In The Name Of God

Generalized Games and Variational Inequality

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Game Theory Course

Generalized Nash Equilibrium

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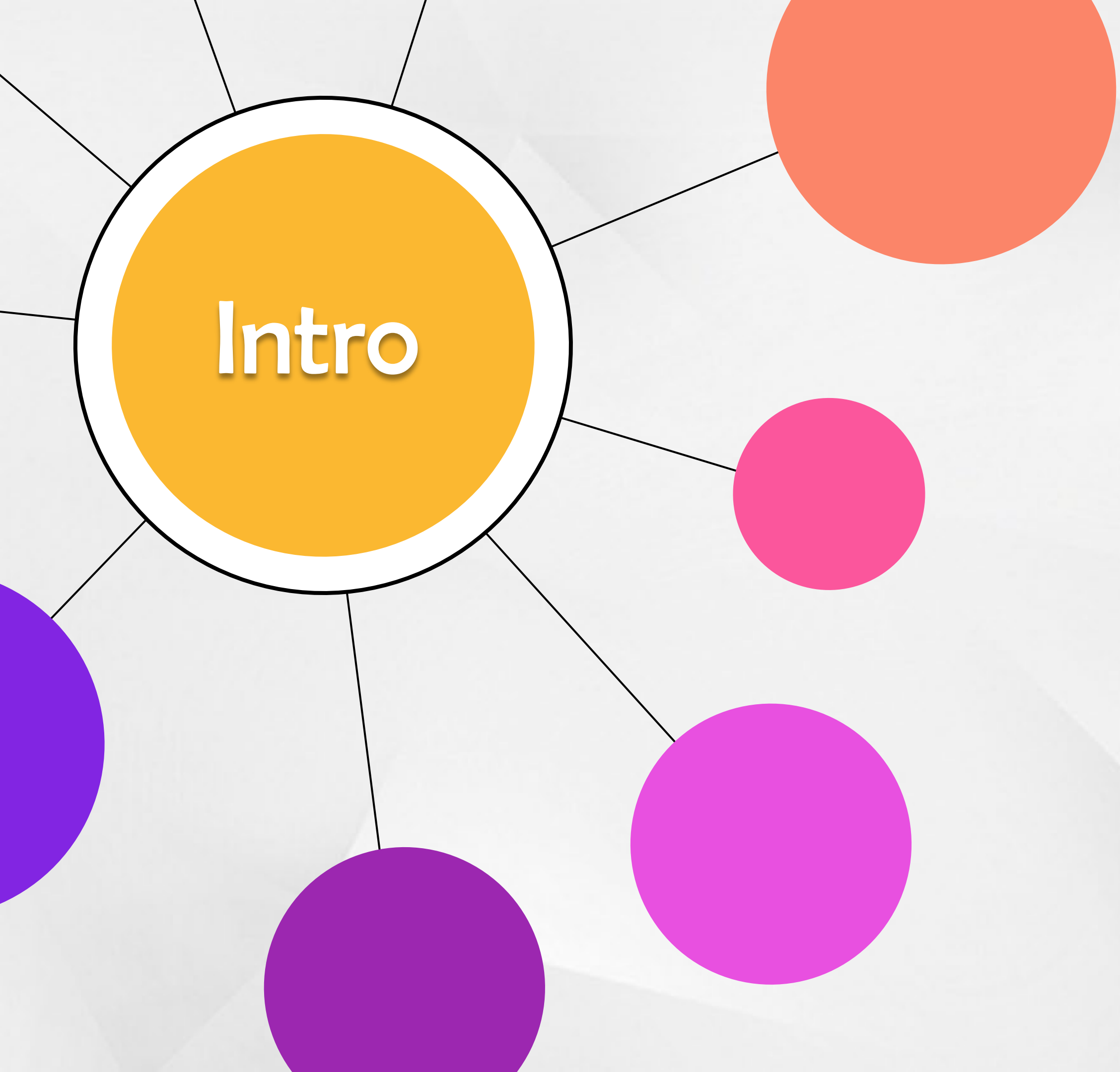
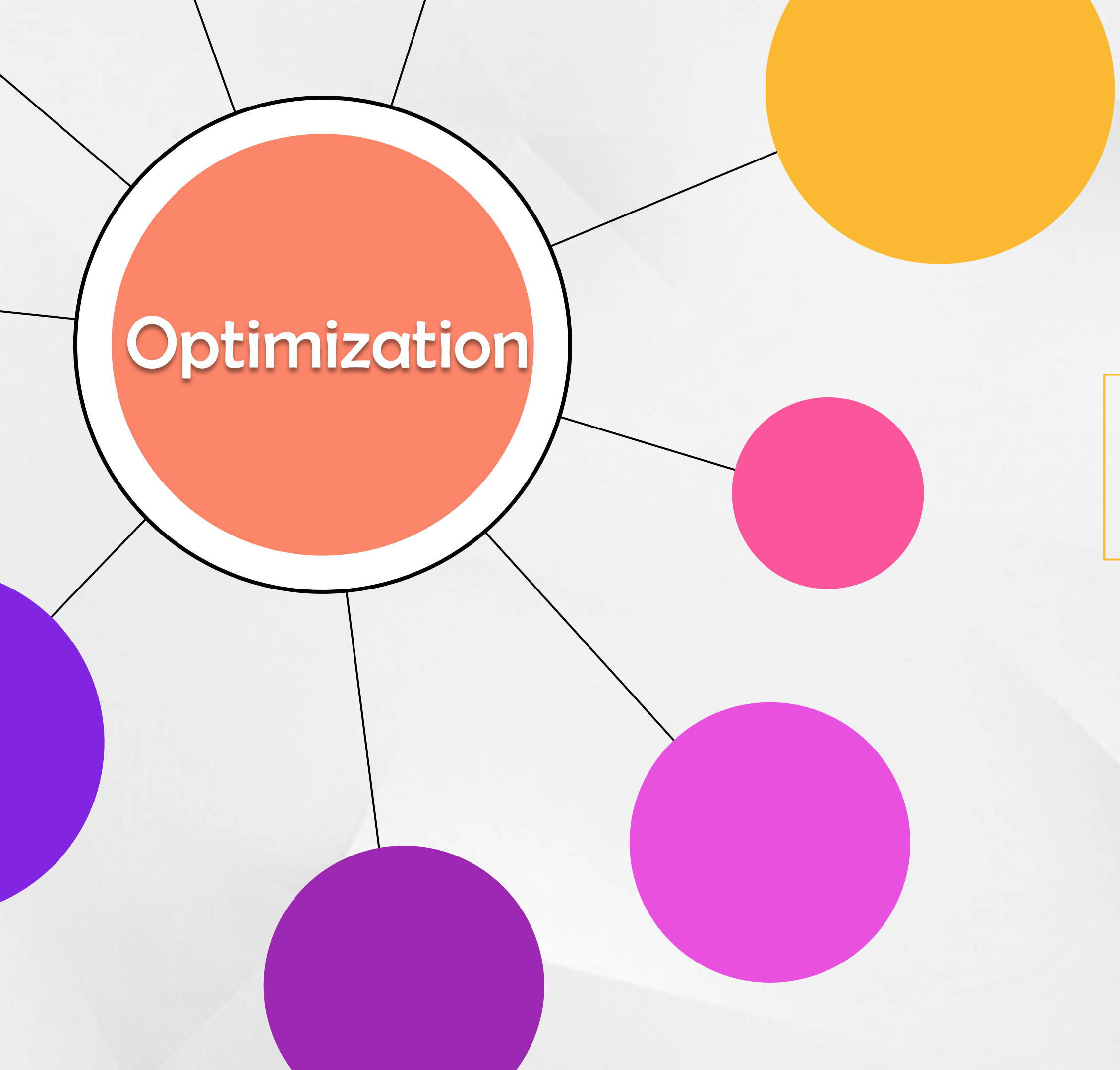


Introduction

Big Picture

- What is Game Theory?
- A game can be represented as a set of coupled optimization problems.
- It shows that Game theory has a strong relationship with Optimization.
- Variational Inequality (VI)
- Why VI is important?



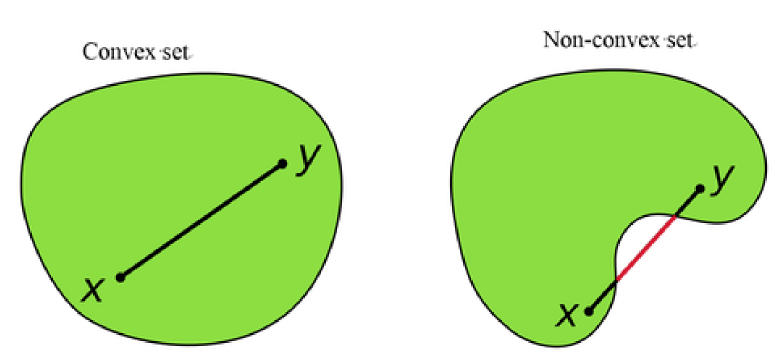
Optimization Theory

What is Convex Set?

CONVEX SETS

A set $\mathcal{K} \subseteq \mathbb{R}^n$ is convex if for any two points $\mathbf{x}, \mathbf{y} \in \mathcal{K}$, the segment joining them belongs to $\mathcal{K}$

$$\alpha \mathbf{x} + (1-\alpha)\mathbf{y} \in \mathcal{K}, \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{K} \text{ and } \alpha \in [0,1]. \quad (1)$$



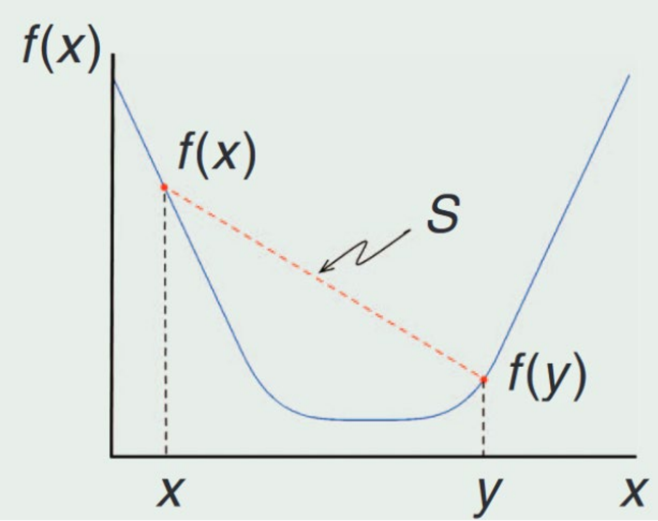
What is convex function?

CONVEX FUNCTIONS

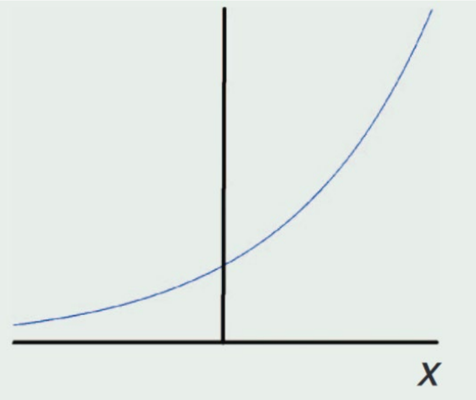
Given a convex set $\mathcal{K} \subseteq \mathbb{R}^n$ and a function $f(\mathrm{x}): \mathcal{K} \to \mathbb{R}$; f is said to be

- *convex* on $\mathcal{K}$ if, $\forall \mathrm{x}, \mathrm{y} \in \mathcal{K}$ and $\alpha \in (0, 1)$,

$$f(\alpha\, \mathrm{x} + (1 - \alpha)\, \mathrm{y}) \leq \alpha\, f(\mathrm{x}) + (1 - \alpha)\, f(\mathrm{y}) \qquad (2)$$



What is strictly convex function?



CONVEX FUNCTIONS

Given a convex set $\mathcal{K} \subseteq \mathbb{R}^n$ and a function $f(\mathrm{x}): \mathcal{K} \rightarrow \mathbb{R}$; f is said to be

- *strictly convex* on $\mathcal{K}$ if the inequality in (2) is strict

$$f(\alpha\,\mathrm{x} + (1-\alpha)\,\mathrm{y}) \leq \alpha\, f(\mathrm{x}) + (1-\alpha)\, f(\mathrm{y}) \qquad (2)$$

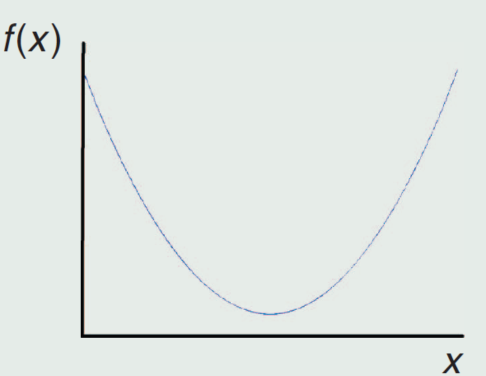
What is strongly convex function?

CONVEX FUNCTIONS

Given a convex set $\mathcal{K} \subseteq \mathbb{R}^n$ and a function $f(\mathrm{x})\colon \mathcal{K} \to \mathbb{R}$; f is said to be

- *strongly convex* on $\mathcal{K}$ if $\forall \mathrm{x}, \mathrm{y} \in \mathcal{K}$ and $\alpha \in (0, 1)$, there exists a constant $c > 0$ such that

$$f(\alpha \mathrm{x} + (1-\alpha)\mathrm{y}) \leq \alpha f(\mathrm{x}) + (1-\alpha) f(\mathrm{y}) - \frac{c}{2}\alpha(1-\alpha)\|\mathrm{x}-\mathrm{y}\|^2. \tag{3}$$



(c)

Optimization Problem

$$\begin{aligned} \underset{\mathrm{x}}{\text{minimize}} \quad & f(\mathrm{x}) \\ \text{subject to} \quad & \mathrm{x} \in \mathcal{K}, \end{aligned} \tag{4}$$

Our assumption throughout of this study:

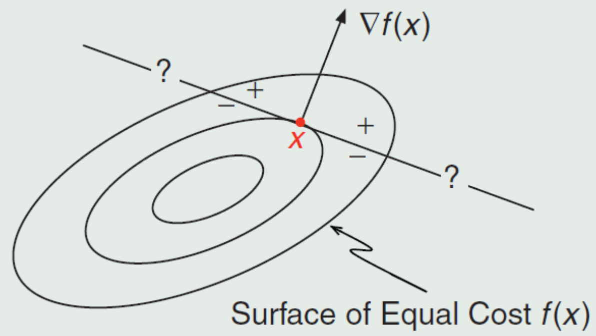
$\mathcal{K}$ is closed and convex

f is convex and continuously differentiable on $\mathcal{K}$

Why this kind:

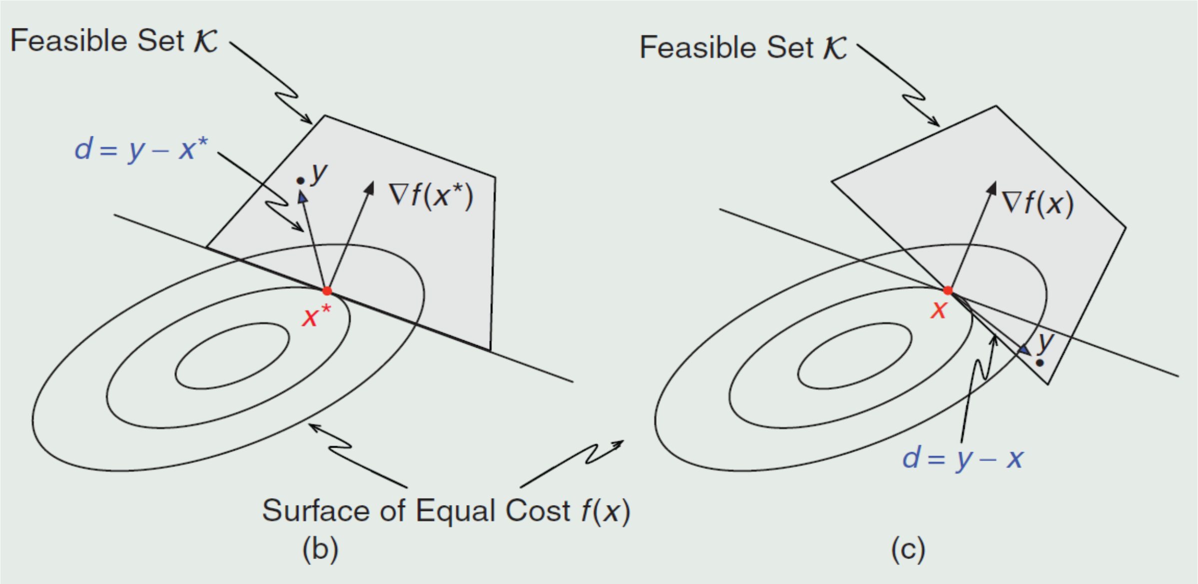
- quite frequently in applications
- powerful analytical and algorithmic tools are available for their study.
- constitute the largest class of tractable optimization problems.

- Conditions of optimal solution (why we use them)
- These conditions constitute the foundations for the theoretical study of the problem and its numerical solution.
- most important one for convex optimization is minimum principle.
- To understand this principle we need to recall gradient of a function, and what it shows (By using the Taylor expansion).
- gradient of a (continuously differentiable) function f represents the direction of maximal ascent of the function.



(a)

The minimum principle essentially just states that if we consider the convex optimization problem (4) and a feasible point x^*, then, if x^* is optimal, the feasible region must not lie in the half space where the function decreases; otherwise the point x^* could not be an optimal solution by definition. It actually turns out that convexity makes this condition also sufficient for optimality.



$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & x \in \mathcal{K}, \end{array} \qquad (4)$$

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & x \in \mathcal{K}, \end{array} \tag{4}$$

MINIMUM PRINCIPLE

Consider the convex optimization problem (4). A feasible point $x^\star \in \mathcal{K}$ is an optimal solution if and only if

$$(y - x^\star)^T \nabla f(x^\star) \geq 0 \quad \forall y \in \mathcal{K}. \tag{5}$$

The case in which the set $\mathcal{K}$ is defined by inequalities and equalities deserves a particular attention. In this case it can be shown that, under some additional conditions, the minimum principle is in fact equivalent to the famous Karush-Kuhn-Tucker (KKT) optimality conditions;

Variational inequality

- VIs constitute a broad class of problems encompassing convex optimization and bearing strong connections to game theory.

VARIATIONAL INEQUALITY PROBLEM

Given a closed and convex set $\mathcal{K} \subseteq \mathbb{R}^n$ and a mapping $\mathrm{F}: \mathcal{K} \to \mathbb{R}^n$, the VI problem, denoted $\mathrm{VI}(\mathcal{K}, \mathrm{F})$, consists in finding a vector $\mathrm{x}^\star \in \mathcal{K}$ (called a solution of the VI) such that [22]:

$$(\mathrm{y} - \mathrm{x}^\star)^T \mathrm{F}(\mathrm{x}^\star) \geq 0, \quad \forall \mathrm{y} \in K. \tag{6}$$

For simplicity consider following assumptions:

F is continuously differentiable on the interior of $\mathcal{K}$

$\mathcal{K}$ is closed and convex.

The relationship of VI with Optimization Problems:

It is clear that if $F = \nabla f$ for some suitable convex function f, $\text{VI}(\mathcal{K}, \nabla f)$ coincides with the problem of finding a point satisfying the minimum principle (5) and therefore with the problem of finding an optimal solution of the convex optimization problem (4).

In fact, we recall that not all continuous functions $\mathbf{F}$ can be expressed as the gradient of a suitable scalar function f. It is well known that this happens if and only if the Jacobian matrix of $\mathbf{F}$ is symmetric for all points in the domain of interest. For example, suppose that $\mathbf{F} = \mathbf{A}\mathbf{x} + \mathbf{b}$ for some suitable square $n \times n$ matrix $\mathbf{A}$ and n-vector $\mathbf{b}$. If $\mathbf{A}$ is symmetric, it is easy to check that $\mathbf{F}(\mathbf{x}) = \nabla f(\mathbf{x})$, with $f(\mathbf{x}) = (1/2)(\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x})$. However, if $\mathbf{A}$ is not symmetric it is impossible to find a function f whose gradient yields $\mathbf{F}$.

Discuss about Existence and Uniqueness of solution of VI:

Given the VI($\mathcal{K}$, F), suppose that

i) the set $\mathcal{K}$ is convex and compact (closed and bounded);

ii) the function F(x) is continuous.

Then, the set of solutions is nonempty and compact. (8)

We try to relax the bounded condition on K and instead impose additional (but not too conservative) conditions on F(x).

Given a convex set $\mathcal{K}$, a mapping $\mathrm{F} : \mathcal{K} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be

- *monotone* on $\mathcal{K}$ if

$$(\mathrm{F}(\mathrm{x}) - \mathrm{F}(\mathrm{y}))^T(\mathrm{x} - \mathrm{y}) \geq 0, \quad \forall \mathrm{x}, \mathrm{y} \in \mathcal{K} \tag{9}$$

- *strictly monotone* on $\mathcal{K}$ if

$$(\mathrm{F}(\mathrm{x}) - \mathrm{F}(\mathrm{y}))^T(\mathrm{x} - \mathrm{y}) > 0, \quad \forall \mathrm{x}, \mathrm{y} \in \mathcal{K} \text{ and } \mathrm{x} \neq \mathrm{y} \tag{10}$$

- *strongly monotone* on $\mathcal{K}$ if there exists a constant $c > 0$ such that

$$(\mathrm{F}(\mathrm{x}) - \mathrm{F}(\mathrm{y}))^T(\mathrm{x} - \mathrm{y}) \geq c\,||\mathrm{x} - \mathrm{y}||^2, \quad \forall \mathrm{x}, \mathrm{y} \in \mathcal{K}. \tag{11}$$

Proposition 8. [FP03, Proposition 2.3.2] Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex. A continuously differentiable operator $F : \mathcal{X} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is strongly monotone with monotonicity constant $\alpha > 0$ if and only if

$$\nabla_x F(x) \succeq \alpha I_n, \qquad \forall x \in \mathcal{X}. \tag{2.9}$$

It is monotone if and only if

$$\nabla_x F(x) \succeq 0, \qquad \forall x \in \mathcal{X}. \tag{2.10}$$

Moreover, if $\mathcal{X}$ is compact then there exists $\alpha > 0$ such that $\nabla_x F(x) \succeq \alpha I_n$ for all $x \in \mathcal{X}$ if and only if $\nabla_x F(x) \succ 0$ for all $x \in \mathcal{X}$. $\square$

if the vector function F is the gradient of a scalar function f, we have:

$$
\begin{array}{lll}
\text{i) } f \text{ convex} & \Leftrightarrow & \nabla f \text{ monotone} \\
\text{ii) } f \text{ strictly convex} & \Leftrightarrow & \nabla f \text{ strictly monotone} \\
\text{iii) } f \text{ strongly convex} & \Leftrightarrow & \nabla f \text{ strongly monotone}
\end{array}
\tag{12}
$$

Variational Inequality

Assume:

- set K is closed and convex
- F is continuous on K

i) If F is monotone on $\mathcal{K}$, the solution set of the VI$(\mathcal{K}, \mathbf{F})$ is closed and convex.

ii) If F is strictly monotone on $\mathcal{K}$, the VI$(\mathcal{K}, \mathbf{F})$ admits at most one solution.

iii) If F is strongly monotone on $\mathcal{K}$, the VI$(\mathcal{K}, \mathbf{F})$ admits a unique solution. (13)

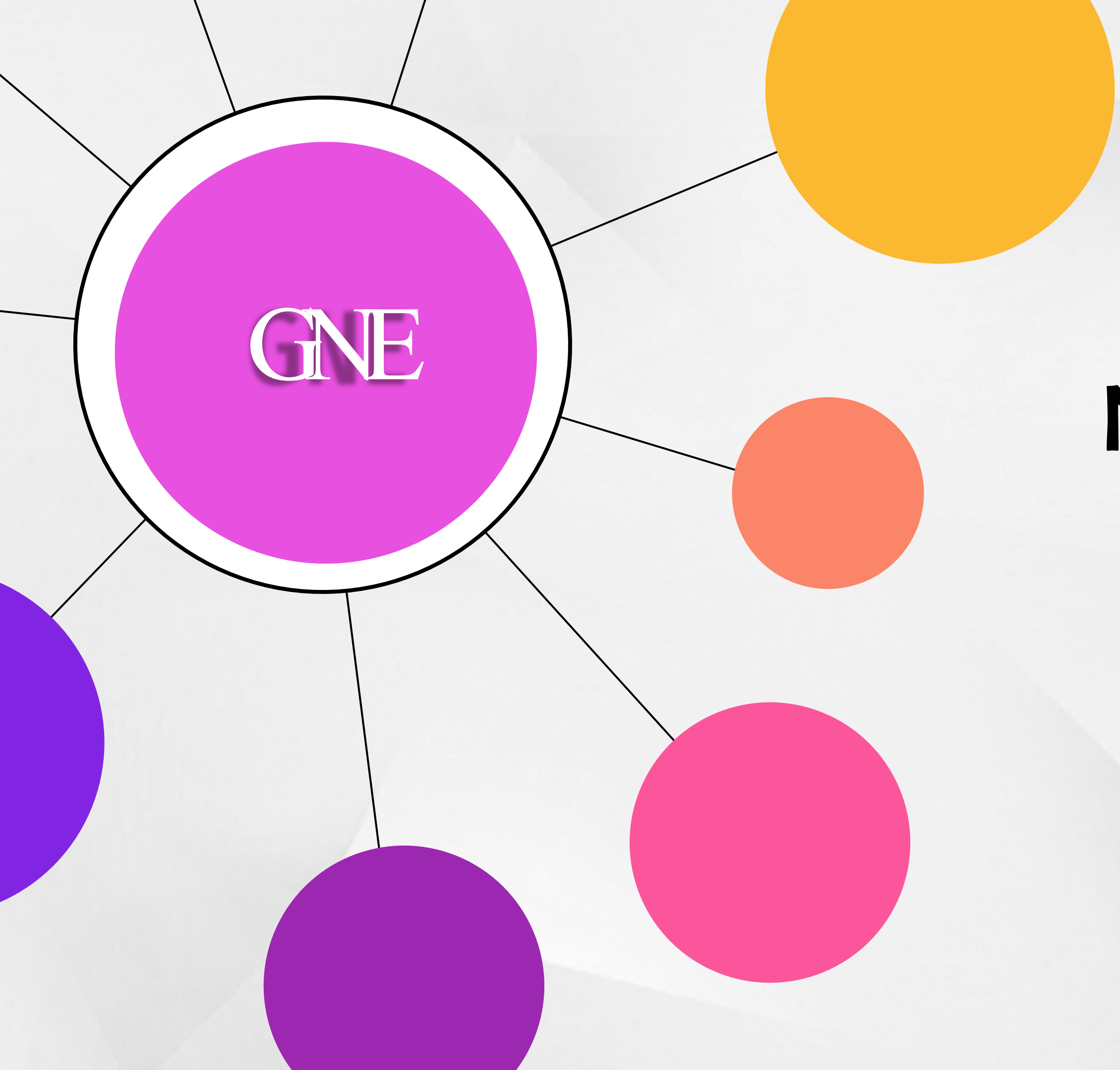
- Several equivalent formulations of the VI problem and thus characterizations of the solution can be found in the literature in terms of systems of equations and/or optimization problems of various kinds.
- One of the is as fixed point problem which paves the way for the development of a large family of iterative methods.
- we need to be familiar with Euclidean projection.

The Euclidean projection of a vector $\mathbf{x}_0$ onto a closed and convex set $\mathcal{K}$, denoted $\Pi_{\mathcal{K}}(\mathbf{x}_0)$, is the unique vector in $\mathcal{K}$ that is closest to $\mathbf{x}_0$ in the Euclidean norm. By definition, $\Pi_{\mathcal{K}}(\mathbf{x}_0)$ is the unique solution of

$$\begin{array}{ll} \underset{\mathbf{y}}{\text{minimize}} & ||\mathbf{y} - \mathbf{x}_0||^2 \\ \text{subject to} & \mathbf{y} \in \mathcal{K}. \end{array} \tag{14}$$

The Relationship between Variational Inequality and Fixed Point Problems:

$$\boxed{x^\star \text{ is a solution of the VI}(\mathcal{K}, F) \;\Leftrightarrow\; x^\star = \Pi_{\mathcal{K}}(x^\star - F(x^\star)). \quad (15)}$$



Generalized Nash Equilibrium Problems

27 Generalized Nash Equilibrium Problems

- In classical Nash equilibrium problems (NEPs), the interactions among players take place at the level of objective functions only.
- In generalized NEPs (GNEPs) where in addition to objective functions we have that the choices available to each player also depend by the actions taken by his rivals.
- In GNEP: coupling among agent : objective function + constraints

- So, the NEP is by far better studied and “easier.” The GNEP has a wider range of applicability but sparser results are available for its study.

Classical Nash Equilibrium:

- Assume there are Q players
- each controlling the variables $\mathbf{x}_i \in \mathbb{R}^{n_i}$.
- $\mathbf{x} \triangleq (\mathbf{x}_1, \ldots, \mathbf{x}_Q)$;
- $\mathbf{x}_{-i} \triangleq (\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_Q)$

aim of player i, given the other players' strategies $\mathbf{x}_{-i}$, is to choose an $\mathbf{x}_i \in \mathcal{Q}_i$ that minimizes his payoff function $f_i(\mathbf{x}_i, \mathbf{x}_{-i})$, i.e.,

$$\begin{array}{ll} \underset{\mathbf{x}_i}{\text{minimize}} & f_i(\mathbf{x}_i, \mathbf{x}_{-i}) \\ \text{subject to} & \mathbf{x}_i \in \mathcal{Q}_i. \end{array} \tag{16}$$

Classical Nash Equilibrium:

aim of player i, given the other players' strategies x_{-i}, is to choose an $\mathrm{x}_i \in \mathcal{Q}_i$ that minimizes his payoff function $f_i(\mathrm{x}_i, \mathrm{x}_{-i})$, i.e.,

$$\begin{array}{ll} \underset{\mathrm{x}_i}{\text{minimize}} & f_i(\mathrm{x}_i, \mathrm{x}_{-i}) \\ \text{subject to} & \mathrm{x}_i \in \mathcal{Q}_i. \end{array} \tag{16}$$

We make the blanket assumption that the objective functions f_i are continuously differentiable and, as a function of x_i alone, convex, while the sets $\mathcal{Q}_i \subseteq R^{n_i}$ are all closed and convex.

Classical Nash Equilibrium:

A (pure strategy) NE, or simply a solution of the NEP, is a feasible point $\mathbf{x}^{\star}$ such that

$$f_i(\mathbf{x}_i^{\star}, \mathbf{x}_{-i}^{\star}) \leq f_i(\mathbf{x}_i, \mathbf{x}_{-i}^{\star}), \quad \forall \mathbf{x}_i \in \mathcal{Q}_i \tag{17}$$

holds for each player $i = 1, \ldots, Q$.

Classical Nash Equilibrium:

خروجی

$$\max_{x_i \in X_i \subseteq \mathbb{R}} u_i(x_i, x_{-i}) = x_i^* = f_i(x_{-i})$$

Best Response Function

ورودی

یک تابع از تصمیم سایر عوامل

Let $\mathcal{B}_i(\mathrm{x}_{-i})$ be the set of optimal solutions of the ith optimization problem (16) and set $\mathcal{B}(\mathrm{x}) \triangleq B_1(\mathrm{x}_{-1}) \times B_2(\mathrm{x}_{-2}) \times \cdots \times B_Q(\mathrm{x}_{-Q})$.

It is clear that a point $\mathrm{x}^\star$ is an NE if and only if it is a fixed point of $\mathcal{B}(\mathrm{x})$, i.e., if and only if $\mathrm{x}^\star \in \mathcal{B}(\mathrm{x}^\star)$.

$(y_i - x_i^\star)^T \nabla_{x_i} f_i(x_i^\star, x_{-i}) \geq 0$

Classical Nash Equilibrium:

In fact, given the equivalence between the VI problem and a convex optimization problem, the following result follows readily from the minimum principle (5) for convex problems.

given a feasible $x^\star$, each $x_i^\star$ is an optimal solution of (16) if and only if it satisfies the minimum principle [see (5)]: $(y_i - x_i^\star)^T \nabla_{x_i} f_i(x_i^\star, x_{-i}^\star) \geq 0$, for all $y_i \in \mathcal{Q}_i$. Summing these conditions and taking into account the Cartesian product structure of $\mathcal{Q}$, leads to the desired equivalence between the NEP and the VI problem.

$\mathcal{Q} \triangleq \prod_{i=1}^{Q} \mathcal{Q}_i$ and $f \triangleq (f_i(x))_{i=1}^{Q}$.

Given the game $\mathcal{G} = \langle \mathcal{Q}, f \rangle$, suppose that for each player i

i) the strategy set $\mathcal{Q}_i$ is closed and convex;

ii) the payoff function $f_i(x_i, x_{-i})$ is continuously differentiable in x and convex in x_i for every fixed x_{-i}.

Then, the game $\mathcal{G}$ is equivalent to the VI$(\mathcal{Q}, F)$, where $F(x) \triangleq (\nabla_{x_i} f_i(x))_{i=1}^{Q}$.

(18)

Given the game $\mathcal{G} = \langle Q, \mathbf{f} \rangle$, suppose that for each player i

i) the strategy set $\mathcal{Q}_i$ is closed and convex;

ii) the payoff function $f_i(\mathbf{x}_i, \mathbf{x}_{-i})$ is continuously differentiable in $\mathbf{x}$ and convex in $\mathbf{x}_i$ for every fixed $\mathbf{x}_{-i}$.

Then, the game $\mathcal{G}$ is equivalent to the VI$(\mathcal{Q}, \mathbf{F})$, where $\mathbf{F}(\mathbf{x}) \triangleq (\nabla_{\mathbf{x}_i} f_i(\mathbf{x}))_{i=1}^{Q}$.

(18)

Given the equivalence between the NEP and the VI problem, conditions guaranteeing the existence of an NE follow readily from the existence of a solution of the VI: Suppose that, in addition to conditions i) and ii) in (18), each player's strategy set $\mathcal{Q}_i$ is compact, then the NEP has a convex and nonempty solution set, thanks to the existence results (13).

Given the game $\mathcal{G} = \langle Q, \mathbf{f} \rangle$, suppose that for each player i

i) the strategy set $\mathcal{Q}_i$ is closed and convex;

ii) the payoff function $f_i(\mathbf{x}_i, \mathbf{x}_{-i})$ is continuously differentiable in $\mathbf{x}$ and convex in $\mathbf{x}_i$ for every fixed $\mathbf{x}_{-i}$.

Then, the game $\mathcal{G}$ is equivalent to the VI($\mathcal{Q}$, F), where $\mathbf{F}(\mathbf{x}) \triangleq (\nabla_{\mathbf{x}_i} f_i(\mathbf{x}))_{i=1}^{Q}$.

(18)

Assuming that the function $\mathbf{F}(\mathbf{x}) \triangleq (\nabla_{\mathbf{x}_i} f_i(\mathbf{x}))_{i=1}^{Q}$ is strongly monotone on $\mathcal{Q}$, we immediately have that $\mathcal{G} = \langle Q, \mathbf{f} \rangle$ has a unique solution. Sufficient conditions easily to be checked that guarantees such a F being strongly monotone on $\mathcal{Q}$ are given in [17] and [28].

ALGORITHM 1: GAUSS-SEIDEL BEST RESPONSE-BASED ALGORITHM

`(S.0)`: Choose any feasible starting point $\mathbf{x}^{(0)} = (\mathbf{x}_i^{(0)})_{i=1}^{Q}$, and set $n = 0$.

`(S.1)`: If $\mathbf{x}^{(n)}$ satisfies a suitable termination criterion: `STOP`

`(S.2)`: `for` $i = 1, \ldots, Q$, compute a solution $\mathbf{x}_i^{(n+1)}$ of

$$\begin{array}{ll} \underset{\mathbf{x}_i}{\text{minimize}} & f_i(\mathbf{x}_1^{(n+1)}, \ldots, \mathbf{x}_{i-1}^{(n+1)}, \mathbf{x}_i, \mathbf{x}_{i+1}^{(n)}, \ldots, \mathbf{x}_Q^{(n)}) \\ \text{subject to} & \mathbf{x}_i \in \mathcal{Q}_i, \end{array} \tag{19}$$

`end`

`(S.3)`: Set $x^{(n+1)} \triangleq (x_i^{(n+1)})_{i=1}^{Q}$ and $n \leftarrow n + 1$; go to `(S.1)`.

The GNEP extends the classical NEP described so far by assuming that each player's strategy set can depend on the rival players' strategies $\mathbf{x}_{-i}$, so we will write $\mathcal{Q}_i(\mathbf{x}_{-i})$ to indicate that we might have a different closed convex set $\mathcal{Q}_i$ for each different $\mathbf{x}_{-i}$.

Analogously to the NEP case, the aim of each player i, given $\mathbf{x}_{-i}$, is to choose a strategy $\mathbf{x}_i \in \mathcal{Q}_i(\mathbf{x}_{-i})$ that solves the problem

$$\begin{array}{ll} \underset{\mathbf{x}_i}{\text{minimize}} & f_i\left(\mathbf{x}_i, \mathbf{x}_{-i}\right) \\ \text{subject to} & \mathbf{x}_i \in \mathcal{Q}_i(\mathbf{x}_{-i}). \end{array} \tag{20}$$

A generalized NE (GNE) is a tuple of strategies $\mathbf{x}^\star = (\mathbf{x}_i^\star, \ldots, \mathbf{x}_Q^\star)$ such that, for all $i = 1, \ldots, Q$,

$$f_i(\mathbf{x}_i^\star, \mathbf{x}_{-i}^\star) \leq f_i(\mathbf{x}_i, \mathbf{x}_{-i}^\star), \quad \forall \mathbf{x}_i \in \mathcal{Q}_i(\mathbf{x}_{-i}^\star). \tag{21}$$

GNEPs with Shared Constraints

A GNEP is termed a GNEP with shared constraints if the feasible sets $\mathcal{Q}_i(\mathbf{x}_{-i})$ are defined as

$$\mathcal{Q}_i(\mathbf{x}_{-i}) \triangleq \{\mathbf{x}_i \in \mathcal{K}_i : \mathbf{g}(\mathbf{x}_i, \mathbf{x}_{-i}) \leq 0\},$$

where $\mathcal{K}_i$ is the (closed and convex) set of individual constraints of player i and $\mathbf{g}(\mathbf{x}_i, \mathbf{x}_{-i}) \leq 0$ represents the set of shared coupling constraints (equal for all the players), with $\mathbf{g} = (g_j)_{j=1}^{m_i}$ assumed to be continuously differentiable and (jointly) convex in $\mathbf{x}$.

$$\mathcal{Q} \triangleq \{\mathbf{x} : \mathbf{g}(\mathbf{x}_i, \mathbf{x}_{-i}) \leq 0, \quad \mathbf{x}_i \in \mathcal{K}_i \ \forall i = 1, \ldots, Q\}. \quad (22)$$

$$\begin{aligned} \mathcal{Q}_i(\mathbf{x}_{-i}) &= \{\mathbf{x}_i \in \mathcal{K}_i : \mathbf{g}(\mathbf{x}_i, \mathbf{x}_{-i}) \leq 0\} \\ &= \{\mathbf{x}_i : (\mathbf{x}_i, \mathbf{x}_{-i}) \in \mathcal{Q}\}. \end{aligned} \quad (23)$$

GNEPs with shared constraints are still very difficult problems, however at least some types of solutions can be studied and calculated relatively easily by using a VI approach. To this end define as usual the function $\mathbf{F}(\mathbf{x}) \triangleq (\nabla_{\mathbf{x}_i} f_i(\mathbf{x}))_{i=1}^{Q}$ and consider the VI$(\mathcal{Q}, \mathbf{F})$, with $\mathcal{Q}$ defined in (22). It can be seen that every solution of this VI is a solution of GNEP with shared constraints, but not vice versa [34], [4]·

Variational solutions are particularly useful in many applications since they have an interesting "economic" interpretation. Indeed, it can be shown that $\bar{\mathrm{x}}$ is a variational solution if and only if $\bar{\mathrm{x}}$, along with a suitable $\bar{\lambda}$ satisfies the NEP defined by

$$\begin{array}{ll} \underset{\mathrm{x}_i}{\text{minimize}} & f_i(\mathrm{x}_i, \mathrm{x}_{-i}) + \sum_{k=1}^{m} \bar{\lambda}_k g_k(\mathrm{x}_i, \mathrm{x}_{-i}) \\ \text{subject to} & \mathrm{x}_i \in \mathcal{K}_i, \end{array} \tag{25}$$

$\forall i = 1, \ldots, Q,$ and furthermore

$$0 \leq \bar{\boldsymbol{\lambda}} \perp \mathrm{g}(\bar{\mathrm{x}}) \leq 0. \tag{26}$$

Thanks for attention