A Crash Course on Data Compression

4. Statistical Compressors

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Overview

- Shannon-Fano
- Huffman, Canonical Huffman
- Arithmetic
- Asymmetric Numeral Systems

The *Statistical* Coding Problem

- **Problem.** We are given a list $L[1..n]$ of n symbols and we are asked to compress it in *as few as possible bits*.
- Without loss of generality, we are going to assume that the symbols are positive integers.
- **Idea.** Build a *statistical model* for L and assign codewords *based on the model*.
- **Simplest statistical model.** Occurrences of each *distinct* symbol $\langle s_1, \ldots, s_m \rangle$ in L.
  We will refer to these occurrences as the *weights* $\langle w_1, \ldots, w_m \rangle$ for the symbols $\langle s_1, \ldots, s_m \rangle$. It follows that $\sum_{i=1}^{m} w_i = n$.

Example for $L = [1,3,1,1,1,5,2,1,7,3,1,2,1,1,1,1]$. We have $n = 16$ and $m = 5$.
The distinct symbols $\langle s_1, \ldots, s_5 \rangle$ are $\langle 1,2,3,5,7 \rangle$ and the weights $\langle w_1, \ldots, w_5 \rangle$ are $\langle 10,2,2,1,1 \rangle$.

The *Minimum-Redundancy* Coding Problem

- **Problem.** We are given a list $L[1..n]$ of n symbols with weights $W = \langle w_1, \ldots, w_m \rangle$, where $m \leq n$ and $\sum_{i=1}^{m} w_i = n$. We are asked to determine a set of *codeword lengths* $T = \langle \ell_1, \ldots, \ell_m \rangle$ such that:

  (1) $\sum_{i=1}^{m} 2^{-\ell_i} \leq 1$ (Kraft-McMillan inequality compliancy)

  (2) the *cost* $C(W, T) = \sum_{i=1}^{m} (w_i \cdot \ell_i)$ bits of the coded L is *minimum*.

  We will refer to the set of codeword lengths $T = \langle \ell_i \rangle$ as the *code*. A code T satisfying **(1)** and **(2)** above is said to be an optimal or *minimum-redundancy* code for W, that is: $C(W, T) \leq C(W, T')$ for any other code $T' = \langle \ell_i' \rangle$.

- Once the set T has been determined, it is easy to assign *prefix-free* codewords.

Assigning *Canonical* Prefix-Free Codewords

- Given the set of codeword lengths $\langle \ell_1, \ldots, \ell_m \rangle$, in non-decreasing order:

- **Algorithm.**
  - The first codeword is $C_1 = 0^{\ell_1}$. Now let $\ell = \ell_1$.
  - For all $i = 2, \ldots, m$:
    1. Let C_i be the smallest unassigned *lexicographic* codewords of ℓ bits (the one "coming after" C_{i-1}). If $\ell_i = \ell$, then return C_i. Otherwise ($\ell_i > \ell$), C_i is padded with possible 0s to the right until a codeword of ℓ_i is obtained.
    2. Set $\ell = \ell_i$.

- This assignment is also known as *canonical* because it allows fast encoding/decoding using simple tables.
(Note that the codewords are *lexicographically* sorted.)

Example for $\langle \ell_i \rangle = \langle 1,2,4,4,4,4 \rangle$.

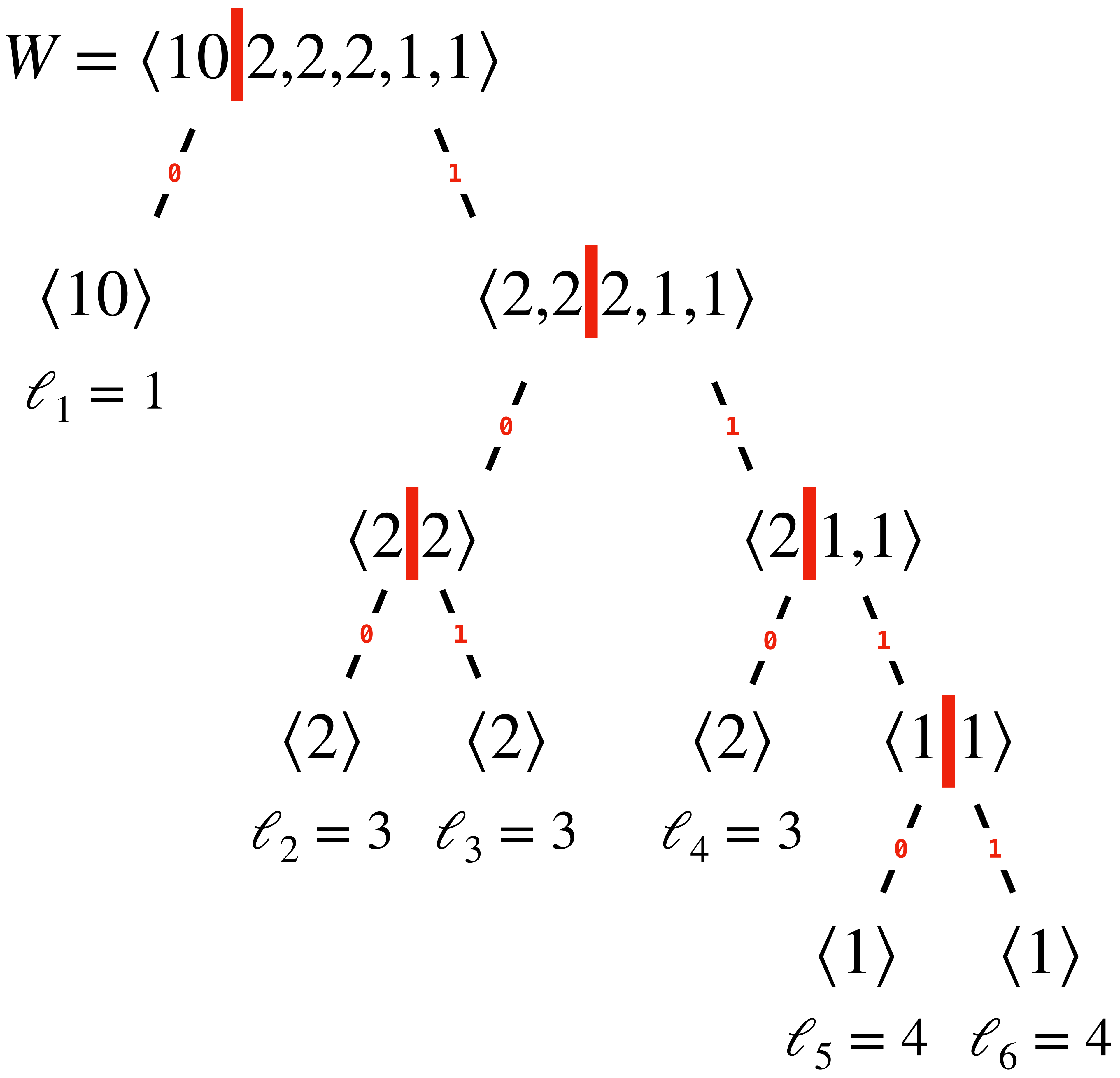
ℓ_i	C_i
1	0
2	10
4	1100
4	1101
4	1110
4	1111

Shannon-Fano

Shannon, 1949 — Fano, 1949

- A greedy, top-down, algorithm to compute a set of codeword lengths.
- It does *not always* produce an optimal prefix-free code.
- Remember the "golden rule" of data compression: assign shorter codewords to more frequent symbols.
- **Algorithm.**
  - Sort the set $\langle w_i \rangle$ in non-increasing order and partition it into two sets $\langle w_i' \rangle$ and $\langle w_i'' \rangle$, of size m' and $m - m'$ respectively, so that $\sum_{i=1}^{m'} w_i' \approx \sum_{i=m'+1}^{m-m'} w_i''$.
  - Proceed recursively on each of the two partitions. Stop when the partition size is 1.
- Informally: the codeword length ℓ_i for the weight w_i is the number of partitions that contain w_i.

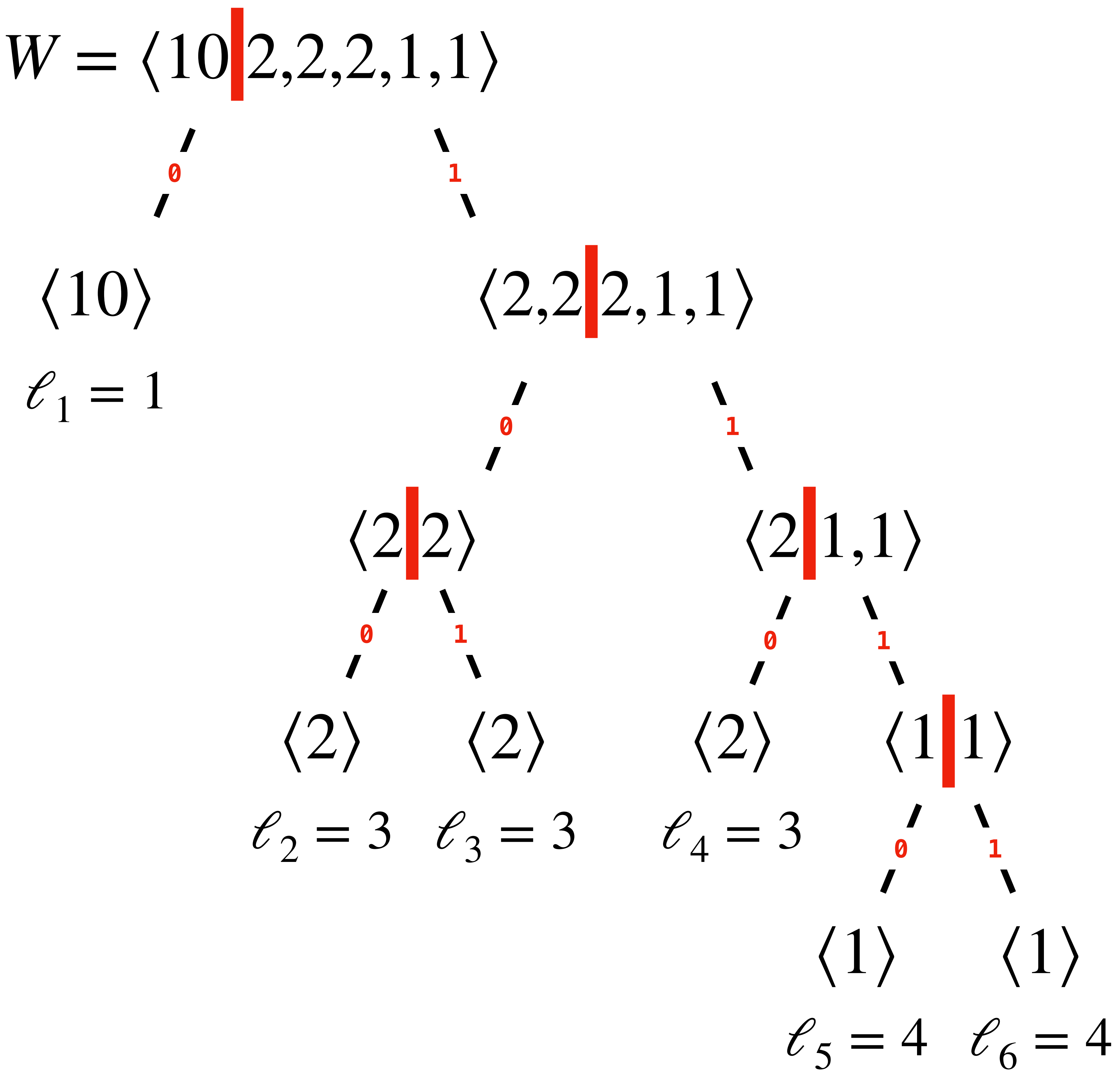
Shannon-Fano — Example



$\langle \ell_i \rangle = \langle 1,3,3,3,4,4 \rangle$

ℓ_i	C_i
1	0
3	100
3	101
3	110
4	1110
4	1111

Shannon-Fano — Example



$\langle \ell_i \rangle = \langle 1,3,3,3,4,4 \rangle$

ℓ_i	C_i
1	`0`
3	`100`
3	`101`
3	`110`
4	`1110`
4	`1111`

- Because of the approximate partitioning rule, the resulting code may not be optimal.
- It is easy to see why the algorithm produces a *prefix-free* code: each symbol is associated to a distinct leaf and two siblings are distinguished by two different symbols, `0` and `1`.

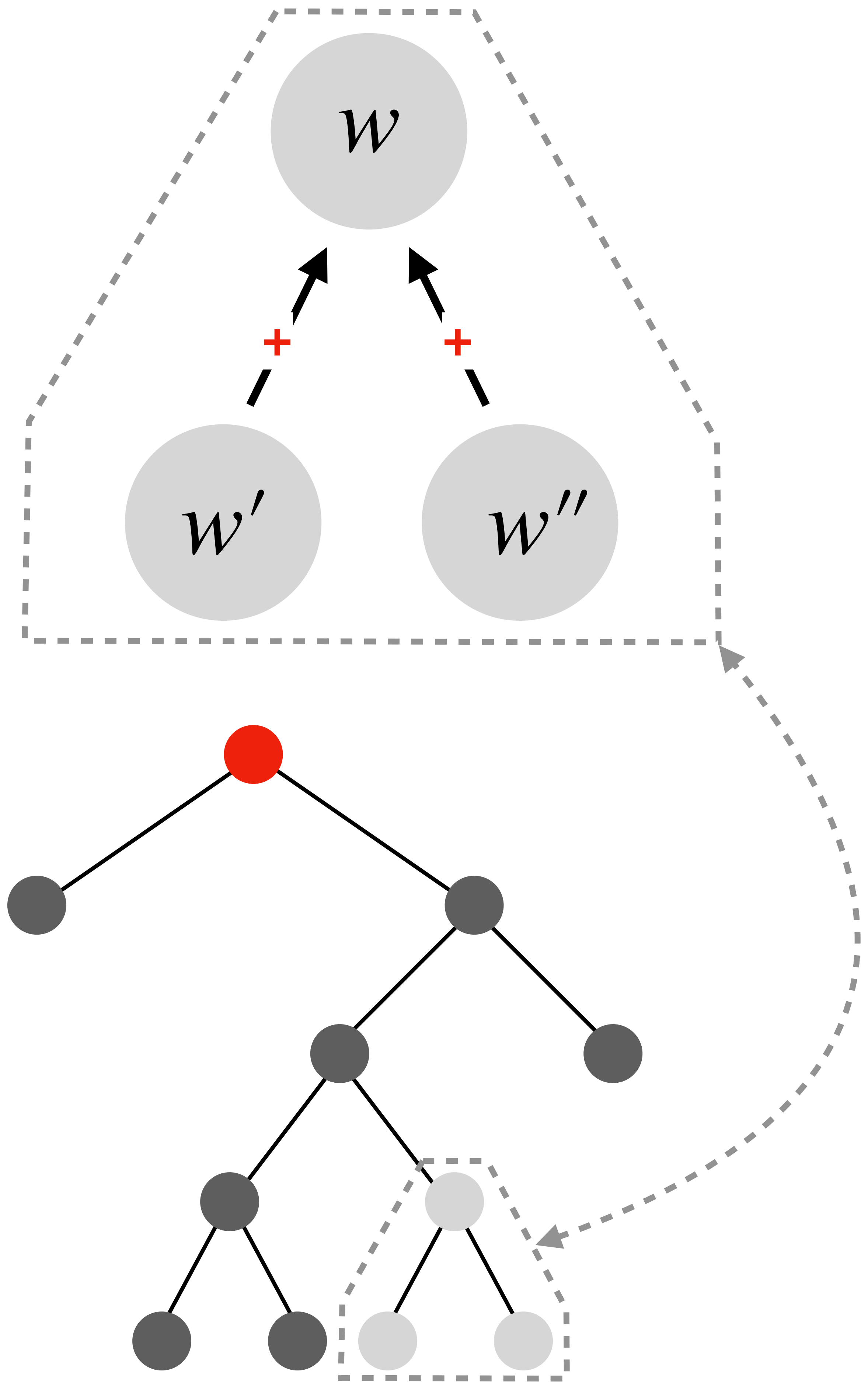
Huffman

Huffman, 1952

- A greedy, bottom-up, algorithm to compute a set of codeword lengths.
  It always produces an *optimal* prefix-free code.

- Invented by Huffman in 1951 at MIT, while attending a class taught by Fano.

- **Algorithm.**
  - Maintain a set of weights S. At the beginning: $S = W = \langle w_i \rangle$.
  - At each step: two smallest weights are removed from S, say w' and w'', and the new weight $w = w' + w''$ is added to S. The sequence of merging operations implicitly defines parent-to-children relationships that it is handy to model as a binary tree.
  - Repeat until $|S| = 1$: at this point the only weight in S is n and it is logically associated to the **root** of the Huffman tree. The tree is used to derive the codeword lengths (and the actual codewords if wanted).



Huffman — Demo

$[1,\langle 10\rangle]$ $[2,\langle 2\rangle]$ $[3,\langle 2\rangle]$ $[4,\langle 2\rangle]$ $[5,\langle 1\rangle]$ $[6,\langle 1\rangle]$

$S = \{[1,\langle 10\rangle], [2,\langle 2\rangle], [3,\langle 2\rangle], [4,\langle 2\rangle], [5,\langle 1\rangle], [6,\langle 1\rangle]\}$

node ids

Huffman — Demo

new
node id

$[7, \langle 2 \rangle]$

0 1

$[1, \langle 10 \rangle]$ $[2, \langle 2 \rangle]$ $[3, \langle 2 \rangle]$ $[4, \langle 2 \rangle]$ $[5, \langle 1 \rangle]$ $[6, \langle 1 \rangle]$

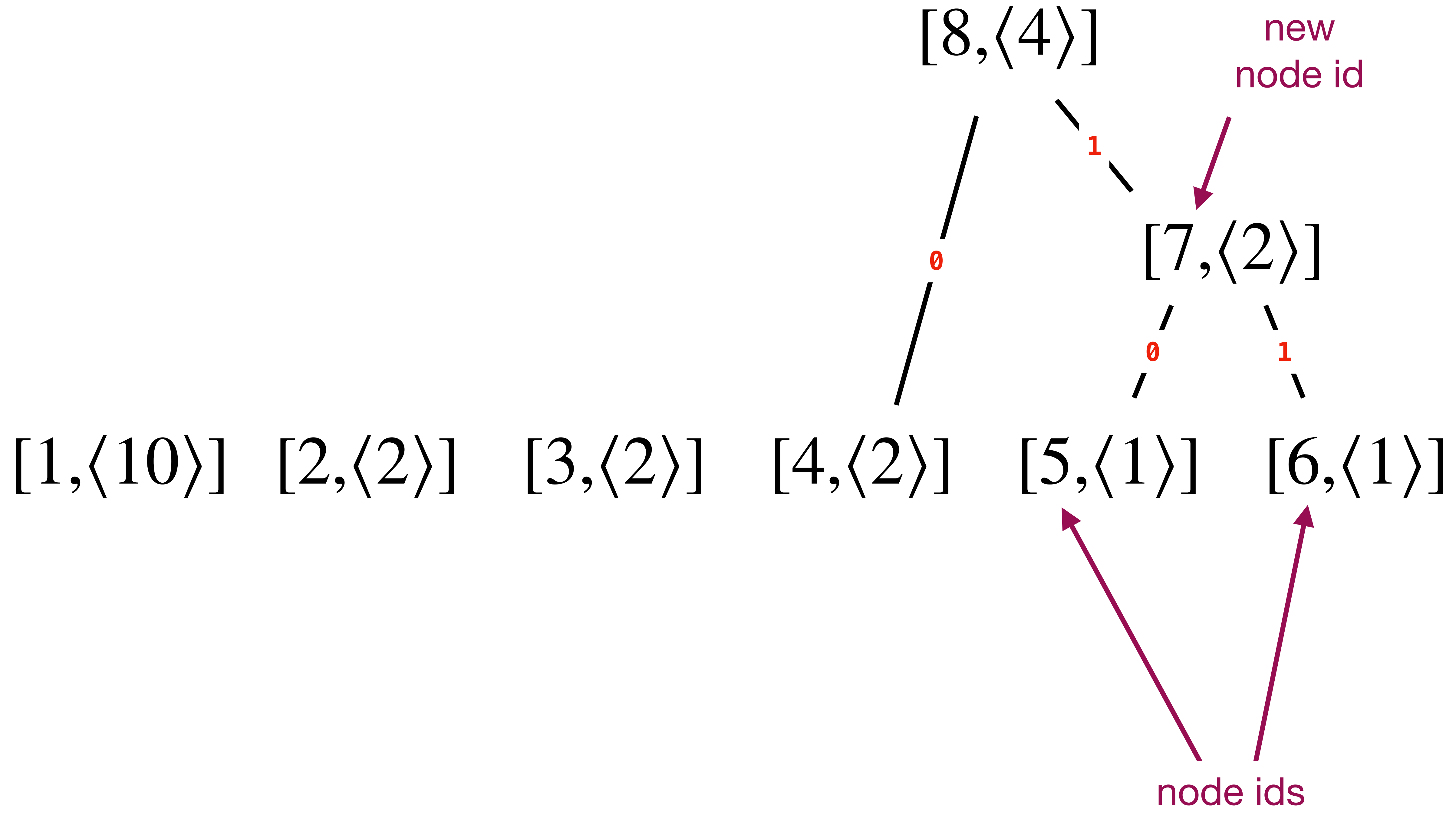
node ids

children of
node 7

$S = \{[1, \langle 10 \rangle], [2, \langle 2 \rangle], [3, \langle 2 \rangle], [4, \langle 2 \rangle], [7, \langle 2 \rangle, (5,6)]\}$

$S = \{[1, \langle 10 \rangle], [2, \langle 2 \rangle], [3, \langle 2 \rangle], [4, \langle 2 \rangle], [5, \langle 1 \rangle], [6, \langle 1 \rangle]\}$

Huffman — Demo



children of node 7

$$S = \{[1,\langle 10\rangle], [2,\langle 2\rangle], [3,\langle 2\rangle], [8,\langle 4\rangle, (4,7)]\}$$

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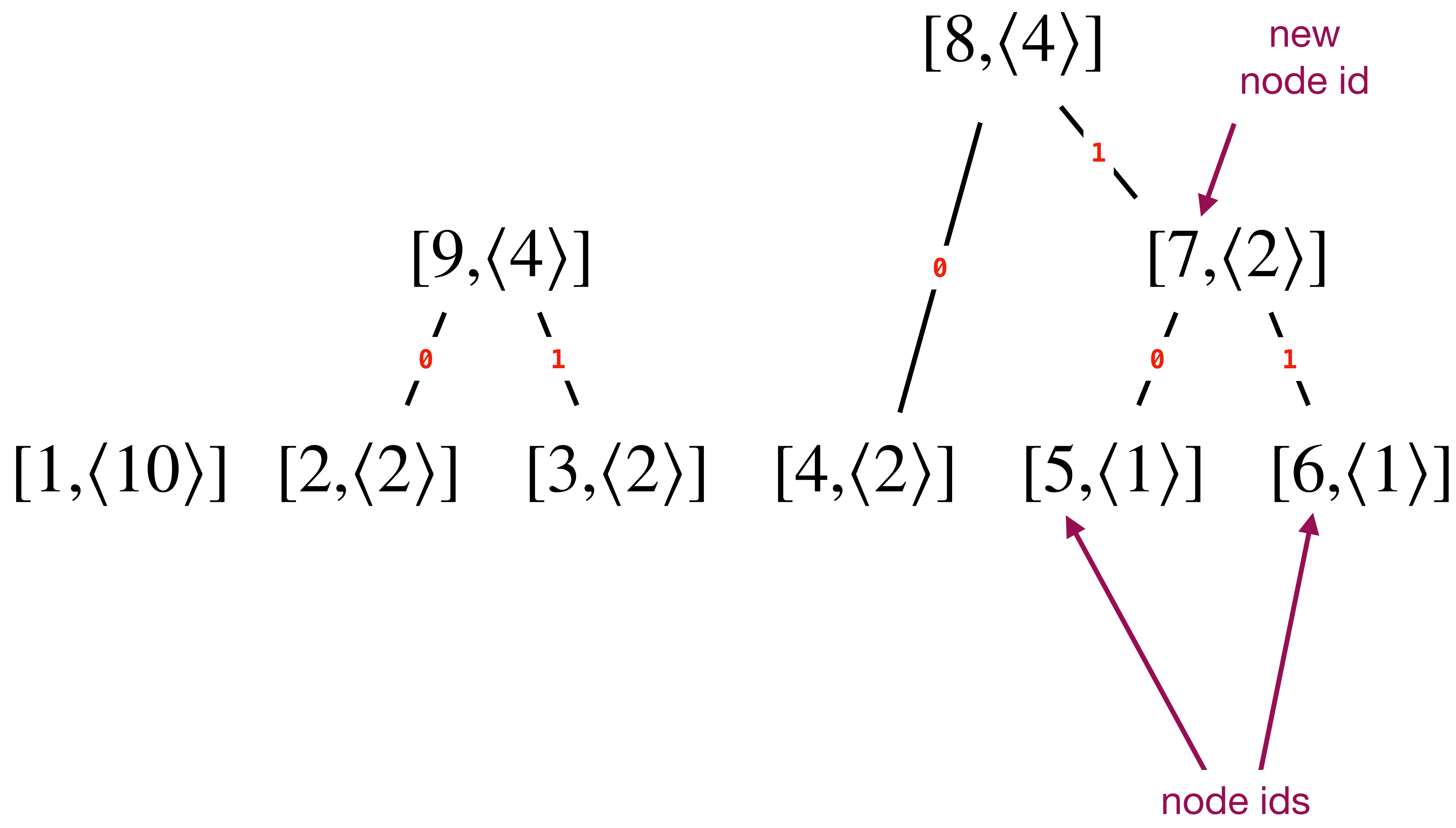
Huffman — Demo



$$S = \{[1,\langle 10\rangle], [9,\langle 4\rangle, (2,3)], [8,\langle 4\rangle, (4,7)]\}$$

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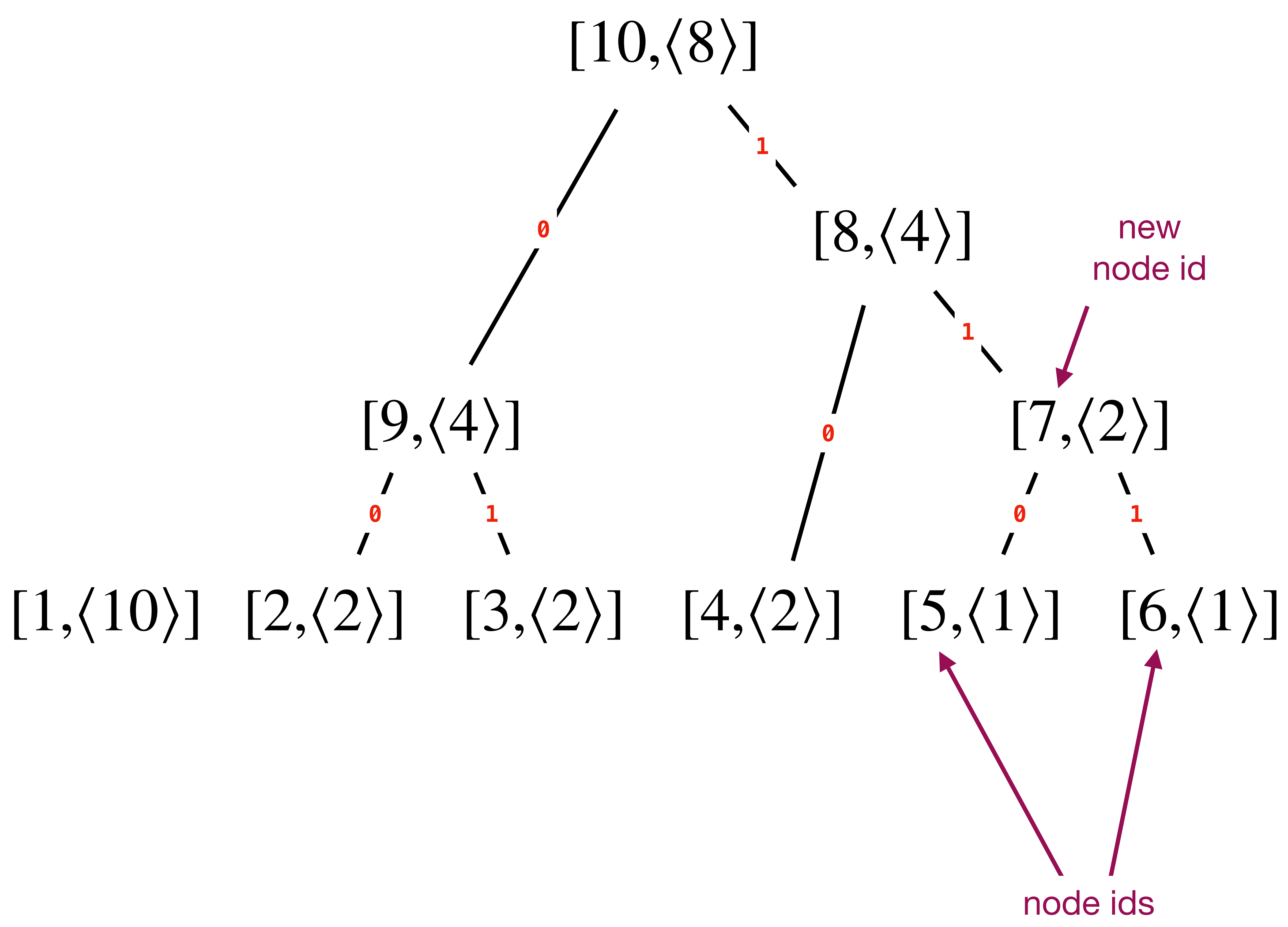

$S = \{[1,\langle 10\rangle], [10,\langle 8\rangle, (9,8)]\}$

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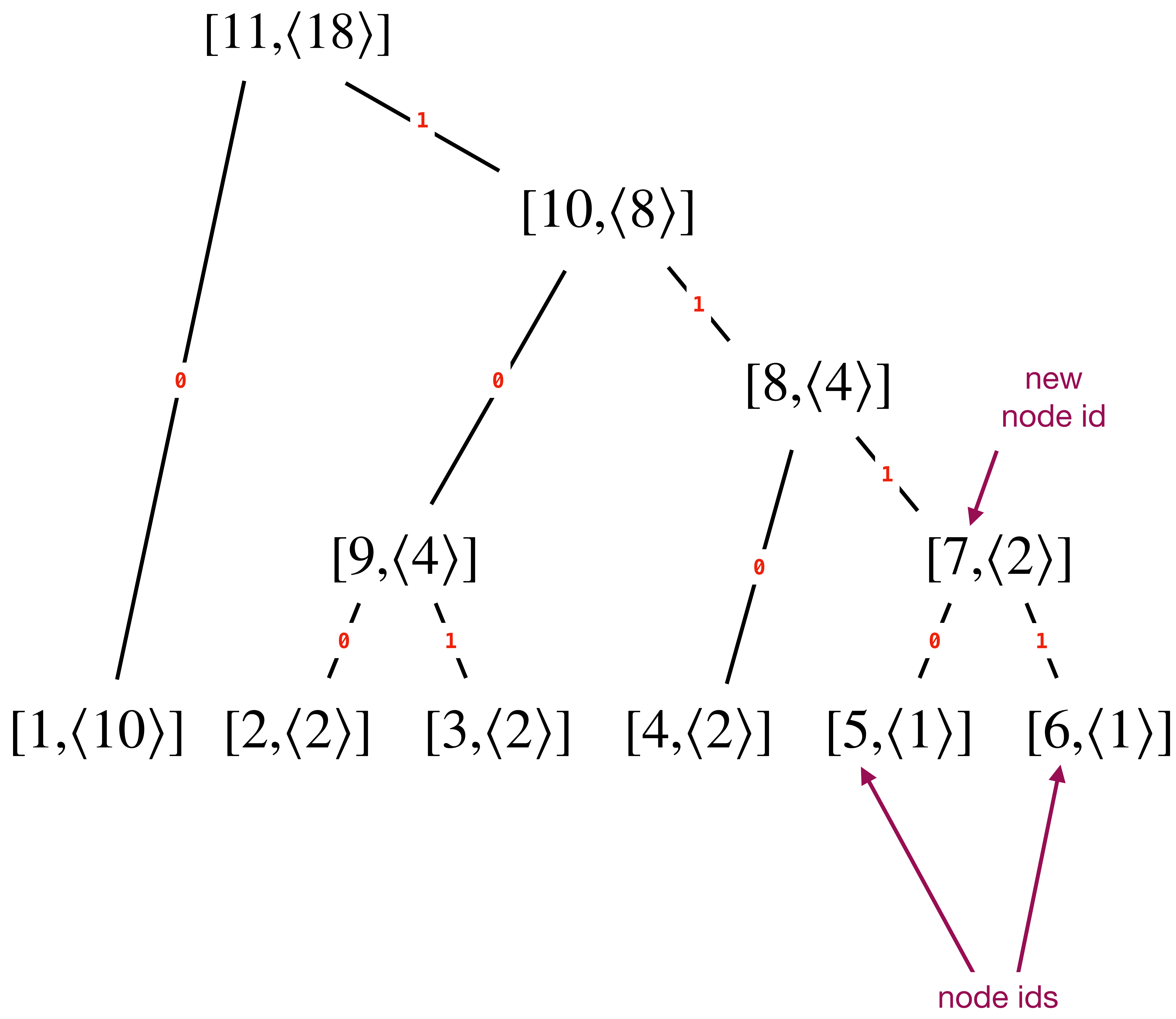

$S = \{[11,\langle 18\rangle,(1,10)]\}$

$S = \{[1,\langle 10\rangle],[10,\langle 8\rangle,(9,8)]\}$

$S = \{[1,\langle 10\rangle],[9,\langle 4\rangle,(2,3)],[8,\langle 4\rangle,(4,7)]\}$

children of node 7

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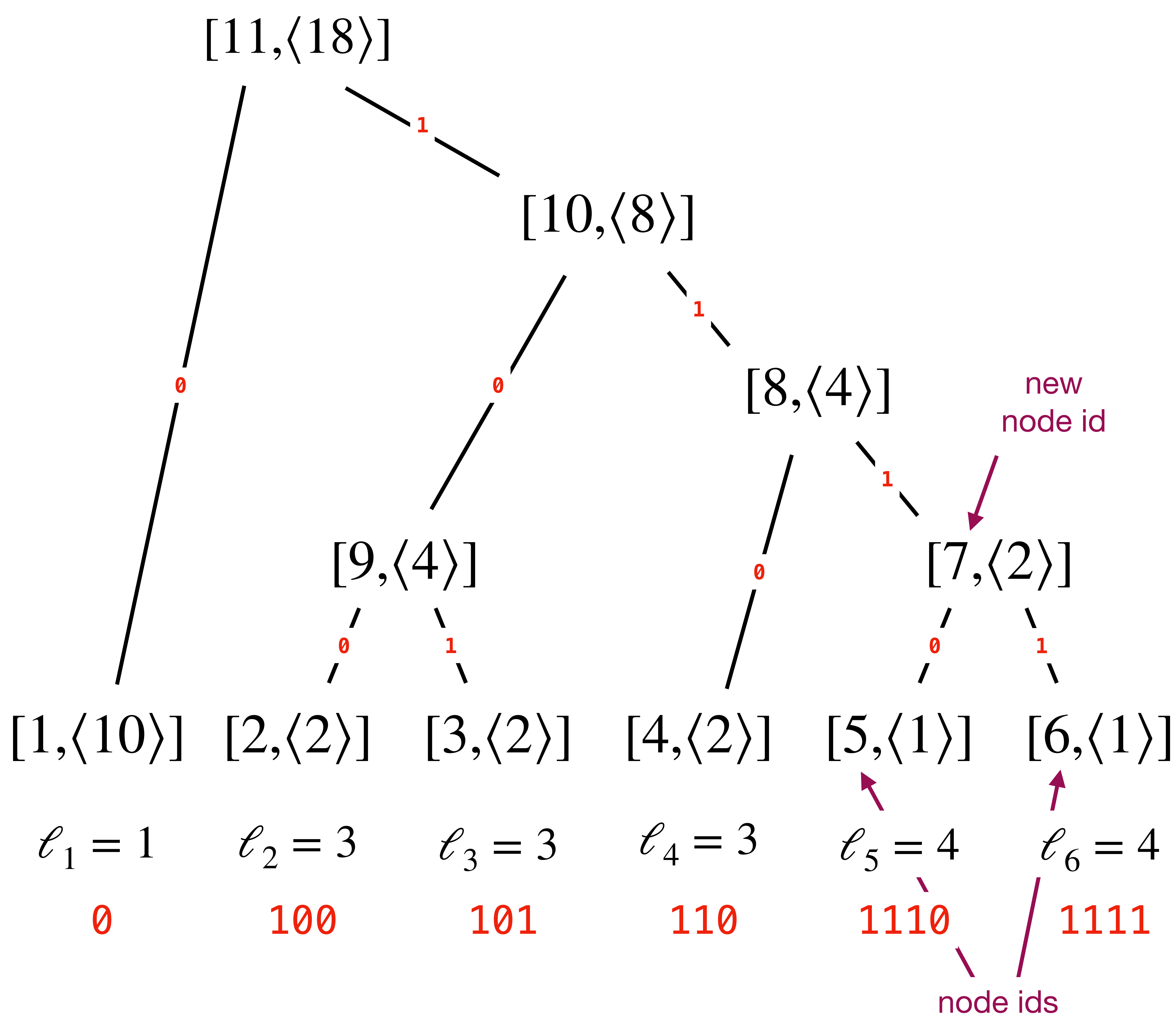
Huffman — Demo



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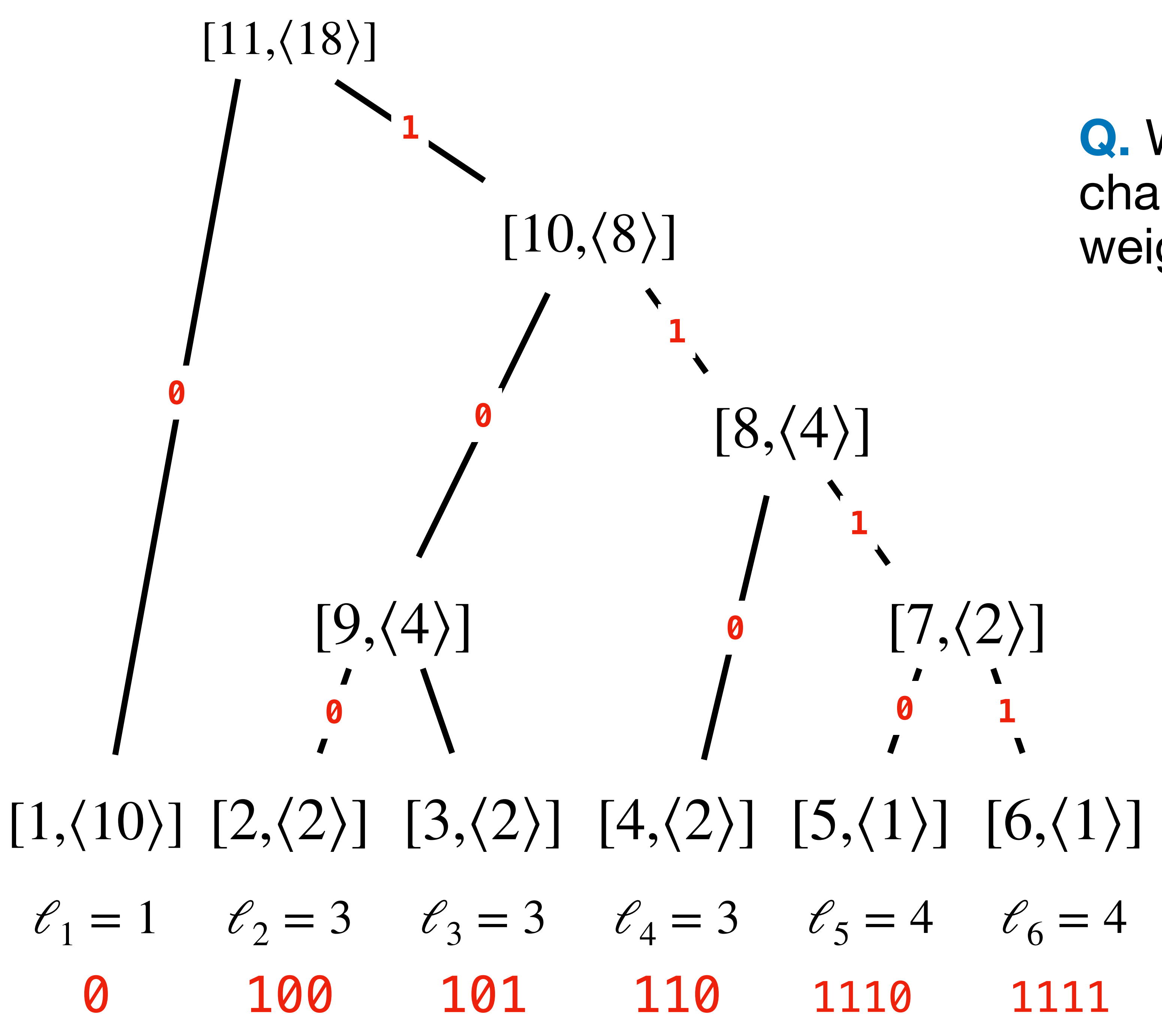
$S = \{[1,\langle 10\rangle], [2,\langle 2\rangle], [3,\langle 2\rangle], [4,\langle 2\rangle], [5,\langle 1\rangle], [6,\langle 1\rangle]\}$

codeword lengths

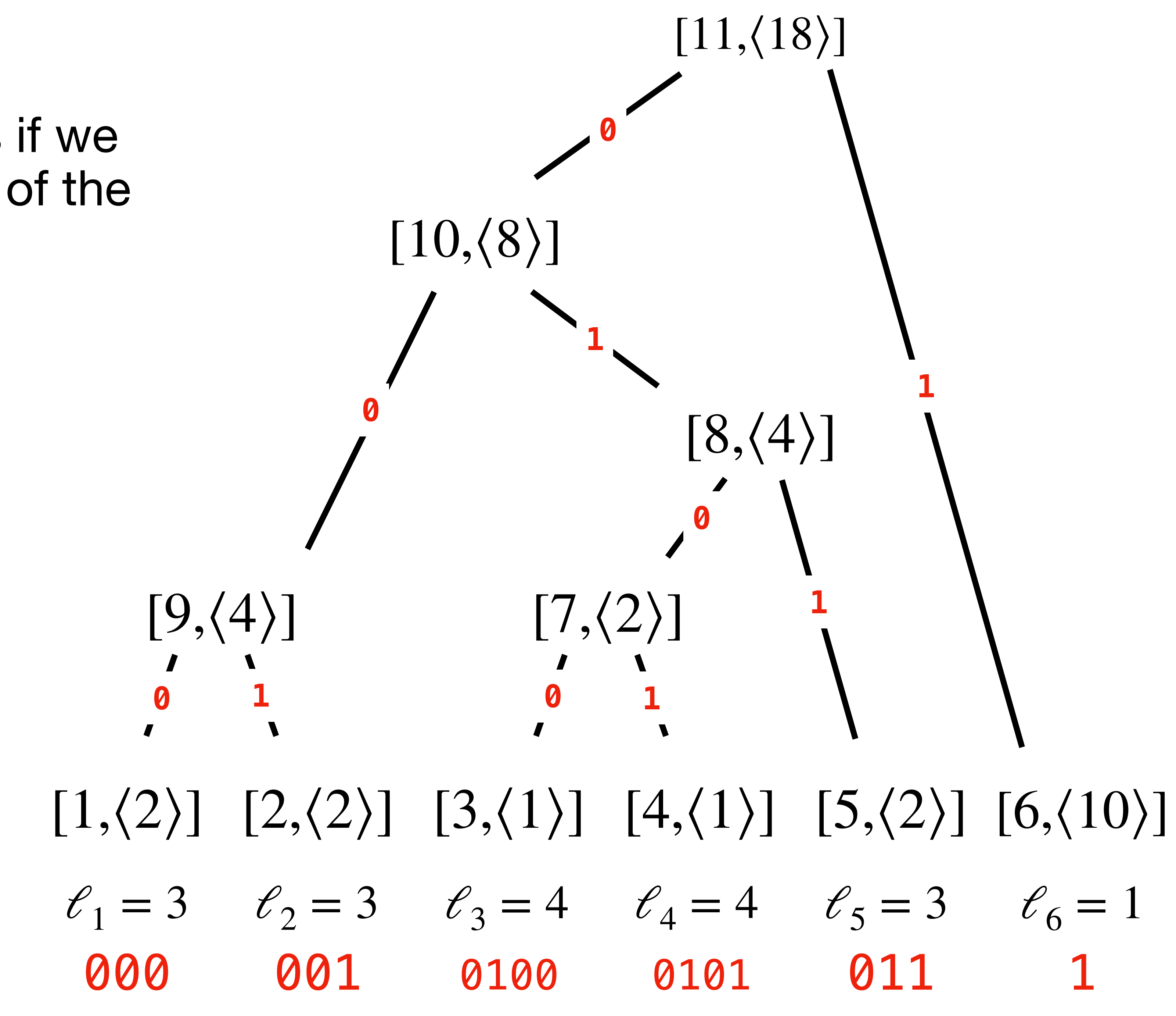
actual codewords

Huffman — Different Orderings of the Weights



(a) **Sorted** Weights

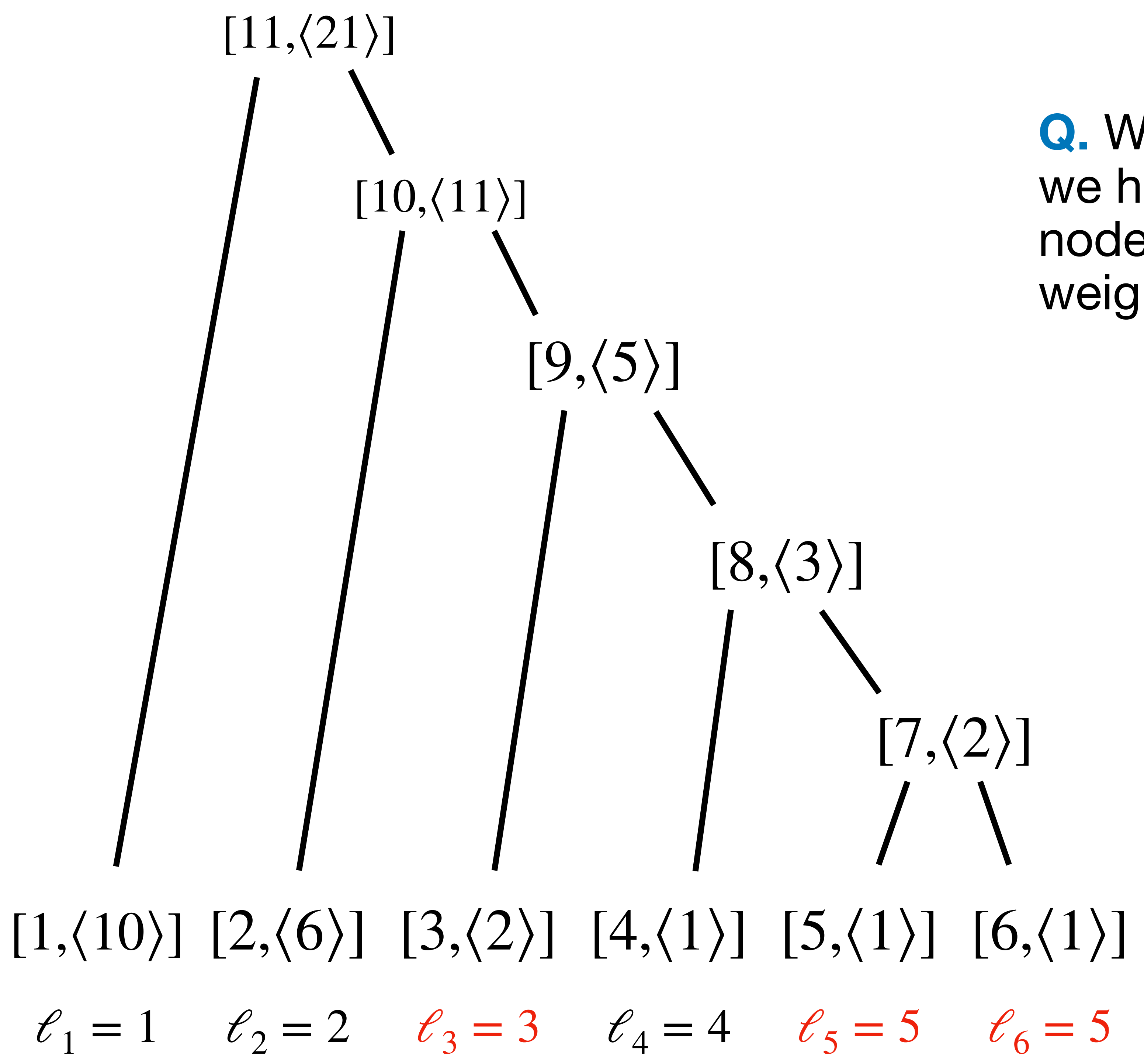
Q. What happens if we change the order of the weights?



(b) **Unsorted** Weights

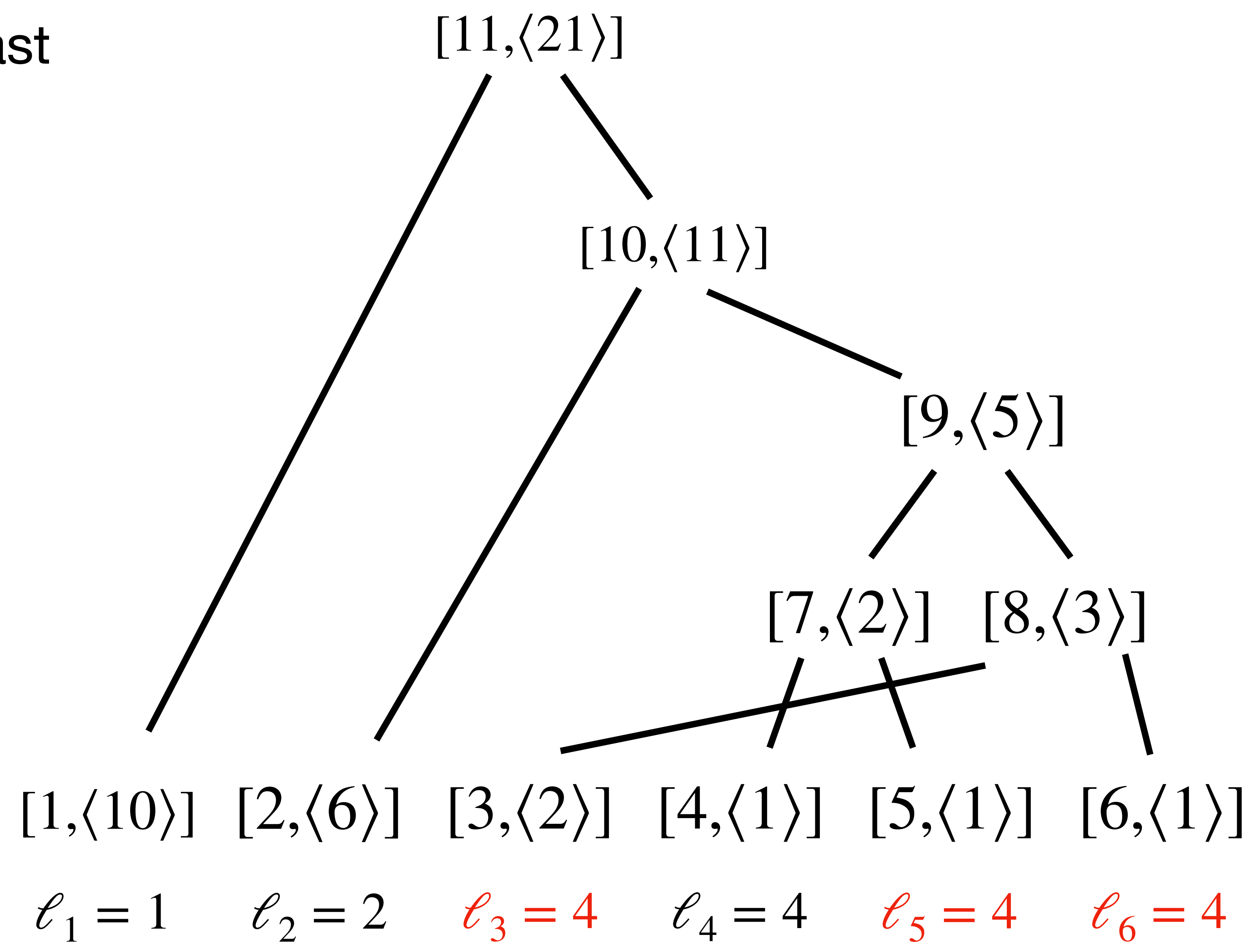
- Same codeword lengths, but different actual codewords.
- Labelling *left*-leaning edges with a `0` and *right*-leaning edges with a `1`, we have that: different orderings of the weights can produce *different* sets of *valid* codewords.

Huffman — Different Merging Strategies



(a) **Higher node ids** first

Q. What happens when we have more than two nodes with the same least weight?



(b) **"Older" nodes first** (created farthest in the past)

if node x has a smaller id than node y, then it was created before y

- Same *average* codeword length, but different *maximum* codeword length.
- Different merging strategies can produce different sets of codeword lengths.
- In the example, we have $\langle 1,2,3,4,5,5 \rangle$ in (a) and $\langle 1,2,4,4,4,4 \rangle$ in (b).
  In both cases the average codeword length is 2.

Canonical Huffman

- If the weights are *sorted in non-increasing order* and we label *left*-leaning edges with a `0` and *right*-leaning edges with a `1`, then we always obtain *lexicographic sorted codewords*.
- We refer to this construction as the *canonical* Huffman code.
- Also, we can always merge nodes that were created farthest in the past to *minimize the maximum codeword length* — a good feature for faster decoding in practice.

Shannon-Fano vs. Huffman

- Both algorithms build prefix-free codes in a greedy manner.
- Shannon-Fano:
  - top-down (divisive)
  - not optimal
- Huffman:
  - bottom-up (aggregative)
  - *optimal*

Shannon-Fano (SF) and Huffman (H) codes built for the 26-letter English alphabet. In this case, Shannon-Fano has an average codeword length of 4.16677 bits; Huffman is optimal with 4.15506 bits.

(a)

symbol	probability	SF	H
A	0.08833	010	1111
B	0.01267	111110	100000
C	0.02081	111010	00000
D	0.04376	10111	11101
E	0.14878	000	110
F	0.02455	11011	00011
G	0.01521	111100	100001
H	0.05831	1001	0111
I	0.05644	1010	0110
J	0.00080	1111111110	0001010111
K	0.00867	11111110	0001011
L	0.04124	11000	11100
M	0.02361	11100	00001

(b)

symbol	probability	SF	H
N	0.06498	1000	1001
O	0.07245	0110	1011
P	0.02575	11010	01000
Q	0.00080	11111111110	000101010
R	0.06872	0111	1010
S	0.05537	10110	0101
T	0.09351	001	001
U	0.02762	11001	01001
V	0.01160	1111110	000100
W	0.01868	111011	100011
X	0.00146	111111110	00010100
Y	0.01521	111101	100010
Z	0.00053	11111111111	0001010110

Probabilities taken from page 174 of:
David Salomon. 2007. *Variable-Length Codes for Data Compression*.
Springer Science & Business Media, ISBN 978-1-84628-959-0.

Huffman — Optimality

- **Optimality.** The Huffman algorithm produces a minimum-redundancy prefix-free code $Huf_m = \langle \ell_i \rangle_m$ for the weights $W_m = \langle w_i \rangle_m$.

- The proof builds on two ingredients:

  1. Max-Depth Leaves Property. If a code T is optimal, then its binary tree must contain two least-weight leaves that are children of the same parent (siblings), hence at the same maximum depth D in the tree.

  2. Inductive process on the number m of weights.

  2.1 Base case for $m = 2$ is always valid since the trivial optimal code is $\langle \ell_1 = 1, \ell_2 = 1 \rangle$ (and the two codewords will always be `0` and `1`).

  2.2 Assume Huf_{m-1} is optimal for W_{m-1} and prove Huf_m is optimal for W_m.

Huffman — Max-Depth Leaves Property

- **Max-Depth Leaves Property.** If the code T is optimal, then its binary tree must contain two least-weight leaves that are children of the same parent (siblings), hence at the same maximum depth D in the tree.

  Proof.

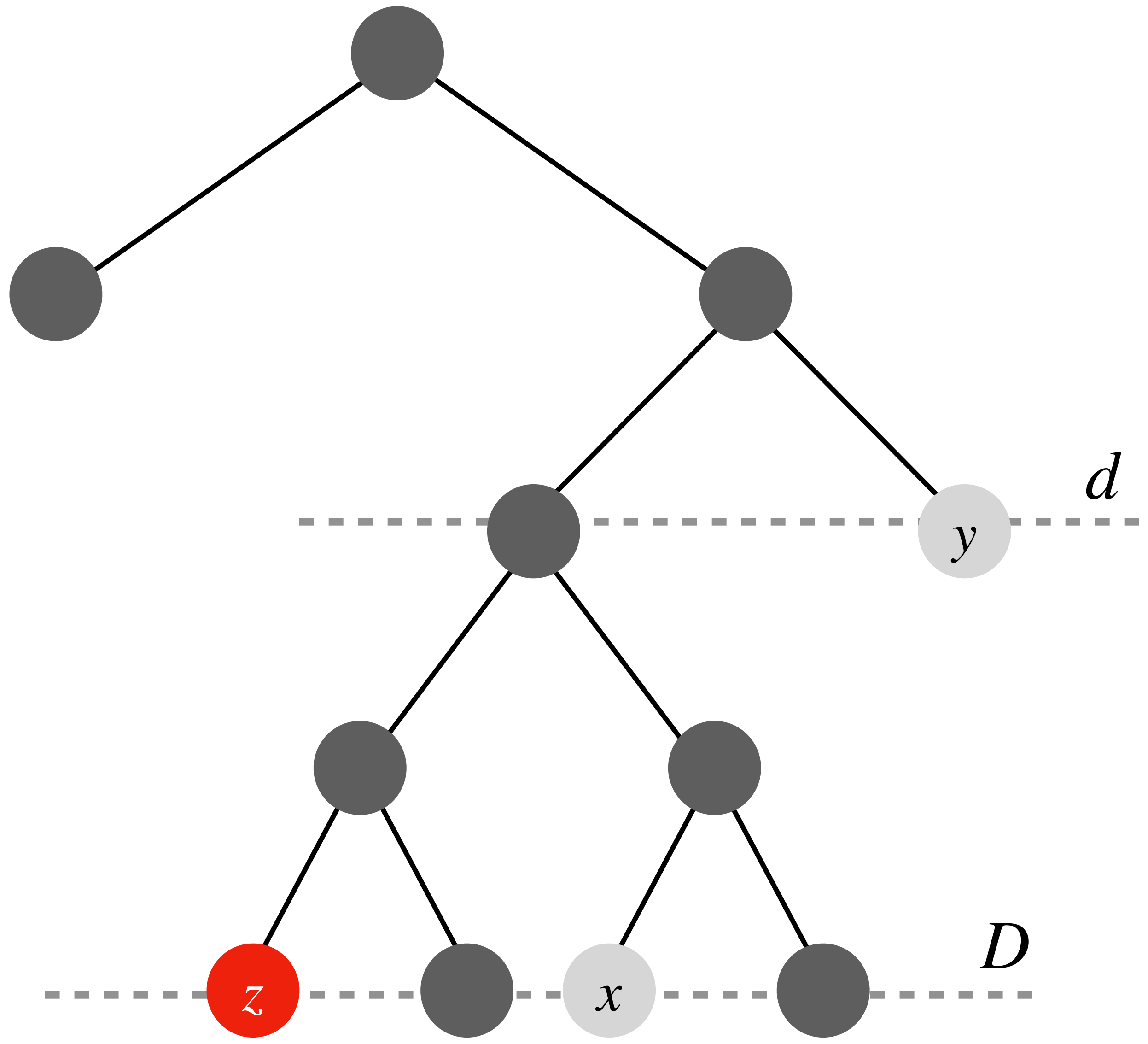
  - Suppose that the two least-weight leaves, say for symbols x and y, are *not* at the same maximum depth D. Let x be at depth D and y at some other depth $d < D$. Since each internal node must have two children, there must be another leaf z at depth D. The weight of z is then $w_z \geq w_y$ because y is a least-weight leaf by assumption.

  - If now we exchange the codeword lengths assigned to z and y, the resulting code T' will have a cost $C(T') \geq C(T)$ because T is optimal by assumption. Since the two codes only differ in the assignment for z and y:

    $$C(T') - C(T) = (w_y D + w_z d) - (w_y d + w_z D) = (w_y - w_z) \cdot (D - d) \geq 0$$

    and since $D - d > 0$, then it must be $w_y \geq w_z$. Therefore we conclude that $w_z = w_y$ and that z is indeed a least-weight leaf at maximum depth D.

  - Now that we are sure the two least-weight leaves are both at depth D, it could be that they are not children of the same parent. If that is the case, just "relabel" the leaves by sorting the weights. ■



Huffman — Induction

- Suppose that Huf_{m-1} is an optimal code for the set W_{m-1} of $m-1$ weights, for $m > 2$. (The case for $m = 2$ is the base case.) And let Opt_m be an optimal code for W_m instead. Therefore, we have

  $C(Opt_m) \leq C(Huf_m)$. **(1)**

- Since Opt_m is optimal, for the *Max-Depth Leaves Property* there must be two leaves, x and y, at the same depth D_m of least-weight, so:

  $C(Opt_m) = C(Opt_{m-1}) + (w_x + w_y) \cdot D_m$ **(2)**

  where Opt_{m-1} is an optimal code on the "reduced" weights $W_{m-1} = W_m \backslash \{w_x, w_y\} \cup \{w_x + w_y\}$.

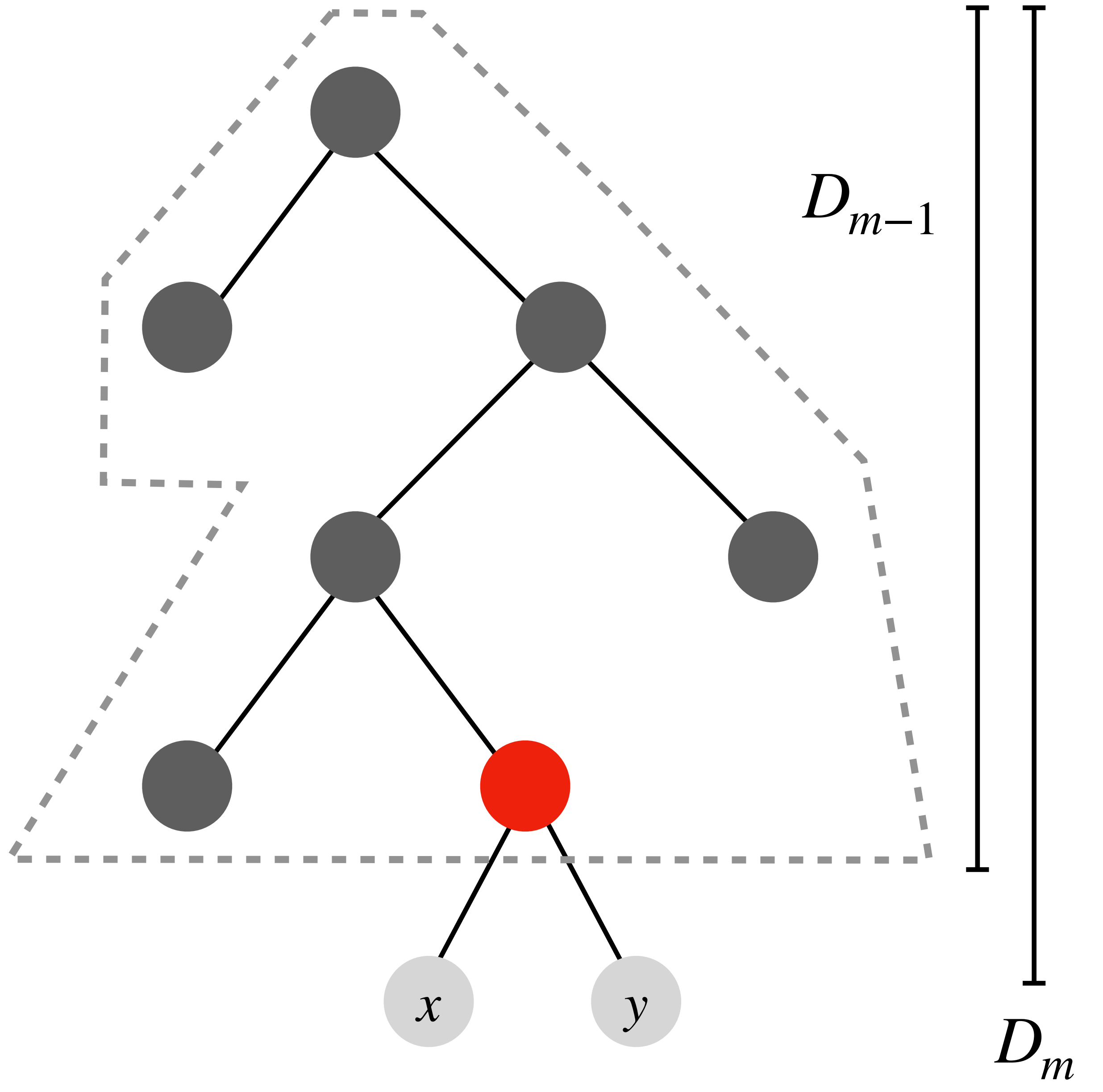
- But for the inductive hypothesis, also Huf_{m-1} is optimal on W_{m-1}, thus it must be

  $C(Huf_{m-1}) \leq C(Opt_{m-1})$. **(3)**

- Therefore, combining **(2)** with **(3)** we have:

  $C(Opt_m) = C(Opt_{m-1}) + (w_x + w_y) \cdot D_m \geq C(Huf_{m-1}) + (w_x + w_y) \cdot D_m = C(Huf_m)$.

  Combining the latter inequality with **(1)**, we obtain that it must be $C(Opt_m) = C(Huf_m)$, hence Huf_m is an optimal code for W_m. ■



Huffman — Relation to Entropy

- Recall the entropy for the probability distribution P:

  $H(P) = \sum_x P(x)\log_2\big(1/P(x)\big)$ is the *expected* codeword length for an optimal code for the distribution P. In this module, we model $P(x) = w_x/n$ using the "weights" $W = \langle w_x \rangle$, so we write

  $H = H(W) = (1/n) \cdot \sum_x w_x \log_2\big(n/w_x\big)$ bits.

- The average codeword length of an Huffman code Huf for W is $L_{Huf} = C(W, Huf)/n$.

- A fundamental result established that $H \leq L_{Huf} < H + 1$, meaning that Huffman can loose **up to 1 bit per symbol** compared to the entropy. **Q.** Is this bad or good?

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- A fundamental result established that $H \leq L_{Huf} < H + 1$, meaning that Huffman can loose **up to 1 bit per symbol** compared to the entropy. **Q.** Is this bad or good?

- **A.** It depends. If H is high, say $H \gg 1$, then the extra bit is *negligible* and Huffman reaches the entropy limit. Otherwise, if $H \ll 1$, the extra bit lost is not negligible and Huffman is *not effective*. Note that this limit is intrinsic of prefix-free codes as they must assign codewords of length ≥ 1.

Huffman — Limitation

Example for $W = \langle 999{,}1 \rangle$, where one symbol is very, very, frequent. Then the entropy is

$$H = (999/1000) \cdot \log_2(1000/999) + (1/1000) \cdot \log_2(1000) \simeq 0.01141 \text{ bits},$$

thus, much less than 1 bit because of the skewed distribution.

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- **Q.** How to overcome this limitation?
  A. Associate a codeword to *more than one symbol*.

Huffman on Blocks

- **Idea.** Compute the Huffman code on *blocks* of L.
  If B is the block size, then the extra bit lost compared to the entropy is amortized among the B symbols, thus the loss is just $1/B$ bits.

- The redundancy becomes smaller for larger B.

- But B cannot grow as much as we want!
  For large B we would incur in high space for the encoding of the Huffman tree and high decoding time.

Arithmetic Coding

Elias, ~1960

- This method offers higher compression ratios than Huffman because it is *not* a prefix-free code.

- This means that we cannot decode *individual* symbols, rather the whole message.

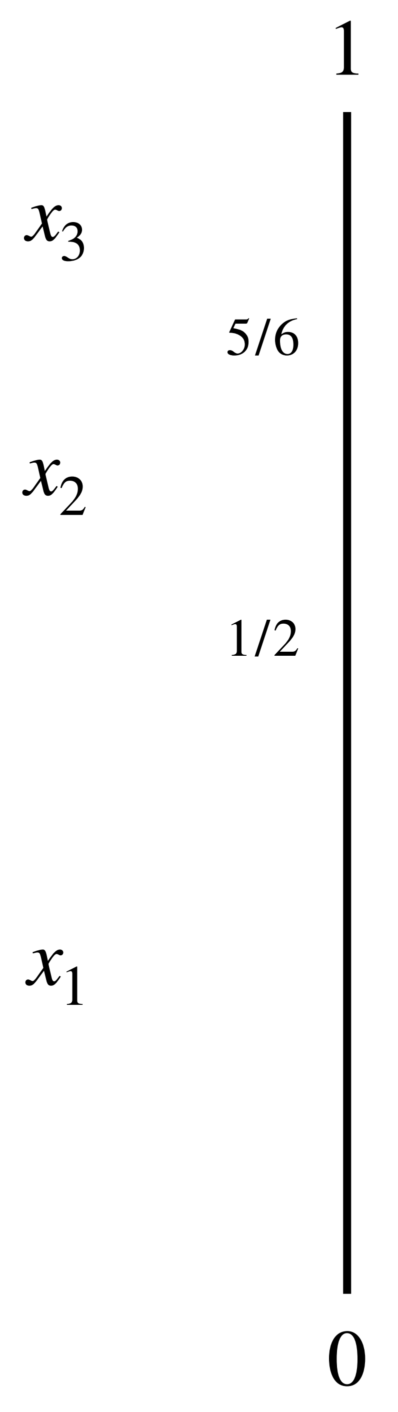
- In practice:
  - slower than (canonical) Huffman to decode;
  - no need to transmit the model (e.g., the Huffman tree).

- **Algorithm.**
  - The real interval $[0.0,1.0)$ is partitioned into segments of length *proportional* to the probabilities $P(x) = w_x/n$ in L and the segment $[l_1, r_1) \subset [0.0,1.0]$ associated to the symbol $L[1]$ is considered.
  - The same partitioning step is applied to $[l_1, r_1)$ and the segment $[l_2, r_2) \subset [l_1, r_1)$ associated to the symbol $L[2]$ is considered.
  - Repeat for all the n symbols and emit a single real number $y \in [l_n, r_n)$.

- The pair $\langle y, n \rangle$ is sufficient to decode the original list L.

Arithmetic Coding — Encoding Demo

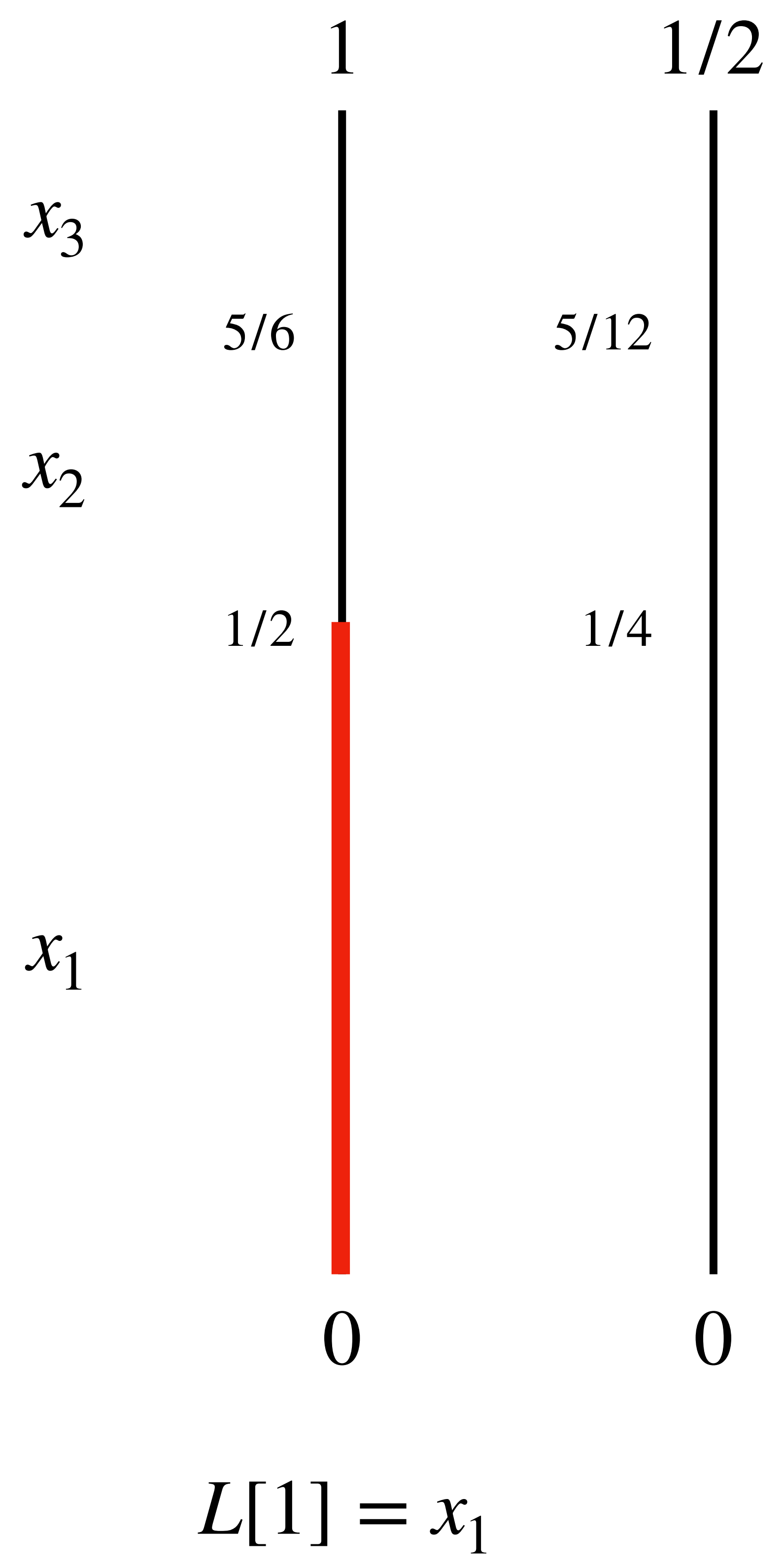
Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $P(x_1) = 1/2$, $P(x_2) = 1/3$, and $P(x_3) = 1/6$.



i	l_i	r_i
0	0.000000	1.000000
1		
2		
3		
4		
5		
6		

Arithmetic Coding — Encoding Demo

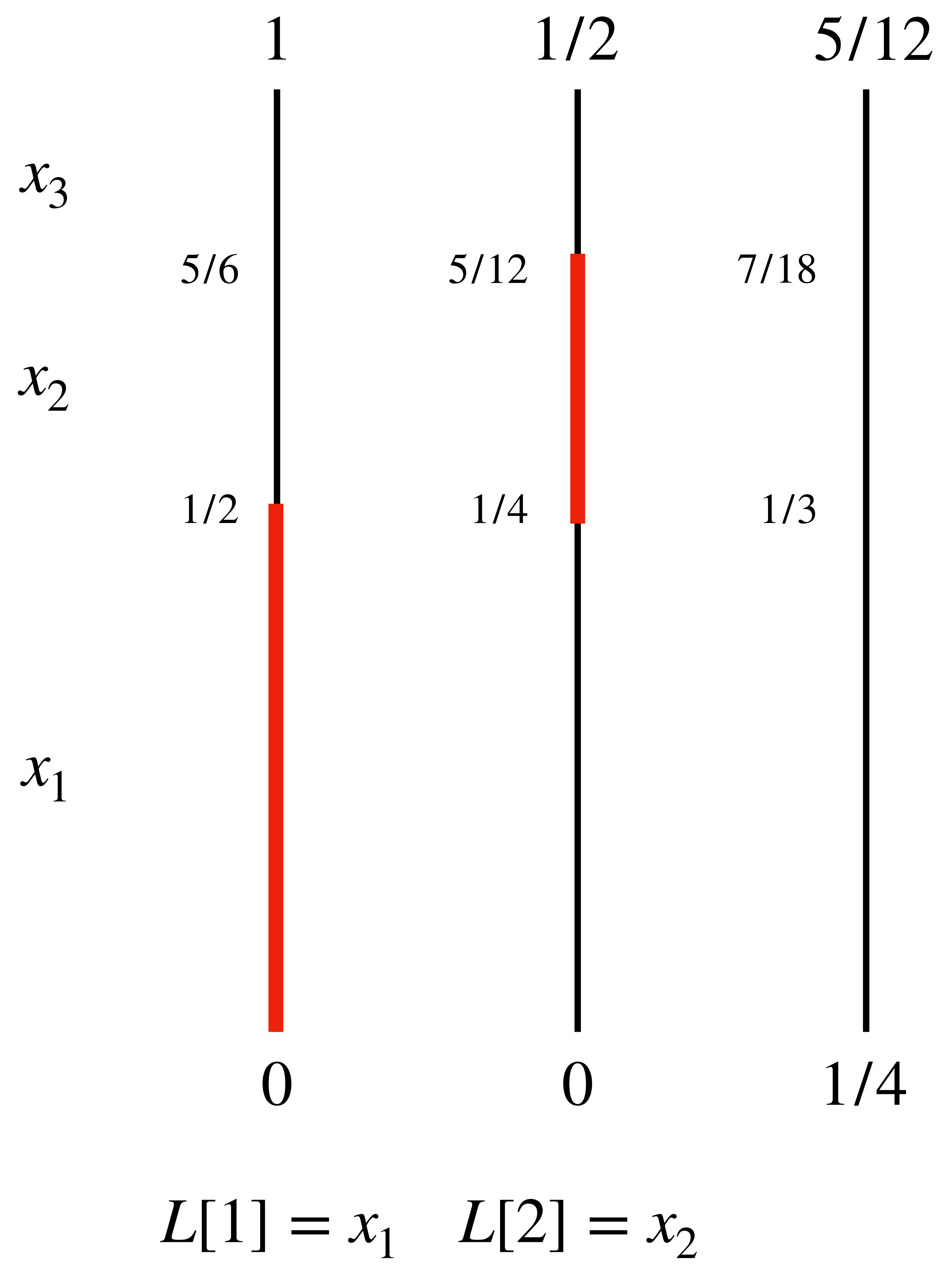
Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $P(x_1) = 1/2$, $P(x_2) = 1/3$, and $P(x_3) = 1/6$.



i	l_i	r_i
0	0.000000	1.000000
1	0.000000	0.500000
2		
3		
4		
5		
6		

Arithmetic Coding — Encoding Demo

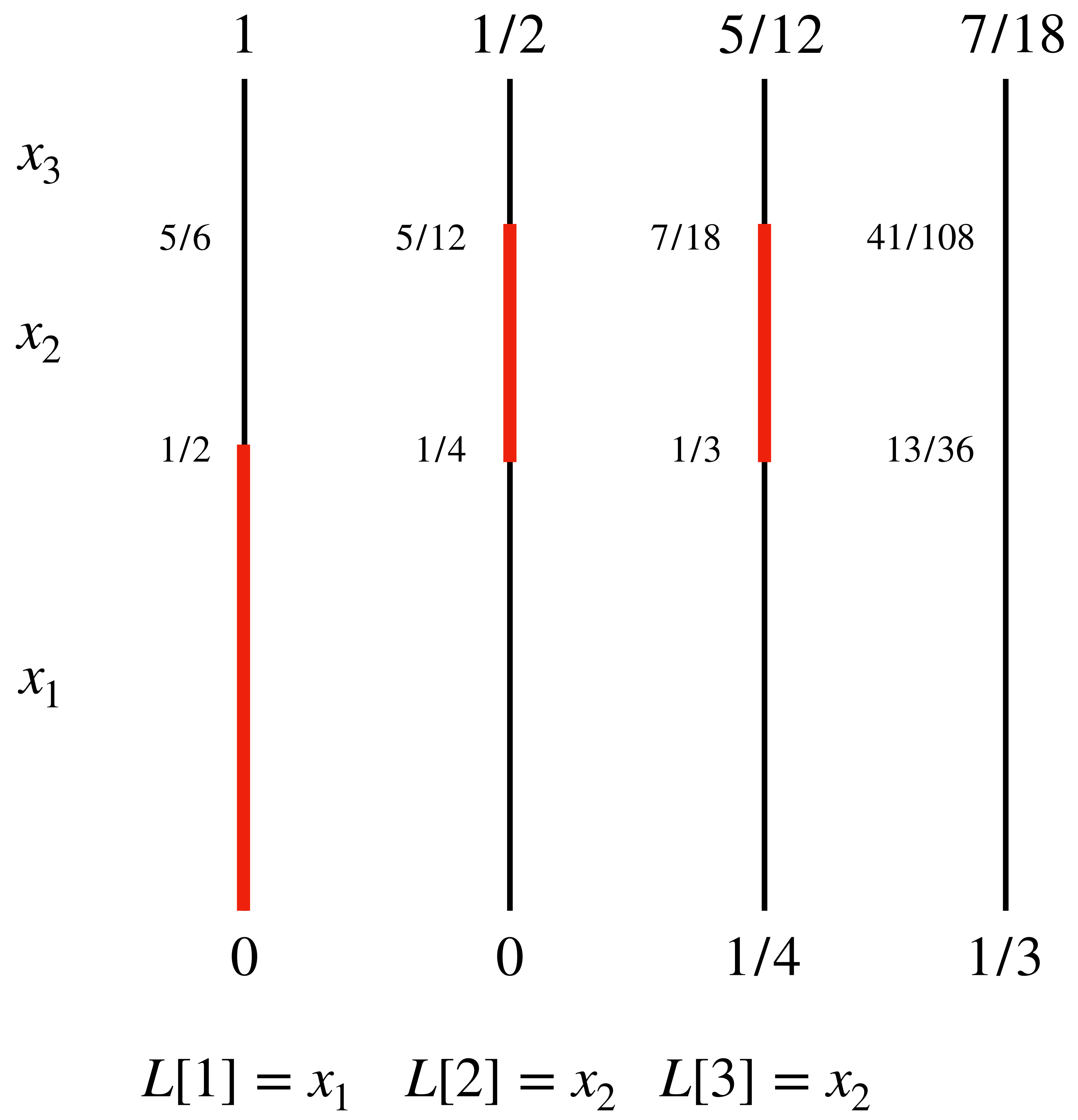
Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $P(x_1) = 1/2$, $P(x_2) = 1/3$, and $P(x_3) = 1/6$.



i	l_i	r_i
0	0.000000	1.000000
1	0.000000	0.500000
2	0.250000	0.416667
3		
4		
5		
6		

Arithmetic Coding — Encoding Demo

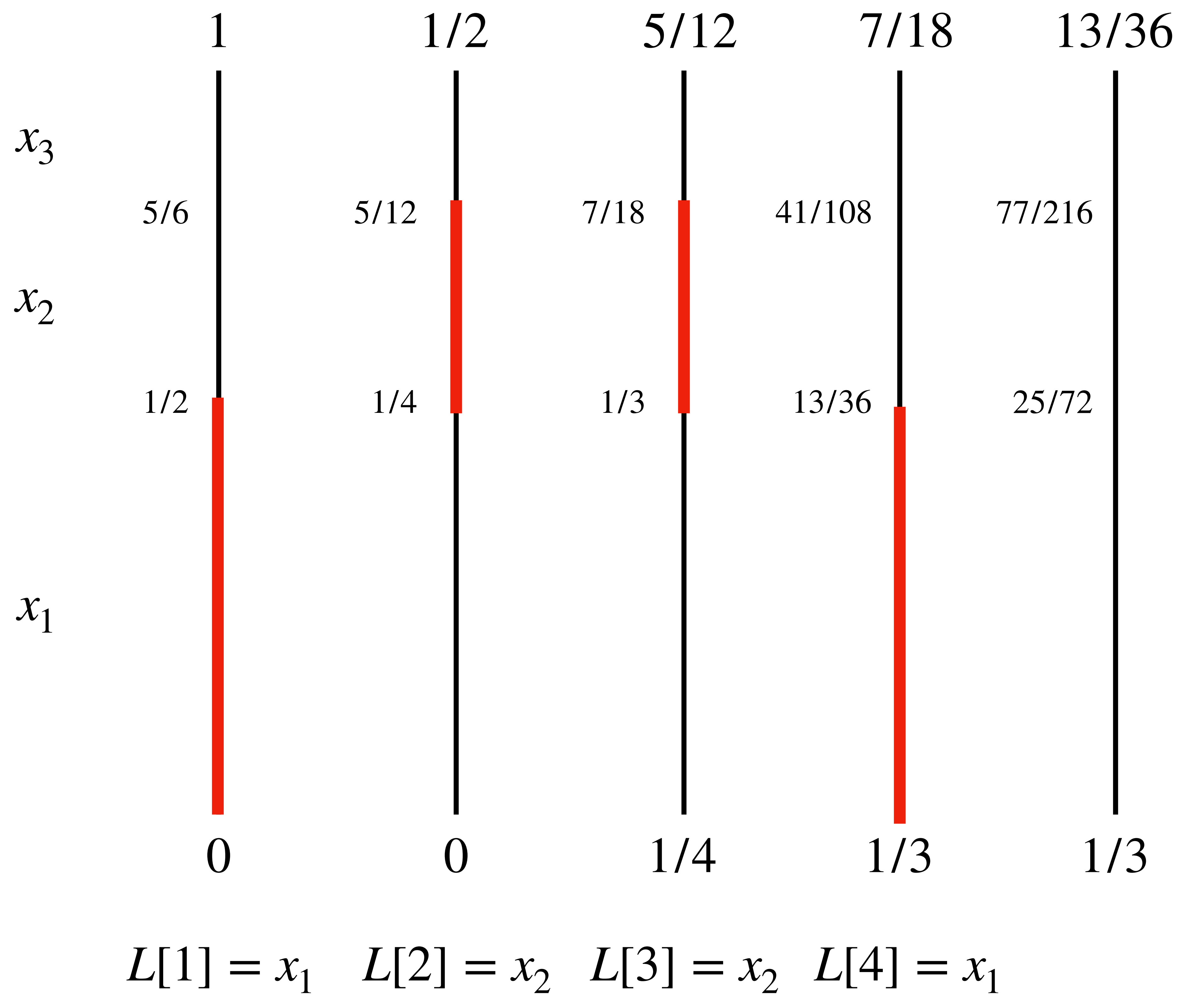
Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $P(x_1) = 1/2$, $P(x_2) = 1/3$, and $P(x_3) = 1/6$.



i	l_i	r_i
0	0.000000	1.000000
1	0.000000	0.500000
2	0.250000	0.416667
3	0.333333	0.388889
4		
5		
6		

Arithmetic Coding — Encoding Demo

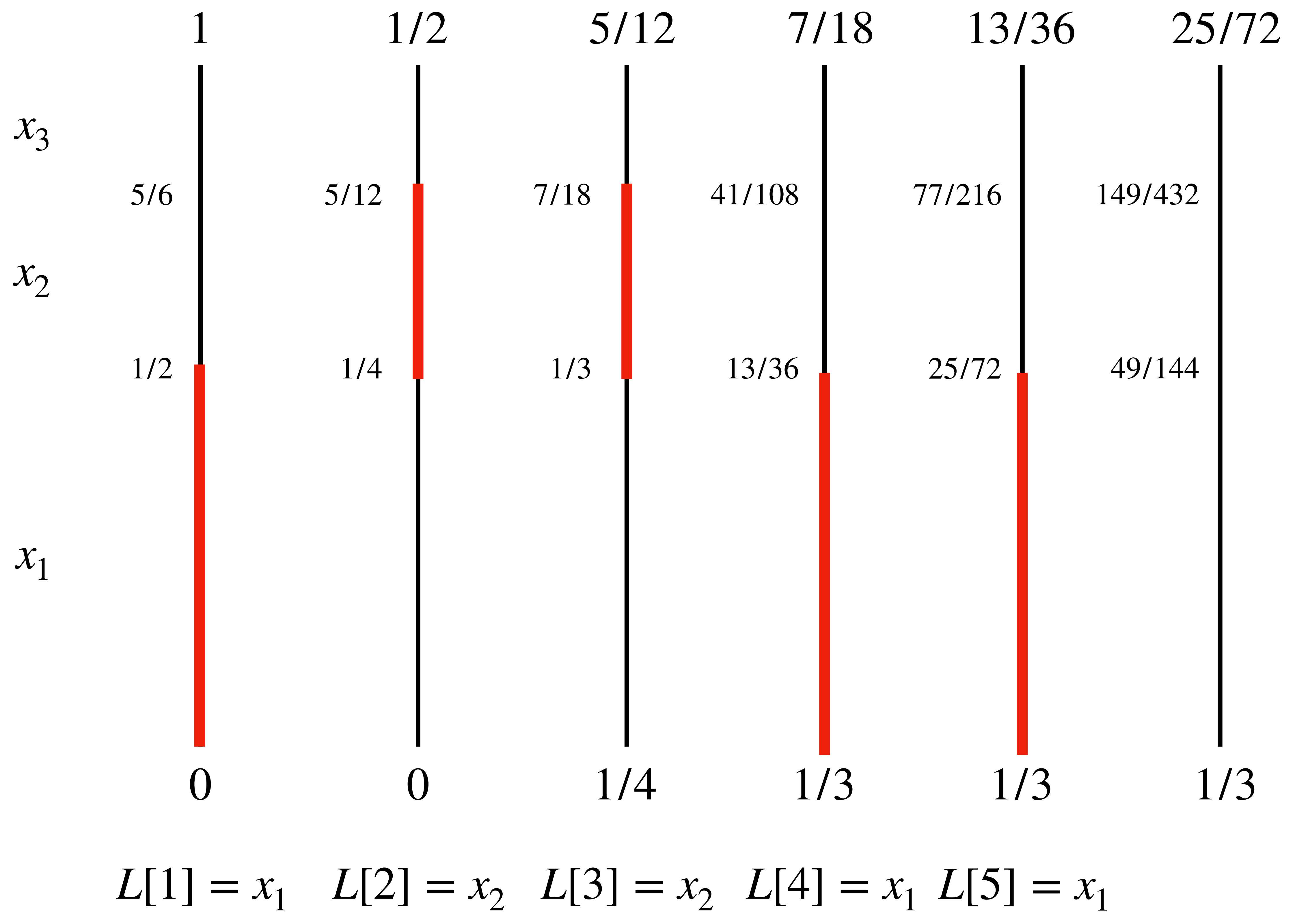
Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $P(x_1) = 1/2$, $P(x_2) = 1/3$, and $P(x_3) = 1/6$.



i	l_i	r_i
0	0.000000	1.000000
1	0.000000	0.500000
2	0.250000	0.416667
3	0.333333	0.388889
4	0.333333	0.361111
5		
6		

Arithmetic Coding — Encoding Demo

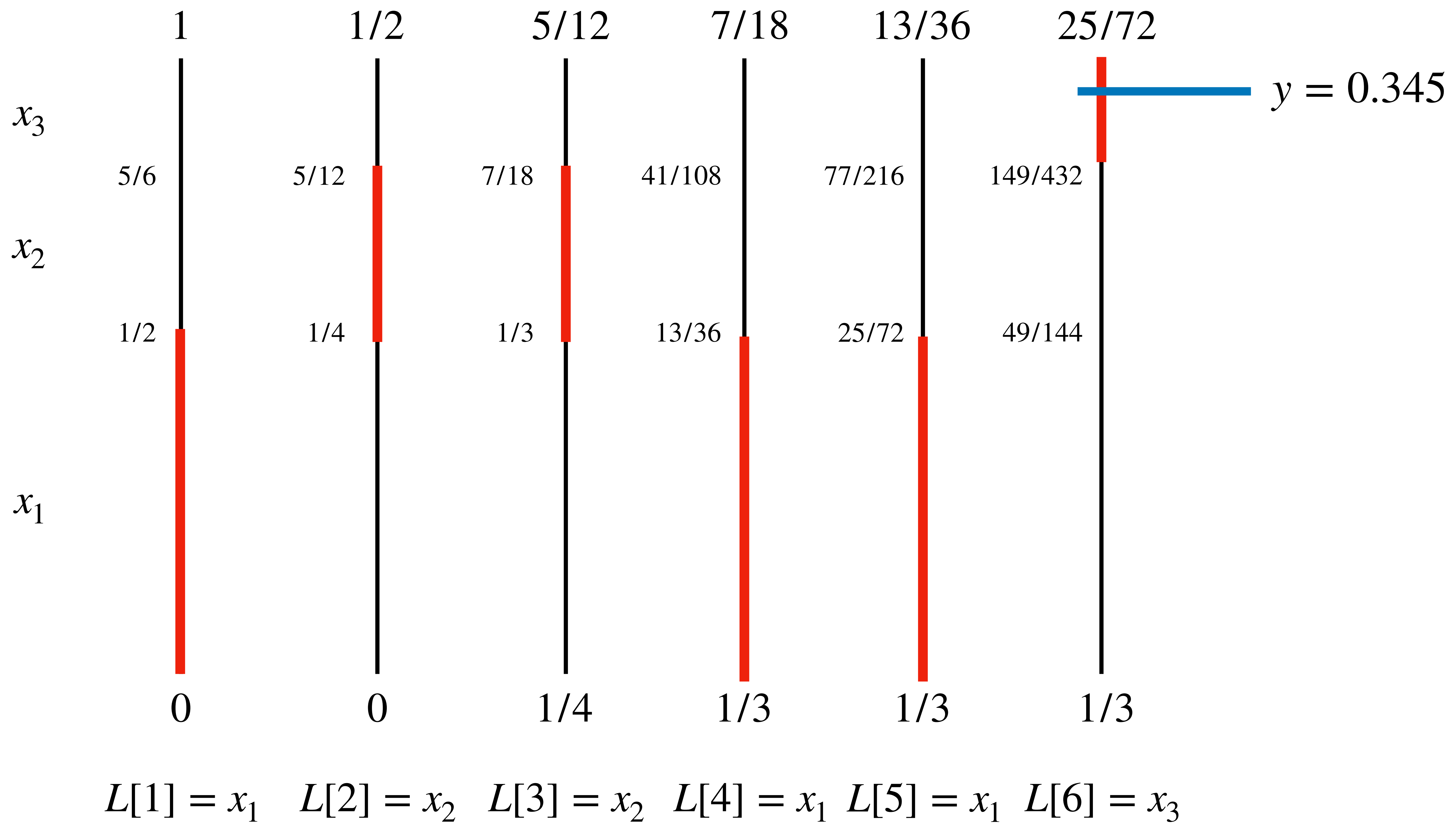
Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $P(x_1) = 1/2$, $P(x_2) = 1/3$, and $P(x_3) = 1/6$.



i	l_i	r_i
0	0.000000	1.000000
1	0.000000	0.500000
2	0.250000	0.416667
3	0.333333	0.388889
4	0.333333	0.361111
5	0.333333	0.347222
6		

Arithmetic Coding — Encoding Demo

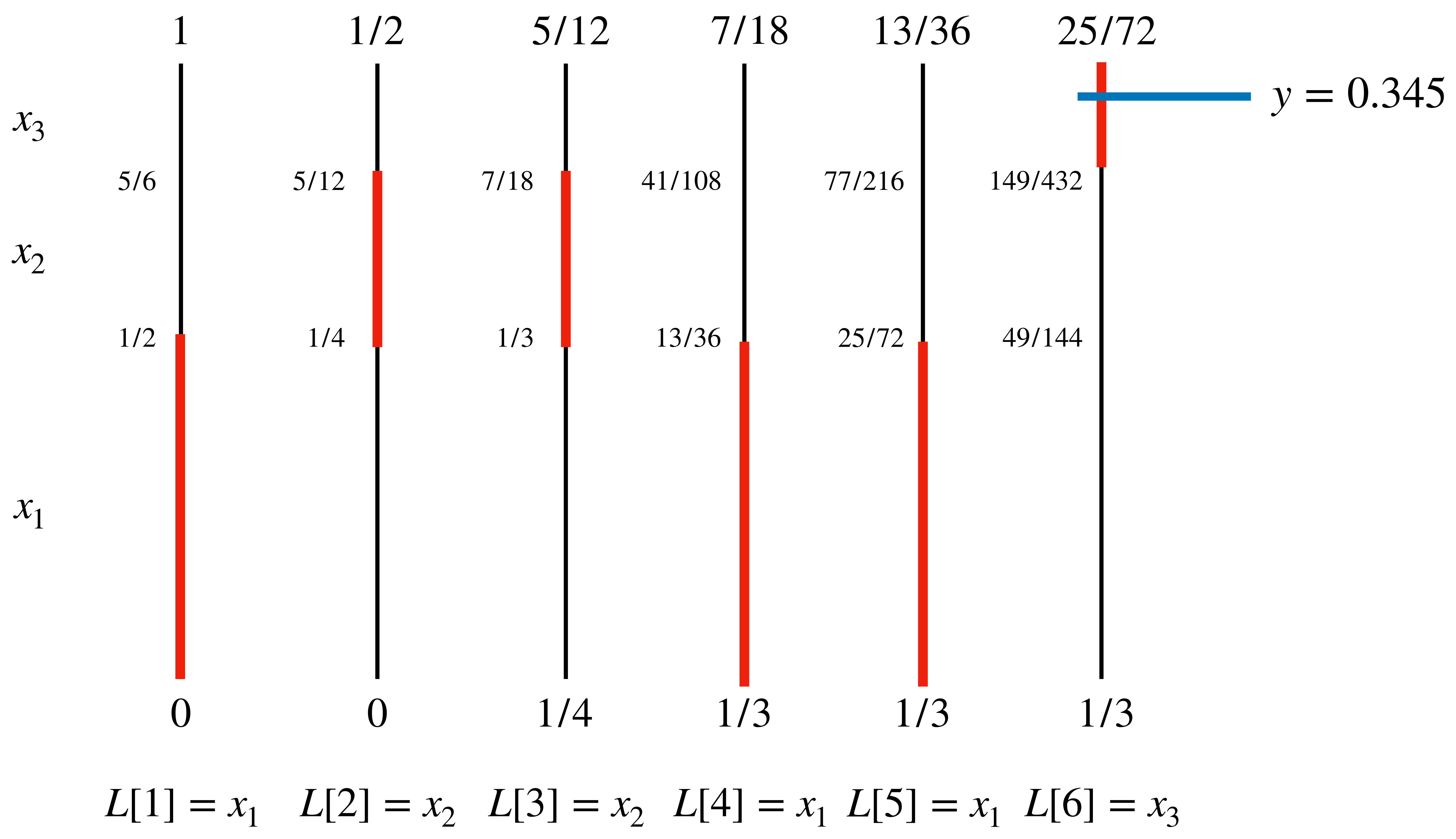
Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $P(x_1) = 1/2$, $P(x_2) = 1/3$, and $P(x_3) = 1/6$.



i	l_i	r_i
0	0.000000	1.000000
1	0.000000	0.500000
2	0.250000	0.416667
3	0.333333	0.388889
4	0.333333	0.361111
5	0.333333	0.347222
6	0.344907	0.347222

Arithmetic Coding — Encoding Demo

Example for $L = [x_1, x_2, x_2, x_1, x_1, x_3]$ with $P(x_1) = 1/2$, $P(x_2) = 1/3$, and $P(x_3) = 1/6$.

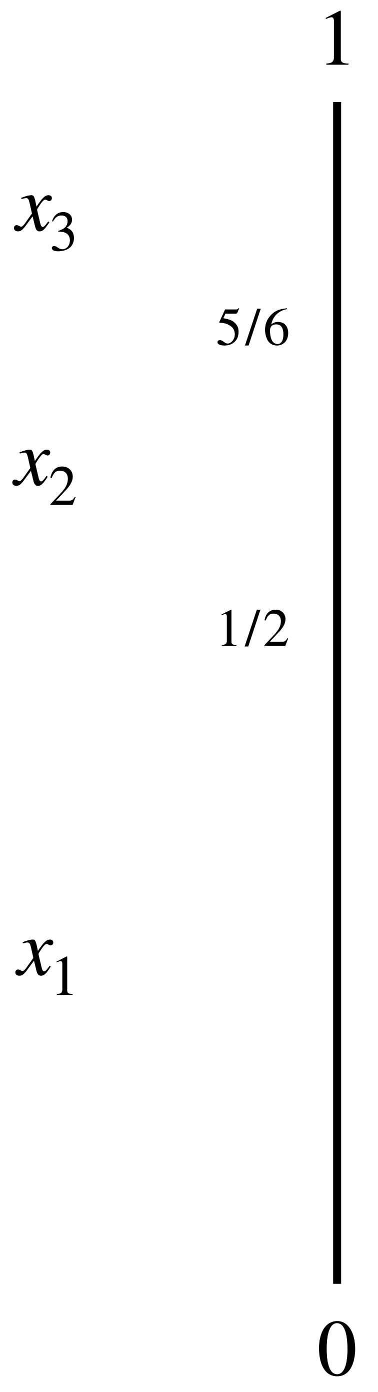


i	l_i	r_i
0	0.000000	1.000000
1	0.000000	0.500000
2	0.250000	0.416667
3	0.333333	0.388889
4	0.333333	0.361111
5	0.333333	0.347222
6	0.344907	0.347222

Note that, by construction, the length $(r_n - l_n)$ of the final interval $[l_n, r_n)$ is $\prod_{i=1}^{n} P(L[i])$.

Arithmetic Coding — Decoding Demo

Decoding. Given $\langle y, n \rangle$, where does y lies in at each of the n steps?
In our example: $y = 0.345$, and $n = 6$.



i	l_i	r_i
0	0.000000	1.000000
1		
2		
3		
4		
5		
6		

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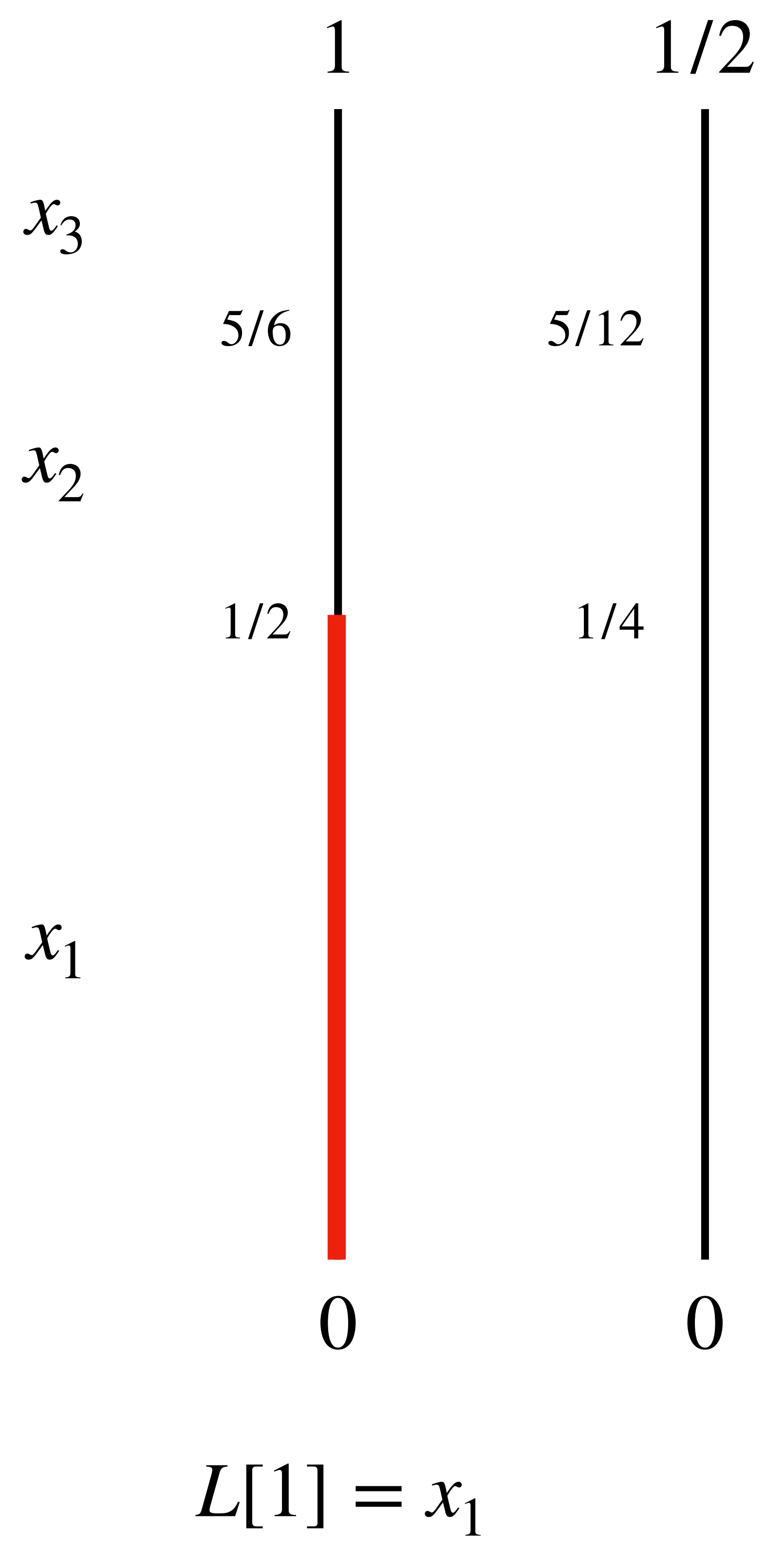

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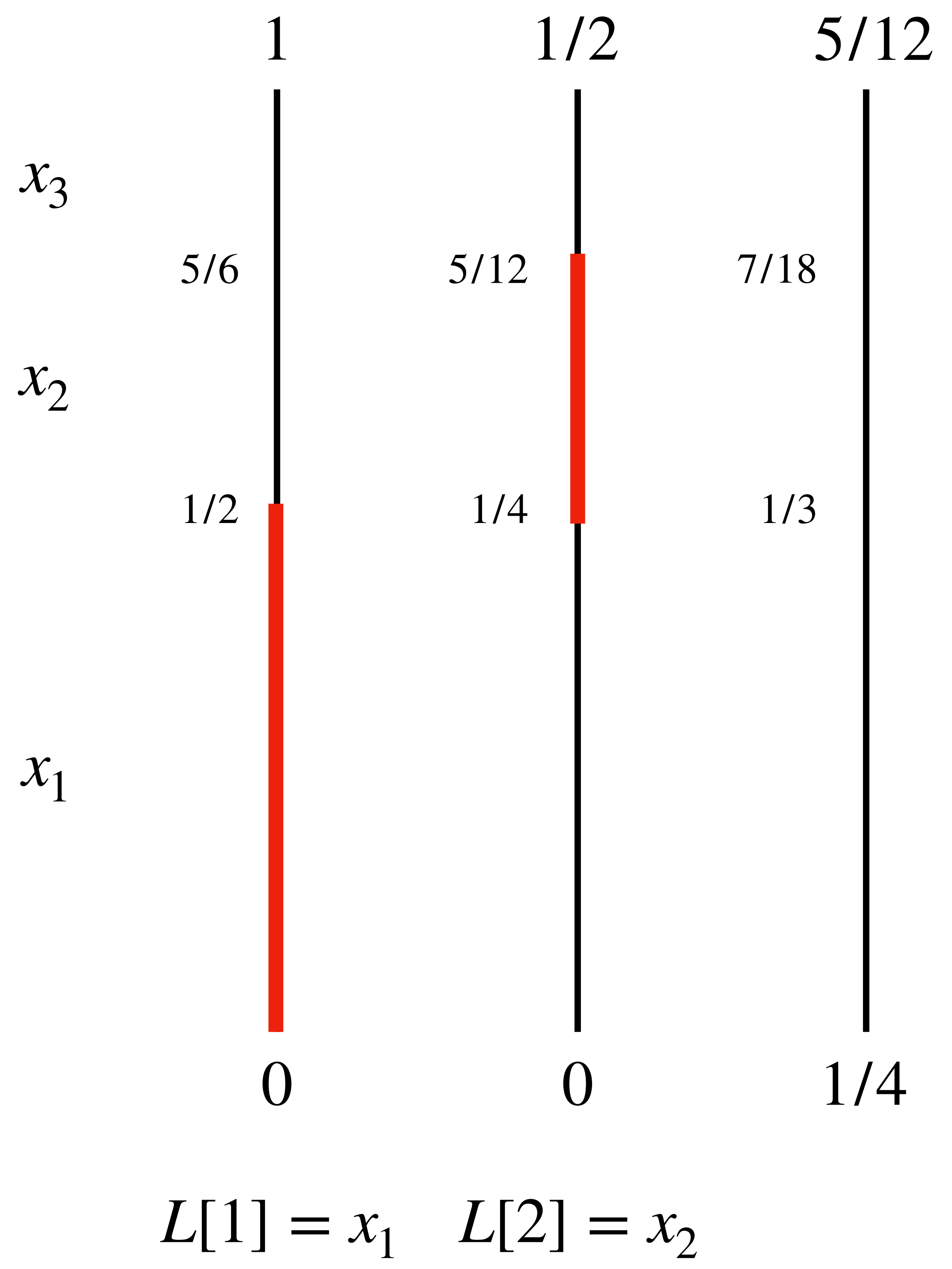

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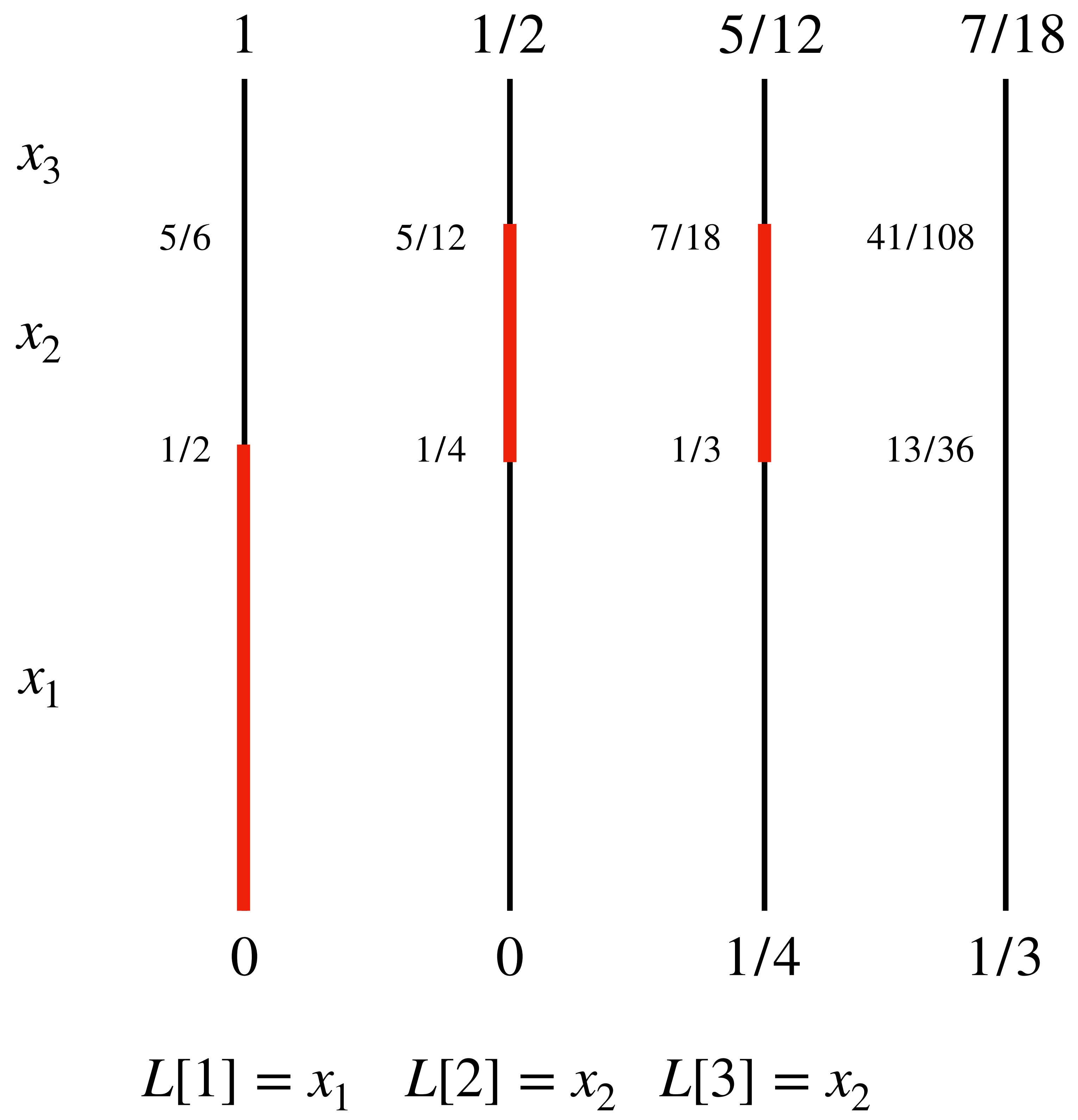

i	l_i	r_i
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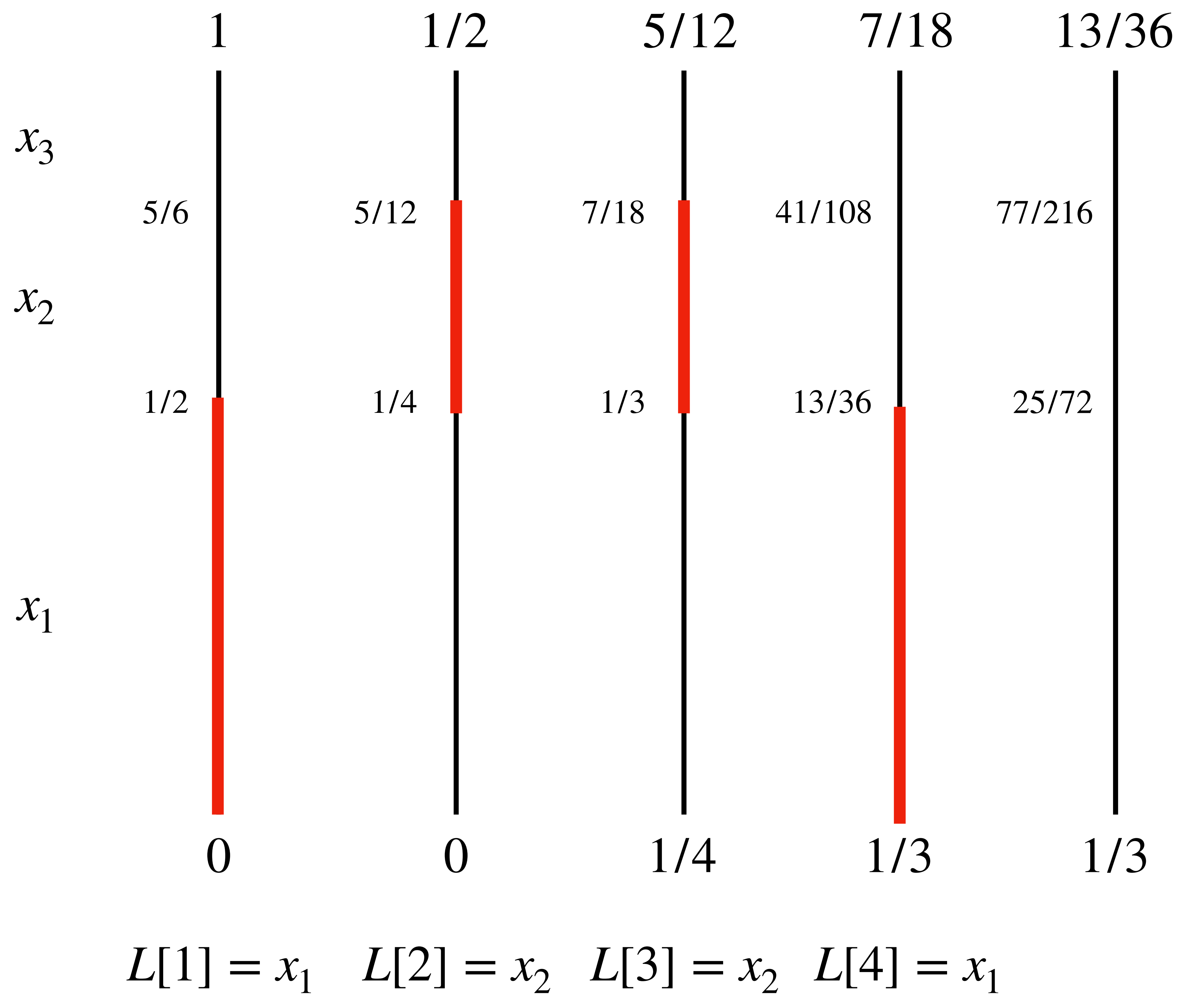

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6		

Arithmetic Coding — Decoding Demo

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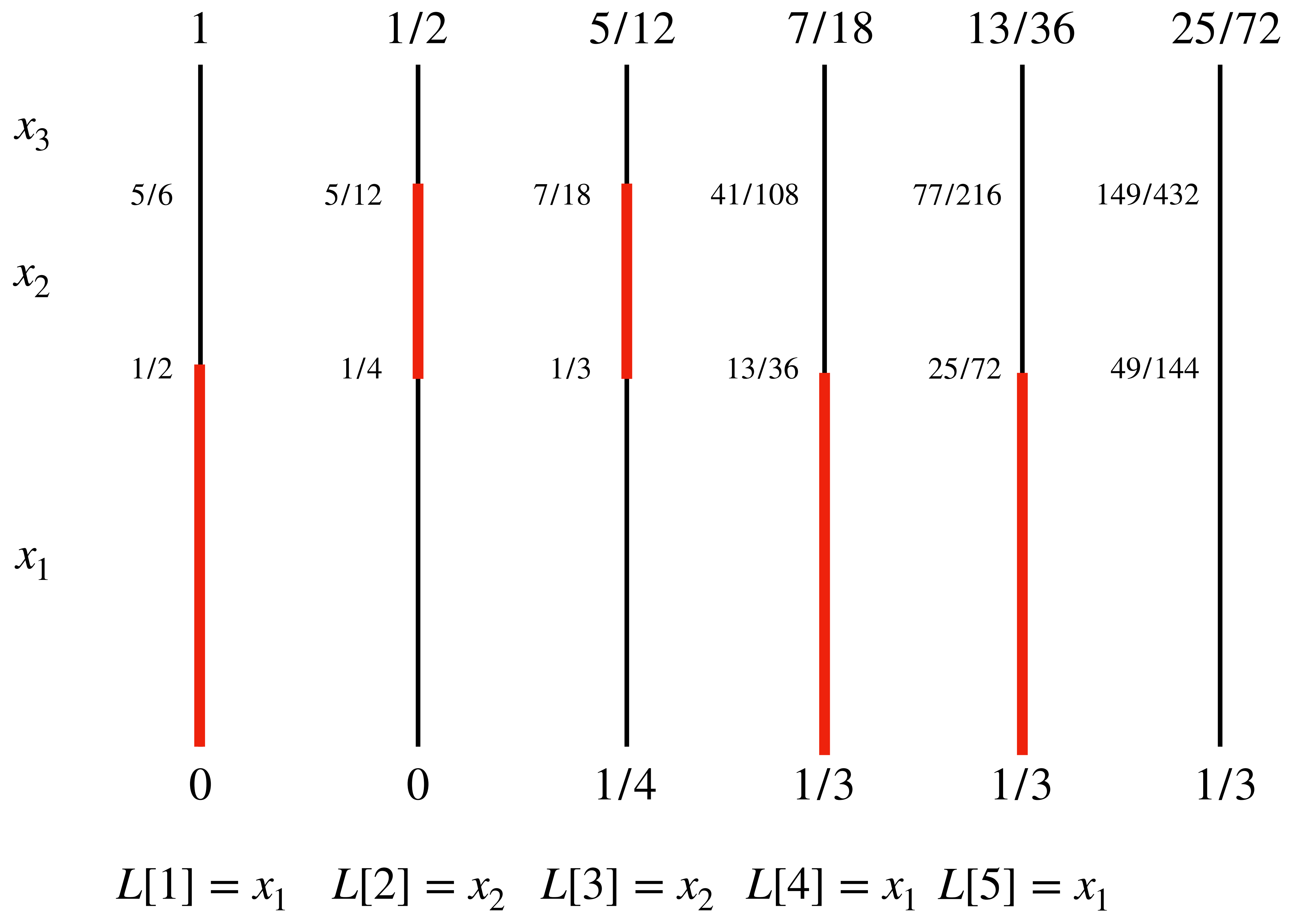

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Arithmetic Coding — Decoding Demo

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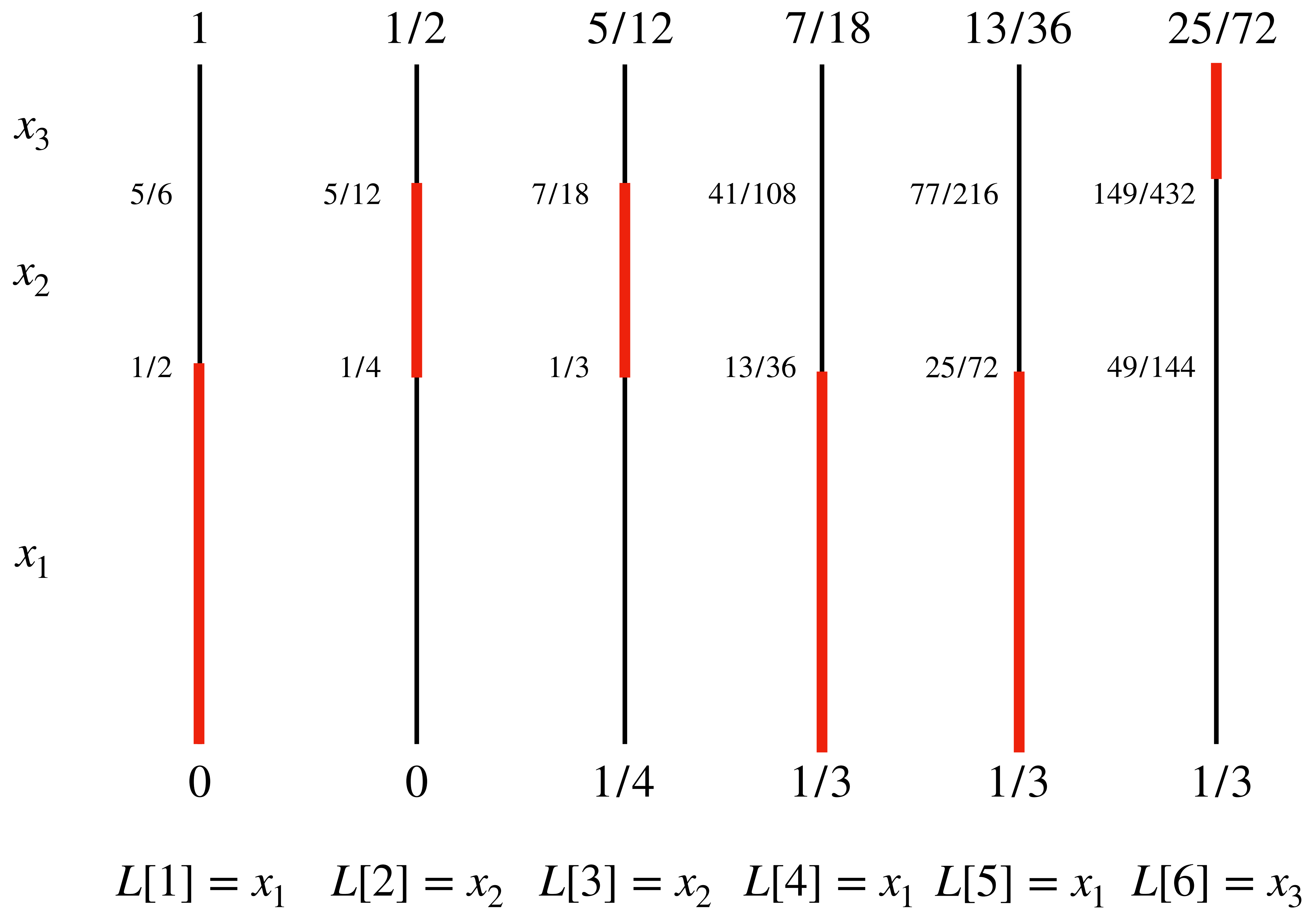

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5	0.333333	0.347222
6	0.344907	0.347222

Arithmetic Coding

- **Q.** How to choose $y \in [l_n, r_n)$ and how to represent it?
- **Dyadic fraction.** In general, a binary string $B[1..k]$ can be used to represent the real value

$$y = \frac{[B[1]\dots B[k]]_{10}}{2^k} = \frac{\sum_{i=1}^{k}\left(B[i]\cdot 2^{k-i}\right)}{2^k} = \sum_{i=1}^{k}\left(B[i]\cdot 2^{-i}\right).$$

- The larger k, the better the approximation.

For example, $B[1..6] = [001011]$ corresponds to $y = [001011]_{10}/2^6 = 11/64 = 0.171875$ and $B[1..7] = [0100111]$ to $y = [0100111]_{10}/2^7 = 39/128 = 0.3046875$.

Arithmetic Coding

- **Finite-precision arithmetic.** Given the binary string $B[1..\infty]$ representing the infinite-precision real number y, $B[1..k]$ represents a "truncated" value $\hat{y} \in [y - 2^{-k}, y]$.

  Geometric series: $\sum_{i=0}^{\infty} \frac{1}{2^i} = 1/(1-1/2) = 2 = \frac{1}{2^0} + \sum_{i=1}^{\infty} \frac{1}{2^i}$

  Proof. Clearly we have $\hat{y} < y$ because $\hat{y}$ is a truncation.

  Therefore: $y - \hat{y} = \sum_{i=1}^{\infty} B[k+i]2^{-(k+i)} \leq 2^{-k} \sum_{i=1}^{\infty} \frac{1}{2^i} = 2^{-k}$, that is $\hat{y} \geq y - 2^{-k}$. ■

- **Arithmetic Coding output bits.** Given $[l_n, r_n)$, the real value $y = (l_n + r_n)/2$ truncated to its first $\left\lceil \log_2\left(\frac{2}{r_n - l_n}\right) \right\rceil$ bits, $\hat{y}$, belongs to the interval $[l_n, r_n)$.

  Proof. We know that $y - 2^{-k} \leq \hat{y} < y$. Setting $k = \lceil \log_2(2/(r_n - l_n)) \rceil$ and $y = (l_n + r_n)/2$, we have that:
  $y - 2^{-k} = (l_n + r_n)/2 - 2^{-\lceil \log_2(2/(r_n - l_n)) \rceil} \geq (l_n + r_n)/2 - (r_n - l_n)/2 = l_n$ and therefore
  $l_n \leq y - 2^{-k} \leq \hat{y} < y = (l_n + r_n)/2 < r_n$. ■

Arithmetic Coding — Optimality

- We have proved that:
  Arithmetic Coding emits $\lceil \log_2(2/(r_n - l_n)) \rceil$ bits, where $r_n - l_n = \prod_{i=1}^{n} P(L[i])$.

- **Optimality.** The number of bits emitted by Arithmetic Coding for encoding $L[1..n]$ is at most $nH + 2$, where H is the entropy of L.

  Proof.

$$\lceil \log_2(2/(r_n - l_n)) \rceil < \log_2 2 - \log_2(r_n - l_n) + 1 = 2 - \log_2\left(\prod_{i=1}^{n} P(L[i])\right) =$$
$$= 2 - \sum_{i=1}^{n} \log_2 P(L[i]) =$$
$$= 2 + \sum_{i=1}^{n} \log_2\left(1/P(L[i])\right) = 2 + nH. \blacksquare$$

The entropy is independent from the **position** of the symbol occurrences, so we can sum over groups of occurrences of the same symbol.

- This means that Arithmetic Coding looses $2/n$ bits per symbol compared to the entropy, which is *negligible* for all practical values of n. Much *better* than Huffman on skewed distributions.

Asymmetric Numeral Systems

Duda, 2009

- Asymmetric Numeral Systems (ANS) approaches the compression ratio of Arithmetic Coding and the speed of Huffman coding. Like Arithmetic Coding, it is *not* a prefix-free code.

- **Idea.** Represent $L[1..i]$ with an *integer state* variable s_i, $i = 1,\ldots,n$.

- A coding table T is used to generate the next state s_i from state s_{i-1}, after the processing of symbol $L[i]$: $s_i = T[L[i], s_{i-1}]$.

- The row $T[i]$ contains integers values assigned in increasing order and in "strides" of length w_i.

- **Algorithm.**
  - At the beginning, set $s_0 = 0$.
  - For each symbol $L[i]$, $i = 1,\ldots,n$, generate a new state $s_i = T[L[i], s_{i-1}]$.
  - Emit $s_n > 0$ encoded with $\lceil \log_2 s_n \rceil$ bits.

Asymmetric Numeral Systems — Demo

Let us assume, without loss of generality, that the symbols are the integer numbers $[1..m]$, so every $L[i] \in [1,m]$ for $i = 1,\dots,n$.
Example for $L[1..n = 6] = [1,2,2,1,1,3]$ with $W[1..m = 3] = [w_1, w_2, w_3] = [3,2,1]$.

W	T									
w_1	1	2	3	7	8	9	13	14	15	19
$j \longrightarrow$ w_2	4	5	10	11	16	17	22	23	28	29
w_3	6	12	18	24	30	36	42	48	54	60
	0	1	2	3	4	5	6	7	8	9

…

$\uparrow$
s_{i-1}

$s_0 = 0$

Asymmetric Numeral Systems — Demo

Let us assume, without loss of generality, that the symbols are the integer numbers $[1..m]$, so every $L[i] \in [1,m]$ for $i = 1,\ldots,n$.
Example for $L[1..n = 6] = [1,2,2,1,1,3]$ with $W[1..m = 3] = [w_1, w_2, w_3] = [3,2,1]$.

	W	T									
	w_1	$\boxed{1}$	2	3	7	8	9	13	14	15	19
$j \longrightarrow$	w_2	4	5	10	11	16	17	22	23	28	29
	w_3	6	12	18	24	30	36	42	48	54	60
		0	$\boxed{1}$	2	3	4	5	6	7	8	9

…

$\uparrow$ s_{i-1}

$s_0 = 0$

$s_1 = T[L[1],0] = 1$

Asymmetric Numeral Systems — Demo

Let us assume, without loss of generality, that the symbols are the integer numbers $[1..m]$, so every $L[i] \in [1,m]$ for $i = 1,\ldots,n$.
Example for $L[1..n = 6] = [1,2,2,1,1,3]$ with $W[1..m = 3] = [w_1, w_2, w_3] = [3,2,1]$.

j →

W	T										
w_1	$\boxed{1}$	2	3	7	8	9	13	14	15	19	
w_2	4	$\boxed{5}$	10	11	16	17	22	23	28	29	$\cdots$
w_3	6	12	18	24	30	36	42	48	54	60	
	0	$\boxed{1}$	2	3	4	$\boxed{5}$	6	7	8	9	

↑
s_{i-1}

$s_0 = 0$

$s_1 = T[L[1],0] = 1$

$s_2 = T[L[2],1] = 5$

Asymmetric Numeral Systems — Demo

Let us assume, without loss of generality, that the symbols are the integer numbers $[1..m]$, so every $L[i] \in [1,m]$ for $i = 1,\ldots,n$.
Example for $L[1..n = 6] = [1,2,2,1,1,3]$ with $W[1..m = 3] = [w_1, w_2, w_3] = [3,2,1]$.

	W	T									
	w_1	$\boxed{1}$	2	3	7	8	9	13	14	15	19
$j \longrightarrow$	w_2	4	$\boxed{5}$	10	11	16	17	22	23	28	29
	w_3	6	12	18	24	30	36	42	48	54	60
		0	$\boxed{1}$	2	3	4	$\boxed{5}$	6	7	8	9

$\cdots$

s_{i-1}

$s_0 = 0$

$s_1 = T[L[1],0] = 1$

$s_2 = T[L[2],1] = 5$

$s_3 = T[L[3],5] = 17$

Asymmetric Numeral Systems — Demo

Let us assume, without loss of generality, that the symbols are the integer numbers $[1..m]$, so every $L[i] \in [1,m]$ for $i = 1,\ldots,n$.
Example for $L[1..n = 6] = [1,2,2,1,1,3]$ with $W[1..m = 3] = [w_1, w_2, w_3] = [3,2,1]$.

	W	T										
	w_1	$\boxed{1}$	2	3	7	8	9	13	14	15	19	
$j \longrightarrow$	w_2	4	$\boxed{5}$	10	11	16	17	22	23	28	29	$\cdots$
	w_3	6	12	18	24	30	36	42	48	54	60	
		0	$\boxed{1}$	2	3	4	$\boxed{5}$	6	7	8	9	

$\uparrow$
s_{i-1}

$s_0 = 0$

$s_1 = T[L[1],0] = 1$

$s_2 = T[L[2],1] = 5$

$s_3 = T[L[3],5] = 17$

$s_4 = T[L[4],17] = 33$

Asymmetric Numeral Systems — Demo

Let us assume, without loss of generality, that the symbols are the integer numbers $[1..m]$, so every $L[i] \in [1,m]$ for $i = 1,\ldots,n$.
Example for $L[1..n=6] = [1,2,2,1,1,3]$ with $W[1..m=3] = [w_1, w_2, w_3] = [3,2,1]$.

W					T						
w_1	$\boxed{1}$	2	3	7	8	9	13	14	15	19	
$j \longrightarrow$ w_2	4	$\boxed{5}$	10	11	16	17	22	23	28	29	…
w_3	6	12	18	24	30	36	42	48	54	60	
	0	$\boxed{1}$	2	3	4	$\boxed{5}$	6	7	8	9	

$\uparrow$
s_{i-1}

$s_0 = 0$

$s_1 = T[L[1],0] = 1$

$s_2 = T[L[2],1] = 5$

$s_3 = T[L[3],5] = 17$

$s_4 = T[L[4],17] = 33$

$s_5 = T[L[5],33] = 67$

Asymmetric Numeral Systems — Demo

Let us assume, without loss of generality, that the symbols are the integer numbers $[1..m]$, so every $L[i] \in [1,m]$ for $i = 1,\ldots,n$.
Example for $L[1..n = 6] = [1,2,2,1,1,3]$ with $W[1..m = 3] = [w_1, w_2, w_3] = [3,2,1]$.

$j \longrightarrow$

W	T									
w_1	$\boxed{1}$	2	3	7	8	9	13	14	15	19
w_2	4	$\boxed{5}$	10	11	16	17	22	23	28	29
w_3	6	12	18	24	30	36	42	48	54	60
	0	$\boxed{1}$	2	3	4	$\boxed{5}$	6	7	8	9

$\cdots$

$\uparrow$
s_{i-1}

$s_0 = 0$

$s_1 = T[L[1],0] = 1$

$s_2 = T[L[2],1] = 5$

$s_3 = T[L[3],5] = 17$

$s_4 = T[L[4],17] = 33$

$s_5 = T[L[5],33] = 67$

$s_6 = T[L[6],67] = 408$

Asymmetric Numeral Systems — Demo

Let us assume, without loss of generality, that the symbols are the integer numbers $[1..m]$,
so every $L[i] \in [1,m]$ for $i = 1,\ldots,n$.
Example for $L[1..n = 6] = [1,2,2,1,1,3]$ with $W[1..m = 3] = [w_1, w_2, w_3] = [3,2,1]$.

	W					T						
	w_1	$\boxed{1}$	2	3	7	8	9	13	14	15	19	
j →	w_2	4	$\boxed{5}$	10	11	16	17	22	23	28	29	$\cdots$
	w_3	6	12	18	24	30	36	42	48	54	60	
		0	$\boxed{1}$	2	3	4	$\boxed{5}$	6	7	8	9	

s_{i-1}

$s_0 = 0$

$s_1 = T[L[1],0] = 1$

$s_2 = T[L[2],1] = 5$

$s_3 = T[L[3],5] = 17$

$s_4 = T[L[4],17] = 33$

$s_5 = T[L[5],33] = 67$

$s_6 = T[L[6],67] = 408$

Actually, no table T is needed:

$$s_i = T[j, s_{i-1}] = R[j] + n\lfloor s_{i-1}/W[j] \rfloor + (s_{i-1} \bmod W[j])$$

where $(x \bmod y) = x - \lfloor x/y \rfloor \in [0,y-1]$ is the rest of the integer division (the "modulo") between x and y and $R[j] = 1 + \sum_{k=1}^{j-1} W[k]$ for $j > 1$ ($R[1] = 1$). We just need the arrays $R[1..m]$ and $W[1..m]$.

Asymmetric Numeral Systems

- We have:
  $s_i = T[j, s_{i-1}] = r_j + n\lfloor s_{i-1}/W[j] \rfloor + (s_{i-1} \bmod W[j])$, where $R[j] = 1 + \sum_{k=1}^{j-1} W[k]$ for $j > 1$ and $R[1] = 1$.

- Note that $s_i = T[j, s_{i-1}] = R[j] + n\lfloor s_{i-1}/W[j] \rfloor + (s_{i-1} \bmod w_j) \leq R[j] + s_{i-1}/P(j) + W[j] - 1$, where $P(j) = W[j]/n$ is the self-probability of symbol j. And since $R[j] + W[j] - 1 \leq n$, we have $s_i \leq n + s_{i-1}/P(j)$.

- **Lemma.**
  The i-th state s_i is such that $s_i < i \cdot \dfrac{n}{\prod_{k=1}^{i} P(L[k])}$.

  Proof. Proceed by induction. For the base case $i = 1$ we have $s_0 = 0$ and $s_1 \leq n + 0/P(L[1]) = n < n/P(L[1])$.
  Now assume $s_{i-1} < (i-1) \cdot \dfrac{n}{\prod_{k=1}^{i-1} P(L[k])}$ is true.
  Then, $s_i \leq n + s_{i-1}/P(L[i]) < n + (i-1) \cdot \dfrac{n}{P(L[i]) \cdot \prod_{k=1}^{i-1} P(L[k])} = n + i \cdot \dfrac{n}{\prod_{k=1}^{i} P(L[k])} - \dfrac{n}{\prod_{k=1}^{i} P(L[k])} < i \cdot \dfrac{n}{\prod_{k=1}^{i} P(L[k])}$,
  because $n < \dfrac{n}{\prod_{k=1}^{i} P(L[k])}$. ■

Asymmetric Numeral Systems — Optimality

- **Optimality.** The number of bits emitted by ANS for encoding $L[1..n]$ is at most $nH + 2\log_2 n + 1$, where H is the entropy of L.

  Proof. For the previous Lemma, we have that $s_n < \dfrac{n^2}{\prod_{i=1}^{n} P(L[i])}$.

  Therefore

  $$\lceil \log_2 s_n \rceil < \log_2 s_n + 1 < \log_2 \frac{n^2}{\prod_{i=1}^{n} P(L[i])} + 1 = 2\log_2 n - \log_2\left(\prod_{i=1}^{n} P(L[i])\right) + 1 =$$

  $$= 2\log_2 n + \sum_{i=1}^{n} \log_2\big(1/P(L[i])\big) + 1 = 2\log_2 n + nH + 1. \blacksquare$$

- Worse than Arithmetic Coding ($2/n$ bits/symb vs. $(2\log_2 n)/n$ bits/symb), but the loss vanishes for growing values of n. Still much better than Huffman.

Further Readings

- Sections 5.1, 5.2, 5.3 of:
Alistair Moffat and Andrew Turpin. 2002. *Compression and coding algorithms*.
Springer Science & Business Media, ISBN 978-1-4615-0935-6.

- Sections 2.1, 2.2, 2.5, 2.6, 5.1, 5.2, 6 of:
Alistair Moffat. 2019. *Huffman Coding*. ACM Computing Surveys. 52, 4, Article 85 (July 2020), 35 pages.
https://doi.org/10.1145/3342555

- Chapter 5.5 (pages 826-838) of:
Robert Sedgewick and Kevin Wayne. 2011. *Algorithms.* 4-th Edition.
Addison-Wesley Professional, ISBN 0-321-57351-X.

- Section 2.6 (about Huffman coding) of:
Gonzalo Navarro. 2016. *Compact Data Structures*. Cambridge University Press, ISBN 978-1-107-15238-0.

- Section 3.8 of:
G. E. P. and Rossano Venturini. 2020. *Techniques for Inverted Index Compression*. ACM Computing Surveys.
53, 6, Article 125 (November 2021), 36 pages. https://doi.org/10.1145/3415148