به نام خدا

تمرین ۴: محاسبات مارکف

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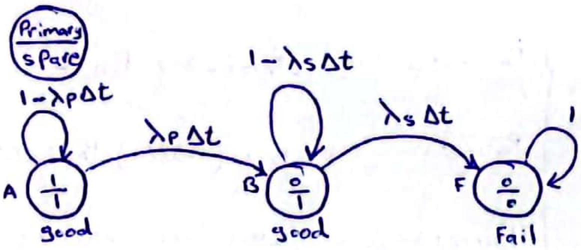
① مدل Cold stand by Spare :



$$\begin{cases} P_A(t+\Delta t) = P_A(t)\cdot(1-\lambda_p\Delta t) \\ P_B(t+\Delta t) = P_B(t)\cdot(1-\lambda_s\Delta t) + P_A(t)\cdot\lambda_p\Delta t \\ P_F(t+\Delta t) = P_B(t)\cdot\lambda_s\Delta t + P_F(t)\cdot 1 \end{cases} \longrightarrow \begin{cases} P'_A(t) = -\lambda_p P_A(t) \\ P'_B(t) = \lambda_p P_A(t) - \lambda_s P_B(t) \\ P'_F(t) = \lambda_s P_B(t) \end{cases}$$

$$\xrightarrow{\mathcal{L}} \begin{cases} sP_A(s) - \cancel{P_A(0)}^{1} = -\lambda_p P_A(s) \\ sP_B(s) - \cancel{P_B(0)}^{0} = \lambda_p P_A(s) - \lambda_s P_B(s) \\ sP_F(s) - \cancel{P_F(0)}^{0} = \lambda_s P_B(s) \end{cases} \Rightarrow \begin{cases} P_A(s) = \dfrac{1}{s+\lambda_p} & (\mathrm{I}) \\ P_B(s) = \dfrac{\lambda_p}{(s+\lambda_p)(s+\lambda_s)} & (\mathrm{II}) \end{cases}$$

$$\mathcal{L}^{-1}\{P_A(s)\} = P_A(t) = e^{-\lambda_p t}$$

$$\mathcal{L}^{-1}\{P_B(s)\} = P_B(t) : \frac{A}{s+\lambda_p} + \frac{B}{s+\lambda_s} = \frac{A(s+\lambda_s)+B(s+\lambda_p)}{(s+\lambda_p)(s+\lambda_s)} \quad \Bigg| \quad A = \frac{-\lambda_p}{-\lambda_s+\lambda_p} \qquad B = \frac{\lambda_p}{\lambda_p-\lambda_s}$$

$$\Rightarrow P_B(s) = \frac{\frac{-\lambda_p}{-\lambda_s+\lambda_p}}{s+\lambda_p} + \frac{\frac{\lambda_p}{\lambda_p-\lambda_s}}{s+\lambda_s} \xrightarrow{\mathcal{L}^{-1}} P_B(t) = \frac{\lambda_p}{\lambda_p-\lambda_s} e^{-\lambda_s t} - \frac{\lambda_p}{\lambda_p-\lambda_s} e^{-\lambda_p t}$$

$$= \frac{\lambda_p}{\lambda_p-\lambda_s}\left(e^{-\lambda_s t} - e^{-\lambda_p t}\right)$$

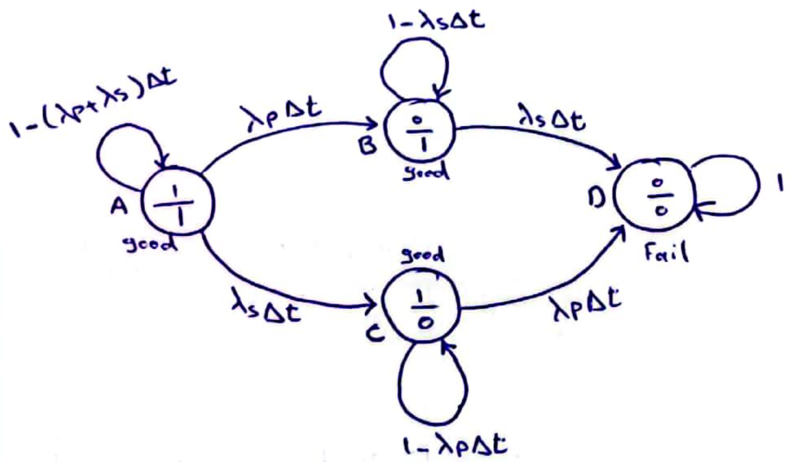
$$\Rightarrow \boxed{R(t) = P_A(t) + P_B(t) = e^{-\lambda_p t} + \frac{\lambda_p}{\lambda_p-\lambda_s}\left(e^{-\lambda_s t} + e^{-\lambda_p t}\right) \quad ; \quad \lambda_p \neq \lambda_s}$$

$$\text{if } \quad \lambda_p = \lambda_s : \quad \begin{cases} P_A(s) = \dfrac{1}{s+\lambda} \\ P_B(s) = \dfrac{1}{(s+\lambda)^2} \\ P_F(s) = \dfrac{\lambda}{s}\cdot\dfrac{\lambda}{(s+\lambda)^2} \end{cases} \xrightarrow{\mathcal{L}^{-1}} \begin{cases} P_A(t) = e^{-\lambda t} \\ P_B(t) = \lambda t e^{-\lambda t} \end{cases}$$

$$\Rightarrow R(t) = P_A(t) + P_B(t) = e^{-\lambda t} + \lambda t e^{-\lambda t}$$

② مدل : Hot stand by spare

← همان مدل موازی است.



* با فرض اینکه $\lambda_s \neq \lambda_p$ مسئله را حل می کنیم

$$P_A(t+\Delta t) = P_{A(t)}\left(1-(\lambda_p+\lambda_s)\Delta t\right)$$
$$P_B(t+\Delta t) = P_{B(t)}\left(1-\lambda_s\Delta t\right) + P_A(\lambda_p\Delta t)$$
$$P_c(t+\Delta t) = P_c(t)\cdot(1-\lambda_p\Delta t) + P_A(t)\lambda_s\Delta t$$
$$P_D(t+\Delta t) = P_D(t)\cdot 1 + P_B(t)\cdot\lambda_s\Delta t + P_B(t)\lambda_p\Delta t$$

$$\longrightarrow \begin{cases} \dfrac{P_A(t+\Delta t)-P_A(t)}{\Delta t} = \dfrac{P_A(t)-\lambda_p\Delta t P_A(t)-\lambda_s\Delta t P_A(t) - P_A(t)}{\Delta t} \\ \dfrac{P_B(t+\Delta t)-P_B(t)}{\Delta t} = \dfrac{P_B(t)(1-\lambda_s\Delta t)+P_A(\lambda_p\Delta t)-P_B(t)}{\Delta t} \\ \dfrac{P_c(t+\Delta t)-P_c(t)}{\Delta t} = \dfrac{P_c(t)(1-\lambda_p\Delta t)+P_A(t)\lambda_s\Delta t - P_c(t)}{\Delta t} \\ \dfrac{P_D(t+\Delta t)-P_D(t)}{\Delta t} = \dfrac{P_D+P_B(t)\lambda_s\Delta t+P_B(t)\lambda_p\Delta t-P_D}{\Delta t} \end{cases}$$

$$\rightarrow \begin{cases} P'_A(t) = -P_A(t)(\lambda_s+\lambda_p) \\ P'_B(t) = P_A(t)\lambda_p - \lambda_s P_B(t) \\ P'_c(t) = P_A(t)\lambda_s - \lambda_p P_c(t) \\ P'_D(t) = P_B(t)(\lambda_s+\lambda_p) \end{cases} \xrightarrow{L} \begin{cases} SP_A(s) - \cancel{P_A(0)}^{1} = -P_A(s)(\lambda_s+\lambda_p) \\ SP_B(s) - \cancel{P_B(0)}^{0} = P_A(s)\lambda_p - \lambda_s P_B(s) \\ SP_c(s) - \cancel{P_c(0)}^{0} = P_A(s)\lambda_s - \lambda_p P_c(s) \\ SP_D(s) - \cancel{P_D(0)}^{0} = P_B(s)(\lambda_s+\lambda_p) \end{cases}$$

$$\Rightarrow \begin{cases} SP_A(s) = -P_A(s)(\lambda_s+\lambda_p) & (\mathrm{I}) \\ SP_B(s) = P_A(s)\lambda_p - \lambda_s P_B(s) & (\mathrm{II}) \\ SP_c(s) = P_A(s)\lambda_s - \lambda_p P_c(s) & (\mathrm{III}) \\ SP_D(s) = P_B(s)(\lambda_s+\lambda_p) & (\mathrm{IV}) \end{cases}$$

$$\Rightarrow SP_A(s) + P_A(s)\lambda_s + P_A(s)\lambda_p = 1 \Rightarrow P_A(s)(S+\lambda_s+\lambda_p) = 1 \Rightarrow \boxed{P_A(s) = \frac{1}{S+\lambda_s+\lambda_p}} \; *$$

$$* \xrightarrow{\text{in (II)}} SP_B(s) + \lambda_s P_B(s) = \frac{\lambda_p}{S+\lambda_s+\lambda_p} \Rightarrow P_B(s)(S+\lambda_s) = \frac{\lambda_p}{S+\lambda_s+\lambda_p}$$

$$\Rightarrow \boxed{P_B = \frac{\lambda_p}{(S+\lambda_s)(S+\lambda_s+\lambda_p)}} \; **$$

$$* \xrightarrow{\text{in (III)}} SP_c(s) + \lambda_p P_c(s) = \frac{\lambda_s}{S+\lambda_s+\lambda_p} \Rightarrow \boxed{P_c(s) = \frac{\lambda_s}{(S+\lambda_p)(S+\lambda_s+\lambda_p)}} \; ***$$

$$** \xrightarrow{\text{in (IV)}} \boxed{P_D(s) = \frac{\lambda_p(\lambda_s+\lambda_p)}{s(S+\lambda_s)(S+\lambda_s+\lambda_p)}}$$

① $P_A(s) = \dfrac{1}{s+(\lambda_s+\lambda_p)} \xleftrightarrow{L^{-1}} \boxed{P_A(t) = e^{-(\lambda_s+\lambda_p)t}}$

② $P_B(s) = \dfrac{\lambda_p}{(s+\lambda_s)(s+\lambda_s+\lambda_p)} = \dfrac{A}{s+\lambda_s} + \dfrac{B}{(s+\lambda_s+\lambda_p)} = \dfrac{A(s+\lambda_s+\lambda_p)+B(s+\lambda_s)}{(s+\lambda_s)(s+\lambda_s+\lambda_p)}$

$= \begin{cases} A = 1 \\ B = -1 \end{cases} \Rightarrow P_B(s) = \dfrac{1}{s+\lambda_s} + \dfrac{-1}{(s+\lambda_s+\lambda_p)} \xleftrightarrow{L^{-1}} \boxed{P_B(t) = e^{-\lambda_s t} - e^{-(\lambda_s+\lambda_p)t}}$

③ $P_c(s) = \dfrac{\lambda_s}{(s+\lambda_p)(s+\lambda_s+\lambda_p)} = \lambda_s \times \dfrac{1}{(s+\lambda_p)(s+\lambda_s+\lambda_p)} = \lambda_s \times \dfrac{1}{s+\lambda_p} \times \dfrac{1}{s+\lambda_p+\lambda_s}$

$= \dfrac{A}{s+\lambda_p} + \dfrac{B}{s+\lambda_s+\lambda_p} = \dfrac{A(s+\lambda_s+\lambda_p)+B(s+\lambda_p)}{(s+\lambda_p)(s+\lambda_s+\lambda_p)}$ $\quad \Big| \; A = 1, \; B = -1$

$\Rightarrow P_c(s) = \dfrac{1}{s+\lambda_p} + \dfrac{-1}{s+\lambda_s+\lambda_p} \xleftrightarrow{L^{-1}} \boxed{P_c(t) = e^{-\lambda_p t} - e^{-(\lambda_s+\lambda_p)t}}$

④ $P_D(s) = \dfrac{\lambda_p(\lambda_s+\lambda_p)}{s(s+\lambda_s)(s+\lambda_s+\lambda_p)} = \dfrac{A}{s} + \dfrac{B}{s+\lambda_s} + \dfrac{C}{s+\lambda_s+\lambda_p}$ $\quad \Big| \; A = \dfrac{\lambda_p}{\lambda_s}, \; B = ?, \; C = ?$

$$\boxed{P_D(t) = \frac{\lambda_p}{\lambda_s} + B e^{-\lambda_s t} + C e^{-(\lambda_s+\lambda_p)t}}$$

$\Rightarrow R(t) = P_A(t) + P_B(t) + P_c(t) = \cancel{e^{-(\lambda_s+\lambda_p)t}} + e^{-\lambda_s t} - e^{-(\lambda_s+\lambda_p)t} + e^{-\lambda_p t} - \cancel{e^{-(\lambda_s+\lambda_p)t}}$

$\cancel{= 2e^{-(\lambda_s+\lambda_p)t}}$

$$= e^{-\lambda_s t} + e^{-\lambda_p t} - e^{-(\lambda_s+\lambda_p)t}$$

* تمامی محاسبات برای سیستم موازی دقیقاً مشابه با Hot stand By Spare می‌شود