Cuadratura de Gauss

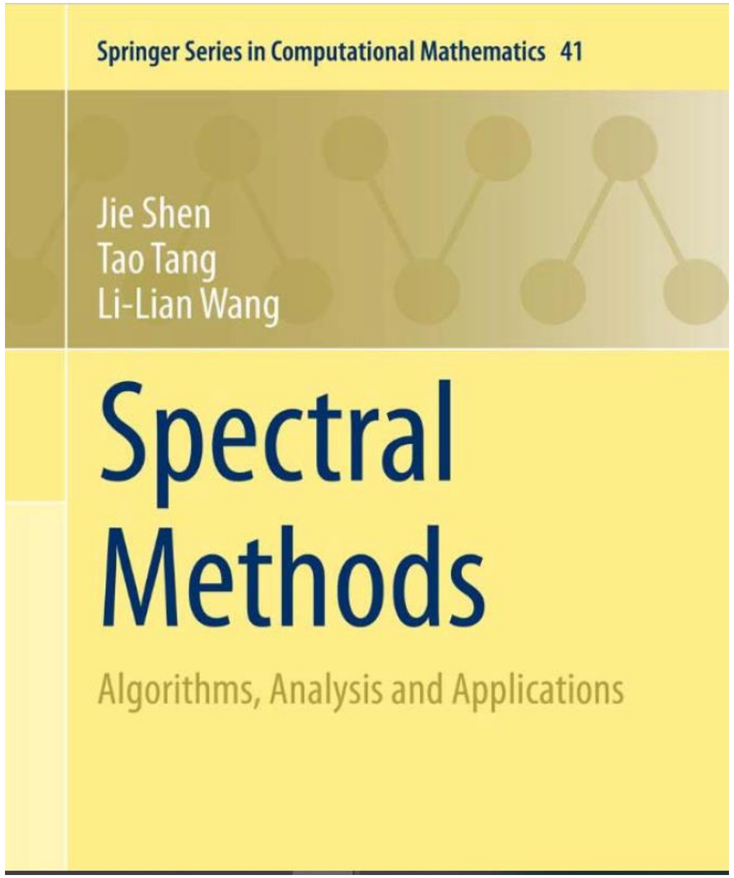
PhD. Alejandro Paredes

Cuadratura de Gauss

Deseamos aproximar *I* a una suma ponderada de evaluaciones $f(x)$ sobre $N+1$ puntos

$$I = \int_a^b \omega(x) f(x) \approx \sum_{j=0}^{N} w_j f(x_j)$$

- Los w_j se denominan funciones de peso y los puntos x_j se escogen apropiadamente.
- Los límites de integración $a, b \in R$ e inclusive pueden ser $\pm\infty$.
- Referencia: Shen,Tang,Wang,Spectral Methods, Algorithms, Analysis and Applications.



Polinomios ortogonales

Sobre $I = (a, b)\ (-\infty\ \leq a < b \leq +\infty)$, la función genérica de peso ω sea tal que

$$\omega(x) > 0, \forall\, x \in I\ \ y\, \omega \in L^1(I)$$

Dos funciones $f(x)$ y $g(x)$ son ortogonales respecto de ω, si

$$(f, g)_\omega = \int_a^b \omega(x) f(x) g(x) dx = 0$$

Un polinomio de grado n

$$p_n(x) = k_n x^n + k_{n-1} x^{n-1} + \cdots + k_0, \qquad k_n \neq 0$$

Una familia de polinomios se dice ortogonales, si

$$(p_n, p_m)_\omega = \int_a^b \omega(x) p_n(x) p_m(x) dx = \gamma_n\, \delta_{mn} \qquad \gamma_n = \|p_n\|_\omega^2 \neq 0$$

Polinomios ortogonales

Polinomio	**Intervalo**	**Función peso $\boldsymbol{\omega(x)}$**	$\boldsymbol{\gamma_n}$
Legendre	$[-1{,}1]$	1	$\dfrac{2}{2n+1}$
Laguerre ($\alpha \succ 1$)	$[0,+\infty]$	$x^{\alpha}e^{-x}$	$\dfrac{\Gamma(n+\alpha+1)}{n!}$
Hermite	$[-\infty,\infty]$	e^{-x^2}	$\sqrt{\pi}2^n n!$
Chebyshev	$[-1{,}1]$	$\dfrac{1}{\sqrt{1-x^2}}$	$\dfrac{c_n\pi}{2}$ $c_0 = 2, c_n = 1\ n \geq 1$

$$\bar{p}_{n+1} = (x-\alpha_n)\bar{p}_n - \beta_n\bar{p}_{n-1} \qquad \boxed{\alpha_n = \frac{(x\bar{p}_n,\bar{p}_n)_\omega}{\|\bar{p}_n\|_\omega^2} \quad n \geq 0} \qquad \boxed{\beta_n = \frac{\|\bar{p}_n\|_\omega^2}{\|\bar{p}_{n-1}\|_\omega^2} \quad n \geq 1}$$

$$\bar{p}_0 = 1\,,\bar{p}_1 = x - \alpha_0$$

Cuadratura de Gauss

Deseamos aproximar la integral utilizando $N+1$ puntos sobre el intervalo $[a,b]$

$$\int_a^b \omega(x) f(x) = \sum_{j=0}^{N} \omega_j f(x_j) + E_N[f]$$

Error de la cuadratura: $$E_N[f] = \frac{1}{(N+1)!}\int_a^b f^{N+1}(\zeta(x)) \prod_{i=0}^{N} (x - x_i)\, dx \qquad \zeta(x) \in [a,b]$$

Si tomamos $f(x) = h_j(x)$ (pol de Lagrange $h(x)$ de grado N interpolado en los puntos x_j)

$$h_j(x) = \prod_{i=0;\, i\neq j}^{N} \frac{(x - x_i)}{(x_j - x_i)} \Rightarrow E_N[f] = 0 \quad f(x_j) = 1$$

$$f(x_i) = 0 \ \forall\, i \neq j$$

$$\Longrightarrow$$

$$\omega_j = \int_a^b h_j\, \omega(x)\, dx \quad 0 \leq j \leq N$$

Cuadratura de Gauss

Teorema : Sean los puntos $\{x_j\}_{j=0}^{N}$ las raíces del polinmio orthogonal (respecto de $\omega(x)$) p_{N+1}. Entonces existe un único conjunto de $\{\omega_j\}_{j=0}^{N}$ definido por

$$\omega_j = \int_a^b h_j\,\omega(x)dx \quad 0 \leq j \leq N$$

tal que

$$\int_a^b \omega(x)p(x) = \sum_{j=0}^{N} \omega_j p(x_j) \qquad \forall\, p \in P_{2N+1}$$

$$\boxed{\int_a^b \omega(x)f(x)dx \approx \sum_{j=0}^{N} \omega_j f(x_j)}$$

Además

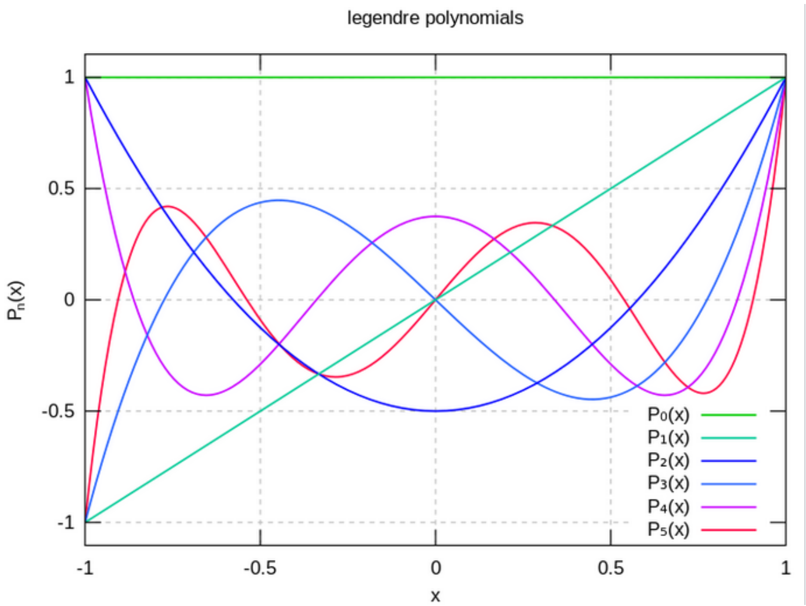
$$\omega_j = \frac{k_{N+1}}{k_N}\frac{\|p_N\|_\omega^2}{p_N(x_j){p'}_{N+1}(x_j)}\; 0 \leq j \leq N$$

Cuadratura de Gauss-Legendre

$N+1$	Raíces $L_{N+1}(x)$		ω	
1	0		2	
2	$\pm\frac{1}{\sqrt{3}}$	±0.57735…	1	
3	0		$\frac{8}{9}$	0.888889…
	$\pm\sqrt{\frac{3}{5}}$	±0.774597…	$\frac{5}{9}$	0.555556…
4	$\pm\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}}$	±0.339981…	$\frac{18+\sqrt{30}}{36}$	0.652145…
	$\pm\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}}$	±0.861136…	$\frac{18-\sqrt{30}}{36}$	0.347855…
5	0		$\frac{128}{225}$	0.568889…
	$\pm\frac{1}{3}\sqrt{5-2\sqrt{\frac{10}{7}}}$	±0.538469…	$\frac{322+13\sqrt{70}}{900}$	0.478629…
	$\pm\frac{1}{3}\sqrt{5+2\sqrt{\frac{10}{7}}}$	±0.90618…	$\frac{322-13\sqrt{70}}{900}$	0.236927…

- Intervalo $[a,b]=[-1,1]$
- $\omega(x)=1$

$$\omega_j = \frac{2}{(1-x_j^2)[L'_{N+1}(x_j)]^2}, \quad 0 \le j \le N.$$



Raíces nunca toman los extremos del intervalo

Cuadratura de Gauss-Radau

Teorema : Sean los puntos $x_0 = a$ y $\{x_j\}_{j=1}^{N}$ las raíces $q_N(x)$

$$q_N(x) = \frac{p_{N+1}(x)+\alpha_N p_N(x)}{x-a} \ , \alpha_N = -\frac{p_{N+1}(a)}{p_N(a)}$$

Entonces $\exists!$ conjunto $\{\omega_j\}_{j=0}^{N}$ definido por $\quad \omega_j = \int_a^b h_j \, \omega(x) dx \quad 0 \le j \le N$ tal que

$$\int_a^b \omega(x) p(x) = \sum_{j=0}^{N} \omega_j p(x_j) \qquad \forall \, p \in P_{2N}$$

$$\boxed{\int_a^b \omega(x) f(x) dx \approx \sum_{j=0}^{N} \omega_j f(x_j)}$$

Cuadratura de Gauss-Radau

Además

$$\omega_0 = \frac{1}{q_N(a)} \int_a^b q_N(x)\omega(x)dx$$

$$\omega_j = \frac{1}{x_j - a} \frac{k_{N+1}}{k_N} \frac{\|q_{N-1}\|_{\widehat{\omega}}^2}{q_{N-1}(x_j)q'_N(x_j)} \; 1 \leq j \leq N \qquad \widehat{\omega} = \omega(x)(x-a)$$

Cuadratura de Gauss-Radau-Legendre

$N+1$	$L_N(x) + L_{N+1}(x)$ Raíces	ω
3	- 1.000000	0.222222
	- 0.289898	1.0249717
	0.689898	0.7528061
4	- 1.000000	0.125000
	- 0.575319	0.657689
	0.181066	0.776387
	0.822824	0.440924
5	- 1.000000	0.080000
	- 0.720480	0.446208
	- 0.167181	0.623653
	0.446314	0.562712
	0.885792	0.287427

- Intervalo $[a,b] = [-1,1]$
- $\omega(x) = 1$

$$\omega_j = \frac{1}{(N+1)^2}\frac{1-x_j}{[L_N(x_j)]^2}, \quad 0 \leq j \leq N$$

Raíces solo toma el extremo izquierdo del intervalo

Cuadratura de Gauss-Lobatto

Teorema : Sean los puntos $x_0 = a, x_N = b$ y $\{x_j\}_{j=1}^{N-1}$ las raíces $z_{N-1}(x)$.

$$z_{N-1}(x) = \frac{p_{N+1}(x) + \alpha_N p_N(x) + \beta_N p_{N-1}}{(x-a)(b-x)}$$

$$p_{N+1}(x) + \alpha_N p_N(x) + \beta_N p_{N-1}(x) = 0 \; para \; x = a, b$$

Entonces $\exists!$ conjunto $\{\omega_j\}_{j=0}^{N}$ definido por $\omega_j = \int_a^b h_j \, \omega(x) dx \quad 0 \le j \le N$ tal que

$$\int_a^b \omega(x) p(x) = \sum_{j=0}^{N} \omega_j p(x_j) \qquad \forall \, p \in P_{2N-1}$$

$$\boxed{\int_a^b \omega(x) f(x) dx \approx \sum_{j=0}^{N} \omega_j f(x_j)}$$

Cuadratura de Gauss-Lobatto

Además
$$\omega_0 = \frac{1}{(b-a)z_{N-1}(a)}\int_a^b (b-x)z_{N-1}(x)\omega(x)dx$$

$$\omega_j = \frac{1}{(x_j-a)(b-x_j)}\frac{k_{N+1}}{k_N}\frac{\|z_{N-2}\|_{\widehat{\omega}}^2}{z_{N-2}(x_j)z\prime_{N-1}(x_j)}\, 1 \le j \le N-1$$

$$\widehat{\omega}\,(x) = \omega(x)(x-a)(x-b)$$

$$\omega_N = \frac{1}{(b-a)z_{N-1}(b)}\int_a^b (x-a)z_{N-1}(x)\omega(x)dx$$

Cuadratura de Gauss-Lobatto-Legendre

$N+1$	Raíces $\pm$ $(1-x^2)L'_N(x)$	ω
2	1.000 000 000 000 000	1.000 000 000 000 000
3	0.000 000 000 000 000	1.333 333 333 333 333
	1.000 000 000 000 000	0.333 333 333 333 333
4	0.447 213 595 499 958	0.833 333 333 333 333
	1.000 000 000 000 000	0.166 666 666 666 667
5	0.000 000 000 000 000	0.711 111 111 111 111
	0.654 653 670 707 977	0.544 444 444 444 444
	1.000 000 000 000 000	0.100 000 000 000 000
6	0.285 231 516 480 645	0.554 858 377 035 486
	0.765 055 323 929 465	0.378 474 956 297 847
	1.000 000 000 000 000	0.066 666 666 666 667
7	0.000 000 000 000 000	0.487 619 047 619 048
	0.468 848 793 470 714	0.431 745 381 209 863
	0.830 223 896 278 567	0.276 826 047 361 566
	1.000 000 000 000 000	0.047 619 047 619 048
8	0.209 299 217 902 479	0.412 458 794 658 704
	0.591 700 181 433 142	0.341 122 692 483 504
	0.871 740 148 509 607	0.210 704 227 143 506
	1.000 000 000 000 000	0.035 714 285 714 286

- Intervalo $[a,b]=[-1,1]$
- $\omega(x)=1$

$$\omega_j = \frac{2}{N(N+1)}\frac{1}{[L_N(x_j)]^2}, \quad 0 \le j \le N.$$

- Las raíces: mínimos y máximos de los polinomios de Legendre
- $\omega(x)=1$

Raíces toman los dos extremos del intervalo

Cuadratura de Gauss- intervalo arbitrario

$$\int_{A}^{B} f(x)\,dx = \frac{B-A}{2}\int_{-1}^{1} f\left(\frac{B-A}{2}\xi + \frac{A+B}{2}\right) d\xi$$

$$\approx \frac{B-A}{2}\sum_{j=0}^{N} \omega_j f\left(\frac{B-A}{2}x_j + \frac{A+B}{2}\right)$$

Ejemplo

Sabemos que $$\int_0^{\pi/2} Cos(x)dx = \frac{\pi}{2}\int_{-1}^{1} f(\xi)\, d\xi = 1 \qquad f(\xi) = \mathrm{Cos}(\frac{\pi}{2}\xi + \frac{\pi}{2})$$

$$I = \int_0^{\pi/2} Cos(x)dx \approx \frac{\pi}{2}(\omega_0 f(x_0) + \omega_1 f(x_1) + \omega_2 f(x_2)\ )$$

GL $x_0 = -\sqrt{\frac{3}{5}}, x_1 = 0, x_2 = \sqrt{\frac{3}{5}}, \omega_0 = \frac{5}{9}\ \omega_1 = \frac{8}{9}\ , \omega_2 = \frac{5}{9}$ $I = \mathbf{1.10710678963935}$

GRL $x_0 = -1.00, x_1 = -0.289898, x_2 = 0.689898, \omega_0 = 0.22222\ \omega_1 = 1.10249717, \omega_2 = 0.75280$
$I = \mathbf{1.10710678963935}$

GLL $x_0 = -1.00, x_1 = 0.000, x_2 = 1.000\ , \omega_0 = 0.33333\ \omega_1 = 1.33333, \omega_2 = 0.33333$
$I = \mathbf{1.00189886753959}$

Ejercicio

Determinel valor de π, utilizando la identidad

$$\pi = \int_0^1 \left(\frac{4}{1+x^2}\right) dx$$

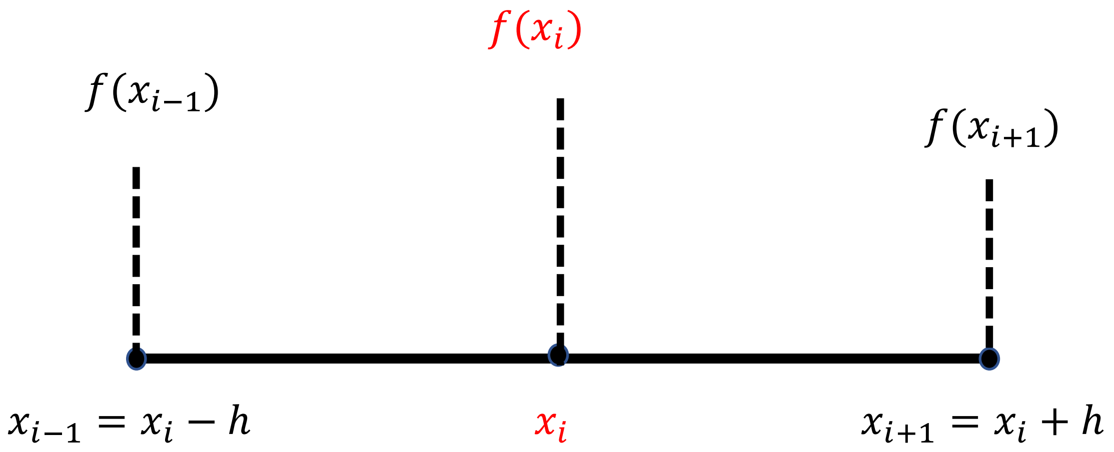
Utilice las cuadraturas de GL, GRL y GLL con 3, 4 y 5 puntos.

Diferenciación numérica

Alejandro Paredes

Desarrollo de Taylor

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2}h^2 + \cdots$$



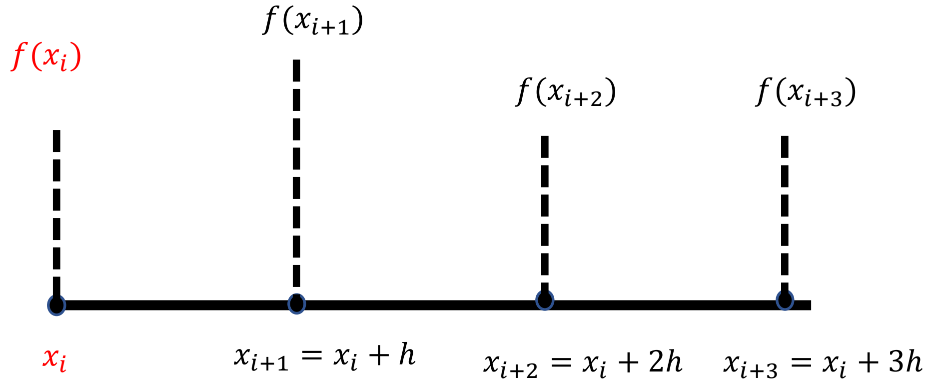
$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{f''(x_i)}{2}h + O(h^2)$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h)$$

$$f''(x_i) = \frac{f(x_{i+2}) - 2\,f(x_{i+1}) + f(x_i)}{h^2} + O(h)$$

Diferenciación forward

	Error
First Derivative	
$f'(x_i) = \dfrac{f(x_{i+1}) - f(x_i)}{h}$	$O(h)$
$f'(x_i) = \dfrac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h}$	$O(h^2)$
Second Derivative	
$f''(x_i) = \dfrac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2}$	$O(h)$
$f''(x_i) = \dfrac{-f(x_{i+3}) + 4f(x_{i+2}) - 5f(x_{i+1}) + 2f(x_i)}{h^2}$	$O(h^2)$
Third Derivative	
$f'''(x_i) = \dfrac{f(x_{i+3}) - 3f(x_{i+2}) + 3f(x_{i+1}) - f(x_i)}{h^3}$	$O(h)$
$f'''(x_i) = \dfrac{-3f(x_{i+4}) + 14f(x_{i+3}) - 24f(x_{i+2}) + 18f(x_{i+1}) - 5f(x_i)}{2h^3}$	$O(h^2)$
Fourth Derivative	
$f''''(x_i) = \dfrac{f(x_{i+4}) - 4f(x_{i+3}) + 6f(x_{i+2}) - 4f(x_{i+1}) + f(x_i)}{h^4}$	$O(h)$
$f''''(x_i) = \dfrac{-2f(x_{i+5}) + 11f(x_{i+4}) - 24f(x_{i+3}) + 26f(x_{i+2}) - 14f(x_{i+1}) + 3f(x_i)}{h^4}$	$O(h^2)$



Diferenciación backwards

First Derivative

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h}$$

$$f'(x_i) = \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2h}$$

Second Derivative

$$f''(x_i) = \frac{f(x_i) - 2f(x_{i-1}) + f(x_{i-2})}{h^2}$$

$$f''(x_i) = \frac{2f(x_i) - 5f(x_{i-1}) + 4f(x_{i-2}) - f(x_{i-3})}{h^2}$$

Third Derivative

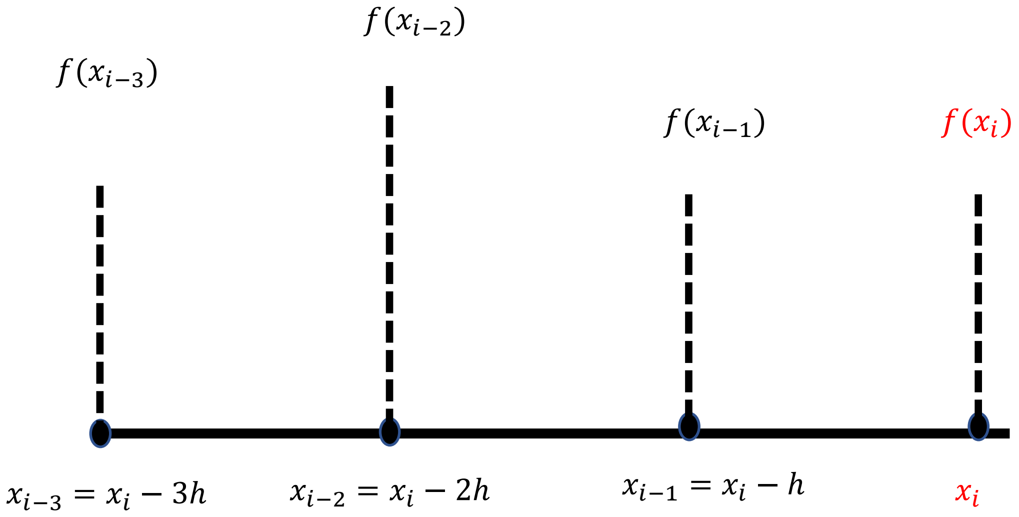
$$f'''(x_i) = \frac{f(x_i) - 3f(x_{i-1}) + 3f(x_{i-2}) - f(x_{i-3})}{h^3}$$

$$f'''(x_i) = \frac{5f(x_i) - 18f(x_{i-1}) + 24f(x_{i-2}) - 14f(x_{i-3}) + 3f(x_{i-4})}{2h^3}$$

Fourth Derivative

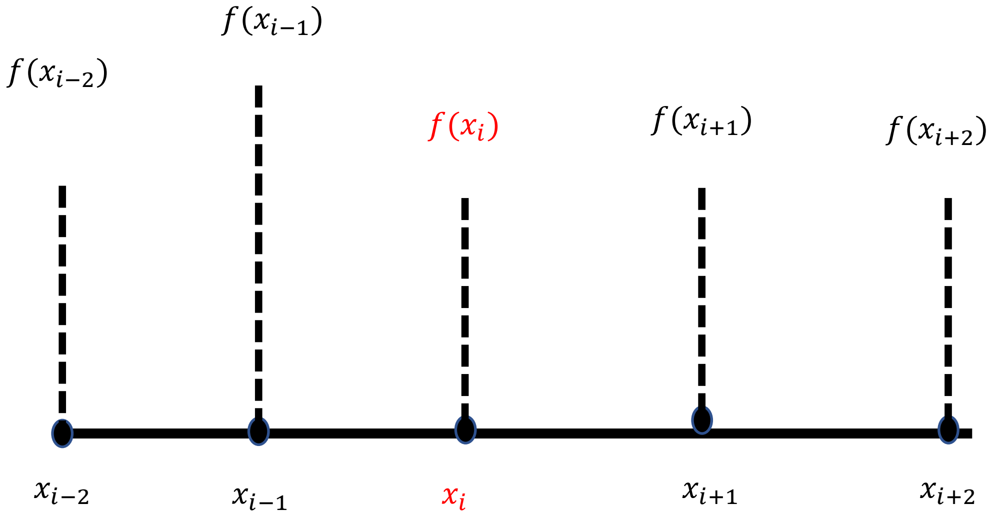
$$f''''(x_i) = \frac{f(x_i) - 4f(x_{i-1}) + 6f(x_{i-2}) - 4f(x_{i-3}) + f(x_{i-4})}{h^4}$$

$$f''''(x_i) = \frac{3f(x_i) - 14f(x_{i-1}) + 26f(x_{i-2}) - 24f(x_{i-3}) + 11f(x_{i-4}) - 2f(x_{i-5})}{h^4}$$



Diferenciación centrada

	Error
First Derivative	
$f'(x_i) = \dfrac{f(x_{i+1}) - f(x_{i-1})}{2h}$	$O(h^2)$
$f'(x_i) = \dfrac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_{i-1}) + f(x_{i-2})}{12h}$	$O(h^4)$
Second Derivative	
$f''(x_i) = \dfrac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$	$O(h^2)$
$f''(x_i) = \dfrac{-f(x_{i+2}) + 16f(x_{i+1}) - 30f(x_i) + 16f(x_{i-1}) - f(x_{i-2})}{12h^2}$	$O(h^4)$
Third Derivative	
$f'''(x_i) = \dfrac{f(x_{i+2}) - 2f(x_{i+1}) + 2f(x_{i-1}) - f(x_{i-2})}{2h^3}$	$O(h^2)$
$f'''(x_i) = \dfrac{-f(x_{i+3}) + 8f(x_{i+2}) - 13f(x_{i+1}) + 13f(x_{i-1}) - 8f(x_{i-2}) + f(x_{i-3})}{8h^3}$	$O(h^4)$
Fourth Derivative	
$f''''(x_i) = \dfrac{f(x_{i+2}) - 4f(x_{i+1}) + 6f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{h^4}$	$O(h^2)$
$f''''(x_i) = \dfrac{-f(x_{i+3}) + 12f(x_{i+2}) + 39f(x_{i+1}) + 56f(x_i) - 39f(x_{i-1}) + 12f(x_{i-2}) + f(x_{i-3})}{6h^4}$	$O(h^4)$



Extrapolación de Richardson

$$I \cong I(h_2) + \frac{1}{(h_1/h_2)^2 - 1}\left[I(h_2) - I(h_1)\right]$$

$$I \cong \frac{4}{3} I(h_2) - \frac{1}{3} I(h_1)$$

$I(h_2), I(h_1)$: derivación centrada de orden h^2

$$D \cong \frac{4}{3} D(h_2) - \frac{1}{3} D(h_1)$$

D Aproximación de orden h^4

Derivadas parciales

$$\left.\frac{\partial u}{\partial x}\right|_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + O[(\Delta x)^2].$$

$$\left.\frac{\partial u}{\partial x}\right|_{i,j} = \frac{1}{2h}\left\{ \underset{i-1,j}{(-1)} \text{---} \underset{i,j}{(0)} \text{---} \underset{i+1,j}{(1)} \right\} + 0(h^2)$$

$$\left.\frac{\partial u}{\partial y}\right|_{i,j} = \frac{1}{2k}\left\{ \begin{array}{c} (1)_{i,j+1} \\ | \\ (0)_{i,j} \\ | \\ (-1)_{i,j-1} \end{array} \right\} + 0(k^2)$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} = \frac{1}{h^2}\left\{ \underset{i-1,j}{(1)} \text{---} \underset{i,j}{(-2)} \text{---} \underset{i+1,j}{(1)} \right\} + 0(h^2)$$

$$\left.\frac{\partial^2 u}{\partial x\, \partial y}\right|_{i,j} = \frac{1}{4h^2}\left\{ \begin{array}{ccc} -1 & 0 & 1 \\ 0 & 0_{i,j} & 0 \\ 1 & 0 & -1 \end{array} \right\} + 0(h^2)$$

$(h = k)$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + O(h^2).$$

$$\left.\nabla^2 u\right|_{i,j} = \frac{1}{h^2}\left\{ \begin{array}{ccc} & 1 & \\ 1 & -4_{i,j} & 1 \\ & 1 & \end{array} \right\} + 0(h^2)$$

$$\left.\nabla^4 u\right|_{i,j} = \frac{1}{h^4}\left\{ \begin{array}{ccccc} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & 20_{i,j} & -8 & 1 \\ & 2 & -8 & 2 & \\ & & 1 & & \end{array} \right\} + 0(h^2)$$

Ejercicio

Un avión es monitoreado por un radar que registra su posición en coordenadas polares cada dos segundos. Calcule los vectores velocidad y aceleración usando diferencias centradas ($O(2)$), diferencias backwards ($O(2)$) y diferencias forwards ($O(2)$).

t, s	200	202	204	206	208	210
θ, rad	0.75	0.72	0.70	0.68	0.67	0.66
r, m	5120	5370	5560	5800	6030	6240

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \qquad \text{y} \qquad \vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta$$