In this notebook I will attempt to find the shortest path between two points $(a, A)$ and $(b, B)$ by using Euler's method of finite differences.

Finding the shortest path between these two points is equivalent to minimizing the functional $J[y] = \int_a^b \sqrt{1 + y'(x)^2}\, dx$ subject to the constraints $y(a) = A$ and $y(b) = B$

In[488]:=

```
DiscretizedFunctional[F_, a_, b_, n_] :=
 Module[{Δx, x},
  Δx = (b - a) / n;
  x[i_] := a + i Δx;

  Sum[F[x[i], y[i], (y[i] - y[i - 1]) / Δx], {i, 1, n + 1}]]
```

In[507]:=

```
F[x_, y_, z_] := Sqrt[1 - z^2]
```

In[508]:=

```
J = DiscretizedFunctional[F, a, b, n]
```

Out[508]=

$$\sum_{i=1}^{n+1} \sqrt{1 - \frac{n^2 (y(i) - y(i-1))^2}{(b-a)^2}}$$

In[509]:=

```
D[J, y[j]] == 0 // Expand
```

Out[509]=

$$\sum_{i=1}^{n+1} \left( -\frac{n^2\, y(i-1)\, \delta_{i-1,j}}{(b-a)^2 \sqrt{1 - \frac{n^2 (y(i)-y(i-1))^2}{(b-a)^2}}} + \frac{n^2\, y(i-1)\, \delta_{i,j}}{(b-a)^2 \sqrt{1 - \frac{n^2 (y(i)-y(i-1))^2}{(b-a)^2}}} + \frac{n^2\, y(i)\, \delta_{i-1,j}}{(b-a)^2 \sqrt{1 - \frac{n^2 (y(i)-y(i-1))^2}{(b-a)^2}}} - \frac{n^2\, y(i)\, \delta_{i,j}}{(b-a)^2 \sqrt{1 - \frac{n^2 (y(i)-y(i-1))^2}{(b-a)^2}}} \right) = 0$$

Because of the presence of the Kronecker-$\delta$ we can break this down into two cases. The first case is when $i = j + 1$ and when $i = j$. So we get the following:

In[476]:=

```
eqn = n^2/(b - a)^2 ((y[j + 1] - y[j])/Sqrt[1 - (n^2 (y[j+1]-y[j])^2)/(b-a)^2] - (y[j] - y[j - 1])/Sqrt[1 - (n^2 (y[j]-y[j-1])^2)/(b-a)^2]) == 0
```

Out[476]=

$$\frac{n^2\left(\frac{y(j+1)-y(j)}{\sqrt{1-\frac{n^2 (y(j+1)-y(j))^2}{(b-a)^2}}}-\frac{y(j)-y(j-1)}{\sqrt{1-\frac{n^2 (y(j)-y(j-1))^2}{(b-a)^2}}}\right)}{(b-a)^2}=0$$

Which is a recurrence equation that can be solved with the boundary conditions $y_0 = A$ and $y_{n+1} = B$.

In[478]:=

```
RSolve[{eqn, y[0] == A, y[n + 1] == B}, y[j], j]
```

Out[478]=

$$\left\{\left\{y(j)\to\frac{A(-1)^j+A(-1)^n-B(-1)^j+B}{(-1)^n+1}\right\},\left\{y(j)\to\frac{A(-j)+A\,n+A+B\,j}{n+1}\right\}\right\}$$

We can see that this is equivalent to $y_j = A + \frac{B - A}{n + 1}\, j$ which describes a straight line. This is what we wanted to show.