Dependable System Design - Fall 2024

Homework 12

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Conditional Reliability Analysis

Problem Statement

Evaluate the conditional probability of the system being operational during the time interval between $[a, t]$, given that the system was functional at time 0, considering repairs. Repairs before time a are acceptable, but the system must remain operational during the interval $[a, t]$ with no repairs allowed.

Definitions

- **Reliability Function:** The reliability function, $R(t)$, is the probability that the random time to failure T exceeds t:

$$R(t) = P(T > t) = 1 - F(t), \tag{1}$$

  where $F(t)$ is the cumulative distribution function (CDF).

- **Conditional Reliability Function:** The conditional reliability function, $m(x)$, is the probability that the system survives an additional time x, given it has already survived up to time t:

$$m(x) = P(T > t + x \mid T > t) = \frac{R(t+x)}{R(t)}. \tag{2}$$

- **Final Conditional Probability:** The conditional probability of the system being operational during $[a, t]$, given it was functional at time 0:

$$P(T > t \text{ in } [a, t] \mid T > 0) = \frac{R(t)}{R(0)}. \tag{3}$$

Solution

1. Define the Required Probability

We need:

$$P(T > t \text{ in } [a, t] \mid T > 0) = P(T > t \mid T > a) \cdot P(T > a \mid T > 0). \tag{4}$$

2. Use the Conditional Reliability Function

From the definition of the conditional reliability function:

$$P(T > t \mid T > a) = \frac{R(t)}{R(a)}. \tag{5}$$

The probability of surviving up to a, given survival at time 0, is:

$$P(T > a \mid T > 0) = \frac{R(a)}{R(0)}. \tag{6}$$

3. Combine Results

Multiplying these probabilities:

$$\begin{aligned} P(T > t \text{ in } [a, t] \mid T > 0) &= P(T > t \mid T > a) \cdot P(T > a \mid T > 0) && (7) \\ &= \frac{R(t)}{R(a)} \cdot \frac{R(a)}{R(0)} && (8) \\ &= \frac{R(t)}{R(0)}. && (9) \end{aligned}$$

Numerical Example

Suppose the reliability function of a system is given by:

$$R(t) = e^{-\lambda t}, \tag{10}$$

where $\lambda = 0.1$ is the failure rate. Let $a = 5$ and $t = 10$, and we want to calculate the conditional probability that the system is operational during $[5, 10]$ given it was operational at $t = 0$.

1. Compute $R(t)$ and $R(0)$

$$R(0) = e^{-0.1 \cdot 0} = 1, \tag{11}$$
$$R(10) = e^{-0.1 \cdot 10} = e^{-1} \approx 0.3679. \tag{12}$$

2. Compute the Conditional Probability

Using the formula:

$$P(T > t \text{ in } [a, t] \mid T > 0) = \frac{R(t)}{R(0)}. \tag{13}$$

Substitute the values:

$$P(T > 10 \text{ in } [5, 10] \mid T > 0) = \frac{R(10)}{R(0)} \tag{14}$$
$$= \frac{0.3679}{1} \tag{15}$$
$$= 0.3679. \tag{16}$$

Thus, the conditional probability that the system remains operational during $[5, 10]$, given it was operational at $t = 0$, is approximately 36.79%.