الف

1) $$\begin{cases} \dot{x}_1 = x_2 = f_1(x_1, x_2, u) \\ \dot{x}_2 = -\dfrac{x_2^4}{x_1^2} + x_1 + \sqrt{u+1} \\ y = \underbrace{x_1^2 + u^2}_{g(\cdot)} \end{cases}$$

$\dot{x}_1 = 0 \Rightarrow \boxed{x_2 = 0}$

$\dot{x}_2 = 0 \Rightarrow 0 + x_1 + \sqrt{u+1} = 0$

$-x_1 = \sqrt{u+1} \Rightarrow \boxed{-\sqrt{u+1} = x_1}$

$\cancel{u = -1 \Rightarrow x_1 = 0}$

ب) $u_{ss} = 3$ $\begin{cases} x_2 = 0 \\ x_1 = -2 \end{cases}$

$$\dot{x}_1 = \frac{\partial f_1}{\partial x_1}\Big|_{\substack{x_{1ss} \\ x_{2ss} \\ u_{ss}}} \Delta x_1 + \frac{\partial f_1}{\partial x_2}\Big|_{ss} \Delta x_2 + \frac{\partial f_1}{\partial u}\Big|_{ss} \Delta u$$

$$\dot{x}_1 = 0 + 1(x_2 - 0) + 0 = \boxed{x_2 = \dot{x}_1}$$

$$\dot{x}_2 = \frac{\partial f_2}{\partial x_1}\Big|_{ss} \Delta x_1 + \frac{\partial f_2}{\partial x_2}\Big|_{ss} \Delta x_2 + \frac{\partial f_1}{\partial u}\Big|_{ss} \Delta u$$

$$\left[1 + \underbrace{\cancel{\left(-x_2^4 (-2) x_1^{-3}\right)}}_{0}\right]_{\substack{-2\\0\\3}} (x_1 + 2) + \underbrace{\cancel{\frac{-4x_2^3}{x_1^2}}}_{0}\Big|_{\substack{-2\\0\\3}} (x_2) + \underbrace{\cancel{\frac{1}{2\sqrt{u+1}}}}_{\frac{1}{4}}\Big|_{\substack{-2\\0\\3}} (u-3)$$

$$\boxed{\dot{x}_2 = x_1 + 2 + \frac{u-3}{4}} = \hat{x}_1 + \hat{u}$$

$$y = \frac{\partial g}{\partial x_1}\Big|_{ss} \Delta x_1 + \frac{\partial g}{\partial x_2}\Big|_{ss} \Delta x_2 + \frac{\partial g}{\partial u}\Big|_{ss} \Delta u$$

$$= -4\underbrace{(x_1 + 2)}_{\hat{x}_1} + 0 + 6\underbrace{(u-3)}_{\hat{u}} = -4\hat{x}_1 + 6\hat{u}$$

$$= -4x_1 - 8 + 6u - 18 = \underbrace{-4x_1 + 6u - 26}_{} = -4\hat{x}_1 + 6\hat{u}$$

الف) نقاط تعادل

$$\dot{x} = 6x - 2x^2 - 2y^2 \Rightarrow f_1(x,y) = x(-y^2 - 2x + 6) \begin{cases} \to x = 0 \\ \to -y^2 - 2x + 6 = 0 \quad (1) \end{cases}$$

$$\dot{y} = 6xy - xy - y^2 \Rightarrow f_2(x,y) = y(-y - x + 6) = 0 \begin{cases} \to y = 0 \\ \to 6 = y + x \quad (2) \end{cases}$$

$$(1),(2) \xrightarrow{-12 + 2y = -2x} -y^2 - 12 + 2y + 6 = 0 \Rightarrow y^2 + 2y + 6 = 0 \Rightarrow y = -1 \pm j\sqrt{5}$$

y حقیقی نیست

if $x = 0 \Rightarrow (2) \Rightarrow y = 6 \quad (0, 6)$

if $y = 0 \Rightarrow (1) \Rightarrow -2x + 6 = 0 \Rightarrow x = 3 \Rightarrow (3, 0)$

or both $x = 0$ and $y = 0 \quad (0, 0)$

ب) حال حول یک نقطه خطی سازی کنیم

برای راحتی کار $(x,y) = (0,0)$ را انتخاب می کنیم

$$\dot{x} = \frac{\partial f_1(x,y)}{\partial x}\Big|_0 (x - 0) + \frac{\partial f_1(x,y)}{\partial y}\Big|_0 (y - 0) = (6 - 4x - y^2)\Big|_0 (x) + (-2xy)\Big|_0 (y)$$

$$= 6x$$

$$\dot{y} = \frac{\partial f_2}{\partial x}\Big|_0 (x - 0) + \frac{\partial f_2(x,y)}{\partial y}\Big|_0 (y - 0) = \cancel{(-y)}\Big|_0 (x) + \cancel{(6 - x - 2y)}\Big|_0 (y)$$

$$= 6y$$

3) الف)

برای بیر در جهت x داریم

$$u = M\frac{d^2x}{dt^2} + m\frac{d^2x_G}{dt^2}$$

برای گوی داریم

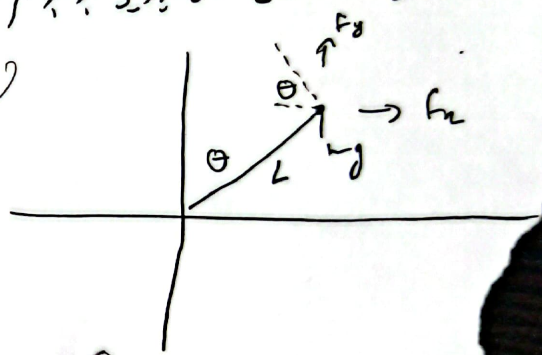
$$\begin{cases} x_G = x + L\sin\theta \\ y_G = L\cos\theta \end{cases}$$

$$\Rightarrow (M+m)\ddot{x} - mL\sin\theta\,\dot{\theta}^2 + mL\cos(\theta)\,\ddot{\theta} = u \quad (AA)$$

حال معادله های گشتاوری را حول جرم m را می نویسیم

$$(F_x\cos\theta)L - (F_y\sin\theta)L = (mg\sin\theta)L$$

$$F_x = m\frac{d^2}{dt^2}(x_G) \quad \& \quad F_y = m\frac{d^2 y_G}{dt^2}$$



$$\Rightarrow m\ddot{x}\cos\theta + mL\ddot{\theta} = mg\sin\theta \quad (A)$$

از دو معادله (A) و (AA) استفاده می کنیم و معادله های $\ddot{x}$ و $\ddot{\theta}$ را بیرون می کشیم داریم

$$\ddot{x} = \frac{u + mL(\sin\theta)\dot{\theta}^2 - mg\cos\theta\sin\theta}{M + m - m\cos^2\theta}$$

$$\ddot{\theta} = \frac{u\cos\theta - (M+m)g\sin\theta + mL(\cos\theta\sin\theta)\dot{\theta}^2}{mL\cos^2\theta - (M+m)L}$$

3) ب)

حال معادلات حالت را حساب می کنیم

داریم $\ddot{\theta}$ و $\ddot{x}$ $\Longrightarrow$

$$
\begin{cases}
\theta = x_1 \\
\dot{\theta} = x_2 = \dot{x}_1 \\
\ddot{\theta} = \dot{x}_2 = \longrightarrow \text{معادله قسمت قبل} \\
x = x_3 \\
\dot{x} = \dot{x}_3 = x_4 \\
\ddot{x} = \dot{x}_4 \longrightarrow \text{معادله قسمت قبل}
\end{cases}
$$

برای تابع خروجی y اگر هم x را بخواهیم و هم θ داریم

$$
y = \begin{bmatrix} \theta \\ x \end{bmatrix} = Cx = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
$$

ج) نقاط تعادل سیستم طرح 1

$x_1 = 0$ $x_3 = 0$ $u = 0$

$x_2 = 0$ $x_4 = 0$

$\dot{x} = 0$

M^2 $\quad x_3 + \delta - g \times 2$

د) معادلات $\dot{x}_3$ و $\dot{x}_1$ سیستم خطی هستند اما معادلات $\dot{x}_4$ و $\dot{x}_2$ غیر خطی هستند.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} & \frac{\partial F_1}{\partial x_4} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} & \frac{\partial F_2}{\partial x_4} \\ \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_2} & \frac{\partial F_3}{\partial x_3} & \frac{\partial F_3}{\partial x_4} \\ \frac{\partial F_4}{\partial x_1} & \frac{\partial F_4}{\partial x_2} & \frac{\partial F_4}{\partial x_3} & \frac{\partial F_4}{\partial x_4} \end{bmatrix} X + \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial u} \\ \frac{\partial f_4}{\partial u} \end{bmatrix}$$

$$\Rightarrow \dot{X} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{g}{L}(1+mM) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -mMg & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ -\frac{1}{Lm} \\ 0 \\ \frac{1}{M} \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} X$$

(۵ ق

$L=1, \quad m=1, \quad M=10$

ما ورودی ها را به صورت مقابل است

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 110 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -100 & 0 \end{bmatrix} \qquad B=\begin{bmatrix} 0 \\ -\frac{1}{10} \\ 0 \\ \frac{1}{10} \end{bmatrix} \qquad C=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$D=0$$

$$(sI-A)^{-1}=\begin{bmatrix} s & -1 & 0 & 0 \\ 0 & s & -110 & 0 \\ 0 & 0 & s & -1 \\ 0 & 0 & 100 & s \end{bmatrix}^{-1}$$

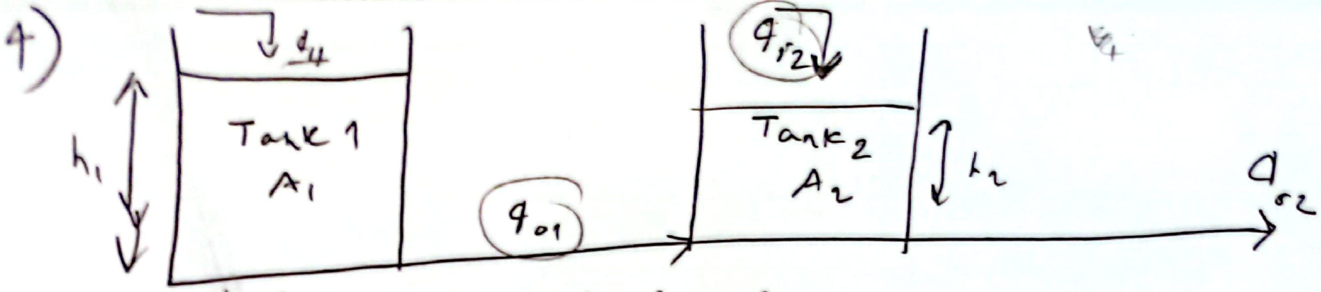
$$\frac{1}{s^2(s^2+100)}\begin{bmatrix} s(s^2+100) & s^2+100 & 110s & 110 \\ 0 & s(s^2+100) & 110s^2 & 110s \\ 0 & 0 & s^3 & s^2 \\ 0 & 0 & -100s^2 & s^3 \end{bmatrix}$$

جواب نهایی

$$\begin{bmatrix} \frac{Y_1(s)}{U(s)} \\ \frac{Y_2(s)}{U(s)} \end{bmatrix} = C(sI-A)^{-1}B+D$$

$$=\begin{bmatrix} \frac{1}{10s^2}+\frac{10}{s^2(s^2-100)} \\ \\ -\frac{1}{10(s^2-110)} \end{bmatrix}$$

4)



اسن) ~~...~~ در این مدل ما ورودی q_{i_2} و q_{i_1} داریم و دو خروجی $\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$

$q_o \propto h^{\frac{1}{2}} \rightarrow q_o = K\sqrt{h}$ (1)

برای Tank 1 می‌دانیم ورودی آن q_{i_1} و خروجی آن q_{o_1} است پس نرخ تغییرات آن

$$V_1 = A_1 h_1 \Rightarrow \underbrace{\frac{dV_1}{dt} = A_1 \frac{dh_1}{dt}}_{**}$$

همچنین داریم

$$\underbrace{q_1 - q_{o1} = \frac{dV_1}{dt}}_{*}$$

$$\xrightarrow[**]{*} A_1 \frac{dh_1}{dt} = q_{i_1} - q_{o_1} \xRightarrow{(1)} A_1 \frac{dh_1}{dt} = q_{i_1} - K\sqrt{h_1}$$

$$\Rightarrow \frac{dh_1}{dt} = \frac{q_{i_1} - K\sqrt{h_1}}{A_1}$$

به طریق مشابه برای tank 2 داریم:

$$\left.\begin{aligned} \frac{dV_2}{dt} &= q_{i_2} + q_{o_1} - q_{o_2} \\ \frac{dV_2}{dt} &= A_2 \frac{dh_2}{dt} \end{aligned}\right\} \Rightarrow A_2 \frac{dh_2}{dt} = q_{i_2} + q_{o_1} - q_{o_2}$$

$$\xrightarrow{(1)} A_2 \frac{dh_2}{dt} = q_{i_2} + K\sqrt{h_1} - K\sqrt{h_2} \Rightarrow$$

$$\frac{dh_2}{dt} = \frac{q_{i_2} + K\sqrt{h_1} - K\sqrt{h_2}}{A_2}$$

متغیرهای حالت: $\left.\begin{aligned} x_1 &= h_1 \\ x_2 &= h_2 \end{aligned}\right\}$

$$\Rightarrow \left\{\begin{aligned} \dot{x}_1 &= \frac{dh_1}{dt} = \frac{q_{i_1}}{A} - \frac{K\sqrt{x_1}}{A} \\ \dot{x}_2 &= \frac{dh_2}{dt} = \frac{q_{i_2}}{A_2} + \frac{K\sqrt{x_1}}{A_2} - \frac{K\sqrt{x_2}}{A_2} \end{aligned}\right\}$$ غیرخطی

برای خروجی داریم $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q_{i_1} \\ q_{i_2} \end{bmatrix}$$

B 4) ب

$x_2^* = h_{2s}$, $x_1^* = h_{1s}$, $q_{is} =$ مقادیر تعادلی

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial q_{i1}} & \frac{\partial f_1}{\partial q_{i1}} \\ \frac{\partial f_2}{\partial q_{1i}} & \frac{\partial f_2}{\partial q_{2i}} \end{bmatrix} \begin{bmatrix} q_{i1} - q_{i1}^* \\ q_{i2} - q_{i2}^* \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{-K}{A_1} \frac{1}{2\sqrt{x_1}} & 0 \\ \frac{K}{A_2} \frac{1}{2\sqrt{x_1}} & \frac{-K}{A_2} \frac{1}{2\sqrt{x_2}} \end{bmatrix} \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix} + \begin{bmatrix} \frac{1}{A_1} & 0 \\ 0 & \frac{1}{A_2} \end{bmatrix} \begin{bmatrix} q_{i1} - q_{i1}^* \\ q_{i2} - q_{i2}^* \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-K}{A_2 2\sqrt{h_{1s}}} & 0 \\ \frac{K}{2A_2\sqrt{h_{1s}}} & \frac{-K}{2A_2\sqrt{h_{2s}}} \end{bmatrix} \begin{bmatrix} x_1 - h_{1s} \\ x_2 - h_{2s} \end{bmatrix} + \begin{bmatrix} \frac{1}{A_1} & 0 \\ 0 & \frac{1}{A_2} \end{bmatrix} \begin{bmatrix} q_{i1} - q_{1s} \\ q_{i2} - q_{2s} \end{bmatrix}$$

خطی سازی شده

حال با توجه به قسمت ب داریم

2) $D = 0$ $\quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} \frac{-K}{2A_1\sqrt{h_{1s}}} & \\ \frac{K}{2A_2\sqrt{h_{1s}}} & \frac{-K}{2A_2\sqrt{h_{2s}}} \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{A_1} & 0 \\ 0 & \frac{1}{A_2} \end{bmatrix}$$

برای تابع تبدیل سیستم داریم:

ماتریس همانی

$$\left.\begin{aligned} \dot{X} &= AX + BQ \\ Y &= CX + DQ \end{aligned}\right\} \Rightarrow \frac{Y(s)}{Q(s)} = C(sI - A)^{-1}B + \cancel{D}$$

$$(sI - A)^{-1} = \begin{bmatrix} \left(s + \frac{K}{2A_2\sqrt{h_{2s}}}\right) & 0 \\ \left(\frac{K_2}{2A_2\sqrt{h_{1s}}}\right) & \left(s + \frac{K}{2A_1\sqrt{h_{1s}}}\right) \end{bmatrix}$$

$$\frac{Y(s)}{Q(s)} = \begin{bmatrix} \frac{1}{A_1}\left(s + \frac{K}{2A_2\sqrt{h_{2s}}}\right) & 0 \\ \frac{1}{A}\left(\frac{K}{2A_2\sqrt{h_{1s}}}\right) & \frac{1}{A_2}\left(s + \frac{K}{2A_1\sqrt{h_{1s}}}\right) \end{bmatrix}$$

5) $\dddot{Y} + 12\ddot{Y} + 2\dot{y} + 16y = 2u$

الف) $y = x_1$

$\dot{y} = \dot{x}_1 = x_2$

$\ddot{y} = \dot{x}_2 = \ddot{x}_1 = x_3$

$\dddot{y} = \dot{x}_3 = \ddot{x}_2$

$\dot{x}_3 = -12\underbrace{\ddot{y}}_{x_3} - 2\underbrace{\dot{y}}_{x_2} - 16\underbrace{y}_{x_1} + 2u$

$$\dot{X} = \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -16 & \cancel{-2}\,-2 & \cancel{-12}\,-12 \end{bmatrix}}^{A} X + \overbrace{\begin{bmatrix} 0 \\ 0 \\ \end{bmatrix}}^{B} u$$

$$Y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C} X \qquad D = 0$$

ب) $G(s) = C(sI - A)^{-1}B + \underbrace{D}_{0}$

$$G(s) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 16 & 2 & s+12 \end{bmatrix}^{-1}_{3\times3} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}_{3\times1}$$

$\det(sI - A) = s^2(s+12) + 16 + 2s = s^3 + 12s^2 + 2s + 16$

$$(sI - A)^{-1} = \frac{1}{s^3 + 12s^2 + 2s + 16} \begin{bmatrix} s^2+12s+2 & s+12 & 1 \\ -16 & s^2+12s & s \\ -16s & -2s-16 & s^2 \end{bmatrix}_{3\times3}$$

$$G(s) = \frac{1}{s^3 + 12s^2 + 2s + 16} \begin{bmatrix} s^2+12s+2 & s+12 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$= \boxed{\frac{2}{s^3 + 12s^2 + 2s + 16} = G(s)}$$