

Semi-Tensor Product of Hypermatrices with Application to Compound Hypermatrices

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Abstract: The semi-tensor product (STP) of matrices is extended to the STP of hypermatrices. Some basic properties of the STP of matrices are extended to the STP of hypermatrices. The hyperdeterminant of hypersquares is introduced. Some algebraic and geometric structures of matrices are extended to hypermatrices. Then the compound hypermatrix is proposed. The STP of hypermatrix is used to compound hypermatrix. Basic properties are proved to be available for compound hypermatrix.

Key Words: Semi-tensor product, d -hypermatrix, general linear group of hypermatrices, compound hypermatrices

1 Introduction

The last two decades have witnessed the rapid development of the semi-tensor product (STP) of matrices [3, 4]. In particular, it has been applied to study Boolean networks and finite valued networks (see survey papers [10, 11, 13, 14]); finite games (see survey paper [7]); finite automata (see survey paper [17]); dimension-varying systems [5, 6], etc. The STP of two matrices is defined as follows:

Definition 1.1 [3] Let $A \in \mathbb{F}^{m \times n}$ and $B \in \mathbb{F}^{p \times q}$ and $t = \text{lcm}(n, p)$. The STP of A and B is defined by

$$A \ltimes B := (A \otimes I_{t/n}) (B \otimes I_{t/p}). \quad (1)$$

Some basic properties of STP are as follows:

Proposition 1.2 (i) (Linearity)

$$\begin{aligned} A \ltimes (\alpha B + \beta C) &= \alpha A \ltimes B + \beta A \ltimes C, \\ (\alpha B + \beta C) \ltimes A &= \alpha B \ltimes A + \beta C \ltimes A, \quad \alpha, \beta \in \mathbb{F}. \end{aligned} \quad (2)$$

(ii) (Associativity)

$$A \ltimes (B \ltimes C) = (A \ltimes B) \ltimes C. \quad (3)$$

Proposition 1.3 (i)

$$(A \ltimes B)^T = B^T \ltimes A^T. \quad (4)$$

(ii) If A and B are two invertible matrices, then

$$(A \ltimes B)^{-1} = B^{-1} \ltimes A^{-1}. \quad (5)$$

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Hypermatrix [12] is a generalized matrix. Roughly speaking, a matrix is a set of data of order 2, while a hypermatrix is a set of data of order $d > 2$. A hypermatrix of order d is closely related to a tensor of covariant order d , which is essentially a multilinear (more precisely d -th order linear) mapping [2]. Meanwhile, it can still be considered as a “matrix” in a certain sense, so that some corresponding properties can be studied, such as determinants (now called the hyperdeterminants) [12], eigenvalues and eigenvectors [15], etc.

The theory of compound matrices has found many applications in systems and control theory [1, 16]. The multiplicative compound matrix is defined as follows.

Definition 1.4 [1] Let $A \in \mathbb{F}^{n \times m}$, $k \leq \min(n, m)$. The k -multiplicative compound of A , denoted by $A^{(k)}$, is a $\binom{n}{k} \times \binom{m}{k}$ matrix containing all k -minors of A in lexicographical order.

Example 1.5 Let

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ -2 & 0 & 1 & -3 \\ 3 & 1 & -2 & 5 \end{bmatrix},$$

Then

$$(i) A^{(1)} = A.$$

$$(ii) A^{(2)} = \begin{bmatrix} 4 & -1 & 5 & 2 & -6 & -1 \\ -5 & 1 & -7 & -3 & 6 & 3 \\ -2 & 1 & -1 & -1 & 3 & -1 \end{bmatrix}.$$

$$(iii) A^{(3)} = \begin{bmatrix} -1 & -3 & 2 & -3 \end{bmatrix}.$$

The additive compound matrix is defined as follows.

Definition 1.6 [1] Let $A \in \mathbb{F}^{n \times n}$, $k \leq n$. The k -additive

compound of A , denoted by $A^{[k]}$, is a $\binom{n}{k} \times \binom{n}{k}$ matrix defined by:

$$A^{[k]} := \frac{d}{d\epsilon} (I_n + \epsilon A)^{(k)} \Big|_{\epsilon=0}. \quad (6)$$

Example 1.7 Consider

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 3 & 1 & -2 \end{bmatrix}$$

Then

(i)

$$\begin{aligned} (I_3 + \epsilon A)^{(1)} &= I_3 + \epsilon A \\ &= \begin{bmatrix} 1+\epsilon & 2\epsilon & -\epsilon \\ -2\epsilon & 1 & \epsilon \\ 3\epsilon & \epsilon & 1-2\epsilon \end{bmatrix} \\ A^{[1]} &= \frac{d}{d\epsilon} (I_3 + \epsilon A) \Big|_{\epsilon=0} = A. \end{aligned}$$

(ii)

$$\begin{aligned} (I_3 + \epsilon A)^{(2)} &= \begin{bmatrix} 1+\epsilon+4\epsilon^2 & \epsilon-\epsilon^2 & \epsilon+2\epsilon^2 \\ -\epsilon-5\epsilon^2 & 1-\epsilon+\epsilon^2 & 2\epsilon-3\epsilon^2 \\ -3\epsilon-2\epsilon^2 & -2\epsilon+\epsilon^2 & 1-2\epsilon-\epsilon^2 \end{bmatrix} \\ A^{[2]} &= \frac{d}{d\epsilon} (I_3 + \epsilon A)^{(2)} \Big|_{\epsilon=0} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ -3 & -2 & -2 \end{bmatrix} \end{aligned}$$

(iii)

$$\begin{aligned} (I_3 + \epsilon A)^{(3)} &= \det(I_3 + \epsilon A) = 1 - \epsilon + O(\epsilon^2). \\ A^{[3]} &= \frac{d}{d\epsilon} (I_3 + \epsilon A)^{(3)} \Big|_{\epsilon=0} = -1. \end{aligned}$$

Some basic properties of compound matrices are listed as follows:

Proposition 1.8 [1]

(i) $A^{(1)} = A$.

(ii) Assume $A \in \mathbb{F}^{n \times n}$, then $A^{(n)} = \det(A)$.

(iii) $(A^{(k)})^T = (A^T)^{(k)}$.

A symmetric $\Rightarrow A^{(k)}$ symmetric.

Theorem 1.9 [1]/Cauchy-Binet Formula] Let $A \in \mathbb{F}^{n \times m}$, $B \in \mathbb{F}^{m \times p}$. Fix a positive integer $k \leq \min(n, m, p)$. Then

$$(AB)^{(k)} = A^{(k)} = B^{(k)}. \quad (7)$$

Corollary 1.10 [1]

(i)

$$(I_n)^{(k)} = I_r, \quad r = \binom{n}{k}. \quad (8)$$

(ii) If $A \in \mathbb{F}^{n \times n}$ is invertible, then $A^{(k)}$ is also invertible, and

$$(A^{(k)})^{-1} = (A^{-1})^{(k)}. \quad (9)$$

(iii) $A, B \in \mathbb{F}^{n \times n}$, then

$$\det(AB) = (AB)^{(n)} = A^{(n)}B^{(n)} = \det(A)\det(B). \quad (10)$$

(iv) If $A \simeq B$, then $A^{(k)} \simeq B^{(k)}$.

Proposition 1.11 [1] Consider $A \in \mathbb{F}^{n \times n}$, let $\lambda_i, i \in [1, n]$ be the eigenvalues of A , and $v_i, i \in [1, n]$ be the corresponding eigenvectors. Then the eigenvalues of $A^{(k)}$ are

$$\left\{ \lambda^\alpha = \prod_{\ell=1}^k \lambda_{i_\ell} \mid \alpha = (i_1, i_2, \dots, i_k) \in Q(n, k) \right\}. \quad (11)$$

Furthermore, if $\alpha = (i_1, i_2, \dots, i_k) \in Q(n, k)$ and

$$W_\alpha := [v_{i_1}, v_{i_2}, \dots, v_{i_k}]^{(k)} \neq 0,$$

then W_α is the eigenvector of $A^{(k)}$ corresponding to λ^α .

Proposition 1.12 [1]

(i)

$$A^{[k]} = \frac{d}{d\epsilon} (\exp(A\epsilon))^{[k]} \Big|_{\epsilon=0}. \quad (12)$$

(ii)

$$(I_n + \epsilon A)^{(k)} = I_r + \epsilon A^{[k]} + o(\epsilon). \quad (13)$$

(iii)

$$\frac{d}{dt} (\exp(At))^{(k)} = A^{[k]} (\exp(At))^{(k)}. \quad (14)$$

(iv) Let $A, T \in \mathbb{F}_{n \times n}$, with T invertible. Then

$$(TAT^{-1})^{[k]} = T^{(k)} A^{[k]} (T^{(k)})^{-1}. \quad (15)$$

Proposition 1.13 [1] Let $A, B \in \mathbb{C}_{n \times n}$. Then

$$(A + B)^{[k]} = A^{[k]} + B^{[k]}. \quad (16)$$

Proposition 1.14 [1] For $A \in \mathbb{F}^{n \times n}$, let $\lambda_i, i \in [1, n]$ be the eigenvalues of A , and $v_i, i \in [1, n]$ be the corresponding eigenvectors. Then the eigenvalues of $A^{[k]}$ are

$$\left\{ \lambda^\alpha = \sum_{\ell=1}^k \lambda_{i_\ell} \mid \alpha = (i_1, i_2, \dots, i_k) \in Q(n, k) \right\}. \quad (17)$$

Furthermore, if $\alpha = (i_1, i_2, \dots, i_k) \in Q(n, k)$ and

$$W_\alpha := [v_{i_1}, v_{i_2}, \dots, v_{i_k}]^{(k)} \neq 0,$$

then W_α is the eigenvector of $A^{[k]}$ corresponding to λ^α .

Finally, we give a list of notations used in this paper.

- 1) \mathbb{F}^n : n dimensional Euclidean space over \mathbb{F} , where \mathbb{F} is a field (Particularly, $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$).
- 2) $\mathbb{F}^{n_1 \times n_2 \times \cdots \times n_d}$: the d -th order hypermatrices of dimensions n_1, n_2, \dots, n_d .
- 3) $\text{lcm}(n, p)$: the least common multiple of n and p .
- 4) δ_n^i : the i -th column of the identity matrix I_n .
- 5) $\Delta_n := \{\delta_n^i | i = 1, \dots, n\}$.
- 6) $\mathbf{1}_\ell := (\underbrace{1, 1, \dots, 1}_\ell)^T$.
- 7) cdet : combinatorial hyperdeterminant.
- 8) ddet : modified combinatorial hyperdeterminant.
- 9) Det : slice-based hyperdeterminant.
- 10) \bowtie : semi-tensor product of matrices.
- 11) \odot : semi-tensor product of hypermatrices.
- 12) $\text{GL}(n^{(d)}, \mathbb{F})$: $n^{(d)}$ -general linear group of hypercubics.
- 13) $\text{GL}(*^{(d)}, \mathbb{F})$: d -th order general linear group of hypercubics.

The purpose of this paper is twofold:

- (i) To generalize the STP of matrices to the STP of hypermatrices, that is, to define a product of two hypermatrices of arbitrary orders and arbitrary dimensions, and to show that some of the above properties can be extended to the STP of hypermatrices; (2) The determinant of matrices is also extended to several types of hyperdeterminants of hypermatrices. The monoid (semigroup with identity) and group structures of matrices are extended to the monoid and group structures of hypermatrices. Finally, the general linear group of hypermatrices is also introduced as a Lie group.
- (ii) Two types of compound hypermatrices are proposed: multiplicative and additive. Using the semi-tensor product of hypermatrices, it is shown that most of the properties of compound matrices can be extended to compound hypermatrices.

2 Matrix Expression of Hypermatrices

Definition 2.1 (i) A set of order d data

$$A := \{a_{i_1, i_2, \dots, i_d} \mid i_s \in [1, n_s], s \in [1, d]\} \in \mathbb{F}^{n_1 \times \cdots \times n_d} \quad (18)$$

is called an d -th order hypermatrix (d -hypermatrix for short) of dimensions $n_1 \times n_2 \times \cdots \times n_d$. The set of d -hypermatrix of dimension $n_1 \times n_2 \times \cdots \times n_d$ is denoted by $\mathbb{F}^{n_1 \times n_2 \times \cdots \times n_d}$, where $a_{i_1, i_2, \dots, i_d} \in \mathbb{F}$ and \mathbb{F} can be \mathbb{R} or \mathbb{C} (or any other field).

- (ii) $A \in \overbrace{\mathbb{F}^n \times n \times \cdots \times n}^d$ is called a d -hypercubic.
- (iii) $A \in \mathbb{F}^{n_1 \times n_2 \times \cdots \times n_d}$ with $n_1 = n_d$ is called a d -hypersquare.

The elements of A can be arranged into a matrix by a particular partition of indices, called the matrix expression of A .

Definition 2.2 Let

$$A = [a_{i_1, i_2, \dots, i_d}] \in \mathbb{F}^{n_1 \times n_2 \times \cdots \times n_d}, \quad (19)$$

$$\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_s\} \subset \{1, 2, \dots, d\} \text{ and } \beta = \{\beta_1, \beta_2, \dots, \beta_t\} \subset \{1, 2, \dots, d\} \text{ and}$$

$$\alpha \cup \beta = \{1, 2, \dots, d\} \quad (20)$$

be a partition, where $s + t = d$. Then

$$M_A^\alpha \in \mathbb{F}^{n_\alpha \times n_\beta} \quad (21)$$

is a matrix of dimension

$$n_\alpha := \prod_{i=1}^s n_{\alpha_i}; \quad n_\beta := \prod_{j=1}^t n_{\beta_j},$$

where its rows are arranged by the index arrangement

$$\text{Id}(\alpha_1, \alpha_2, \dots, \alpha_s; n_{\alpha_1}, n_{\alpha_2}, \dots, n_{\alpha_s}) \in Q(s, d),$$

and its columns are arranged by the index

$$\text{Id}(\beta_1, \beta_2, \dots, \beta_t; n_{\beta_1}, n_{\beta_2}, \dots, n_{\beta_t}) \in Q(t, d),$$

where $Q(m, n)$ is the set of m sub-indices of the set of n indices. M_A^α is called the matrix expression of A with respect to the partition $\{\alpha, \beta = \alpha^c\}$.

Remark 2.3 (i) For simplicity, we always assume $\alpha_1 < \alpha_2 < \dots < \alpha_s, \beta_1 < \beta_2 < \dots < \beta_t$.

(ii) Index arrangement Id means the indexed data are arranged in alphabetic order [4].

Example 2.4 Given $A = [a_{i_1, i_2, i_3}] \in \mathbb{F}^{2 \times 3 \times 2}$. Then

(i)

$$M_A^\emptyset = [a_{111}, a_{112}, a_{121}, a_{122}, a_{131}, a_{132}, a_{211}, a_{212}, a_{221}, a_{222}, a_{231}, a_{232}].$$

(ii)

$$M_A^{\{1\}} = \begin{bmatrix} a_{111} & a_{112} & a_{121} & a_{122} & a_{131} & a_{132} \\ a_{211} & a_{212} & a_{221} & a_{222} & a_{231} & a_{232} \end{bmatrix};$$

$$M_A^{\{2\}} = \begin{bmatrix} a_{111} & a_{112} & a_{211} & a_{212} \\ a_{121} & a_{122} & a_{221} & a_{222} \\ a_{131} & a_{132} & a_{231} & a_{232} \end{bmatrix}; \quad \text{etc.}$$

(iii)

$$M_A^{\{1,2\}} = \begin{bmatrix} a_{111} & a_{112} \\ a_{121} & a_{122} \\ a_{131} & a_{132} \\ a_{211} & a_{212} \\ a_{221} & a_{222} \\ a_{231} & a_{232} \end{bmatrix};$$

$$M_A^{\{1,3\}} = \begin{bmatrix} a_{111} & a_{121} & a_{131} \\ a_{112} & a_{122} & a_{132} \\ a_{211} & a_{221} & a_{231} \\ a_{212} & a_{222} & a_{232} \end{bmatrix}; \quad \text{etc.}$$

(iv)

$$M_A^{\{1,2,3\}} = (M_A^\emptyset)^T.$$

Denote

$$V_A := M_A^\emptyset; \quad M_A := M_A^{\{1\}}.$$

In the sequel, M_A plays a particularly important role. Let $A \in \mathbb{F}^{n_1 \times n_2 \times \dots \times n_d}$. Then

$$M_A = [A_1, A_2, \dots, A_q], \quad (22)$$

where $A_i \in \mathbb{F}^{n_1 \times n_d}$, $i \in [1, q]$, ($q = \prod_{i=2}^{d-1} n_i$) are called slices of A .

Proposition 2.5 Given $A \in \mathbb{F}^{n_1 \times n_2 \times \dots \times n_d}$, $\alpha \in Q(s, d)$ and $\beta \in Q(t, d)$ ($s + t = d$) be a partition of $[1, d]$. Then A can be considered as a multi-linear mapping $\pi_A^{(\alpha)} : \mathbb{F}^n \rightarrow \mathbb{F}^m$, where $m = \prod_{i \in \alpha} n_i$ and $n = \prod_{i \in \beta} n_i$, and

$$\pi_A^\alpha(x) := M_A^\alpha x \in \mathbb{F}^m, \quad x \in \mathbb{F}^n. \quad (23)$$

Definition 2.6 [12]

(i) Given a d -hypermatrix $A = [a_{j_1, j_2, \dots, j_d}] \in \mathbb{F}^{n_1 \times n_2 \times \dots \times n_d}$, and assume $\sigma \in \mathbf{S}_d$. The σ -transpose of A is

$$A^\sigma := [a_{j_{\sigma(1)}, \dots, j_{\sigma(d)}}] \in \mathbb{F}^{n_{\sigma(1)} \times \dots \times n_{\sigma(d)}}. \quad (24)$$

(ii) A d -hypercubic $A \in \overbrace{\mathbb{F}^n \times \dots \times \mathbb{F}^n}^d$ is called symmetric if $\forall \sigma \in \mathbf{S}_d$, $A^\sigma = A$; A is called skew-symmetric if $\forall \sigma \in \mathbf{S}_d$, $A^\sigma = \text{sgn}(\sigma)A$.

Proposition 2.7 A 2-hypercubic $A \in \mathbb{F}^{n \times n}$ is (skew-)symmetric, if and only if, M_A is (skew-)symmetric.

Remark 2.8 The above arguments stand true even when \mathbb{F} is the set of perfect hypercomplex numbers (PHNs) [8]. In fact, most of arguments throughout this paper also hold for PHNs.

3 Semi-tensor Product of Hypermatrices

Definition 3.1 Let $A \in \mathbb{F}^{m \times s \times n}$ and $B \in \mathbb{F}^{p \times s \times q}$ be two 3-hypermatrices, $t = \text{lcm}(n, p)$. Then

$$M_A = [A_1, A_2, \dots, A_s],$$

$$M_B = [B_1, B_2, \dots, B_s],$$

where

$$A_i \in \mathbb{F}^{m \times n}, \quad B_i \in \mathbb{F}^{p \times q}, \quad i \in [1, s].$$

Then the SPTH of A and B is defined by

$$A \odot B := C, \quad (25)$$

where

$$M_C = [C_1, C_2, \dots, C_s] \in \mathbb{F}^{(mt/n) \times s \times (qt/p)},$$

and

$$C_i = A_i \ltimes B_i, \quad i \in [1, s].$$

Example 3.2 Given $A \in \mathbb{F}^{2 \times 2 \times 3}$ and $B \in \mathbb{F}^{2 \times 2 \times 2}$ with

$$M_A = \begin{bmatrix} a_{111} & a_{112} & a_{113} & a_{121} & a_{122} & a_{123} \\ a_{211} & a_{212} & a_{213} & a_{221} & a_{222} & a_{223} \end{bmatrix} := [A_1, A_2],$$

$$M_B = \begin{bmatrix} b_{111} & b_{112} & b_{121} & b_{122} \\ b_{211} & b_{212} & b_{221} & a_{222} \end{bmatrix} := [B_1, B_2].$$

Let

$$C = A \odot B \in \mathbb{F}^{4 \times 2 \times 6}.$$

Then

$$M_C = [A_1 \ltimes B_1, A_2 \ltimes B_2].$$

Next, we extend the STPH to general cases:

- Case 1, $d > 3$:

Assume $A \in \mathbb{F}^{n_1 \times n_2 \times \dots \times n_d}$. Denote $s = \prod_{i=2}^{d-1} n_i$, and define $\pi : \mathbb{F}^{n_1 \times n_2 \times \dots \times n_d} \rightarrow \mathbb{F}^{n_1 \times s \times n_d}$ by

$$\pi : a_{i_1, i_2, \dots, i_d} \mapsto b_{i_1, t, i_d}, \quad (26)$$

where

$$\begin{aligned} t &= (i_2 - 1)n_3 n_4 \cdots n_{d-1} + (i_3 - 1)n_4 n_5 \cdots n_{d-1} + \\ &\quad \cdots + (i_{d-2} - 1)n_{d-1} + i_{d-1}. \end{aligned} \quad (27)$$

Then it is easy to see that π is a bijective mapping. Hence we can extend the STPH defined in Definition 3.1 to more general cases.

Definition 3.3 Assume $A \in \mathbb{F}^{m \times s_2 \times \dots \times s_{d-1} \times n}$ and $B \in \mathbb{F}^{p \times s_2 \times \dots \times s_{d-1} \times q}$, then

$$A \odot B := \pi^{-1}(\pi(A) \odot \pi(B)) \in \mathbb{F}^{(mt/n) \times s_2 \times \dots \times s_{d-1} \times (qt/p)}. \quad (28)$$

- Case 2, $d = 3$ and $s_1 \neq s_2$:

Definition 3.4 Assume $A \in \mathbb{F}^{m \times s_1 \times n}$ and $B \in \mathbb{F}^{p \times s_2 \times q}$ and $\text{lcm}(s_1, s_2) = s$, construct

$$M_{\tilde{A}} := \mathbf{1}_{s/s_1}^T \otimes M_A; \\ M_{\tilde{B}} := \mathbf{1}_{s/s_2}^T \otimes M_B.$$

Then $\tilde{A} \in \mathbb{F}^{m \times s \times n}$ and $\tilde{B} \in \mathbb{F}^{p \times s \times q}$. Define

$$A \odot B := \tilde{A} \odot \tilde{B} \in \mathbb{F}^{(mt/n) \times s \times (qt/p)}. \quad (29)$$

Combining Definitions 3.1, 3.3, and 3.4, one sees easily that the STPH of two arbitrary hypermatrices is properly defined. Hence it is enough to consider the case of Definition 3.1.

Example 3.5 Given $A \in \mathbb{F}^{m \times 3 \times n}$ and $B \in \mathbb{F}^{p \times 2 \times 2 \times q}$. Express

$$M_A = [A_1, A_2, A_3], \\ M_B = [B_{11}, B_{12}, B_{21}, B_{22}].$$

Then $C = A \odot B$ can be calculated by

$$\begin{aligned} M_C &= [\mathbf{1}_4^T \otimes M_A] \odot [\mathbf{1}_3^T \otimes M_B] \\ &= [A_1 \ltimes B_{11}, A_2 \ltimes B_{12}, A_3 \ltimes B_{21}, A_1 \ltimes B_{22} \\ &\quad A_2 \ltimes B_{11}, A_3 \ltimes B_{12}, A_1 \ltimes B_{21}, A_2 \ltimes B_{22} \\ &\quad A_3 \ltimes B_{11}, A_1 \ltimes B_{12}, A_2 \ltimes B_{21}, A_3 \ltimes B_{22}]. \end{aligned}$$

Next, we show some basic properties of STPH.

Denote by

$$\mathbb{F}^{\infty^\infty} = \sum_{d=1}^{\infty} \sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} \mathbb{F}^{n_1 \times n_2 \times \cdots \times n_d}.$$

To include \mathbb{F} , $\mathbb{F}^{n_1 \times n_2}$, we consider

$$a = a(1 \times 1 \times 1);$$

and $A \in \mathbb{F}^{n_1 \times n_2}$ as

$$A \in \mathbb{F}^{n_1 \times 1 \times n_2}.$$

Then the STP of hypermatrices is also applicable to \mathbb{F} , $\mathbb{F}^{n_1 \times n_2}$.

Definition 3.6 Let $a \in \mathbb{F}$ and $B \in \mathbb{F}^{p \times s \times q}$. Then

$$a \odot B := aB. \quad (30)$$

Let $A \in \mathbb{F}^{m \times n}$ and $B \in \mathbb{F}^{p \times s \times q}$, and $t = \text{lcm}(n, p)$. Then

$$A \odot B := C \in \mathbb{F}^{mt/n \times s \times qt/p}, \quad (31)$$

where

$$M_C = [A \ltimes B_1, A \ltimes B_2, \dots, A \ltimes B_s].$$

Proposition 3.7 Assume $A, B, C \in \mathbb{F}^{\infty^\infty}$, then

$$A \odot (B \odot C) = (A \odot B) \odot C. \quad (32)$$

Proposition 3.8 Assume $A, B \in \mathbb{F}^{n_1 \times n_2 \times \cdots \times n_d}$ and $C \in \mathbb{F}^{\infty^\infty}$, then

$$\begin{aligned} (A + B) \odot C &= A \odot C + B \odot C, \\ C \odot (A + B) &= C \odot A + C \odot B. \end{aligned} \quad (33)$$

Proposition 3.9 Assume A and B are two invertible hypersquares, then

$$(A \odot B)^{-1} = B^{-1} \odot A^{-1}. \quad (34)$$

4 Hyperdeterminants

Definition 4.1 [12] Let $A \in \overbrace{\mathbb{F}^n \times \cdots \times n}^d$ be a d -hypercubic. The combinatorial hyperdeterminant (CH-determinant) is defined by

$$\text{cdet}(A) = \frac{1}{n!} \sum_{\sigma_1, \dots, \sigma_d \in \mathbf{S}_n} \prod_{j=1}^d \text{sgn}(\sigma_j) \prod_{i=1}^n a_{\sigma_1(i), \dots, \sigma_d(i)}. \quad (35)$$

Remark 4.2 Combinatorial hyperdeterminant has some nice properties. Unfortunately, for an odd order d , the combinatorial hyperdeterminant of a d -hypercubic is identically zero [12]. So it is not suitable for our approach where, mostly, $d = 3$. So we provide a modification as follows.

Definition 4.3 Let A be a d -hypercubic with dimension $n_i = n$, $i \in [1, d]$. The modified combinatorial hyperdeterminant (MCH-determinant) of A is defined by

$$\begin{aligned} \text{ddet}(A) = \\ \sum_{\sigma_1, \dots, \sigma_{d-1} \in \mathbf{S}_n} \prod_{j=1}^{d-1} \text{sgn}(\sigma_j) \prod_{i=1}^n a_{i, \sigma_1(i), \dots, \sigma_{d-1}(i)}. \end{aligned} \quad (36)$$

Proposition 4.4 Assume $d = 2$, then

$$\text{ddet}(A) = \det(M_A). \quad (37)$$

Proposition 4.5 When d is even

$$\text{ddet}(A) = \text{cdet}(A). \quad (38)$$

The following definition is more suitable for our purpose:

Definition 4.6 (i) Let A be a d -hypersquare with dimension $n_1 = n_d = n$, $d \geq 3$. Denote $s = \prod_{i=2}^{d-1} n_i$, and

$$M_A := [A_1, A_2, \dots, A_s].$$

Then the slice-based hyperdeterminant (SH-determinant) of A is defined by

$$\text{Det}(A) := \prod_{i=1}^s \det(A_i). \quad (39)$$

(ii) d -hypersquare A is called non-singular (invertible), if

$$\text{Det}(A) \neq 0.$$

(iii) The d -hypersquare B is called the inverse of A , if

$$M_B := [A_1^{-1}, A_2^{-1}, \dots, A_s^{-1}].$$

Example 4.7 Assume $A \in \mathbb{F}^{2 \times 2 \times 2}$ with

$$M_A = \begin{bmatrix} a_{111} & a_{112} & a_{121} & a_{122} \\ a_{211} & a_{212} & a_{221} & a_{222} \end{bmatrix} \quad (40)$$

(i)

$$\begin{aligned} & \text{cdet}(A) \\ &= \frac{1}{2} \sum_{\sigma_1, \sigma_2, \sigma_3=1}^2 a_{\sigma_1(1)\sigma_2(1)\sigma_3(1)} \times a_{\sigma_1(2)\sigma_2(2)\sigma_3(2)} \\ &= \frac{1}{2} (a_{111}a_{222} - a_{112}a_{221} - a_{121}a_{212} + a_{122}a_{211} \\ &\quad - a_{211}a_{122} + a_{212}a_{121} + a_{221}a_{112} - a_{222}a_{111}) \\ &= 0. \end{aligned}$$

(ii)

$$\begin{aligned} \text{ddet}(A) &= \sum_{\sigma_1=1}^2 \sum_{\sigma_2=1}^2 a_{1\sigma_1(1)\sigma_2(1)} a_{2\sigma_1(2)\sigma_2(2)} \\ &= a_{111}a_{222} - a_{112}a_{221} - a_{121}a_{212} + a_{122}a_{211}. \end{aligned}$$

(iii)

$$\begin{aligned} \text{Det}(A) &= \det(A_1) \det(A_2) \\ &= (a_{111}a_{212} - a_{112}a_{211})(a_{121}a_{222} - a_{122}a_{211}). \end{aligned}$$

Definition 4.8 (i) Let $A \in \mathbb{F}^{m \times n}$ and $m \leq n$. Denote by $A^{(m)} = (a_1, a_2, \dots, a_r)$, where $r = \binom{n}{m}$. Then

$$\text{Det}(A) := \prod_{i=1}^r a_i. \quad (41)$$

(ii) Let $A \in \mathbb{F}^{n_1 \times n_2 \times \dots \times n_d}$. Denote by $M_A = [A_1, A_2, \dots, A_s]$, where $s = \prod_{i=2}^{d-1} n_i$. Then

$$\text{Det}(A) := \prod_{i=1}^s \text{Det}(A_i). \quad (42)$$

Next, with the newly introduced notions of STPH and hyperdeterminant, we reveal some basic algebraic structure of hypermatrices.

Proposition 4.9 $(\mathbb{F}^{\infty^\infty}, \odot)$ is a monoid (semi-group with identity), with identity $1 \in \mathbb{F}$.

Denote $J_n^s \in \mathbb{F}^{n \times s \times n}$ as

$$M_{J_n^s} = \underbrace{[I_n, I_n, \dots, I_n]}_s.$$

Definition 4.10 Let $\mathbb{F}^{n \times s \times n}$ be the set of hypersquare, where $s = (s_2, s_3, \dots, s_{d-1})$, $d \geq 2$. Consider

$$G = \{A \in \mathbb{F}^{n \times s \times n} \mid \text{Det}(A) \neq 0\}.$$

Denote by

$$\text{GL}(n^s, \mathbb{F}) := (G, \odot). \quad (43)$$

Then $\text{GL}(n^s, \mathbb{F})$ is a group with identity J_n^s , called the general linear group of hypermatrices.

Proposition 4.11 $\text{GL}(n^s, \mathbb{F})$ is a Lie-group with natural manifold structure.

Definition 4.12 Define

$$\text{GL}(\infty^\infty, \mathbb{F}) := \bigcup_{n=1}^{\infty} \bigcup_{d=1}^{\infty} \text{GL}(n^s, \mathbb{F}). \quad (44)$$

Then $\text{GL}(\infty^\infty, \mathbb{F})$ is called the dimension-free general linear group of hypermatrices.

Proposition 4.13 $\text{GL}(\infty^\infty, \mathbb{F})$ is a dimension-free Lie-group with dimension-free manifold structure [9].

5 Compound Hypermatrices

The extension of compound matrices to compound hypermatrices is straightforward:

Definition 5.1 Let $A \in \mathbb{F}^{n_1, n_2, \dots, n_d}$, $k \leq \min(n_1, n_d)$. Denote

$$M_A := [A_1, A_2, \dots, A_s],$$

where $s = \prod_{i=2}^{d-1} n_i$.

(i) The k -multiplicative compound hypermatrix of A , denoted by $A^{(k)}$, is defined by

$$M_{A^{(k)}} := [A_1^{(k)}, A_2^{(k)}, \dots, A_s^{(k)}]. \quad (45)$$

(ii) The k -additive compound hypermatrix of A , denoted by $A^{[k]}$, is defined by

$$M_{A^{[k]}} := [A_1^{[k]}, A_2^{[k]}, \dots, A_s^{[k]}]. \quad (46)$$

The following properties are a direct consequence of Definition 5.1 and the corresponding properties of compound matrices.

Proposition 5.2 (i)

$$A^{(1)} = A. \quad (47)$$

(ii) Assume A is a hypersquare, that is, $A \in \mathbb{F}^{n_1 \times n_2 \times \dots \times n_d}$ and $n_1 = n_d$, then

$$A^{(n)} = [\det(A_1), \det(A_2), \dots, \det(A_s)]. \quad (48)$$

Theorem 5.3 (Cauchy-Binet Formula) Let $A \in \mathbb{F}^{n_1 \times n_2 \times \dots \times n_d}$, $B \in \mathbb{F}^{m_1 \times m_2 \times \dots \times m_d}$, where $n_d = m_1$ and $n_i = m_i$, $i \in [2, d-1]$. Fix a positive integer $k \leq \min(n_1, n_d, m_1)$, then

$$(A \odot B)^{(k)} = A^{(k)} \odot B^{(k)}. \quad (49)$$

Remark 5.4 When $A \in \mathbb{F}^{m \times n}$, $B \in \mathbb{F}^{n \times r}$. Fix a positive integer $k \leq \min(m, n, r)$, then

$$(AB)^{(k)} = A^{(k)} B^{(k)}. \quad (50)$$

This is the classical Cauchy-Binet Formula.

Corollary 5.5 (i)

$$(J_n^d)^{(k)} = (J_r^d)^{(k)}, \quad r = \binom{n}{k}. \quad (51)$$

Particularly,

$$(I_n)^{(k)} = I_r, \quad r = \binom{n}{k}. \quad (52)$$

(ii) If $A \in \mathbb{F}_{n_1 \times n_2 \times \dots \times n_d}$, $n_1 = n_d = n$, is invertible, then $A^{(k)}$ is also invertible, and

$$(A^{(k)})^{-1} = (A^{-1})^{(k)}. \quad (53)$$

Definition 5.6 Let $A, B \in \mathbb{F}_{n_1 \times n_2 \times \dots \times n_d}$, $n_1 = n_d = n$. A and B are said to be similar, denoted by $A \simeq B$, if there exists a nonsingular $T \in \mathbb{F}_{n_1 \times n_2 \times \dots \times n_d}$ such that

$$T^{-1} \odot A \odot T = B. \quad (54)$$

Proposition 5.7 Let $A, B \in \mathbb{F}_{n_1 \times n_2 \times \dots \times n_d}$, $n_1 = n_d = n$. If $A \simeq B$, then

(i)

$$A^{(k)} \sim B^{(k)}, \quad k \leq n; \quad (55)$$

(ii)

$$A^{[k]} \sim B^{[k]}, \quad k \leq n; \quad (56)$$

Definition 5.8 Assume $A \in \mathbb{F}^{n_1 \times n_2 \times \dots \times n_d}$, $n_1 = n_d = n$, and $X \in \mathbb{F}^{n_1 \times n_2 \times \dots \times n_{d-1} \times 1}$. Moreover,

$$M_X = [X_1, X_2, \dots, X_s], \quad s = \prod_{i=2}^{d-1} n_i,$$

and $X_i \neq 0$, $i \in [1, s]$. If there exists $\lambda = (\lambda_1, \dots, \lambda_s) \in \mathbb{F}^s$ such that

$$A \odot X = \lambda \odot X, \quad (57)$$

then λ is called the eigenvalue of A with eigenvector X .

Using Definition 5.8, Propositions 1.11 and 1.14 can be extended to compound hypermatrices easily.

6 Conclusion

In this paper the STP of matrices was extended to the STP of two arbitrary hypermatrices. We showed that almost fundamental properties of the STP of matrices can be extended to that of hypermatrices. Three determinants of hypermatrices, namely the CH-determinant, the MCH-determinant and the SH-determinant, are introduced and studied. The monoid and the group of hypermatrices are also introduced. Then the general linear group of hypermatrices, as a Lie group, is introduced and studied. Finally, the compound hypermatrix is presented and some interesting properties are revealed.

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