Probability Theory Review

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Outline

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Definitions, Axioms, and Corollaries

- Performing an **experiment** $\rightarrow$ **outcome**
- **Sample Space** (S): set of all possible outcomes of an experiment
- **Event** (E): a subset of $S(E \subseteq S)$
- **Probability** (**Bayesian** definition): a number between 0 and 1 to which we ascribe meaning
- **Frequentist** definition of probability

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

Axiom 1: $0 \leq P(E) \leq 1$
Axiom 2: $P(S) = 1$
Axiom 3: if E and F are mutually exclusive $(E \cap F = 0)$, then $P(E) + P(F) = P(E \cup F)$ Corollary 1:
$P(E^C) = 1 - P(E) \qquad (= P(S) - P(E))$
Corollary 2: $E \subseteq F$, then $P(E) \leq P(F)$
Corollary 3: $P(E \cup F) = P(E) + P(F) - P(EF)$ (Inclusion-Exclusion Principle)
General Inclusion-Exclusion Principle:

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1 < \ldots < i_r} P(E_{i1} E_{i2} \ldots E_{ir})$$

Equally Likely Outcomes: Define S as a sample space with equally likely outcomes. Then $P(E) = \frac{|E|}{|S|}$

Conditional Probability and Bayes' Rule

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Let's apply conditional probability to obtain **Bayes' Rule**!

$$P(B \mid A) := \frac{P(A \cap B)}{P(A)} = \frac{P(A \cap B)}{P(A)} = \frac{P(B)P(A \mid B)}{P(A)}$$

Conditioned Bayes' Rule: given events A, B, C

$$P(A \mid B, C) = \frac{P(B \mid A, C)P(A \mid C)}{P(B \mid C)}$$

Law of Total Probability

Let $B_1, \ldots, B_n$ be n disjoint events whose union is the entire sample space. Then, for any event A,

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$$= \sum_{i=1}^{n} P(A \mid B_i) P(B_i)$$

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P(B_k \mid A) &= \frac{P(B_k) P(A \mid B_k)}{P(A)} \\
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\end{aligned}
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Example

Treasure chest A holds 100 gold coins. Treasure chest B holds 60 gold and 40 silver coins. Choose a treasure chest uniformly at random, and pick a coin from that chest uniformly at random. If the coin is gold, then what is the probability that you chose chest A?

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Solution:

$$\begin{aligned} P(A \mid G) &= \frac{P(A)P(G \mid A)}{P(A)P(G \mid A) + P(B)P(G \mid B)} \\ &= \frac{0.5 \times 1}{0.5 \times 1 + 0.5 \times 0.6} \\ &= 0.625 \end{aligned}$$

Chain Rule

For any n events $A_1, \ldots, A_n$, the joint probability can be expressed as a product of conditionals:

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)P(A_2 \mid A_1) \ldots P(A_n \mid A_{n-1} \cap A_{n-2} \cap \ldots \cap A_1)$$

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Implication: If two events are independent, observing one event does not change the probability that the other event occurs. **In general**: events $A_1, \ldots, A_n$ are **mutually independent** if

$$P\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} P(A_i)$$

for any subset $S \subseteq \{1, \ldots n\}$

Outline

Random Variables

- A **random variable** X is a variable that probabilistically takes on different values. It maps outcomes to real values
- X takes on values in $Val(X) \subseteq \mathbb{R}$ or Support $\mathrm{Sup}(X)$
- $X = k$ is the **event** that random variable X takes on value k

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Discrete RVs:

- $Val(X)$ is a set
- $P(X = k)$ can be nonzero

Continuous RVs:

- $Val(X)$ is a range
- $P(X = k) = 0$ for all k. $P(a \leq X \leq b)$ can be nonzero

Probability Mass Function (PMF)

Given a **discrete** RV X, a PMF maps values of X to probabilities.

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For a valid PMF, $\sum_{x \in Val(x)} p_X(x) = 1$

Cumulative Distribution Function (CDF)

A CDF maps a continuous RV to a probability (i.e. $\mathbb{R} \rightarrow [0,\ 1]$)

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A CDF must fulfill the following:

- $\lim_{x \to -\infty} F_X(x) = 0$
- $\lim_{x \to \infty} F_X(x) = 1$
- If $a \leq b$, then $F_X(a) \leq F_X(b)$ (i.e. CDF must be nondecreasing)

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Also note: $P(a \leq X \leq b) = F_X(b) - F_X(a)$.

Probability Density Function (PDF)

PDF of a continuous RV is simply the derivative of the CDF.

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Thus,

$$P(a \leq X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(x)\ dx$$

A valid PDF must be such that

- for all real numbers x, $f_X(x) > 0$
- $\int_{-\infty}^{\infty} f_X(x)\ dx = 1$

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Expectation

Let g be an arbitrary real-valued function:

- if X is a discrete RV with PMF p_X:

$$\mathbb{E}\left[g(X)\right] := \sum_{x \in Val(X)} g(x) P_X(x)$$

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Intuitively, expectation is a weighted average of the values of $g(x)$, weighted by the probability of x.

Properties of Expectation

For any constant $a \in \mathbb{R}$ and arbitrary real function f:

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Linearity of Expectation

Given n real-valued functions $f_1(X), \ldots, f_n(X)$,

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Law of Total Expectation

Given two RVs $X,\ Y$:

$$\mathbb{E}\left[\mathbb{E}[X \mid Y]\right]$$

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Law of Total Expectation

Given two RVs $X,\ Y$:

$$\mathbb{E}\left[\mathbb{E}[X \mid Y]\right]$$

N.B. $\mathbb{E}[X \mid Y] = \sum_{x \in Val(x)} x p_{X|Y}(x \mid y)$ is a function of Y.

Example of Law of Total Expectation

El Goog sources two batteries, A and B, for its phone. A phone with battery A runs on average 12 hours on a single charge, but only 8 hours on average with battery B. El puts battery A in 80% of its phones and battery B in the rest. If you buy a phone from El, how many hours do you expect it to run on a single charge?

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Solution: Let L be the time your phone runs on a single charge.

- $p_X(A) = 0.8,\ p_X(B) = 0.2$
- $\mathbb{E}[L \mid A] = 12,\ \mathbb{E}[L \mid B] = 8$

$$\begin{aligned}
\mathbb{E}[L] &= \mathbb{E}\left[\mathbb{E}[L \mid X]\right] \\
&= \sum_{X \in \{A,\ B\}} \mathbb{E}[L \mid X] p_X(x) \\
&= \mathbb{E}[L \mid A] p_X(A) + \mathbb{E}[L \mid B] p_X(B) \\
&= 12 \times 0.8 + 8 \times 0.2 = 11.2
\end{aligned}$$

Variance

The variance of a RV X measures how concentrated the distribution of X is around its mean.

$$\begin{aligned} Var(X) &:= \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \\ &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \end{aligned}$$

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Interpretation: $Var(X)$ is the expected deviation of X from $\mathbb{E}[X]$
Properties: For any constant $a \in \mathbb{R}$, real-valued function $f(X)$

- $Var[a] = 0$
- $Var[af(X)] = a^2 Var[f(X)]$

Example Distributions

Distribution	PDF or PMF	Mean	Variance
$Bernoulli(p)$	$\begin{cases} p, & \text{if } x = 1 \\ 1-p, & \text{if } x = 0. \end{cases}$	p	$p(1-p)$
$Binomial(n, p)$	$\binom{n}{k} p^k (1-p)^{n-k}$ for $k = 0, 1, ..., n$	np	$np(1-p)$
$Geometric(p)$	$p(1-p)^{k-1}$ for $k = 1, 2, ...$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
$Poisson(\lambda)$	$\frac{e^{-\lambda}\lambda^k}{k!}$ for $k = 0, 1, ...$	λ	λ
$Uniform(a, b)$	$\frac{1}{b-a}$ for all $x \in (a, b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
$Gaussian(\mu, \sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ for all $x \in (-\infty, \infty)$	μ	σ^2
$Exponential(\lambda)$	$\lambda e^{-\lambda x}$ for all $x \geq 0, \lambda \geq 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$

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Joint and Marginal Distributions

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- **Marginal PMF** of X, given joint PMF of $X,\ Y$:

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Joint and Marginal Distributions for Multiple RVs

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These definitions can be extended to multiple random variables in the same way as in the previous slide. For example, for n continuous RV's $X_1, \ldots, X_n$ and function $g : \mathbb{R}^n \to \mathbb{R}$:

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$$\mathbb{E}[g(X)] := \int \int \ldots \int g(x_1, \ldots, \ x_n) f_{X_1, \ldots, \ X_n}(x_1, \ldots, \ x_n) \ dx_1 \ldots \ dx_n$$

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Covariance

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- If $Cov[X,\ Y] < 0$, then X and Y are negatively correlated
- If $Cov[X,\ Y] > 0$, then X and Y are positively correlated
- If $Cov[X,\ Y] < 0$, then X and Y are uncorrelated

Properties Involving Covariance

- If $X \perp Y$, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$. Thus,

$$Cov[X,\ Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0$$

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- **Special Case:**

$$Cov[X,\ X] = \mathbb{E}[XX] - \mathbb{E}[X]\mathbb{E}[X] = Var[X]$$

Outline

Conditional distributions for RVs

Works the same way with RV's as with events:

- For discrete $X, \ Y$:

$$p_{Y|X}(y \mid x) = \frac{p_{XY}(x, \ y)}{p_X(x)}$$

- For continuous $X, \ Y$:

$$f_{Y|X}(y \mid x) = \frac{f_{XY}(x, \ y)}{f_X(x)}$$

- In general, for continuous $X_1, \ \ldots, \ X_n$:

$$f_{X_1|X_2, \ \ldots, \ X_n}(x_1 \mid x_2, \ \ldots, \ x_n) = \frac{f_{X_1, \ X_2, \ \ldots, \ X_n}(x_1, \ x_2, \ \ldots, \ x_n)}{f_{X_2, \ \ldots, \ X_n}(x_2, \ \ldots, \ x_n)}$$

Bayes' Rule for RVs

Also works the same way for RV 's as with events:

- For discrete $X, \; Y$:

$$p_{Y|X}(y \mid x) = \frac{p_{X|Y}(x \mid y) p_Y(y)}{\sum_{y' \in Val(Y)} p_{X|Y}(x \mid y') p_Y(y')}$$

- For continuous $X, \; Y$:

$$f_{Y|X}(y \mid x) = \frac{f_{X|Y}(x \mid y) f_Y(y)}{\int_{-\infty}^{\infty} f_{X|Y}(x \mid y') f_Y(y')}$$

Chain Rule for RVs

Also works the same way for RV 's as with events:

$$
\begin{aligned}
f(x_1,\ x_2,\ \ldots,\ x_n) &= f(x_1) f(x_2 \mid x_1)\ \ldots\ f(x_n \mid x_1,\ x_2,\ \ldots,\ x_{n-1}) \\
&= f(x_1) \prod_{i=2}^{n} f(x_i \mid x_1,\ x_2,\ \ldots,\ x_{i-1})
\end{aligned}
$$

Independence for RVs

- For $X \perp Y$ to hold, it must be that $F_{XY}(x,\ y) = F_X(x)F_Y(y)$ **FOR ALL VALUES** of $x,\ y$.

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- For $X \perp Y$ to hold, it must be that $F_{XY}(x,\ y) = F_X(x)F_Y(y)$ **FOR ALL VALUES** of $x,\ y$.
- Since $f_{Y|X}(y \mid x) = f_Y(y)$ if $X \perp Y$, chain rule for mutually independent $X_1, \dots, X_n$ is:

$$f(x_1,\ x_2,\ \dots,\ x_n) = f(x_1)f(x_2)\dots f(x_n) = \prod_{i=1}^{n} f(x_i)$$

(very important assumption for a Naive Bayes classifier!)

Outline

Random Vectors

Given n RV's $X_1, \ldots, X_n$, we can define a random vector X s.t.

$$\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

Note: all the notions of joint PDF/CDF will apply to X.

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$$g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_m(x) \end{bmatrix}, \quad \mathbb{E}\left[g(X)\right] = \begin{bmatrix} \mathbb{E}\left[g_1(X)\right] \\ \mathbb{E}\left[g_2(X)\right] \\ \vdots \\ \mathbb{E}\left[g_m(X)\right] \end{bmatrix}$$

Covariance Matrices

For a random vector $X \in \mathbb{R}^n$, we define its **covariance matrix** Σ as the $n \times n$ matrix whose ij-th entry contains the covariance between X_i and X_j.

$$\Sigma = \begin{bmatrix} Cov[X_1, X_1] & \dots & Cov[X_1, X_n] \\ \vdots & \ddots & \vdots \\ Cov[X_n, X_1] & \dots & Cov[X_n, X_n] \end{bmatrix}$$

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Properties:

- Σ is symmetric and PSD
- If $X_i \perp X_j$ for all i, j then $\Sigma = diag(Var[X_1], \dots, Var[X_n])$

Outline

Multivariate Gaussian

The multivariate Gaussian $X \sim \mathcal{N}(\mu,\ \Sigma),\ X \in \mathbb{R}^n$:

$$p(x;\ \mu,\ \Sigma) = \frac{1}{det(\Sigma)^{\frac{1}{2}}(2\pi)^{\frac{n}{2}}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$$

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$$p(x;\ \mu,\ \sigma^2) = \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

Notice that if $\Sigma \in \mathbb{R}^{1\times 1}$, then $\Sigma = Var[X_1] = \sigma^2$, and so $\Sigma^{-1} = \frac{1}{\sigma^2}$ and $det(\Sigma)^{\frac{1}{2}} = \sigma$

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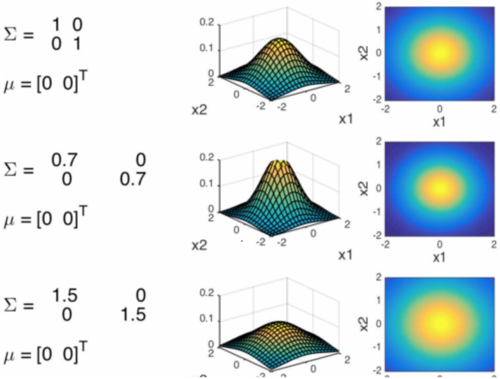
- In general, the isocontours of a MV Gaussian are n-dimensional ellipsoids with principal axes in the directions of the eigenvectors of covariance matrix Σ (remember, Σ is PSD, so all n eigenvalues are non-negative). The axes' relative lengths depend on the eigenvalues of Σ.

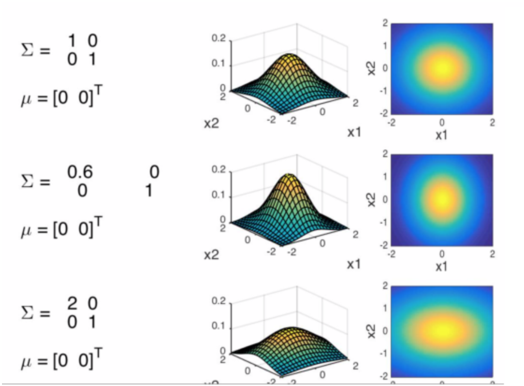
Visualization of Multivariate Gaussian

Effect of changing variance:



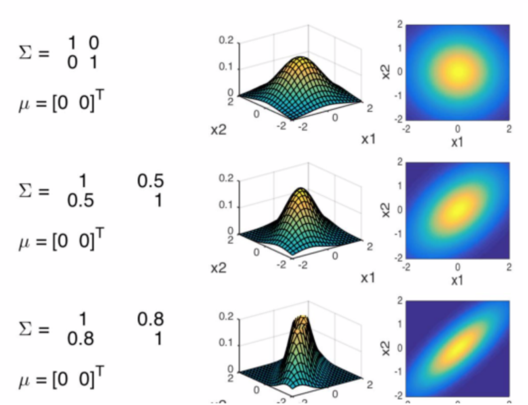
Visualization of Multivariate Gaussian

If $Var[X_1] \neq Var[X_2]$



Visualization of Multivariate Gaussian

If X_1 and X_2 are positively correlated:



Visualization of Multivariate Gaussian

If X_1 and X_2 are negatively correlated: