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Novel Adaptive IIR Filter for Frequency Estimation and Tracking

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In many applications, a sinusoidal signal may be subjected to nonlinear effects in which possible harmonic frequency components are generated. In such an environment, we may want to estimate (track) the signal's fundamental frequency as well as any harmonic frequencies. Using a second-order notch filter to estimate fundamental and harmonic frequencies is insufficient since it only accommodates one frequency component [1], [2]. On the other hand, applying a higher-order infinite impulse response (IIR) notch filter may not be efficient due to adopting multiple adaptive filter coefficients.

In this article, we present a novel adaptive harmonic IIR notch filter with a single adaptive coefficient to efficiently perform frequency estimation and tracking in a harmonic frequency environment. Furthermore, we devise a simple scheme to select the initial filter coefficient to insure algorithm convergence to its global minimum error.

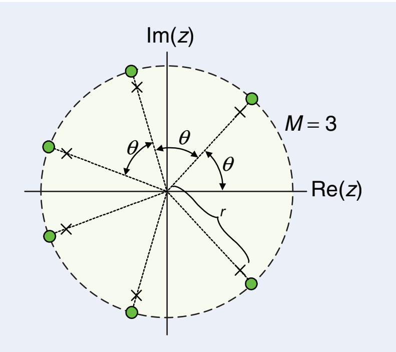
HARMONIC NOTCH FILTER STRUCTURE

Consider frequency estimation of a measured signal $x(n)$ containing a fundamental frequency component and its harmonics up to Mth order as

$$x(n)=\sum_{m=1}^{M}A_m\sin[2\pi(mf)nT+\phi_m]+v(n) \quad (1)$$

where A_m, mf, and ϕ_m are the magnitude, frequency (hertz), and phase angle of the mth harmonic component, respectively. $v(n)$ is a white Gaussian noise. Notice that n and T designate the time index and sampling period respectively. To estimate frequency in such a harmonic frequency environment, we propose a harmonic IIR notch filter whose z-domain pole-zero plot is shown in Figure 1 for the case of $M=3$ (the fundamental and two harmonics).

The idea is to place constrained pole-zero pairs with their angles equal to $\pm m\theta$ (multiples of the fundamental frequency angle θ) relative to the horizontal axis on the pole-zero plot for $m=1$, $2,\ldots,M$, respectively, to construct a multinotch filter transfer function [1]–[3]. The zeroes on the unit circle give us infinite-depth notches, and the parameter r controls the bandwidth of the notches. Parameter r is chosen to be close to, but less than, one to achieve narrowband notches and avoid any filter stability problems.



[FIG1] **Pole-zero plot for the harmonic IIR notch filter for $M=3$.**

Hence, once θ is adapted to the angle corresponding to the fundamental frequency, each $m\theta$ ($m=2,\ldots,M$) will automatically adapt to its harmonic frequency. We construct the filter transfer function in a cascaded form as

$$H(z)=\frac{Y(z)}{X(z)}=\prod_{m=1}^{M}H_m(z) \quad (2)$$

where the transfer function $H_m(z)$ at the mth second-order IIR section is defined as

$$H_m(z)=\frac{1-2z^{-1}\cos(m\theta)+z^{-2}}{1-2rz^{-1}\cos(m\theta)+r^2z^{-2}}. \quad (3)$$

From (2) and (3), the transfer function has only one adaptive coefficient θ. We determine the filter output $y_m(n)$ at the mth section as

$$\begin{aligned}y_m(n)=&\,y_{m-1}(n)\\&-2\cos(m\theta)y_{m-1}(n-1)\\&+y_{m-1}(n-2)\\&+2r\cos(m\theta)y_m(n-1)\\&-r^2y_m(n-2)\end{aligned} \quad (4)$$

and $y_0(n)=x(n)$.

Once the adaptive parameter θ has converged to its fundamental frequency, the harmonic IIR notch filter will filter out all the $x(n)$ input signal's fundamental and harmonic frequency components. Thus the last second-order subfilter output $y_M(n)$ is expected to be $y_M(n)\approx 0$ for a noiseless $x(n)$ input signal. Our objective, then, is to minimize the power of the last subfilter output, $E[y_M^2(n)]$, at which time the converged parameter θ is our desired result. Here, we define the filter output $y_M(n)$ as the error signal $e(n)$, that is, $e(n)=y_M(n)$, in a sense that its power is to be minimized. Hence, we can express the power of the last subfilter output via the mean square error (MSE) function [1] as in (5),

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$$E[e^2(n)] = E[y_M^2(n)] = \frac{1}{2\pi j}\oint \left| \prod_{m=1}^{M} \frac{1 - 2z^{-1}\cos(m\theta) + z^{-2}}{1 - 2rz^{-1}\cos(m\theta) + r^2z^{-2}} \right|^2 \Phi_{xx}\frac{dz}{z} \quad (5)$$

shown above, where Φ_{xx} is the power spectrum of the input signal. Equation (5) tells us that the MSE function is a nonlinear function of the adaptive parameter θ, and it may contain local minima. Selection of an initial value θ is critical for a global convergence of the adaptive algorithm. Fortunately, we need not evaluate (5) to obtain the MSE of $e(n)$. Instead, we estimate the MSE function for each given θ as follows:

$$\text{MSE} = E[e^2(n,\theta)] \approx \frac{1}{N}\sum_{n=1}^{N} y_M^2(n,\theta)$$
$$0 \le \theta \le \pi/M, \quad (6)$$

where N is the number of filter output samples averaged as θ varies over the range $0 \le \theta \le \pi/M$ rad. The cyclic frequency in hertz corresponding to θ in radians is $f = \theta f_s/(2M)$, where f_s is the sampling rate in hertz. To ensure the global minimum is at the fundamental frequency in case there are more than two global minima, we estimate another MSE function from the first subfilter corresponding to the fundamental frequency as follows:

$$\text{MSE1} = E[e_1^2(n,\theta)] \approx \frac{1}{N}\sum_{n=1}^{N} y_1^2(n,\theta)$$
$$0 \le \theta \le \pi/M. \quad (7)$$

It is well known that the MSE1$= E[e_1^2(n,\theta)]$ from this $m=1$ subfilter always has a global minimum [1]. Hence, once the region of the global minimum is identified by verifying MSE functions from (6) and (7), we could determine a frequency capture range based on the plotted MSE function in (6) as described in the following example.

Figure 2 shows an example of the MSE and MSE1 functions for $M = 3$ and $r = 0.95$, where $f_s = 8{,}000$ Hz, the fundamental frequency of the input signal is 1,000 Hz, the input signal to noise power ratio (SNR) is 18 dB, and $N = 400$ filter output samples were averaged for each θ.

Note that θ's range of $0 \le \theta \le \pi/M$ rad is based on the assumption that the $x(n)$ input's fundamental component and its $M-1$ harmonic frequencies are less than $f_s/2$. Therefore, the corresponding cyclic frequency range in Figure 2 is from 0 Hz to $4{,}000/M = 1333.33$ Hz for $M = 3$. For our Figure 2 example, there are four local minima of the MSE function from (6), where one global minimum is located at 1,000 Hz. Clearly, the MSE1 function from (7) verifies the global minimum at the fundamental frequency. If we let the adaptive algorithm initially start from any point within the global minimum capture range in the MSE function from (6), the adaptive harmonic IIR notch filter parameter θ will converge to the global minimum of the MSE function, which corresponds to the fundamental frequency of the input signal.

ALGORITHM DEVELOPMENT

As discussed in the last section, we develop the adaptive algorithm to contain two major steps. The first step is to determine the initial value of the adaptive filter $\theta(0)$ coefficient, while the second step is to apply the least mean-square (LMS) algorithm to obtain the desired precise value for the filter's θ coefficient.

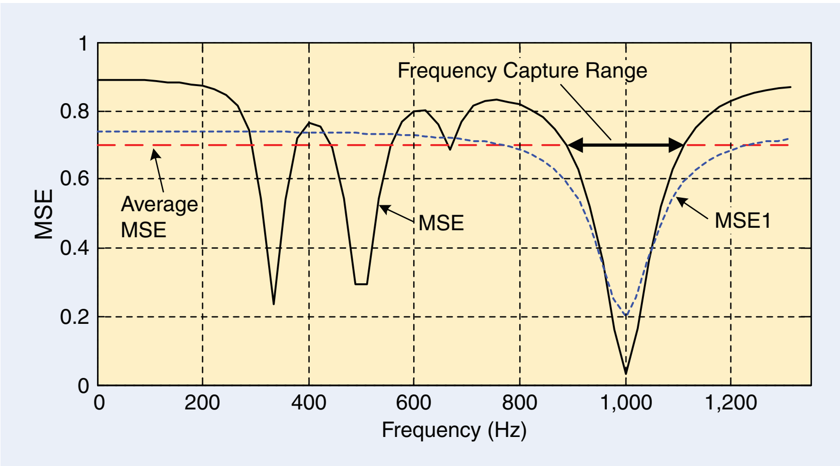
With a help of Figure 2, we can describe a strategy for selecting the initial value $\theta(0)$ as follows. After estimating MSE function values using (6) for $0 \le \theta \le \pi/M$, corresponding to $0 \le f \le f_s/(2/M)$ Hz, we compute the average value of MSE (horizontal line in Figure 2). Next we use that average value to determine two intersection points in the global minimum valley of the MSE function to obtain a frequency capture range (the frequency range between two intersection points), as depicted in Figure 2. Notice that the best initial $\theta(0)$ value should be the point of f (Hz) on the horizontal axis, at which the MSE has the largest deviation below its average value, such as $f = 1{,}000$ Hz. However, the LMS algorithm starting at any $\theta(0)$ point within the frequency capture range will converge to its global minimum.

For the second step, we apply the LMS algorithm to obtain the filter update equation to obtain the desired precise value for the filter's θ coefficient. Taking the derivative of $e^2(n) = [y_M(n)]^2$ and substituting the result to zero, we achieve

$$\theta(n+1) = \theta(n) - 2\mu y_M(n)\beta_M(n) \quad (8)$$

where μ is a convergence step-size parameter and the gradient term $\beta_m(n)$ at subfilter section m is defined as

$$\beta_m(n) = \frac{\partial y_m(n)}{\partial \theta(n)} \quad (9)$$



[FIG2] **Notch filter MSE and MSE1 functions for $M = 3$ and $r = 0.95$.**

with

$$\beta_0(n) = \frac{\partial y_0(n)}{\partial \theta(n)} = \frac{\partial x(n)}{\partial \theta(n)} = 0. \quad (10)$$

Using (4), $\beta_m(n)$ for $m = 1, 2, \ldots, M$ can be recursively computed using the following equation:

$$\begin{aligned}\beta_m(n) = {} & \beta_{m-1}(n) - 2\cos[m\theta(n)] \\ & \times \beta_{m-1}(n-1) \\ & + 2m\sin[m\theta(n)]y_{m-1}(n-1) \\ & + \beta_{m-1}(n-2) \\ & + 2r\cos[m\theta(n)]\beta_m(n-1) \\ & - r^2\beta_m(n-2) \\ & - 2rm\sin[m\theta(n)]y_m(n-1).\end{aligned} \quad (11)$$

Again, for the first subfilter (when $m = 1$), $\beta_0(n) = \beta_0(n-1) = \beta_0(n-2) = 0$.

ALGORITHM STEPS

We summarize the adaptive harmonic notch filter algorithm below.

Step 1: Determine the initial adaptive coefficient $\theta(0)$:

- Calculate and plot MSE $= E[y_M^2(n)]_\theta$ with (6) and MSE1 $= E[y_1^2(n)]_\theta$ with (7) using N filter outputs as θ is varied (scanned) over the range $0 \le \theta \le \pi/M$.
- Determine the MSE function's average and global minimum values.
- Based on the average and global minimum of the MSE, determine the frequency capture range.
- Choose the initial coefficient $\theta(0)$ to be within the frequency capture range.

Step 2: Apply the LMS algorithm for frequency estimation and tracking:

- For $m = 1, 2, \ldots, M$, apply (4) and (11).
- Apply the LMS process in (8) to obtain $\theta(n+1)$. Note that μ should be a very small value ($\mu << 1$), and chosen empirically to guarantee the algorithm convergence.
- Convert $\theta(n)$ in radians to the desired estimated fundamental frequency in Hz using

$$f(n) = \frac{\theta(n)}{2\pi} \times f_s \text{ (Hz)}. \quad (12)$$

COMPUTER SIMULATIONS

In our software simulations, the $x(n)$ input signal of 1000 Hz plus two harmonics, sampled at $f_s = 8{,}000$ Hz, is given by

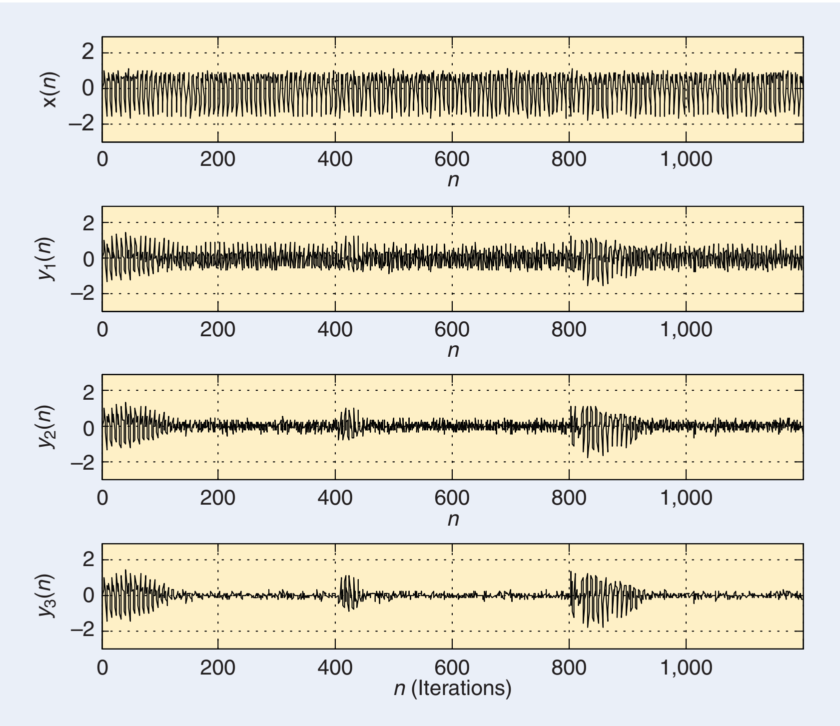
$$\begin{aligned}x(n) = {} & \sin[2\pi \times 1000 \times n/f_s] \\ & + 0.5\cos[2\pi \times (2 \times 1000) \\ & \times n/f_s] \\ & - 0.25\cos[2\pi \times (3 \times 1000) \\ & \times n/f_s] + v(n).\end{aligned} \quad (13)$$

The input signal-to-noise ratio (SNR) SNR is set to 18 dB by controlling the variance of the Gaussian $v(n)$ noise sequence in (13). We used $M = 3$ and $r = 0.95$ for constructing the harmonic notch filter. The MSE (6) and MSE1 (7) functions from Step 1 are plotted in Figure 2. The frequency capture range from 900 Hz to 1100 Hz (0.225π to 0.275π rad) was determined, and we chose $\theta(0) = 0.225\pi$ rad (900 Hz) and $\mu = 0.0001$ for the LMS algorithm.

To show the frequency tracking capabilities of the harmonic notch filter, Figure 3 shows the time-domain input and output at each filter subsection. The input fundamental frequency switches from 1,000 Hz to 1,075 Hz at time $n = 400$, while at $n = 800$ the fundamental frequency switches to 975 Hz. As depicted in Figure 3, the algorithm converges to the global minimums after 150 LMS iterations.

Figure 4(a) shows our adaptive harmonic notch filter's frequency magnitude response, which has the null points located at the fundamental and harmonic frequencies. That figure indicates why we call our filter a "harmonic notch filter." Figure 4(b) illustrates frequency tracking of the desired fundamental frequency $f(n)$ defined in (12), where the dashed curve indicates the true input fundamental frequencies. For this frequency tracking example, we can identify the fundamental frequencies after 150 LMS iterations.

As is well known, the LMS algorithm experiences reduced output fluctuations, but extended convergence time, when the value of μ is reduced. The harmonic notch filter exhibits similar behavior when the r bandwidth parameter is reduced to $r = 0.85$, as shown in Figure 4(c). (When parameter r is reduced, the global valley of the MSE function from (6) becomes less steep so that the LMS algorithm converges to its global minimum with less sensitivity.) For this $r = 0.85$ scenario, we can identify the fundamental frequencies after roughly 200 LMS iterations.

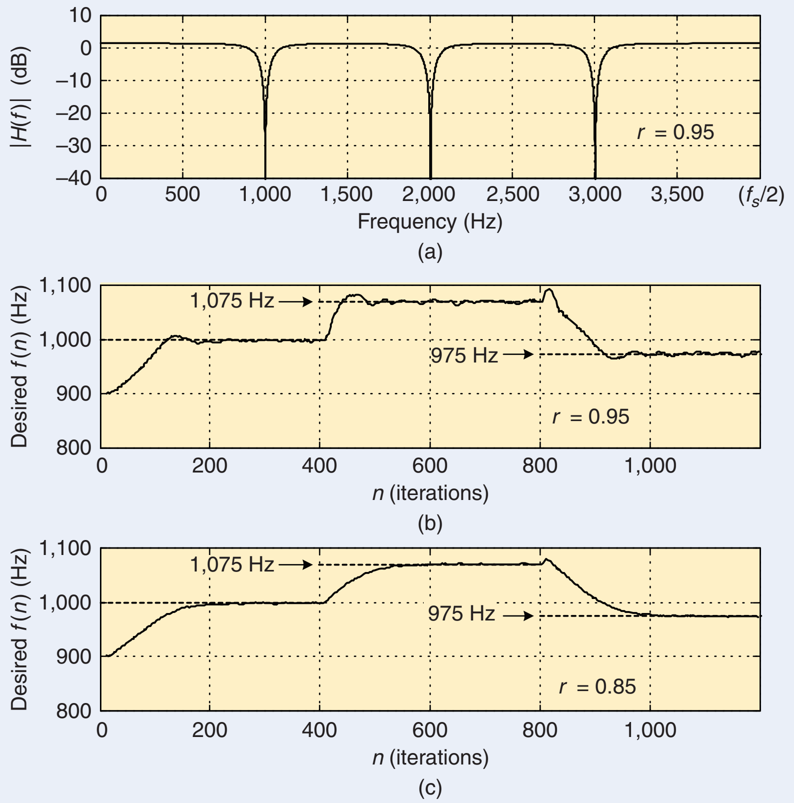


[FIG3] Input and outputs of the $M = 3$ IIR notch subfilters.

TYPICAL ALGORITHM PERFORMANCE

As expected, the adaptive harmonic IIR notch filter algorithm's frequency estimation performance improves with increased input signal SNRs. Figure 5 shows our algorithm's performance for frequency estimation and tracking under various noise conditions, where the frequency deviation from the fundamental frequency of 1,000 Hz was plotted versus the input signal's SNR. To obtain the results in Figure 5, we selected $\theta(0) = 0.225\pi$ rad (900 Hz), and used the M, r, and μ values shown in that figure. Next we generated $N = 20{,}000$ input samples using (13) at various SNRs, and for each SNR data point we simply computed the standard deviation using the last 50 tracked $f(n)$ frequency values. As shown in Figure 5, for the worst case in our simulations (SNR $= -5$ dB) standard deviation of our frequency estimation was 0.55 Hz (less than 1 Hz). We observed that the algorithm performed well under high SNR conditions.

Our experiments also demonstrated that the harmonic notch filter possesses frequency tracking capability when the input signal contained fewer harmonics than we anticipated. On the other hand, if the number of harmonics in the input signal is greater than the number of second-order IIR subfilters, the harmonic notch filter still tracks the fundamental signal frequency but with some performance degradation.



[FIG4] Harmonic notch filter: (a) frequency magnitude response, (b) $f(n)$ frequency tracking with $r = 0.95$, and (c) reduced-noise tracking with $r = 0.85$.

CONCLUSIONS

In this article, we developed a novel adaptive harmonic IIR notch filter for frequency estimation and tracking in a multiharmonic frequency environment. This frequency estimation algorithm requires only a single adaptive coefficient and efficiently estimates fundamental and harmonic frequencies simultaneously.

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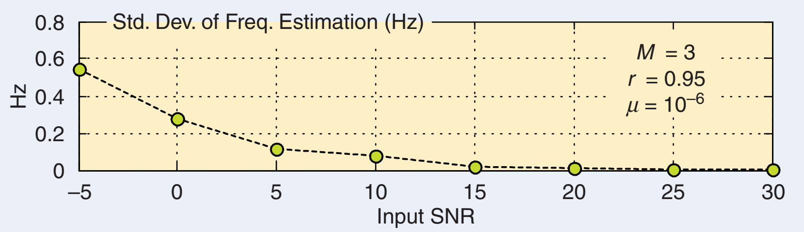

[FIG5] Standard deviation of frequency estimates versus input SNR.

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