1-2

Mathematical Induction

A proof of ~~Mathematical~~ Induction.

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$\frac{(k+1)(k+1+1)}{2}$$

$$\frac{(k+1)(k+2)}{2}$$

Basic Step

For $n=1$

$$1+2+3+\cdots 1 = \frac{1(1+1)}{2}$$

$$1+2+3+\cdots 1 = \frac{1(\cancel{2})}{\cancel{2}}$$

$$1+2+3+\cdots \boxed{1=1}$$

Inductive Step

let it is true for $n=k$

$$1+2+3+\cdots k = \frac{k(k+1)}{2}$$

We have to prove that it is true for $n=k+1$. So add $k+1$ on b/s

$$1+2+3+\cdots k+(k+1) = \frac{k(k+1)}{2}+(k+1)$$

$$1+2+3+\cdots k+\underbrace{(k+1)}_{n} = \frac{k^2+k+2k+2}{2}$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)((k+1)+1)}{2}$$

Q3 Geometric Series

$$\sum_{i=0}^{n} r^n = \frac{r^{n+1} - 1}{r - 1} \quad \text{for } n \geq 0$$

Basic Step

$$r^0 = \frac{r^{0+1} - 1}{r-1} = \frac{r^1 - 1}{r-1} = \cancel{\frac{r-1}{r-1}}$$

$$1 = 1$$

Inductive Step

Let it is true for $n = k$

$$\sum_{i=0}^{n} r^k = \frac{r^{k+1} - 1}{r-1}$$

We have to prove that it is true for $n = k+1$

$$\sum_{i=0}^{k} r^k + r^{k+1} = \frac{r^{k+1} - 1}{r-1} + r^{k+1}$$

$$\sum_{i=0}^{k+1} r^{k+1} = \frac{r^{k+1} - 1 + r^{k+1}(r-1)}{r-1}$$

$$= \frac{\cancel{r^{k+1}} - 1 + r^{k+2} - \cancel{r^{k+1}}}{r-1}$$

$$= \frac{r^{k+2} - 1}{r-1}$$

$$= \frac{r^{(k+1)+1} - 1}{r-1}$$

$1+3+5+\ldots\ldots(2n-1)=n^2$

Basic Step

~~for n=1~~

$\cancel{1+3+5+\ldots\ldots(2n-1)=n^2}$

$1+3+5+\ldots\ldots(2(1)-1)=1^2$

$1=1$

Inductive Step

let it's true for $n=k$

$1+3+5+\ldots\ldots(2k-1)=k^2$

we have to prove, that it is true for $n=k+1$

$1+3+5+\ldots\ldots(2k-1)+\left(2(k+1)+1\right)=(k+1)^2$

$2k+2+1=k^2+2k+1$

$2k+3=(k+1)^2$

$1+6+11+\ldots\ldots 5n-4=\frac{n(5n-3)}{2}$

Basic Step

$1+6+11+\ldots\ldots 5(1)-4=\frac{1(5(1)-3)}{2}$

$1=\frac{1(\cancel{2})}{\cancel{2}}$

$\boxed{1=1}$

Inductive Step

let true for $n = k$

$$1 + 6 + 11 + \cdots + 5k - 4 = \frac{k(5k-3)}{2}$$

For $n = k+1$

$$1 + 6 + 11 + \cdots 5k - 4 + (5(k+1) - 4) = \frac{k(5k-3)}{2} + (5(k+1) - 4)$$

$$5k+1 = \frac{5k^2 - 3k + 2(5k+1)}{2}$$

$$5k+1 = \frac{5k^2 - 3k + 10k + 2}{2}$$

$$5k+1 = \frac{5k^2 + 7k + 2}{2}$$

$$5k+1 = \frac{5k^2 + 5k + 2k + 2}{2}$$

$$5k+1 = \frac{5k(k+1) + 2(k+1)}{2}$$

$$5k+1 = \frac{(k+1)(5k+2)}{2}$$

$$5k+1 = \frac{(k+1)(5k+5-3)}{2}$$

$$5k+1 = \frac{(k+1)(5(k+1)-3)}{2}$$

$$= \frac{(k+1)(5(k+1)-3)}{2}$$

$$= \frac{(k+1)(5k+5-3)}{2}$$

$$= \frac{(k+1)(5k+2)}{2}$$

$\cancel{\frac{18}{3}} = \cancel{3r}$

$\frac{18}{3} = r$

$18 = 3r$

$p(n) = 2^{2n} - 1$ is divisible by 3

Basic Step

$P(0) = 2^{0} - 1$

$= 1 - 1$

$= 0 = 0/3 = 0$

Inductive Step

let: $p(k) = 2^{2k} - 1$ is also divisible by 3

We have to prove that $p(k+1)$ is also divisible by 3

let: $2^{2k} - 1 = 3r \rightarrow (1)$

$p(k+1) = 2^{2(k+1)} - 1$

$p(k+1) = 2^{2k+2} - 1$

$= 2^{2k} \cdot 2^{2} - 1$

$= 2^{2k} \cdot 4 - 1$

$= 2^{2k} \times (3+1) - 1$

$= 3 \cdot 2^{2k} + \boxed{2^{2k} - 1}$

$= 3 \cdot 2^{2k} + 3r$

$= 3(2^{2k} + r)$

Well Ordering Principle

- A set of all non-negative integers must have a smallest element.

- Every non empty set S of non-negative integers contains a least element.

  $S = \{2, 4, 6, 5, 8\}$ → form of mathematical induction.

- Fibonacci Sequence
  0 1 1 2 3 5 8 13 . . . .

- Recursive Sequence
  → write next term in terms of previous term

$$\boxed{F_n = F_{n-1} + F_{n-2}}$$

→ . Recursive Relation

. Recursion / Recursive Sequences

⇒ Initial Conditions

$F_0 = 0$

$F_1 = 1$

→ Recursive Relative is in the form of equation.

Q:

$C_k = C_{k-1} + k\,C_{k-2} + 1$ for $k \geq 2$

and

$C_1 = 1,\ C_2 = 2$

$C_3, C_4, C_5 = ?$

$C_3 = C_{3-1} + 3\,C_{3-2} + 1$

$C_3 = C_2 + 3\,C_2 + 1$

$= 2 + 3(1) + 1$

$\cancel{2+1+3}$

$C_3 = 2 + 3 + 1 = \boxed{6}$

$C_4 = C_{4-1} + 4\,C_{4-2} + 1$

$= C_3 + 4\,C_2 + 1$

$= 6 + 4(2) + 1$

$= 6 + 8 + 1$

$\boxed{C_4 = 15}$

$C_5 = C_4 + 5\,C_3 + 1$

$= 15 + 5(6) + 1$

$\boxed{C_5 = 46}$

Q: $F_n = F_{n-1} + F_{n-2}$

$F_0 = 1,\ F_1 = 1$

$F_3 = ?$

$F_2 = F_{2-1} + F_{2-2}$

$= F_1 + F_0$

$F_2 = 1 + 1 = \boxed{2}$

$F_3 = F_{3-1} + F_{3-2}$

$= F_2 + F_1$

$= 2 + 1$

$F_3 \boxed{= 3}$

~~(-4)(Sr Sher~~

Q:1 Find first four terms of following Relation.

$a_k = 2a_{k-1} + k$ for all $k \geq 2$

$a_1 = 1$

$a_2 = 4$

$a_3 = 11$

$a_4 = 26$

$a_2 = 2a_{2-1} + 2$
$= 2(1) + 2$
$= 4$

$S_k = S_{k-1} + 2S_{k-2}$ for $k \geq 2$

$S_0 = 1$

$S_1 = 1$

$S_2 = 3$

$S_3 = 5$

$S_4 = ?$

$S_4 = S_3 + 2S_2$
$= 5 + 2(3)$
$= 5 + 6$
$\boxed{S_4 = 11}$

Does an Explicit Formula Satisfies a Recursive Relation (R.R)

let:

$a_n = 3n+1$ for $n \geq 0$

so that it satisfies R.R

$a_k = a_{k-1} + 3$ for $k \geq 1$

$a_k = \boxed{3k+1}$

$a_{k-1} = 3(k-1)+1$
$= 3k-3+1$
$= 3k-2$

R.H.S

$= a_{k-1} + 3$
$= 3k-2+3$
$\boxed{= 3k+1}$

L.H.S = R.H.S

let $b_n = 4^n$ for $n \geq 0$
show that it satisfies

R.R

$b_k = 4b_{k-1}$ for $k \geq 1$

$b_k = 4^k \quad | \quad b_{k-1} = 4^{k-1}$

$$\begin{aligned} b_k &= 4b_{k-1} \\ &= 4(4^{k-1}) \\ &= 4^{k-1+1} \\ &= 4^k \end{aligned}$$

let: $t_n = 2+n$

so that is satisfies

R.R $\longrightarrow t_k = 2t_{k-1} - t_{k-2}$

~~2+k = 2~~

$t_k = \boxed{2+k} \quad | \quad t_{k-1} = 2 + k - 1 = \boxed{k+1}$

$t_{k-2} = 2 + k - 2 = \boxed{k}$

$$\begin{aligned} t_k &= 2t_{k-1} - t_{k-2} \\ &= 2(k+1) - k \\ &= 2k + 2 - k \\ &= \boxed{2+k} \end{aligned}$$

Closed form from given Recursive Sequence

⇒ used for analysis of algorithm

Solve Recurrence Eq find closed form/Explicit formula

① Iterative Method

Q

$a_0, a_1, a_2 \ldots\ldots$

$a_k = a_{k-1} + 2$

$a_0 = 1$

So we want to guess explicit formula

$a_1 = 1+2 = 3$	$1+2$
$a_2 = 3+2 = 5$	$1+2+2$
$a_3 = 5+2 = 7$	$1+2+2+2$
$a_4 = 7+2 = 9$	$1+2+2+2+2$

$1, 3, 5, 7, \ldots\ldots$

$a_0 = 1+2(0)$

$a_1 = 1+2(1)$

$a_2 = 1+2(2)$

$a_3 = 1+2(3)$

$a_4 = 1+2(4)$

$\boxed{a_n = 1+2n}$

↳ closed form

Q

let

$a_k = r a_{k-1}$ for $k \geq 1$

$a_0 = a$

$a_1 = r a_0 = ar$

$a_2 = r a_1 = r(ra) = ar^2$

$a_3 = ar^3$

$a_4 = ar^4$

$\boxed{a_n = ar^n}$ → closed form

Q

$c_k = 3c_{k-1} + 1$ for $k \geq 2$

$c_1 = 1$

$c_2 = 3c_1 + 1 = 4$

$c_3 = 13$

$c_4 = 40$

$c_5 = 121$

$1, 4, 13, 40, 121, \ldots$

$c_1 = 1$

$c_2 = 3 + 1$

$c_3 = 3(3+1)+1 = 3^2 + 3 + 1$

$c_4 = 3(3^2 + 3 + 1) + 1$

$= 3^3 + 3^2 + 3 + 1 = 40$

$c_n = 3^{n-1} + 3^{n-2} + \cdots + 3 + 1$

$$\sum_{i=0}^{n-1} 3^i$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{1(3^n - 1)}{3 - 1}$$

$$\boxed{= \frac{3^n - 1}{2}} \rightarrow \text{closed form}$$

Q A worker is promised bonus if he can increase productivity by 2 units a day everyday for 30 days. If his productivity is 170 on day 0, what is the productivity on day 30.

$d_{30} = ?$

$d_0 = 170$

$d_1 = 170 + 2 = 172$

$d_2 = 172 + 2 = 174$

$d_3 = 176$

$d_4 = 178$

170
170 + 2
170 + 2 + 2
170 + 2 + 2 + 2
170 + 2 + 2 + 2 + 2

$d_n = 170 + 2(n)$

$d_{30} = 170 + 2(30)$

$d_{30} = 170 + 60$

$\boxed{d_{30} = 230}$

Q A runner targets himself to improve on a course 3 seconds a day. On day 0, he runs course (track) in 3 min. How fast will he run on day 14.

$3\text{ min} = 3 \times 60\text{ sec}$
$= 180\text{ sec}$

$d_0 = 180$

$d_1 = 180 - 3 = 177$

$d_2 = 174$

$d_3 = 171$

180
180 − 3
180 − 3 − 3
180 − 3 − 3 − 3

$d_n = 180 - 3(n)$

$d_{14} = 180 - 3(14)$

$= 138$

Tower of Hanoi

$T_1 = 1 \quad = 1$

$T_2 = 3 \quad = 2+1$

$T_3 = 7 \quad = 2(2+1)+1$

$= 2^2+2+1$

$T_4 = 13 \quad 2(2^2+2+1)+1$

2^3+2^2+2+1

$T_k = 2T_{k-1} - 1$

$T_n = 2^{n-1} + 2^{n-2} + \cdots + 2 + 1$

$$T_n = \sum_{i=0}^{n-1} 2^i$$

Geometric Series

$$= \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{1(2^n - 1)}{2 - 1}$$

$$= 2^n - 1$$

$$T_{64} = 2^{64} - 1$$

$$= 1.844 \times 10^{19} \text{ sec}$$