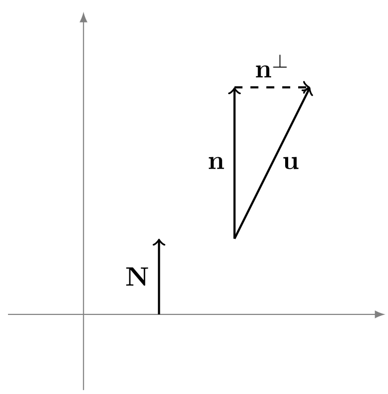
1. What is the mirror image of a point? In more mathematical terms this corresponds to asking what is the *reflection* of the point about a plane (the mirror). The following problem is actually this in disguise and is crucial for working with lighting graphics.

   Given a plane in terms of its normal, $\mathbf{N}$, and a very shiny point p on the plane that we bounce light off, if we know s, the location of a light source, what will be the direction vector of the reflected light?

   Answer:
   As a pre-cursor to the answer consider the figure below. It is obvious (from vector addition) that

$$\mathbf{u} = \mathbf{n} + \mathbf{n}^{\perp}.$$



   Suppose $\mathbf{N}$ is of unit length pointing in the same direction as $\mathbf{n}$. Clearly $\mathbf{n}$ is some multiple of $\mathbf{N}$. Say $\mathbf{n} = \beta\mathbf{N}$. What happens if we take the dot product of $\mathbf{u}$ with $\mathbf{N}$?

$$\begin{aligned}
\mathbf{N} \cdot \mathbf{u} &= \mathbf{N} \cdot \mathbf{n} + \mathbf{N} \cdot \mathbf{n}^{\perp} \\
&= \mathbf{N} \cdot \mathbf{n} + 0 \\
&= \mathbf{N} \cdot (\beta\mathbf{N}) \\
&= \beta\mathbf{N} \cdot \mathbf{N} \\
&= \beta
\end{aligned}$$

   So since $\mathbf{n} = \beta\mathbf{N}$, $\mathbf{n}$ is the component of $\mathbf{u}$ in the direction $\mathbf{N}$.

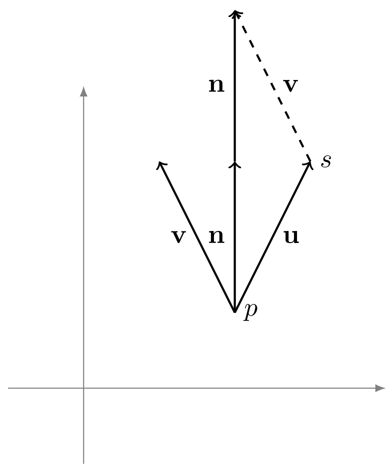
   With that as background we let $\mathbf{u} = s - p$. As earlier we say that

$$\mathbf{n} = \mathbf{N} \cdot \mathbf{u}.$$

We want to find another vector, $\mathbf{v}$, such that

$$\mathbf{N} \cdot \mathbf{v} = \mathbf{n}$$

also.



Then from the figure above adding the vectors $\mathbf{u}$ and $\mathbf{v}$ gives

$$\mathbf{u} + \mathbf{v} = \mathbf{n} + \mathbf{n} = 2\mathbf{n} = 2(\mathbf{N} \cdot \mathbf{u})\mathbf{N}$$

and

$$\mathbf{v} = 2(\mathbf{N} \cdot \mathbf{u})\mathbf{N} - \mathbf{u}.$$

2. In a First Person Shooter (FPS) game one of the most basic operations in our game will be the need to find what object gets hit when we fire the gun. If we have a gun at some location, firing in some direction, develop a test that detects whether the bullet fired from the gun will pierce a polygon. The polygon will be given as an array of points that lie on some plane and we will also be given the direction the normal points in.

   First, develop a quick test to eliminate back facing polygons. Then, if a polygon is front-facing, proceed with the main test. The main test will involve finding the point where the bullet pierces the plane that the polygon lies on and then testing if that point lies inside the polygon. (Some interesting discussions of this last part of the problem can be found here.)

   In the next few lectures we will see the importance of "ray tracing" for implementing realistic scene lighting. The idea will be the same as what we have considered here.

   Answer:
   The polygon the bullet hits should be facing back towards the gunman. So we can *cull* away any polygon whose normal points in the *same* direction as we are firing. If the polygon's (length-one) normal is $\mathbf{N}$ and we fire in a

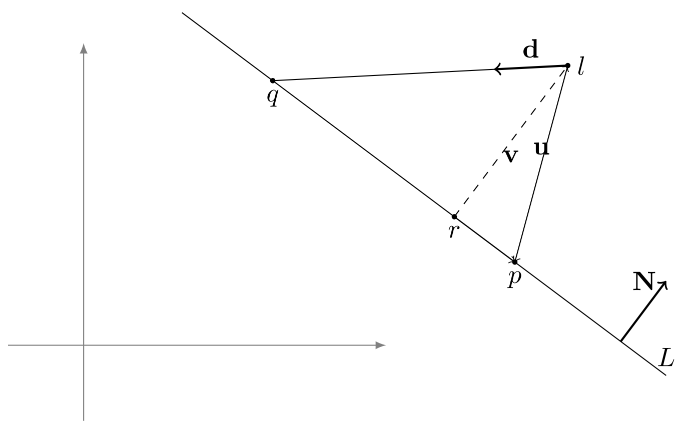
direction d (vector usually not of length one) then we can eliminate from further consideration a polygon when

$$\mathbf{N} \cdot \mathbf{d} \geq 0.$$

Now what? Now we have to test if the bullet, travelling in a direction $\mathbf{d}$, pierces the polygon P. This is a two-step process. Firstly we need to find q where the bullet pierces the plane that P lies on and, secondly, we need to ask is this point within the confines of the polygon.

Initially all we know is l the location of the "gun" (or, later, origin of the ray), $\mathbf{d}$ the direction of fire and the plane that the polygon is contained in (via its normal $\mathbf{N}$ and a point on that plane – any of the points of the polygon's outline will do).

The picture below illustrates the 2-D version of the first issue: we ask what is the extension of $\mathbf{d}$ from the location l on to the line L.



If we let $\mathbf{u} = p - l$ be the vector from l to p then we can find $\mathbf{v} = r - l$ the vector from l to r. From the last question this is just

$$\mathbf{v} = (\mathbf{N} \cdot \mathbf{u})\mathbf{N}.$$

Now consider the vector $\mathbf{w} = q - l$ from l to q. It must *also* be that

$$\mathbf{v} = (\mathbf{N} \cdot \mathbf{w})\mathbf{N}.$$

So

$$\mathbf{N} \cdot \mathbf{u} = \mathbf{N} \cdot \mathbf{w}.$$

We don't know $\mathbf{w}$ but we *do* know that it is some multiple of $\mathbf{d}$, say, β; that is,

$$\mathbf{w} = \beta\mathbf{d}.$$

Then

$$\mathbf{N} \cdot \mathbf{u} = \mathbf{N} \cdot (\beta\mathbf{d}) = \beta\mathbf{N} \cdot \mathbf{d}.$$

This means

$$\beta = \frac{\mathbf{N} \cdot \mathbf{u}}{\mathbf{N} \cdot \mathbf{d}}$$

and, so

$$\begin{aligned} q &= l + \mathbf{w} \\ &= l + \frac{\mathbf{N} \cdot \mathbf{u}}{\mathbf{N} \cdot \mathbf{d}} \mathbf{d} \end{aligned}$$

This gives us the point where the ray originating from l and in the directino d pierces the plane that our polygon lies in. Now all we need to do is check if this point is contained within the confines of the polygon.

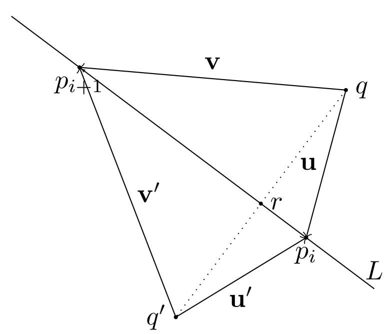
The link mentioned in the tute sheet has a good discussion of approaches to take and we will discuss two now.

Since we are given the vertices that make up the polygon's outline in an anti-clockwise ordering we could, in theory, find the equations of the lines between each successive pair of points. More usefully from this we could find the outward pointing normal for each polygon edge. This will allow us to tell which side of the edge the point lies on. The point q will be contained within the polygon P only if it is on the negative side of every edge of the polygon.

A second method also considers each edge of the polygon in turn but this time it uses the cross-product to tell if the point q is to the left or the right of the line formed by two end points of the edge. If we consider the edges in counter-clockwise order then the q is inside the poly if it is *to the left* of every edge.

From q we form vectors $\mathbf{u}$ and $\mathbf{v}$ to the two end points and we take their cross product. Recall that 1) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$, and 2) if $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ then $|\mathbf{w}|$ (magnitude of vector) denotes the area of the parallelogram made by the vectors $\mathbf{u}$ and $\mathbf{v}$. What these two properties amount to is being able to tell whether q is on the left or the right of the edge.

In the picture below q' is a reflection of the point q so $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ will be the same magnitude as $\mathbf{w}' = \mathbf{u}' \times \mathbf{v}'$ they will be of opposite sign: $\mathbf{w} = -\mathbf{w}'$. So we can tell which side of the edge $p_i p_{i+1}$ the point q lies on.



$\mathbf{v}$
p_{i+1}
q
$\mathbf{u}$
$\mathbf{v}'$
r
p_i
L
q'
$\mathbf{u}'$