Dependable System Design - Fall 2024

Homework 11

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Conditional Probability of System Reliability

Problem Statement

Calculate the conditional probability of the system being operational within the time interval $[a, t]$, given that the system was operational at time $t = 0$.

Solution

We aim to calculate:

$$P(a \leq T \leq t \mid T > 0), \tag{1}$$

where:

- T: Time-to-failure random variable,
- $[a, t]$: Time interval under consideration,
- $T > 0$: The system was operational at $t = 0$.

1. Conditional Probability Formula

The conditional probability formula is given by:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}. \tag{2}$$

Here, A corresponds to the event $a \leq T \leq t$, and B corresponds to the event $T > 0$. Substituting:

$$P(a \leq T \leq t \mid T > 0) = \frac{P(a \leq T \leq t \cap T > 0)}{P(T > 0)}. \tag{3}$$

Since $T > 0$ is already guaranteed, this simplifies to:

$$P(a \leq T \leq t \mid T > 0) = \frac{P(a \leq T \leq t)}{P(T > 0)}. \tag{4}$$

2. Reliability Function

The reliability function $R(t)$ is defined as:

$$R(t) = P(T > t). \tag{5}$$

Its complement is the cumulative distribution function (CDF), $F(t)$, which gives the probability of failure by time t:

$$F(t) = P(T \leq t) = 1 - R(t). \tag{6}$$

For the interval probability $P(a \leq T \leq t)$, we have:

$$P(a \leq T \leq t) = P(T > a) - P(T > t). \tag{7}$$

Substituting the reliability function:

$$P(a \leq T \leq t) = R(a) - R(t). \tag{8}$$

3. Substituting into Conditional Probability Formula

Using the relationships above, the conditional probability becomes:

$$P(a \leq T \leq t \mid T > 0) = \frac{P(a \leq T \leq t)}{P(T > 0)}. \tag{9}$$

From the reliability assumption, at $t = 0$, $R(0) = P(T > 0) = 1$ because the system starts operational. Thus:

$$P(a \leq T \leq t \mid T > 0) = P(a \leq T \leq t). \tag{10}$$

Substituting $P(a \leq T \leq t) = R(a) - R(t)$:

$$P(a \leq T \leq t \mid T > 0) = R(a) - R(t). \tag{11}$$

Final Expression

The conditional probability of the system being operational within the interval $[a, t]$, given that it was functional at $t = 0$, is:

$$\boxed{P(a \leq T \leq t \mid T > 0) = R(a) - R(t).} \tag{12}$$

Numerical Example

Let us assume the reliability function $R(t)$ is given by an exponential distribution:

$$R(t) = e^{-\lambda t}, \tag{13}$$

where $\lambda > 0$ is the failure rate. Suppose $\lambda = 0.1$, $a = 2$, and $t = 5$.

1. Calculate $R(a)$ and $R(t)$

$$R(a) = e^{-\lambda a} = e^{-0.1 \cdot 2} = e^{-0.2} \approx 0.8187, \tag{14}$$

$$R(t) = e^{-\lambda t} = e^{-0.1 \cdot 5} = e^{-0.5} \approx 0.6065. \tag{15}$$

2. Compute the Conditional Probability

Using the formula:

$$P(a \leq T \leq t \mid T > 0) = R(a) - R(t), \tag{16}$$

we substitute the values of $R(a)$ and $R(t)$:

$$\begin{aligned} P(a \leq T \leq t \mid T > 0) &= 0.8187 - 0.6065 \quad (17) \\ &= 0.2122. \quad (18) \end{aligned}$$

Final Answer

The conditional probability that the system is operational within the interval $[2, 5]$, given it was operational at $t = 0$, is approximately:

$$P(2 \leq T \leq 5 \mid T > 0) \approx 0.2122. \tag{19}$$