Linear Algebra

Vectors:

A vector is a point in space. It is represented in the form of $\begin{bmatrix} a \\ b \end{bmatrix}$ (column matrix).

Magnitude

$$\vec{V} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$|\vec{V}| = \sqrt{a^2 + b^2}$$

It is represented by $|\vec{V}|$ and also called length or norm.

$$\vec{V} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$|\vec{V}| = \sqrt{(3)^2 + (4)^2} = \sqrt{9 + 16} = \sqrt{25}$$

$$|\vec{V}| = 5$$

Unit Vector/Normal Vector

A vector whose norm is always 1 is called unit vector.

It tells us about the direction

of a vector.
It is represented by $\hat{v}$.

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} \qquad \text{if } \vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\hat{v} = \frac{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}{5}$$

$$\hat{v} = \frac{1}{5} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\hat{v} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$$

Vector Addition:

Vector addition is possible if both vectors have same dimensions.

$$\vec{q} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{p} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad \vec{r} \begin{bmatrix} 5 & 6 \end{bmatrix}$$

Vector addition of $\vec{p}$ and $\vec{q}$ is possible while with $\vec{r}$ it is not possible.

$$\vec{q} + \vec{p} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1+3 \\ 2+4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

Scalar Multiplication:

Multiplication of a vector with a scalar quantity.

$a = 3$, $\vec{r} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$a \times \vec{r} = 3 \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$

Linear Combination:

It has only two operations.

Vector Addition

Scalar Multiplication

$a = 5$ $\vec{r} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ $\vec{n} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$

$a(\vec{r} + \vec{n}) = 5\left(\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}\right) = 5\left(\begin{bmatrix} 4 \\ 8 \\ 9 \end{bmatrix}\right)$

$a(\vec{r} \times \vec{n}) = \begin{bmatrix} 20 \\ 40 \\ 45 \end{bmatrix}$

In vectors, linear combination is very important.

Any vector in a plane is represented in the form of a vector combination/linear combination.

Dot product (Inner product)

$$\vec{V} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \vec{U} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{V}^t = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$\vec{V} \cdot \vec{U} = \vec{V}^t \cdot \vec{U} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

Orthogonal:

Two vectors are said to be orthogonal to each other if their dot product is equal to zero.

$$\vec{n} = \begin{bmatrix} 2 \\ 1 \\ -2 \\ 4 \end{bmatrix} \qquad \vec{m} = \begin{bmatrix} 3 \\ -6 \\ 4 \\ 2 \end{bmatrix}$$

$$\vec{n}^t = \begin{bmatrix} 2 & 1 & -2 & 4 \end{bmatrix}$$

$$\vec{n} \cdot \vec{m} = \vec{n}^t \circ \vec{m} = \begin{bmatrix} 2 & 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -6 \\ 4 \\ 2 \end{bmatrix}$$

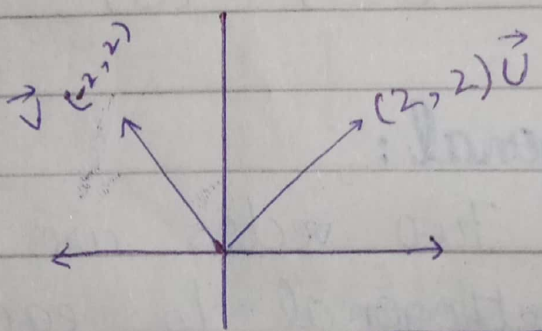
$$= [6 - 6 - 8 + 8]$$

$$= [0]$$

Both vectors are orthogonal t
each other.

Orthonormal:

If two unit vectors
are perpendicular to each oth
than they are called orthono



$$\vec{U} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow |\vec{U}| = \sqrt{(2)^2 + (2)^2} = \sqrt{4+}$$

$$|\vec{U}| = \sqrt{8} = 2\sqrt{2}$$

$$\hat{U} = \frac{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}{2\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$\vec{V} = [-2 \quad 2]$, $|\vec{V}| = \sqrt{(-2)^2 + (2)^2} = \sqrt{4+4}$

$\vec{V} = \sqrt{8} = 2\sqrt{2}$

$\hat{V} = \dfrac{\begin{bmatrix} -2 \\ 2 \end{bmatrix}}{2\sqrt{2}} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

For orthonormal:

$\hat{U} \cdot \hat{V} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\hat{U}^t \cdot \hat{V} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$= \left[-\frac{1}{2} + \frac{1}{2} \right] = [0]$

Hence proved that U and V are orthonormal to each other.

Matrices:

Matrices are a sort of functions.

They transform vectors.

dimension:

The dimension of a matrix is represented by $m \times n$ where m is no. of rows and n is no. of columns.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 5 \end{bmatrix}$$

dimension = 2×3

Square Matrix:

A matrix of dimens[ion] $m \times m$ is called square matri[x]

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

dimension = 2×2

Transpose of a matrix:

Converting rows into columns or columns into ro[ws]

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}, \quad A^t = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$$

Symmetric matrix:

If $A^t = A$ then matri[x] A is called symmetric matrix

$$A = \begin{bmatrix} 12 & 7 \\ 7 & 12 \end{bmatrix}, \quad A^t = \begin{bmatrix} 12 & 7 \\ 7 & 12 \end{bmatrix}$$

$$\boxed{A = A^t}$$

Multiplication:

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \\ 3 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

$$\vec{A}\cdot\vec{B} = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \\ 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 2+6+28 \\ 1+4+35 \\ 3+6+7 \end{bmatrix}$$

$$\vec{A}\cdot\vec{B} = \begin{bmatrix} 36 \\ 40 \\ 16 \end{bmatrix}$$

A vector '$\vec{B}$' is transformed by matrix A.

Finding unknowns

$$A = \begin{array}{c} \begin{matrix} 1 & \;\;1 & \;\;0 \\ \downarrow & \;\;\downarrow & \;\;\downarrow \end{matrix} \\ \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 6 & 1 \end{bmatrix} \end{array} \begin{bmatrix} \; \\ \; \\ \; \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \\ 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} \; \\ \; \\ \; \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \\ 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix}$$

Matrix - Matrix multiplication:

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \\ 3 & 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 7 \\ 7 & 2 & 5 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 2+6+28 & 6+12+8 & 4+21+20 \\ 1+4+35 & 3+8+10 & 2+14+25 \\ 3+6+7 & 9+12+2 & 6+21+5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 36 & 26 & 45 \\ 40 & 21 & 41 \\ 16 & 23 & 32 \end{bmatrix}$$

Determinants:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$|A| = (2)(4) - (1)(3)$$

$$= 8 - 3 = 5$$

Singular Matrix:

A matrix whose determinant is equal to zero is called singular matrix.

$$A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$$

$$|A| = 3(2) - 1(6)$$
$$= 6 - 6$$
$$= 0$$

∴ Inverse of a singular matrix does not exist.

- Columns are linearly dependent.

Rank:

Number of linearly independent columns is called rank of a matrix.

It cannot be zero.

linear dependent columns mean that two or more columns are adding together to make other columns...

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \\ 3 & 3 & 1 \end{bmatrix}$$

Rank is 3

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 4 & 1 & 5 \end{bmatrix}$$

As one column is dependent on other two so the rank of this matrix is **2**.

Inverse of a matrix:

Inverse of a 2x2 matrix is defined as:

$$A^{-1} = \frac{Adj\ A}{|A|}$$

$\because A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

where $Adj\ A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

Let:

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$$

$$|A| = 3(5) - 2(1)$$

$$= 15 - 2 = 13$$

$$\text{Adj } A = \begin{bmatrix} 5 & -1 \\ -2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

$$A^{-1} = \frac{\begin{bmatrix} 5 & -1 \\ -2 & 3 \end{bmatrix}}{13} = \begin{bmatrix} 5/13 & -1/13 \\ -2/13 & 3/13 \end{bmatrix}$$

Identity Matrix:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$