# Equation of Lines, Planes, and Hyperplanes

Machine Learning Foundations

## 1 Introduction

These concepts are fundamental for algorithms like Logistic Regression and Support Vector Machines (SVM). We'll cover:

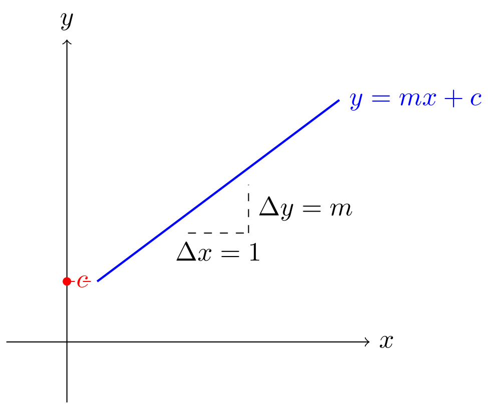
- 2D line equations
- 3D plane equations
- $n$-dimensional hyperplanes
- Geometric interpretations

## 2 Equation of a Straight Line in 2D

### 2.1 Standard Forms

- **Slope-intercept form**:
$$y = mx + c$$
  - $m$ = Slope (rate of change)
  - $c$ = Intercept (y-axis intersection)
- **Alternative notation**:
$$y = \beta_0 + \beta_1 x$$
- **General form**:
$$ax + by + c = 0$$

### 2.2 Components Visualized




- **Slope** $m$: Ratio of vertical change ($\Delta y$) to horizontal change ($\Delta x$)
$$m = \frac{\Delta y}{\Delta x}$$
- **Intercept** $c$: Value where line crosses y-axis ($x = 0$)

### 2.3 Equivalence of Forms

Convert general form to slope-intercept:

$$\begin{aligned} ax + by + c = 0 \\ by &= -ax - c \\ y &= -\frac{a}{b}x - \frac{c}{b} \end{aligned}$$

where $m = -\frac{a}{b}$ and $c = -\frac{c}{b}$.

## 3 Vector Notation

### 3.1 2D Line Equation

$$w_1x_1 + w_2x_2 + b = 0$$

- $x_1, x_2$: Input features
- $w_1, w_2$: Weights (coefficients)
- $b$: Bias term (intercept)

### 3.2 Compact Vector Form

$$\mathbf{w}^T\mathbf{x} + b = 0$$

where $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

## 4 Extension to Higher Dimensions

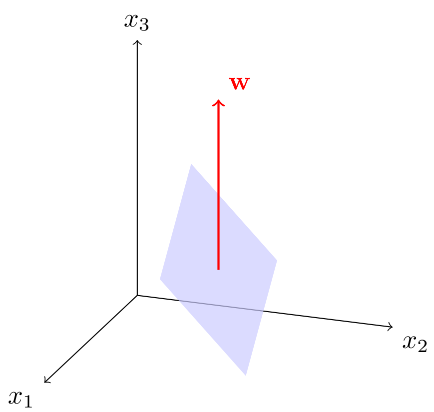
### 4.1 3D Plane Equation

$$w_1x_1 + w_2x_2 + w_3x_3 + b = 0$$

or in vector form:

$$\mathbf{w}^T\mathbf{x} + b = 0$$

with $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$




### 4.2 $n$-D Hyperplane

$$w_1x_1 + w_2x_2 + \cdots + w_nx_n + b = 0$$

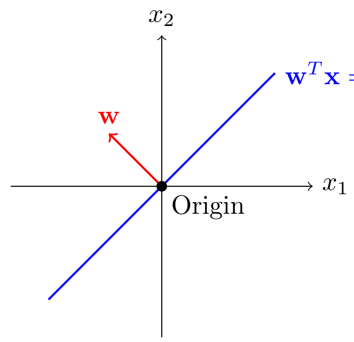
$$\boxed{\mathbf{w}^T\mathbf{x} + b = 0}$$

where $\mathbf{w}, \mathbf{x} \in \mathbb{R}^n$

# 5 Special Case: Origin-Passing Planes

When the hyperplane passes through the origin ($\mathbf{0}$):

$$\mathbf{w}^T\mathbf{x} = 0$$




# 6 Geometric Interpretation

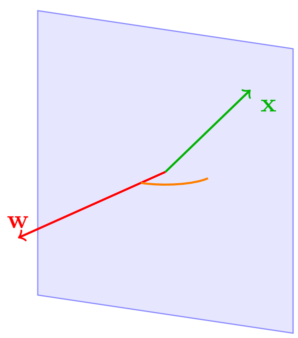
The weight vector $\mathbf{w}$ is **normal** (perpendicular) to the plane/hyperplane.

## 6.1 Mathematical Proof

$$\mathbf{w}^T\mathbf{x} = \|\mathbf{w}\|\|\mathbf{x}\|\cos\theta = 0$$

- $\|\mathbf{w}\|$ and $\|\mathbf{x}\|$ are magnitudes (always $> 0$)
- $\cos\theta = 0 \implies \theta = 90°$
- $\therefore$ $\mathbf{w}$ is perpendicular to all vectors $\mathbf{x}$ in the plane

## 6.2 Visualization in 3D




# 7 Key Takeaways

1. **2D Line**: $y = mx + c$ or $\mathbf{w}^T\mathbf{x} + b = 0$
2. **3D Plane**: $w_1x_1 + w_2x_2 + w_3x_3 + b = 0$
3. **$n$-D Hyperplane**: $\mathbf{w}^T\mathbf{x} + b = 0$
4. **Origin-passing**: $\mathbf{w}^T\mathbf{x} = 0$
5. $\mathbf{w}$ is **always perpendicular** to the plane
6. Essential for understanding:
   - Logistic Regression decision boundaries
   - SVM margin optimization
   - Neural network activation patterns