$$\underset{\sim}{R}\begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{a}{2} \\ 0 & \frac{5}{2} & -\frac{19}{2} & b-\frac{3a}{2} \\ 0 & 0 & 0 & c+\frac{5a}{2}+3b-\frac{9a}{2} \end{bmatrix}, \text{ by } R_3+3R_2$$

The given system of equations will be inconsistent if

$$\text{rank } A < \text{rank } A_b .$$

Here we have rank $A = 2$ and rank A_b should be 3 in order to meet the condition rank $A <$ rank A_b, if

$$c+\frac{5a}{2}+3b-\frac{9a}{2} \neq 0$$

or $$c-2a+3b \neq 0$$

$\Rightarrow$ $$c \neq 2a-b.$$

5. A soap manufacture decides to spend 600,000 rupees on radio, magazine and TV advertising. If he spends as much on TV advertising as on magazines and radio together, and the amount spent on magazines and TV combined equals five times that spent on radio, what is the amount to be spent on each type of advertising?

Sol. Let x, y, z be the amounts in rupees spent on radio, magazines and TV advertising respectively. Then according to the given conditions, we have

$$x+y+z = 600{,}000 \qquad (1)$$

$$z = x+y$$

$\Rightarrow$ $$x+y-z = 0 \qquad (2)$$

$$y+z = 5x$$

$\Rightarrow$ $$5x-y-z = 0 \qquad (3)$$

To solve the equations (1), (2) and (3), we use Gauss–Jordan method and have

$$A_b = \begin{bmatrix} 1 & 1 & 1 & 600{,}000 \\ 1 & 1 & -1 & 0 \\ 5 & -1 & -1 & 0 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 0 & 0 & 2 & 600{,}000 \\ 1 & 1 & -1 & 0 \\ 0 & -6 & 4 & 0 \end{bmatrix}, \text{ by } R_1-R_2, R_3-5R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 0 & 0 & 2 & 600{,}000 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & -2/3 & 0 \end{bmatrix}, \text{ by } -\frac{1}{6}R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 0 & 0 & 1 & 300{,}000 \\ 1 & 0 & -1/3 & 0 \\ 0 & 1 & -2/3 & 0 \end{bmatrix}. \text{ by } R_2-R_3, \frac{1}{2}R_1$$

$$\underset{\sim}{R}\begin{bmatrix} 0 & 0 & 1 & 300{,}000 \\ 1 & 0 & 0 & 100{,}000 \\ 0 & 1 & 0 & 200{,}000 \end{bmatrix}, \text{ by } R_2+\frac{1}{3}R_1, R_3+\frac{2}{3}R_1$$

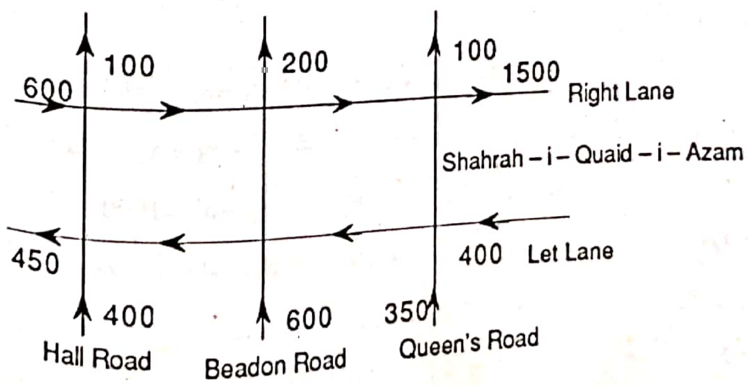
$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & 100{,}000 \\ 0 & 0 & 1 & 300{,}000 \\ 0 & 1 & 0 & 200{,}000 \end{bmatrix}, \text{ by } R_{12}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & 100{,}000 \\ 0 & 1 & 0 & 200{,}000 \\ 0 & 0 & 1 & 300{,}000 \end{bmatrix}, \text{ by } R_{23}$$

Clearly, we see that rank A = rank A_b = 3 (the number of unknowns). Hence the system has a unique solution given by

$$x = 100{,}000,\ y = 200{,}000,\ z = 300{,}000$$

6. Traffic counters submitted the following information for 23 March from 7 P.M to 8 P.M. on the following roads of Lahore.

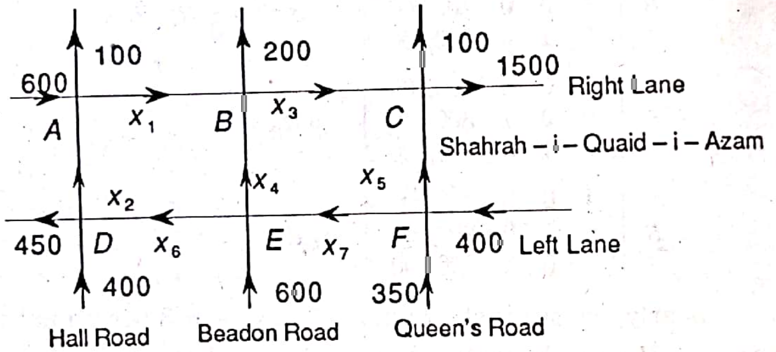


(i) Construct a mathematical model that describes this system, carefully labelling the variables you introduce.

(ii) Show that there must be at least 50 vehicles travelling on the section of let lane to Hall Road from Beadon Road during the count.

(iii) The city planners are inclined to take this traffic count as typical rush hour evening traffic in this area. In their planning of the annual closure of left lane between Queen's and Beadon for repair, how much traffic can be expected on right lane between Queen's Road and on Beadon Road.

Sol. (i) Let x_1, x_2, x_3, x_4, x_5 denote the number of vehicles along different sections of various roads, as shown in the trafic chart.



The required mathematical model can be described by the following equations.

The incoming trafic at A is x_2, 600 vehicles and balance the sum of $100, x_1$. Thus

$$x_2 + 600 = x_1 + 100 \Rightarrow x_1 - x_2 = 500 \quad \text{(i)}$$

Similarly at B: $x_1 + x_4 = x_3 + 200 \Rightarrow x_1 - x_3 + x_4 = 200$ (ii)

At C: $x_3 + x_5 = 1500 + 100 \Rightarrow x_3 + x_5 = 1600$ (iii)

At D: $x_6 + 400 = x_2 + 450 \Rightarrow x_2 - x_6 = -50$ (iv)

At E: $x_7 + 600 = x_4 + x_6 \Rightarrow x_4 + x_6 - x_7 = 600$ (v)

At F: $x_5 + x_7 = 400 + 350 \Rightarrow x_5 + x_7 = 750$ (vi)

From the above six equations, the augmented matrix is given as

$$A_b = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 500 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1600 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -50 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 600 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 750 \end{bmatrix}$$

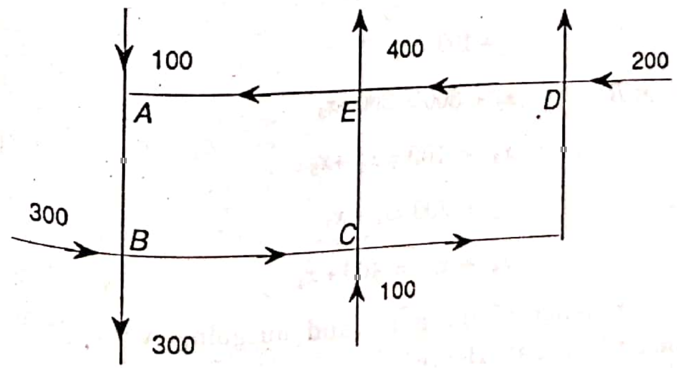
(ii) Equation (iv) implies that $x_6 = x_2 + 50$. This shows that if $x_2 = 0$, then least number of vehicles travelling on the section of the left lane to Hall Road from Beadon Road during the count is 50.

(iii) On account of the closure of left lane between Queen's and Beadon for repair, we must have $x_7 = 0$. Then x_3, the number of vehicles expected on right lane between Queen's Road and on Beadon Road can be obtained from equations (iii) and (vi). Eq. (vi) implies $x_5 = 750$, and so substituting this in eq. (iii), we get

$$x_3 + 750 = 1600$$

$$\Rightarrow x_3 = 1600 - 750 = 850.$$

7. One part of Lahore's network of traffic is given with the number of vehicles that enter and leave during a typical rush hour as shown below. All the lanes are one–way in the direction indicated by the arrows.

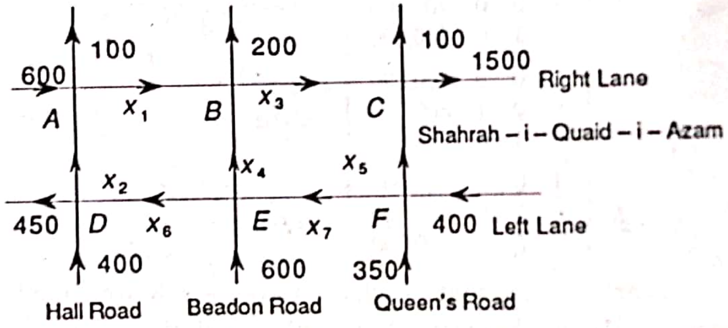


(i) Construct a mathematical model that describes this system, carefully labelling the variables you introduce.

(ii) Show that there must be at least 50 vehicles travelling on the section of let lane to Hall Road from Beadon Road during the count.

(iii) The city planners are inclined to take this traffic count as typical rush hour evening traffic in this area. In their planning of the annual closure of left lane between Queen's and Beadon for repair, how much traffic can be expected on right lane between Queen's Road and on Beadon Road.

Sol. (i) Let x_1, x_2, x_3, x_4, x_5 denote the number of vehicles along different sections of various roads, as shown in the trafic chart.



The required mathematical model can be described by the following equations.

The incoming trafic at A is x_2, 600 vehicles and balance the sum of 100,x_1. Thus

$$x_2 + 600 = x_1 + 100 \Rightarrow x_1 - x_2 = 500 \quad \text{(i)}$$

Similarly at B: $x_1 + x_4 = x_3 + 200 \Rightarrow x_1 - x_3 + x_4 = 200$ (ii)

At C: $x_3 + x_5 = 1500 + 100 \Rightarrow x_3 + x_5 = 1600$ (iii)

At D: $x_6 + 400 = x_2 + 450 \Rightarrow x_2 - x_6 = -50$ (iv)

At E: $x_7 + 600 = x_4 + x_6 \Rightarrow x_4 + x_6 - x_7 = 600$ (v)

At F: $x_5 + x_7 = 400 + 350 \Rightarrow x_5 + x_7 = 750$ (vi)

From the above six equations, the augmented matrix is given as

$$A_b = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 500 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1600 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -50 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 600 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 750 \end{bmatrix}$$

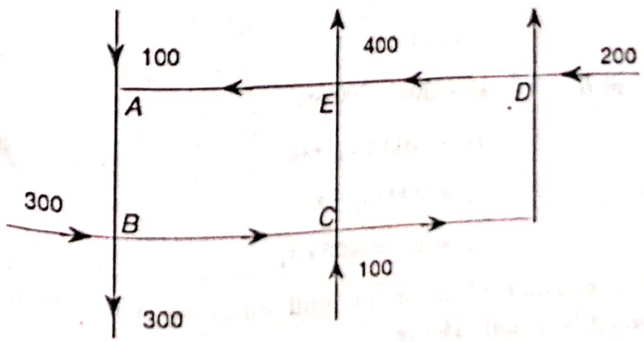
(ii) Equation (iv) implies that $x_6 = x_2 + 50$. This shows that if $x_2 = 0$, then least number of vehicles travelling on the section of the left lane to Hall Road from Beadon Road during the count is 50.

(iii) On account of the closure of left lane between Queen's and Beadon for repair, we must have $x_7 = 0$. Then x_3, the number of vehicles expected on right lane between Queen's Road and on Beadon Road can be obtained from equations (iii) and (vi). Eq. (vi) implies $x_5 = 750$, and so substituting this in eq. (iii), we get

$$x_3 + 750 = 1600$$

$$\Rightarrow \quad x_3 = 1600 - 750 = 850.$$

7. One part of Lahore's network of traffic is given with the number of vehicles that enter and leave during a typical rush hour as shown below. All the lanes are one-way in the direction indicated by the arrows.

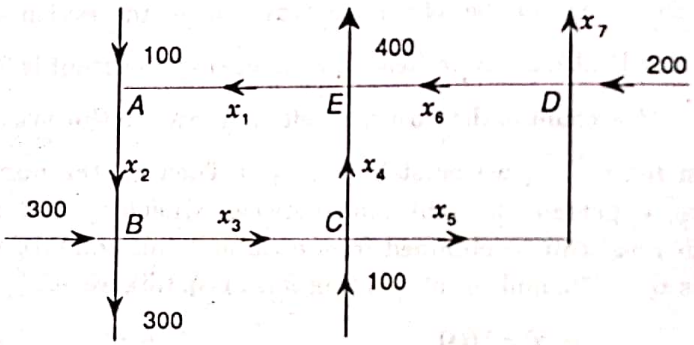


(i) Construct a linear mathematical model that describes this system.

(ii) If the stretch EA is closed for repair, what will be the traffic flow along the other stretches.

(iii) If only 100 vehicles are allowed to pass during the rush hour through EA, how will that affect on other branches?

Sol. (i) Let $x_1, x_2, x_3, x_4, x_5, x_7$ denote the number of vehicles along different sections of various roads in the rush hour is shown in the trafic chart.



The required mathematical model can be described by the following linear equations.

The trafic load coming at A is x_1 and 100 whose sum is equal to x_2 vehicles.

Thus at A :

$$x_1 + 100 = x_2 \quad \text{(i)}$$

Similarly at B : $x_2 + 300 = 300 + x_3$ (ii)

At C : $x_3 + 100 = x_4 + x_5$ (iii)

At D : $x_5 + 200 = x_6 + x_7$ (iv)

At E : $x_6 + x_4 = 400 + x_1$ (v)

The number of incoming and outgoing vehicles in the network must be equal . Hence

$$x_7 + 700 = 700 \Rightarrow x_7 = 0 \quad \text{(vi)}$$

From the above six equations, the augmented matrix is given as

$$A_b = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -100 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & -100 \\ 0 & 0 & 0 & 0 & 1 & -1 & -200 \\ 1 & 0 & 0 & -1 & 0 & -1 & -400 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Since, rank of the matrix $A \neq$ rank of the matrix A_b, so we can put $x_6 = a$.

(ii) When the section EA is closed for repair, $x_1 = 0$.

Equations (i) and (ii) imply that $x_2 = 100$ and $x_3 = 100$.

Eq. (v) implies, $x_4 = 400 - a$, where $a \leq 400$ (vii)

Eq. (iii) implies, $x_5 = a$ (viii)

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 700 \quad \text{(ix)}$$

$\Rightarrow$ $0 + 100 + 100 + 400 - a + a + a = 700$

or $a = 100$ i.e., $x_5 = a = 100$

and $x_4 = 400 - a \Rightarrow x_4 = 300$

(iii) In this case $x_1 = 100$.

Equations (i) and (ii) imply that $x_2 = 200$ and $x_3 = 200$.

Eq. (v) implies, $x_4 + x_6 = 500$ (x)

Eq. (iii) implies, $x_4 + x_5 = 300$ (xi)

$\Rightarrow$ $x_4 + a = 300 \Rightarrow x_4 = 300 - a, \; 0 \leq a \leq 300$

Eq. (iv) implies, $x_5 + 200 = a$ i.e., $x_5 = a - 200, \; 0 \leq a \leq 200$

Thus, $0 \leq a \leq 300$ and $0 \leq a \leq 200 \Rightarrow 0 \leq a \leq 200$.

Assigning arbitrary value to a such that $0 \leq a \leq 200$, we can get infinite number of solutions.