1.5 completing the square of quadratic equation

In this section, we will learn how we can rewrite any quadratic equation of the form

$$ax^2 + bx + c = 0$$

to be in the form

$$(x - h)^2 = k$$

This rewriting process is called completing the square of quadratic equation $ax^2 + bx + c = 0$.

Note that by completing the square process, the equation becomes easy to solve by extracting roots for both sides of the equation.

Definition 1.5.1 The trinomial $ax^2 + bx + c$ is a called perfect square trinomial if and only if

$$(\frac{b}{2})^2 = c$$

(R) It is important to point out that the leading coefficient must be equal to 1 for this to be true.

Example 1.13 Complete the square:

$$x^2 + 8x + \ ? \ = (x + \ ?)^2$$

.

In this example, the coefficient of the middle term $b = 8$, so find the value that completes the square as follows:

$$\begin{array}{lll} x^2+ & 8x+ & 16 = (x+2)^2 \\ & \downarrow & \uparrow \\ & (\frac{8}{2})^2 = & 16 \end{array}$$

The value that completes the square is 16.

■

We can use this technique to solve quadratic equations. The idea is to take any quadratic equation in standard form and complete the square so that we can solve it by extracting roots. The following are general steps for solving a quadratic equation in standard form by completing the square.

R Completing the square for the polynomials is a powerful technique to solve any quadratic equation, even it is a prime equation with complex roots. Moreover, it is the foundation of the quadratic formula, Our subject to the next section. The following algorithm shows how to solve by completing the square and how the quadratic formula is derived.

$ax^2+bx+c=0$	Divide by a, the leading coefficient.
$x^2+\frac{b}{a}x+\frac{c}{a}=0$	Transpose the constant term to the RHS.
$x^2+\frac{b}{a}x=-\frac{c}{a}$	Add $\left(\frac{1}{2}\left(\frac{b}{a}\right)\right)^2$ to both sides
$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}$	Factor the LHS and simplify the RHS.
$\left(x+\frac{b}{a}\right)^2=\frac{b^2-4ac}{4a^2}$	Take the square root of both sides.
$\lvert x+\frac{b}{a}\rvert=\frac{\sqrt{b^2-4ac}}{2a}$	Solve for$x$.
$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	**Quadratic Formula**.

In the following example we will explain above steps in a numerical example.

■ **Example 1.14** Solve by completing the square:

$$2x^2+16x+46=0$$

Step 1: Divide the equation by $a=2$, the leading coefficient.

$$x^2+8x+23=0$$

Step 2: Transpose the constant term to the right hand side of the equation.

$$x^2+8x=-23$$

Step 3: Add the $\left(\frac{b}{2}\right)^2$ to both sides of the equation, to make the left hand side a complete square trinomial. Here $\left(\frac{8}{2}\right)^2=8^2=64$. So,

$$x^2+8x+64=-23+64$$

Step 4: Factor the left hand side and simplify the right hand side.

$$(x+4)^2=41$$

Step 5: Take the square root of both sides.

$$|x+4|=\sqrt{41}$$

Step 6: Solve for the absolute value equation for x.

$$x+4=\sqrt{41} \qquad \text{Or} \qquad x+4=-\sqrt{41}$$
$$x=-4+\sqrt{41} \qquad\qquad x=-4-\sqrt{41}$$

■

■ **Example 1.15**

$2y^2+7y-4=0$	Divide by 2, the leading coefficient.
$x^2+\frac{7}{2}x+2=0$	Transpose the constant term to the RHS.
$x^2+\frac{7}{2}x=-2$	Add $\left(\frac{1}{2}\left(\frac{7}{2}\right)\right)^2$ to both sides
$x^2+\frac{7}{2}x+\frac{49}{16}=-2+\frac{49}{16}$	Factor the LHS and simplify the RHS.
$\left(x+\frac{7}{2}\right)^2=\frac{49-32}{16}$	Take the square root of both sides.
$\lvert x+\frac{7}{2}\rvert=\frac{\sqrt{17}}{4}$	Solve for$x$.
$x=\frac{-7\pm\sqrt{17}}{4}$	**Quadratic Formula**.

■

Exercise 1.1 Part A: Completing the Square

Complete the square.

1. $x^2+6x+\ ?\ =(x+\ ?)^2$
2. $x^2+8x+\ ?\ =(x+\ ?)^2$
3. $x^2-2x+\ ?\ =(x-\ ?)^2$
4. $x^2-4x+\ ?\ =(x-\ ?)^2$
5. $x^2+7x+\ ?\ =(x+\ ?)^2$
6. $x^2+3x+\ ?\ =(x+\ ?)^2$
7. $x^2+\frac{2}{3}x+\ ?\ =(x+\ ?)^2$
8. $x^2+\frac{4}{5}x+\ ?\ =(x+\ ?)^2$
9. $x^2+\frac{3}{4}x+\ ?\ =(x+\ ?)^2$
10. $x^2+\frac{5}{3}x+\ ?\ =(x+\ ?)^2$

■

Exercise 1.2 *Solve by factoring and then solve by completing the square. Check answers.*

11. $x^2+2x-8=0$
12. $x^2-8x+15=0$
13. $y^2+2y-24=0$
14. $y^2-12y+11=0$
15. $t^2+3t-28=0$
16. $t^2-7t+10=0$
17. $2x^2+3x-2=0$
18. $3x^2-x-2=0$
19. $2y^2-y-1=0$
20. $2y^2+7y-4=0$

■

Exercise 1.3 *Solve by completing the square.*

21. $x^2+6x-1=0$
22. $x^2+8x+10=0$
23. $x^2-2x-7=0$
24. $x^2-6x-3=0$
25. $x^2-2x+4=0$
26. $x^2-4x+9=0$
27. $t^2+10t-75=0$
28. $t^2+12t-108=0$
29. $x^2-4x-1=15$
30. $x^2-12x+8=-10$
31. $y^2-20y=-25$
32. $y^2+18y=-53$
33. $x^2-0.6x-0.27=0$
34. $x^2-1.6x-0.8=0$
35. $x^2-\frac{2}{3}x-\frac{1}{3}=0$
36. $x^2-\frac{4}{5}x-\frac{1}{5}=0$
37. $x^2+x-1=0$
38. $x^2+x-3=0$
39. $y^2+3y-2=0$
40. $y^2+5y-3=0$
41. $x^2+3x+5=0$
42. $x^2+x+1=0$
43. $x^2-7x+\frac{11}{2}=0$
44. $x^2-9x+\frac{3}{2}=0$
45. $t^2-\frac{1}{2}t-1=0$
46. $t^2-\frac{1}{3}t-2=0$
47. $x^2-1.7x-0.0875=0$
48. $x^2+3.3x-1.2775=0$
49. $4x^2-8x-1=0$
50. $2x^2-4x-3=0$
51. $3x^2+6x+1=0$
52. $5x^2+10x+2=0$
53. $3x^2+2x-3=0$
54. $5x^2+2x-5=0$
55. $4x^2-12x-15=0$
56. $2x^2+4x-43=0$
57. $2x^2-4x+10=0$
58. $6x^2-24x+42=0$
59. $2x^2-x-2=0$
60. $2x^2+3x-1=0$
61. $3x^2+2x-2=0$
62. $3x^2-x-1=0$
63. $x(x+1)-11(x-2)=0$
64. $(x+1)(x+7)-4(3x+2)=0$
65. $y^2=(2y+3)(y-1)-2(y-1)$
66. $(2y+5)(y-5)-y(y-8)=-24$
67. $(t+2)^2=3(3t+1)$
68. $(3t+2)(t-4)-(t-8)=1-10t$ ■

Exercise 1.4 *Solve by completing the square and round off the solutions to the nearest hundredth.*

69. $(2x-1)^2 = 2x$
70. $(3x-2)^2 = 5-15x$
71. $(2x+1)(3x+1) = 9x+4$
72. $(3x+1)(4x-1) = 17x-4$
73. $9x(x-1) - 2(2x-1) = -4x$
74. $(6x+1)^2 - 6(6x+1) = 0$

Exercise 1.5 **Part B: Discussion Board** 75. Research and discuss the Hindu method for completing the square. 76. Explain why the technique for completing the square described in this section requires that the leading coefficient be equal to 1.

Solutions: To get the solution sheet. (Click here)