# Distance from a Point to a Plane

Machine Learning Foundations

## 1 Introduction

Calculating the distance from a point to a plane/hyperplane is crucial for:

- Support Vector Machines (margin optimization)
- Logistic Regression (decision boundaries)
- Anomaly detection (distance to separation surface)

For a plane $\pi$ passing through origin: $\mathbf{w}^T\mathbf{x} = 0$ where $\mathbf{w}$ is the normal vector.

## 2 Distance Formula

The distance $d$ from point $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ to the plane is:

$$\boxed{d = \frac{|\mathbf{w}^T\mathbf{s}|}{\|\mathbf{w}\|}}$$

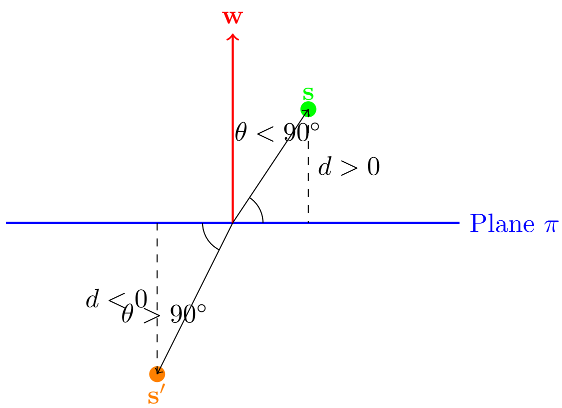
- $\mathbf{w}^T\mathbf{s}$: Dot product (projection)
- $\|\mathbf{w}\|$: Magnitude of normal vector

## 3 Geometric Interpretation

$$\mathbf{w}^T\mathbf{s} = \|\mathbf{w}\|\|\mathbf{s}\|\cos\theta$$

where $\theta$ is the angle between vectors $\mathbf{w}$ and $\mathbf{s}$.

### 3.1 Side Determination




- **Above plane** ($d > 0$): $\theta < 90° \Rightarrow \cos\theta > 0$
- **Below plane** ($d < 0$): $\theta > 90° \Rightarrow \cos\theta < 0$
- Distance magnitude: $|d| = \dfrac{|\mathbf{w}^T\mathbf{s}|}{\|\mathbf{w}\|}$

## 4 Vector Form Derivation

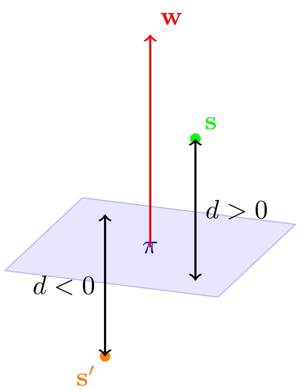
The distance is the projection of $\mathbf{s}$ onto $\mathbf{w}$:

$$d = \frac{\mathbf{w}^T \mathbf{s}}{\|\mathbf{w}\|} = \|\mathbf{s}\| \cos \theta$$

## 5 Machine Learning Significance

- **SVM**: Maximizes margin $\frac{2}{\|\mathbf{w}\|}$ using point-plane distances
- **Classification**: Signed distance indicates class membership
- **Confidence**: $|d|$ measures certainty of classification

## 6 3D Visualization




## 7 Key Formulas Summary

$$\begin{aligned} \text{Plane equation} \quad & \mathbf{w}^T \mathbf{x} = 0 \\ \text{Distance} \quad & d = \frac{\mathbf{w}^T \mathbf{s}}{\|\mathbf{w}\|} \\ \text{Magnitude} \quad & |d| = \frac{|\mathbf{w}^T \mathbf{s}|}{\|\mathbf{w}\|} \\ \text{Side determination} \quad & \operatorname{sign}(d) = \operatorname{sign}(\mathbf{w}^T \mathbf{s}) \end{aligned}$$

## 8 Applications in ML

- **SVM**: Uses $d$ to define margin $\dfrac{2}{\|\mathbf{w}\|}$
- **Logistic Regression**: Signed distance inputs to sigmoid function
- **Outlier Detection**: Large $|d|$ indicates anomalies
- **Multi-class**: Distances to multiple hyperplanes