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Chapter 8: Factoring Polynomials

Among all the topics we will cover in class, factoring polynomials is probably the most important topic. There are many sections in later chapters where the first step will be to factor a polynomial. So, if you can't factor the polynomial then you won't be able to even start the problem let alone finish it.

Let's start out by talking a little bit about just what factoring is.

Definition 0.0.1 ***Factoring*** *is the process to find two or more quantities whose product equals the original quantity.*

We do this all the time with numbers. For instance, here are a variety of ways to factor 12.

$$\begin{array}{ccc} 12 = (2)(6) & 12 = (-2)(-6) & 12 = (3)(4) \\ 12 = (\frac{1}{2})(24) & 12 = (-2)(-2)(3) & 12 = (2)(2)(3) \end{array}$$

There are many more possible ways to factor 12, but these are representative of many of them.

Definition 0.0.2 *A* ***prime number*** *is an integer whose only positive factors are 1 and itself. For example, 2, 3, 5, and 7 are all examples of prime numbers. Examples of numbers that are not prime are 4, 6, and 12 to pick a few.*

A common method of factoring numbers is to **completely factor (prime factorization)** the number into *positive prime factors*.

If we completely factor a number into positive prime factors there will be a unique representation for that number up to order. That is the reason for factoring things in this way. For our example above with 12, the complete factorization is,

$$12 = (2)(3)(2)$$

Factoring polynomials is done in pretty much the same manner. We determine all the terms that were multiplied together to get the given polynomial. We then try to factor each of the terms we found in the first step. This continues until we simply can't factor anymore. When we can't do any more factoring we will say that the polynomial is completely factored.

Here are a couple of examples.

$$x^2 - 16 = (x + 4)(x - 4)$$

This is completely factored since neither of the two factors on the right can be further factored. Likewise,

$$x^4 - 16 = (x^2 + 4)(x^2 - 4)$$

is not completely factored because the second factor can be further factored. Note that the first factor is completely factored however. Here is the complete factorization of this polynomial.

$$x^4 - 16 = (x^2 + 4)(x + 2)(x - 2)$$

The purpose of this section is to familiarize ourselves with many of the techniques for factoring polynomials.

0.1 The GCF and Factoring by Grouping

0.1.1 Greatest Common Factor (GCF)

The first method for factoring polynomials will be factoring out the **greatest common factor (GCF)**. When factoring in general this will also be the first method that we should try as it will often simplify the problem.

Problem 0.1.1 ***So, how we find the greatest common factor of two or more monomials??***

- Write the prime factorization of each coefficient in exponent notation.
- Choose the least power of each common prime factor that appears in all coefficients.

The GCF is the product of those prime factors.

Example 0.1.2 *Find the GCF of monomials*
$6a^3c, 12a^2c^2, and - 36a^3c$
$6a^3c = (2)(3)a^3c$
$12a^2c^2 = (2)(2)(3)a^2c^2 = (2^2)(3)\mathrm{a}^2c^2$
$-36a^3c = (-1)(2)(2)(3)(3)a^3c = (-1)(2^2)(3^2)a^3\mathrm{c}$
The GCF $= (2)(3)\mathrm{a}^2\mathrm{c}{=}6\mathrm{a}^2\mathrm{c}$

Problem 0.1.3 ***How to factorize the polynomials by using the GCF??***

1. Find the GCF of each term and
2. Divide the original polynomial by GCF to obtain the second factor.

The second factor will be a polynomial.

Remark 0.1.4 *Note that Factoring is the reverse process of multiplying.*

Example 0.1.5 *Factor each polynomial using the greatest monomial factor.*
$12x^3z^5 + 72x^5y^2z - 60x^2y$
Finding GCF
$12x^3z^5 = 2^2(3)\ x^3z^5$
$72x^5y^2z = (3^2)(2^3)x^5y^2z$
$60x^2y = (3)(5)(2^2)\text{x}^2\text{y}$
$GCF = (2^2)(3)x^2 = 12x^2$
(Hint: you can use mental math to find out the greatest number divides all the terms of the polynomial).
$12x^3z^5y + 72x^5y^2z - 60x^2y$
$= 12x^2(xz^5y + 6x^3y^2z - 5y)$

Try these(Q8):

$$10a^3b^3 + 25a^2b^3 - 5a^2b$$
$$= (5a^2b)(ab^2 + 5b^2 - 1)$$

0.1.2 Factor a polynomial with four terms by grouping.

In some cases there is not a GCF for ALL the terms in a polynomial. If you have four terms with no GCF, then try factoring by grouping.

Step 1: Group the first two terms together and then the last two terms together.

Step 2: Factor out a GCF from each of the paired factors. If there is not a GCF, factor out a "1".

Step 3: The remaining terms inside the two sets of parenthesis should be identical. This is one factor of the trinomial. The other factor is formed by combining the GCF's into a second set of parenthesis.

Example 0.1.6 *Factor* $3xy + 2x + 6y + 3$

Step 1: *Group the first two terms together and then the last two terms together.*

$(3xy + 2x) + (9y + 6)$

Step 2: Factor out a GCF *from each separate binomial.*

$x(3y+2)+3(3y+2)$

Step 3: Factor out the common binomial.

$(3y+2)(x+3)$

Example 0.1.7 *Factor* $2x^3-6x^2+x-3$

Note how there is not a GCF for ALL the terms. So let's go ahead and factor this by grouping.

Step 1: Group the first two terms together and then the last two terms together.

Example 0.1.8 $(2x^3-6x^2)+(x-3)$

Step 2: *Factor out a* GCF *from each separate binomial.*

$2x^2(x-3)+1(x-3)$

Step 3: *Factor out the common binomial.*

$(x-3)(2x^2+1)$

0.2 Factoring Trinomials

After completing this section, you should be able to:
Factor a trinomial of the form x^2+bx+c.
Factor a trinomial of the form $ax^2+bx+c, a \neq 1$
Indicate if a polynomial is a prime polynomial.

0.2.1 Factor Trinomials: x^2+bx+c

Basically, we are reversing the FOIL method to get our factored form. We are looking for two binomials that when you multiply them you get the given trinomial.

Example 0.2.1 x^2-x-6

Step 1: Set up a product of two ( ) where each will hold two terms.

It will look like this: (   )(   )

Step 2: Find the factors that go in the first positions.

To get the x^2 (which is the F in FOIL), we would have to have an x in the first positions in each ( ).

So it would look like this: $(x \qquad)(x \qquad)$.

Step 3: Find the factors that go in the last positions.

The factors that would go in the last position would have to be two expressions such that**:**

- **Their product equals c (the constant)**
- **Their sum equals b (number in front of x term).**

As you are finding these factors, you have to consider the sign of the expressions:

- If c is positive, your factors are going to both have the same sign depending on b's sign.
- If c is negative, your factors are going to have opposite signs depending on b's sign.

Example 0.2.2 *Factor* $16d + d^2 + 28$
First reorder the terms of polynomial to be
$d^2 + 16d + 28$
Since we have d squared as our first term, we will need the following:
$(d \qquad)(d \qquad)$
Now, we need two numbers whose product is 28 and sum is 16. That would have to be 14 and 2.
Putting that into our factors we get:
$(d\ +2\ )(d + 14\ \ )$
By calculater:
$x = -2, x = -14$
$(x + 2)(x + 14)$

Example 0.2.3
$$p^2 - 5rp - 24r^2$$
$$= (p + 3r)(p - 8r)$$

0.2.2 Factor Trinomials: $ax^2 + bx + c, a \neq 1$

If a Trinomial of the form $ax^2 + bx + c$ is factorable, it can be done using the **AC Method.**

Step 1. Make sure the trinomial is in standard form $(ax^2 + bx + c)$.

Step 2. Factor out a GCF (Greatest Common Factor) if applicable.

Step 3. Multiply "$a \cdot c$" and identify "b".

Step 4. Begin listing factor pairs of "$a \cdot c$" . Continue until you find the pair of numbers thier product equal " $a \cdot c$ " ,and their sum equal "b".

Step 5. Use the two numbers found in Step 4 to rewrite the trinomial as a 4 term polynomial by breaking up the middle term into two parts.

Step 6. Factor the resulting polynomial using the Grouping Method by grouping the first two terms together and grouping the second two terms together.

Step 7. Check the answer - Multiply the factors to verify that you get the original trinomial.

Example 0.2.4 $11y + 2y^2 - 6$

Step 1. Make sure the trinomial is in standard form $(ax^2 + bx + c)$.

Example 0.2.5 $2y^2 + 11y - 6$

Step 3. Multiply "$a \cdot c$" and identify "b".

Example 0.2.6 $ac = -12,\ b = 11$

Step 4. *Begin listing factor pairs of "$a \cdot c$" . Continue until you find the* ***pair of numbers thier product equal " $a \cdot c$ " ,and their sum equal "b".***

First number$= 12$, *Second number* $= -1$

Step 5. *Use the two numbers found in Step 4 to rewrite the trinomial as a 4 term polynomial by breaking up the middle term into two parts.*

$2y^2 + 11\text{y} - 6$
$2y^2$+12y-y -6

Step 6. *Factor the resulting polynomial using the Grouping Method.*

$2y^2+11y-6$
$2y^2+12y-y\ -6$
$(2y^2+12y)+(-y\ -6)$
$2y(y+6)-1(y+6)$
$(y+6)(2y-1)$
By calculator
$y=\frac{1}{2}\Longrightarrow 2y=1\Longrightarrow 2y-1$
$y=-6\Longrightarrow y+6$
$(x+6)(2x-1)$

Q8:(Try these) $6x^2-19x+15$

$a\cdot c=15\cdot 6=90, \qquad b=-19$

First number=-10, Second number=-9
$6x^2-19x+15$
$6x^2-9x-10x+15$
$(6x^2-9x)+(-10x+15)$
$3x(2x-3)-5(2x-3)$
$(2x-3)(3x-5)$

Q5:(Try These) $-12y+9+4y^2$

$4y^2-12y+9$
$a\cdot c=36, \quad b=-12$
$First=-6 \quad Second=-6$

$4y^2-12y+9$
$4y^2-6y-6y+9$
$(4y^2-6y)+(-6y+9)$
$2y(2y-3)-3(2y-3)$
$(2y-3)(2y-3)$

By calculator
$y=\frac{3}{2}\Longrightarrow y-\frac{3}{2}=0\Longrightarrow 2y-3=0$

Remark 0.2.7 *If a polynomial can be written as a product of factors, then it is called* ***factorable*** *if not its called* ***prime polynomial.*** *Sometimes you will*

not know it is prime until you start looking for factors of it. Once you have exhausted all possibilities, then you can call it prime. ***Be careful. Do not think because you could not factor it on the first try that it is prime. You must go through ALL possibilities first before declaring it prime.***

0.3 Factoring Special Products

After completing this section, you should be able to:

- Factor a perfect square trinomial.
- Factor a difference of squares.
- Factor a sum or difference of cubes.

0.3.1 Factoring a Perfect Square Trinomial

Perfect square trinomial

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

$$(First \pm Second)^2 = (First)^2 \pm 2(First)(Second) + (Second)^2$$

Depending on the above special product:

- we will learn how we can perform the above formula to square any binomial.
- we will learn how we can determine if any trinomial is a perfect square or not.

Example 0.3.1 *Simplify* $(3x + 2y)^2$

$$\begin{aligned}(3x + 2y)^2 &= 9x^2 + 2(3x)(2y) + 4y^2\\ &= 9x^2 + 12xy + 4y^2\end{aligned}$$

Simplify $(6x - 2)^2$

$$\begin{aligned}(6x - 2)^2 &= 36x^2 - 2(6x)(2) + 4\\ &= 36x^2 - 24x + 4\end{aligned}$$

$$[5x + (-4y)]^2 = (5x - 4y)^2 = 25x^2 - 2(5x)(4y) + 16y^2 = 25x^2 - 40xy + 16y^2$$

Exercise 0.3.2 (Try these) *Determine if each trinomial is a perfect square.*

Example 0.3.3 $4x^2 + 20x + 25$
$\stackrel{?}{=} (2x)^2 + 2(2x)(5) + 5^2$
$= (2x + 5)^2$
Is a perfect square.
$36x^2 + 108x + 81$
$\stackrel{?}{=} (6x)^2 + 2(6x)(9) + 9^2$
$= (6x + 9)^2$
Is a perfect square.
$9m^2 + 15mn + 25n^2$
$\stackrel{?}{=} (3m)^2 + 2(3m)(5n) + (5n)^2$
Not a perfect square, since $2(3m)(5n) \neq 15mn$

0.3.2

0.3.3 Factoring a Difference of Two Squares

Difference of squares
$$a^2 - b^2 = (a + b)(a - b)$$

Note that the sum of two squares DOES NOT factor.

Just like the perfect square trinomial, the difference of two squares has to be exactly in this form to use this rule. When you have the difference of two bases being squared, it factors as the product of the sum and difference of the bases that are being squared.

Example 0.3.4 *Factor* $x^2 - 9$
$x^2 - 9 = (x + 3)(x - 3)$

Example 0.3.5 *Factor* $64y^2 - 81$
$64y^2 - 81 = (8y + 9)(8y - 9)$

Example 0.3.6 *Multiply* $(\frac{1}{5}w + \frac{1}{7}k)(\frac{1}{5}w - \frac{1}{7}k)$
$(\frac{1}{5}w + \frac{1}{7}k)(\frac{1}{5}w - \frac{1}{7}k) = \frac{1}{25}w^2 - \frac{1}{49}k^2$

0.4 Factoring Strategy

1. GCF: Always check for the GCF first, no matter what.
2. Binomials: check which one of the following cases
   (a) difference of squares
   (b) sum of cubes
   (c) difference of cubes
3. Trinomials:
   (a) First, check if its perfect square trinomial:
   (b) $x^2 + bx + c$: search for two numbers their product c and their sum b
   (c) $ax^2 + bx + c, a \neq 1$: use AC method
4. Polynomials with four terms: Factor by grouping

Example 0.4.1 *Factor* $39m^2n^2 + 65mn^2 + 26n^2$

$= 13n^2(3m^2 + 5m + 2)$ *Take out the GCF ($13n^2$)*

$= 13n^2(3m^2 + 3m + 2m + 2)$ *rewrite the middle term as sum of two terms(their product is a*

$= 13n^2[3m(m + 1) + 2(m + 1)]$ *Grouping, take out the GCF from each group*

$= 13n^2\,(m + 1)\,(3m + 2)$ *takeout the GCF $(m + 1)$*

Example 0.4.2 $48t^3 + 88t^2 + 24t$

$= 8t(6t^2 + 11t + 3)$ *Take out the GCF (8t)*

$= 8t(6t^2 + 9t + 2t + 3)$ *rewrite the middle term as sum of two terms(their product is ac)*

$= 8t[3t(2t + 3) + (2t + 3)]$ *Grouping, take out the GCF from each group*

$= 8t(2t + 3)(3t + 1)$ *takeout the GCF $2t + 1)$*

$x = -\frac{3}{2} \Longrightarrow x + \frac{3}{2} = 0 \Longrightarrow 2x + 3$

$x = -\frac{1}{3} \Longrightarrow x + \frac{1}{3} = 3x + 1$

Example 0.4.3 $64x^4 + 64x^3 - 324x^2 - 324x$

$= 4x(16x^3 + 16x^2 - 81x - 81)$ *Take out the GCF (8t)*

$= 4x[16x^2(x + 1) - 81(x + 1)]$ *Grouping, take out the GCF from each group*

$= 4x(x + 1)(16x^2 - 81)$ *takeout the GCF $(x + 1)$*

$4x(x + 1)(4x + 9)(4x - 9)$ *difference of squares*

Example 0.4.4 $27a^4 + 27a^3 - 12a^2 - 12a$

$= 3a(9a^3 + 9a^2 - 4a - 4)$ *Take out the GCF (3a)*

$= 3a[9a^2(a + 1) - 4(a + 1)]$ *Grouping, take out the GCF from each group*

$= 3a(a + 1)(9a^2 - 4)$ *takeout the GCF $(a + 1)$*

$= 3a(a + 1)(3a - 2)(3a + 2)$ *Difference of squares*

Example 0.4.5 $140m^3 + 133m^2 + 21m$
$= 7m(20m^2 + 19m + 3)$ *Take out the GCF (3a)*
$= 7m(20m^2 + 15\text{m}+4\text{m} + 3)$ *rewrite the middle term as sum of two terms(their product is ac)*
$= 7m[5m(4m + 3) + (4m + 3)]$ *Grouping, take out the GCF from each group*
$= 7m(4m + 3)(5m + 1)$ *takeout the GCF* $(4m + 3)$

Example 0.4.6 $a^4b - 5a^2b + 4b$
$= b((a^2)^2 - 5a^2 + 4)$ *Take out the GCF (3a)*
$= b(x^2 - 5x + 4)$ *Let* $x = a^2 \Longrightarrow x^2 = a^4$, *and substitute*
$= b(x - 4)(x - 1)$ *Factoring trinomial* $x^2 + bx + c$
$= b(a^2 - 4)(a^2 - 1)$ *Substitute back the value of* x
$b(a - 2)(a + 2)(a + 1)(a + 1)$ *Difference of squares for* $(a^2 - 4)$ *and* $(a^2 - 1)$

0.5 Factor a sum or difference of cubes.

Factor a sum or difference of cubes.
$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$

Example 0.5.1 *Factor* $27x^3 - 64$
$27x^3 \pm 64 = (3x \pm 4)(9x^2 \mp 12x + 16)$

Example 0.5.2 $x^8 - 1$
$= (\text{x}^4\text{-}1)(x^4 + 1)$ *difference of squares.*
$= (\text{x}^2\text{-}1)(x^2 + 1)(x^4 + 1)$ *difference of squares.*
$= (x - 1)(x + 1)(x^2 + 1)(x^4 + 1)$ *Difference of squares*

Example 0.5.3 $x^6 - 1$
$= (\text{x}^2\text{-}1)(x^4 + x^2 + 1)$ *difference of cubes.*
$= (x - 1)(x + 1)(x^4 + x^2 + 1)$ *difference of squares.*

Example 0.5.4 $y = \frac{5}{4}x + 3$
$4y = 5x + 12$
$-5x + 4y = 12$
$5x - 4y = -12$