

A Crash Course on Data Compression

4. Statistical Compressors

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Overview

- Shannon-Fano
- Huffman, Canonical Huffman
- Arithmetic
- Asymmetric Numeral Systems

The *Statistical* Coding Problem

- **Problem.** We are given a list $L[1..n]$ of n symbols and we are asked to compress it in as few as possible bits.
- Without loss of generality, we are going to assume that the symbols are positive integers.
- **Idea.** Build a *statistical model* for L and assign codewords *based on the model*.
- **Simplest statistical model.** Occurrences of each *distinct* symbol $\langle s_1, \dots, s_m \rangle$ in L . We will refer to these occurrences as the *weights* $\langle w_1, \dots, w_m \rangle$ for the symbols $\langle s_1, \dots, s_m \rangle$. It follows that $\sum_{i=1}^m w_i = n$.

Example for $L = [1, 3, 1, 1, 1, 5, 2, 1, 7, 3, 1, 2, 1, 1, 1, 1]$. We have $n = 16$ and $m = 5$.
The distinct symbols $\langle s_1, \dots, s_5 \rangle$ are $\langle 1, 2, 3, 5, 7 \rangle$ and the weights $\langle w_1, \dots, w_5 \rangle$ are $\langle 10, 2, 2, 1, 1 \rangle$.

The *Minimum-Redundancy* Coding Problem

- **Problem.** We are given a list $L[1..n]$ of n symbols with weights $W = \langle w_1, \dots, w_m \rangle$, where $m \leq n$ and $\sum_{i=1}^m w_i = n$. We are asked to determine a set of *codeword lengths* $T = \langle \ell_1, \dots, \ell_m \rangle$ such that:

- (1) $\sum_{i=1}^m 2^{-\ell_i} \leq 1$ (Kraft-McMillan inequality compliancy)
- (2) the cost $C(W, T) = \sum_{i=1}^m (w_i \cdot \ell_i)$ bits of the coded L is *minimum*.

We will refer to the set of codeword lengths $T = \langle \ell_i \rangle$ as the *code*. A code T satisfying **(1)** and **(2)** above is said to be an optimal or *minimum-redundancy* code for W , that is:
 $C(W, T) \leq C(W, T')$ for any other code $T' = \langle \ell'_i \rangle$.

- Once the set T has been determined, it is easy to assign *prefix-free* codewords.

Assigning *Canonical* Prefix-Free Codewords

- Given the set of codeword lengths $\langle \ell_1, \dots, \ell_m \rangle$, in non-decreasing order:
- Algorithm.**
 - The first codeword is $C_1 = 0^{\ell_1}$. Now let $\ell = \ell_1$.
 - For all $i = 2, \dots, m$:
 - Let C_i be the smallest unassigned *lexicographic* codewords of ℓ bits (the one “coming after” C_{i-1}). If $\ell_i = \ell$, then return C_i . Otherwise ($\ell_i > \ell$), C_i is padded with possible 0s to the right until a codeword of ℓ_i is obtained.
 - Set $\ell = \ell_i$.
- This assignment is also known as *canonical* because it allows fast encoding/decoding using simple tables.
(Note that the codewords are *lexicographically* sorted.)

Example for $\langle \ell_i \rangle = \langle 1, 2, 4, 4, 4, 4 \rangle$.

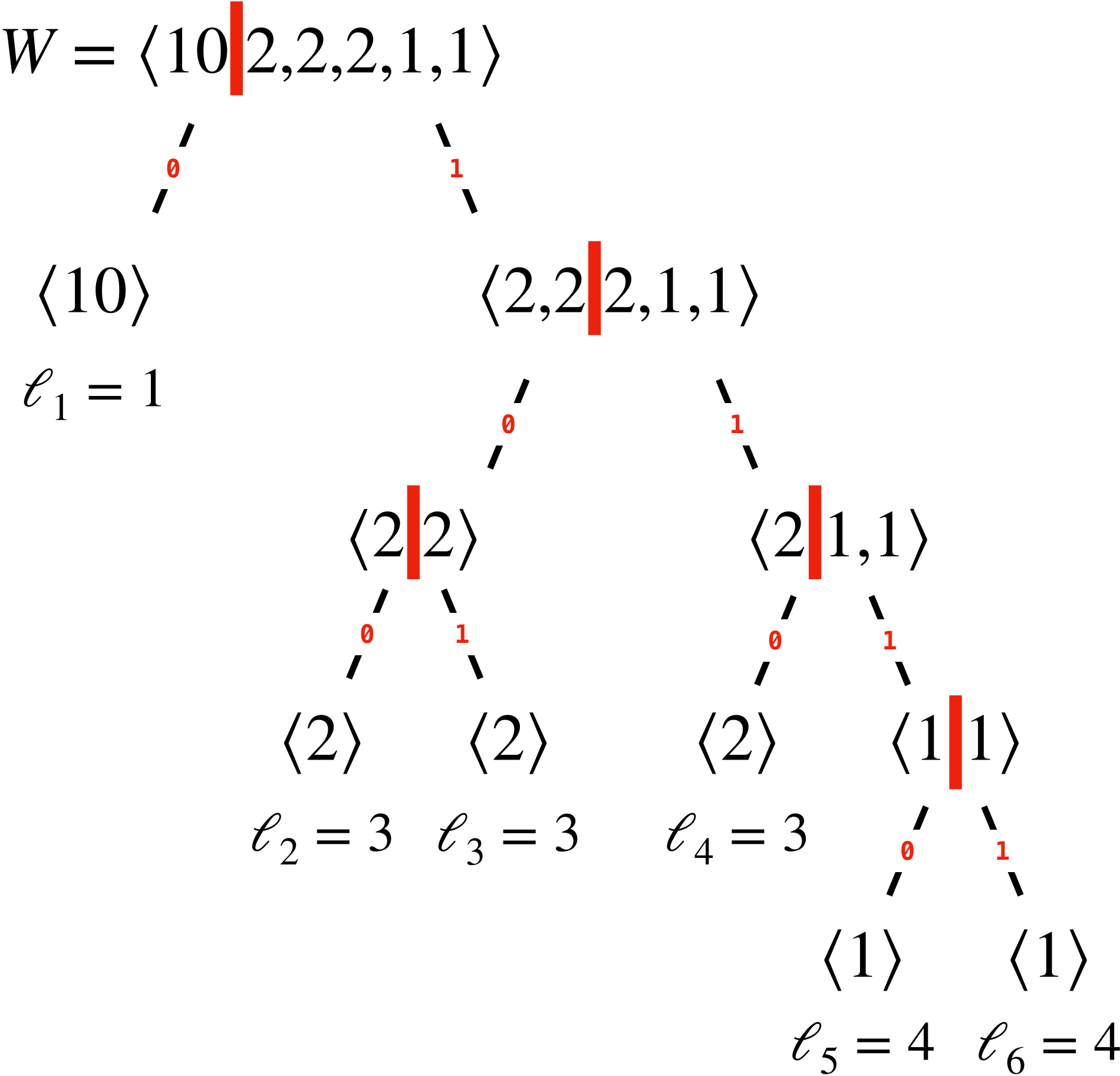
ℓ_i	C_i
1	0
2	10
4	1100
4	1101
4	1110
4	1111

Shannon-Fano

Shannon, 1949 — Fano, 1949

- A greedy, top-down, algorithm to compute a set of codeword lengths.
- It does *not always* produce an optimal prefix-free code.
- Remember the “golden rule” of data compression: assign shorter codewords to more frequent symbols.
- **Algorithm.**
 - Sort the set $\langle w_i \rangle$ in non-increasing order and partition it into two sets $\langle w'_i \rangle$ and $\langle w''_i \rangle$, of size m' and $m - m'$ respectively, so that $\sum_{i=1}^{m'} w'_i \approx \sum_{i=m'+1}^{m-m'} w''_i$.
 - Proceed recursively on each of the two partitions. Stop when the partition size is 1.
- Informally: the codeword length ℓ_i for the weight w_i is the number of partitions that contain w_i .

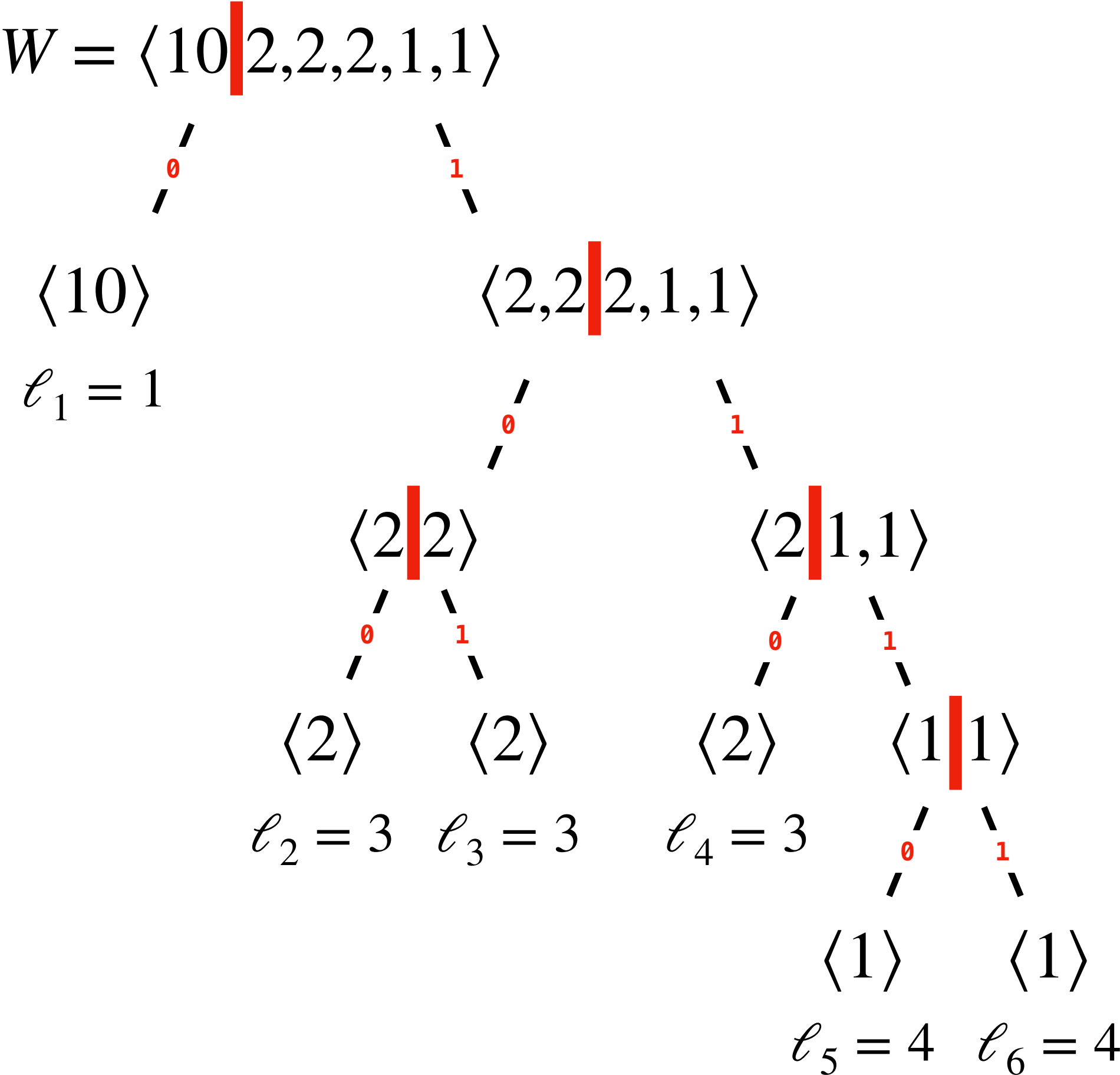
Shannon-Fano — Example



$$\langle \ell_i \rangle = \langle 1, 3, 3, 3, 4, 4 \rangle$$

ℓ_i	C_i
1	0
3	100
3	101
3	110
4	1110
4	1111

Shannon-Fano — Example



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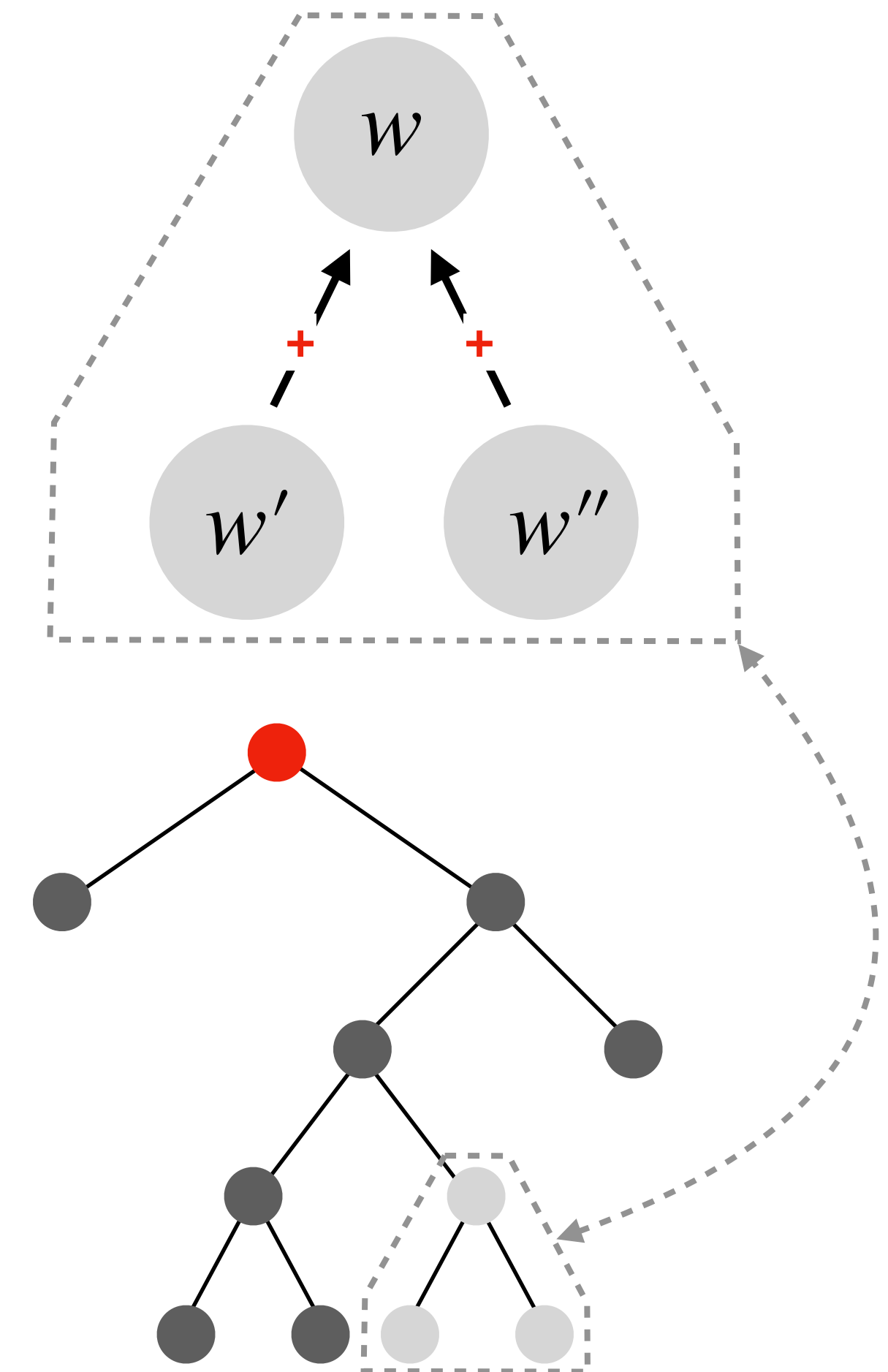
ℓ_i	C_i
1	0
3	100
3	101
3	110
4	1110
4	1111

- Because of the approximate partitioning rule, the resulting code may not be optimal.
- It is easy to see why the algorithm produces a *prefix-free* code: each symbol is associated to a distinct leaf and two siblings are distinguished by two different symbols, 0 and 1.

Huffman

Huffman, 1952

- A greedy, bottom-up, algorithm to compute a set of codeword lengths. It always produces an *optimal* prefix-free code.
- Invented by Huffman in 1951 at MIT, while attending a class taught by Fano.
- **Algorithm.**
 - Maintain a set of weights S . At the beginning: $S = W = \langle w_i \rangle$.
 - At each step: two smallest weights are selected from S , say w' and w'' , and a new weight $w = w' + w''$ is added to S . The sequence of merging operations implicitly defines parent-to-children relationships that it is handy to model as a binary tree.
 - Repeat until $|S| = 1$: at this point the only weight in S is n and it is logically associated to the **root** of the Huffman tree. The tree is used to derive the codeword lengths (and the actual codewords if wanted).



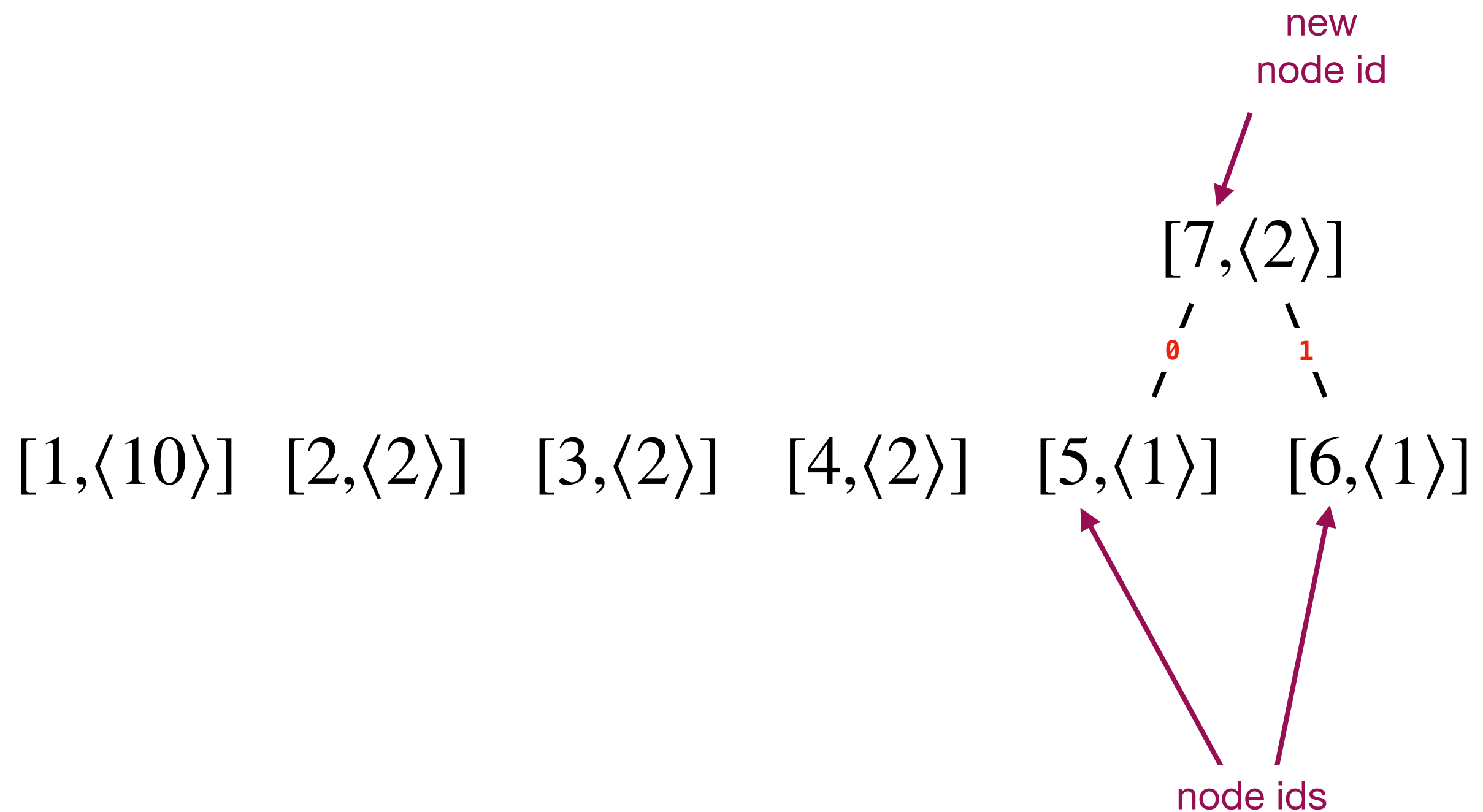
Huffman — Demo

$[1, \langle 10 \rangle]$ $[2, \langle 2 \rangle]$ $[3, \langle 2 \rangle]$ $[4, \langle 2 \rangle]$ $[5, \langle 1 \rangle]$ $[6, \langle 1 \rangle]$ $S = \{[1, \langle 10 \rangle], [2, \langle 2 \rangle], [3, \langle 2 \rangle], [4, \langle 2 \rangle], [5, \langle 1 \rangle], [6, \langle 1 \rangle]\}$

node ids



Huffman — Demo

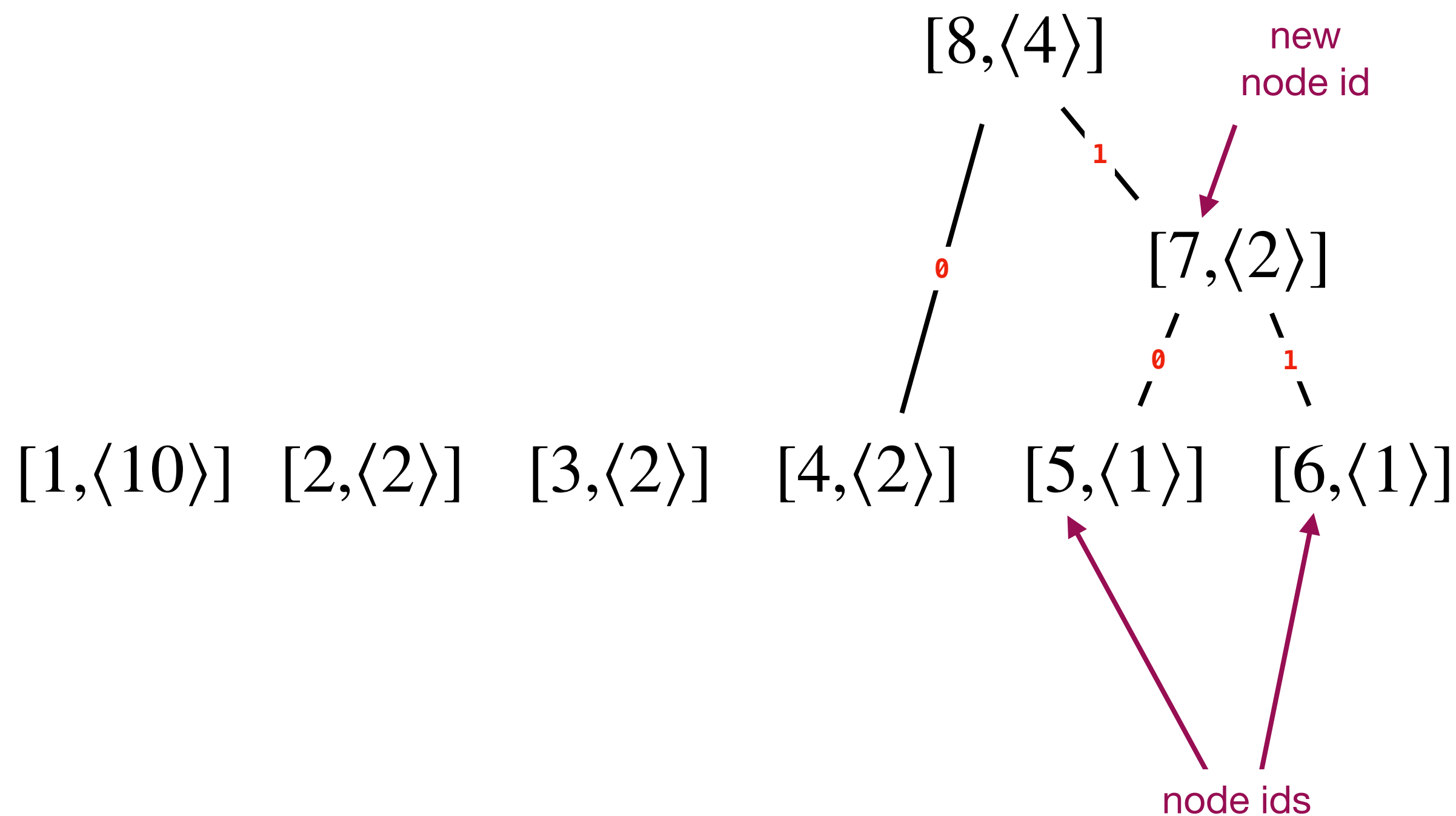


children of node 7

$S = \{[1, \langle 10 \rangle], [2, \langle 2 \rangle], [3, \langle 2 \rangle], [4, \langle 2 \rangle], [7, \langle 2 \rangle], (5, 6)]\}$

$S = \{[1, \langle 10 \rangle], [2, \langle 2 \rangle], [3, \langle 2 \rangle], [4, \langle 2 \rangle], [5, \langle 1 \rangle], [6, \langle 1 \rangle]]\}$

Huffman — Demo

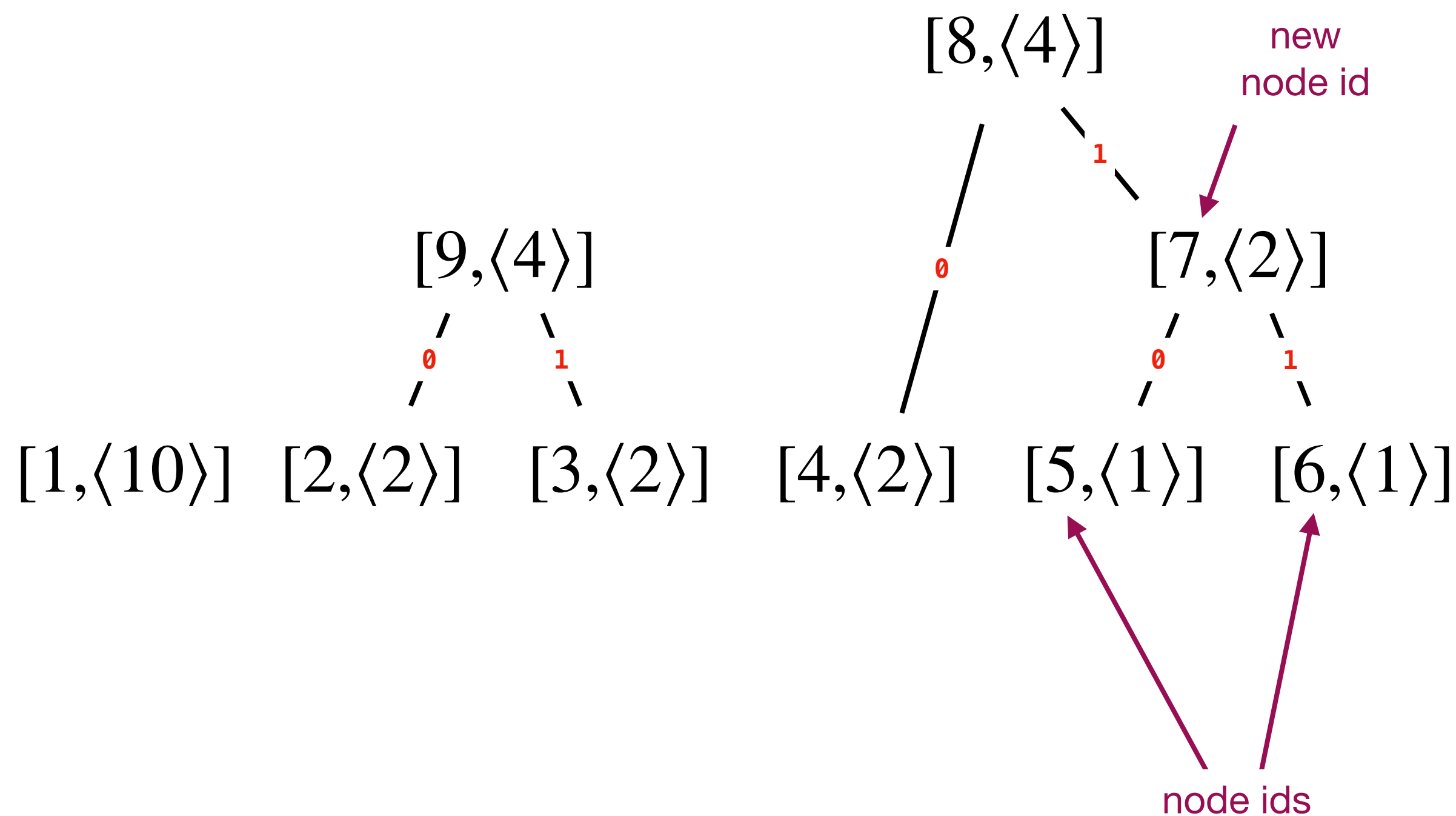


$S = \{[1, \langle 10 \rangle], [2, \langle 2 \rangle], [3, \langle 2 \rangle], [8, \langle 4 \rangle], (4, 7)]\}$

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Huffman — Demo



$$S = \{[1, \langle 10 \rangle], [9, \langle 4 \rangle, (2, 3)], [8, \langle 4 \rangle, (4, 7)]\}$$

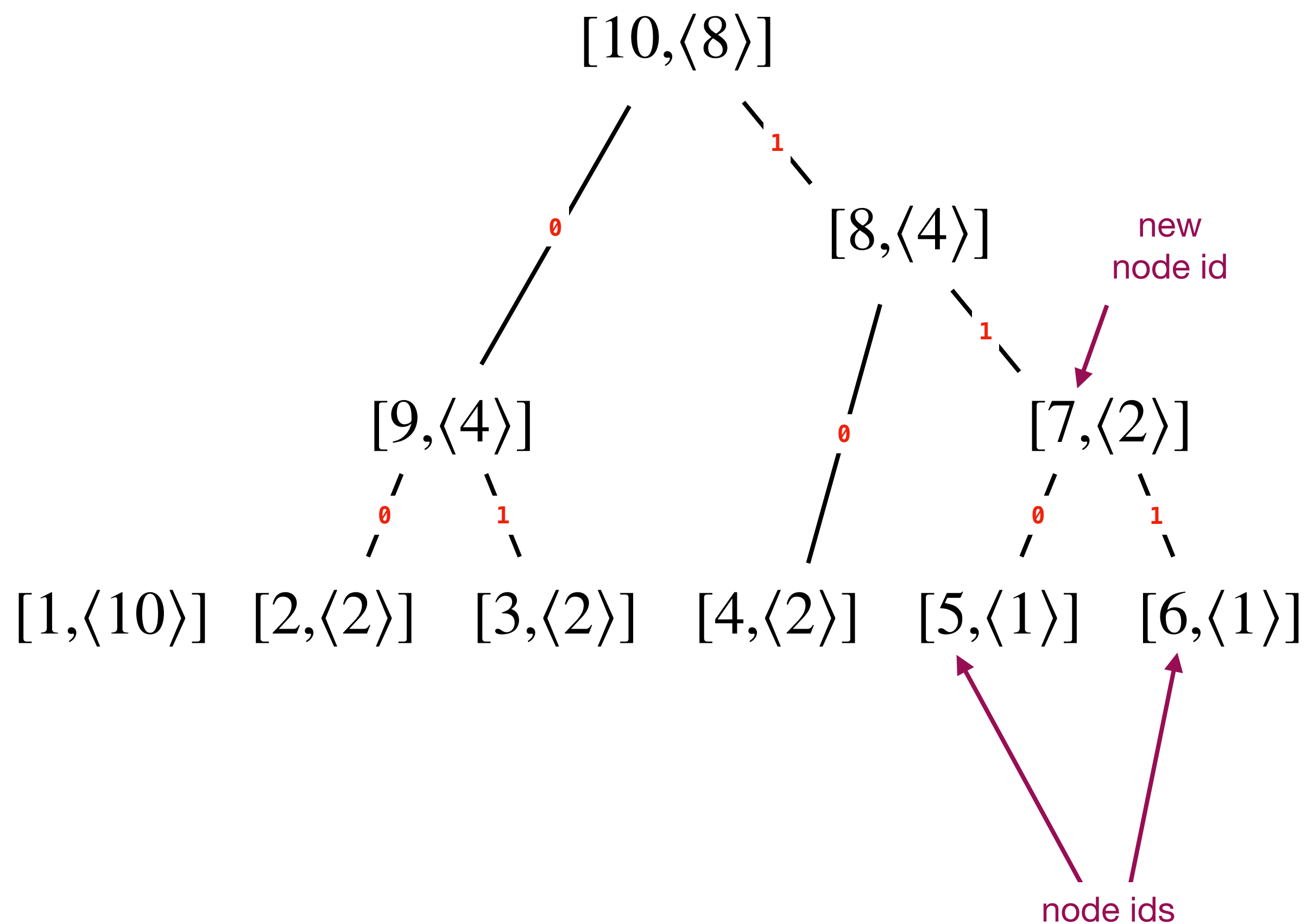
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Huffman — Demo



$$S = \{[1, \langle 10 \rangle], [10, \langle 8 \rangle], (9, 8)]\}$$

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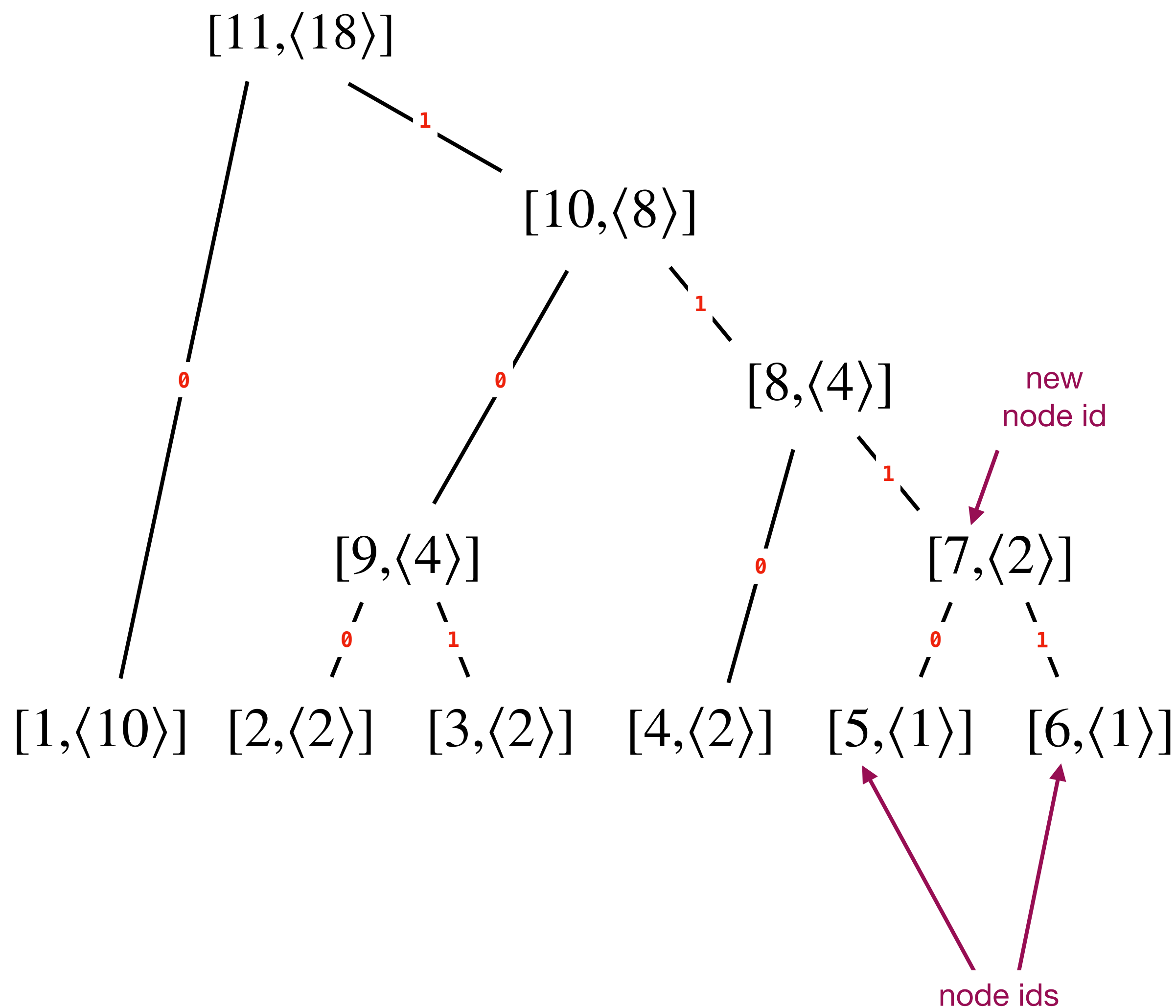
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children of
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Huffman — Demo



$$S = \{[11, \langle 18 \rangle], (1, 10)\}$$

$$S = \{[1, \langle 10 \rangle], [10, \langle 8 \rangle], (9, 8)\}$$

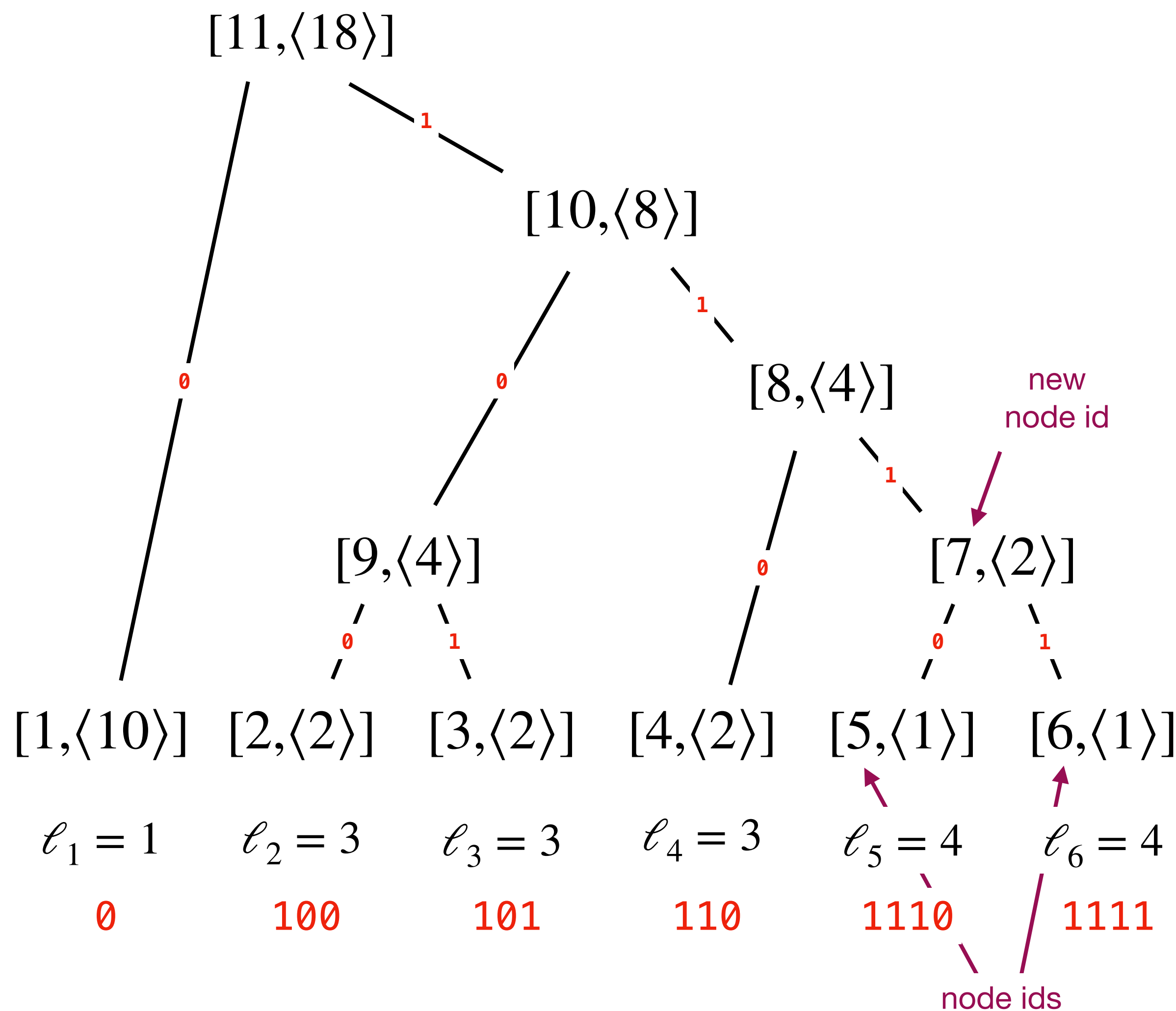
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Huffman — Demo



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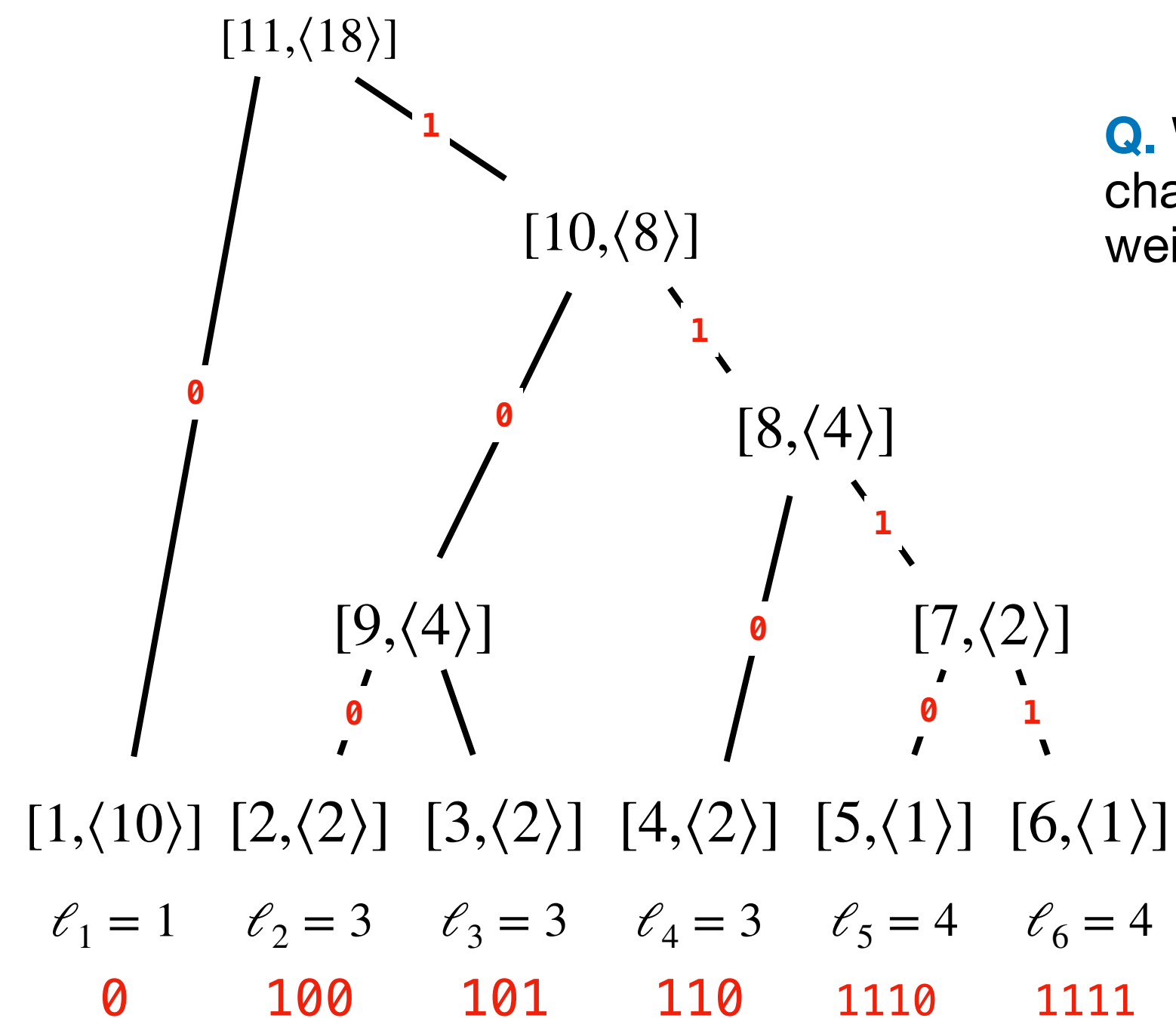
$$S = \{[1, \langle 10 \rangle], [2, \langle 2 \rangle], [3, \langle 2 \rangle], [4, \langle 2 \rangle], [5, \langle 1 \rangle], [6, \langle 1 \rangle]\}$$

← codeword lengths

← actual codewords

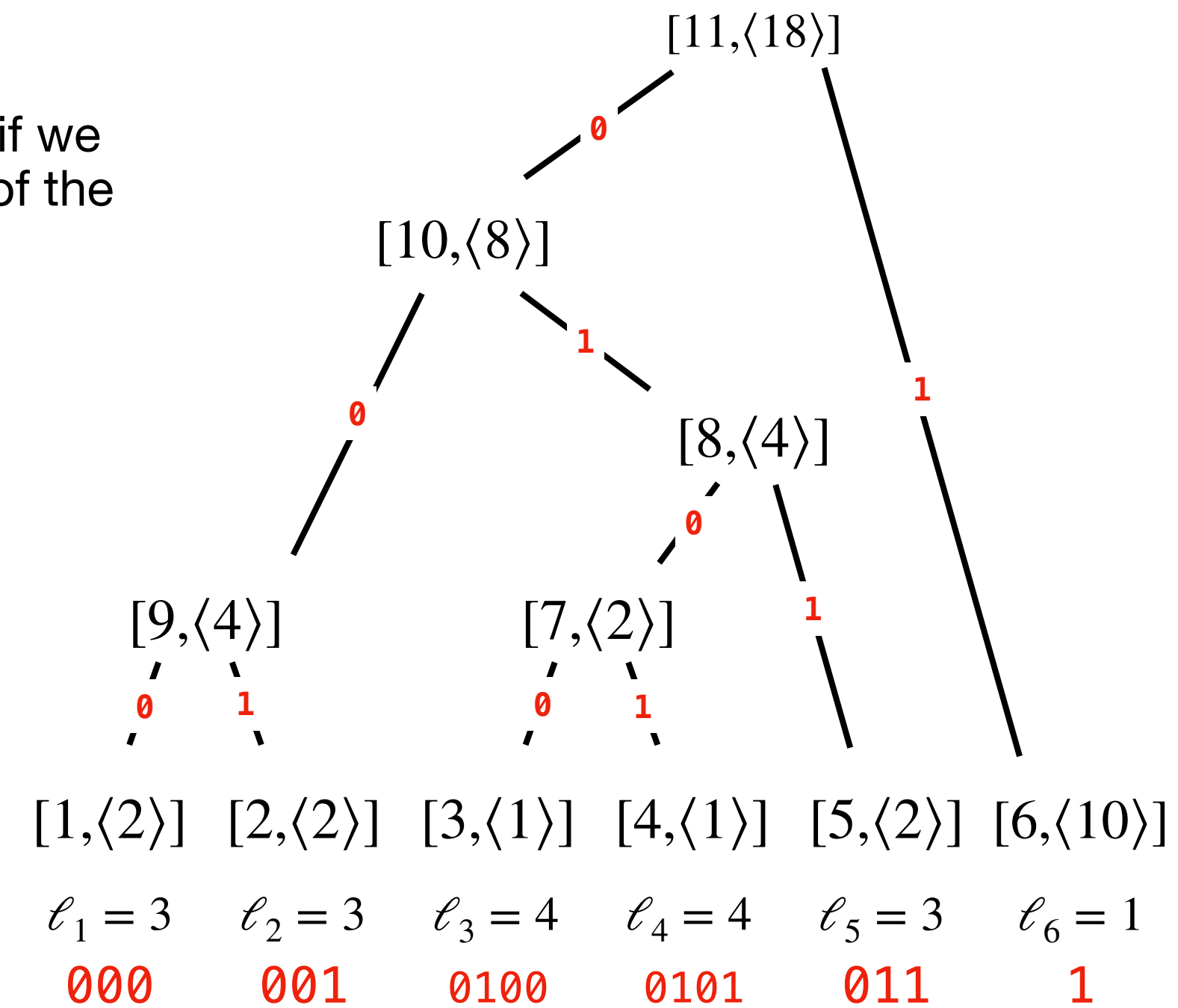
children of
node 7

Huffman — Different Orderings of the Weights



(a) **Sorted** Weights

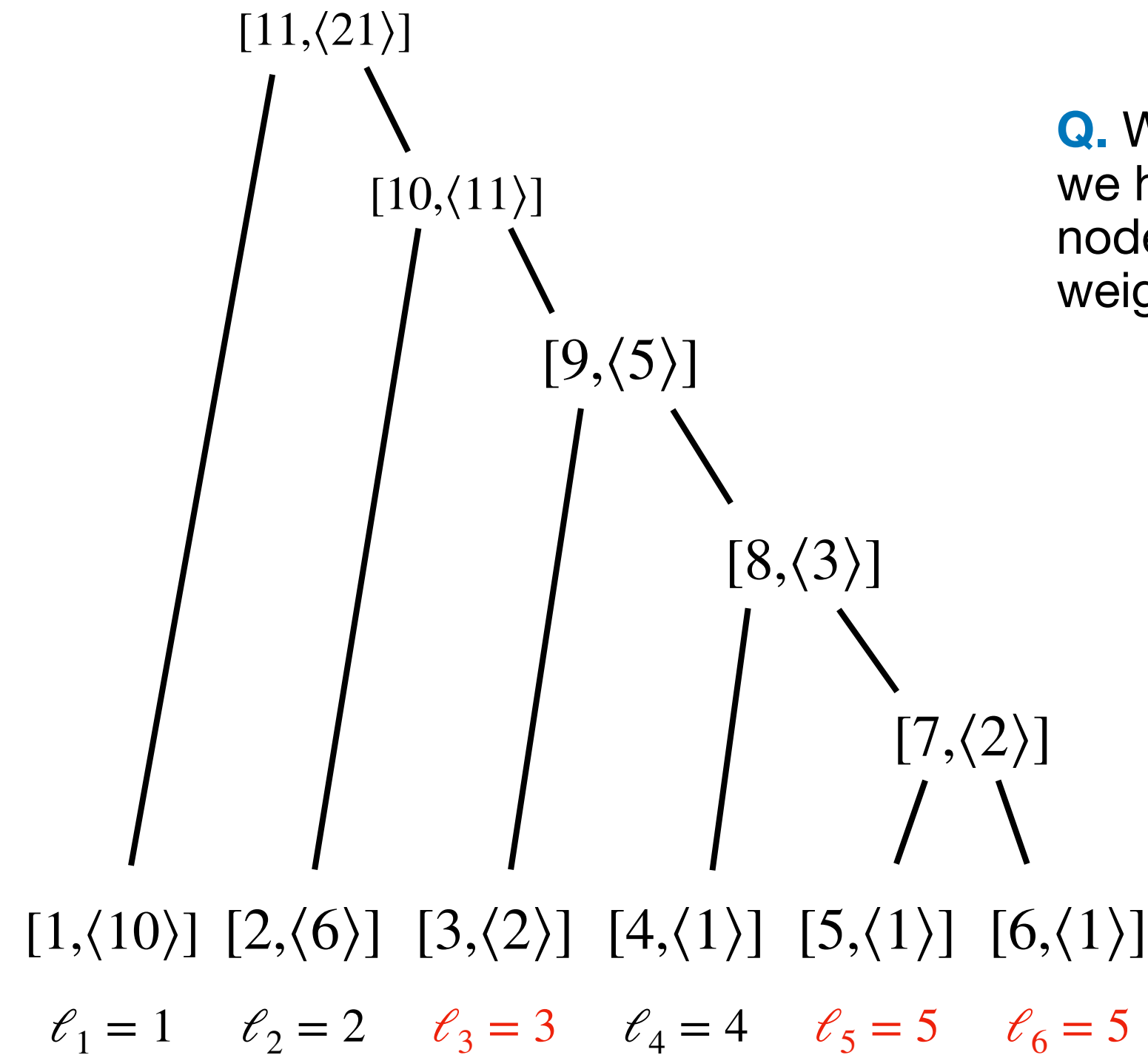
Q. What happens if we change the order of the weights?



(b) **Unsorted** Weights

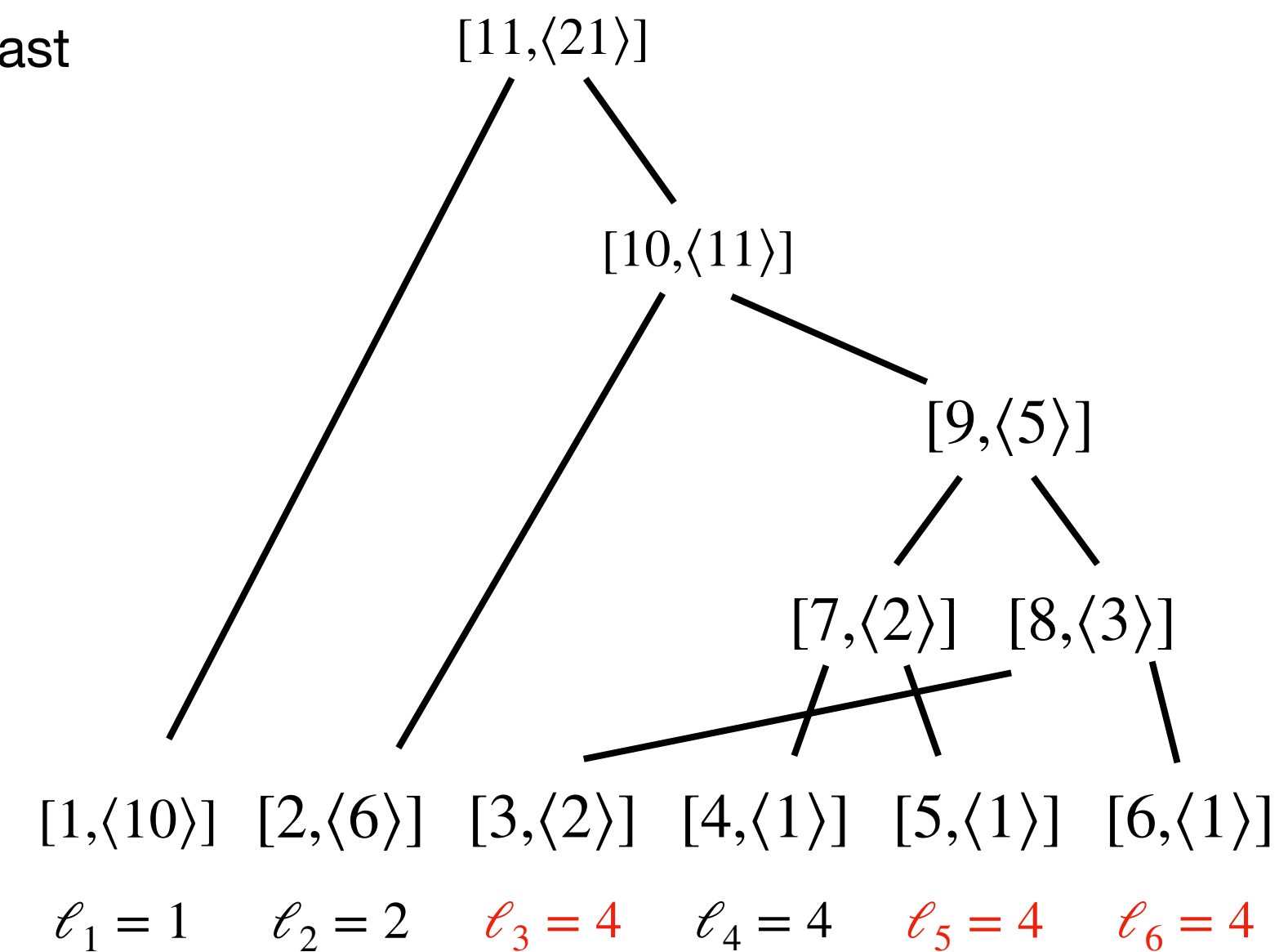
- Same codeword lengths, but different actual codewords.
- Labelling *left*-leaning edges with a 0 and *right*-leaning edges with a 1, we have that: different orderings of the weights can produce *different* sets of *valid* codewords.

Huffman — Different Merging Strategies



(a) **Higher node ids** first

Q. What happens when we have more than two nodes with the same least weight?



(b) **“Older” nodes first** (created farthest in the past)

- Same *average* codeword length, but different *maximum* codeword length.
- Different merging strategies can produce different sets of codeword lengths.
- In the example, we have $\langle 1, 2, 3, 4, 5, 5 \rangle$ in (a) and $\langle 1, 2, 4, 4, 4, 4 \rangle$ in (b). In both cases the average codeword length is 2.

if node x has a smaller id than node y, then it was created before y

Canonical Huffman

- If the weights are *sorted in non-increasing order* and we label *left*-leaning edges with a 0 and *right*-leaning edges with a 1, then we always obtain *lexicographic sorted codewords*.
- We refer to this construction as the *canonical* Huffman code.
- Also, we can always merge nodes that were created farthest in the past to *minimize the maximum codeword length* — a good feature for faster decoding in practice.

Shannon-Fano vs. Huffman

Shannon-Fano (SF) and Huffman (H) codes built for the 26-letter English alphabet. In this case, Shannon-Fano has an average codeword length of 4.16677 bits; Huffman is optimal with 4.15506 bits.

- Both algorithms build prefix-free codes in a greedy manner.
- Shannon-Fano:
 - top-down (divisive)
 - not optimal
- Huffman:
 - bottom-up (aggregative)
 - *optimal*

(a)				(b)			
symbol	probability	SF	H	symbol	probability	SF	H
A	0.08833	010	1111	N	0.06498	1000	1001
B	0.01267	111110	100000	O	0.07245	0110	1011
C	0.02081	111010	00000	P	0.02575	11010	01000
D	0.04376	10111	11101	Q	0.00080	11111111110	000101010
E	0.14878	000	110	R	0.06872	0111	1010
F	0.02455	11011	00011	S	0.05537	10110	0101
G	0.01521	111100	100001	T	0.09351	001	001
H	0.05831	1001	0111	U	0.02762	11001	01001
I	0.05644	1010	0110	V	0.01160	1111110	000100
J	0.00080	1111111110	0001010111	W	0.01868	111011	100011
K	0.00867	11111110	0001011	X	0.00146	111111110	00010100
L	0.04124	11000	11100	Y	0.01521	111101	100010
M	0.02361	11100	00001	Z	0.00053	11111111111	0001010110

Probabilities taken from page 174 of:
David Salomon. 2007. *Variable-Length Codes for Data Compression*.
Springer Science & Business Media, ISBN 978-1-84628-959-0.

Huffman — Optimality

- **Optimality.** The Huffman algorithm produces a minimum-redundancy prefix-free code $Huf_m = \langle \ell_i \rangle_m$ for the weights $W_m = \langle w_i \rangle_m$.
- The proof builds on two ingredients:
 1. **Max-Depth Leaves Property.** If a code T is optimal, then its binary tree must contain two least-weight leaves that are children of the same parent (siblings), hence at the same maximum depth D in the tree.
 2. Inductive process on the number m of weights.
 - 2.1 Base case for $m = 2$ is always valid since the trivial optimal code is $\langle \ell_1 = 1, \ell_2 = 1 \rangle$ (and the two codewords will always be 0 and 1).
 - 2.2 Assume Huf_{m-1} is optimal for W_{m-1} and prove Huf_m is optimal for W_m .

Huffman — Max-Depth Leaves Property

- **Max-Depth Leaves Property.** If the code T is optimal, then its binary tree must contain two least-weight leaves that are children of the same parent (siblings), hence at the same maximum depth D in the tree.

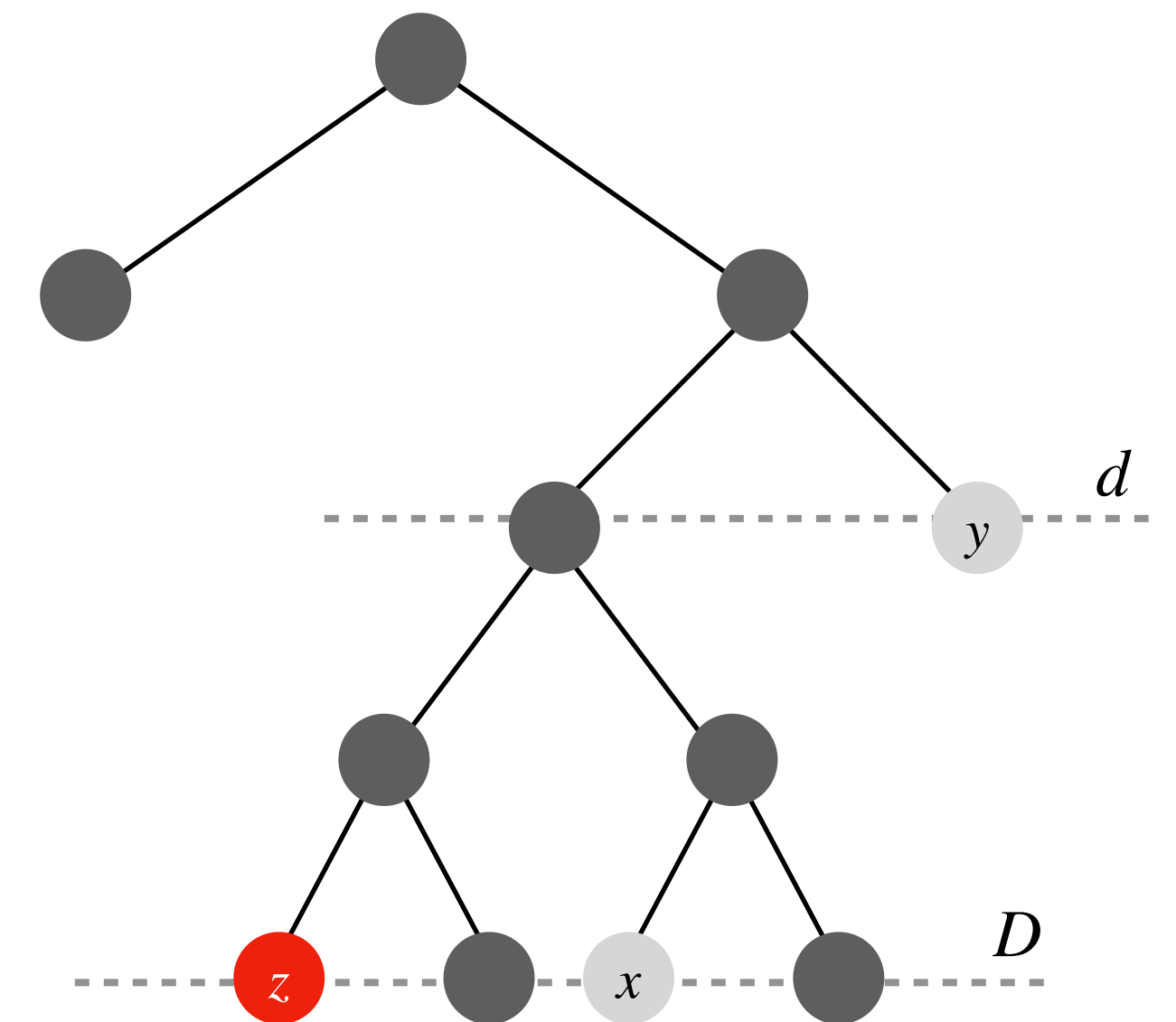
Proof.

- Suppose that the two least-weight leaves, say for symbols x and y , are *not* at the same maximum depth D . Let x be at depth D and y at some other depth $d < D$. Since each internal node must have two children, there must be another **leaf z** at depth D . The weight of z is then $w_z \geq w_y$ because y is a least-weight leaf by assumption.
- If now we exchange the codeword lengths assigned to z and y , the resulting code T' will have a cost $C(T') \geq C(T)$ because T is optimal by assumption. Since the two codes only differ in the assignment for z and y :

$$C(T') - C(T) = (w_y D + w_z d) - (w_y d + w_z D) = (w_y - w_z) \cdot (D - d) \geq 0$$

and since $D - d > 0$, then it must be $w_y \geq w_z$. Therefore we conclude that $w_z = w_y$ and that z is indeed a least-weight leaf at maximum depth D .

- Now that we are sure the two least-weight leaves are both at depth D , it could be that they are not children of the same parent. If that is the case, just “relabel” the leaves by sorting the weights. ■



Huffman — Induction

- Suppose that Huf_{m-1} is an optimal code for the set W_{m-1} of $m - 1$ weights, for $m > 2$. (The case for $m = 2$ is the base case.) And let Opt_m be an optimal code for W_m instead. Therefore, we have

$$C(Opt_m) \leq C(Huf_m). \quad (1)$$

- Since Opt_m is optimal, for the *Max-Depth Leaves Property* there must be two leaves, x and y , at the same depth D_m of least-weight, so:

$$C(Opt_m) = C(Opt_{m-1}) + (w_x + w_y) \cdot D_m \quad (2)$$

where Opt_{m-1} is an optimal code on the “reduced” weights $W_{m-1} = W_m \setminus \{w_x, w_y\} \cup \{w_x + w_y\}$.

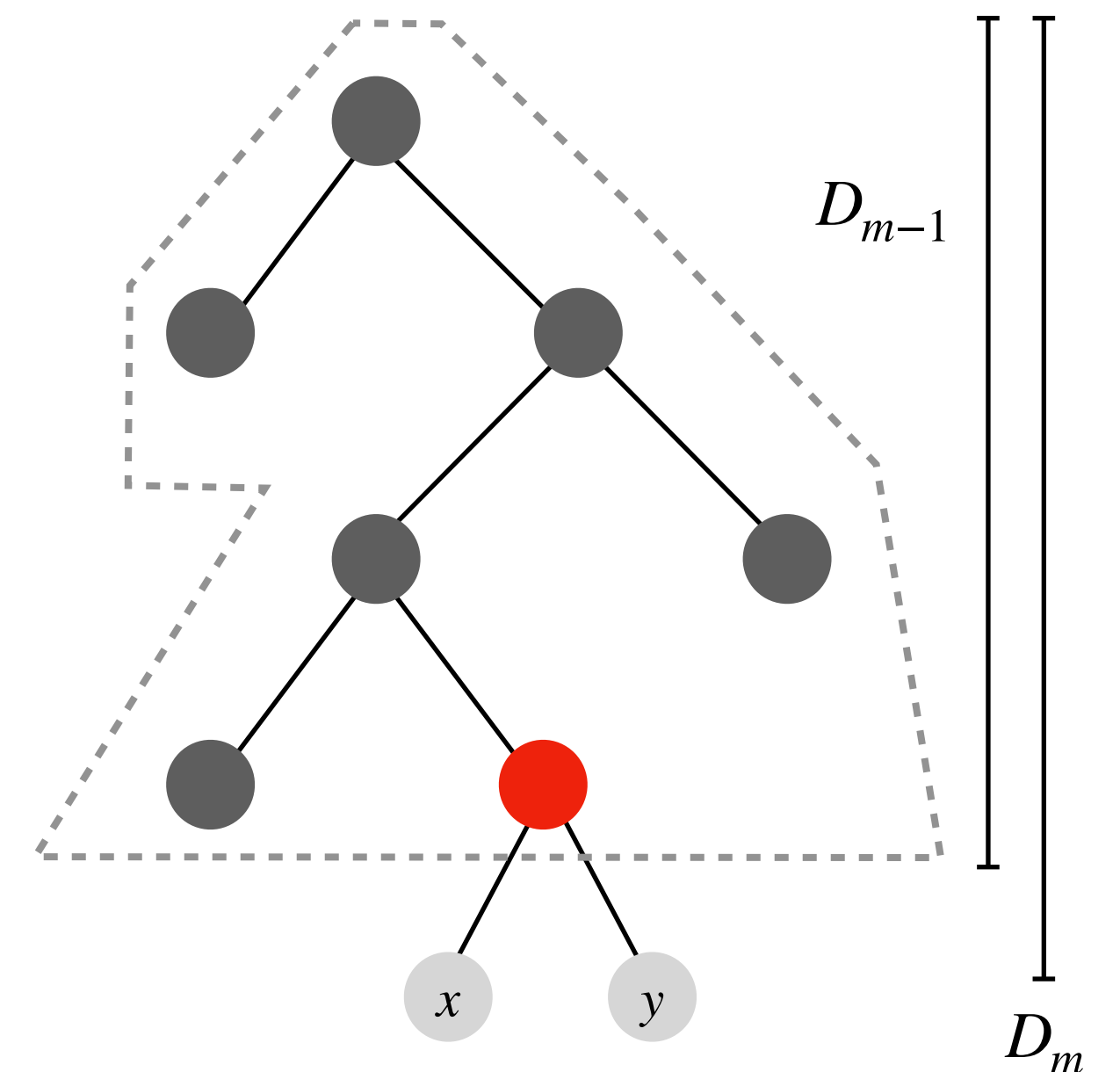
- But for the inductive hypothesis, also Huf_{m-1} is optimal on W_{m-1} , thus it must be

$$C(Huf_{m-1}) \leq C(Opt_{m-1}). \quad (3)$$

- Therefore, combining **(2)** with **(3)** we have:

$$C(Opt_m) = C(Opt_{m-1}) + (w_x + w_y) \cdot D_m \geq C(Huf_{m-1}) + (w_x + w_y) \cdot D_m = C(Huf_m).$$

Combining the latter inequality with **(1)**, we obtain that it must be $C(Opt_m) = C(Huf_m)$, hence Huf_m is an optimal code for W_m . ■



Huffman — Relation to Entropy

- Recall the entropy for the probability distribution P :

$H(P) = \sum_x P(x) \log_2(1/P(x))$ is the *expected* codeword length for an optimal code for the distribution P . In this module, we model $P(x) = w_x/n$ using the “weights” $W = \langle w_x \rangle$, so we write

$$H = H(W) = (1/n) \cdot \sum_x w_x \log_2(n/w_x) \text{ bits.}$$

- The average codeword length of an Huffman code Huf for W is $L_{Huf} = C(W, Huf)/n$.
- A fundamental result established that $H \leq L_{Huf} < H + 1$, meaning that Huffman can loose **up to 1 bit per symbol** compared to the entropy. **Q.** Is this bad or good?

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- A fundamental result established that $H \leq L_{Huf} < H + 1$, meaning that Huffman can loose **up to 1 bit per symbol** compared to the entropy. **Q.** Is this bad or good?
- A.** It depends. If H is high, say $H \gg 1$, then the extra bit is *negligible* and Huffman reaches the entropy limit. Otherwise, if $H \ll 1$, the extra bit lost is not negligible and Huffman is *not effective*. Note that this limit is intrinsic of prefix-free codes as they must assign codewords of length ≥ 1 .

Huffman — Limitation

Example for $W = \langle 999, 1 \rangle$, where one symbol is very, very, frequent. Then the entropy is

$$H = (999/1000) \cdot \log_2(1000/999) + (1/1000) \cdot \log_2(1000) \simeq 0.01141 \text{ bits},$$

thus, much less than 1 bit because of the skewed distribution.

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- **Q.** How to overcome this limitation?
A. Associate a codeword to *more than one symbol*.

Huffman on Blocks

- **Idea.** Compute the Huffman code on *blocks* of L .
If B is the block size, then the extra bit lost compared to the entropy is amortized among the B symbols, thus the loss is just $1/B$ bits.
- The redundancy becomes smaller for larger B .
- But B cannot grow as much as we want!
For large B we would incur in high space for the encoding of the Huffman tree and high decoding time.

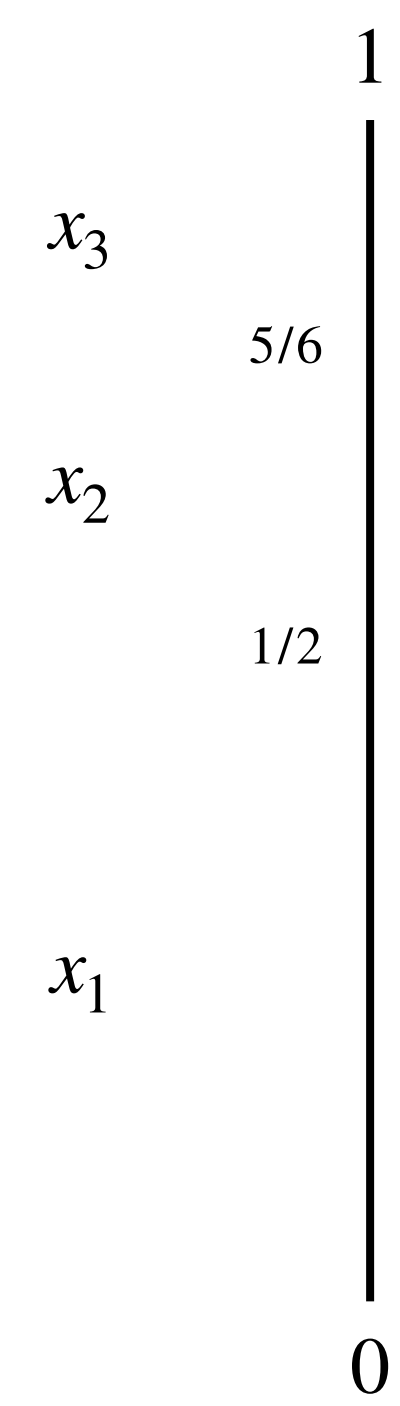
Arithmetic Coding

Elias, ~1960

- This method offers higher compression ratios than Huffman because it is *not* a prefix-free code.
- This means that we cannot decode *individual* symbols, rather the whole message.
- In practice:
 - slower than (canonical) Huffman to decode;
 - no need to transmit the model (e.g., the Huffman tree).
- **Algorithm.**
 - The real interval $[0.0, 1.0)$ is partitioned into segments of length *proportional* to the probabilities $P(x) = w_x/n$ in L and the segment $[l_1, r_1) \subset [0.0, 1.0]$ associated to the symbol $L[1]$ is considered.
 - The same partitioning step is applied to $[l_1, r_1)$ and the segment $[l_2, r_2) \subset [l_1, r_1)$ associated to the symbol $L[2]$ is considered.
 - Repeat for all the n symbols and emit a single real number $y \in [l_n, r_n)$.
- The pair $\langle y, n \rangle$ is sufficient to decode the original list L .

Arithmetic Coding — Encoding Demo

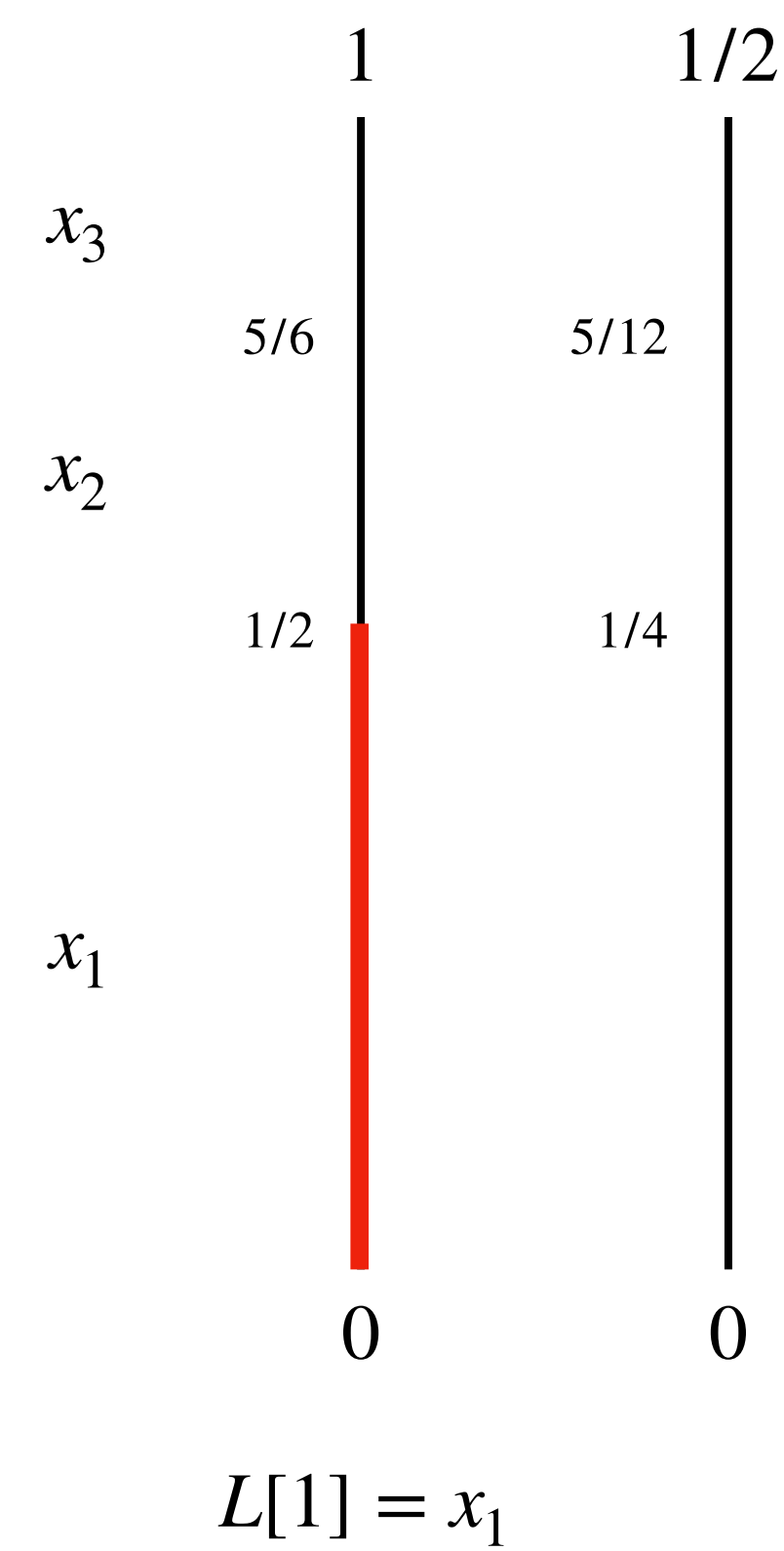
Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $P(x_1) = 1/2$, $P(x_2) = 1/3$, and $P(x_3) = 1/6$.



i	l_i	r_i
0	0.000000	1.000000
1		
2		
3		
4		
5		
6		

Arithmetic Coding — Encoding Demo

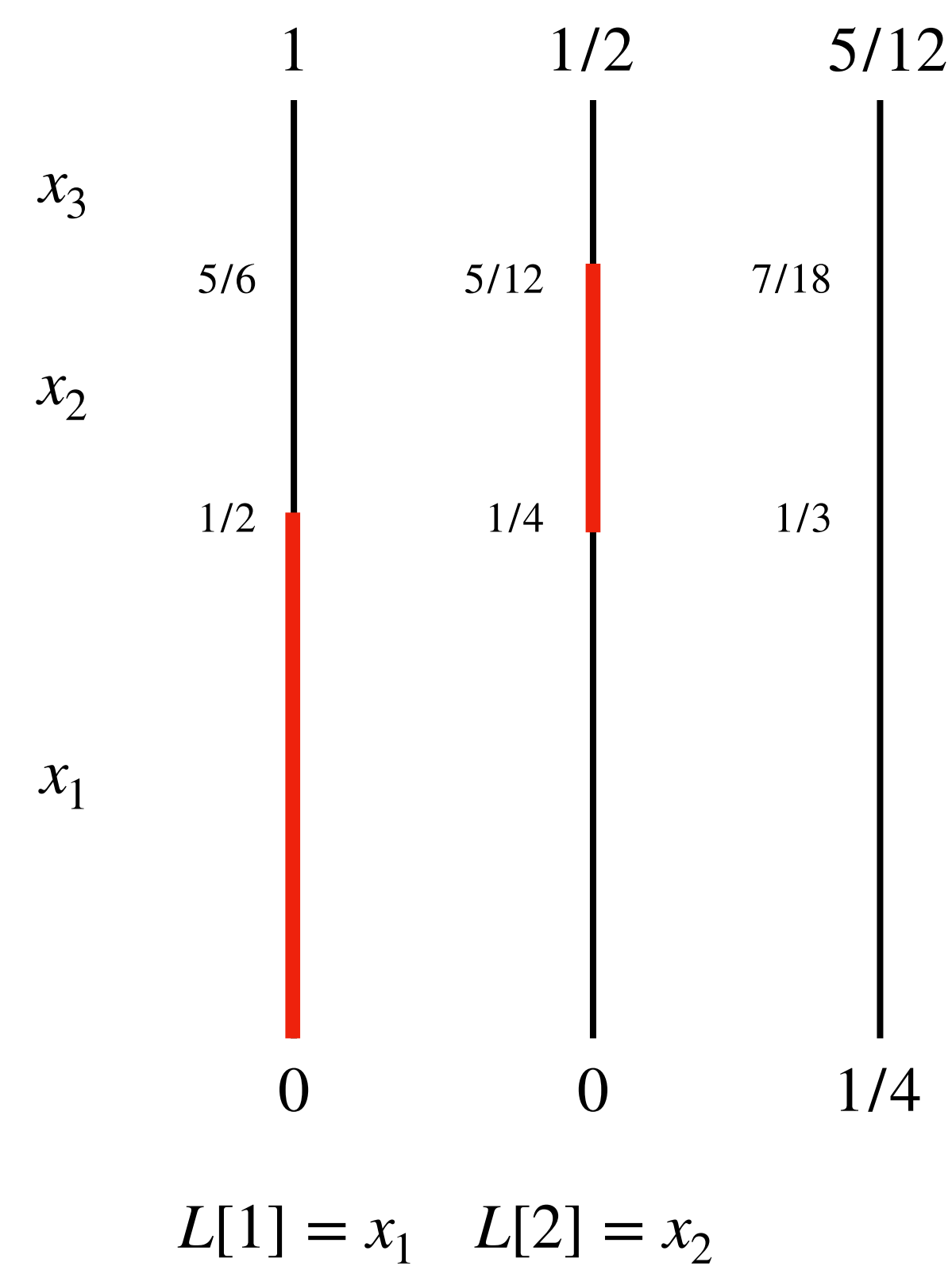
Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $P(x_1) = 1/2$, $P(x_2) = 1/3$, and $P(x_3) = 1/6$.



i	l_i	r_i
0	0.000000	1.000000
1	0.000000	0.500000
2		
3		
4		
5		
6		

Arithmetic Coding — Encoding Demo

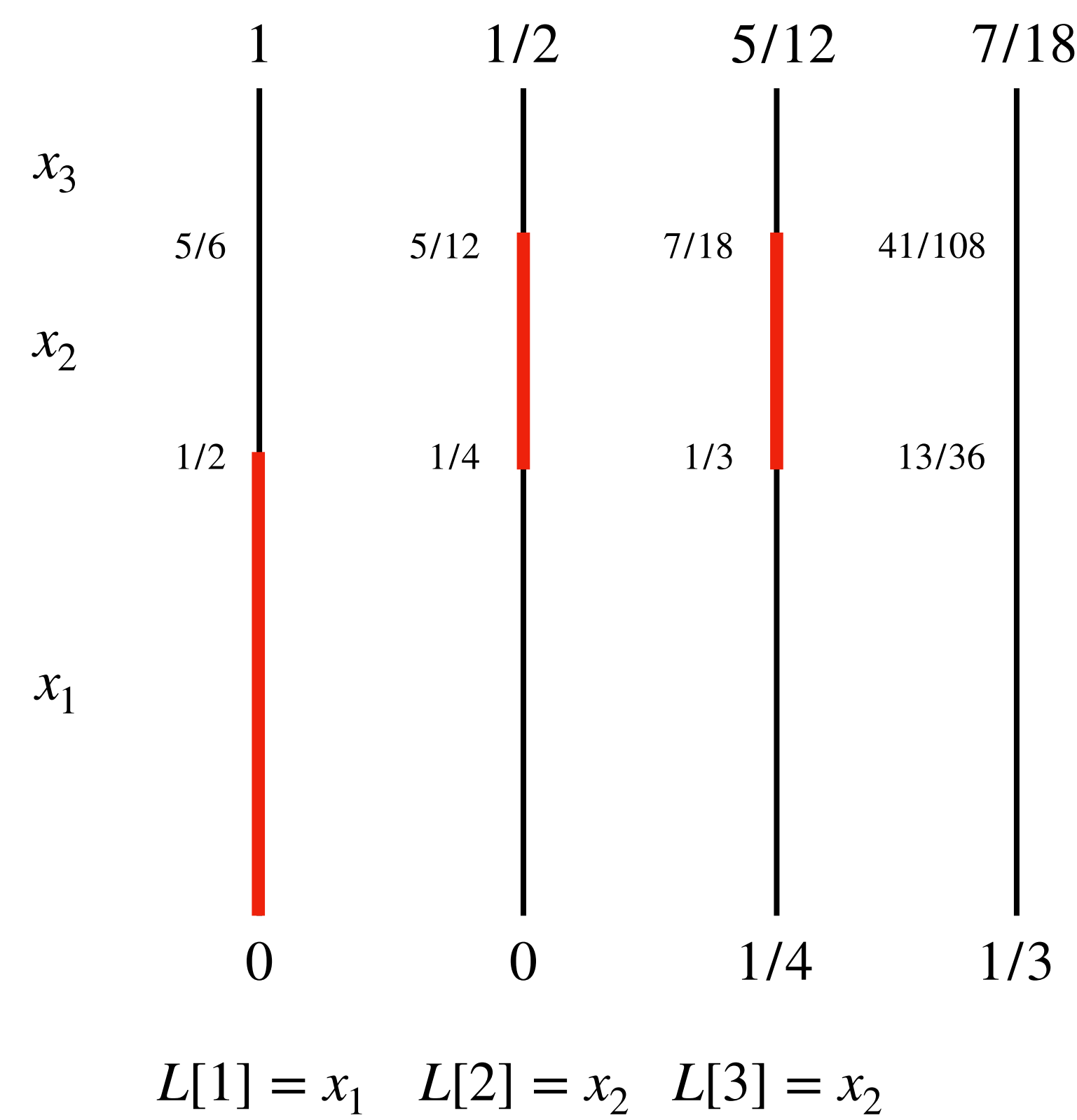
Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $P(x_1) = 1/2$, $P(x_2) = 1/3$, and $P(x_3) = 1/6$.



i	l_i	r_i
0	0.000000	1.000000
1	0.000000	0.500000
2	0.250000	0.416667
3		
4		
5		
6		

Arithmetic Coding — Encoding Demo

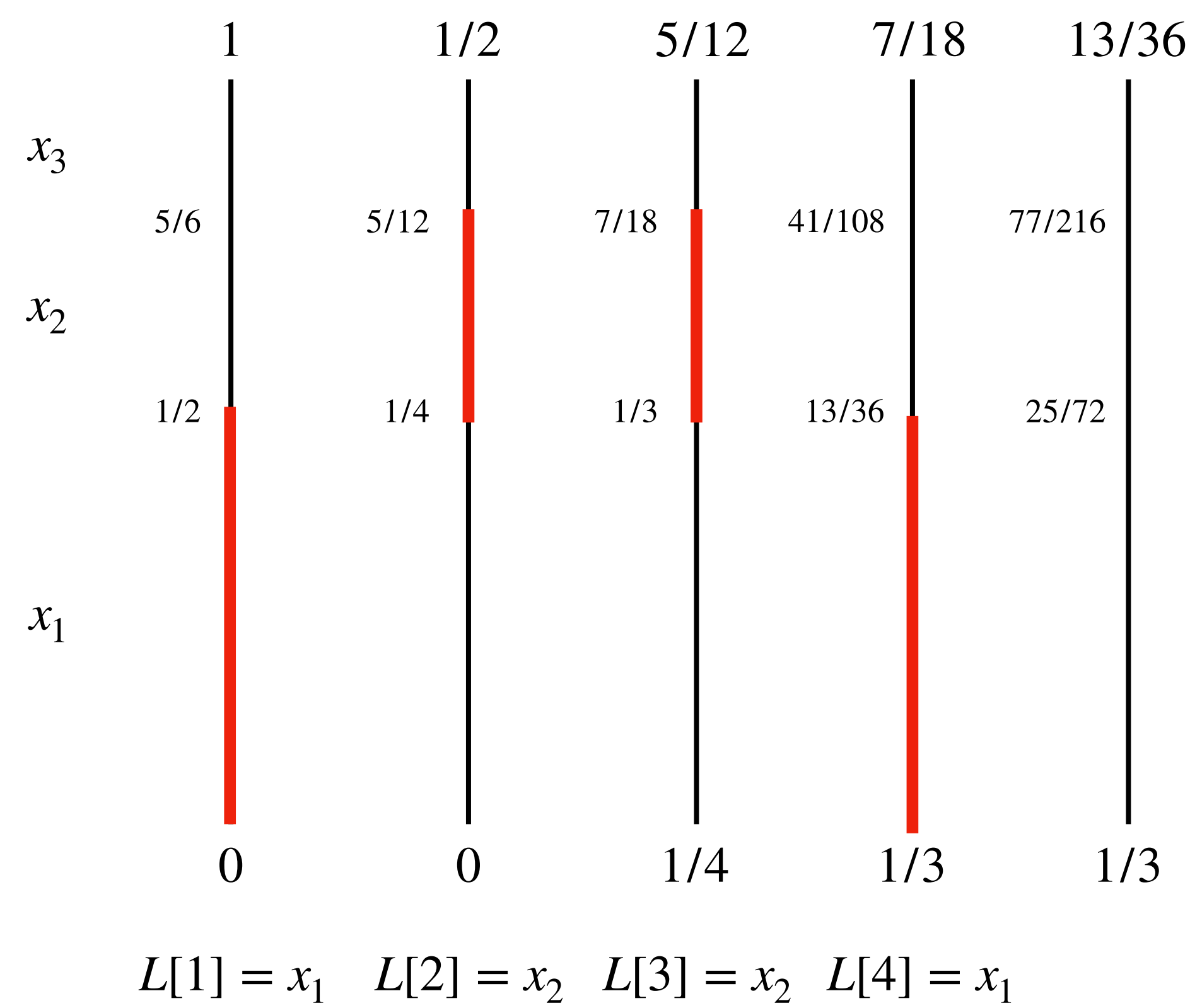
Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $P(x_1) = 1/2$, $P(x_2) = 1/3$, and $P(x_3) = 1/6$.



i	l_i	r_i
0	0.000000	1.000000
1	0.000000	0.500000
2	0.250000	0.416667
3	0.333333	0.388889
4		
5		
6		

Arithmetic Coding — Encoding Demo

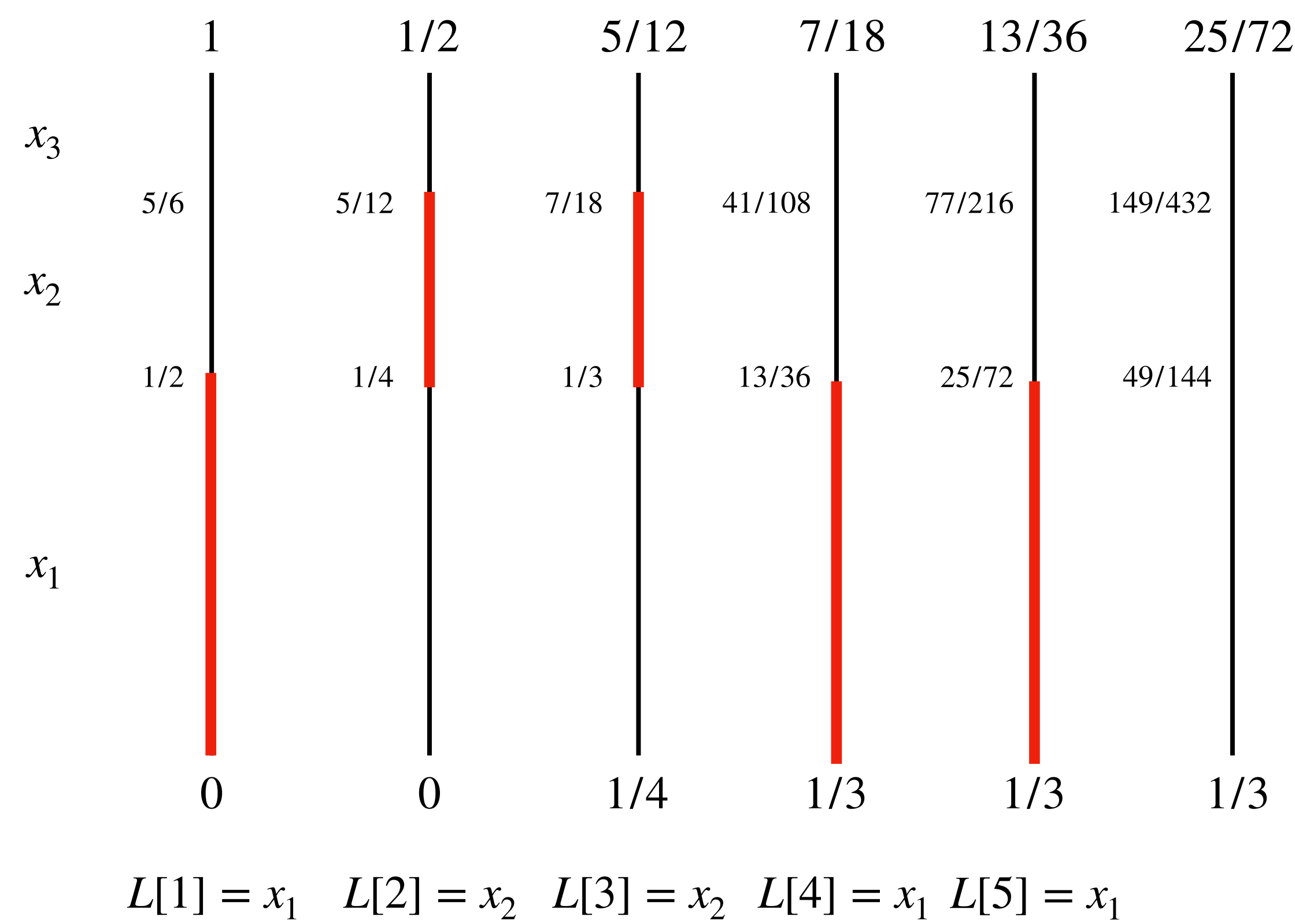
Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $P(x_1) = 1/2$, $P(x_2) = 1/3$, and $P(x_3) = 1/6$.



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0	0.000000	1.000000
1	0.000000	0.500000
2	0.250000	0.416667
3	0.333333	0.388889
4	0.333333	0.361111
5		
6		

Arithmetic Coding — Encoding Demo

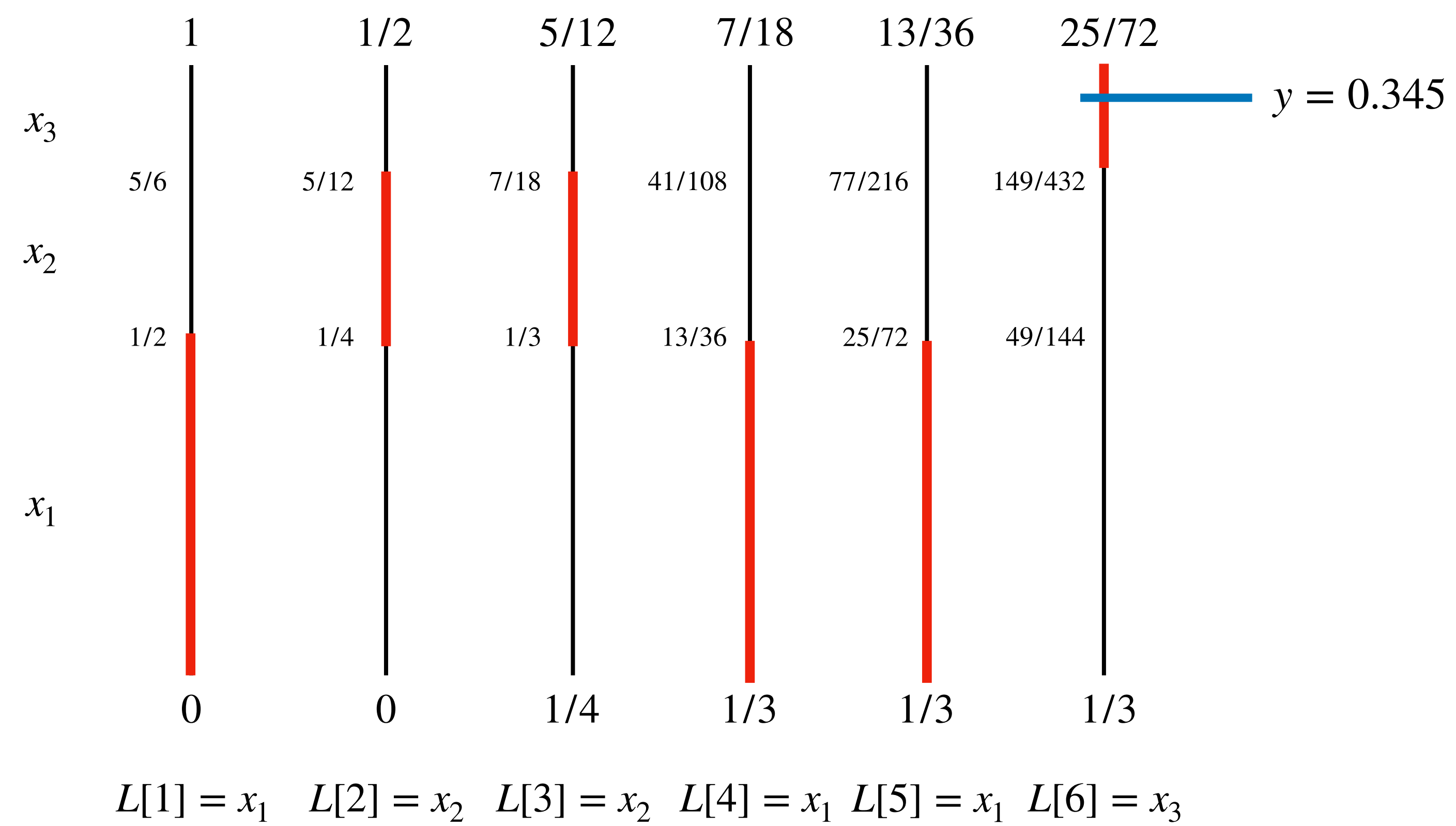
Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $P(x_1) = 1/2$, $P(x_2) = 1/3$, and $P(x_3) = 1/6$.



i	l_i	r_i
0	0.000000	1.000000
1	0.000000	0.500000
2	0.250000	0.416667
3	0.333333	0.388889
4	0.333333	0.361111
5	0.333333	0.347222
6		

Arithmetic Coding — Encoding Demo

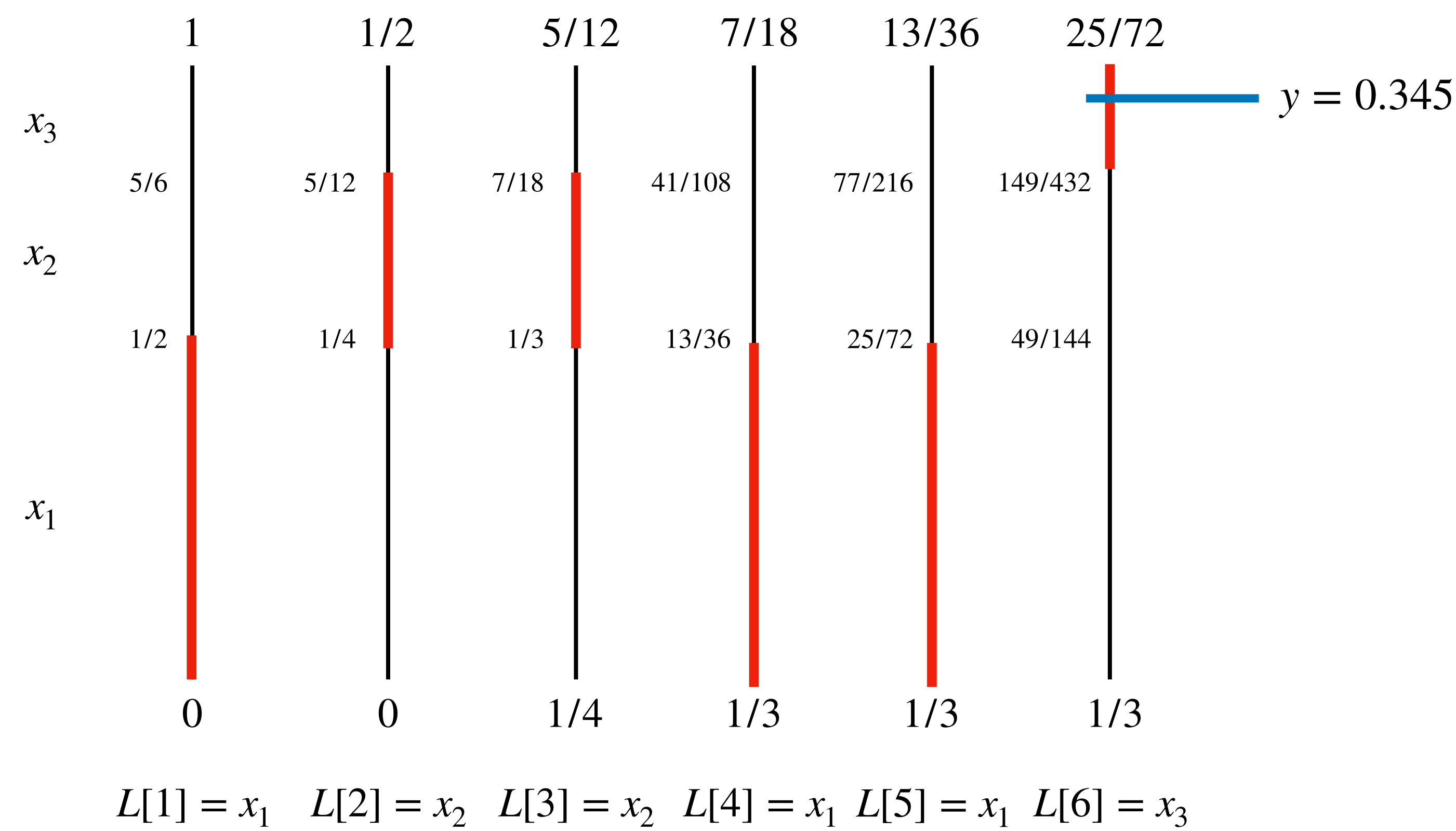
Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $P(x_1) = 1/2$, $P(x_2) = 1/3$, and $P(x_3) = 1/6$.



i	l_i	r_i
0	0.000000	1.000000
1	0.000000	0.500000
2	0.250000	0.416667
3	0.333333	0.388889
4	0.333333	0.361111
5	0.333333	0.347222
6	0.344907	0.347222

Arithmetic Coding — Encoding Demo

Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $P(x_1) = 1/2$, $P(x_2) = 1/3$, and $P(x_3) = 1/6$.

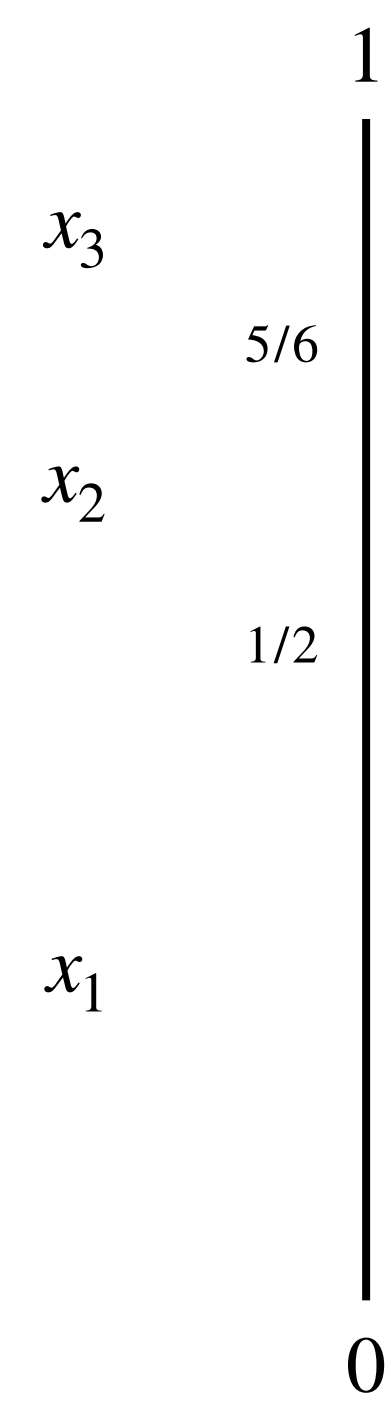


<i>i</i>	<i>l_i</i>	<i>r_i</i>
0	0.000000	1.000000
1	0.000000	0.500000
2	0.250000	0.416667
3	0.333333	0.388889
4	0.333333	0.361111
5	0.333333	0.347222
6	0.344907	0.347222

Note that, by construction, the length $(r_n - l_n)$ of the final interval $[l_n, r_n)$ is $\prod_{i=1}^n P(L[i])$.

Arithmetic Coding — Decoding Demo

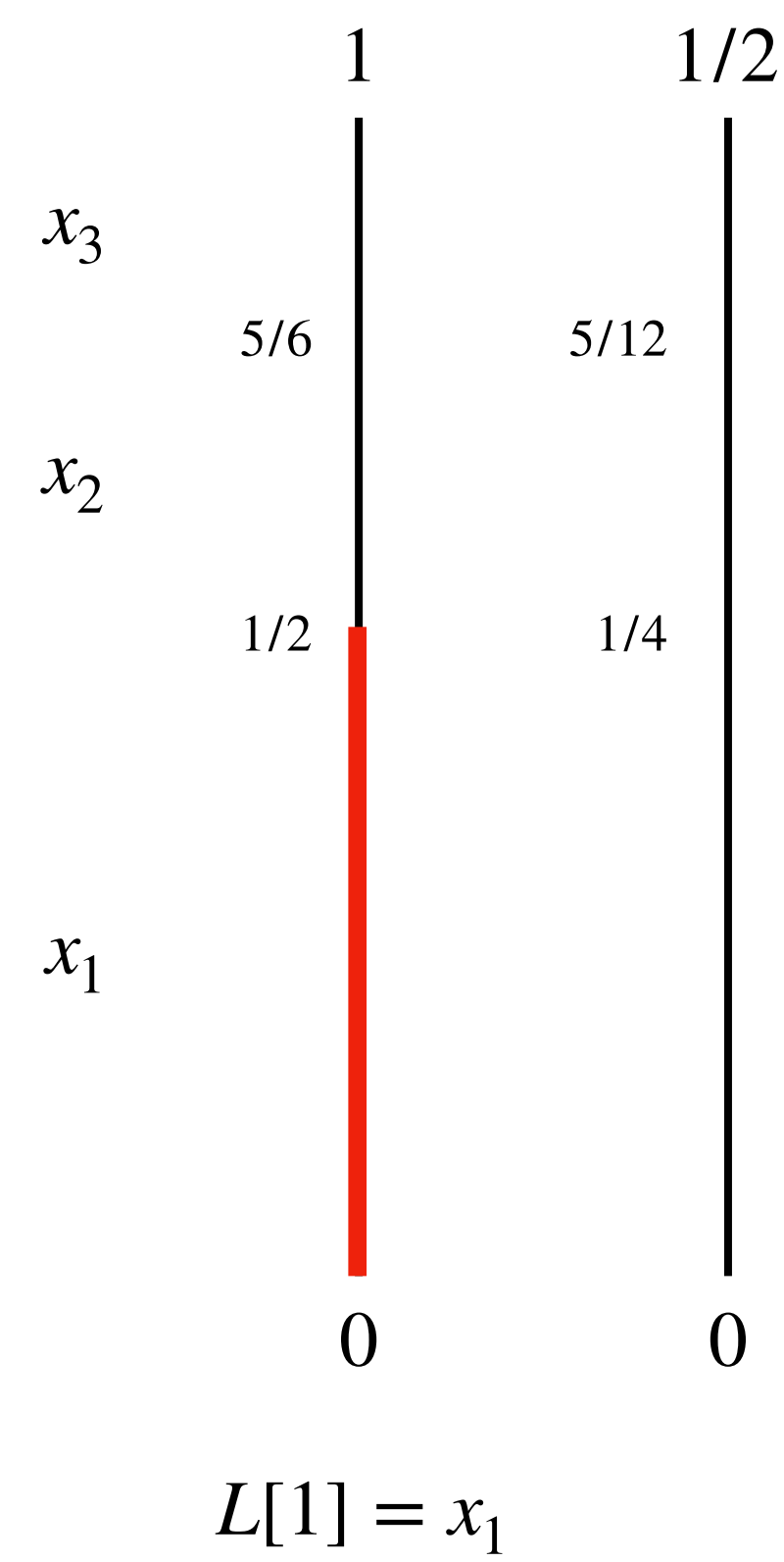
Decoding. Given $\langle y, n \rangle$, where does y lies in at each of the n steps?
In our example: $y = 0.345$, and $n = 6$.



i	l_i	r_i
0	0.000000	1.000000
1		
2		
3		
4		
5		
6		

Arithmetic Coding — Decoding Demo

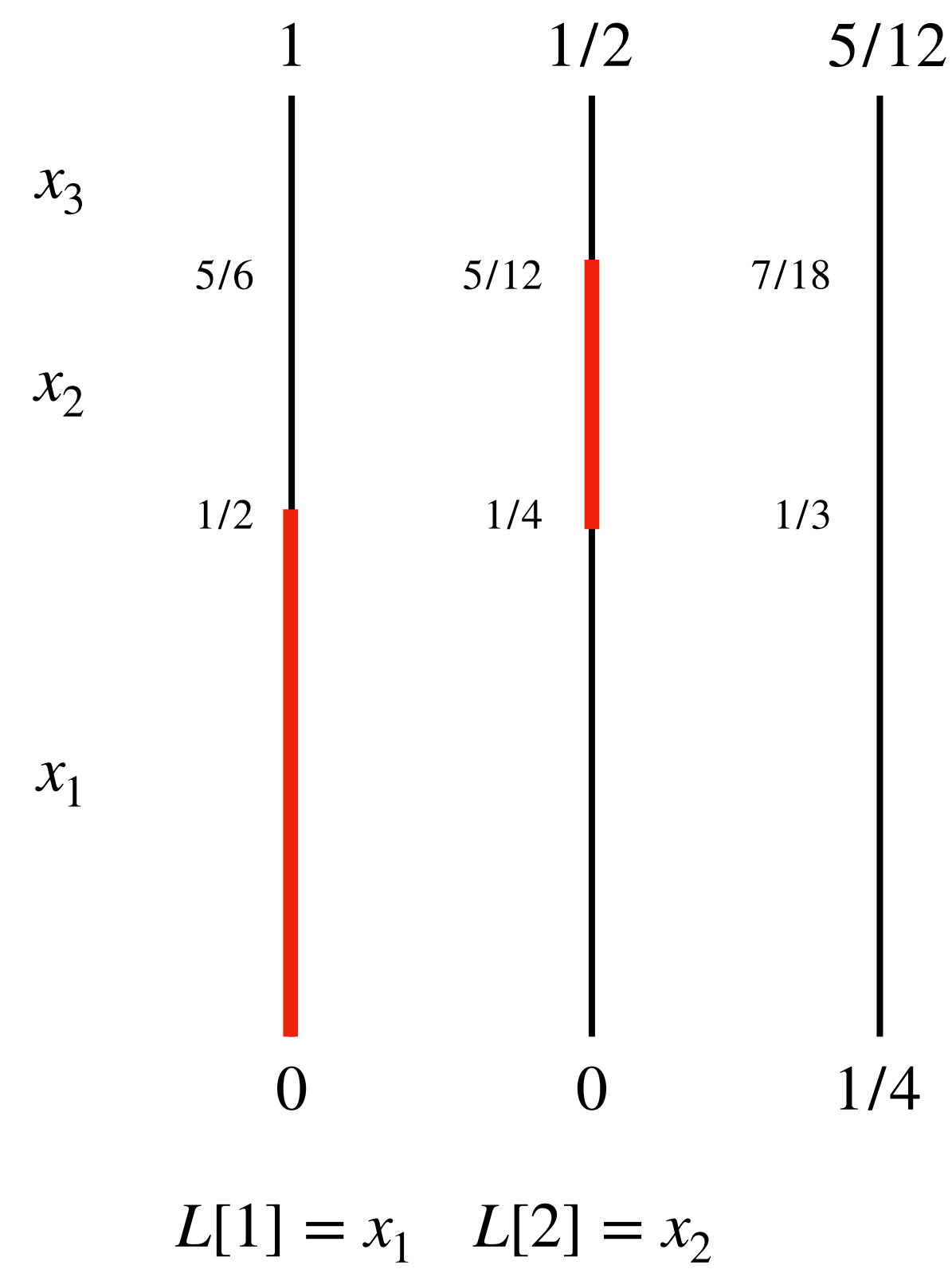
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Arithmetic Coding — Decoding Demo

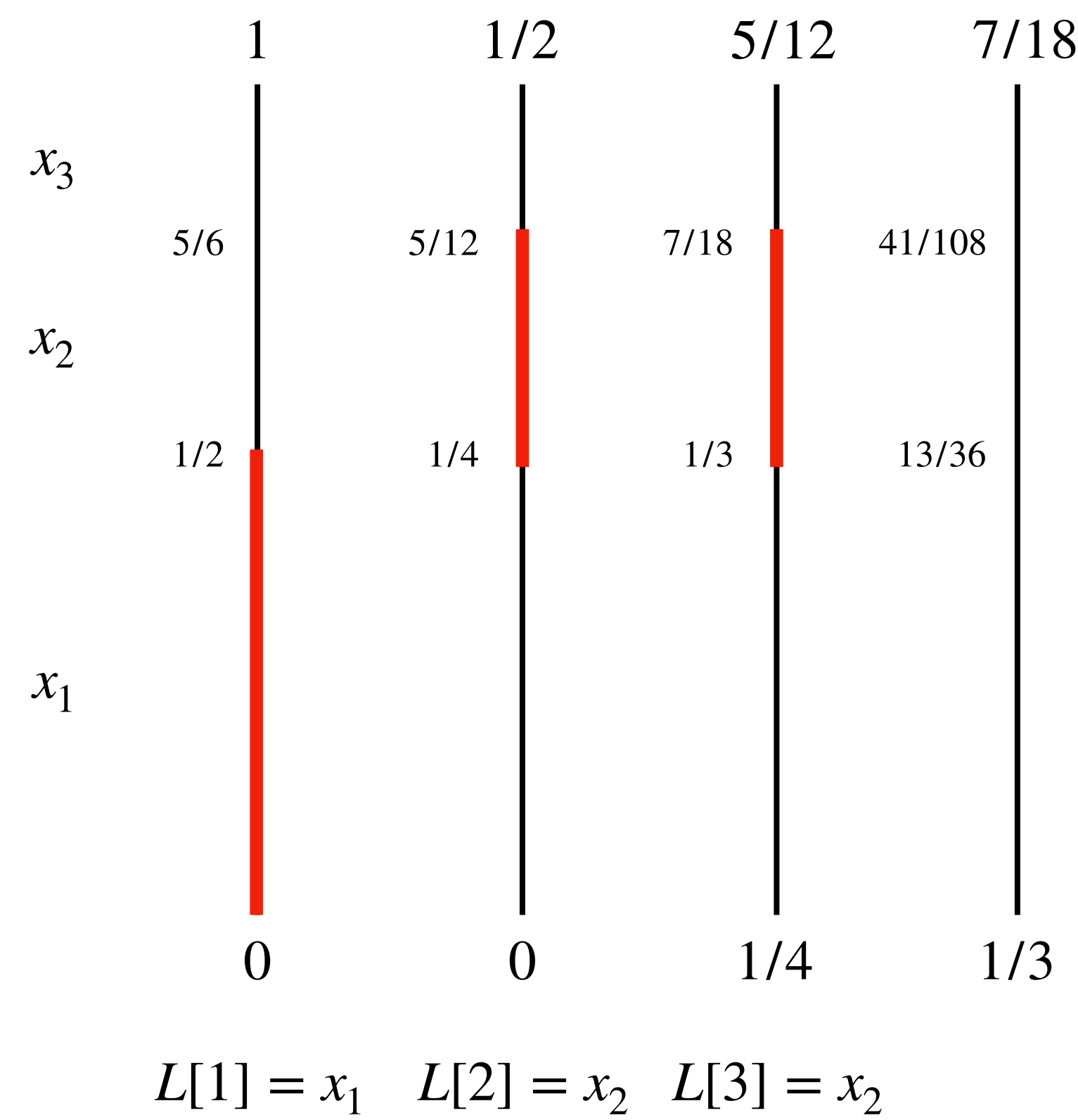
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Arithmetic Coding — Decoding Demo

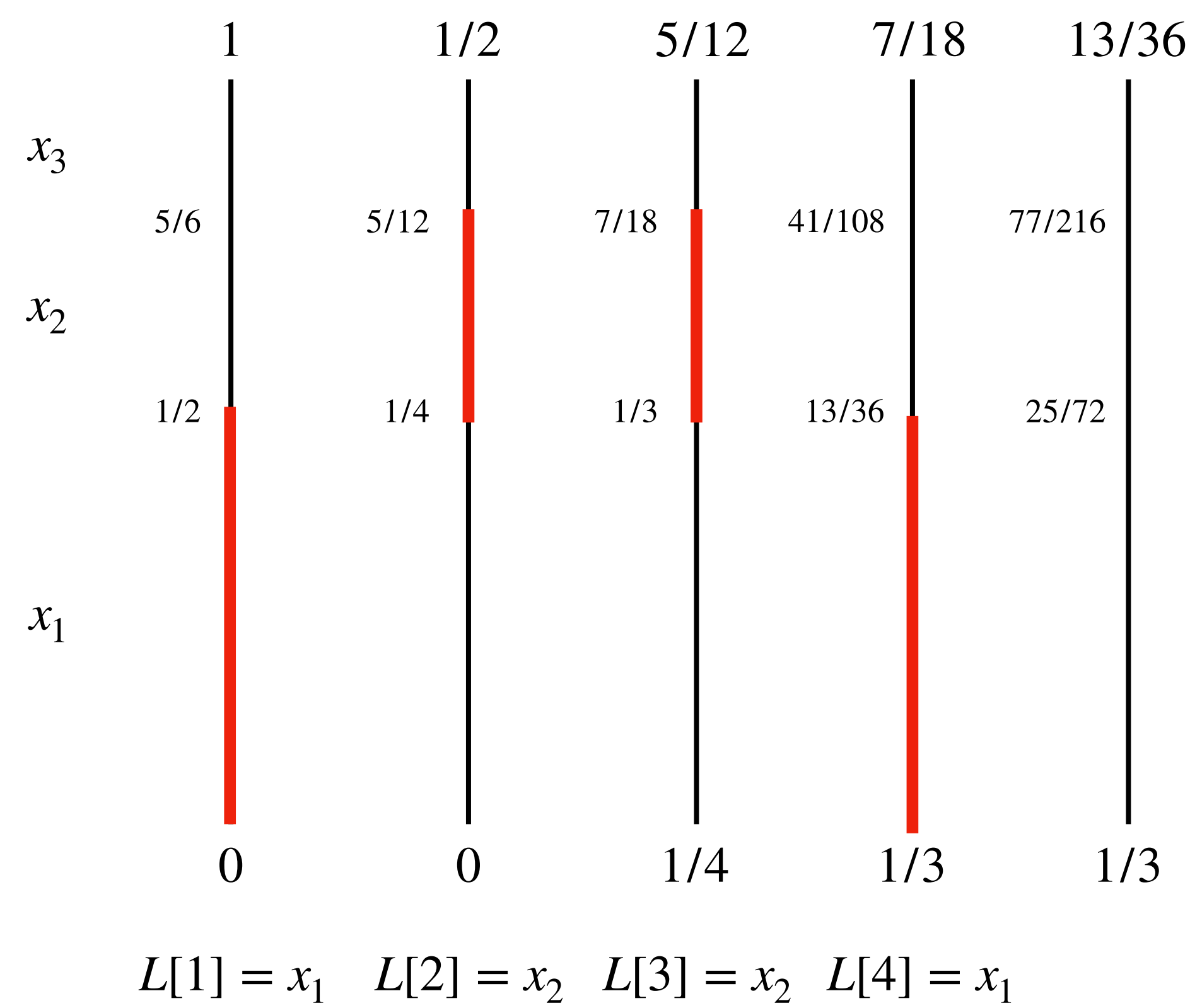
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Arithmetic Coding — Decoding Demo

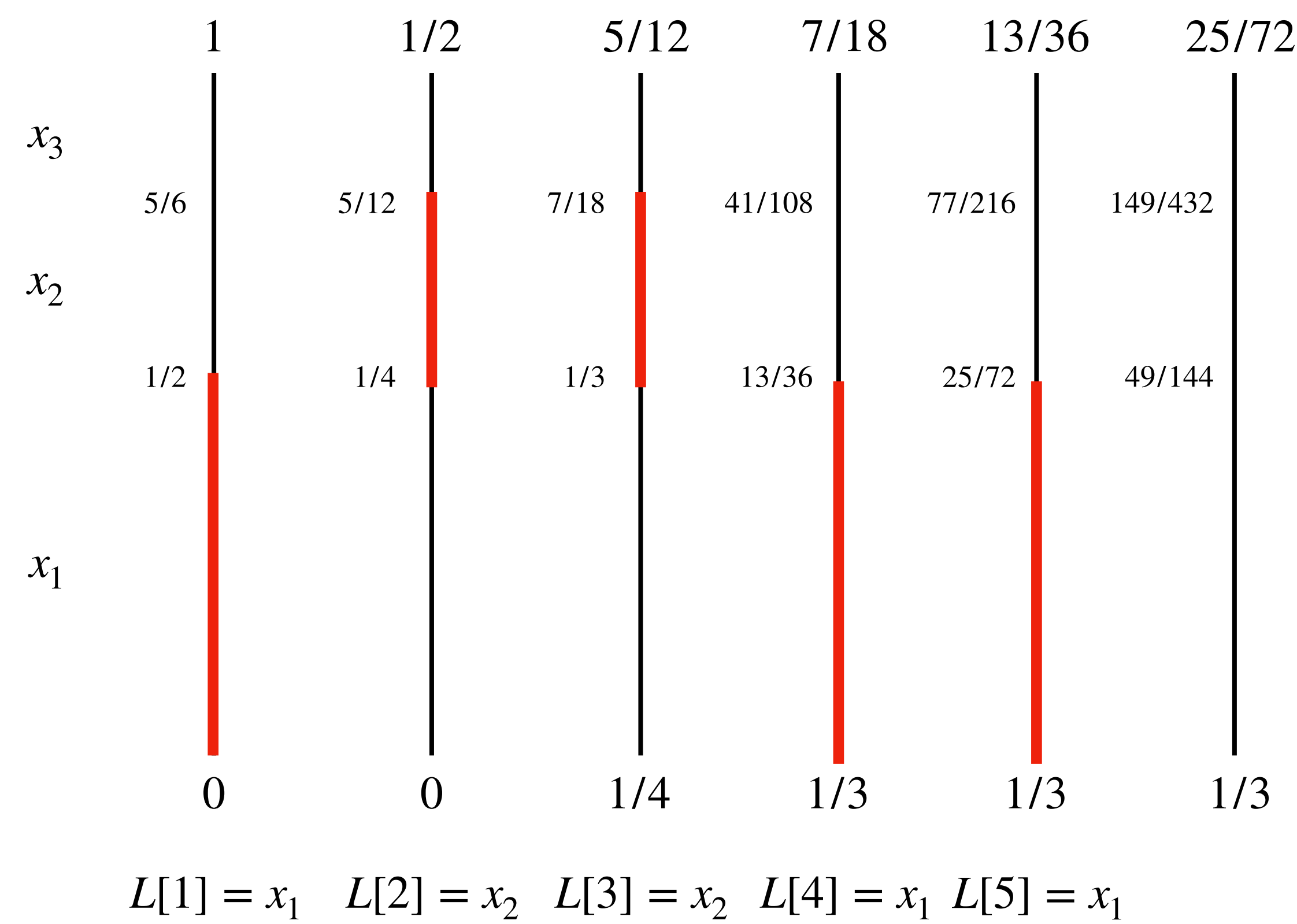
Decoding. Given $\langle y, n \rangle$, where does y lies in at each of the n steps?
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Arithmetic Coding — Decoding Demo

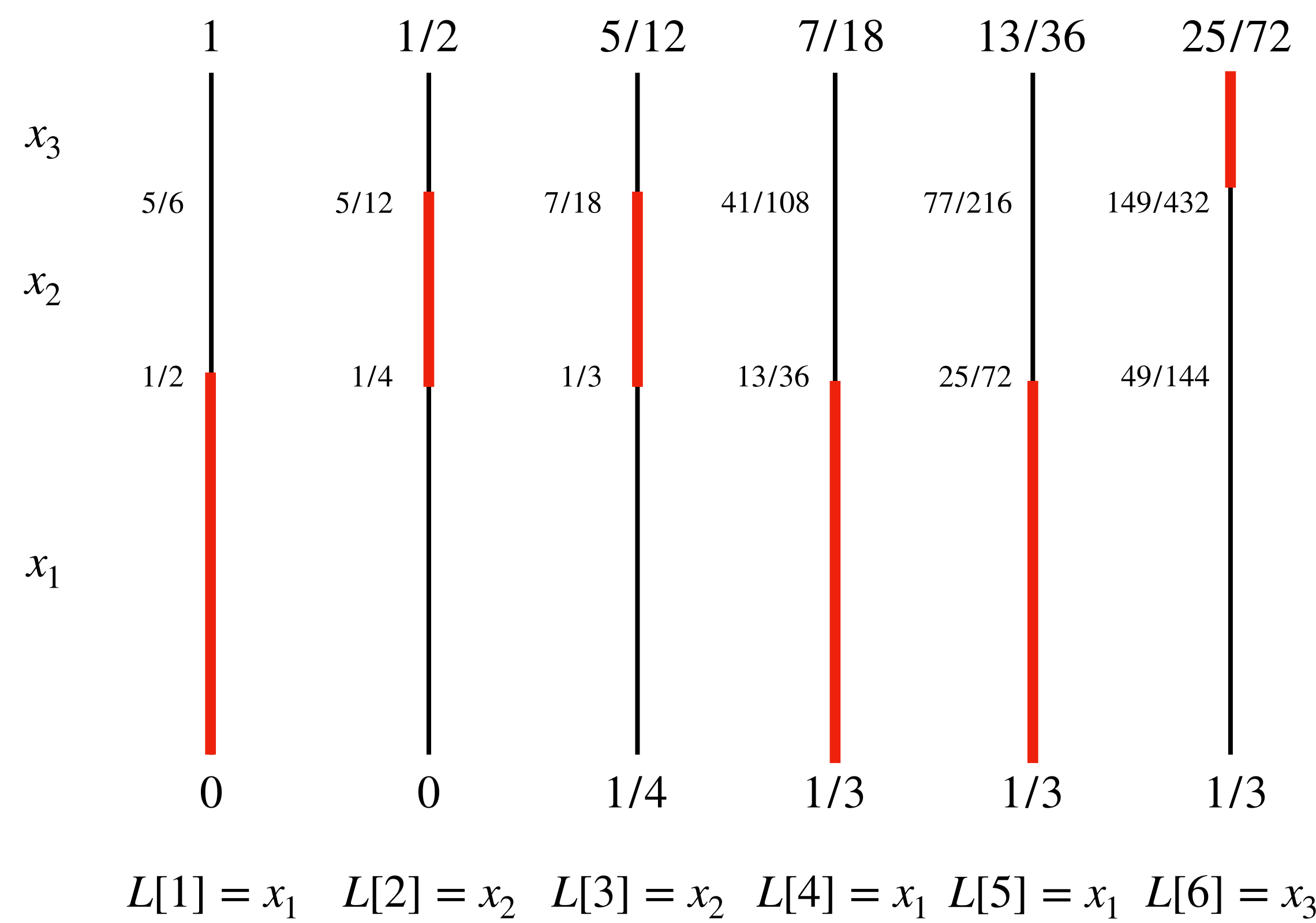
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Arithmetic Coding — Decoding Demo

Decoding. Given $\langle y, n \rangle$, where does y lies in at each of the n steps?
In our example: $y = 0.345$, and $n = 6$.



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Arithmetic Coding

- **Q.** How to choose $y \in [l_n, r_n)$ and how to represent it?
- **Dyadic fraction.** In general, a binary string $B[1..k]$ can be used to represent the real value

$$y = \frac{[B[1] \dots B[k]]_{10}}{2^k} = \frac{\sum_{i=1}^k (B[i] \cdot 2^{k-i})}{2^k} = \sum_{i=1}^k (B[i] \cdot 2^{-i}).$$

- The larger k , the better the approximation.

For example, $B[1..6] = [001011]$ corresponds to $y = [001011]_{10}/2^6 = 11/64 = 0.171875$ and $B[1..7] = [0100111]$ to $y = [0100111]_{10}/2^7 = 39/128 = 0.3046875$.

Arithmetic Coding

- **Finite-precision arithmetic.** Given the binary string $B[1..\infty]$ representing the infinite-precision real number y , $B[1..k]$ represents a “truncated” value $\hat{y} \in [y - 2^{-k}, y]$.

Proof. Clearly we have $\hat{y} < y$ because \hat{y} is a truncation.

Geometric series: $\sum_{i=0}^{\infty} \frac{1}{2^i} = 1/(1 - 1/2) = 2 = \frac{1}{2^0} + \sum_{i=1}^{\infty} \frac{1}{2^i}$

Therefore: $y - \hat{y} = \sum_{i=1}^{\infty} B[k+i]2^{-(k+i)} \leq 2^{-k} \sum_{i=1}^{\infty} \frac{1}{2^i} = 2^{-k}$, that is $\hat{y} \geq y - 2^{-k}$. ■

- **Arithmetic Coding output bits.** Given $[l_n, r_n)$, the real value $y = (l_n + r_n)/2$ truncated to its first $\left\lceil \log_2 \left(\frac{2}{r_n - l_n} \right) \right\rceil$ bits, \hat{y} , belongs to the interval $[l_n, r_n)$.

Proof. We know that $y - 2^{-k} \leq \hat{y} < y$. Setting $k = \lceil \log_2(2/(r_n - l_n)) \rceil$ and $y = (l_n + r_n)/2$, we have that:

$$y - 2^{-k} = (l_n + r_n)/2 - 2^{-\lceil \log_2(2/(r_n - l_n)) \rceil} \geq (l_n + r_n)/2 - (r_n - l_n)/2 = l_n \text{ and therefore}$$

$$l_n \leq y - 2^{-k} \leq \hat{y} < y = (l_n + r_n)/2 < r_n. \quad \blacksquare$$

Arithmetic Coding — Optimality

- We have proved that:

Arithmetic Coding emits $\lceil \log_2(2/(r_n - l_n)) \rceil$ bits, where $r_n - l_n = \prod_{i=1}^n P(L[i])$.

- **Optimality.** The number of bits emitted by Arithmetic Coding for encoding $L[1..n]$ is at most $nH + 2$, where H is the entropy of L .

Proof.

$$\begin{aligned} \lceil \log_2(2/(r_n - l_n)) \rceil &< \log_2 2 - \log_2(r_n - l_n) + 1 = 2 - \log_2\left(\prod_{i=1}^n P(L[i])\right) = \\ &= 2 - \sum_{i=1}^n \log_2 P(L[i]) = \\ &= 2 + \sum_{i=1}^n \log_2(1/P(L[i])) = 2 + nH. \blacksquare \end{aligned}$$

The entropy is independent from the **position** of the symbol occurrences, so we can sum over groups of occurrences of the same symbol.

- This means that Arithmetic Coding loses $2/n$ bits per symbol compared to the entropy, which is *negligible* for all practical values of n . Much *better* than Huffman on skewed distributions.

Asymmetric Numeral Systems

Duda, 2009

- Asymmetric Numeral Systems (ANS) approaches the compression ratio of Arithmetic Coding and the speed of Huffman coding. Like Arithmetic Coding, it is *not* a prefix-free code.
- **Idea.** Represent $L[1..i]$ with an *integer state* variable s_i , $i = 1, \dots, n$.
- A coding table T is used to generate the next state s_i from state s_{i-1} , after the processing of symbol $L[i]$: $s_i = T[L[i], s_{i-1}]$.
- The row $T[i]$ contains integers values assigned in increasing order and in “strides” of length w_i .
- **Algorithm.**
 - At the beginning, set $s_0 = 0$.
 - For each symbol $L[i]$, $i = 1, \dots, n$, generate a new state $s_i = T[L[i], s_{i-1}]$.
 - Emit $s_n > 0$ encoded with $\lceil \log_2 s_n \rceil$ bits.

Asymmetric Numeral Systems — Demo

Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $W = \langle w_1, w_2, w_3 \rangle = \langle 3, 2, 1 \rangle$ and $n = 6$.

W	T									
	0	1	2	3	4	5	6	7	8	9
w_1	1	2	3	7	8	9	13	14	15	19
w_2	4	5	10	11	16	17	22	23	28	29
w_3	6	12	18	24	30	36	42	48	54	60

...

$s_0 = 0$

Asymmetric Numeral Systems — Demo

Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $W = \langle w_1, w_2, w_3 \rangle = \langle 3, 2, 1 \rangle$ and $n = 6$.

W	T									
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w_3	6	12	18	24	30	36	42	48	54	60
	0	1	2	3	4	5	6	7	8	9

...

$s_0 = 0$

$s_1 = T[L[1],0] = 1$

Asymmetric Numeral Systems — Demo

Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $W = \langle w_1, w_2, w_3 \rangle = \langle 3, 2, 1 \rangle$ and $n = 6$.

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w_2	4	5	10	11	16	17	22	23	28	29
w_3	6	12	18	24	30	36	42	48	54	60
	0	1	2	3	4	5	6	7	8	9

...

$$s_0 = 0$$

$$s_1 = T[L[1], 0] = 1$$

$$s_2 = T[L[2], 1] = 5$$

Asymmetric Numeral Systems — Demo

Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $W = \langle w_1, w_2, w_3 \rangle = \langle 3, 2, 1 \rangle$ and $n = 6$.

W	T									
w_1	1	2	3	7	8	9	13	14	15	19
w_2	4	5	10	11	16	17	22	23	28	29
w_3	6	12	18	24	30	36	42	48	54	60
	0	1	2	3	4	5	6	7	8	9

...

$s_0 = 0$

$s_1 = T[L[1],0] = 1$

$s_2 = T[L[2],1] = 5$

$s_3 = T[L[3],5] = 17$

Asymmetric Numeral Systems — Demo

Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $W = \langle w_1, w_2, w_3 \rangle = \langle 3, 2, 1 \rangle$ and $n = 6$.

W	T									
w_1	1	2	3	7	8	9	13	14	15	19
w_2	4	5	10	11	16	17	22	23	28	29
w_3	6	12	18	24	30	36	42	48	54	60
	0	1	2	3	4	5	6	7	8	9

...

$s_0 = 0$
 $s_1 = T[L[1], 0] = 1$
 $s_2 = T[L[2], 1] = 5$
 $s_3 = T[L[3], 5] = 17$
 $s_4 = T[L[4], 17] = 33$

Asymmetric Numeral Systems — Demo

Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $W = \langle w_1, w_2, w_3 \rangle = \langle 3, 2, 1 \rangle$ and $n = 6$.

W	T									
w_1	1	2	3	7	8	9	13	14	15	19
w_2	4	5	10	11	16	17	22	23	28	29
w_3	6	12	18	24	30	36	42	48	54	60
	0	1	2	3	4	5	6	7	8	9

...

$s_0 = 0$
 $s_1 = T[L[1], 0] = 1$
 $s_2 = T[L[2], 1] = 5$
 $s_3 = T[L[3], 5] = 17$
 $s_4 = T[L[4], 17] = 33$
 $s_5 = T[L[5], 33] = 67$

Asymmetric Numeral Systems — Demo

Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $W = \langle w_1, w_2, w_3 \rangle = \langle 3, 2, 1 \rangle$ and $n = 6$.

W	T									
w_1	1	2	3	7	8	9	13	14	15	19
w_2	4	5	10	11	16	17	22	23	28	29
w_3	6	12	18	24	30	36	42	48	54	60
	0	1	2	3	4	5	6	7	8	9

...

$s_0 = 0$
 $s_1 = T[L[1], 0] = 1$
 $s_2 = T[L[2], 1] = 5$
 $s_3 = T[L[3], 5] = 17$
 $s_4 = T[L[4], 17] = 33$
 $s_5 = T[L[5], 33] = 67$
 $s_6 = T[L[6], 67] = 408$

Asymmetric Numeral Systems — Demo

Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $W = \langle w_1, w_2, w_3 \rangle = \langle 3, 2, 1 \rangle$ and $n = 6$.

W	T									
w_1	1	2	3	7	8	9	13	14	15	19
w_2	4	5	10	11	16	17	22	23	28	29
w_3	6	12	18	24	30	36	42	48	54	60
	0	1	2	3	4	5	6	7	8	9

...

$$s_0 = 0$$

$$s_1 = T[L[1], 0] = 1$$

$$s_2 = T[L[2], 1] = 5$$

$$s_3 = T[L[3], 5] = 17$$

$$s_4 = T[L[4], 17] = 33$$

$$s_5 = T[L[5], 33] = 67$$

$$s_6 = T[L[6], 67] = 408$$

Actually, no table T is needed:

$$s_i = T[L[i], s_{i-1}] = r_i + n \lfloor s_{i-1} / w_i \rfloor + s_{i-1} \bmod w_i$$

where $(x \bmod y) = x - \lfloor x/y \rfloor y \leq y - 1$ is the rest of the integer division (the “modulo”) between x and y , $r_i = 1 + \sum_{j=1}^i w_j$ for $i > 1$ and $r_1 = 1$. We just need the array $R[1..m] = [r_1, \dots, r_m]$.

Asymmetric Numeral Systems

- Actually, no table T is needed:

$$s_i = T[L[i], s_{i-1}] = r_i + n \lfloor s_{i-1} / w_i \rfloor + s_{i-1} \bmod w_i, \text{ where } r_i = 1 + \sum_{j=1}^i w_j \text{ for } i > 1 \text{ and } r_1 = 1.$$

- Note that $s_i = T[L[i], s_{i-1}] = r_i + n \lfloor s_{i-1} / w_i \rfloor + s_{i-1} \bmod w_i \leq r_i + s_{i-1} / P(L[i]) + w_i - 1$ and since $r_i + w_i - 1 \leq n$, we have $s_i \leq n + s_{i-1} / P(L[i])$.

- **Lemma.**

$$\text{The } i\text{-th state } s_i \text{ is such that } s_i < i \cdot \frac{n}{\prod_{j=1}^i P(L[j])}.$$

Proof. Proceed by induction. For the base case $i = 1$ we have $s_0 = 0$ and $s_1 \leq n + 0 / P(L[1]) = n < n / P(L[1])$.

Now assume $s_{i-1} < (i-1) \cdot \frac{n}{\prod_{j=1}^{i-1} P(L[j])}$ is true.

$$\text{Then, } s_i \leq n + s_{i-1} / P(L[i]) < n + (i-1) \cdot \frac{n}{P(L[i]) \cdot \prod_{j=1}^{i-1} P(L[j])} < i \cdot \frac{n}{\prod_{j=1}^i P(L[j])}, \text{ since } n < \frac{n}{\prod_{j=1}^i P(L[j])}. \blacksquare$$

Asymmetric Numeral Systems — Optimality

- **Optimality.** The number of bits emitted by ANS for encoding $L[1..n]$ is at most $nH + 2 \log_2 n + 1$, where H is the entropy of L .

Proof. For the previous lemma, we have that $s_n < \frac{n^2}{\prod_{i=1}^n P(L[i])}$.

Therefore

$$\begin{aligned} \lceil \log_2 s_n \rceil &< \log_2 s_n + 1 < \log_2 \frac{n^2}{\prod_{i=1}^n P(L[i])} + 1 = 2 \log_2 n - \log_2 \left(\prod_{i=1}^n P(L[i]) \right) + 1 = \\ &= 2 \log_2 n + \sum_{i=1}^n \log_2 (1/P(L[i])) + 1 = 2 \log_2 n + nH + 1. \blacksquare \end{aligned}$$

- Worse than Arithmetic Coding ($2/n$ bits/symb vs. $(2 \log_2 n)/n$ bits/symb), but the loss vanishes for growing values of n . Still much better than Huffman.

Further Readings

- Sections 5.1, 5.2, 5.3 of:
Alistair Moffat and Andrew Turpin. 2002. *Compression and coding algorithms*. Springer Science & Business Media, ISBN 978-1-4615-0935-6.
- Sections 2.1, 2.2, 2.5, 2.6, 5.1, 5.2, 6 of:
Alistair Moffat. 2019. *Huffman Coding*. ACM Computing Surveys. 52, 4, Article 85 (July 2020), 35 pages. <https://doi.org/10.1145/3342555>
- Chapter 5.5 (pages 826-838) of:
Robert Sedgewick and Kevin Wayne. 2011. *Algorithms*. 4-th Edition. Addison-Wesley Professional, ISBN 0-321-57351-X.
- Section 2.6 (about Huffman coding) of:
Gonzalo Navarro. 2016. *Compact Data Structures*. Cambridge University Press, ISBN 978-1-107-15238-0.
- Section 3.8 of:
G. E. P. and Rossano Venturini. 2020. *Techniques for Inverted Index Compression*. ACM Computing Surveys. 53, 6, Article 125 (November 2021), 36 pages. <https://doi.org/10.1145/3415148>