Department of Electrical Engineering, IIT Bombay

AUTUMN SEMESTER: JUL-NOV 2024

Semester-End Examination: EE 229 – Signal Processing I – (B. Tech.)

Maximum Marks: 80 (40 percent weight)

Date: Monday 11 Nov. 2024 **Time: 13:30 to 16:30 hours**

Instructions:

1. Please begin the answer to each **main question, that is,** Q1, Q2, Q3, Q4 on a **fresh page** of the answer booklet.
2. This is a **closed book, closed notes** examination.
3. Show your reasoning and important steps clearly.
4. Unless otherwise stated,
   (a) treat x as the input to the system and y, the output (b) Stability is understood in the 'Bounded input, bounded output' (BIBO) sense. (c) $j = \sqrt{-1}$

Q1. (20 marks) In this question, let the time t be measured in milliseconds.

(a) Obtain the ratio of the Laplace Transform of the voltage across each of the following elements to the Laplace Transform of the current flowing in the element, as a function of the Laplace variable s. (i) A resistor of value R ohms. (ii) An inductor of value L Henry. (iii) A capacitor of value C Farads. These ratios are called Laplace Domain Impedances of the elements, which allow us to solve circuits in the s-domain, in a manner analogous to phasor circuits.

(b) Hence, obtain the Laplace Transform of the voltage across the capacitor C_2 and the Laplace Transform of the current in the capacitor C_2, in the circuit of Fig. Q1, given that the input at terminals 'ab' is a unit step voltage. Obtain the time domain expressions for the voltage and the current associated with the capacitor C_2.

(c) Obtain the time domain expressions for the voltage and the current associated with the capacitor C_2, in the circuit of Fig. Q1, given that the input at terminals 'ab' is the steady sinusoidal voltage described by $100\cos(2\pi t)$ Volts.

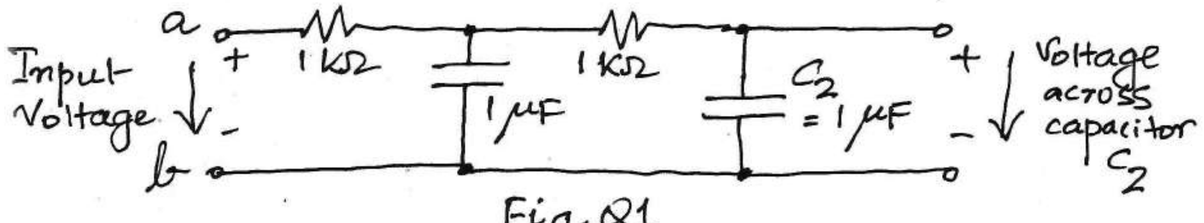


Fig. Q1

Q2. (20 marks): The algebraic expression for the system function of a discrete time linear, shift-invariant (LSI) system S is given by:

$$H(z) = \frac{(2\,z^{-1} + 3)}{(1 + 2\,z^{-1})(1 - 0.5\,z^{-1})^2}$$

(a) Identify the possible Regions of Convergence for this system function.

(b) For each of these Regions of Convergence, obtain the impulse response of the corresponding underlying discrete time LSI system. In each case, state and explain, whether the underlying system is causal and whether the underlying system is stable.

(c) Consider the stable discrete LSI system in Q2(b). Obtain the output of the system, when the input sequence is $x[n] = ((-1)^n)\ for\ all\ n.$
Hint: Is this a discrete phasor sequence?

(d) Consider the causal discrete LSI system in Q2(b). Identify one input sequence with exactly 2 consecutive non-zero samples, which, when applied to the system(s), produces an output which is absolutely summable. Obtain the ratio of the absolute sum of this output to the absolute sum of the input. Show that this ratio is proportional to the value of the z-transform of the output at a particular value of z, in this particular case. Indicate which value of z it is and explain why it is so.

3. **(20 marks)** The continuous time signal $x(t) = 100\,\{\cos(2\pi t) + \cos(3\pi t) + \cos(4\pi t)\}$, is applied as the input to two system configurations A and B, as described below, in two independent experiments. Obtain, with proper explanation, the output signal in each of these two system configurations. Explain the practical significance of these two idealized experiments. The time t is measured in seconds and $u(t)$ represents the unit step.

A. $x(t)$ is applied to an ideal sampler with sampling rate 10 Hz, the output of this sampler is given as input to a linear shift invariant (LSI) system with the impulse response: $h(t) = u(t) - u(t - 0.03)$. The output of this LSI system is applied to an ideal lowpass filter, with a cutoff frequency of 15 Hz.

B. $x(t)$ is multiplied by the periodic signal $p(t) = \sum_{k=-\infty}^{k=+\infty}\{\,u(t + 0.1\,k) - u(t + 0.1\,k - 0.03)\}$. and the product signal is applied to an ideal lowpass filter, with a cutoff frequency of 15 Hz.

Q4. (20 marks). The continuous time signal $x_1(t) = e^{-2t}u(t)$ has the Fourier Transform $X_1(j\Omega)$. The continuous time signal $x_2(t) = e^{-5t}u(t)$ has the Fourier Transform $X_2(j\Omega)$. Obtain:

(a) The inverse Fourier Transform of $X_1(j\Omega)\overline{X_2(j\Omega)}$.

(b) The inverse Fourier Transform of $X_1(j\Omega)X_2(j\Omega)$.

(c) The inverse Fourier Transform of $|X_1(j\Omega)|^2$.

(d) The inverse Fourier Transform of $\Omega^2 X_1(j\Omega)X_2(j\Omega)$.

(e) $\int_{-\infty}^{+\infty} X_1(j\Omega)\overline{X_2(j\Omega)}d\Omega$

(f) $\int_{-\infty}^{+\infty} |X_1(j\Omega)|^2\overline{X_2(j\Omega)}\,d\Omega$

End of question paper – with best wishes.

Q1 -

(a) (i) $v_R(t) = R\, i_R(t) \Rightarrow V_R(s) = R I_R(s)$

$$\Rightarrow \frac{V_R(s)}{I_R(s)} = R$$

(ii) $v_L(t) = L \dfrac{d i_L(t)}{dt} \Rightarrow V_L(s) = s L I_L(s)$

$$\Rightarrow \frac{V_L(s)}{I_L(s)} = Ls.$$

(iii) $i_C(t) = C \dfrac{d v_C(t)}{dt} \Rightarrow I_C(s) = Cs V_C(s)$

$$\Rightarrow \frac{V_C(s)}{I_C(s)} = \frac{1}{Cs}.$$

(b) Let us draw the network of Fig. Q1 in the Laplace domain.

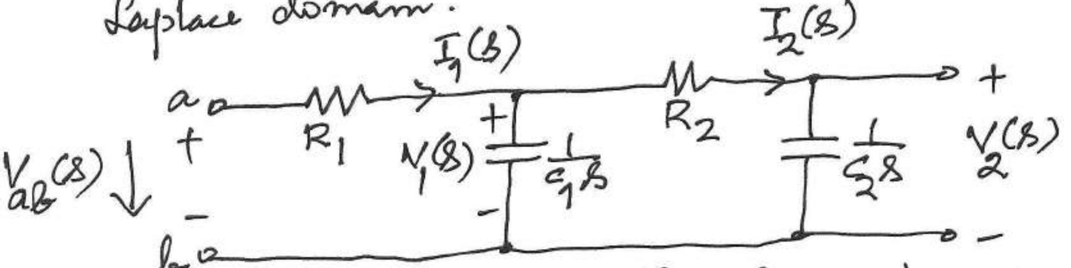


Fig. Q1 in the Laplace Domain.

$R_1 = 1\,k\Omega$, $R_2 = 1\,k\Omega$, $C_1 = 1\,\mu F$, $C_2 = 1\,\mu F$.

It would be convenient to proceed from $V_2(s)$ towards $V_{ab}(s)$ using network analysis.

$$I_2(s) = C_2 s \cdot V_2(s).$$

$$V_1(s) = \left(R_2 + \frac{1}{C_2 s}\right) I_2(s) = \left(R_2 C_2 s + 1\right) V_2(s).$$

$$I_1(s) = V_1(s) . C_1 s + I_2(s).$$

$$= \left(1 + R_2 C_2 s\right) C_1 s . V_2(s) + C_2 s V_2(s)$$

$$= \left(C_1 s + R_2 C_2 s C_1 s + C_2 s\right) V_2(s)$$

$$V_{ab}(s) = R_1 I_1(s) + V_1(s).$$

$$V_{ab}(s) = R_1(C_1 s + R_2 C_2 C_1 s^2 + C_2 s)V_2(s) + (1 + R_2 C_2 s)V_2(s)$$

$$= (R_1 C_1 s + R_1 R_2 C_1 C_2 s^2 + R_1 C_2 s + 1 + R_2 C_2 s)V_2(s)$$

$$\Rightarrow \frac{V_2(s)}{V_{ab}(s)} = \frac{1}{1 + R_1 C_1 s + R_1 R_2 C_1 C_2 s^2 + R_1 C_2 s + R_2 C_2 s}$$

We are measuring time, hence time constants, in milliseconds - implying frequency is in kHz.

$R_1 C_1 = 1$, $R_1 R_2 C_1 C_2 = 1$, $R_1 C_2 = 1$, $R_2 C_2 = 1$
in this unit.

Accordingly $$\frac{V_2(s)}{V_{ab}(s)} = \frac{1}{1 + s + s^2 + s + s}$$

$$= \frac{1}{1 + 3s + s^2}$$

When a unit step voltage is applied at $V_{ab}(s)$, we have $V_{ab}(s) = \frac{1}{s}$.

Whereupon $$V_2(s) = \frac{1}{s} \cdot \frac{1}{1 + 3s + s^2}$$

$$= \frac{1}{s} + \frac{(\quad)}{1 + 3s + s^2}$$

To find the second term, we subtract

$$\frac{1}{s(1 + 3s + s^2)} - \frac{1}{s} = \frac{1 - (1 + 3s + s^2)}{s(1 + 3s + s^2)}$$

$$= \frac{-(3s + s^2)}{s(1 + 3s + s^2)} = \frac{-(3 + s)}{1 + 3s + s^2}$$

Decompose $-\dfrac{(3+s)}{1+3s+s^2}$: Roots of denominator

$$s^2+3s+1 = 0$$

$$\Rightarrow s = \frac{-3 \pm \sqrt{9-4}}{2}$$

$$= \frac{-3 \pm \sqrt{5}}{2}$$

$$= \underset{(\alpha)}{\frac{-3+\sqrt{5}}{2}}, \underset{(\beta)}{\frac{-3-\sqrt{5}}{2}}.$$

$$-\frac{3+s}{(s-\alpha)(s-\beta)} = -\frac{s-(\alpha+\beta)}{(s-\alpha)(s-\beta)}$$

$$= +\frac{\beta/(\alpha-\beta)}{s-\alpha} + \frac{\alpha/\beta-\alpha}{s-\beta}$$

$$= \frac{1}{\alpha-\beta}\left\{\frac{\beta}{s-\alpha} - \frac{\alpha}{s-\beta}\right\}.$$

Thus

$$v_2(s) = \frac{1}{s} + \frac{1}{(\alpha-\beta)}\left\{\frac{\beta}{s-\alpha} - \frac{\alpha}{s-\beta}\right\}$$

In the time domain, we have

$$v_2(t) = u(t) + \frac{1}{(\alpha-\beta)}\left\{\beta e^{\alpha t} - \alpha e^{\beta t}\right\} u(t).$$

Note that both α and β are negative and hence the second term is a combination of two decaying exponentials.

The current in the capacitor C_2

$$= 1 . \frac{dv_2(t)}{dt} = \delta(t) + \frac{1}{(\alpha-\beta)}\left\{\beta\alpha e^{\alpha t} - \alpha\beta e^{\beta t}\right\} u(t)$$

$$+ \frac{1}{(\alpha-\beta)}\left\{\beta e^{\alpha t} - \alpha e^{\beta t}\right\} \delta(t).$$

$$= \delta(t) + \frac{1}{(\alpha-\beta)} \beta\alpha (e^{\alpha t} - e^{\beta t}) u(t) + \left(\frac{\beta-\alpha}{\alpha-\beta}\right)\delta(t)$$

$$= \delta(t) - \delta(t) + \frac{1}{(\alpha-\beta)} \alpha\beta (e^{\alpha t} - e^{\beta t}) u(t)$$

$$= \frac{\alpha\beta}{\alpha-\beta} (e^{\alpha t} - e^{\beta t}) u(t).$$

Here, again, it is a combination of two decaying exponentials.

Q1- (c) Here we must employ sinusoidal steady state analysis by putting $s = j\Omega_0$, $\Omega_0 = 2\pi \times 1$

$$\left.\frac{V_2(s)}{V_{ab}(s)}\right|_{s=j\Omega_0} = \left.\frac{1}{3s + s^2 + 1}\right|_{s = j2\pi}$$

$$= \frac{1}{[1 - (2\pi)^2] + j3(2\pi)} = \frac{1}{[1-(2\pi)^2] + j6\pi}$$

Steady state sinusoidal voltage across C_2

$$v_2(t) = +\sqrt{\frac{1}{(1-(2\pi)^2)^2 + (6\pi)^2}} \cdot 100 \cos(2\pi t + \phi_0)$$

$$\phi_0 = -\tan^{-1} \frac{6\pi}{1-(2\pi)^2}$$

Steady state sinusoidal current in C_2

$$= 1 \cdot \frac{dv_2(t)}{dt} \text{ which is sinusoidal.}$$

(It is adequate, in part (b), (c) to have indicated this much or equivalent, without actually completing all further numerical work)

Q2 - $$H(z) = \frac{2z^{-1}+3}{(1+2z^{-1})(1-\frac{1}{2}z^{-1})^2}$$

Poles at $z=-2$, $z=\frac{1}{2}$ (repeated of multiplicity 2).

(a) Possible regions of convergence:

$|z|<\frac{1}{2}$, $\frac{1}{2}<|z|<2$, $|z|>2$.

(b) We must decompose $H(z)$ into partial fractions.

$$H(z) = \frac{(3-1)/(1+\frac{1}{4})^2}{1+2z^{-1}} + \frac{(\quad)}{(1-\frac{1}{2}z^{-1})^2}$$

The second term can be found by subtraction.

$$H(z) - \frac{(3-1)/(1+\frac{1}{4})^2}{1+2z^{-1}} = H(z) - \frac{\left(\frac{4}{5}\right)^2 \cdot 2}{1+2z^{-1}}$$

$$= \frac{2z^{-1}+3}{(1+2z^{-1})(1-\frac{1}{2}z^{-1})^2} - \frac{2\times\left(\frac{4}{5}\right)^2}{1+2z^{-1}}$$

$$= \left\{\frac{(2z^{-1}+3)}{(1-\frac{1}{2}z^{-1})^2} - 2\left(\frac{16}{25}\right)\right\}\cdot\frac{1}{1+2z^{-1}}$$

$$= \frac{2z^{-1}+3-2\left(\frac{16}{25}\right)(1-\frac{1}{2}z^{-1})^2}{(1-\frac{1}{2}z^{-1})^2}\cdot\frac{1}{1+2z^{-1}}$$

$$= \frac{2z^{-1} + 3 - \frac{32}{25}\left(1 - z^{-1} + \frac{1}{4}z^{-2}\right)}{\left(1 - \frac{1}{2}z^{-1}\right)^2\left(1 + 2z^{-1}\right)}$$

$$= 2z^{-1} + 3 - \frac{32}{25} + \frac{32}{25}z^{-1} - \frac{8}{25}z^{-2}$$

$$= \frac{\frac{82}{25}z^{-1} + \frac{75-32}{25} - \frac{8}{25}z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)^2\left(1 + 2z^{-1}\right)}$$

$$= \frac{1}{25} \cdot \frac{43 + 82z^{-1} - 8z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)^2\left(1 + 2z^{-1}\right)}$$

$$\begin{array}{r|l}
2z^{-1}+1 \;) & -8z^{-2} + 82z^{-1} + 43 \;(-4z^{-1} + 43 \\
 & -8z^{-2} - 4z^{-1} \\
\hline
 & \qquad\quad 86z^{-1} + 43 \\
 & \qquad\quad 86z^{-1} + 43 \\
\hline
 & \qquad\qquad \times \\
\hline
\end{array}$$

$$= \frac{1}{25} \cdot \frac{-4z^{-1} + 43}{\left(1 - \frac{1}{2}z^{-1}\right)^2}$$

Now we must evaluate the inverse z-transform term by term in each case.

$$H(z) = 2\left(\frac{4}{5}\right)^2 \frac{1}{1+2z^{-1}} + \frac{-4z^{-1}+43}{\left(1-\frac{1}{2}z^{-1}\right)^2}$$

To find the inverse z-transform of

$\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)^2}$, note $\frac{d}{dz}\left(\frac{1}{1-\frac{1}{2}z^{-1}}\right)$

$$= \frac{(-1)\left(-\frac{1}{2}\right)(-1)z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)^2}$$

$$\Rightarrow (-z)\frac{d}{dz}\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)^2} = \frac{\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2}$$

$-z\frac{d}{dz}(\cdot)$ amounts to multiplication by n in time domain.

Let us consider that we have the following inverse z-transforms

$$\frac{1}{1+2z^{-1}} : \begin{array}{ll} |z|>2 : & (-2)^n u[n] \triangleq h_1[n] \\ |z|<2 : & -(-2)^n u[-n-1] \triangleq h_2[n] \end{array}$$

$$\frac{\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2} : \begin{array}{ll} |z|>\frac{1}{2} : & n\cdot\left(\frac{1}{2}\right)^n u[n] \triangleq h_3[n] \\ |z|<\frac{1}{2} : & -n\cdot\left(\frac{1}{2}\right)^n u[-n-1] \triangleq h_4[n] \end{array}$$

Thus we have the following impulse responses :

$$H(z) = 2\left(\frac{4}{5}\right)^2 \cdot \frac{1}{1+2z^{-1}} + (-8)\frac{\left(\frac{1}{2}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)^2}$$

$$+ (86z)\frac{\left(\frac{1}{2}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)^2}$$

System with $|z| < \frac{1}{2}$.

Impulse response

$$= 2\left(\frac{4}{5}\right)^2 \cdot h_2[n] + (-8)\, h_4[n] + 86 \cdot h_4[n+1]$$

The system is neither causal nor stable, since the $h_4[n]$ terms are exponentially growing and are nonzero for $n<0$.

System with $\frac{1}{2} < |z| < 2$

Impulse response

$$= 2\left(\frac{4}{5}\right)^2 h_2[n] + (-8)\, h_3[n] + 86\, h_3[n+1]$$

The system is stable since both $h_2[n]$ and $h_3[n]$ lead to absolutely summable sequences.

However, the system is non causal due to the $h_2[n]$ term.

System with $|z| > 2$

Impulse response

$$= 2\left(\frac{4}{5}\right)^2 h_1[n] + (-8) h_3[n] + 8 h_3[n+1]$$

The system is causal as all terms $h_1[n]$, $h_3[n]$, $h_3[n+1]$ are $0 \; \forall n < 0$

Please note $n u[n]$ is 0 for $n = 0$.

Hence $(n+1) u[n+1] = 0$ for $n = -1$.

However the system is not stable due to the exponentially growing term $h_1[n]$.

Q2- (c) The stable system has $\frac{1}{2} < |z| < 2$.

$$x[n] = (-1)^n \quad \forall n$$

$$= e^{j\pi n} \quad \forall n.$$

This is an eigenfunction of the system with eigenvalue $H(z)\Big|_{z = e^{j\pi} = -1}$

Thus the output would be

$$H(-1) \cdot (-1)^n \quad \forall n$$

$$= \frac{(3-2)}{(1-2)\left(1+\frac{1}{2}\right)^2} \cdot (-1)^n \quad \forall n$$

Q2-(d) The causal system has $|z|>2$.

If the input must produce an absolutely summable output, it must 'destroy' the pole giving an exponentially rising term. Let this input be $x_1[n]$ with Z-transform $X_1(z)$, R_1. The algebraic expression for the Z-transform of the output would be

$$X_1(z)\,H(z) \triangleq Y_1(z)$$

$$X_1(z) \equiv A_0(1+2z^{-1}).$$

Thus $$Y_1(z) = \frac{A_0(2z^{-1}+3)}{(1-0.5z^{-1})^2}$$

$$= \frac{A_0(+4)\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})^2} + 6A_0 z \cdot \frac{\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})^2}$$

Now $$\frac{\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})^2} \xrightarrow[\substack{|z|>2 \\ \text{or} \\ |z|>\frac{1}{2} \\ \text{equivalently}}]{\mathcal{Z}^{-1}} n\left(\frac{1}{2}\right)^n u[n].$$

Thus $$y_1[n] = 4A_0 n\left(\frac{1}{2}\right)^n u[n] + 6A_0(n+1)\left(\frac{1}{2}\right)^{n+1} u[n+1]$$

Both the terms are non-negative for each n.

Thus $$|y_1[n]| = 4|A_0|\left|n\cdot\left(\frac{1}{2}\right)^n u[n]\right| + 6|A_0|\left|(n+1)\cdot\left(\frac{1}{2}\right)^{n+1} u[n+1]\right|$$

Now $\sum_{n \in \mathbb{Z}} |y_1[n]| = \left| \sum_{n \in \mathbb{Z}} |y_1[n]| \, z^{-n} \Big|_{z=1} \right|$

$$= \left| \sum_{n \in \mathbb{Z}} y_1[n] \, z^{-n} \Big|_{z=1} \right|$$

Since $y_1[n]$ is nonnegative (or atleast of the same sign) $\forall n$.

Thus the absolute sum of the output $= |Y_1(1)|$

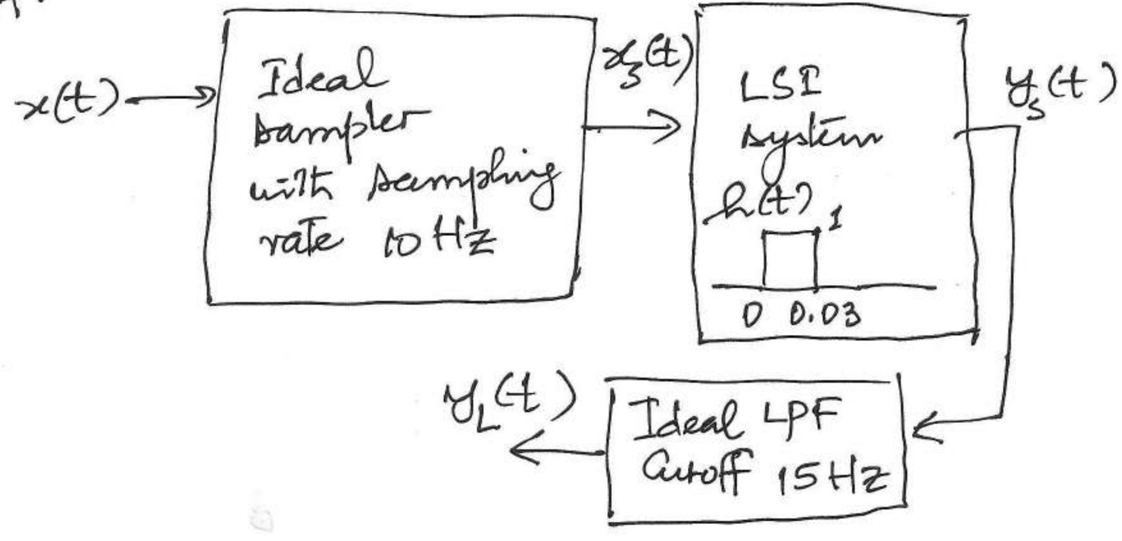
The ratio $\dfrac{\sum_{n \in \mathbb{Z}} |y_1[n]|}{\sum_{n \in \mathbb{Z}} |x_1[n]|} = \dfrac{|Y_1(1)|}{|A_0|(1+2)}$

$$= \frac{|A_0| \cdot \dfrac{2+3}{\left(1-\frac{1}{2}\right)^2}}{|A_0| \, 3} = \frac{5}{\left(\frac{1}{2}\right)^2 3}$$

$$= \frac{5}{3} \times 4 = \frac{20}{3}$$

Q3 -

A.



When $x(t)$ is ideally sampled with a sampling rate of 10 Hz, the spectrum of $x(t)$ is translated to every multiple of 10 Hz and these translated spectra are added. We may regard the whole pattern of these translated and summed spectra to be multiplied by a constant B_0. This gives us the spectrum of $x_s(t)$.

The spectrum of $x_s(t)$ is multiplied by the frequency response of the LSI system with impulse response $h(t)$ to get the spectrum of $y_s(t)$. The spectrum of $y_s(t)$ is retained upto frequencies of magnitude less than or equal to 15 Hz and the rest of the spectrum of $y_s(t)$ is brought to zero, in the spectrum of $y_L(t)$.

$$x(t) = 100\{\cos(2\pi f_1 t) + \cos(2\pi f_2 t) + \cos(2\pi f_3 t)\}$$

where $f_1 = 1$, $f_2 = \frac{3}{2}$, $f_3 = \frac{4}{2} = 2$ Hz.

Note that the time is measured in seconds and hence frequency in Hz.

$x_s(t)$ will, thus have sinusoidal components at all frequencies of the form

$$10k \pm 1, \quad 10k \pm \frac{3}{2}, \quad 10k \pm 2$$

$$k = 1, \cdots, +\infty$$

in addition to the frequencies $1, \frac{3}{2}, 2$.

All of these will be of the same amplitude, since all the components in $x(t)$ are of the same amplitude and the aliases would not overlap with each other or the original spectrum. This is evident from all the original spectral components being $< \frac{10}{2} = 5$ Hz.

Frequency response of the LSI system with impulse response $h(t)$

$$= \int_0^{\tau} e^{-j\Omega t}\, dt = \left.\frac{e^{-j\Omega t}}{-j\Omega}\right|_0^{\tau} ; \quad \tau = 0.03$$

$$= \frac{1 - e^{-j\Omega\tau}}{j\Omega} = \tau e^{-j\Omega\frac{\tau}{2}} \cdot \frac{2j \sin \Omega\frac{\tau}{2}}{j\Omega\tau}$$

$$= \tau . e^{-j\Omega\frac{\tau}{2}} \cdot \frac{2j . \sin \Omega\frac{\tau}{2}}{2j . \Omega\frac{\tau}{2}} = \tau e^{-j\Omega\frac{\tau}{2}} \operatorname{sinc}\left(\frac{\Omega\tau}{2\pi}\right)$$

$$= \tau e^{-j2\pi f\frac{\tau}{2}} . \operatorname{sinc}(f\tau)$$

$y_L(t)$ is thus a linear combination of

$\cos 2\pi t$, $\cos 3\pi t$, $\cos 4\pi t$, $\cos(2\pi \times 9t)$,
$\cos\left(2\pi \times \frac{17}{2}t\right)$, $\cos(2\pi \times 8t)$, $\cos(2\pi \times 11t)$,
$\cos\left(2\pi \times \frac{23}{2}t\right)$, $\cos(2\pi \times 12t)$

since frequencies > 15 Hz are removed.
The amplitudes of these components will be in the ratio $|\text{sinc } f\tau|$, $\tau = 0.03$, f = Hz frequency of component.

The phase of $-\pi f\tau$ (radians), $\tau = 0.03$ is added in the frequency component f, along with an additional phase of π in case $\text{sinc}(f\tau)$ is negative at that frequency.
$\text{sinc}(f\tau)$ would be negative for $1 < f\tau < 2$

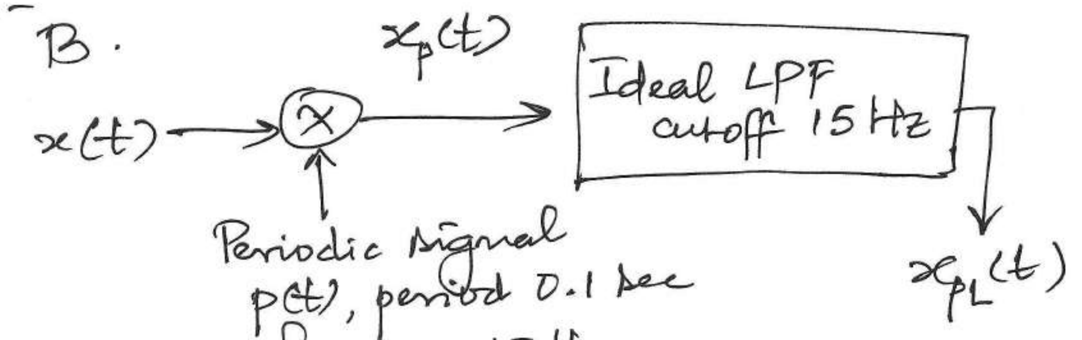
or $\frac{1}{\tau} < f < \frac{2}{\tau}$ or $\frac{1}{0.03} < f < \frac{2}{0.03}$

But none of the frequencies upto 12 Hz are in this range. So only a phase of $-\pi f\tau$ radians is added.

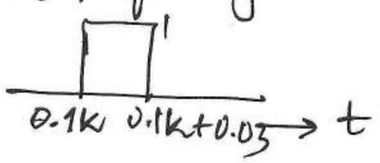
(This reasoning is to be checked in evaluation, without being too fastidious in nitty-gritties!)

Practical significance of experiment A:
We are doing what is practically known as sample and hold, but the 'hold' operates for only a part of the sample interval.

Q3 – B.



$\Rightarrow$ frequency 10 Hz



The spectrum of $x(t)$ is convolved with the spectrum of periodic signal $p(t)$; which latter is a train of spectral impulses with strengths proportional to the Fourier components of $p(t)$; located at all multiples of the fundamental frequency 10 Hz.

$x_{pL}(t)$ cuts off all spectral components above 15 Hz. Thus all spectral impulses in the spectrum of $p(t)$, beyond the fundamental at 10 Hz and the zero frequency impulse are irrelevant at $x_{pL}(t)$.

Zero frequency spectral component $= \frac{1}{0.1}\int_0^{0.03} dt = \frac{0.03}{0.1}$
$= \underline{0.3}$

Fundamental spectral component

$$= \frac{1}{0.1}\int_0^{0.03} e^{-j\frac{2\pi}{0.1}t}\,dt = \frac{1}{0.1}\left.\frac{e^{-j\frac{2\pi}{0.1}t}}{-j\frac{2\pi}{0.1}}\right|_0^{0.03}$$

$$= 10.\ \frac{1-e^{-j\frac{2\pi}{0.1}0.03}}{j\cdot 10(2\pi)} = \frac{\cancel{10}\, e^{-j\frac{2\pi}{0.1}\frac{0.03}{2}}\left(2\cancel{j}\sin\frac{\cancel{2}\pi\times 0.03}{0.1\times\cancel{2}}\right)}{\cancel{j}\cdot\cancel{10}(\cancel{2}\pi)}$$

$=$

$$= \frac{1}{\pi} e^{-j(0.3\pi)} \sin(0.3\pi)$$

Thus $x_{pL}(t)$ is also a linear combination of the same sinusoidal components that make up $x_L(t)$ in A.

The amplitudes and phases are different; though, $\cos 2\pi t$, $\cos 3\pi t$, $\cos 4\pi t$ have an amplitude proportional to 0.3.

$\cos(2\pi \times 9t)$, $\cos(2\pi \times \frac{17}{2} t)$, $\cos(2\pi \times 8t)$, $\cos(2\pi \times 11t)$, $\cos(2\pi \times \frac{23}{2} t)$, $\cos(2\pi \times 12t)$ all have an amplitude proportional to $\frac{\sin 0.3\pi}{\pi}$ and a phase of -0.3π radians uniformly added.

Note that, both in configuration A and in configuration B, the same spectral components appear at the output, but the relative amplitudes and phases are quite different.

Practical significance of configuration B:
This represents natural sampling with realistic pulses instead of impulses in ideal sampling.

Q4-

(a) If $x_2(t)$ has the Fourier Transform $X_2(j\Omega)$,

then $X_2(j\Omega) = \int_{-\infty}^{+\infty} x_2(t) e^{-j\Omega t} dt$

$$\overline{X_2(j\Omega)} = \int_{-\infty}^{+\infty} \overline{x_2(t)}\, \overline{e^{-j\Omega t}}\, dt$$

$$= \int_{-\infty}^{+\infty} \overline{x_2(t)}\, e^{j\Omega t} dt = \int_{-\infty}^{+\infty} \overline{x_2(t)}\, e^{-j\Omega(-t)} dt$$

Put $-t = \lambda$ $\quad d\lambda = -dt$

$$\overline{X_2(j\Omega)} = \int_{-\infty}^{+\infty} \overline{x_2(-\lambda)}\, e^{-j\Omega\lambda} d\lambda$$

Thus $\overline{x_2(-t)}$ has the Fourier Transform $\overline{X_2(j\Omega)}$.

In this case $\overline{x_2(-t)} = e^{5t} u(-t)$.

When we multiply two Fourier Transforms, the corresponding signals are convolved. Let us not forget the '2π' factor.

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X_1(j\Omega)\overline{X_2(j\Omega)} e^{j\Omega t} d\Omega = x_1(t) * \overline{x_2(-t)}$$

$$= \int_{-\infty}^{+\infty} x_1(\lambda) \overline{x_2(-(t-\lambda))}\, d\lambda$$

$$= \int_{-\infty}^{+\infty} x_1(\lambda) \cdot \overline{x_2(\lambda - t)}\, d\lambda = \int_{-\infty}^{+\infty} e^{-2\lambda} u(\lambda) \cdot e^{-5(\lambda - t)} u(\lambda - t)\, d\lambda$$

$$= \int_{-\infty}^{+\infty} e^{-2\lambda} e^{-5\lambda} e^{5t} u(\lambda) u(\lambda - t)\, d\lambda$$

$$\left\{ u(\lambda - t) = 1 \text{ for } \lambda - t \geq 0 \Rightarrow t \leq \lambda \right\}$$

$$= e^{5t} \int_{-\infty}^{+\infty} e^{-7\lambda} u(\lambda) u(\lambda - t)\, d\lambda$$

$$= e^{5t} \int_{\max(0,t)}^{+\infty} e^{-7\lambda}\, d\lambda = e^{5t} \left(\frac{e^{-7\lambda}}{-7} \right)_{\max(0,t)}^{+\infty}$$

$$= e^{5t} \frac{e^{-7t_1}}{7} \Bigg|_{t_1 \leftarrow \max(0,t)}$$

Thus if $t < 0$, $t_1 = 0$

whereupon we have $\frac{1}{7} e^{5t}$

If $t > 0$, $t_1 = t$ whereupon we have $\frac{e^{5t} e^{-7t}}{7}$

$$= \frac{1}{7} e^{-2t}$$

The required inverse Fourier Transform is

$$\frac{1}{7} \left\{ e^{-2t} u(t) + e^{5t} u(-t) \right\}$$

Q4 (b) $\frac{1}{2\pi}\int_{-\infty}^{+\infty} X_1(j\Omega) X_2(j\Omega) e^{j\Omega t} d\Omega$

$$= x_1(t) * x_2(t)$$

$$= \int_{-\infty}^{+\infty} x_1(\lambda) x_2(t-\lambda) d\lambda$$

$$= \int_{-\infty}^{+\infty} e^{-2\lambda} u(\lambda) e^{-5(t-\lambda)} u(t-\lambda) d\lambda$$

$$= \left\{ \int_0^t e^{-2\lambda} e^{-5t} e^{5\lambda} d\lambda \right\} u(t)$$

$$= e^{-5t} \int_0^t e^{3\lambda} d\lambda \; u(t)$$

$$= e^{-5t} \left. \frac{e^{3\lambda}}{3} \right|_0^t u(t)$$

$$= \frac{1}{3} e^{-5t} (e^{3t} - 1) u(t) = \frac{1}{3}(e^{-2t} - e^{-5t}) u(t)$$

(c) $|X_1(j\Omega)|^2 = X_1(j\Omega) \overline{X_1(j\Omega)}$

The inverse Fourier Transform would be

$$x_1(t) * \overline{x_1(-t)}$$

$$= \int_{-\infty}^{+\infty} x_1(\lambda) . \overline{x_1(-(t-\lambda))} d\lambda$$

$$= \int_{-\infty}^{+\infty} e^{-2\lambda} u(\lambda) \cdot e^{-2(\lambda - t)} u(\lambda - t) d\lambda$$

$$= e^{2t} \int_{\max(0,t)}^{+\infty} e^{-2\lambda} \cdot e^{-2\lambda} d\lambda$$

$$= e^{2t} \cdot \int_{\max(0,t)}^{+\infty} e^{-4\lambda} d\lambda = e^{2t} \left. \frac{e^{-4\lambda}}{-4} \right|_{\max(0,t)}^{+\infty}$$

$$= \frac{1}{4} e^{2t} e^{-4(\max(0,t))}$$

$$= \frac{1}{4} \cdot e^{2t} \left\{ e^{-4t} u(t) + e^{0} u(-t) \right\}$$

$$= \frac{1}{4} e^{2t} \left\{ e^{-4t} u(t) + u(-t) \right\}$$

$$= \frac{1}{4} \left\{ e^{-2t} u(t) + e^{2t} u(-t) \right\}$$

In using this expression, we have to be cautions at $t = 0$. We take only one of the two terms when we evaluate the integral.

Q4(d) $\Omega^2 = -(j\Omega)^2 X_1(j\Omega) X_2(j\Omega)$

The inverse Fourier Transform of this is

$$-\frac{d^2}{dt^2} \left\{ \text{Inverse Fourier Transform of } X_1(j\Omega) X_2(j\Omega) \right\}$$

which is

$$-\frac{d^2}{dt^2}\left\{\frac{1}{3}\left(e^{-2t}-e^{-5t}\right)u(t)\right\}$$

$$= -\frac{d}{dt}\left\{\frac{1}{3}\left(-2e^{-2t}+5e^{-5t}\right)u(t) + \frac{1}{3}\left(e^{-2t}-e^{-5t}\right)\delta(t)\right\}$$

$$= -\frac{d}{dt}\left\{\frac{1}{3}\left(-2e^{-2t}+5e^{-5t}\right)u(t)\right\}$$

$$= -\frac{1}{3}\left\{+4e^{-2t}-25e^{-5t}\right\}u(t) - \frac{1}{3}\left(-2e^{-2t}+5e^{-5t}\right)\delta(t)$$

$$= \left(\frac{25}{3}e^{-5t}-\frac{4}{3}e^{-2t}\right)u(t)-\delta(t).$$

Q4 – (e) $\displaystyle\int_{-\infty}^{+\infty} X_1(j\Omega)\overline{X_2(j\Omega)}\,d\Omega = 2\pi\int_{-\infty}^{+\infty} x_1(t)\overline{x_2(t)}\,dt$

(Parseval's theorem)

$$= 2\pi\int_0^{\infty} e^{-2t}e^{-5t}\,dt = 2\pi\frac{e^{-7t}}{-7}\Big|_0^{\infty}$$

$$= 2\pi/7$$

Also $2\pi\left(x_1(t) * \overline{x_2(-t)}\right)\Big|_{t=0} = 2\pi/7$

from Q4(a).

Q4 (f) $\int_{-\infty}^{+\infty} |X_1(j\Omega)|^2 \overline{X_2(j\Omega)}\, d\Omega$

$$= 2\pi \int_{-\infty}^{+\infty} \frac{1}{4}\left(e^{-2t}u(t) + e^{2t}u(-t)\right) e^{-5t}u(t)\, dt$$

from Parseval's theorem and Q4(c)

$$= \frac{2\pi}{4}\int_{0}^{\infty} e^{-2t}e^{-5t}\, dt = \frac{2\pi}{4}\int_{0}^{\infty} e^{-7t}\, dt$$

$$= \frac{2\pi}{4}\left.\frac{e^{-7t}}{-7}\right|_0^{\infty} = \frac{2\pi}{4\times 7} = \frac{\pi}{14}$$