CS4815 Week02 Tutorial (Partial) Solutions

Look over the vector review material `http://garryowen.csisdmz.ul.ie/~cs4815/resources/oth2.pdf` and use this information to solve the following problems.

1. Show that the normal to the line $2x - y - 4 = 0$ is the vector $\mathbf{u} = (2, -1)^T$. Then show that, in general, the normal to the line $ax + by + c = 0$ is the vector $\mathbf{u} = (a, b)^T$.

   Answer:
   Consider two points, $p = (p_x, p_y)$ and $q = (q_x, q_y)$ on the line $ax + by + c = 0$. Then $ap_x + bp_y + c = 0$ and $aq_x + bq_y + c = 0$. Let the distance from p_x to q_x be δ_x and let the distance from p_y to q_y be δ_y. That is, $q_x = p_x + \delta_x$ and $q_y = p_y + \delta_y$. Then

   $$\begin{aligned} aq_x + bq_y + c = 0 &= a(p_x + \delta_x) + b(p_y + \delta_y) + c \\ &= ap_x + bp_y + c + a\delta_x + b\delta_y \\ &= a\delta_x + b\delta_y, \text{since} \quad ap_x + bp_y + c = 0 \end{aligned}$$

   Since p and q are points we can talk about the vector $\mathbf{v} = q - p$; and, further, since they both are on the line then $\mathbf{v}$ lies on the line.

   $$\begin{aligned} \mathbf{v} &= \begin{pmatrix} q_x \\ q_y \end{pmatrix} - \begin{pmatrix} p_x \\ p_y \end{pmatrix} \\ &= \begin{pmatrix} p_x + \delta_x \\ p_y + \delta_y \end{pmatrix} - \begin{pmatrix} p_x \\ p_y \end{pmatrix} \\ &= \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} \end{aligned}$$

   Two vectors are perpendicular if $\mathbf{u} \cdot \mathbf{v} = 0$. With $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix}$ we showed previously that $\mathbf{u} \cdot \mathbf{v} = a\delta_x + b\delta_y = 0$.

   What is the importance of this? With the line $ax + by + c = 0$, the vector that is perpendicular to it, its **normal**, is $\mathbf{u} = (a, b)^T$.

2. Use vector methods to find

   - the equation of the line through $p = (2, 3)$ and perpendicular to the line $x + 2y + 5 = 0$.

     Answer:
     We'll call L the line $x+2y+5 = 0$ and we'll call the line perpendicular to L that goes through p, L'.
     From the previous exercise we can say that the vector at right angles to L – its normal – is $\mathbf{u} = (1, 2)^T$ since $a = 1$ and $b = 2$. (Note that

this vector will lie *on* the line L'.) Finding the normal is the key to finding the equation of a line, so we need $\mathbf{v}$, the normal to L'. Let $\mathbf{v} = (a', b')^T$.

Since the two lines are perpendicular their normals must be perpendicular also. That is $\mathbf{u} \cdot \mathbf{v} = 0$. So

$$1a' + 2b' = 0.$$

Any values of a' and b' will do as long as the above equation holds so we will use $a' = 2$ and $b' = -1$ meaning $\mathbf{v} = (2, -1)^T$.

This is the normal to L' so we can say that the line perpendicular to L is of the form $2x - y + c = 0$. *Every* line perpendicular to L will be of that form but we want the line that goes through $p = (2, 3)$ so $(2, 3)$ must satisfy the line equation with equality. That is, $2 \times 2 - 1 \times 3 + c = 0$. From this we find that $c = -1$.

So the equation of L' is

$$2x - y - 1 = 0.$$

- the equation of the line through $p_1 = (2, 3)$ and $p_2 = (5, -1)$

  Answer:

  p_1, p_2 are both on the line L so the vector

  $$\mathbf{v} = (p_2 - p_1) = \begin{pmatrix} 5 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} = (3, -4)^T$$

  is aligned with L.

  In order to find the equation of a line we need its *normal*. What is normal to $\mathbf{v}$? It is a vector $\mathbf{u} = (a, b)^T$ so that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v} = 0$. So

  $$\mathbf{u}^T\mathbf{v} = (a, b) \begin{pmatrix} 3 \\ -4 \end{pmatrix} = 0 = 3a - 4b$$

  We can choose a to be anything so long as $3a - 4b = 0$ so choose $a = 4$. Then $b = 3$. So the vector $\mathbf{u} = (4, 3)^T$ is normal to the line L.

  Now along with $p_1 = (2, 3)$, a general point on the line is $q = (x, y)$ so the vector $\mathbf{w} = (q - p_1)$ is aligned with L and is normal to $\mathbf{u} = (4, 3)^T$.

  $$\mathbf{u} \cdot \mathbf{w} = 0 = \mathbf{u}^T(q - p_1) = \mathbf{u}^T q - \mathbf{u}^T p_1$$

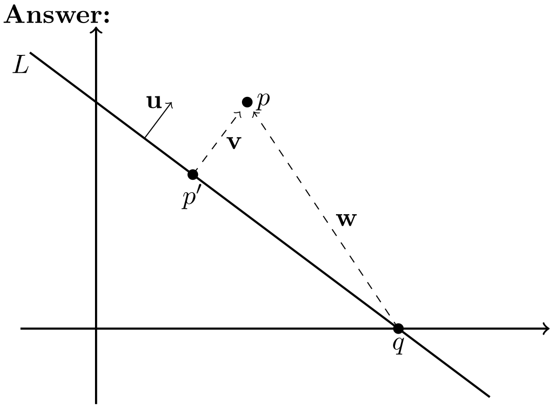
  So

  $$\begin{aligned} \mathbf{u}^T q &= \mathbf{u}^T p_1 \\ (4, 3)^T \begin{pmatrix} x \\ y \end{pmatrix} &= (4, 3)^T \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ 4x + 3y &= 8 + 9 = 17 \end{aligned}$$

  Then L is $4x + 3y - 17 = 0$.

- use vector methods to find the distance of the point $p = (2, 3)$ from the line $3x + 4y - 12 = 0$

  Answer:

  The key to answering this is the dot product operation.

  The distance d from p to L is from the closest point on L to p. This is the vector $\mathbf{v} = (p - p')$ and since it is perpendicular to L it will be some multiple of the normal $\mathbf{u}$. So we can write $\mathbf{v} = \alpha\mathbf{u}$. Taking the dot product we can say

  $$\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot (\alpha\mathbf{u}) = \alpha\mathbf{u} \cdot \mathbf{u} = \alpha||\mathbf{u}||^2 = \alpha(3, 4)^T \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 25\alpha$$

  The point $q = (4, 0)$ is also on L. Looking at the vector $\mathbf{w} = (p - q)$ from q to p

  $$\mathbf{w} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

  and taking the dot product here

  $$\begin{aligned} \mathbf{u} \cdot \mathbf{w} &= \mathbf{u}^T(p - q) \\ &= (3, 4) \begin{pmatrix} -2 \\ 3 \end{pmatrix} \\ &= 6 \end{aligned}$$

  But we can also say $\mathbf{w} = (p - q) = (p' - q) + (p - p')$ or

  $$\mathbf{w} = \mathbf{v}^\perp + \mathbf{v}$$

  where $\mathbf{v}^\perp$ is the vector $(p' - q)$ that is perpendicular to $\mathbf{v}$. Taking dot products here

  $$\mathbf{u} \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} + \mathbf{v}^\perp) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}^\perp$$

  Since $\mathbf{u}$ is parallel to $\mathbf{v}$ then it is also perpendicular to $\mathbf{v}^\perp$ meaning $\mathbf{u} \cdot \mathbf{v}^\perp = 0$. So $\mathbf{u} \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{v}$.

From before $\mathbf{u} \cdot \mathbf{w} = 6$ and $\mathbf{u} \cdot \mathbf{v} = 25\alpha$. Therefore $\alpha = 6/25$ and

$$\mathbf{v} = \frac{6}{25}\mathbf{u}$$

.

The length of $\mathbf{v}$, which is what we want is

$$||\mathbf{v}|| = \frac{6}{25}||\mathbf{u}|| = \frac{6}{25}\sqrt{3^2 + 4^2} = \frac{6}{5}$$

3. Resolve a vector $\mathbf{a}$ into two components $\mathbf{a_1}$ and $\mathbf{a_2}$ that are, respectively, parallel and perpendicular to another vector $\mathbf{b}$. That is, find vectors $\mathbf{a_1}$ and $\mathbf{a_2}$ so that

   - $\mathbf{a_1} = \alpha\mathbf{b}$, where α is a scalar (number)
   - $\mathbf{a_2} \cdot \mathbf{b} = 0$

   It will be of particular interest to us to know what happens if $\mathbf{b}$ happens to be one of the basis vectors of some co-ordinate system. So what is the formula for $\mathbf{a_1}, \mathbf{a_2}$ when $\mathbf{b}$ has length 1?

Answer:

This is identical to what we did in the last question, just a bit more abstract. First we write $\mathbf{a}$ as the sum of two vectors $\mathbf{a_1}$ and $\mathbf{a_2}$

$$\mathbf{a} = \mathbf{a_1} + \mathbf{a_2}$$

where $\mathbf{a_1}$ is parallel to $\mathbf{b}$ and $\mathbf{a_2}$ is perpendicular to it. With $\mathbf{a_1}$ parallel to $\mathbf{b}$ it can be written as $\mathbf{a_1} = \alpha\mathbf{b}$. And since $\mathbf{a_2}$ is perpendicular to it it must be that $\mathbf{b} \cdot \mathbf{a_2} = 0$.

Then

$$\mathbf{b} \cdot \mathbf{a} = \mathbf{b} \cdot (\mathbf{a_1} + \mathbf{a_2}) = \mathbf{b} \cdot \mathbf{a_1} + \mathbf{b} \cdot \mathbf{a_2} = \mathbf{b} \cdot \mathbf{a_1} + \mathbf{0} = \mathbf{b} \cdot \mathbf{a_1}$$

And since we arranged for $\mathbf{a_1}$ to be parallel to $\mathbf{b}$ we get

$$\mathbf{b} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a_1} = \mathbf{b} \cdot (\alpha\mathbf{b}) = \alpha\mathbf{b} \cdot \mathbf{b} = \alpha||\mathbf{b}||^2.$$

So

$$\alpha = \frac{\mathbf{b} \cdot \mathbf{a}}{||\mathbf{b}||^2}$$

and

$$\mathbf{a_1} = \frac{\mathbf{b} \cdot \mathbf{a}}{||\mathbf{b}||^2}\mathbf{b}.$$

Note that if we had another vector $\mathbf{b}^\perp$ that was perpendicular to $\mathbf{b}$ then we could go through the exact same steps to get

$$\mathbf{a_2} = \frac{\mathbf{b}^\perp \cdot \mathbf{a}}{||\mathbf{b}^\perp||^2}\mathbf{b}^\perp.$$

The significance of this is that if (as we will see later) the vectors $\mathbf{a_1}$ and $\mathbf{a_2}$ were the coordinate axes of a camera system then we have been able to express an arbitrary vector $\mathbf{a}$ in terms of that coordinate system and so convert from one set of coordinates to another.