol{y}) = \boldsymbol{h}'\boldsymbol{Q}'\boldsymbol{b}$ and $\boldsymbol{T}\boldsymbol{h} \in V_r$.
- $E(\boldsymbol{T}'\boldsymbol{y}) = \boldsymbol{T}'\boldsymbol{X}\boldsymbol{b} \equiv \boldsymbol{Q}'\boldsymbol{b} \qquad \forall \boldsymbol{b}$.

Therefore

$$\boxed{\boldsymbol{Q}' = \boldsymbol{T}'\boldsymbol{X}} \qquad Q = X^T T$$

Further

$$\boxed{\text{rank}(AB) \leq \min(\text{rank}\ A, \text{rank}\ B)}$$

$$\text{rank}(\boldsymbol{Q}) = \text{rank}(\boldsymbol{T}'\boldsymbol{X}) \leq \text{rank}\ (\boldsymbol{T}).$$

Also, $\boldsymbol{T}'\boldsymbol{y} = \boldsymbol{Q}'\widehat{\boldsymbol{b}} = \boldsymbol{Q}'(\boldsymbol{X}'\boldsymbol{X})^*\boldsymbol{X}'\boldsymbol{y}$ for all values of $\boldsymbol{y}$,

$$\underbrace{\boldsymbol{T}' = \boldsymbol{Q}'(\boldsymbol{X}'\boldsymbol{X})^*\boldsymbol{X}'}$$

and $\text{rank}\ (\boldsymbol{T}) \leq \text{rank}\ (\boldsymbol{Q})$.

It follows that $\text{rank}\ (\boldsymbol{Q}) = \text{rank}\ (\boldsymbol{T})$. $\qquad Cov(AZ) = A\,Cov(Z)A^T$

For the covariance matrix of $\boldsymbol{T}'\boldsymbol{y}$, the best estimator of $\boldsymbol{Q}'\boldsymbol{b}$, it follows that:

$Cov(y) = \sigma^2 I$

$$\begin{aligned}
Cov\ (\boldsymbol{T}'\boldsymbol{y}) &= \boldsymbol{T}'\boldsymbol{T}\boldsymbol{\sigma}^2 \\
&= \boldsymbol{Q}'(\boldsymbol{X}'\boldsymbol{X})^*\boldsymbol{X}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{*\prime}\boldsymbol{Q}\boldsymbol{\sigma}^2 \\
&= \boldsymbol{T}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^*\boldsymbol{X}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{*\prime}\boldsymbol{X}'\boldsymbol{T}\boldsymbol{\sigma}^2 \\
&= \boldsymbol{T}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^*\boldsymbol{X}'\boldsymbol{T}\boldsymbol{\sigma}^2 \text{ since } \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^*\boldsymbol{X}' \text{ is idempotent} \\
Q^T = T^T X \quad \longrightarrow &= \underbrace{\boldsymbol{Q}'(\boldsymbol{X}'\boldsymbol{X})^*\boldsymbol{Q}}\boldsymbol{\sigma}^2.
\end{aligned}$$

If $rank\ (\boldsymbol{X}) = p$ then $\boldsymbol{b}$ is estimable and

$$Cov\ (\widehat{\boldsymbol{b}}) = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{\sigma}^2.$$

$$E(y)=E\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=\begin{pmatrix}\alpha+\beta-\gamma\\\alpha+\beta-\gamma\\\alpha+\beta+\gamma\end{pmatrix}\qquad b=\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$$

$$\begin{pmatrix}\alpha+\beta-\gamma\\\alpha+\beta+\gamma\end{pmatrix}=\begin{pmatrix}q_1^Tb\\q_2^Tb\end{pmatrix}=Q^Tb \quad\Rightarrow\quad Q^T\hat b=\begin{pmatrix}q_1^T\hat b\\q_2^T\hat b\end{pmatrix}=\begin{pmatrix}\hat\alpha+\hat\beta-\hat\gamma\\\hat\alpha+\hat\beta+\hat\gamma\end{pmatrix}$$

$$\Rightarrow \hat\beta=0 \qquad \Rightarrow \hat\alpha+\hat\beta-\hat\gamma=\tfrac{1}{2}y_1+\tfrac{1}{2}y_2=\tfrac{1}{2}(y_1+y_2)$$

$$\hat\gamma=-\tfrac{1}{4}y_1-\tfrac{1}{4}y_2+\tfrac{1}{2}y_3$$

$$\hat\alpha=\tfrac{1}{4}y_1+\tfrac{1}{4}y_2+\tfrac{1}{2}y_3$$

$$\Rightarrow Q^T\hat b=\begin{pmatrix}\frac{1}{2}(y_1+y_2)\\y_3\end{pmatrix}=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}&0\\0&0&1\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=T^Ty$$

$h^TQ^Tb$ is estimable with estimator $h^TT^Ty$

$$q_1^Tb+q_2^Tb=\underline{2\alpha+2\beta}=(1\quad 1)\begin{pmatrix}q_1^Tb\\q_2^Tb\end{pmatrix}$$

$$h=\begin{pmatrix}1\\1\end{pmatrix}$$

$$h^TT^Ty=(1\quad 1)\begin{pmatrix}\frac{1}{2}&\frac{1}{2}&0\\0&0&1\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$$

$$=\tfrac{1}{2}y_1+\tfrac{1}{2}y_2+y_3=2\hat\alpha+2\hat\beta$$

$$Cov(T^T y) = Q^T (X^T X)^* Q \sigma^2$$

Example $\quad Q^T b = \begin{pmatrix} \alpha+\beta-\gamma \\ \alpha+\beta+\gamma \end{pmatrix} \qquad T^T y = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$Cov(T^T y) = T^T T \sigma^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \sigma^2 = \underbrace{\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}}_{\text{diagonal}} \sigma^2$$

If we assume $y \sim N$

We know $\quad Q^T \hat{b} = \begin{pmatrix} \hat\alpha+\hat\beta-\hat\gamma \\ \hat\alpha+\hat\beta+\hat\gamma \end{pmatrix} \Rightarrow Cov(Q^T \hat{b}) = Q^T (X^T X)^* Q \sigma^2$

$$= \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \underbrace{(X^T X)^*}_{\begin{pmatrix} \frac{3}{8} & 0 & \frac{1}{8} \\ 0 & 0 & 0 \\ \frac{1}{8} & 0 & \frac{3}{8} \end{pmatrix}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} \sigma^2$$

$$E(y) = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

Normal equations $\qquad = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \sigma^2 = Cov(T^T y)$

$$X^T X \hat{b} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \hat\alpha \\ \hat\beta \\ \hat\gamma \end{pmatrix} = \begin{pmatrix} 3 & 3 & -1 \\ 3 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} \hat\alpha \\ \hat\beta \\ \hat\gamma \end{pmatrix} = \begin{pmatrix} 3\hat\alpha + 3\hat\beta - \hat\gamma \\ 3\hat\alpha + 3\hat\beta - \hat\gamma \\ -\hat\alpha - \hat\beta + 3\hat\gamma \end{pmatrix}$$

$$X^T y = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1+y_2+y_3 \\ y_1+y_2+y_3 \\ -y_1-y_2+y_3 \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_1 \\ Y_2 \end{pmatrix}$$

$\hat\beta = 0 \qquad R1: \hat\alpha = \frac{1}{3}(\hat\gamma + y_1+y_2+y_3) = \frac{1}{3}(\hat\gamma + Y_1)$

$R2+R3: \quad 2\hat\alpha + 2\hat\gamma = Y_1 + Y_2 \qquad \& \quad \hat\gamma = \frac{1}{2}(Y_1+Y_2) - \frac{1}{3}(\hat\gamma + Y_1)$

$\Rightarrow \frac{4}{3}\hat\gamma = \frac{1}{2}Y_1 + \frac{1}{2}Y_2 - \frac{1}{3}Y_1 = \frac{1}{6}Y_1 + \frac{1}{2}Y_2$

$\Rightarrow \hat\alpha = \frac{3}{8}Y_1 + \frac{1}{8}Y_2 \qquad \hat{b} = (X^T X)^* X^T y$

$$\Rightarrow \begin{pmatrix} \hat\alpha \\ \hat\beta \\ \hat\gamma \end{pmatrix} = \begin{pmatrix} \frac{3}{8}Y_1 + \frac{1}{8}Y_2 \\ 0 \\ \frac{1}{8}Y_1 + \frac{3}{8}Y_2 \end{pmatrix} = (X^T X)^* X^T y = \begin{pmatrix} \quad \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_1 \\ Y_2 \end{pmatrix}$$

## Examples

1.

$$\underbrace{X}\underset{\sim}{b} = E(\boldsymbol{y}) = \begin{pmatrix} \mu+\alpha+\beta+\gamma \\ \mu+\alpha+\beta+\gamma \\ \mu-\alpha-\beta+\gamma \\ \mu-\alpha-\beta+\gamma \\ \mu+\alpha+\beta-\gamma \\ \mu+\alpha+\beta-\gamma \\ \mu-\alpha-\beta-\gamma \\ \mu-\alpha-\beta-\gamma \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & +1 & +1 & +1 \\ 1 & +1 & +1 & +1 \\ 1 & -1 & -1 & +1 \\ 1 & -1 & -1 & +1 \\ 1 & +1 & +1 & -1 \\ 1 & +1 & +1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix}}^{X} \underset{\sim}{\overset{b}{\begin{pmatrix} \mu \\ \alpha \\ \beta \\ \gamma \end{pmatrix}}} \qquad q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$$

and $cov\ (\boldsymbol{y}) = \sigma^2 \boldsymbol{I}$. Consider the linear function $\boldsymbol{q}'\boldsymbol{b} = q_1\mu + q_2\alpha + q_3\beta + q_4\gamma$.

$$\operatorname{rank}\begin{pmatrix} q^T \\ X \end{pmatrix} = \operatorname{rank} X$$

$$rank \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \\ 1 & +1 & +1 & +1 \\ 1 & +1 & +1 & +1 \\ 1 & -1 & -1 & +1 \\ 1 & -1 & -1 & +1 \\ 1 & +1 & +1 & -1 \\ 1 & +1 & +1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix} = rank \begin{pmatrix} q_1 & q_2 & q_2 - q_3 & q_4 \\ 1 & +1 & 0 & +1 \\ 1 & +1 & 0 & +1 \\ 1 & -1 & 0 & +1 \\ 1 & -1 & 0 & +1 \\ 1 & +1 & 0 & -1 \\ 1 & +1 & 0 & -1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & 0 & -1 \end{pmatrix} = rank\ (\boldsymbol{X}) = 3$$

$\mu = (1\ \underline{0\ 0}\ 0)b$ ✓

$\alpha = (0\ \underline{1\ 0}\ 0)b$ ✗

$\mu+\gamma = (1\ \underline{0\ 0}\ 1)b$ ✓

$\gamma = (0\ \underline{0\ 0}\ 1)b$ ✓

$\beta-\gamma = (0\ \underline{0\ 1}\ -1)b$ ✗

if and only if $q_2 = q_3$. $\boldsymbol{q}'\boldsymbol{b}$ is therefore estimable if and only if $q_2 = q_3$.

$(X^TX)\hat{b} = X^Ty$

$$\boldsymbol{X}'\boldsymbol{X} = \begin{pmatrix} 8 & 0 & 0 & 0 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

and $$\boldsymbol{X}'\boldsymbol{y} = \begin{pmatrix} y_1+y_2+y_3+y_4+y_5+y_6+y_7+y_8 \\ y_1+y_2-y_3-y_4+y_5+y_6-y_7-y_8 \\ y_1+y_2-y_3-y_4+y_5+y_6-y_7-y_8 \\ y_1+y_2+y_3+y_4-y_5-y_6-y_7-y_8 \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_2 \\ Y_3 \end{pmatrix}$$

$8\hat{\mu} = Y_1 \Rightarrow \hat{\mu} = \frac{1}{8}Y_1$

$8\hat{\gamma} = Y_3 \Rightarrow \hat{\gamma} = \frac{1}{8}Y_3$

$r(X) = 3$ & 4 parameters & 3 equation

A solution of the normal equations is:

$$\hat{\mu} = Y_1/8,\ \hat{\alpha} = Y_2/8,\ \hat{\beta} = 0,\ \hat{\gamma} = Y_3/8.$$

$\hat{\alpha} = 0$ OR $\hat{\beta} = 0$

The best estimator of $\boldsymbol{q}'\boldsymbol{b}$ is $q^T\hat{b}$

$$q_1\hat{\mu} + q_2\hat{\alpha} + q_3\hat{\beta} + q_4\hat{\gamma} = q_1Y_1/8 + q_2Y_2/8 + q_4Y_3/8.$$

Furthermore,

$$SSR = \widehat{\boldsymbol{b}}'\boldsymbol{X}'\boldsymbol{y} = \frac{1}{8}Y_1^2 + \frac{1}{8}Y_2^2 + \frac{1}{8}Y_3^2$$
$$SSE = \boldsymbol{y}'\boldsymbol{y} - SSR$$

and

$$MSE = \frac{1}{5}SSE$$

2. Suppose that each combination of three out of four objects with weights $w_1, w_2, w_3$ and $w_4$ is weighed twice. A constant error, $g$, is attributed to the scale used and a random error $\boldsymbol{e}$ is made at each observation, such that $\mathrm{var}\,(e) = \sigma^2$.

$$E(\boldsymbol{y}) = \begin{pmatrix} g + w_1 + w_2 + w_3 \\ g + w_1 + w_2 + w_3 \\ g + w_1 + w_2 + w_4 \\ g + w_1 + w_2 + w_4 \\ g + w_1 + w_3 + w_4 \\ g + w_1 + w_3 + w_4 \\ g + w_2 + w_3 + w_4 \\ g + w_2 + w_3 + w_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} g \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$$

and $\mathbf{cov}\,(\boldsymbol{y}) = \sigma^2\boldsymbol{I}$. Consider the linear function $\boldsymbol{q}'\boldsymbol{b} = q_0 g + q_1 w_1 + q_2 w_2 + q_3 w_3 + q_4 w_4$.

$$rank \begin{pmatrix} q_0 & q_1 & q_2 & q_3 & q_4 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix} = rank \begin{pmatrix} 3q_0 - q_1 - q_2 - q_3 - q_4 & q_1 & q_2 & q_3 & q_4 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} = rank\,(\boldsymbol{X}) = 4$$

if and only if $3q_0 = q_1 + q_2 + q_3 + q_4$. Therefore $\boldsymbol{q}'\boldsymbol{b}$ is estimable if and only if $3q_0 = q_1 + q_2 + q_3 + q_4$.

$$\boldsymbol{X}'\boldsymbol{X} = \begin{pmatrix} 8 & 6 & 6 & 6 & 6 \\ 6 & 6 & 4 & 4 & 4 \\ 6 & 4 & 6 & 4 & 4 \\ 6 & 4 & 4 & 6 & 4 \\ 6 & 4 & 4 & 4 & 6 \end{pmatrix}$$

$$\text{and } \boldsymbol{X'y} = \begin{pmatrix} y_1+y_2+y_3+y_4+y_5+y_6+y_7+y_8 \\ y_1+y_2+y_3+y_4+y_5+y_6 \\ y_1+y_2+y_3+y_4+y_7+y_8 \\ y_1+y_2+y_5+y_6+y_7+y_8 \\ y_3+y_4+y_5+y_6+y_7+y_8 \end{pmatrix} = \begin{pmatrix} G \\ W_1 \\ W_2 \\ W_3 \\ W_4 \end{pmatrix}$$

where $W_i$ is the sum of the weights when the $i$-th subject is weighed. The normal equations are:

$$\begin{aligned} 8g+6w_1+6w_2+6w_3+6w_4 &= G \\ 6g+6w_1+4w_2+4w_3+4w_4 &= W_1 \\ 6g+4w_1+6w_2+4w_3+4w_4 &= W_2 \\ 6g+4w_1+4w_2+6w_3+4w_4 &= W_3 \\ 6g+4w_1+4w_2+4w_3+6w_4 &= W_4 \end{aligned}$$

These normal equations can be reduced to 4 equations in 5 unknowns. To find a solution, set

$$g+w_1+w_2+w_3+w_4=0.$$

(It should not be an estimable function). A solution of the normal equations is:

$$\hat{g}=G/2,\ \hat{w}_i=W_i/2-G/2$$

The best estimator of $\boldsymbol{q'b}$ is

$$\frac{1}{2}\big(q_0G+q_1(W_1-G)+q_2(W_2-G)+q_3(W_3-G)+q_4(W_4-G)\big)$$

It follows, for instance, that the best estimator of $w_1-w_2$ is:

$$\hat{w}_1-\hat{w}_2=\frac{1}{2}(W_1-W_2)=\frac{1}{2}(y_5+y_6-y_7-y_8).$$

Furthermore,

$$\begin{aligned} SSR &= \boldsymbol{\hat{b}'X'y} \\ &= \frac{1}{2}G^2+W_1(W_1/2-G/2)+W_2(W_2/2-G/2)+W_3(W_3/2-G/2)+W_4(W_4/2-G/2) \\ &= \frac{1}{2}\sum_{i=1}^{4} W_i^2-G^2 \end{aligned}$$

$$SSE=\boldsymbol{y'y}-SSR$$

and $MSE=\frac{1}{4}SSE$.