# 21–7 Electric Field Calculations for Continuous Charge Distributions

In many cases we can treat charge as being distributed continuously.[†] We can divide up a charge distribution into infinitesimal charges $dQ$, each of which will act as a tiny point charge. The contribution to the electric field at a distance $r$ from each $dQ$ is

$$dE = \frac{1}{4\pi\epsilon_0}\frac{dQ}{r^2}. \qquad \text{(21–6a)}$$

Then the electric field, $\vec{\mathbf{E}}$, at any point is obtained by summing over all the infinitesimal contributions, which is the integral

$$\vec{\mathbf{E}} = \int d\vec{\mathbf{E}}. \qquad \text{(21–6b)}$$

Note that $d\vec{\mathbf{E}}$ is a vector (Eq. 21–6a gives its magnitude). [In situations where Eq. 21–6b is difficult to evaluate, other techniques (discussed in the next two Chapters) can often be used instead to determine $\vec{\mathbf{E}}$. Numerical integration can also be used in many cases.]


FIGURE 21–28 Example 21–9.

**EXAMPLE 21–9** **A ring of charge.** A thin, ring-shaped object of radius $a$ holds a total charge $+Q$ distributed uniformly around it. Determine the electric field at a point P on its axis, a distance $x$ from the center. See Fig. 21–28. Let $\lambda$ be the charge per unit length (C/m).

**APPROACH AND SOLUTION** We explicitly follow the steps of the Problem Solving Strategy on page 571.

1. **Draw a careful diagram.** The **direction** of the electric field due to one infinitesimal length $d\ell$ of the charged ring is shown in Fig. 21–28.
2. **Apply Coulomb's law.** The electric field, $d\vec{\mathbf{E}}$, due to this particular segment of the ring of length $d\ell$ has magnitude
$$dE = \frac{1}{4\pi\epsilon_0}\frac{dQ}{r^2}.$$
The whole ring has length (circumference) of $2\pi a$, so the charge on a length $d\ell$ is
$$dQ = Q\left(\frac{d\ell}{2\pi a}\right) = \lambda\, d\ell$$
where $\lambda = Q/2\pi a$ is the charge per unit length. Now we write $dE$ as
$$dE = \frac{1}{4\pi\epsilon_0}\frac{\lambda\, d\ell}{r^2}.$$
3. **Add vectorially and use symmetry:** The vector $d\vec{\mathbf{E}}$ has components $dE_x$ along the $x$ axis and $dE_\perp$ perpendicular to the $x$ axis (Fig. 21–28). We are going to sum (integrate) around the entire ring. We note that an equal-length segment diametrically opposite the $d\ell$ shown will produce a $d\vec{\mathbf{E}}$ whose component perpendicular to the $x$ axis will just cancel the $dE_\perp$ shown. This is true for all segments of the ring, so by symmetry $\vec{\mathbf{E}}$ will have zero $y$ component, and so we need only sum the $x$ components, $dE_x$. The total field is then
$$E = E_x = \int dE_x = \int dE\cos\theta = \frac{1}{4\pi\epsilon_0}\lambda\int\frac{d\ell}{r^2}\cos\theta.$$
Since $\cos\theta = x/r$, where $r = (x^2 + a^2)^{\frac{1}{2}}$, we have
$$E = \frac{\lambda}{(4\pi\epsilon_0)}\frac{x}{(x^2 + a^2)^{\frac{3}{2}}}\int_0^{2\pi a} d\ell = \frac{1}{4\pi\epsilon_0}\frac{\lambda x(2\pi a)}{(x^2 + a^2)^{\frac{3}{2}}} = \frac{1}{4\pi\epsilon_0}\frac{Qx}{(x^2 + a^2)^{\frac{3}{2}}}.$$
4. **To check reasonableness,** note that at great distances, $x \gg a$, this result reduces to $E = Q/(4\pi\epsilon_0 x^2)$. We would expect this result because at great distances the ring would appear to be a point charge ($1/r^2$ dependence). Also note that our result gives $E = 0$ at $x = 0$, as we might expect because all components will cancel at the center of the circle.

PROBLEM SOLVING

Use symmetry when possible

PROBLEM SOLVING

Check result by noting that at a great distance the ring looks like a point charge

[†] Because we believe there is a minimum charge ($e$), the treatment here is only for convenience; it is nonetheless useful and accurate since $e$ is usually very much smaller than macroscopic charges.

Note in this Example three important problem-solving techniques that can be used elsewhere: (1) using symmetry to reduce the complexity of the problem; (2) expressing the charge $dQ$ in terms of a charge density (here linear, $\lambda = Q/2\pi a$); and (3) checking the answer at the limit of large $r$, which serves as an indication (but not proof) of the correctness of the answer—if the result does not check at large $r$, your result has to be wrong.

PROBLEM SOLVING

**CONCEPTUAL EXAMPLE 21–10** **Charge at the center of a ring.** Imagine a small positive charge placed at the center of a nonconducting ring carrying a uniformly distributed negative charge. Is the positive charge in equilibrium if it is displaced slightly from the center along the axis of the ring, and if so is it stable? What if the small charge is negative? Neglect gravity, as it is much smaller than the electrostatic forces.

**RESPONSE** The positive charge is in equilibrium because there is no net force on it, by symmetry. If the positive charge moves away from the center of the ring along the axis in either direction, the net force will be back towards the center of the ring and so the charge is in *stable* equilibrium. A negative charge at the center of the ring would feel no net force, but is in *unstable* equilibrium because if it moved along the ring's axis, the net force would be away from the ring and the charge would be pushed farther away.

**EXAMPLE 21–11** **Long line of charge.** Determine the magnitude of the electric field at any point P a distance $x$ from the midpoint 0 of a very long line (a wire, say) of uniformly distributed positive charge, Fig. 21–29. Assume $x$ is much smaller than the length of the wire, and let $\lambda$ be the charge per unit length (C/m).

**APPROACH** We set up a coordinate system so the wire is on the $y$ axis with origin 0 as shown. A segment of wire $dy$ has charge $dQ = \lambda\, dy$. The field $d\vec{\mathbf{E}}$ at point P due to this length $dy$ of wire (at $y$) has magnitude

$$dE = \frac{1}{4\pi\epsilon_0}\frac{dQ}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\lambda\, dy}{(x^2 + y^2)},$$

where $r = (x^2 + y^2)^{\frac{1}{2}}$ as shown in Fig. 21–29. The vector $d\vec{\mathbf{E}}$ has components $dE_x$ and $dE_y$ as shown where $dE_x = dE\cos\theta$ and $dE_y = dE\sin\theta$.

**SOLUTION** Because 0 is at the midpoint of the wire, the $y$ component of $\vec{\mathbf{E}}$ will be zero since there will be equal contributions to $E_y = \int dE_y$ from above and below point 0:

$$E_y = \int dE\sin\theta = 0.$$

Thus we have

$$E = E_x = \int dE\cos\theta = \frac{\lambda}{4\pi\epsilon_0}\int\frac{\cos\theta\, dy}{x^2 + y^2}.$$

The integration here is over $y$, along the wire, with $x$ treated as constant. We must now write $\theta$ as a function of $y$, or $y$ as a function of $\theta$. We do the latter: since $y = x\tan\theta$, then $dy = x\, d\theta/\cos^2\theta$. Furthermore, because $\cos\theta = x/\sqrt{x^2 + y^2}$, then $1/(x^2 + y^2) = \cos^2\theta/x^2$ and our integrand above is $(\cos\theta)(x\, d\theta/\cos^2\theta)(\cos^2\theta/x^2) = \cos\theta\, d\theta/x$. Hence

$$E = \frac{\lambda}{4\pi\epsilon_0}\frac{1}{x}\int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta = \frac{\lambda}{4\pi\epsilon_0 x}(\sin\theta)\bigg|_{-\pi/2}^{\pi/2} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{x},$$

where we have assumed the wire is extremely long in both directions ($y \rightarrow \pm\infty$) which corresponds to the limits $\theta = \pm\pi/2$. Thus the field near a long straight wire of uniform charge decreases inversely as the first power of the distance from the wire.

**NOTE** This result, obtained for an infinite wire, is a good approximation for a wire of finite length as long as $x$ is small compared to the distance of P from the ends of the wire.

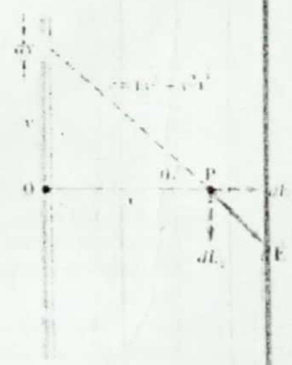


FIGURE 21–29 Example 21–11.



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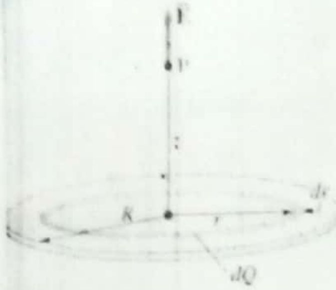



FIGURE 21–30 Example 21–12: uniformly charged flat disk of radius $R$.

**EXAMPLE 21–12** **Uniformly charged disk.** Charge is distributed uniformly over a thin circular disk of radius $R$. The charge per unit area (C/m²) is $\sigma$. Calculate the electric field at a point P on the axis of the disk, a distance $z$ above its center, Fig. 21–30.

**APPROACH** We can think of the disk as a set of concentric rings. We can then apply the result of Example 21–9 to each of these rings, and then sum over all the rings.

**SOLUTION** For the ring of radius $r$ shown in Fig. 21–30, the electric field has magnitude (see result of Example 21–9)

$$dE = \frac{1}{4\pi\epsilon_0}\frac{z\,dQ}{(z^2 + r^2)^{\frac{3}{2}}}$$

where we have written $dE$ (instead of $E$) for this thin ring of total charge $dQ$. The ring has area $(dr)(2\pi r)$ and charge per unit area $\sigma = dQ/(2\pi r\,dr)$. We solve this for $dQ$ $(= \sigma\, 2\pi r\,dr)$ and insert it in the equation above for $dE$:

$$dE = \frac{1}{4\pi\epsilon_0}\frac{z\sigma 2\pi r\,dr}{(z^2 + r^2)^{\frac{3}{2}}} = \frac{z\sigma r\,dr}{2\epsilon_0(z^2 + r^2)^{\frac{3}{2}}}.$$

Now we sum over all the rings, starting at $r = 0$ out to the largest with $r = R$:

$$E = \frac{z\sigma}{2\epsilon_0}\int_0^R \frac{r\,dr}{(z^2 + r^2)^{\frac{3}{2}}} = \frac{z\sigma}{2\epsilon_0}\left[-\frac{1}{(z^2 + r^2)^{\frac{1}{2}}}\right]_0^R$$

$$= \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{(z^2 + R^2)^{\frac{1}{2}}}\right].$$

This gives the magnitude of $\vec{\mathbf{E}}$ at any point $z$ along the axis of the disk. The direction of each $d\vec{\mathbf{E}}$ due to each ring is along the $z$ axis (as in Example 21–9), and therefore the direction of $\vec{\mathbf{E}}$ is along $z$. If $Q$ (and $\sigma$) are positive, $\vec{\mathbf{E}}$ points away from the disk; if $Q$ (and $\sigma$) are negative, $\vec{\mathbf{E}}$ points toward the disk.

If the radius of the disk in Example 21–12 is much greater than the distance of our point P from the disk (i.e., $z \ll R$) then we can obtain a very useful result: the second term in the solution above becomes very small, so

$$E = \frac{\sigma}{2\epsilon_0}. \qquad \text{[infinite plane]} \qquad (21\text{–}7)$$

This result is valid for any point above (or below) an infinite plane of any shape holding a uniform charge density $\sigma$. It is also valid for points close to a finite plane, as long as the point is close to the plane compared to the distance to the edges of the plane. Thus the field near a large uniformly charged plane is uniform, and directed outward if the plane is positively charged.

It is interesting to compare here the distance dependence of the electric field due to a point charge ($E \sim 1/r^2$), due to a very long uniform line of charge ($E \sim 1/r$), and due to a very large uniform plane of charge ($E$ does not depend on $r$).

**EXAMPLE 21–13** **Two parallel plates.** Determine the electric field between two large parallel plates or sheets, which are very thin and are separated by a distance $d$ which is small compared to their height and width. One plate carries a uniform surface charge density $\sigma$ and the other carries a uniform surface charge density $-\sigma$, as shown in Fig. 21–31 (the plates extend upward and downward beyond the part shown).

**APPROACH** From Eq. 21–7, each plate sets up an electric field of magnitude $\sigma/2\epsilon_0$. The field due to the positive plate points away from that plate whereas the field due to the negative plate points toward that plate.

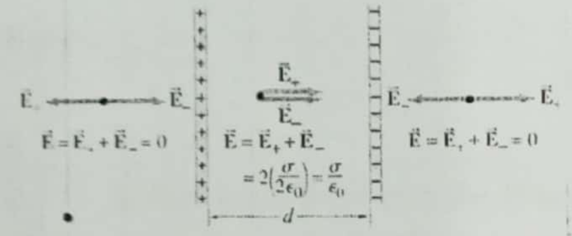



FIGURE 21–31 Example 21–13. (Only the center portion of these large plates is shown: their dimensions are large compared to their separation $d$.)

**SOLUTION** In the region between the plates, the fields add together as shown:

$$E = E_+ + E_- = \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0}.$$

The field is uniform, since the plates are very large compared to their separation, so this result is valid for any point, whether near one or the other of the plates, or midway between them as long as the point is far from the ends. Outside the plates, the fields cancel,

$$E = E_+ + E_- = \frac{\sigma}{2\epsilon_0} - \frac{\sigma}{2\epsilon_0} = 0,$$

as shown in Fig. 21–31. These results are valid ideally for infinitely large plates; they are a good approximation for finite plates if the separation is much less than the dimensions of the plate and for points not too close to the edge.

**NOTE:** These useful and extraordinary results illustrate the principle of superposition and its great power.

## 21–8 Field Lines

Since the electric field is a vector, it is sometimes referred to as a **vector field**. We could indicate the electric field with arrows at various points in a given situation, such as at A, B, and C in Fig. 21–32. The directions of $\vec{E}_A$, $\vec{E}_B$, and $\vec{E}_C$ are the same as that of the forces shown earlier in Fig. 21–22, but the magnitudes (arrow lengths) are different since we divide $\vec{F}$ in Fig. 21–22 by $q$ to get $\vec{E}$. However, the relative lengths of $\vec{E}_A$, $\vec{E}_B$, and $\vec{E}_C$ are the same as for the forces since we divide by the same $q$ each time. To indicate the electric field in such a way at *many* points, however, would result in many arrows, which might appear complicated or confusing. To avoid this, we use another technique, that of field lines.

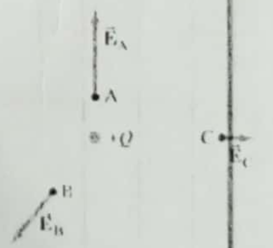
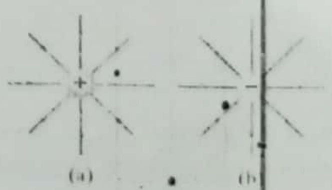



FIGURE 21–32 Electric field vector shown at three points due to a single point charge $Q$. (Compare to Fig. 21–22.)

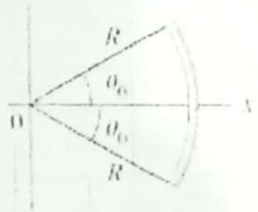
To visualize the electric field, we draw a series of lines to indicate the direction of the electric field at various points in space. These **electric field lines** (sometimes called **lines of force**) are drawn so that they indicate the direction of the force due to the given field on a positive test charge. The lines of force due to a single isolated positive charge are shown in Fig. 21–33a, and for a single isolated negative charge in Fig. 21–33b. In part (a) the lines point radially outward from the charge, and in part (b) they point radially inward toward the charge because that is the direction the force would be on a positive test charge in each case (as in Fig. 21–25). Only a few representative lines are shown. We could just as well draw lines in between those shown since the electric field exists there as well. We can draw the lines so that the *number of lines starting on a positive charge, or ending on a negative charge, is proportional to the magnitude of the charge.* Notice that nearer the charge, where the electric field is greater ($F \propto 1/r^2$), the lines are closer together. This is a general property of electric field lines: *the closer together the lines are, the stronger the electric field in that region.* In fact, field lines can be drawn so that the number of lines crossing unit area perpendicular to $\vec{E}$ is proportional to the magnitude of the electric field.

FIGURE 21–33 Electric field lines (a) near a single positive point charge, (b) near a single negative point charge.



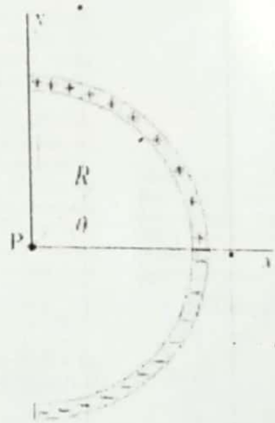
## Problems

1. A thin rod is bent into the shape of an arc of a circle of radius R carries a uniform charge per unit length $\lambda$. The arc subtends a total angle $2\theta_o$. Symmetric about the x –axis as shown in figure.




Determine the electric field E at point O.

2. A thin glass rod is a semicircle of radius R as shown below




A charge is no uniformly distributed along the rod with a linear charge density given by $\lambda = \lambda_o Sin\theta$, where $\lambda_o$ is a positive constant and point P lies at the centre of the ring.

Up to now we have not been concerned with an ambiguity in the direction of the vector $\vec{\mathbf{A}}$ or $d\vec{\mathbf{A}}$ that represents a surface. For example, in Fig. 22–1c, the vector $\vec{\mathbf{A}}$ could point upward and to the right (as shown) or downward to the left and still be perpendicular to the surface. For a closed surface, we define (arbitrarily) the direction of $\vec{\mathbf{A}}$, or of $d\vec{\mathbf{A}}$, to point *outward* from the enclosed volume, Fig. 22–4. For an electric field line leaving the enclosed volume (on the right in Fig. 22–4), the angle $\theta$ between $\vec{\mathbf{E}}$ and $d\vec{\mathbf{A}}$ must be less than $\pi/2$ (= 90°); hence $\cos\theta > 0$. For a line entering the volume (on the left in Fig. 22–4) $\theta > \pi/2$; hence $\cos\theta < 0$. Hence, *flux entering the enclosed volume is negative* ($\int E\cos\theta\, dA < 0$), whereas *flux leaving the volume is positive*. Consequently, Eq. 22–3 gives the net flux *out* of the volume. If $\Phi_E$ is negative, there is a net flux *into* the volume.

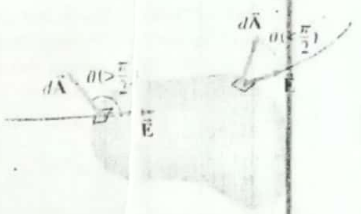
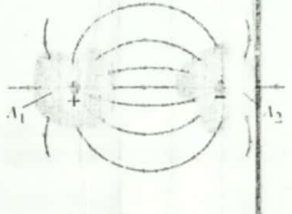



FIGURE 22–4 The direction of an element of area $d\vec{\mathbf{A}}$ is taken to point outward from an enclosed surface.

In Figs. 22–3 and 22–4, each field line that enters the volume also leaves the volume. Hence $\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = 0$. There is no net flux into or out of this enclosed surface. The flux, $\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$, will be nonzero only if one or more lines start or end within the surface. Since electric field lines start and stop only on electric charges, the flux will be nonzero only if the surface encloses a net charge. For example, the surface labeled $A_1$ in Fig. 22–5 encloses a positive charge and there is a net outward flux through this surface ($\Phi_E > 0$). The surface $A_2$ encloses an equal magnitude negative charge and there is a net inward flux ($\Phi_E < 0$). For the configuration shown in Fig. 22–6, the flux through the surface shown is negative (count the lines). The value of $\Phi_E$ depends on the charge enclosed by the surface, and this is what Gauss's law is all about.

FIGURE 22–5 An electric dipole. Flux through surface $A_1$ is positive. Flux through $A_2$ is negative.

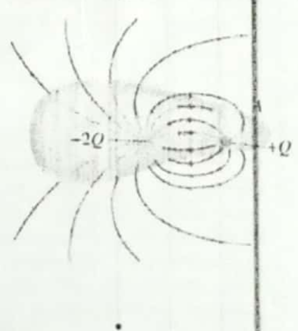



## 22–2 Gauss's Law

The precise relation between the electric flux through a closed surface and the net charge $Q_{encl}$ enclosed within that surface is given by **Gauss's law**:

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{encl}}{\epsilon_0}, \tag{22–4}$$

FIGURE 22–6 Net flux through surface $A$ is negative.




where $\epsilon_0$ is the same constant (permittivity of free space) that appears in Coulomb's law. The integral on the left is over the value of $\vec{\mathbf{E}}$ on any closed surface, and we choose that surface for our convenience in any given situation. The charge $Q_{encl}$ is the net charge *enclosed* by that surface. It doesn't matter where or how the charge is distributed within the surface. Any charge outside this surface must not be included. A charge outside the chosen surface may affect the position of the electric field lines, but will not affect the net number of lines entering or leaving the surface. For example, $Q_{encl}$ for the gaussian surface $A_1$ in Fig. 22–5 would be the positive charge enclosed by $A_1$; the negative charge does contribute to the electric field at $A_1$ but it is *not* enclosed by surface $A_1$ and so is not included in $Q_{encl}$.

Now let us see how Gauss's law is related to Coulomb's law. First, we show that Coulomb's law follows from Gauss's law. In Fig. 22–7 we have a single isolated charge $Q$. For our "gaussian surface," we choose an imaginary sphere of radius $r$ centered on the charge. Because Gauss's law is supposed to be valid for any surface, we have chosen one that will make our calculation easy. Because of the symmetry of this (imaginary) sphere about the charge at its center, we know that $\vec{\mathbf{E}}$ must have the same magnitude at any point on the surface, and that $\vec{\mathbf{E}}$ points radially outward (inward for a negative charge) parallel to $d\vec{\mathbf{A}}$, an element of the surface area. Hence, we write the integral in Gauss's law as

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint E\, dA = E\oint dA = E(4\pi r^2)$$

since the surface area of a sphere of radius $r$ is $4\pi r^2$, and the magnitude of $\vec{\mathbf{E}}$ is the same at all points on this spherical surface. Then Gauss's law becomes, with $Q_{encl} = Q$,

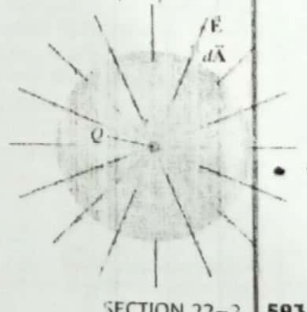
$$\frac{Q}{\epsilon_0} = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E(4\pi r^2)$$

because $\vec{\mathbf{E}}$ and $d\vec{\mathbf{A}}$ are both perpendicular to the surface at each point, and $\cos\theta = 1$. Solving for $E$ we obtain

$$E = \frac{Q}{4\pi\epsilon_0 r^2},$$

which is the electric field form of Coulomb's law, Eq. 21–4b.

FIGURE 22–7 A single point charge $Q$ at the center of an imaginary sphere of radius $r$ (our "gaussian surface"—that is, the closed surface we choose to use for applying Gauss's law in this case).



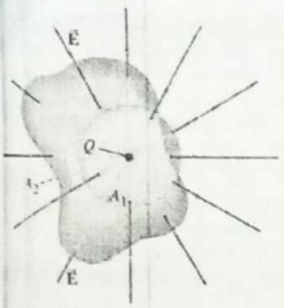



FIGURE 22–8 A single point charge surrounded by a spherical surface, $A_1$, and an irregular surface, $A_2$.

Now let us do the reverse, and derive Gauss's law from Coulomb's law for static electric charges[1]. First we consider a single point charge $Q$ surrounded by an imaginary spherical surface as in Fig. 22–7(and shown again, green, in Fig. 22–8). Coulomb's law tells us that the electric field at the spherical surface is $E = (1/4\pi\epsilon_0)(Q/r^2)$. Reversing the argument we just used, we have

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\,dA = \frac{Q}{4\pi\epsilon_0 r^2}(4\pi r^2) = \frac{Q}{\epsilon_0}.$$

This is Gauss's law, with $Q_{\text{encl}} = Q$, and we derived it for the special case of a spherical surface enclosing a point charge at its center. But what about some other surface, such as the irregular surface labeled $A_2$ in Fig. 22–8? The same number of field lines (due to our charge $Q$) pass through surface $A_2$, as pass through the spherical surface, $A_1$. Therefore, because the flux through a surface is proportional to the number of lines through it as we saw in Section 22–1, the flux through $A_2$ is the same as through $A_1$,

$$\oint_{A_2} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint_{A_1} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q}{\epsilon_0}.$$

Hence, we can expect that

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q}{\epsilon_0}$$

would be valid for *any* surface surrounding a single point charge $Q$.

Finally, let us look at the case of more than one charge. For each charge, $Q_i$, enclosed by the chosen surface,

$$\oint \vec{\mathbf{E}}_i \cdot d\vec{\mathbf{A}} = \frac{Q_i}{\epsilon_0},$$

where $\vec{\mathbf{E}}_i$ refers to the electric field produced by $Q_i$ alone. By the superposition principle for electric fields (Section 21–6), the total field $\vec{\mathbf{E}}$ is equal to the sum of the fields due to each separate charge, $\vec{\mathbf{E}} = \Sigma\vec{\mathbf{E}}_i$. Hence

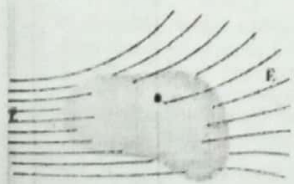
$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint (\Sigma\vec{\mathbf{E}}_i) \cdot d\vec{\mathbf{A}} = \Sigma\frac{Q_i}{\epsilon_0} = \frac{Q_{\text{encl}}}{\epsilon_0},$$

where $Q_{\text{encl}} = \Sigma Q_i$ is the total net charge enclosed within the surface. Thus we see, based on this simple argument, that Gauss's law follows from Coulomb's law for any distribution of static electric charge enclosed within a closed surface of any shape.

The derivation of Gauss's law from Coulomb's law is valid for electric fields produced by static electric charges. We will see later that electric fields can also be produced by changing magnetic fields. Coulomb's law cannot be used to describe such electric fields. But Gauss's law *is* found to hold also for electric fields produced in any of these ways. Hence *Gauss's law is a more general law than Coulomb's law*. It holds for any electric field whatsoever.

Even for the case of static electric fields that we are considering in this Chapter, it is important to recognize that $\vec{\mathbf{E}}$ on the left side of Gauss's law is not necessarily due only to the charge $Q_{\text{encl}}$ that appears on the right. For example, in Fig. 22–9 there is an electric field $\vec{\mathbf{E}}$ at all points on the imaginary gaussian surface, but it is not due to the charge enclosed by the surface (which is $Q_{\text{encl}} = 0$ in this case). The electric field $\vec{\mathbf{E}}$ which appears on the left side of Gauss's law is the *total* electric field at each point, on the gaussian surface chosen, not just that due to the charge $Q_{\text{encl}}$, which appears on the right side. Gauss's law has been found to be valid for the total field at any surface. It tells us that any *difference* between the input and output flux of the electric field over any surface is due to charge within that surface.

FIGURE 22–9 Electric flux through a closed surface. (Same as Fig. 22–3.) No electric charge is enclosed by this surface ($Q_{\text{encl}} = 0$).




[1]Note that Gauss's law would look more complicated in terms of the constant $k = 1/4\pi\epsilon_0$ that we originally used in Coulomb's law (Eqs. 21–1 or 21–4a):

| Coulomb's law | Gauss's law |
|---|---|
| $E = k\dfrac{Q}{r^2}$ | $\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = 4\pi kQ$ |
| $E = \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r^2}$ | $\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \dfrac{Q}{\epsilon_0}$ |

Gauss's law has a simpler form using $\epsilon_0$; Coulomb's law is simpler using $k$. The normal convention is to use $\epsilon_0$ rather than $k$ because Gauss's law is considered more general and therefore it is preferable to have it in simpler form.

CONCEPTUAL EXAMPLE 22–2 **Flux from Gauss's law.** Consider the two gaussian surfaces, $A_1$ and $A_2$, shown in Fig. 22–10. The only charge present is the charge $Q$ at the center of surface $A_1$. What is the net flux through each surface, $A_1$ and $A_2$?

**RESPONSE** The surface $A_1$ encloses the charge $+Q$. By Gauss's law, the net flux through $A_1$ is then $Q/\epsilon_0$. For surface $A_2$, the charge $+Q$ is outside the surface. Surface $A_2$ encloses zero net charge, so the net electric flux through $A_2$ is zero, by Gauss's law. Note that all field lines that enter the volume enclosed by surface $A_2$ also leave it.

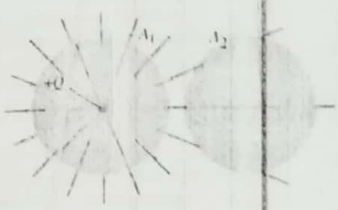



FIGURE 22–10 Example 22–2. Two gaussian surfaces.

**EXERCISE B** A point charge $Q$ is at the center of a spherical gaussian surface $A$. When a second charge $Q$ is placed just outside $A$, the total flux through this spherical surface $A$ is (*a*) unchanged, (*b*) doubled, (*c*) halved, (*d*) none of these.

**EXERCISE C** Three 2.95 $\mu$C charges are in a small box. What is the net flux leaving the box? (*a*) $3.3 \times 10^{12}\ \mathrm{N \cdot m^2/C}$, (*b*) $3.3 \times 10^{5}\ \mathrm{N \cdot m^2/C}$, (*c*) $1.0 \times 10^{12}\ \mathrm{N \cdot m^2/C}$, (*d*) $1.0 \times 10^{6}\ \mathrm{N \cdot m^2/C}$, (*e*) $6.7 \times 10^{6}\ \mathrm{N \cdot m^2/C}$.

We note that the integral in Gauss's law is often rather difficult to carry out in practice. We rarely need to do it except for some fairly simple situations that we now discuss.

## 22–3 Applications of Gauss's Law

Gauss's law is a very compact and elegant way to write the relation between electric charge and electric field. It also offers a simple way to determine the electric field when the charge distribution is simple and/or possesses a high degree of symmetry. In order to apply Gauss's law, however, we must choose the "gaussian" surface very carefully (for the integral on the left side of Gauss's law) so we can determine $\vec{\mathbf{E}}$. We normally try to think of a surface that has just the symmetry needed so that $E$ will be constant on all or on parts of its surface. Sometimes we choose a surface so the flux through part of the surface is zero.

EXAMPLE 22–3 **Spherical conductor.** A thin spherical shell of radius $r_0$ possesses a total net charge $Q$ that is uniformly distributed on it (Fig. 22–11). Determine the electric field at points (*a*) outside the shell, and (*b*) inside the shell. (*c*) What if the conductor were a solid sphere?

**APPROACH** Because the charge is distributed symmetrically, the electric field must also be symmetric. Thus the field outside the sphere must be directed radially outward (inward if $Q < 0$) and must depend only on $r$, not on angle (spherical coordinates).

**SOLUTION** (*a*) The electric field will have the same magnitude at all points on an imaginary gaussian surface, if we choose that surface as a sphere of radius $r$ ($r > r_0$) concentric with the shell, and shown in Fig. 22–11 as the dashed circle $A_1$. Because $\vec{\mathbf{E}}$ is perpendicular to this surface, the cosine of the angle between $\vec{\mathbf{E}}$ and $d\vec{\mathbf{A}}$ is always 1. Gauss's law then gives (with $Q_{\mathrm{encl}} = Q$ in Eq. 22–4)

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E(4\pi r^2) = \frac{Q}{\epsilon_0},$$

where $4\pi r^2$ is the surface area of our sphere (gaussian surface) of radius $r$. Thus

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}. \qquad [r > r_0]$$

Thus the field outside a uniformly charged spherical shell is the same as if all the charge were concentrated at the center as a point charge.

(*b*) Inside the shell, the electric field must also be symmetric. So $E$ must again have the same value at all points on a spherical gaussian surface ($A_2$ in Fig. 22–11) concentric with the shell. Thus $E$ can be factored out of the integral and, with $Q_{\mathrm{encl}} = 0$ because the charge enclosed within the sphere $A_2$ is zero, we have

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E(4\pi r^2) = 0.$$

Hence

$$E = 0 \qquad [r < r_0]$$

inside a uniform spherical shell of charge.

(*c*) These same results also apply to a uniformly charged solid spherical conductor, since all the charge would lie in a thin layer at the surface.

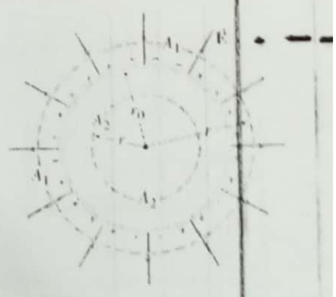



FIGURE 22–11 Cross-sectional drawing of a thin spherical shell of radius $r_0$, carrying a net charge $Q$ uniformly distributed. $A_1$ and $A_2$ represent two gaussian surfaces we use to determine $\vec{\mathbf{E}}$. Example 22–3.

**EXERCISE D** A charge $Q$ is placed on a hollow metal ball. We saw in Chapter 21 that the charge is all on the surface of the ball because metal is a conductor. How does the charge distribute itself on the ball? (*a*) Half on the inside surface and half on the outside surface. (*b*) Part on each surface in inverse proportion to the two radii. (*c*) Part on each surface but with a more complicated dependence on the radii than in answer (*b*). (*d*) All on the inside surface. (*e*) All on the outside surface.

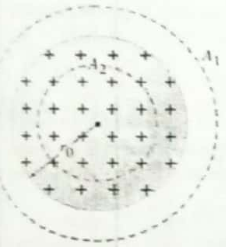



FIGURE 22–12 A solid sphere of uniform charge density. Example 22–4.

**EXAMPLE 22–4 Solid sphere of charge.** An electric charge $Q$ is distributed uniformly throughout a nonconducting sphere of radius $r_0$, Fig. 22–12. Determine the electric field (*a*) outside the sphere ($r > r_0$) and (*b*) inside the sphere ($r < r_0$).

**APPROACH** Since the charge is distributed symmetrically in the sphere, the electric field at all points must again be symmetric. $\vec{\mathbf{E}}$ depends only on $r$ and is directed radially outward (or inward if $Q < 0$).

**SOLUTION** (*a*) For our gaussian surface we choose a sphere of radius $r$ ($r > r_0$), labeled $A_1$ in Fig. 22–12. Since $E$ depends only on $r$, Gauss's law gives, with $Q_{\text{encl}} = Q$,

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E(4\pi r^2) = \frac{Q}{\epsilon_0}$$

or

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}.$$

Again, the field outside a spherically symmetric distribution of charge is the same as that for a point charge of the same magnitude located at the center of the sphere.

(*b*) Inside the sphere, we choose for our gaussian surface a concentric sphere of radius $r$ ($r < r_0$), labeled $A_2$ in Fig. 22–12. From symmetry, the magnitude of $\vec{\mathbf{E}}$ is the same at all points on $A_2$, and $\vec{\mathbf{E}}$ is perpendicular to the surface, so

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E(4\pi r^2).$$

We must equate this to $Q_{\text{encl}}/\epsilon_0$ where $Q_{\text{encl}}$ is the charge enclosed by $A_2$. $Q_{\text{encl}}$ is not the total charge $Q$ but only a portion of it. We define the **charge density**, $\rho_E$, as the charge per unit volume $(\rho_E = dQ/dV)$, and here we are given that $\rho_E$ = constant. So the charge enclosed by the gaussian surface $A_2$, a sphere of radius $r$, is

$$Q_{\text{encl}} = \left(\frac{\frac{4}{3}\pi r^3 \rho_E}{\frac{4}{3}\pi r_0^3 \rho_E}\right) Q = \frac{r^3}{r_0^3} Q.$$

Hence, from Gauss's law,

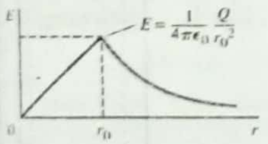
$$E(4\pi r^2) = \frac{Q_{\text{encl}}}{\epsilon_0} = \frac{r^3}{r_0^3}\frac{Q}{\epsilon_0}$$

or

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r_0^3}\, r. \qquad [r < r_0]$$

Thus the field increases linearly with $r$, until $r = r_0$. It then decreases as $1/r^2$, as plotted in Fig. 22–13.

FIGURE 22–13 Magnitude of the electric field as a function of the distance $r$ from the center of a uniformly charged solid sphere.




**EXERCISE E** Return to the Chapter-Opening Question, page 591, and answer it again now. Try to explain why you may have answered differently the first time.

The results in Example 22–4 would have been difficult to obtain from Coulomb's law by integrating over the sphere. Using Gauss's law and the symmetry of the situation, this result is obtained rather easily, and shows the great power of Gauss's law. However, its use in this way is limited mainly to cases where the charge distribution has a high degree of symmetry. In such cases, we *choose* a simple surface on which $E$ = constant, so the integration is simple. Gauss's law holds, of course, for any surface.

**EXAMPLE 22–5** **Nonuniformly charged solid sphere.** Suppose the charge density of the solid sphere in Fig. 22–12, Example 22–4, is given by $\rho_E = \alpha r^2$, where $\alpha$ is a constant. (*a*) Find $\alpha$ in terms of the total charge $Q$ on the sphere and its radius $r_0$. (*b*) Find the electric field as a function of $r$ inside the sphere.

**APPROACH** We divide the sphere up into concentric thin shells of thickness $dr$ as shown in Fig. 22–14, and integrate (*a*) setting $Q = \int \rho_E \, dV$ and (*b*) using Gauss's law.

**SOLUTION** (*a*) A thin shell of radius $r$ and thickness $dr$ (Fig. 22–14) has volume $dV = 4\pi r^2 \, dr$. The total charge is given by

$$Q = \int \rho_E \, dV = \int_0^{r_0} (\alpha r^2)(4\pi r^2 \, dr) = 4\pi\alpha \int_0^{r_0} r^4 \, dr = \frac{4\pi\alpha}{5} r_0^5.$$

Thus $\alpha = 5Q/4\pi r_0^5$.

(*b*) To find $E$ inside the sphere at distance $r$ from its center, we apply Gauss's law to an imaginary sphere of radius $r$ which will enclose a charge

$$Q_{\text{encl}} = \int_0^r \rho_E \, dV = \int_0^r (\alpha r^2) 4\pi r^2 \, dr = \int_0^r \left(\frac{5Q}{4\pi r_0^5} r^2\right) 4\pi r^2 \, dr = Q \frac{r^5}{r_0^5}.$$

By symmetry, $E$ will be the same at all points on the surface of a sphere of radius $r$, so Gauss's law gives

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{\text{encl}}}{\epsilon_0}$$

$$(E)(4\pi r^2) = Q \frac{r^5}{\epsilon_0 r_0^5},$$

so

$$E = \frac{Qr^3}{4\pi\epsilon_0 r_0^5}.$$

FIGURE 22–14 Example 22–5.

**EXAMPLE 22–6** **Long uniform line of charge.** A very long straight wire possesses a uniform positive charge per unit length, $\lambda$. Calculate the electric field at points near (but outside) the wire, far from the ends.

**APPROACH** Because of the symmetry, we expect the field to be directed radially outward and to depend only on the perpendicular distance, $R$, from the wire. Because of the cylindrical symmetry, the field will be the same at all points on a gaussian surface that is a cylinder with the wire along its axis, Fig. 22–15. $\vec{\mathbf{E}}$ is perpendicular to this surface at all points. For Gauss's law, we need a closed surface, so we include the flat ends of the cylinder. Since $\vec{\mathbf{E}}$ is parallel to the ends, there is no flux through the ends (the cosine of the angle between $\vec{\mathbf{E}}$ and $d\vec{\mathbf{A}}$ on the ends is $\cos 90° = 0$).

**SOLUTION** For our chosen gaussian surface Gauss's law gives

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E(2\pi R \ell) = \frac{Q_{\text{encl}}}{\epsilon_0} = \frac{\lambda \ell}{\epsilon_0},$$

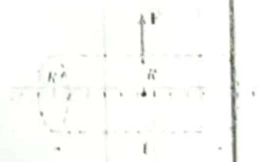
where $\ell$ is the length of our chosen gaussian surface ($\ell \ll$ length of wire), and $2\pi R$ is its circumference. Hence

$$E = \frac{1}{2\pi\epsilon_0} \frac{\lambda}{R}.$$

**NOTE** This is the same result we found in Example 21–11 using Coulomb's law (we used $x$ there instead of $R$), but here it took much less effort. Again we see the great power of Gauss's law.[†]

**NOTE** Recall from Chapter 10, Fig. 10–2, that we use $R$ for the distance of a particle from an axis (cylindrical symmetry), but lower case $r$ for the distance from a point (usually the origin 0).

FIGURE 22–15 Calculation of $\vec{\mathbf{E}}$ due to a very long line of charge. Example 22–6.



[†]But note that the method of Example 21–11 allows calculation of $E$ also for a short line of charge by using the appropriate limits for the integral, whereas Gauss's law is not readily adapted due to lack of symmetry.



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**EXAMPLE 22–7 Infinite plane of charge.** Charge is distributed uniformly, with a surface charge density $\sigma$ ($\sigma$ = charge per unit area = $dQ/dA$), over a very large but very thin nonconducting flat plane surface. Determine the electric field at points near the plane.

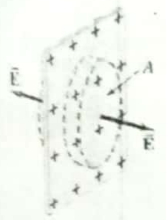


FIGURE 22–16 Calculation of the electric field outside a large uniformly charged nonconducting plane surface. Example 22–7.

**APPROACH** We choose as our gaussian surface a small closed cylinder whose axis is perpendicular to the plane and which extends through the plane as shown in Fig. 22–16. Because of the symmetry, we expect $\vec{\mathbf{E}}$ to be directed perpendicular to the plane on both sides as shown, and to be uniform over the end caps of the cylinder, each of whose area is $A$.

**SOLUTION** Since no flux passes through the curved sides of our chosen cylindrical surface, all the flux is through the two end caps. So Gauss's law gives

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = 2EA = \frac{Q_{\text{encl}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0},$$

where $Q_{\text{encl}} = \sigma A$ is the charge enclosed by our gaussian cylinder. The electric field is then

$$E = \frac{\sigma}{2\epsilon_0}.$$

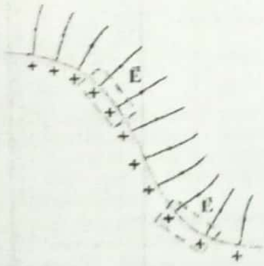
**NOTE** This is the same result we obtained much more laboriously in Chapter 21, Eq. 21–7. The field is uniform for points far from the ends of the plane, and close to its surface.

**EXAMPLE 22–8 Electric field near any conducting surface.** Show that the electric field just outside the surface of any good conductor of arbitrary shape is given by

$$E = \frac{\sigma}{\epsilon_0},$$

where $\sigma$ is the surface charge density on the conductor's surface at that point.

FIGURE 22–17 Electric field near surface of a conductor. Example 22–8.



**APPROACH** We choose as our gaussian surface a small cylindrical box, as we did in the previous Example. We choose the cylinder to be very small in height, so that one of its circular ends is just above the conductor (Fig. 22–17). The other end is just below the conductor's surface, and the sides are perpendicular to it.

**SOLUTION** The electric field is zero inside a conductor and is perpendicular to the surface just outside it (Section 21–9), so electric flux passes only through the outside end of our cylindrical box; no flux passes through the short sides or inside end. We choose the area $A$ (of the flat cylinder end) small enough so that $E$ is essentially uniform over it. Then Gauss's law gives

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = \frac{Q_{\text{encl}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0},$$

so that

$$E = \frac{\sigma}{\epsilon_0}. \qquad \text{[at surface of conductor]} \quad (22\text{–}5)$$

**NOTE** This useful result applies for a conductor of any shape.

**CAUTION**
When is $E = \sigma/\epsilon_0$ and when is $E = \sigma/2\epsilon_0$?

Why is it that the field outside a large plane nonconductor is $E = \sigma/2\epsilon_0$ (Example 22–7) whereas outside a conductor it is $E = \sigma/\epsilon_0$ (Example 22–8)? The reason for the factor of 2 comes not from conductor verses nonconductor. It comes instead from how we define charge per unit area $\sigma$. For a thin flat nonconductor, Fig. 22–16, the charge may be distributed throughout the volume (not only on the surface, as for a conductor). The charge per unit area $\sigma$ represents all the charge throughout the thickness of the thin nonconductor. Also our gaussian surface has its ends outside the nonconductor on each side, so as to include all this charge.

3. (II) A cube of side $\ell$ is placed in a uniform field $E_0$ with edges parallel to the field lines. (*a*) What is the net flux through the cube? (*b*) What is the flux through each of its six faces?

4. (II) A uniform field $\vec{\mathbf{E}}$ is parallel to the axis of a hollow hemisphere of radius $r$, Fig. 22–25. (*a*) What is the electric flux through the hemispherical surface? (*b*) What is the result if $\vec{\mathbf{E}}$ is instead perpendicular to the axis?


FIGURE 22–25 Problem 4.

## 22–2 Gauss's Law

5. (I) The total electric flux from a cubical box 28.0 cm on a side is $1.84 \times 10^3\ \mathrm{N \cdot m^2/C}$. What charge is enclosed by the box?

6. (I) Figure 22–26 shows five closed surfaces that surround various charges in a plane, as indicated. Determine the electric flux through each surface, $S_1$, $S_2$, $S_3$, $S_4$, and $S_5$. The surfaces are flat "pillbox" surfaces that extend only slightly above and below the plane in which the charges lie.

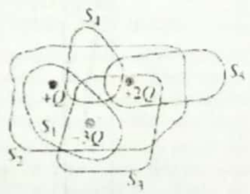



FIGURE 22–26 Problem 6.

7. (II) In Fig. 22–27, two objects, $O_1$ and $O_2$, have charges $+1.0\ \mu C$ and $-2.0\ \mu C$ respectively, and a third object, $O_3$, is electrically neutral. (*a*) What is the electric flux through the surface $A_1$ that encloses all the three objects? (*b*) What is the electric flux through the surface $A_2$ that encloses the third object only?

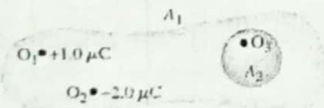



FIGURE 22–27 Problem 7.

8. (II) A ring of charge with uniform charge density is completely enclosed in a hollow donut shape. An exact copy of the ring is completely enclosed in a hollow sphere. What is the ratio of the flux out of the donut shape to that out of the sphere?

9. (II) In a certain region of space, the electric field is constant in direction (say horizontal, in the $x$ direction), but its magnitude decreases from $E = 560\ \mathrm{N/C}$ at $x = 0$ to $E = 410\ \mathrm{N/C}$ at $x = 25\ \mathrm{m}$. Determine the charge within a cubical box of side $\ell = 25\ \mathrm{m}$, where the box is oriented so that four of its sides are parallel to the field lines (Fig. 22–28).

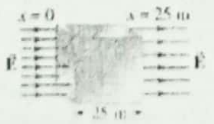



FIGURE 22–28 Problem 9.

10. (II) A point charge $Q$ is placed at the center of a cube of side $\ell$. What is the flux through one face of the cube?

11. (II) A 15.0-cm-long uniformly charged plastic rod is sealed inside a plastic bag. The total electric flux leaving the bag is $7.3 \times 10^5\ \mathrm{N \cdot m^2/C}$. What is the linear charge density on the rod?

## 22–3 Applications of Gauss's Law

12. (I) Draw the electric field lines around a negatively charged metal egg.

13. (I) The field just outside a 3.50-cm-radius metal ball is $6.25 \times 10^2\ \mathrm{N/C}$ and points toward the ball. What charge resides on the ball?

14. (I) Starting from the result of Example 22–3, show that the electric field just outside a uniformly charged spherical conductor is $E = \sigma/\epsilon_0$, consistent with Example 22–8.

15. (I) A long thin wire, hundreds of meters long, carries a uniformly distributed charge of $-7.2\ \mu C$ per meter of length. Estimate the magnitude and direction of the electric field at points (*a*) 5.0 m and (*b*) 1.5 m perpendicular from the center of the wire.

16. (I) A metal globe has 1.50 mC of charge put on it at the north pole. Then $-3.00$ mC of charge is applied to the south pole. Draw the field lines for this system after it has come to equilibrium.

17. (II) A nonconducting sphere is made of two layers. The innermost section has a radius of 6.0 cm and a uniform charge density of $-5.0\ \mathrm{C/m^3}$. The outer layer has a uniform charge density of $+8.0\ \mathrm{C/m^3}$ and extends from an inner radius of 6.0 cm to an outer radius of 12.0 cm. Determine the electric field for (*a*) $0 < r < 6.0$ cm, (*b*) $6.0\ \mathrm{cm} < r < 12.0$ cm, and (*c*) $12.0\ \mathrm{cm} < r < 50.0$ cm. (*d*) Plot the magnitude of the electric field for $0 < r < 50.0$ cm. Is the field continuous at the edges of the layers?

18. (II) A solid metal sphere of radius 3.00 m carries a total charge of $-5.50\ \mu C$. What is the magnitude of the electric field at a distance from the sphere's center of (*a*) 0.250 m, (*b*) 2.90 m, (*c*) 3.10 m, and (*d*) 8.00 m? How would the answers differ if the sphere were (*e*) a thin shell, or (*f*) a solid nonconductor uniformly charged throughout?

19. (II) A 15.0-cm-diameter nonconducting sphere carries a total charge of $2.25\ \mu C$ distributed uniformly throughout its volume. Graph the electric field $E$ as a function of the distance $r$ from the center of the sphere from $r = 0$ to $r = 30.0$ cm.

20. (II) A flat square sheet of thin aluminum foil, 25 cm on a side, carries a uniformly distributed 275 nC charge. What, approximately, is the electric field (*a*) 1.0 cm above the center of the sheet and (*b*) 15 m above the center of the sheet?

21. (II) A spherical cavity of radius 4.50 cm is at the center of a metal sphere of radius 18.0 cm. A point charge $Q = 5.50\ \mu C$ rests at the very center of the cavity, whereas the metal conductor carries no net charge. Determine the electric field at a point (*a*) 3.00 cm from the center of the cavity, (*b*) 6.00 cm from the center of the cavity, (*c*) 30.0 cm from the center.

22. (II) A point charge $Q$ rests at the center of an uncharged thin spherical conducting shell. What is the electric field $E$ as a function of $r$ (*a*) for $r$ less than the radius of the shell, (*b*) inside the shell, and (*c*) beyond the shell? (*d*) Does the shell affect the field due to $Q$ alone? Does the charge $Q$ affect the shell?

23. (II) A solid metal cube has a spherical cavity at its center as shown in Fig. 22–29. At the center of the cavity there is a point charge $Q = +8.00\,\mu C$. The metal cube carries a net charge $q = -6.10\,\mu C$ (not including $Q$). Determine (*a*) the total charge on the surface of the spherical cavity and (*b*) the total charge on the outer surface of the cube.

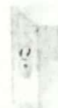


FIGURE 22–29 Problem 23.

24. (II) Two large, flat metal plates are separated by a distance that is very small compared to their height and width. The conductors are given equal but opposite uniform surface charge densities $\pm\sigma$. Ignore edge effects and use Gauss's law to show (*a*) that for points far from the edges, the electric field between the plates is $E = \sigma/\epsilon_0$ and (*b*) that outside the plates on either side the field is zero. (*c*) How would your results be altered if the two plates were nonconductors? (See Fig. 22–30).

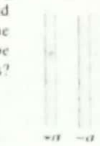


FIGURE 22–30 Problems 24, 25, and 26.

25. (II) Suppose the two conducting plates in Problem 24 have the *same* sign and magnitude of charge. What then will be the electric field (*a*) between them and (*b*) outside them on either side? (*c*) What if the plates are nonconducting?

26. (II) The electric field between two square metal plates is 160 N/C. The plates are 1.0 m on a side and are separated by 3.0 cm, as in Fig. 22–30. What is the charge on each plate? Neglect edge effects.

27. (II) Two thin concentric spherical shells of radii $r_1$ and $r_2$ ($r_1 < r_2$) contain uniform surface charge densities $\sigma_1$ and $\sigma_2$, respectively (see Fig. 22–31). Determine the electric field for (*a*) $0 < r < r_1$, (*b*) $r_1 < r < r_2$, and (*c*) $r > r_2$. (*d*) Under what conditions will $E = 0$ for $r > r_2$? (*e*) Under what conditions will $E = 0$ for $r_1 < r < r_2$? Neglect the thickness of the shells.

FIGURE 22–31 Two spherical shells (Problem 27).

28. (II) A spherical rubber balloon carries a total charge $Q$ uniformly distributed on its surface. At $t = 0$ the nonconducting balloon has radius $r_0$ and the balloon is then slowly blown up so that $r$ increases linearly to $2r_0$ in a time $t$. Determine the electric field as a function of time (*a*) just outside the balloon surface and (*b*) at $r = 3.2r_0$.

29. (II) Suppose the nonconducting sphere of Example 22–4 has a spherical cavity of radius $r_1$ centered at the sphere's center (Fig. 22–32). Assuming the charge $Q$ is distributed uniformly in the "shell" (between $r = r_1$ and $r = r_0$), determine the electric field as a function of $r$ for (*a*) $0 < r < r_1$, (*b*) $r_1 < r < r_0$, and (*c*) $r > r_0$.

FIGURE 22–32 Problems 29, 30, 31, and 44.

30. (II) Suppose in Fig. 22–32, Problem 29, there is also a charge $q$ at the center of the cavity. Determine the electric field for (*a*) $0 < r < r_1$, (*b*) $r_1 < r < r_0$, and (*c*) $r > r_0$.

31. (II) Suppose the thick spherical shell of Problem 29 is a conductor. It carries a total net charge $Q$ and at its center there is a point charge $q$. What total charge is found on (*a*) the inner surface of the shell and (*b*) the outer surface of the shell? Determine the electric field for (*c*) $0 < r < r_1$, (*d*) $r_1 < r < r_0$, and (*e*) $r > r_0$.

32. (II) Suppose that at the center of the cavity inside the shell (charge $Q$) of Fig. 22–11 (and Example 22–3), there is a point charge $q$ ($\neq \pm Q$). Determine the electric field for (*a*) $0 < r < r_0$, and for (*b*) $r > r_0$. What are your answers if (*c*) $q = Q$ and (*d*) $q = -Q$?

33. (II) A long cylindrical shell of radius $R_0$ and length $\ell$ ($R_0 \ll \ell$) possesses a uniform surface charge density (charge per unit area) $\sigma$ (Fig. 22–33). Determine the electric field at points (*a*) outside the cylinder ($R > R_0$) and (*b*) inside the cylinder ($0 < R < R_0$); assume the points are far from the ends and not too far from the shell ($R \ll \ell$). (*c*) Compare to the result for a long line of charge, Example 22–6. Neglect the thickness of shell.

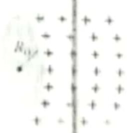


FIGURE 22–33 Problem 33.

34. (II) A very long solid nonconducting cylinder of radius $R_0$ and length $\ell$ ($R_0 \ll \ell$) possesses a uniform volume charge density $\rho_E$ (C/m³), Fig. 22–34. Determine the electric field at points (*a*) outside the cylinder ($R > R_0$) and (*b*) inside the cylinder ($R < R_0$). Do only for points far from the ends and for which $R \ll \ell$.

FIGURE 22–34 Problem 34

35. (II) A thin cylindrical shell of radius $R_1$ is surrounded by a second concentric cylindrical shell of radius $R_2$ (Fig. 22–35). The inner shell has a total charge $+Q$ and the outer shell $-Q$. Assuming the length $\ell$ of the shells is much greater than $R_1$ or $R_2$, determine the electric field as a function of $R$ (the perpendicular distance from the common axis of the cylinders) for (*a*) $0 < R < R_1$, (*b*) $R_1 < R < R_2$, and (*c*) $R > R_2$. (*d*) What is the kinetic energy of an electron if it moves between (and concentric with) the shells in a circular orbit of radius $(R_1 + R_2)/2$? Neglect thickness of shells.

FIGURE 22–35 Problems 35, 36, and 37.

36. (II) A thin cylindrical shell of radius $R_1 = 6.5$ cm is surrounded by a second cylindrical shell of radius $R_2 = 9.0$ cm, as in Fig. 22–35. Both cylinders are 5.0 m long and the inner one carries a total charge $Q_1 = -0.88\,\mu C$ and the outer one $Q_2 = +1.56\,\mu C$. For points far from the ends of the cylinders, determine the electric field at a radial distance $r$ from the central axis of (*a*) 3.0 cm, (*b*) 7.0 cm, and (*c*) 12.0 cm.

37. (II) (*a*) If an electron ($m = 9.1 \times 10^{-31}$ kg) escaped from the surface of the inner cylinder in Problem 36 (Fig. 22–35) with negligible speed, what would be its speed when it reached the outer cylinder? (*b*) If a proton ($m = 1.67 \times 10^{-27}$ kg) revolves in a circular orbit of radius $r = 7.0$ cm about the axis (i.e., between the cylinders), what must be its speed?

38. (II) A very long solid nonconducting cylinder of radius $R_1$ is uniformly charged with a charge density $\rho_E$. It is surrounded by a concentric cylindrical tube of inner radius $R_2$ and outer radius $R_3$ as shown in Fig. 22–36, and it too carries a uniform charge density $\rho_E$. Determine the electric field as a function of the distance $R$ from the center of the cylinders for (*a*) $0 < R < R_1$, (*b*) $R_1 < R < R_2$, (*c*) $R_2 < R < R_3$, and (*d*) $R > R_3$. (*e*) If $\rho_E = 15\,\mu\text{C/m}^3$ and $R_1 = \frac{1}{2}R_2 = \frac{1}{3}R_3 = 5.0\,\text{cm}$, plot $E$ as a function of $R$ from $R = 0$ to $R = 20.0\,\text{cm}$. Assume the cylinders are very long compared to $R_3$.

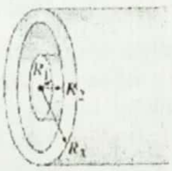



FIGURE 22–36 Problem 38.

39. (II) A nonconducting sphere of radius $r_0$ is uniformly charged with volume charge density $\rho_E$. It is surrounded by a concentric metal (conducting) spherical shell of inner radius $r_1$ and outer radius $r_2$, which carries a net charge $+Q$. Determine the resulting electric field in the regions (*a*) $0 < r < r_0$, (*b*) $r_0 < r < r_1$, (*c*) $r_1 < r < r_2$, and (*d*) $r > r_2$ where the radial distance $r$ is measured from the center of the nonconducting sphere.

40. (II) A very long solid nonconducting cylinder of radius $R_1$ is uniformly charged with charge density $\rho_E$. It is surrounded by a cylindrical metal (conducting) tube of inner radius $R_2$ and outer radius $R_3$, which has no net charge (cross-sectional view shown in Fig. 22–37). If the axes of the two cylinders are parallel, but displaced from each other by a distance $d$, determine the resulting electric field in the region $R > R_3$, where the radial distance $R$ is measured from the metal cylinder's axis. Assume $d < (R_2 - R_1)$.

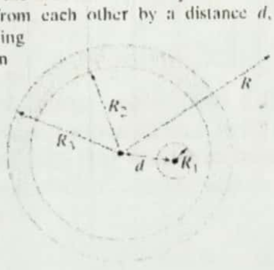



FIGURE 22–37 Problem 40.

41. (II) A flat ring (inner radius $R_0$, outer radius $4R_0$) is uniformly charged. In terms of the total charge $Q$, determine the electric field on the axis at points (*a*) $0.25R_0$ and (*b*) $75R_0$ from the center of the ring. [*Hint*: The ring can be replaced with two oppositely charged superposed disks.]

42. (II) An uncharged solid conducting sphere of radius $r_0$ contains two spherical cavities of radii $r_1$ and $r_2$, respectively. Point charge $Q_1$ is then placed within the cavity of radius $r_1$ and point charge $Q_2$ is placed within the cavity of radius $r_2$ (Fig. 22–38). Determine the resulting electric field (magnitude and direction) at locations outside the solid sphere ($r > r_0$), where $r$ is the distance from its center.

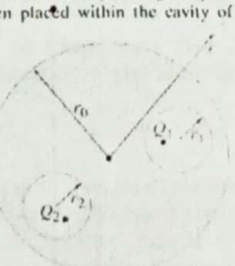



FIGURE 22–38 Problem 42.

43. (III) A very large (i.e., assume infinite) flat slab of nonconducting material has thickness $d$ and a uniform volume charge density $+\rho_E$. (*a*) Show that a uniform electric field exists outside of this slab. Determine its magnitude $E$ and its direction (relative to the slab's surface). (*b*) As shown in Fig. 22–39, the slab is now aligned so that one of its surfaces lies on the line $y = x$. At time $t = 0$, a pointlike particle (mass $m$, charge $+q$) is located at position $\vec{r} = +y_0\hat{\mathbf{j}}$ and has velocity $\vec{v} = v_0\hat{\mathbf{i}}$. Show that the particle will collide with the slab if $v_0 \geq \sqrt{\sqrt{2}qy_0\rho_E d/\epsilon_0 m}$. Ignore gravity.

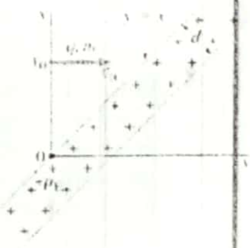



FIGURE 22–39 Problem 43.

44. (III) Suppose the density of charge between $r_1$ and $r_0$ of the hollow sphere of Problem 29 (Fig. 22–32) varies as $\rho_E = \rho_0 r_1/r$. Determine the electric field as a function of $r$ for (*a*) $0 < r < r_1$, (*b*) $r_1 < r < r_0$, and (*c*) $r > r_0$. (*d*) Plot $E$ versus $r$ from $r = 0$ to $r = 2r_0$.

45. (III) Suppose two thin flat plates measure $1.0\,\text{m} \times 1.0\,\text{m}$ and are separated by 5.0 mm. They are oppositely charged with $\pm 15\,\mu\text{C}$. (*a*) Estimate the total force exerted by one plate on the other (ignore edge effects). (*b*) How much work would be required to move the plates from 5.0 mm apart to 1.00 cm apart?

46. (III) A flat slab of nonconducting material (Fig. 22–40) carries a uniform charge per unit volume, $\rho_E$. The slab has thickness $d$ which is small compared to the height and breadth of the slab. Determine the electric field as a function of $x$ (*a*) inside the slab and (*b*) outside the slab (at distances much less than the slab's height or breadth). Take the origin at the center of the slab.


FIGURE 22–40 Problem 46.

47. (III) A flat slab of nonconducting material has thickness $2d$, which is small compared to its height and breadth. Define the $x$ axis to be along the direction of the slab's thickness with the origin at the center of the slab (Fig. 22–41). If the slab carries a volume charge density $\rho_E(x) = -\rho_0$ in the region $-d \leq x < 0$ and $\rho_E(x) = +\rho_0$ in the region $0 < x \leq +d$, determine the electric field $\vec{\mathbf{E}}$ as a function of $x$ in the regions (*a*) outside the slab, (*b*) $0 < x \leq +d$, and (*c*) $-d \leq x < 0$. Let $\rho_0$ be a positive constant.

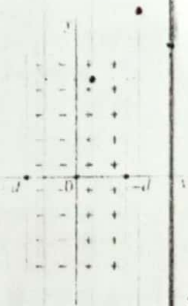


FIGURE 22–41 Problem 47.

## 25-3 Calculating the Capacitance

Our goal here is to calculate the capacitance of a capacitor once we know its geometry. Because we shall consider a number of different geometrics, it seems wise to develop a general plan to simplify the work. In brief our plan is as follows: (1) Assume a charge $q$ on the plates; (2) calculate the electric field $\vec{E}$ between the plates in terms of this charge, using Gauss' law; (3) knowing $\vec{E}$, calculate the potential difference $V$ between the plates from Eq. 24-18; (4) calculate $C$ from Eq. 25-1.

Before we start, we can simplify the calculation of both the electric field and the potential difference by making certain assumptions. We discuss each in turn.

We use Gauss' law to relate $q$ and $E$. Then we integrate the $E$ to get the potential difference.

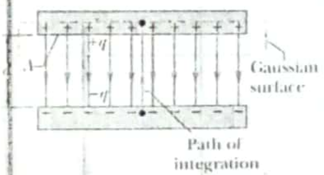



**Fig. 25-5** A charged parallel-plate capacitor. A Gaussian surface encloses the charge on the positive plate. The integration of Eq. 25-6 is taken along a path extending directly from the negative plate to the positive plate.

### Calculating the Electric Field

To relate the electric field $\vec{E}$ between the plates of a capacitor to the charge $q$ on either plate, we shall use Gauss' law:

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = q. \tag{25-3}$$

Here $q$ is the charge enclosed by a Gaussian surface and $\oint \vec{E} \cdot d\vec{A}$ is the net electric flux through that surface. In all cases that we shall consider, the Gaussian surface will be such that whenever there is an electric flux through it, $\vec{E}$ will have a uniform magnitude $E$ and the vectors $\vec{E}$ and $d\vec{A}$ will be parallel. Equation 25-3 then reduces to

$$q = \varepsilon_0 EA \quad \text{(special case of Eq. 25-3).} \tag{25-4}$$

in which $A$ is the area of that part of the Gaussian surface through which there is a flux. For convenience, we shall always draw the Gaussian surface in such a way that it completely encloses the charge on the positive plate; see Fig. 25-5 for an example.

### Calculating the Potential Difference

In the notation of Chapter 24 (Eq. 24-18), the potential difference between the plates of a capacitor is related to the field $\vec{E}$ by

$$V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{s}, \tag{25-5}$$

in which the integral is to be evaluated along any path that starts on one plate and ends on the other. We shall always choose a path that follows an electric field line, from the negative plate to the positive plate. For this path, the vectors $\vec{E}$ and $d\vec{s}$ will have opposite directions; so the dot product $\vec{E} \cdot d\vec{s}$ will be equal to $-E\,ds$. Thus, the right side of Eq. 25-5 will then be positive. Letting $V$ represent the difference $V_f - V_i$, we can then recast Eq. 25-5 as

$$V = \int_-^+ E\,ds \quad \text{(special case of Eq. 25-5).} \tag{25-6}$$

in which the $-$ and $+$ remind us that our path of integration starts on the negative plate and ends on the positive plate.

We are now ready to apply Eqs. 25-4 and 25-6 to some particular cases.

### A Parallel-Plate Capacitor

We assume, as Fig. 25-5 suggests, that the plates of our parallel-plate capacitor are so large and so close together that we can neglect the fringing of the electric field

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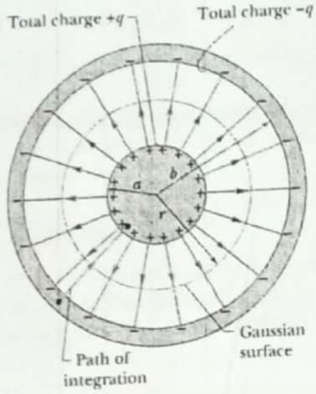


**Fig. 25-6** A cross section of a long cylindrical capacitor, showing a cylindrical Gaussian surface of radius $r$ (that encloses the positive plate) and the radial path of integration along which Eq. 25-6 is to be applied. This figure also serves to illustrate a spherical capacitor in a cross section through its center.

at the edges of the plates, taking $\vec{E}$ to be constant throughout the region between the plates.

We draw a Gaussian surface that encloses just the charge $q$ on the positive plate, as in Fig. 25-5. From Eq. 25-4 we can then write

$$q = \varepsilon_0 EA, \tag{25-7}$$

where $A$ is the area of the plate.

Equation 25-6 yields

$$V = \int_-^+ E\,ds = E\int_0^d ds = Ed. \tag{25-8}$$

In Eq. 25-8, $E$ can be placed outside the integral because it is a constant; the second integral then is simply the plate separation $d$.

If we now substitute $q$ from Eq. 25-7 and $V$ from Eq. 25-8 into the relation $q = CV$ (Eq. 25-1), we find

$$C = \frac{\varepsilon_0 A}{d} \quad \text{(parallel-plate capacitor)}. \tag{25-9}$$

Thus, the capacitance does indeed depend only on geometrical factors—namely, the plate area $A$ and the plate separation $d$. Note that $C$ increases as we increase area $A$ or decrease separation $d$.

As an aside, we point out that Eq. 25-9 suggests one of our reasons for writing the electrostatic constant in Coulomb's law in the form $1/4\pi\varepsilon_0$. If we had not done so, Eq. 25-9—which is used more often in engineering practice than Coulomb's law—would have been less simple in form. We note further that Eq. 25-9 permits us to express the permittivity constant $\varepsilon_0$ in a unit more appropriate for use in problems involving capacitors; namely,

$$\varepsilon_0 = 8.85 \times 10^{-12}\ \text{F/m} = 8.85\ \text{pF/m}. \tag{25-10}$$

We have previously expressed this constant as

$$\varepsilon_0 = 8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2. \tag{25-11}$$

## A Cylindrical Capacitor

Figure 25-6 shows, in cross section, a cylindrical capacitor of length $L$ formed by two coaxial cylinders of radii $a$ and $b$. We assume that $L \gg b$ so that we can neglect the fringing of the electric field that occurs at the ends of the cylinders. Each plate contains a charge of magnitude $q$.

As a Gaussian surface, we choose a cylinder of length $L$ and radius $r$, closed by end caps and placed as is shown in Fig. 25-6. It is coaxial with the cylinders and encloses the central cylinder and thus also the charge $q$ on that cylinder. Equation 25-4 then relates that charge and the field magnitude $E$ as

$$q = \varepsilon_0 EA = \varepsilon_0 E(2\pi rL),$$

in which $2\pi rL$ is the area of the curved part of the Gaussian surface. There is no flux through the end caps. Solving for $E$ yields

$$E = \frac{q}{2\pi\varepsilon_0 Lr}. \tag{25-12}$$

Substitution of this result into Eq. 25-6 yields

$$V = \int_-^+ E\,ds = -\frac{q}{2\pi\varepsilon_0 L}\int_b^a \frac{dr}{r} = \frac{q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right), \tag{25-13}$$

where we have used the fact that here $ds = -dr$ (we integrated radially inward).

From the relation $C = q/V$, we then have

$$C = 2\pi\varepsilon_0 \frac{L}{\ln(b/a)} \quad \text{(cylindrical capacitor).} \tag{25-14}$$

We see that the capacitance of a cylindrical capacitor, like that of a parallel-plate capacitor, depends only on geometrical factors, in this case the length $L$ and the two radii $b$ and $a$.

## A Spherical Capacitor

Figure 25-6 can also serve as a central cross section of a capacitor that consists of two concentric spherical shells, of radii $a$ and $b$. As a Gaussian surface we draw a sphere of radius $r$ concentric with the two shells; then Eq. 25-4 yields

$$q = \varepsilon_0 EA = \varepsilon_0 E(4\pi r^2),$$

in which $4\pi r^2$ is the area of the spherical Gaussian surface. We solve this equation for $E$, obtaining

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}, \tag{25-15}$$

which we recognize as the expression for the electric field due to a uniform spherical charge distribution (Eq. 23-15).

If we substitute this expression into Eq. 25-6, we find

$$V = \int_-^+ E\,ds = -\frac{q}{4\pi\varepsilon_0}\int_b^a \frac{dr}{r^2} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{q}{4\pi\varepsilon_0}\frac{b-a}{ab}, \tag{25-16}$$

where again we have substituted $-dr$ for $ds$. If we now substitute Eq. 25-16 into Eq. 25-1 and solve for $C$, we find

$$C = 4\pi\varepsilon_0 \frac{ab}{b-a} \quad \text{(spherical capacitor).} \tag{25-17}$$

## An Isolated Sphere

We can assign a capacitance to a *single* isolated spherical conductor of radius $R$ by assuming that the "missing plate" is a conducting sphere of infinite radius. After all, the field lines that leave the surface of a positively charged isolated conductor must end somewhere; the walls of the room in which the conductor is housed can serve effectively as our sphere of infinite radius.

To find the capacitance of the conductor, we first rewrite Eq. 25-17 as

$$C = 4\pi\varepsilon_0 \frac{a}{1 - a/b}.$$

If we then let $b \to \infty$ and substitute $R$ for $a$, we find

$$C = 4\pi\varepsilon_0 R \quad \text{(isolated sphere).} \tag{25-18}$$

Note that this formula and the others we have derived for capacitance (Eqs. 25-9, 25-14, and 25-17) involve the constant $\varepsilon_0$ multiplied by a quantity that has the dimensions of a length.

### CHECKPOINT 2

For capacitors charged by the same battery, does the charge stored by the capacitor increase, decrease, or remain the same in each of the following situations? (a) The plate separation of a parallel-plate capacitor is increased. (b) The radius of the inner cylinder of a cylindrical capacitor is increased. (c) The radius of the outer spherical shell of a spherical capacitor is increased.

Sample Problem

### Charging the plates in a parallel-plate capacitor

In Fig. 25-7*a*, switch S is closed to connect the uncharged capacitor of capacitance $C = 0.25\ \mu F$ to the battery of potential difference $V = 12$ V. The lower capacitor plate has thickness $L = 0.50$ cm and face area $A = 2.0 \times 10^{-4}\ m^2$, and it consists of copper, in which the density of conduction electrons is $n = 8.49 \times 10^{28}$ electrons/m³. From what depth $d$ within the plate (Fig. 25-7*b*) must electrons move to the plate face as the capacitor becomes charged?

KEY IDEA

The charge collected on the plate is related to the capacitance and the potential difference across the capacitor by Eq. 25-1 ($q = CV$).

***Calculations:*** Because the lower plate is connected to the negative terminal of the battery, conduction electrons move up to the face of the plate. From Eq. 25-1, the total charge magnitude that collects there is

$$q = CV = (0.25 \times 10^{-6}\ F)(12\ V) = 3.0 \times 10^{-6}\ C.$$

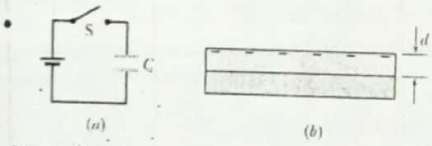



**Fig. 25-7** (*a*) A battery and capacitor circuit. (*b*) The lower capacitor plate.

Dividing this result by $e$ gives us the number $N$ of conduction electrons that come up to the face:

$$N = \frac{q}{e} = \frac{3.0 \times 10^{-6}\ C}{1.602 \times 10^{-19}\ C} = 1.873 \times 10^{13}\ \text{electrons}.$$

These electrons come from a volume that is the product of the face area $A$ and the depth $d$ we seek. Thus, from the density of conduction electrons (number per volume), we can write

$$n = \frac{N}{Ad},$$

or

$$d = \frac{N}{An} = \frac{1.873 \times 10^{13}\ \text{electrons}}{(2.0 \times 10^{-4}\ m^2)(8.49 \times 10^{28}\ \text{electrons/m}^3)}$$

$$= 1.1 \times 10^{-12}\ m = 1.1\ \text{pm}. \qquad \text{(Answer)}$$

In common speech, we would say that the battery charges the capacitor by supplying the charged particles. But what the battery really does is set up an electric field in the wires and plate such that electrons very close to the plate face move up to the negative face.

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## 25-4 Capacitors in Parallel and in Series

When there is a combination of capacitors in a circuit, we can sometimes replace that combination with an **equivalent capacitor**—that is, a single capacitor that has the same capacitance as the actual combination of capacitors. With such a replacement, we can simplify the circuit, affording easier solutions for unknown quantities of the circuit. Here we discuss two basic combinations of capacitors that allow such a replacement.

### Capacitors in Parallel

Figure 25-8*a* shows an electric circuit in which three capacitors are connected *in parallel* to battery B. This description has little to do with how the capacitor plates are drawn. Rather, "in parallel" means that the capacitors are directly wired together at one plate and directly wired together at the other plate, and that the same potential difference $V$ is applied across the two groups of wired-together plates. Thus, each capacitor has the same potential difference $V$, which produces charge on the capacitor. (In Fig. 25-8*a*, the applied potential $V$ is maintained by the battery.) In general, When a potential difference $V$ is applied across several capacitors connected in parallel, that potential difference $V$ is applied across each capacitor. The total charge $q$ stored on the capacitors is the sum of the charges stored on all the capacitors.

When we analyze a circuit of capacitors in parallel, we can simplify it with this mental replacement:

Capacitors connected in parallel can be replaced with an equivalent capacitor that has the same *total* charge $q$ and the same potential difference $V$ as the actual capacitors.

(You might remember this result with the nonsense word "par-V," which is close to "party," to mean "capacitors in parallel have the same $V$.") Figure 25-8*b* shows the equivalent capacitor (with equivalent capacitance $C_{eq}$) that has replaced the three capacitors (with actual capacitances $C_1$, $C_2$, and $C_3$) of Fig. 25-8*a*.

To derive an expression for $C_{eq}$ in Fig. 25-8*b*, we first use Eq. 25-1 to find the charge on each actual capacitor:

$$q_1 = C_1V, \quad q_2 = C_2V, \quad \text{and} \quad q_3 = C_3V.$$

The total charge on the parallel combination of Fig. 25-8*a* is then

$$q = q_1 + q_2 + q_3 = (C_1 + C_2 + C_3)V.$$

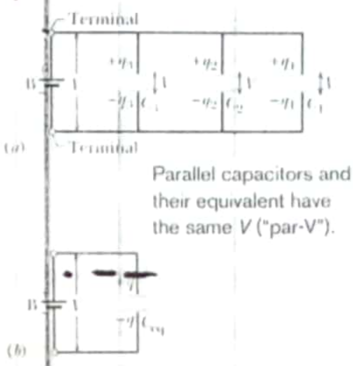
The equivalent capacitance, with the same total charge $q$ and applied potential difference $V$ as the combination, is then

$$C_{eq} = \frac{q}{V} = C_1 + C_2 + C_3,$$

a result that we can easily extend to any number $n$ of capacitors, as

$$C_{eq} = \sum_{j=1}^{n} C_j \qquad (n \text{ capacitors in parallel}). \qquad (25\text{-}19)$$

Thus, to find the equivalent capacitance of a parallel combination, we simply add the individual capacitances.




Parallel capacitors and their equivalent have the same $V$ ("par-V").

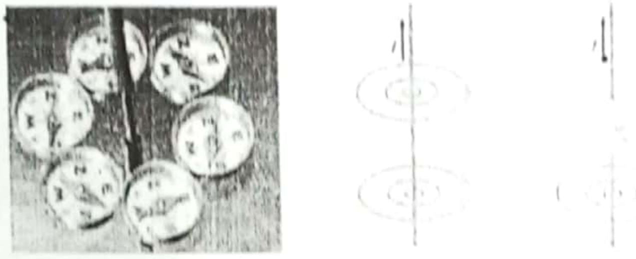
**Fig. 25-8** (*a*) Three capacitors connected in parallel to battery B. The battery maintains potential difference $V$ across its terminals and thus across *each* capacitor. (*b*) The equivalent capacitor, with capacitance $C_{eq}$, replaces the parallel combination.

## Capacitors in Series

Figure 25-9*a* shows three capacitors connected *in series* to battery B. This description has little to do with how the capacitors are drawn. Rather, "in series" means that the capacitors are wired serially, one after the other, and that a potential difference $V$ is applied across the two ends of the series. (In Fig. 25-9*a*, this potential difference $V$ is maintained by battery B.) The potential differences that then exist across the capacitors in the series produce identical charges $q$ on them.

When a potential difference $V$ is applied across several capacitors connected in series, the capacitors have identical charge $q$. The sum of the potential differences across all the capacitors is equal to the applied potential difference $V$.

We can explain how the capacitors end up with identical charge by following a *chain reaction* of events, in which the charging of each capacitor causes the charging of the next capacitor. We start with capacitor 3 and work upward to capacitor 1. When the battery is first connected to the series of capacitors, it



(a) (b) (c)

FIGURE 27-8 (a) Deflection of compass needles near a current-carrying wire, showing the presence and direction of the magnetic field. (b) Magnetic field lines around an electric current in a straight wire. See also the Chapter-Opening photo. (c) Right-hand rule for remembering the direction of the magnetic field: when the thumb points in the direction of the conventional current, the fingers wrapped around the wire point in the direction of the magnetic field.

## 27–2 Electric Currents Produce Magnetic Fields

During the eighteenth century, many scientists sought to find a connection between electricity and magnetism. A stationary electric charge and a magnet were shown to have no influence on each other. But in 1820, Hans Christian Oersted (1777–1851) found that when a compass needle is placed near a wire, the needle deflects as soon as the two ends of the wire are connected to the terminals of a battery and the wire carries an electric current. As we have seen, a compass needle is deflected by a magnetic field. So Oersted's experiment showed that **an electric current produces a magnetic field**. He had found a connection between electricity and magnetism.

A compass needle placed near a straight section of current-carrying wire experiences a force, causing the needle to align tangent to a circle around the wire, Fig. 27–8a. Thus, the magnetic field lines produced by a current in a straight wire are in the form of circles with the wire at their center, Fig. 27–8b. The direction of these lines is indicated by the north pole of the compasses in Fig. 27–8a. There is a simple way to remember the direction of the magnetic field lines in this case. It is called a **right-hand rule**: grasp the wire with your right hand so that your thumb points in the direction of the conventional (positive) current; then your fingers will encircle the wire in the direction of the magnetic field, Fig. 27–8c.

The magnetic field lines due to a circular loop of current-carrying wire can be determined in a similar way using a compass. The result is shown in Fig. 27–9. Again the right-hand rule can be used, as shown in Fig. 27–10. Unlike the uniform field shown in Fig. 27–7, the magnetic fields shown in Figs. 27–8 and 27–9 are *not* uniform—the fields are different in magnitude and direction at different points.

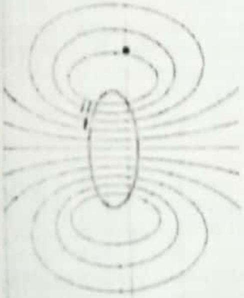
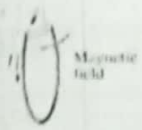


FIGURE 27-9 Magnetic field lines due to a circular loop of wire.

*Right-hand-rule 1: magnetic field direction produced by electric current*

FIGURE 27-10 Right-hand rule for determining the direction of the magnetic field relative to the current.




**EXERCISE B** A straight wire carries a current directly toward you. In what direction are the magnetic field lines surrounding the wire?

## 27–3 Force on an Electric Current in a Magnetic Field; Definition of $\vec{\mathbf{B}}$

In Section 27–2 we saw that an electric current exerts a force on a magnet, such as a compass needle. By Newton's third law, we might expect the reverse to be true as well: we should expect that *a magnet exerts a force on a current-carrying wire*. Experiments indeed confirm this effect, and it too was first observed by Oersted.

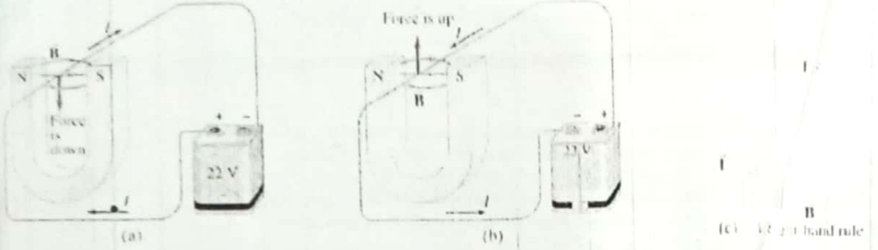



FIGURE 27–11 (a) Force on a current-carrying wire placed in a magnetic field **B**; (b) same, but current reversed; (c) right-hand rule for setup in (b).

Suppose a straight wire is placed in the magnetic field between the poles of a horseshoe magnet as shown in Fig. 27–11. When a current flows in the wire, experiment shows that a force is exerted on the wire. But this force is *not* toward one or the other pole of the magnet. Instead, the force is directed at right angles to the magnetic field direction, downward in Fig. 27–11a. If the current is reversed in direction, the force is in the opposite direction, upward as shown in Fig. 27–11b. Experiments show that *the direction of the force is always perpendicular to the direction of the current and also perpendicular to the direction of the magnetic field,* $\vec{\mathbf{B}}$.

The direction of the force is given by another **right-hand rule**, as illustrated in Fig. 27–11c. Orient your right hand until your outstretched fingers can point in the direction of the conventional current $I$, and when you bend your fingers they point in the direction of the magnetic field lines, $\vec{\mathbf{B}}$. Then your outstretched thumb will point in the direction of the force $\vec{\mathbf{F}}$ on the wire.

*Right-hand rule 2: force on current exerted by B*

This right-hand rule describes the direction of the force. What about the magnitude of the force on the wire? It is found experimentally that the magnitude of the force is directly proportional to the current $I$ in the wire, and to the length $\ell$ of wire exposed to the magnetic field (assumed uniform). Furthermore, if the magnetic field $B$ is made stronger, the force is found to be proportionally greater. The force also depends on the angle $\theta$ between the current direction and the magnetic field (Fig. 27–12), being proportional to $\sin\theta$. Thus, the force on a wire carrying a current $I$ with length $\ell$ in a uniform magnetic field $B$ is given by

$$F \propto I\ell B\sin\theta.$$

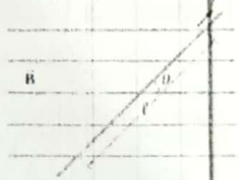



FIGURE 27–12 Current-carrying wire in a magnetic field. Force on the wire is directed into the page.

When the current is perpendicular to the field lines ($\theta = 90°$), the force is strongest. When the wire is parallel to the magnetic field lines ($\theta = 0°$), there is no force at all.

Up to now we have not defined the magnetic field strength precisely. In fact, the magnetic field $B$ can be conveniently defined in terms of the above proportion so that the proportionality constant is precisely 1. Thus we have

$$F = I\ell B\sin\theta \tag{27–1}$$

If the direction of the current is perpendicular to the field $\vec{\mathbf{B}}$ ($\theta = 90°$), then the force is

$$F_{max} = I\ell B. \qquad [\text{current} \perp \vec{\mathbf{B}}] \tag{27–2}$$

If the current is parallel to the field ($\theta = 0°$), the force is zero. The magnitude of $\vec{\mathbf{B}}$ can be defined using Eq. 27–2 as $B = F_{max}/I\ell$, where $F_{max}$ is the magnitude of the force on a straight length $\ell$ of wire carrying a current $I$ when the wire is perpendicular to $\vec{\mathbf{B}}$.

The relation between the force $\vec{\mathbf{F}}$ on a wire carrying current $I$, and the magnetic field $\vec{\mathbf{B}}$ that causes the force, can be written as a vector equation. To do so we recall that the direction of $\vec{\mathbf{F}}$ is given by the right-hand rule (Fig. 27–11c), and the magnitude by Eq. 27–1. This is consistent with the definition of the vector cross product (see Section 11–2), so we can write

$$\vec{\mathbf{F}} = I\vec{\boldsymbol{\ell}} \times \vec{\mathbf{B}}; \tag{27–3}$$

here, $\vec{\boldsymbol{\ell}}$ is a vector whose magnitude is the length of the wire and its direction is along the wire (assumed straight) in the direction of the conventional (positive) current.

Equation 27–3 applies if the magnetic field is uniform and the wire is straight. If $\vec{\mathbf{B}}$ is not uniform, or if the wire does not everywhere make the same angle $\theta$ with $\vec{\mathbf{B}}$, then Eq. 27–3 can be written more generally as

$$d\vec{\mathbf{F}} = I\, d\vec{\boldsymbol{\ell}} \times \vec{\mathbf{B}}, \tag{27–4}$$

where $d\vec{\mathbf{F}}$ is the infinitesimal force acting on a differential length $d\vec{\boldsymbol{\ell}}$ of the wire. The total force on the wire is then found by integrating.

Equation 27–4 can serve (just as well as Eq. 27–2 or 27–3) as a practical definition of $\vec{\mathbf{B}}$. An equivalent way to define $\vec{\mathbf{B}}$, in terms of the force on a moving electric charge, is discussed in the next Section.

**EXERCISE C** A wire carrying current $I$ is perpendicular to a magnetic field of strength $B$. Assuming a fixed length of wire, which of the following changes will result in decreasing the force on the wire by a factor of 2? (*a*) Decrease the angle from 90° to 45°; (*b*) decrease the angle from 90° to 30°; (*c*) decrease the current in the wire to $I/2$; (*d*) decrease the magnetic field strength to $B/2$; (*e*) none of these will do it.

The SI unit for magnetic field $B$ is the **tesla** (T). From Eqs. 27–1, 2, 3, or 4, we see that $1\,\mathrm{T} = 1\,\mathrm{N/A \cdot m}$. An older name for the tesla is the "weber per meter squared" ($1\,\mathrm{Wb/m^2} = 1\,\mathrm{T}$). Another unit sometimes used to specify magnetic field is a cgs unit, the **gauss** (G): $1\,\mathrm{G} = 10^{-4}\,\mathrm{T}$. A field given in gauss should always be changed to teslas before using with other SI units. To get a "feel" for these units, we note that the magnetic field of the Earth at its surface is about $\frac{1}{2}$ G or $0.5 \times 10^{-4}\,\mathrm{T}$. On the other hand, the field near a small magnet attached to your refrigerator may be 100 G (0.01 T) whereas strong electromagnets can produce fields on the order of 2 T and superconducting magnets can produce over 10 T.

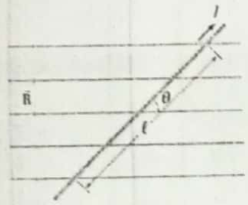



FIGURE 27–12 (Repeated for Example 27–1.) Current-carrying wire in a magnetic field. Force on the wire is directed into the page.

**EXAMPLE 27–1** **Magnetic force on a current-carrying wire.** A wire carrying a 30-A current has a length $\ell = 12$ cm between the pole faces of a magnet at an angle $\theta = 60°$ (Fig. 27–12). The magnetic field is approximately uniform at 0.90 T. We ignore the field beyond the pole pieces. What is the magnitude of the force on the wire?

**APPROACH** We use Eq. 27–1, $F = I\ell B \sin\theta$.

**SOLUTION** The force $F$ on the 12-cm length of wire within the uniform field $B$ is

$$F = I\ell B \sin\theta = (30\,\mathrm{A})(0.12\,\mathrm{m})(0.90\,\mathrm{T})(0.866) = 2.8\,\mathrm{N}.$$

**EXERCISE D** A straight power line carries 30 A and is perpendicular to the Earth's magnetic field of $0.50 \times 10^{-4}\,\mathrm{T}$. What magnitude force is exerted on 100 m of this power line?

On a diagram, when we want to represent an electric current or a magnetic field that is pointing out of the page (toward us) or into the page, we use $\odot$ or $\times$, respectively. The $\odot$ is meant to resemble the tip of an arrow pointing directly toward the reader, whereas the $\times$ or $\otimes$ resembles the tail of an arrow moving away. (See Figs. 27–13 and 27–14.)

**EXAMPLE 27–2** **Measuring a magnetic field.** A rectangular loop of wire hangs vertically as shown in Fig. 27–13. A magnetic field $\vec{\mathbf{B}}$ is directed horizontally, perpendicular to the wire, and points out of the page at all points as represented by the symbol $\odot$. The magnetic field $\vec{\mathbf{B}}$ is very nearly uniform along the horizontal portion of wire ab (length $\ell = 10.0$ cm) which is near the center of the gap of a large magnet producing the field. The top portion of the wire loop is free of the field. The loop hangs from a balance which measures a downward magnetic force (in addition to the gravitational force) of $F = 3.48 \times 10^{-2}$ N when the wire carries a current $I = 0.245$ A. What is the magnitude of the magnetic field $B$?

**APPROACH** Three straight sections of the wire loop are in the magnetic field: a horizontal section and two vertical sections. We apply Eq. 27–1 to each section and use the right-hand rule.

**SOLUTION** The magnetic force on the left vertical section of wire points to the left; the force on the vertical section on the right points to the right. These two forces are equal and in opposite directions and so add up to zero. Hence, the net magnetic force on the loop is that on the horizontal section ab, whose length is $\ell = 0.100$ m. The angle $\theta$ between $\vec{\mathbf{B}}$ and the wire is $\theta = 90°$, so $\sin\theta = 1$. Thus Eq. 27–1 gives

$$B = \frac{F}{I\ell} = \frac{3.48 \times 10^{-2}\,\text{N}}{(0.245\,\text{A})(0.100\,\text{m})} = 1.42\,\text{T}.$$

**NOTE** This technique can be a precise means of determining magnetic field strength.

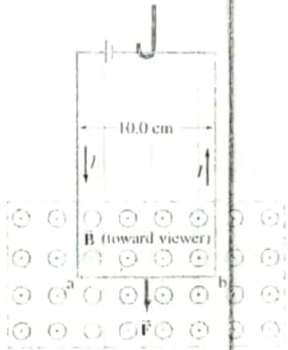



FIGURE 27–13 Measuring a magnetic field $\vec{\mathbf{B}}$. Example 27–2.

**EXAMPLE 27–3** **Magnetic force on a semicircular wire.** A rigid wire, carrying a current $I$, consists of a semicircle of radius $R$ and two straight portions as shown in Fig. 27–14. The wire lies in a plane perpendicular to a uniform magnetic field $\vec{\mathbf{B}}_0$. Note choice of $x$ and $y$ axis. The straight portions each have length $\ell$ within the field. Determine the net force on the wire due to the magnetic field $\vec{\mathbf{B}}_0$.

**APPROACH** The forces on the two straight sections are equal $(= I\ell B_0)$ and in opposite directions, so they cancel. Hence the net force is that on the semicircular portion.

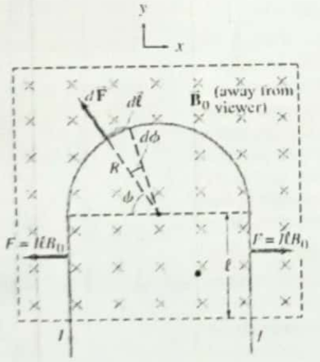



FIGURE 27–14 Example 27–3.

**SOLUTION** We divide the semicircle into short lengths $d\ell = R\,d\phi$ as indicated in Fig. 27–14, and use Eq. 27–4, $d\vec{\mathbf{F}} = I\,d\vec{\boldsymbol{\ell}} \times \vec{\mathbf{B}}$, to find

$$dF = IB_0R\,d\phi,$$

where $dF$ is the force on the length $d\ell = R\,d\phi$, and the angle between $d\vec{\boldsymbol{\ell}}$ and $\vec{\mathbf{B}}_0$ is 90° (so $\sin\theta = 1$ in the cross product). The $x$ component of the force $d\vec{\mathbf{F}}$ on the segment $d\vec{\boldsymbol{\ell}}$ shown, and the $x$ component of $d\vec{\mathbf{F}}$ for a symmetrically located $d\vec{\boldsymbol{\ell}}$ on the other side of the semicircle, will cancel each other. Thus for the entire semicircle there will be no $x$ component of force. Hence we need be concerned only with the $y$ components, each equal to $dF\sin\phi$, and the total force will have magnitude

$$F = \int_0^{\pi} dF\sin\phi = IB_0R\int_0^{\pi}\sin\phi\,d\phi = -IB_0R\cos\phi\Big|_0^{\pi} = 2IB_0R,$$

with direction vertically upward along the $y$ axis in Fig. 27–14.

# 27–4 Force on an Electric Charge Moving in a Magnetic Field

We have seen that a current-carrying wire experiences a force when placed in a magnetic field. Since a current in a wire consists of moving electric charges, we might expect that freely moving charged particles (not in a wire) would also experience a force when passing through a magnetic field. Indeed, this is the case.

From what we already know we can predict the force on a single moving electric charge. If $N$ such particles of charge $q$ pass by a given point in time $t$, they constitute a current $I = Nq/t$. We let $t$ be the time for a charge $q$ to travel a distance $\ell$ in a magnetic field $\vec{\mathbf{B}}$; then $\vec{\boldsymbol{\ell}} = \vec{\mathbf{v}}t$ where $\vec{\mathbf{v}}$ is the velocity of the particle. Thus, the force on these $N$ particles is, by Eq. 27–3, $\vec{\mathbf{F}} = I\vec{\boldsymbol{\ell}} \times \vec{\mathbf{B}} = (Nq/t)(\vec{\mathbf{v}}t) \times \vec{\mathbf{B}} = Nq\vec{\mathbf{v}} \times \vec{\mathbf{B}}$. The force on *one* of the $N$ particles is then

$$\vec{\mathbf{F}} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}. \tag{27–5a}$$

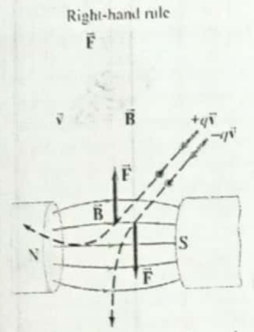



**FIGURE 27–15** Force on charged particles due to a magnetic field is perpendicular to the magnetic field direction. If $\vec{\mathbf{v}}$ is horizontal, then $\vec{\mathbf{F}}$ is vertical.

This basic and important result can be considered as an alternative way of defining the magnetic field $\vec{\mathbf{B}}$, in place of Eq. 27–4 or 27–3. The magnitude of the force in Eq. 27–5a is

$$F = qvB\sin\theta. \tag{27–5b}$$

This gives the magnitude of the force on a particle of charge $q$ moving with velocity $\vec{\mathbf{v}}$ at a point where the magnetic field has magnitude $B$. The angle between $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$ is $\theta$. The force is greatest when the particle moves perpendicular to $\vec{\mathbf{B}}$ $(\theta = 90°)$:

$$F_{max} = qvB. \qquad [\vec{\mathbf{v}} \perp \vec{\mathbf{B}}]$$

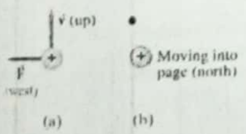
The force is *zero* if the particle moves *parallel* to the field lines $(\theta = 0°)$. The *direction* of the force is perpendicular to the magnetic field $\vec{\mathbf{B}}$ and to the velocity $\vec{\mathbf{v}}$ of the particle. It is given again by a **right-hand rule** (as for any cross product): you orient your right hand so that your outstretched fingers point along the direction of the particle's velocity ($\vec{\mathbf{v}}$), and when you bend your fingers they must point along the direction of $\vec{\mathbf{B}}$. Then your thumb will point in the direction of the force. This is true only for *positively* charged particles, and will be "up" for the positive particle shown in Fig. 27–15. For negatively charged particles, the force is in exactly the opposite direction, "down" in Fig. 27–15.

**CONCEPTUAL EXAMPLE 27–4** **Negative charge near a magnet.** A negative charge $-Q$ is placed at rest near a magnet. Will the charge begin to move? Will it feel a force? What if the charge were positive, $+Q$?

**RESPONSE** No to all questions. A charge at rest has velocity equal to zero. Magnetic fields exert a force only on moving electric charges (Eqs. 27–5).

**EXERCISE E** Return to the Chapter-Opening Question, page 707, and answer it again now. Try to explain why you may have answered differently the first time.

**FIGURE 27–16** Example 27–5.




**EXAMPLE 27–5** **Magnetic force on a proton.** A magnetic field exerts a force of $8.0 \times 10^{-14}$ N toward the west on a proton moving vertically upward at a speed of $5.0 \times 10^{6}$ m/s (Fig. 27–16a). When moving horizontally in a northerly direction, the force on the proton is zero (Fig. 27–16b). Determine the magnitude and direction of the magnetic field in this region. (The charge on a proton is $q = +e = 1.6 \times 10^{-19}$ C.)

**APPROACH** Since the force on the proton is zero when moving north, the field must be in a north–south direction. In order to produce a force to the west when the proton moves upward, the right-hand rule tells us that $\vec{\mathbf{B}}$ must point toward the north. (Your thumb points west and the outstretched fingers of your right hand point upward only when your bent fingers point north.) The magnitude of $\vec{\mathbf{B}}$ is found using Eq. 27–5b.

**SOLUTION** Equation 27–5b with $\theta = 90°$ gives

$$B = \frac{F}{qv} = \frac{8.0 \times 10^{-14}\ \text{N}}{(1.6 \times 10^{-19}\ \text{C})(5.0 \times 10^{6}\ \text{m/s})} = 0.10\ \text{T}.$$

**EXAMPLE 27–6** ESTIMATE **Magnetic force on ions during a nerve pulse.** Estimate the magnetic force due to the Earth's magnetic field on ions crossing a cell membrane during an action potential. Assume the speed of the ions is $10^{-2}$ m/s (Section 25–10).

**APPROACH** Using $F = qvB$, set the magnetic field of the Earth to be roughly $B \approx 10^{-4}$ T, and the charge $q \approx e \approx 10^{-19}$ C.

**SOLUTION** $F \approx (10^{-19}\,\mathrm{C})(10^{-2}\,\mathrm{m/s})(10^{-4}\,\mathrm{T}) = 10^{-25}\,\mathrm{N}$.

**NOTE** This is an extremely small force. Yet it is thought migrating animals do somehow detect the Earth's magnetic field, and this is an area of active research.

The path of a charged particle moving in a plane perpendicular to a uniform magnetic field is a circle as we shall now show. In Fig. 27–17 the magnetic field is directed *into* the paper, as represented by ×'s. An electron at point P is moving to the right, and the force on it at this point is downward as shown (use the right-hand rule and reverse the direction for negative charge). The electron is thus deflected toward the page bottom. A moment later, say, when it reaches point Q, the force is still perpendicular to the velocity and is in the direction shown. Because the force is always perpendicular to $\vec{\mathbf{v}}$, the magnitude of $\vec{\mathbf{v}}$ does not change—the electron moves at constant speed. We saw in Chapter 5 that if the force on a particle is always perpendicular to its velocity $\vec{\mathbf{v}}$, the particle moves in a circle and has a centripetal acceleration $a = v^2/r$ (Eq. 5–1). Thus a charged particle moves in a circular path with constant centripetal acceleration in a uniform magnetic field (see Example 27–7). The electron moves clockwise in Fig. 27–17. A positive particle in this field would feel a force in the opposite direction and would thus move counterclockwise.

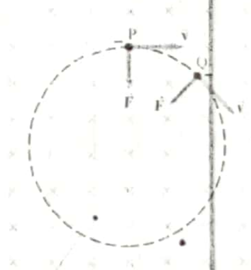



FIGURE 27–17 Force exerted by a uniform magnetic field on a moving charged particle (in this case, an electron) produces a circular path.

**EXAMPLE 27–7** **Electron's path in a uniform magnetic field.** An electron travels at $2.0 \times 10^7$ m/s in a plane perpendicular to a uniform 0.010-T magnetic field. Describe its path quantitatively.

**APPROACH** The electron moves at speed $v$ in a curved path and so must have a centripetal acceleration $a = v^2/r$ (Eq. 5–1). We find the radius of curvature using Newton's second law. The force is given by Eq. 27–5b with $\sin\theta = 1$: $F = qvB$.

**SOLUTION** We insert $F$ and $a$ into Newton's second law:

$$\Sigma F = ma$$

$$qvB = \frac{mv^2}{r}.$$

We solve for $r$ and find

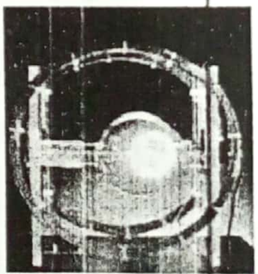
$$r = \frac{mv}{qB}.$$

Since $\vec{\mathbf{F}}$ is perpendicular to $\vec{\mathbf{v}}$, the magnitude of $\vec{\mathbf{v}}$ doesn't change. From this equation we see that if $\vec{\mathbf{B}}$ = constant, then $r$ = constant, and the curve must be a circle as we claimed above. To get $r$ we put in the numbers:

$$r = \frac{(9.1 \times 10^{-31}\,\mathrm{kg})(2.0 \times 10^{7}\,\mathrm{m/s})}{(1.6 \times 10^{-19}\,\mathrm{C})(0.010\,\mathrm{T})} = 1.1 \times 10^{-2}\,\mathrm{m} = 1.1\,\mathrm{cm}.$$

**NOTE** See Fig. 27–18.

FIGURE 27–18 The blue ring inside the glass tube is the glow of a beam of electrons that ionize the gas molecules. The red coils of current-carrying wire produce a nearly uniform magnetic field, illustrating the circular path of charged particles in a uniform magnetic field.



The time $T$ required for a particle of charge $q$ moving with constant speed $v$ to make one circular revolution in a uniform magnetic field $\vec{\mathbf{B}}$ $(\perp \vec{\mathbf{v}})$ is $T = 2\pi r/v$, where $2\pi r$ is the circumference of its circular path. From Example 27–7, $r = mv/qB$, so

$$T = \frac{2\pi m}{qB}.$$

Since $T$ is the period of rotation, the frequency of rotation is

$$f = \frac{1}{T} = \frac{qB}{2\pi m}. \tag{27–6}$$

This is often called the **cyclotron frequency** of a particle in a field because this is the frequency at which particles revolve in a cyclotron (see Problem 66).

# 27–5 Torque on a Current Loop; Magnetic Dipole Moment

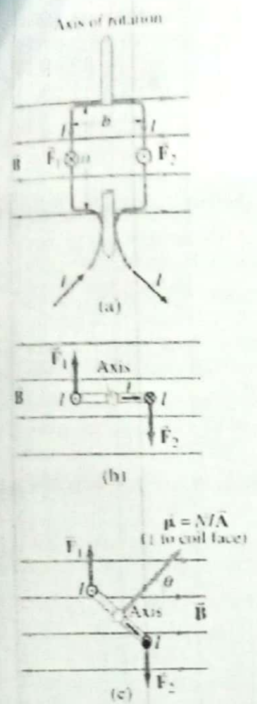



**FIGURE 27–22** Calculating the torque on a current loop in a magnetic field $\vec{B}$. (a) Loop face parallel to $\vec{B}$ field lines; (b) top view; (c) loop makes an angle to $\vec{B}$, reducing the torque since the lever arm is reduced.

When an electric current flows in a closed loop of wire placed in an external magnetic field, as shown in Fig. 27–22, the magnetic force on the current can produce a torque. This is the principle behind a number of important practical devices, including motors and analog voltmeters and ammeters, which we discuss in the next Section. The interaction between a current and a magnetic field is important in other areas as well, including atomic physics.

Current flows through the rectangular loop in Fig. 27–22a, whose face we assume is parallel to $\vec{B}$. $\vec{B}$ exerts no force and no torque on the horizontal segments of wire because they are parallel to the field and $\sin\theta = 0$ in Eq. 27–1. But the magnetic field does exert a force on each of the vertical sections of wire as shown, $\vec{F}_1$ and $\vec{F}_2$ (see also top view, Fig. 27–22b). By right-hand-rule-2 (Fig. 27–11c or Table 27–1) the direction of the force on the upward current on the left is in the opposite direction from the equal magnitude force $\vec{F}_2$ on the downward current on the right. These forces give rise to a net torque that acts to rotate the coil about its vertical axis.

Let us calculate the magnitude of this torque. From Eq. 27–2 (current ⊥ $\vec{B}$), the force $F = IaB$, where $a$ is the length of the vertical arm of the coil. The lever arm for each force is $b/2$, where $b$ is the width of the coil and the "axis" is at the midpoint. The torques produced by $\vec{F}_1$ and $\vec{F}_2$ act in the same direction, so the total torque is the sum of the two torques:

$$\tau = IaB\frac{b}{2} + IaB\frac{b}{2} = IabB = IAB,$$

where $A = ab$ is the area of the coil. If the coil consists of $N$ loops of wire, the current is then $NI$, so the torque becomes

$$\tau = NIAB.$$

If the coil makes an angle $\theta$ with the magnetic field, as shown in Fig. 27–22c, the forces are unchanged, but each lever arm is reduced from $\frac{1}{2}b$ to $\frac{1}{2}b\sin\theta$. Note that the angle $\theta$ is taken to be the angle between $\vec{B}$ and the perpendicular to the face of the coil, Fig. 27–22c. So the torque becomes

$$\tau = NIAB\sin\theta. \tag{27–9}$$

This formula, derived here for a rectangular coil, is valid for any shape of flat coil.

The quantity $NIA$ is called the **magnetic dipole moment** of the coil and is considered a vector:

$$\vec{\mu} = NI\vec{A}, \tag{27–10}$$

where the direction of $\vec{A}$ (and therefore of $\vec{\mu}$) is *perpendicular* to the plane of the coil (the green arrow in Fig. 27–22c) consistent with the right-hand rule (cup your right hand so your fingers wrap around the loop in the direction of current flow, then your thumb points in the direction of $\vec{\mu}$ and $\vec{A}$). With this definition of $\vec{\mu}$, we can rewrite Eq. 27–9 in vector form:

$$\vec{\tau} = NI\vec{A} \times \vec{B}$$

or

$$\vec{\tau} = \vec{\mu} \times \vec{B}, \tag{27–11}$$

which gives the correct magnitude and direction for the torque $\vec{\tau}$.

Equation 27–11 has the same form as Eq. 21–9b for an electric dipole (with electric dipole moment $\vec{p}$) in an electric field $\vec{E}$, which is $\vec{\tau} = \vec{p} \times \vec{E}$. And just as an electric dipole has potential energy given by $U = -\vec{p}\cdot\vec{E}$ when in an electric field, we expect a similar form for a magnetic dipole in a magnetic field. In order to rotate a current loop (Fig. 27–22) so as to increase $\theta$, we must do work against the torque due to the magnetic field

Hence the potential energy depends on angle (see Eq. 10–22, the work-energy principle for rotational motion) as

$$U = \int \tau \, d\theta = \int NIAB \sin\theta \, d\theta = -\mu B \cos\theta + C.$$

If we choose $U = 0$ at $\theta = \pi/2$, then the arbitrary constant $C$ is zero and the potential energy is

$$U = -\mu B \cos\theta = -\vec{\mu} \cdot \vec{B}, \tag{27-12}$$

as expected (compare Eq. 21–10). Bar magnets and compass needles, as well as current loops, can be considered as magnetic dipoles. Note the striking similarities of the fields produced by a bar magnet and a current loop, Figs. 27–4b and 27–9.

**EXAMPLE 27–11** **Torque on a coil.** A circular coil of wire has a diameter of 20.0 cm and contains 10 loops. The current in each loop is 3.00 A, and the coil is placed in a 2.00-T external magnetic field. Determine the maximum and minimum torque exerted on the coil by the field.

**APPROACH** Equation 27–9 is valid for any shape of coil, including circular loops. Maximum and minimum torque are determined by the angle $\theta$ the coil makes with the magnetic field.

**SOLUTION** The area of one loop of the coil is

$$A = \pi r^2 = \pi(0.100\,\text{m})^2 = 3.14 \times 10^{-2}\,\text{m}^2.$$

The maximum torque occurs when the coil's face is parallel to the magnetic field, so $\theta = 90°$ in Fig. 27–22c, and $\sin\theta = 1$ in Eq. 27–9:

$$\tau = NIAB \sin\theta = (10)(3.00\,\text{A})(3.14 \times 10^{-2}\,\text{m}^2)(2.00\,\text{T})(1) = 1.88\,\text{N}\cdot\text{m}.$$

The minimum torque occurs if $\sin\theta = 0$, for which $\theta = 0°$, and then $\tau = 0$ from Eq. 27–9.

**NOTE** If the coil is free to turn, it will rotate toward the orientation with $\theta = 0°$.

**EXAMPLE 27–12** **Magnetic moment of a hydrogen atom.** Determine the magnetic dipole moment of the electron orbiting the proton of a hydrogen atom at a given instant, assuming (in the Bohr model) it is in its ground state with a circular orbit of radius $0.529 \times 10^{-10}$ m. [This is a very rough picture of atomic structure, but nonetheless gives an accurate result.]

**APPROACH** We start by setting the electrostatic force on the electron due to the proton equal to $ma = mv^2/r$ since the electron's acceleration is centripetal.

**SOLUTION** The electron is held in its orbit by the coulomb force, so Newton's second law, $F = ma$, gives

$$\frac{e^2}{4\pi\epsilon_0 r^2} = \frac{mv^2}{r};$$

so

$$v = \sqrt{\frac{e^2}{4\pi\epsilon_0 m r}}$$

$$= \sqrt{\frac{(8.99 \times 10^9\,\text{N}\cdot\text{m}^2/\text{C}^2)(1.60 \times 10^{-19}\,\text{C})^2}{(9.11 \times 10^{-31}\,\text{kg})(0.529 \times 10^{-10}\,\text{m})}} = 2.19 \times 10^6\,\text{m/s}.$$

Since current is the electric charge that passes a given point per unit time, the revolving electron is equivalent to a current

$$I = \frac{e}{T} = \frac{ev}{2\pi r},$$

where $T = 2\pi r/v$ is the time required for one orbit. Since the area of the orbit is $A = \pi r^2$, the magnetic dipole moment is

$$\mu = IA = \frac{ev}{2\pi r}(\pi r^2) = \tfrac{1}{2} evr$$

$$= \tfrac{1}{2}(1.60 \times 10^{-19}\,\text{C})(2.19 \times 10^6\,\text{m/s})(0.529 \times 10^{-10}\,\text{m}) = 9.27 \times 10^{-24}\,\text{A}\cdot\text{m}^2,$$

or $9.27 \times 10^{-24}$ J/T.

Thomson believed that an electron was not an atom, but rather a constituent, or part, of an atom. Convincing evidence for this came soon with the determination of the charge and the mass of the cathode rays. Thomson's student J. S. Townsend made the first direct (but rough) measurements of $e$ in 1897. But it was the more refined **oil-drop experiment** of Robert A. Millikan (1868–1953) that yielded a precise value for the charge on the electron and showed that charge comes in discrete amounts. In this experiment, tiny droplets of mineral oil carrying an electric charge were allowed to fall under gravity between two parallel plates, Fig. 27–31. The electric field $E$ between the plates was adjusted until the drop was suspended in midair. The downward pull of gravity, $mg$, was then just balanced by the upward force due to the electric field. Thus $qE = mg$, so the charge $q = mg/E$. The mass of the droplet was determined by measuring its terminal velocity in the absence of the electric field. Sometimes the drop was charged negatively, and sometimes positively, suggesting that the drop had acquired or lost electrons (by friction, leaving the atomizer). Millikan's painstaking observations and analysis presented convincing evidence that any charge was an integral multiple of a smallest charge, $e$, that was ascribed to the electron, and that the value of $e$ was $1.6 \times 10^{-19}\,C$. This value of $e$, combined with the measurement of $e/m$, gives the mass of the electron to be $(1.6 \times 10^{-19}\,C)/(1.76 \times 10^{11}\,C/kg) = 9.1 \times 10^{-31}\,kg$. This mass is less than a thousandth the mass of the smallest atom, and thus confirmed the idea that the electron is only a part of an atom. The accepted value today for the mass of the electron is $m_e = 9.11 \times 10^{-31}\,kg$.


FIGURE 27–31 Millikan's oil-drop experiment.

## CRT, Revisited

The cathode ray tube (CRT), which can serve as the picture tube of TV sets, oscilloscopes, and computer monitors, was discussed in Chapter 23. There, in Fig. 23–22, we saw a design using electric deflection plates to maneuver the electron beam. Many CRTs, however, make use of the magnetic field produced by coils to maneuver the electron beam. They operate much like the coils shown in Fig. 27–30.

# *27–8 The Hall Effect

When a current-carrying conductor is held fixed in a magnetic field, the field exerts a sideways force on the charges moving in the conductor. For example, if electrons move to the right in the rectangular conductor shown in Fig. 27–32a, the inward magnetic field will exert a downward force on the electrons $\vec{\mathbf{F}}_B = -e\vec{\mathbf{v}}_d \times \vec{\mathbf{B}}$, where $\vec{\mathbf{v}}_d$ is the drift velocity of the electrons (Section 25–8). Thus the electrons will tend to move nearer to face D than face C. There will thus be a potential difference between faces C and D of the conductor. This potential difference builds up until the electric field $\vec{\mathbf{E}}_H$ that it produces exerts a force, $e\vec{\mathbf{E}}_H$, on the moving charges that is equal and opposite to the magnetic force. This effect is called the **Hall effect** after E. H. Hall, who discovered it in 1879. The difference of potential produced is called the **Hall emf**.

The electric field due to the separation of charge is called the *Hall field*, $\vec{\mathbf{E}}_H$, and points downward in Fig. 27–32a, as shown. In equilibrium, the force due to this electric field is balanced by the magnetic force $ev_d B$, so

$$eE_H = ev_d B.$$

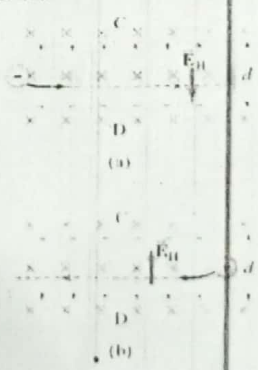
Hence $E_H = v_d B$. The Hall emf is then (Eq. 23–4b, assuming the conductor is long and thin so $E_H$ is uniform)

$$\mathscr{E}_H = E_H d = v_d B d, \qquad (27\text{–}14)$$

where $d$ is the width of the conductor.

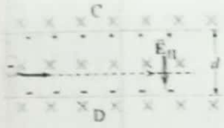
A current of negative charges moving to the right is equivalent to positive charges moving to the left, at least for most purposes. But the Hall effect can distinguish these two. As can be seen in Fig. 27–32b, positive particles moving to the left are deflected downward, so that the bottom surface is positive relative to the top surface. This is the reverse of part (a). Indeed, the direction of the emf in the Hall effect first revealed that it is negative particles that move in metal conductors.

FIGURE 27–32 The Hall effect. (a) Negative charges moving to the right as the current. (b) Positive charges moving to the left as the current.



The magnitude of the Hall emf is proportional to the strength of the magnetic field. The Hall effect can thus be used to measure magnetic field strengths. First the conductor, called a *Hall probe*, is calibrated with known magnetic fields. Then, for the same current, its emf output will be a measure of $B$. Hall probes can be made very small and are convenient and accurate to use.

The Hall effect can also be used to measure the drift velocity of charge carriers when the external magnetic field $B$ is known. Such a measurement also allows us to determine the density of charge carriers in the material.



(a)

FIGURE 27–32a (Repeated here for Example 27–13.)

**EXAMPLE 27–13 Drift velocity using the Hall effect.** A long copper strip 1.8 cm wide and 1.0 mm thick is placed in a 1.2-T magnetic field as in Fig. 27–32a. When a steady current of 15 A passes through it, the Hall emf is measured to be 1.02 $\mu$V. Determine the drift velocity of the electrons and the density of free (conducting) electrons (number per unit volume) in the copper.

**APPROACH** We use Eq. 27–14 to obtain the drift velocity, and Eq. 25–13 of Chapter 25 to find the density of conducting electrons.

**SOLUTION** The drift velocity (Eq. 27–14) is

$$v_d = \frac{\mathscr{E}_H}{Bd} = \frac{1.02 \times 10^{-6}\,\mathrm{V}}{(1.2\,\mathrm{T})(1.8 \times 10^{-2}\,\mathrm{m})} = 4.7 \times 10^{-5}\,\mathrm{m/s}.$$

The density of charge carriers $n$ is obtained from Eq. 25–13, $I = nev_d A$, where $A$ is the cross-sectional area through which the current $I$ flows. Then

$$n = \frac{I}{ev_d A} = \frac{15\,\mathrm{A}}{(1.6 \times 10^{-19}\,\mathrm{C})(4.7 \times 10^{-5}\,\mathrm{m/s})(1.8 \times 10^{-2}\,\mathrm{m})(1.0 \times 10^{-3}\,\mathrm{m})}$$
$$= 11 \times 10^{28}\,\mathrm{m}^{-3}.$$

This value for the density of free electrons in copper, $n = 11 \times 10^{28}$ per m$^3$, is the experimentally measured value. It represents *more* than one free electron per atom, which as we saw in Example 25–14 is $8.4 \times 10^{28}\,\mathrm{m}^{-3}$.

## *27–9 Mass Spectrometer

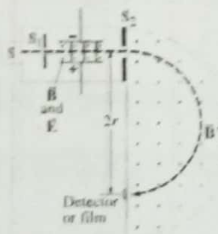
PHYSICS APPLIED

A **mass spectrometer** is a device to measure masses of atoms. It is used today not only in physics but also in chemistry, geology, and medicine, often to identify atoms (and their concentration) in given samples. As shown in Fig. 27–33, ions are produced by heating, or by an electric current, in the source or sample S. The particles, of mass $m$ and electric charge $q$, pass through slit $S_1$ and enter crossed electric and magnetic fields. Ions follow a straight-line path in this "velocity selector" (as in Example 27–10) if the electric force $qE$ is balanced by the magnetic force $qvB$: that is, if $qE = qvB$, or $v = E/B$. Thus only those ions whose speed is $v = E/B$ will pass through undeflected and emerge through slit $S_2$. In the semicircular region, after $S_2$, there is only a magnetic field, $B'$, so the ions follow a circular path. The radius of the circular path is found from their mark on film (or detectors) if $B'$ is fixed; or else $r$ is fixed by the position of a detector and $B'$ is varied until detection occurs. Newton's second law, $\Sigma F = ma$, applied to an ion moving in a circle under the influence only of the magnetic field $B'$ gives $qvB' = mv^2/r$. Since $v = E/B$, we have

$$m = \frac{qB'r}{v} = \frac{qBB'r}{E}.$$

All the quantities on the right side are known or can be measured, and thus $m$ can be determined.

FIGURE 27–33 Bainbridge-type mass spectrometer. The magnetic fields $B$ and $B'$ point out of the paper (indicated by the dots), for positive ions.



## 15–2 Types of Waves: Transverse and Longitudinal

When a wave travels down a cord—say, from left to right as in Fig. 15–1—the particles of the cord vibrate up and down in a direction transverse (that is, perpendicular) to the motion of the wave itself. Such a wave is called a **transverse wave** (Fig. 15–4a). There exists another type of wave known as a **longitudinal wave**. In a longitudinal wave, the vibration of the particles of the medium is *along* the direction of the wave's motion. Longitudinal waves are readily formed on a stretched spring or Slinky by alternately compressing and expanding one end. This is shown in Fig. 15–4b, and can be compared to the transverse wave in Fig. 15–4a.

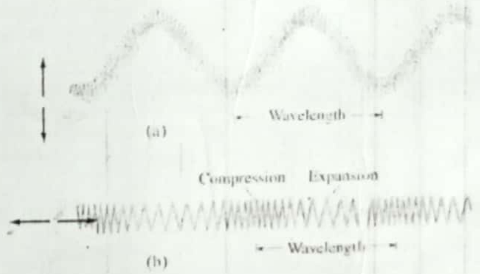



FIGURE 15–4 (a) Transverse wave; (b) longitudinal wave.

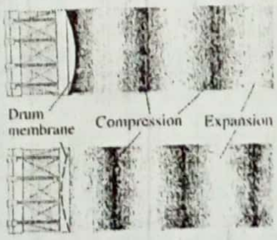
A series of compressions and expansions propagate along the spring. The *compressions* are those areas where the coils are momentarily close together. *Expansions* (sometimes called *rarefactions*) are regions where the coils are momentarily far apart. Compressions and expansions correspond to the crests and troughs of a transverse wave.

An important example of a longitudinal wave is a sound wave in air. A vibrating drumhead, for instance, alternately compresses and rarefies the air in contact with it, producing a longitudinal wave that travels outward in the air, as shown in Fig. 15–5.

As in the case of transverse waves, each section of the medium in which a longitudinal wave passes oscillates over a very small distance, whereas the wave itself can travel large distances. Wavelength, frequency, and wave velocity all have meaning for a longitudinal wave. The wavelength is the distance between successive compressions (or between successive expansions), and frequency is the number of compressions that pass a given point per second. The wave velocity is the velocity with which each compression appears to move; it is equal to the product of wavelength and frequency, $v = \lambda f$ (Eq. 15–1).

A longitudinal wave can be represented graphically by plotting the density of air molecules (or coils of a Slinky) versus position at a given instant, as shown in Fig. 15–6. Such a graphical representation makes it easy to illustrate what is happening. Note that the graph looks much like a transverse wave.

FIGURE 15–5 Production of a sound wave, which is longitudinal, shown at two moments in time about a half period ($\frac{1}{2}T$) apart.



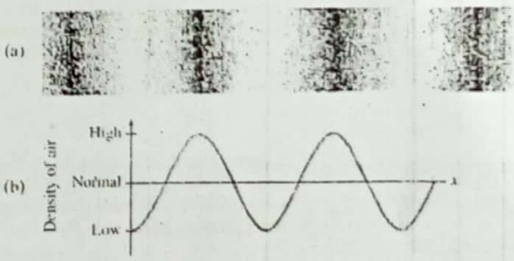




FIGURE 15–6 (a) A longitudinal wave with (b) its graphical representation at a particular instant in time.



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## Velocity of Transverse Waves

The velocity of a wave depends on the properties of the medium in which it travels. The velocity of a transverse wave on a stretched string or cord, for example, depends on the tension in the cord, $F_T$, and on the mass per unit length of the cord, $\mu$ (the Greek letter mu, where here $\mu = m/\ell$). For waves of small amplitude, the relationship is

$$v = \sqrt{\frac{F_T}{\mu}}. \qquad \left[\begin{array}{l}\text{transverse} \\ \text{wave on a cord}\end{array}\right] \qquad (15\text{–}2)$$

Before giving a derivation of this formula, it is worth noting that at least qualitatively it makes sense on the basis of Newtonian mechanics. That is, we do expect the tension to be in the numerator and the mass per unit length in the denominator. Why? Because when the tension is greater, we expect the velocity to be greater since each segment of cord is in tighter contact with its neighbor. And, the greater the mass per unit length, the more inertia the cord has and the more slowly the wave would be expected to propagate.

**EXERCISE C** A wave starts at the left end of a long cord (see Fig. 15–1) when someone shakes the cord back and forth at the rate of 2.0 Hz. The wave is observed to move to the right at 4.0 m/s. If the frequency is increased from 2.0 to 3.0 Hz, the new speed of the wave is (*a*) 1.0 m/s, (*b*) 2.0 m/s, (*c*) 4.0 m/s, (*d*) 8.0 m/s, (*e*) 16.0 m/s.

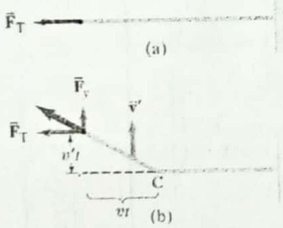



**FIGURE 15–7** Diagram of simple wave pulse on a cord for derivation of Eq. 15–2. The vector shown in (b) as the resultant of $\vec{F}_T + \vec{F}_y$ has to be directed along the cord because the cord is flexible. (Diagram is not to scale: we assume $v' \ll v$; the upward angle of the cord is exaggerated for visibility.)

We can make a simple derivation of Eq. 15–2 using a simple model of a cord under a tension $F_T$ as shown in Fig. 15–7a. The cord is pulled upward at a speed $v'$ by the force $F_y$. As shown in Fig. 15–7b all points of the cord to the left of point C move upward at the speed $v'$, and those to the right are still at rest. The speed of propagation, $v$, of this wave pulse is the speed of point C, the leading edge of the pulse. Point C moves to the right a distance $vt$ in a time $t$, whereas the end of the cord moves upward a distance $v't$. By similar triangles we have the approximate relation

$$\frac{F_T}{F_y} = \frac{vt}{v't} = \frac{v}{v'},$$

which is accurate for small displacements ($v't \ll vt$) so that $F_T$ does not change appreciably. As we saw in Chapter 9, the impulse given to an object is equal to its change in momentum. During the time $t$ the total upward impulse is $F_y t = (v'/v)F_T t$. The change in momentum of the cord, $\Delta p$, is the mass of cord moving upward times its velocity. Since the upward moving segment of cord has mass equal to the mass per unit length $\mu$ times its length $vt$ we have

$$F_y t = \Delta p$$

$$\frac{v'}{v} F_T t = (\mu v t)v'.$$

Solving for $v$ we find $v = \sqrt{F_T/\mu}$ which is Eq. 15–2. Although it was derived for a special case, it is valid for any wave shape since other shapes can be considered to be made up of many tiny such lengths. But it is valid only for small displacements (as was our derivation). Experiment is in accord with this result derived from Newtonian mechanics.

**EXAMPLE 15–2** **Pulse on a wire.** An 80.0-m-long, 2.10-mm-diameter copper wire is stretched between two poles. A bird lands at the center point of the wire, sending a small wave pulse out in both directions. The pulses reflect at the ends and arrive back at the bird's location 0.750 seconds after it landed. Determine the tension in the wire.

**APPROACH** From Eq. 15–2, the tension is given by $F_T = \mu v^2$. The speed $v$ is distance divided by the time. The mass per unit length $\mu$ is calculated from the density of copper and the dimensions of the wire.

**SOLUTION** Each wave pulse travels 40.0 m to the pole and back again (= 80.0 m) in 0.750 s. Thus their speed is $v = (80.0\,\mathrm{m})/(0.750\,\mathrm{s}) = 107\,\mathrm{m/s}$. We take (Table 13–1) the density of copper as $8.90 \times 10^3\,\mathrm{kg/m^3}$. The volume of copper in the wire is the cross-sectional area $(\pi r^2)$ times the length $\ell$, and the mass of the wire is the volume times the density: $m = \rho(\pi r^2)\ell$ for a wire of radius $r$ and length $\ell$. Then $\mu = m/\ell$ is

$$\mu = \rho\pi r^2\ell/\ell = \rho\pi r^2 = (8.90 \times 10^3\,\mathrm{kg/m^3})\pi(1.05 \times 10^{-3}\,\mathrm{m})^2 = 0.0308\,\mathrm{kg/m}.$$

Thus, the tension is $F_T = \mu v^2 = (0.0308\,\mathrm{kg/m})(107\,\mathrm{m/s})^2 = 353\,\mathrm{N}$.

## Velocity of Longitudinal Waves

The velocity of a longitudinal wave has a form similar to that for a transverse wave on a cord (Eq. 15–2); that is,

$$v = \sqrt{\frac{\text{elastic force factor}}{\text{inertia factor}}}.$$

In particular, for a longitudinal wave traveling down a long solid rod,

$$v = \sqrt{\frac{E}{\rho}}, \qquad \left[\begin{array}{l}\text{longitudinal}\\ \text{wave in a long rod}\end{array}\right] \tag{15–3}$$

where $E$ is the elastic modulus (Section 12–4) of the material and $\rho$ is its density. For a longitudinal wave traveling in a liquid or gas,

$$v = \sqrt{\frac{B}{\rho}}, \qquad \left[\begin{array}{l}\text{longitudinal wave}\\ \text{in a fluid}\end{array}\right] \tag{15–4}$$

where $B$ is the bulk modulus (Section 12–4) and $\rho$ again is the density.

PHYSICS APPLIED
Space perception by animals, using sound waves

**EXAMPLE 15–3** **Echolocation.** Echolocation is a form of sensory perception used by animals such as bats, toothed whales, and dolphins. The animal emits a pulse of sound (a longitudinal wave) which, after reflection from objects, returns and is detected by the animal. Echolocation waves can have frequencies of about 100,000 Hz. (*a*) Estimate the wavelength of a sea animal's echolocation wave. (*b*) If an obstacle is 100 m from the animal, how long after the animal emits a wave is its reflection detected?

**APPROACH** We first compute the speed of longitudinal (sound) waves in sea water, using Eq. 15–4 and Tables 12–1 and 13–1. The wavelength is $\lambda = v/f$.

**SOLUTION** (*a*) The speed of longitudinal waves in sea water, which is slightly more dense than pure water, is

$$v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{2.0 \times 10^9\,\mathrm{N/m^2}}{1.025 \times 10^3\,\mathrm{kg/m^3}}} = 1.4 \times 10^3\,\mathrm{m/s}.$$

Then, using Eq. 15–1, we find

$$\lambda = \frac{v}{f} = \frac{(1.4 \times 10^3\,\mathrm{m/s})}{(1.0 \times 10^5\,\mathrm{Hz})} = 14\,\mathrm{mm}.$$

(*b*) The time required for the round-trip between the animal and the object is

$$t = \frac{\text{distance}}{\text{speed}} = \frac{2(100\,\mathrm{m})}{1.4 \times 10^3\,\mathrm{m/s}} = 0.14\,\mathrm{s}.$$

**NOTE** We shall see later that waves can be used to "resolve" (or detect) objects only if the wavelength is comparable to or smaller than the object. Thus, a dolphin can resolve objects on the order of a centimeter or larger in size.

## Deriving Velocity of Wave in a Fluid

We now derive Eq. 15–4. Consider a wave pulse traveling in a fluid in a long tube, so that the wave motion is one dimensional. The tube is fitted with a piston at the end and is filled with a fluid which, at $t = 0$, is of uniform density $\rho$ and at uniform pressure $P_0$, Fig. 15–8a. At this moment the piston is abruptly made to start moving to the right with speed $v'$, compressing the fluid in front of it. In the (short) time $t$ the piston moves a distance $v't$. The compressed fluid itself also moves with speed $v'$, but the leading edge of the compressed region moves to the right at the characteristic speed $v$ of compression waves in that fluid; we assume the wave speed $v$ is much larger than the piston speed $v'$. The leading edge of the compression (which at $t = 0$ was at the piston face) thus moves a distance $vt$ in time $t$ as shown in Fig. 15–8b. Let the pressure in the compression be $P_0 + \Delta P$, which is $\Delta P$ higher than in the uncompressed fluid. To move the piston to the right requires an external force $(P_0 + \Delta P)S$ acting to the right, where $S$ is the cross-sectional area of the tube. ($S$ for "surface area"; we save $A$ for amplitude.) The *net* force on the compressed region of the fluid is

$$F_{\text{net}} = (P_0 + \Delta P)S - P_0 S = S\,\Delta P$$

since the uncompressed fluid exerts a force $P_0 S$ to the left at the leading edge. Hence the impulse given to the compressed fluid, which equals its change in momentum, is

$$F_{\text{net}}\, t = \Delta m v'$$
$$S\,\Delta P\, t = (\rho S v t)v',$$

where $(\rho S v t)$ represents the mass of fluid which is given the speed $v'$ (the compressed fluid of area $S$ moves a distance $vt$, Fig. 15–8, so the volume moved is $Svt$). Hence we have

$$\Delta P = \rho v v'.$$

From the definition of the bulk modulus, $B$ (Eq. 12–7):

$$B = -\frac{\Delta P}{\Delta V/V_0} = -\frac{\rho v v'}{\Delta V/V_0},$$

where $\Delta V/V_0$ is the fractional change in volume due to compression. The original volume of the compressed fluid is $V_0 = Svt$ (see Fig. 15–8), and it has been compressed by an amount $\Delta V = -Sv't$ (Fig. 15–8b). Thus

$$B = -\frac{\rho v v'}{\Delta V/V_0} = -\rho v v'\left(\frac{Svt}{-Sv't}\right) = \rho v^2,$$

and so

$$v = \sqrt{\frac{B}{\rho}},$$

which is what we set out to show, Eq. 15–4.

The derivation of Eq. 15–3 follows similar lines, but takes into account the expansion of the sides of a rod when the end of the rod is compressed.

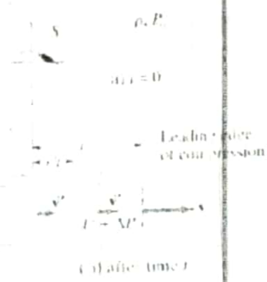


FIGURE 15–8 Determining the speed of a one-dimensional longitudinal wave in a fluid contained in a long narrow tube.

## Other Waves

PHYSICS APPLIED
*Earthquake waves*

Both transverse and longitudinal waves are produced when an **earthquake** occurs. The transverse waves that travel through the body of the Earth are called S waves (S for shear), and the longitudinal waves are called P waves (P for pressure) or *compression* waves. Both longitudinal and transverse waves can travel through a solid since the atoms or molecules can vibrate about their relatively fixed positions in any direction. But only longitudinal waves can propagate through a fluid, because any transverse motion would not experience any restoring force since a fluid is readily deformable. This fact was used by geophysicists to infer that a portion of the Earth's core must be liquid: after an earthquake, longitudinal waves are detected diametrically across the Earth, but not transverse waves.

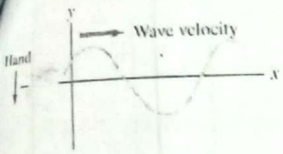



FIGURE 15–14 Example 15–5. The wave at $t = 0$ (the hand is falling). Not to scale.

**EXAMPLE 15–5 A traveling wave.** The left-hand end of a long horizontal stretched cord oscillates transversely in SHM with frequency $f = 250\,\text{Hz}$ and amplitude 2.6 cm. The cord is under a tension of 140 N and has a linear density $\mu = 0.12\,\text{kg/m}$. At $t = 0$, the end of the cord has an upward displacement of 1.6 cm and is falling (Fig. 15–14). Determine (*a*) the wavelength of waves produced and (*b*) the equation for the traveling wave.

**APPROACH** We first find the phase velocity of the transverse wave from Eq. 15–2; then $\lambda = v/f$. In (*b*), we need to find the phase $\phi$ using the initial conditions.

**SOLUTION** (*a*) The wave velocity is

$$v = \sqrt{\frac{F_T}{\mu}} = \sqrt{\frac{140\,\text{N}}{0.12\,\text{kg/m}}} = 34\,\text{m/s}.$$

Then

$$\lambda = \frac{v}{f} = \frac{34\,\text{m/s}}{250\,\text{Hz}} = 0.14\,\text{m} \quad \text{or} \quad 14\,\text{cm}.$$

(*b*) Let $x = 0$ at the left-hand end of the cord. The phase of the wave at $t = 0$ is not zero in general as was assumed in Eqs. 15–9, 10, and 13. The general form for a wave traveling to the right is

$$D(x, t) = A\sin(kx - \omega t + \phi),$$

where $\phi$ is the phase angle. In our case, the amplitude $A = 2.6\,\text{cm}$; and at $t = 0$, $x = 0$, we are given $D = 1.6\,\text{cm}$. Thus

$$1.6 = 2.6\sin\phi,$$

so $\phi = \sin^{-1}(1.6/2.6) = 38° = 0.66\,\text{rad}$. We also have $\omega = 2\pi f = 1570\,\text{s}^{-1}$ and $k = 2\pi/\lambda = 2\pi/0.14\,\text{m} = 45\,\text{m}^{-1}$. Hence

$$D = (0.026\,\text{m})\sin\left[(45\,\text{m}^{-1})x - (1570\,\text{s})t + 0.66\right]$$

which we can write more simply as

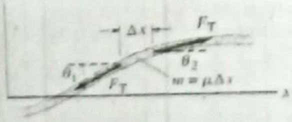
$$D = 0.026\sin(45x - 1570t + 0.66),$$

and we specify clearly that $D$ and $x$ are in meters and $t$ in seconds.

## *15–5 The Wave Equation

Many types of waves satisfy an important general equation that is the equivalent of Newton's second law of motion for particles. This "equation of motion for a wave" is called the **wave equation**, and we derive it now for waves traveling on a stretched horizontal string.

FIGURE 15–15 Deriving the wave equation from Newton's second law: a segment of string under tension $F_T$.




We assume the amplitude of the wave is small compared to the wavelength so that each point on the string can be assumed to move only vertically and the tension in the string, $F_T$, does not vary during a vibration. We apply Newton's second law, $\Sigma F = ma$, to the vertical motion of a tiny section of the string as shown in Fig. 15–15. The amplitude of the wave is small, so the angles $\theta_1$ and $\theta_2$ that the string makes with the horizontal are small. The length of this section is then approximately $\Delta x$, and its mass is $\mu\,\Delta x$, where $\mu$ is the mass per unit length of the string. The net vertical force on this section of string is $F_T\sin\theta_2 - F_T\sin\theta_1$. So Newton's second law applied to the vertical ($y$) direction gives

$$\Sigma F_y = ma_y$$

$$F_T\sin\theta_2 - F_T\sin\theta_1 = (\mu\,\Delta x)\frac{\partial^2 D}{\partial t^2}. \quad \text{(i)}$$

We have written the acceleration as $a_y = \partial^2 D/\partial t^2$ since the motion is only vertical, and we use the partial derivative notation because the displacement $D$ is a function of both $x$ and $t$.

Because the angles $\theta_1$ and $\theta_2$ are assumed small, $\sin\theta \approx \tan\theta$ and $\tan\theta$ is equal to the slope $s$ of the string at each point:

$$\sin\theta \approx \tan\theta = \frac{\partial D}{\partial x} = s.$$

Thus our equation (i) at the bottom of the previous page becomes

$$F_T(s_2 - s_1) = \mu\,\Delta x\,\frac{\partial^2 D}{\partial t^2}$$

or

$$F_T\frac{\Delta s}{\Delta x} = \mu\frac{\partial^2 D}{\partial t^2}, \qquad \text{(ii)}$$

where $\Delta s = s_2 - s_1$ is the difference in the slope between the two ends of our tiny section. Now we take the limit of $\Delta x \to 0$, so that

$$F_T\lim_{\Delta x\to 0}\frac{\Delta s}{\Delta x} = F_T\frac{\partial s}{\partial x}$$
$$= F_T\frac{\partial}{\partial x}\left(\frac{\partial D}{\partial x}\right) = F_T\frac{\partial^2 D}{\partial x^2}$$

since the slope $s = \partial D/\partial x$, as we wrote above. Substituting this into the equation labeled (ii) above gives

$$F_T\frac{\partial^2 D}{\partial x^2} = \mu\frac{\partial^2 D}{\partial t^2}$$

or

$$\frac{\partial^2 D}{\partial x^2} = \frac{\mu}{F_T}\frac{\partial^2 D}{\partial t^2}.$$

We saw earlier in this Chapter (Eq. 15–2) that the velocity of waves on a string is given by $v = \sqrt{F_T/\mu}$, so we can write this last equation as

$$\frac{\partial^2 D}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 D}{\partial t^2}. \qquad \text{(15–16)}$$

This is the **one-dimensional wave equation**, and it can describe not only small amplitude waves on a stretched string, but also small amplitude longitudinal waves (such as sound waves) in gases, liquids, and elastic solids, in which case $D$ can refer to the pressure variations. In this case, the wave equation is a direct consequence of Newton's second law applied to a continuous elastic medium. The wave equation also describes electromagnetic waves for which $D$ refers to the electric or magnetic field, as we shall see in Chapter 31. Equation 15–16 applies to waves traveling in one dimension only. For waves spreading out in three dimensions, the wave equation is the same, with the addition of $\partial^2 D/\partial y^2$ and $\partial^2 D/\partial z^2$ to the left side of Eq. 15–16.

The wave equation is a *linear* equation: the displacement $D$ appears singly in each term. There are no terms that contain $D^2$, or $D(\partial D/\partial x)$, or the like in which $D$ appears more than once. Thus, if $D_1(x,t)$ and $D_2(x,t)$ are two different solutions of the wave equation, then the linear combination

$$D_3(x,t) = aD_1(x,t) + bD_2(x,t),$$

where $a$ and $b$ are constants, is also a solution. This is readily seen by direct substitution into the wave equation. This is the essence of the *superposition principle*, which we discuss in the next Section. Basically it says that if two waves pass through the same region of space at the same time, the actual displacement is the sum of the separate displacements. For waves on a string, or for sound waves, this is valid only for small-amplitude waves. If the amplitude is not small enough, the equations for wave propagation may become nonlinear and the principle of superposition would not hold and more complicated effects may occur.

**EXAMPLE 15–6** **Wave equation solution.** Verify that the sinusoidal wave of Eq. 15–10c, $D(x, t) = A\sin(kx - \omega t)$, satisfies the wave equation.

**APPROACH** We substitute Eq. 15–10c into the wave equation, Eq. 15–16.

**SOLUTION** We take the derivative of Eq. 15–10c twice with respect to $t$:

$$\frac{\partial D}{\partial t} = -\omega A\cos(kx - \omega t)$$

$$\frac{\partial^2 D}{\partial t^2} = -\omega^2 A\sin(kx - \omega t).$$

With respect to $x$, the derivatives are

$$\frac{\partial D}{\partial x} = kA\cos(kx - \omega t)$$

$$\frac{\partial^2 D}{\partial x^2} = -k^2 A\sin(kx - \omega t).$$

If we now divide the second derivatives we get

$$\frac{\partial^2 D/\partial t^2}{\partial^2 D/\partial x^2} = \frac{-\omega^2 A\sin(kx - \omega t)}{-k^2 A\sin(kx - \omega t)} = \frac{\omega^2}{k^2}.$$

From Eq. 15–12 we have $\omega^2/k^2 = v^2$, so we see that Eq. 15–10 does satisfy the wave equation (Eq. 15–16).

## 15–6 The Principle of Superposition

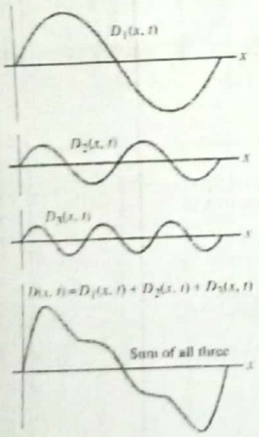
When two or more waves pass through the same region of space at the same time, it is found that for many waves *the actual displacement is the vector (or algebraic) sum of the separate displacements*. This is called the **principle of superposition**. It is valid for mechanical waves as long as the displacements are not too large and there is a linear relationship between the displacement and the restoring force of the oscillating medium.[1] If the amplitude of a mechanical wave, for example, is so large that it goes beyond the elastic region of the medium, and Hooke's law is no longer operative, the superposition principle is no longer accurate.[2] For the most part, we will consider systems for which the superposition principle can be assumed to hold.

One result of the superposition principle is that if two waves pass through the same region of space, they continue to move independently of one another. You may have noticed, for example, that the ripples on the surface of water (two-dimensional waves) that form from two rocks striking the water at different places will pass through each other.

Figure 15–16 shows an example of the superposition principle. In this case there are three waves present, on a stretched string, each of different amplitude and frequency. At any time, such as at the instant shown, the actual amplitude at any position $x$ is the algebraic sum of the amplitude of the three waves at that position. The actual wave is not a simple sinusoidal wave and is called a *composite* (or *complex*) *wave*. (Amplitudes are exaggerated in Fig. 15–16.)

It can be shown that any complex wave can be considered as being composed of many simple sinusoidal waves of different amplitudes, wavelengths, and frequencies. This is known as *Fourier's theorem*. A complex periodic wave of period $T$ can be represented as a sum of pure sinusoidal terms whose frequencies are integral multiples of $f = 1/T$. If the wave is not periodic, the sum becomes an integral (called a *Fourier integral*). Although we will not go into the details here, we see the importance of considering sinusoidal waves (and simple harmonic motion): because any other wave shape can be considered a sum of such pure sinusoidal waves.

FIGURE 15–16 The superposition principle for one-dimensional waves. Composite wave formed from three sinusoidal waves of different amplitudes and frequencies ($f_0$, $2f_0$, $3f_0$) at a certain instant in time. The amplitude of the composite wave at each point in space, at any time, is the algebraic sum of the amplitudes of the component waves. Amplitudes are shown exaggerated; for the superposition principle to hold, they must be small compared to the wavelengths.




[1] For electromagnetic waves in vacuum, Chapter 31, the superposition principle always holds.

[2] Intermodulation distortion in high-fidelity equipment is an example of the superposition principle not holding when two frequencies do not combine linearly in the electronics.

The phenomenon of beats can occur with any kind of wave and is a very sensitive method for comparing frequencies. For example, to tune a piano, a piano tuner listens for beats produced between his standard tuning fork and that of a particular string on the piano, and knows it is in tune when the beats disappear. The members of an orchestra tune up by listening for beats between their instruments and that of a standard tone (usually A above middle C at 440 Hz) produced by a piano or an oboe. A beat frequency is perceived as an intensity modulation (a wavering between loud and soft) for beat frequencies below 20 Hz or so, and as a separate low tone for higher beat frequencies (audible if the tones are strong enough).

PHYSICS APPLIED

**EXAMPLE 16–13** **Beats.** A tuning fork produces a steady 400-Hz tone. When this tuning fork is struck and held near a vibrating guitar string, twenty beats are counted in five seconds. What are the possible frequencies produced by the guitar string?

**APPROACH** For beats to occur, the string must vibrate at a frequency different from 400 Hz by whatever the beat frequency is.

**SOLUTION** The beat frequency is

$$f_{\text{beat}} = 20\ \text{vibrations}/5\ \text{s} = 4\ \text{Hz}.$$

This is the difference of the frequencies of the two waves. Because one wave is known to be 400 Hz, the other must be either 404 Hz or 396 Hz.

## 16–7 Doppler Effect

You may have noticed that you hear the pitch of the siren on a speeding fire truck drop abruptly as it passes you. Or you may have noticed the change in pitch of a blaring horn on a fast-moving car as it passes by you. The pitch of the engine noise of a racecar changes as the car passes an observer. When a source of sound is moving toward an observer, the pitch the observer hears is higher than when the source is at rest; and when the source is traveling away from the observer, the pitch is lower. This phenomenon is known as the **Doppler effect**[†] and occurs for all types of waves. Let us now see why it occurs, and calculate the difference between the perceived and source frequencies when there is relative motion between source and observer.

Consider the siren of a fire truck at rest, which is emitting sound of a particular frequency in all directions as shown in Fig. 16–18a. The sound waves are moving at the speed of sound in air, $v_{\text{snd}}$, which is independent of the velocity of the source or observer. If our source, the fire truck, is moving, the siren emits sound at the same frequency as it does at rest. But the sound wavefronts it emits forward, in front of it, are closer together than when the fire truck is at rest, as shown in Fig. 16–18b. This is because the fire truck, as it moves, is "chasing" the previously emitted wavefronts, and emits each crest closer to the previous one. Thus an observer on the sidewalk in front of the truck will detect more wave crests passing per second, so the frequency heard is higher. The wavefronts emitted behind the truck, on the other hand, are farther apart than when the truck is at rest because the truck is speeding away from them. Hence, fewer wave crests per second pass by an observer behind the moving truck (Fig. 16–18b) and the perceived pitch is lower.

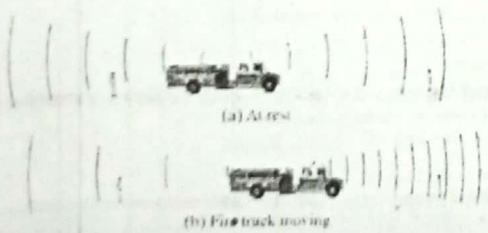



FIGURE 16–18 (a) Both observers on the sidewalk hear the same frequency from a fire truck at rest. (b) Doppler effect: observer toward whom the fire truck moves hears a higher-frequency sound, and observer behind the fire truck hears a lower-frequency sound.

[†] After J. C. Doppler (1803–1853).

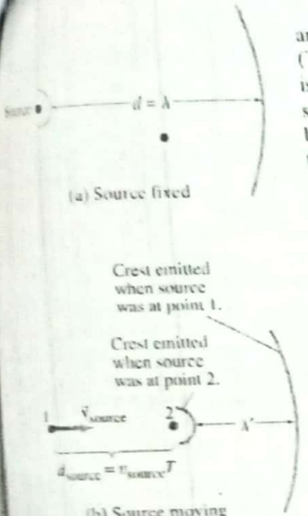



FIGURE 16–19 Determination of the frequency shift in the Doppler effect (see text). The red dot is the source.

We can calculate the frequency shift perceived by making use of Fig. 16–19, and we assume the air (or other medium) is at rest in our reference frame. (The stationary observer is off to the right.) In Fig. 16–19a, the source of the sound is shown as a red dot, and is at rest. Two successive wave crests are shown, the second of which has just been emitted and so is still near the source. The distance between these crests is $\lambda$, the wavelength. If the frequency of the source is $f$, then the time between emissions of wave crests is

$$T = \frac{1}{f} = \frac{\lambda}{v_{snd}}.$$

In Fig. 16–19b, the source is moving with a velocity $v_{source}$ toward the observer. In a time $T$ (as just defined), the first wave crest has moved a distance $d = v_{snd}T = \lambda$, where $v_{snd}$ is the velocity of the sound wave in air (which is the same whether the source is moving or not). In this same time, the source has moved a distance $d_{source} = v_{source}T$. Then the distance between successive wave crests, which is the wavelength $\lambda'$ the observer will perceive, is

$$\begin{aligned}\lambda' &= d - d_{source} \\ &= \lambda - v_{source}T \\ &= \lambda - v_{source}\frac{\lambda}{v_{snd}} \\ &= \lambda\left(1 - \frac{v_{source}}{v_{snd}}\right).\end{aligned}$$

We subtract $\lambda$ from both sides of this equation and find that the shift in wavelength, $\Delta\lambda$, is

$$\Delta\lambda = \lambda' - \lambda = -\lambda\frac{v_{source}}{v_{snd}}.$$

So the shift in wavelength is directly proportional to the source speed $v_{source}$. The frequency $f'$ that will be perceived by our stationary observer on the ground is given by

$$f' = \frac{v_{snd}}{\lambda'} = \frac{v_{snd}}{\lambda\left(1 - \frac{v_{source}}{v_{snd}}\right)}.$$

Since $v_{snd}/\lambda = f$, then

$$f' = \frac{f}{\left(1 - \frac{v_{source}}{v_{snd}}\right)}. \qquad \left[\begin{array}{l}\text{source moving toward}\\ \text{stationary observer}\end{array}\right] \qquad (16\text{–}9a)$$

Because the denominator is less than 1, the observed frequency $f'$ is greater than the source frequency $f$. That is, $f' > f$. For example, if a source emits a sound of frequency 400 Hz when at rest, then when the source moves toward a fixed observer with a speed of 30 m/s, the observer hears a frequency (at 20°C) of

$$f' = \frac{400\ \text{Hz}}{1 - \frac{30\ \text{m/s}}{343\ \text{m/s}}} = 438\ \text{Hz}.$$

Now consider a source moving *away* from the stationary observer at a speed $v_{source}$. Using the same arguments as above, the wavelength $\lambda'$ perceived by our observer will have the minus sign on $d_{source}$ (second equation on this page) changed to plus:

$$\begin{aligned}\lambda' &= d + d_{source} \\ &= \lambda\left(1 + \frac{v_{source}}{v_{snd}}\right).\end{aligned}$$

The difference between the observed and emitted wavelengths will be $\Delta\lambda = \lambda' - \lambda = +\lambda(v_{source}/v_{snd})$. The observed frequency of the wave, $f' = v_{snd}/\lambda'$, will be

$$f' = \frac{f}{\left(1 + \frac{v_{source}}{v_{snd}}\right)}. \qquad \left[\begin{array}{l}\text{source moving away from}\\ \text{stationary observer}\end{array}\right] \qquad (16\text{–}9b)$$

If a source emitting at 400 Hz is moving away from a fixed observer at 30 m/s, the observer hears a frequency $f' = (400\ \text{Hz})/[1 + (30\ \text{m/s})/(343\ \text{m/s})] = 368\ \text{Hz}$.

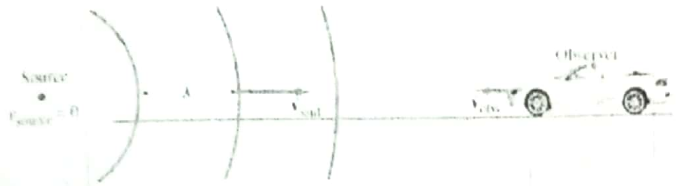


FIGURE 16–20 Observer moving with speed $v_{\text{obs}}$ toward a stationary source detects wave crests passing at speed $v' = v_{\text{snd}} + v_{\text{obs}}$ where $v_{\text{snd}}$ is the speed of the sound waves in air.

The Doppler effect also occurs when the source is at rest and the observer is in motion. If the observer is traveling *toward* the source, the pitch heard is higher than that of the emitted source frequency. If the observer is traveling *away* from the source, the pitch heard is lower. Quantitatively the change in frequency is different than for the case of a moving source. With a fixed source and a moving observer, the distance between wave crests, the wavelength $\lambda$, is not changed. But the velocity of the crests with respect to the observer *is* changed. If the observer is moving toward the source, Fig. 16–20, the speed $v'$ of the waves relative to the observer is a simple addition of velocities: $v' = v_{\text{snd}} + v_{\text{obs}}$, where $v_{\text{snd}}$ is the velocity of sound in air (we assume the air is still) and $v_{\text{obs}}$ is the velocity of the observer. Hence, the frequency heard is

$$f' = \frac{v'}{\lambda} = \frac{v_{\text{snd}} + v_{\text{obs}}}{\lambda}.$$

Because $\lambda = v_{\text{snd}}/f$, then

$$f' = \frac{(v_{\text{snd}} + v_{\text{obs}})f}{v_{\text{snd}}},$$

or

$$f' = \left(1 + \frac{v_{\text{obs}}}{v_{\text{snd}}}\right)f. \qquad \left[\begin{array}{l}\text{observer moving toward}\\ \text{stationary source}\end{array}\right] \qquad \text{(16–10a)}$$

If the observer is moving away from the source, the relative velocity is $v' = v_{\text{snd}} - v_{\text{obs}}$, so

$$f' = \left(1 - \frac{v_{\text{obs}}}{v_{\text{snd}}}\right)f. \qquad \left[\begin{array}{l}\text{observer moving away}\\ \text{from stationary source}\end{array}\right] \qquad \text{(16–10b)}$$

**EXAMPLE 16–14** **A moving siren.** The siren of a police car at rest emits at a predominant frequency of 1600 Hz. What frequency will you hear if you are at rest and the police car moves at 25.0 m/s (*a*) toward you, and (*b*) away from you?

**APPROACH** The observer is fixed, and the source moves, so we use Eqs. 16–9. The frequency you (the observer) hear is the emitted frequency $f$ divided by the factor $(1 \pm v_{\text{source}}/v_{\text{snd}})$ where $v_{\text{source}}$ is the speed of the police car. Use the minus sign when the car moves toward you (giving a higher frequency); use the plus sign when the car moves away from you (lower frequency).

**SOLUTION** (*a*) The car is moving toward you, so (Eq. 16–9a)

$$f' = \frac{f}{\left(1 - \dfrac{v_{\text{source}}}{v_{\text{snd}}}\right)} = \frac{1600\ \text{Hz}}{\left(1 - \dfrac{25.0\ \text{m/s}}{343\ \text{m/s}}\right)} = 1726\ \text{Hz} \approx 1730\ \text{Hz}.$$

(*b*) The car is moving away from you, so (Eq. 16–9b)

$$f' = \frac{f}{\left(1 + \dfrac{v_{\text{source}}}{v_{\text{snd}}}\right)} = \frac{1600\ \text{Hz}}{\left(1 + \dfrac{25.0\ \text{m/s}}{343\ \text{m/s}}\right)} = 1491\ \text{Hz} \approx 1490\ \text{Hz}.$$

**EXERCISE G** Suppose the police car of Example 16–14 is at rest and emits at 1600 Hz. What frequency would you hear if you were moving at 25.0 m/s (*a*) toward it, and (*b*) away from it?

When a sound wave is reflected from a moving obstacle, the frequency of the reflected wave will, because of the Doppler effect, be different from that of the incident wave. This is illustrated in the following Example.

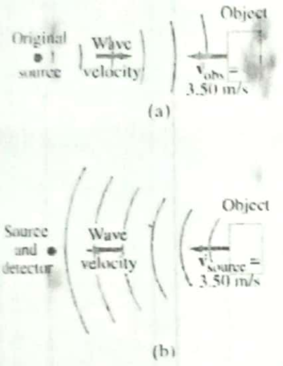



FIGURE 16–21 Example 16–15.

**EXAMPLE 16–15** **Two Doppler shifts.** A 5000-Hz sound wave is emitted by a stationary source. This sound wave reflects from an object moving 3.50 m/s toward the source (Fig. 16–21). What is the frequency of the wave reflected by the moving object as detected by a detector at rest near the source?

**APPROACH** There are actually two Doppler shifts in this situation. First, the moving object acts like an observer moving toward the source with speed $v_{obs} = 3.50\,\mathrm{m/s}$ (Fig. 16–21a) and so "detects" a sound wave of frequency (Eq. 16–10a) $f' = f[1 + (v_{obs}/v_{snd})]$. Second, reflection of the wave from the moving object is equivalent to the object reemitting the wave, acting effectively as a moving source with speed $v_{source} = 3.50\,\mathrm{m/s}$ (Fig. 16–21b). The final frequency detected, $f''$, is given by $f'' = f'/[1 - v_{source}/v_{snd}]$, Eq. 16–9a.

**SOLUTION** The frequency $f'$ that is "detected" by the moving object is (Eq. 16–10a):

$$f' = \left(1 + \frac{v_{obs}}{v_{snd}}\right)f = \left(1 + \frac{3.50\,\mathrm{m/s}}{343\,\mathrm{m/s}}\right)(5000\,\mathrm{Hz}) = 5051\,\mathrm{Hz}.$$

The moving object now "emits" (reflects) a sound of frequency (Eq. 16–9a)

$$f'' = \frac{f'}{\left(1 - \dfrac{v_{source}}{v_{snd}}\right)} = \frac{5051\,\mathrm{Hz}}{\left(1 - \dfrac{3.50\,\mathrm{m/s}}{343\,\mathrm{m/s}}\right)} = 5103\,\mathrm{Hz}.$$

Thus the frequency shifts by 103 Hz.

**NOTE** Bats use this technique to be aware of their surroundings. This is also the principle behind Doppler radar as speed-measuring devices for vehicles and other objects.

PHYSICS APPLIED
*Doppler blood-flow meter and other medical uses*

The incident wave and the reflected wave in Example 16–15, when mixed together (say, electronically), interfere with one another and beats are produced. The beat frequency is equal to the difference in the two frequencies, 103 Hz. This Doppler technique is used in a variety of medical applications, usually with ultrasonic waves in the megahertz frequency range. For example, ultrasonic waves reflected from red blood cells can be used to determine the velocity of blood flow. Similarly, the technique can be used to detect the movement of the chest of a young fetus and to monitor its heartbeat.

For convenience, we can write Eqs. 16–9 and 16–10 as a single equation that covers all cases of both source and observer in motion:

$$f' = f\left(\frac{v_{snd} \pm v_{obs}}{v_{snd} \mp v_{source}}\right). \qquad \left[\begin{array}{l}\text{source and}\\ \text{observer moving}\end{array}\right] \qquad \text{(16–11)}$$

PROBLEM SOLVING
*Getting the signs right*

To get the signs right, recall from your own experience that the frequency is higher when observer and source approach each other, and lower when they move apart. Thus the upper signs in numerator and denominator apply if source and/or observer move toward each other; the lower signs apply if they are moving apart.

**EXERCISE H** How fast would a source have to approach an observer for the observed frequency to be one octave above (twice) the produced frequency? (*a*) $\frac{1}{2}v_{snd}$, (*b*) $v_{snd}$, (*c*) $2v_{snd}$, (*d*) $4v_{snd}$.

## Doppler Effect for Light

The Doppler effect occurs for other types of waves as well. Light and other types of electromagnetic waves (such as radar) exhibit the Doppler effect: although the formulas for the frequency shift are not identical to Eqs. 16–9 and 16–10, as discussed in Chapter 44, the effect is similar. One important application is for weather forecasting using radar. The time delay between the emission of radar pulses and their reception after being reflected off raindrops gives the position of precipitation. Measuring the Doppler shift in frequency (as in Example 16–15) tells how fast the storm is moving and in which direction.

PHYSICS APPLIED

Another important application is to astronomy, where the velocities of distant galaxies can be determined from the Doppler shift. Light from distant galaxies is shifted toward lower frequencies, indicating that the galaxies are moving away from us. This is called the **redshift** since red has the lowest frequency of visible light. The greater the frequency shift, the greater the velocity of recession. It is found that the farther the galaxies are from us, the faster they move away. This observation is the basis for the idea that the universe is expanding, and is one basis for the idea that the universe began as a great explosion, affectionately called the "Big Bang" (Chapter 44).

PHYSICS APPLIED

## *16–8 Shock Waves and the Sonic Boom

An object such as an airplane traveling faster than the speed of sound is said to have a **supersonic speed**. Such a speed is often given as a **Mach**[*] number, which is defined as the ratio of the speed of the object to the speed of sound in the surrounding medium. For example, a plane traveling 600 m/s high in the atmosphere, where the speed of sound is only 300 m/s, has a speed of Mach 2.

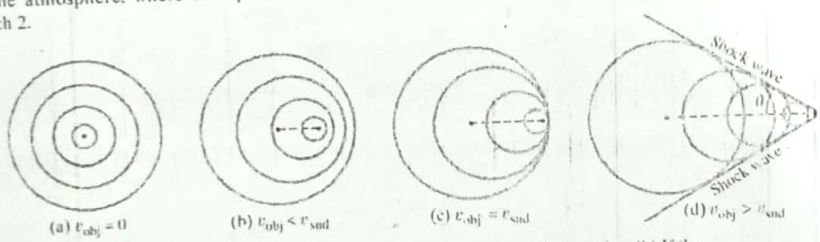
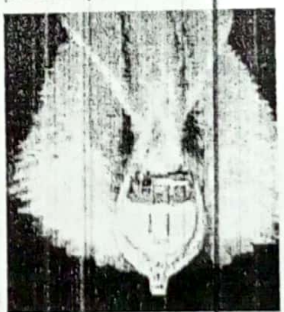



FIGURE 16–22 Sound waves emitted by an object (a) at rest or (b, c, and d) moving. (b) If the object's velocity is less than the velocity of sound, the Doppler effect occurs; (d) if its velocity is greater than the velocity of sound, a shock wave is produced.

When a source of sound moves at subsonic speeds (less than the speed of sound), the pitch of the sound is altered as we have seen (the Doppler effect); see also Fig. 16–22a and b. But if a source of sound moves faster than the speed of sound, a more dramatic effect known as a **shock wave** occurs. In this case, the source is actually "outrunning" the waves it produces. As shown in Fig. 16–22c, when the source is traveling *at* the speed of sound, the wave fronts it emits in the forward direction "pile up" directly in front of it. When the object moves faster, at a supersonic speed, the wave fronts pile up on one another along the sides, as shown in Fig. 16–22d. The different wave crests overlap one another and form a single very large crest which is the shock wave. Behind this very large crest there is usually a very large trough. A shock wave is essentially the result of constructive interference of a large number of wave fronts. A shock wave in air is analogous to the bow wave of a boat traveling faster than the speed of the water waves it produces, Fig. 16–23.

FIGURE 16–23 Bow waves produced by a boat.



[*] After the Austrian physicist Ernst Mach (1838–1916).

# Lectures

**BS Computer Science and Software Engineering**

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| **Session:** | **2020** |
| **Subject:** | **Applied Physics** |
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