Department of Electrical Engineering, IIT Bombay

AUTUMN SEMESTER: JUL-NOV 2024

Mid-Semester Examination: EE 229 – Signal Processing I – (B. Tech.)

Maximum Marks: 50 (25 percent weight)

Date: Saturday 14 Sept. 2024 **Time: 13:30 to 15:30 hours**

Instructions:

1. Please begin the answer to each **main question** on a **fresh page** of the answer booklet.
2. This is a **closed book, closed notes** examination.
3. Show your reasoning and important steps clearly.
4. Unless otherwise stated, (a)treat x as the input to the system and y, the output (b)Stability is understood in the 'Bounded input, bounded output' (BIBO) sense. (c) $j = \sqrt{-1}$

Q1. (3 X 5 = 15 marks):

In each of the parts *(a)* to *(e)* below, **either** provide a satisfactory **proof** of the statement A, if it is true, independent of example, ignoring the system S **or disprove** the statement A, **using the system S as a counter-example**, with proper explanation. State clearly, what you are doing.

(a) A: *A discrete linear system is stable in the 'Bounded input, bounded output' (BIBO) sense, if its impulse response is absolutely summable.*
S: $y[n] = (n+3)x[n] + (n+2)x[n-1]$

(b) A: *A linear system always produces the output* $y[n] = 0 \;\; \forall\, n$*, when the input is* $x[n] = 0 \;\; \forall\, n$
S: $y[n] = 3\,x[n] + 2\,x[n-1] + 9$

(c) A: *A shift invariant system always produces the output* $y[n] = 0 \;\; \forall\, n$*, when the input is* $x[n] = 0 \;\; \forall\, n$
S: $y[n] = 3\,x[n] + 2\,x[n-1] + 9$

(d) A: *An additive system must also be homogeneous.*
S: $y[n] = 3(complex\ conjugate\ of\ x[n]) + 4\,(Real\, part\, of\, x[n-1]) + x[n-2]$

(e) A: *A homogeneous system must also be additive.*
S: $y[n] = 2\,\frac{x^2[n]}{x[n-1]},\ if\ x[n-1] \neq 0;\ \ y[n] = 0, if\ x[n-1] = 0$

Q2. (5 x 3 = 15 marks):

S1 and S2 are two continuous time, linear shift invariant systems, whose impulse responses, respectively, are: $e^{-2t}u(t)$ and $e^{-5t}u(t)$, where $u(t)$ is the unit step function.

(a) Let S1 and S2 be connected in series to form a composite linear shift invariant system. Obtain the impulse response of this composite linear shift invariant system. Why does the order, in which the two systems are connected in series, **not** matter?
(b) Let an input $x(t)$ be applied to the individual systems S1 and S2, as also to the composite series connected system described in Q2(a). Show that the output of the composite system is a linear combination of the outputs of the individual systems S1 and S2 and obtain the coefficients in this linear combination.
(c) State and explain whether (I) the individual systems S1 and S2, as also (II) the composite series connected system of Q2(a), are (i) causal (ii) stable.

Q3. (10 marks) A linear, shift invariant, continuous time system has the frequency response:

$$H(j\Omega) = 7j, \forall\ (0 < \Omega < \pi);\ \ (-7j), \forall(-\pi < \Omega < 0);\ \ 0, \forall\ (|\Omega| > \pi)$$

(a) Obtain its impulse response.
(b) Obtain the output of this system, when the input is $x(t) = 3\cos(2t)$.

Q4. (10 marks) A linear, shift invariant, causal discrete time system is described by:

$y[n] = \alpha\, y[n-1] + x[n]$, where α is a real constant.

(a) Obtain its impulse response.
(b) Obtain the condition on α, for the system to be BIBO stable. Explain.
(c) Obtain the response of the system to the unit step input.

End of question paper – with best wishes.

Q1 -

(a) We shall ~~dis~~prove the statement 'A' using the system 'S' as counter-example

The impulse response of system 'S' is

$$h[n] = (n+3)\delta[n] + (n+2)\delta[n-1]$$

$$= 3\delta[n] + 2\delta[n-1]$$

This impulse response is absolutely summable.

Indeed $\sum_{n\in\mathbb{Z}} |h[n]| = 3+2 = 5.$

However a bounded input, say $x[n] = u[n]$ produces the unbounded output

$$(n+3)u[n] + (n+2)u[n-1]$$

$$= 3\delta[n] + (n+3+n+2)u[n-1]$$

$$= 3\delta[n] + (2n+5)u[n-1]$$

This discrete linear system is NOT BIBO stable. This has happened as it is not shift-invariant.

(b) We shall prove the statement 'A' independent of example.

Indeed with input $x[n] = 0 \ \forall n$, output $y[n]$,

$$2x[n] = x[n] = 0 \quad \forall n$$

whereupon, linearity of the system

$$\Rightarrow 2x[n] \text{ produces } 2y[n]$$

and hence $2y[n] = y[n] \ \forall n$

$$\Rightarrow y[n] = 0 \quad \forall n$$

Q1 -

(c) We shall disprove statement A'', using system 'S' as counter-example.

Indeed, S is a shift-invariant system since $x[n-D] \longrightarrow 3x[n-D] + 2x[n-D-1] + 9$

$$= y[n-D]$$

However the input $x[n] = 0 \;\; \forall n$ produces $y[n] = 9 \;\; \forall n$, not $y(n) = 0 \;\; \forall n$.

In fact, 'S' is not linear.

(d) We shall disprove statement A', using system 'S' as counter-example.

Indeed let $x[n] \xrightarrow{S} y[n]$

$$j\,x[n] \xrightarrow{S} 3(\overline{j\,x[n]}) + 4\,Re(j\,x[n-1]) + j\,x[n-2]$$

$$\neq j\{3\,\bar{x}[n] + 4\,Re(x[n-1]) + x[n-2]\}.$$

The system is not homogeneous, as exemplified by scaling by j

(e) We shall disprove statement 'A' using system 'S' as counter-example

Indeed let $x[n] \longrightarrow y[n]$

For any nonzero constant c, $c\,x[n] \rightarrow c\,y[n]$

On the other hand $x[n] = 0 \;\; \forall n \Rightarrow y[n] = 0 \;\; \forall n$

The system 'S' is homogeneous. It is clearly not additive as $x_{1,2}[n] \rightarrow y_{1,2}[n]$

$$\not\Rightarrow x_1[n] + x_2[n] \longrightarrow y_1[n] + y_2[n]$$

Q2 -

(a) The impulse response of this composite linear shift-invariant system is the convolution of $e^{-2t}u(t)$ and $e^{-5t}u(t)$.
It does <u>not</u> matter, in which order they are connected, since convolution is <u>commutative</u>.
The composite system has impulse response

$$h(t) = \int_{-\infty}^{+\infty} e^{-2\lambda} u(\lambda)\, e^{-5(t-\lambda)} u(t-\lambda)\, d\lambda$$

$$= \left\{ \int_0^t e^{-2\lambda} \cdot e^{-5t} \cdot e^{5\lambda}\, d\lambda \right\} u(t)$$

$$= e^{-5t} u(t) \cdot \int_0^t e^{3\lambda}\, d\lambda$$

$$= e^{-5t} u(t) \left. \frac{e^{3\lambda}}{3} \right|_0^t$$

$$= \frac{1}{3} e^{-5t} u(t) \left(e^{3t} - 1 \right) = \frac{1}{3} e^{-2t} u(t) - \frac{1}{3} e^{-5t} u(t),$$

$$= \frac{1}{3} (\text{Impulse response of } S_1) - \frac{1}{3} (\text{Impulse response of } S_2)$$

Q2 -

(b) clearly $x(t) \to \boxed{S_1} \to x(t) * e^{-2t} u(t)$

$$x(t) \to \boxed{S_2} \to x(t) * e^{-5t} u(t)$$

$$x(t) \to \boxed{S_1} \to \boxed{S_2} \to x(t) * h(t)$$

However

$$x(t) * h(t)$$

$$= \frac{1}{3}\left[x(t) * e^{-2t} u(t)\right] - \frac{1}{3}\left[x(t) * e^{-5t} u(t)\right]$$

$$= \frac{1}{3}(\text{output of } S_1) - \frac{1}{3}(\text{output of } S_2)$$

which proves the statement
with coefficients
$\frac{1}{3}$ associated with output of S_1
$-\frac{1}{3}$ associated with output of S_2.

Q2 -
(c)

Each of the systems

S_1, S_2, Composite series connected system

is causal
as the impulse response is
zero $\forall\, t < 0$.

The impulse responses of S_1, S_2
are absolutely integrable

Indeed $\int_{-\infty}^{+\infty} |e^{-2t} u(t)|\, dt$

$$= \int_0^{\infty} e^{-2\lambda} d\lambda = \left.\frac{e^{-2\lambda}}{-2}\right|_0^{\infty}$$

$$= \frac{1}{2}.$$

$$\int_{-\infty}^{+\infty} |e^{-5t} u(t)|\, dt = \int_0^{\infty} e^{-5\lambda} d\lambda = \left.\frac{e^{-5\lambda}}{-5}\right|_0^{\infty}$$

$$= \frac{1}{5}.$$

As regards the composite system

$$\int_{-\infty}^{+\infty} |h(t)|\, dt = \int_{-\infty}^{+\infty} \left|\frac{1}{3} e^{-2t} u(t) - \frac{1}{3} e^{-5t} u(t)\right| dt$$

$$\leq \int_0^{\infty} \frac{1}{3} e^{-2\lambda} d\lambda + \int_0^{\infty} \frac{1}{3} e^{-5\lambda} d\lambda = \frac{1}{3}\left(\frac{1}{2} + \frac{1}{5}\right) < \infty$$

Whereupon $h(t)$ is absolutely integrable
and the composite system,
BIBO stable

Q3 - $H(j\Omega) = 7j \quad \forall\ 0 < \Omega < \pi$
$= -7j \quad \forall\ -\pi < \Omega < 0$
$= 0 \quad \forall\ |\Omega| > \pi.$

(a) Impulse response =
Inverse Fourier Transform of $H(j\Omega)$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$= \frac{1}{2\pi} 7j \cdot \left\{ \int_0^{\pi} e^{j\Omega t}\, d\Omega - \int_{-\pi}^{0} e^{j\Omega t}\, d\Omega \right\}$$

$$= \frac{1}{2\pi}\cdot 7j \left\{ \int_0^{\pi} e^{j\Omega t}\, d\Omega + \int_{\pi}^{0} e^{-j\Omega_1 t}\, d\Omega_1 \right\}$$

$$= \frac{1}{2\pi}\cdot 7j \left\{ \int_0^{\pi} e^{j\Omega t}\, d\Omega - \int_0^{\pi} e^{-j\Omega t}\, d\Omega \right\}.$$

$$= \frac{7j}{2\pi}\int_0^{\pi} (e^{j\Omega t} - e^{-j\Omega t})\, d\Omega$$

$$= \frac{7j}{2\pi}\cdot \int_0^{\pi} 2j \cdot \sin \Omega t\, d\Omega = \frac{-7j \cdot 2j}{2\pi} \left.\frac{\cos \Omega t}{t}\right|_0^{\pi}$$

$$= 7\left\{\frac{\cos \pi t - 1}{t}\right\} = -\frac{7}{t}\, 2 \sin^2 \frac{\pi}{2} t$$

$$= -\frac{14}{t} \sin^2 \frac{\pi}{2} t .$$

Q3 - (b) $x(t) = 3\cos 2t$

$$= \frac{3}{2}\left\{e^{j2t} + e^{-j2t}\right\}$$

$$0 < 2 < \pi, \qquad -\pi < -2 < 0$$

Accordingly, from the frequency response, the output $= \frac{7j}{2}\cdot 3e^{j2t} - \frac{7j}{2}\cdot 3e^{-j2t}$

$$= -(7\times 3)\left(\frac{e^{j2t} - e^{-j2t}}{2j}\right)$$

$$= -21\sin 2t.$$

Q4 -
(a) Let the impulse response be $h[n]$
Then from the causality of the system, $h[n] = 0 \ \forall \ n<0$

Further $h[n] = \alpha h[n-1] + x[n] \Big|_{x[n] = \delta[n]}$

For $n=0$, $h[0] = \delta[0]$
$= 1$

$n=1$, $h(1) = \alpha h[0] + 0$
$= \alpha$

$n>1$ $h[n] = \alpha^n$

The impulse response $= \alpha^n u[n]$

(b) For the system to be stable,
$h[n]$ must be absolutely summable.

$$\Rightarrow \sum_{n=-\infty}^{+\infty} |h[n]| = \sum_{n=0}^{\infty} |\alpha|^n < \infty$$

$$\Rightarrow \qquad |\alpha| < 1.$$

(c) with a unit step input $u[n]$, the output would be $\sum_{k=-\infty}^{+\infty} \alpha^k u[k] u[n-k]$

$$= \left(\sum_{k=0}^{n} \alpha^k \right) u[n] = \frac{1-\alpha^{n+1}}{1-\alpha} \cdot u[n]$$