Contents

Chapter 6: Systems of Linear Equations and Inequalities.

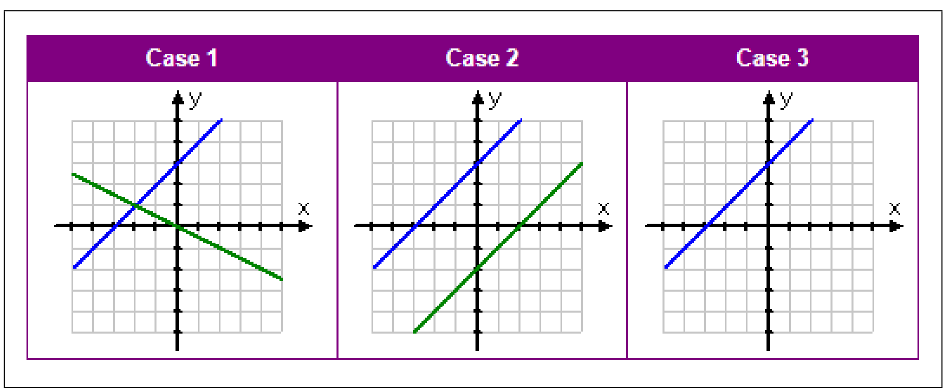
0.1 Solve Systems of Linear Equations Graphically.

Definition 0.1.1 *A system of linear equations is just a set of two or more linear equations. In two variables (x and y), the graph of a system of two equations is a pair of lines in the plane.*

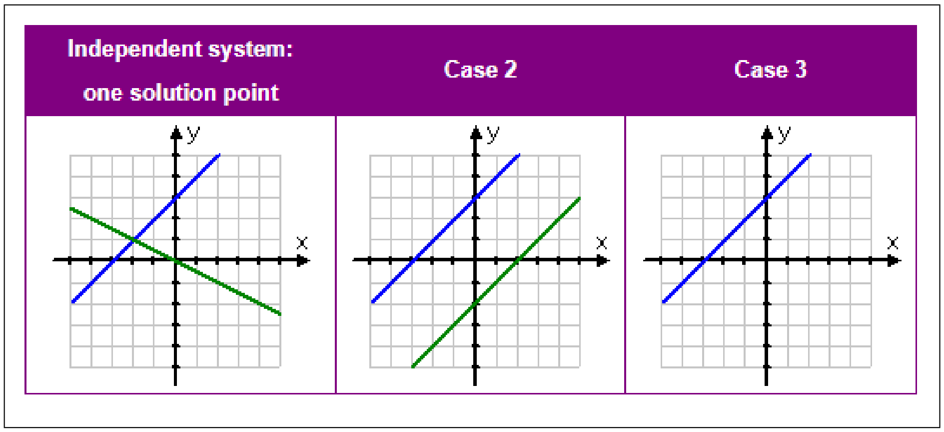
Remark 0.1.2 *When you are solving systems of equations (linear or otherwise), you are, in terms of the equations' related graphed lines, finding any intersection points of those lines.*

For two-variable linear systems of equations, there are then three possible types of solutions to the systems, which correspond to three different types of graphs of two straight lines.

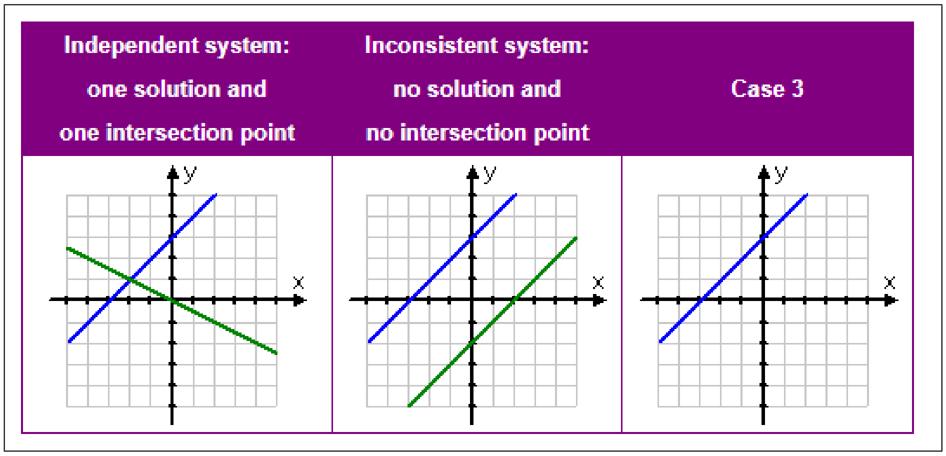
These three cases are illustrated below:



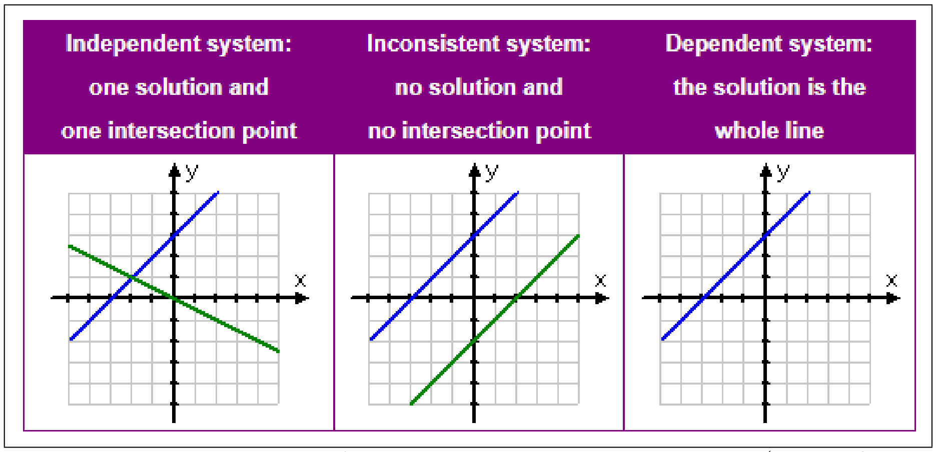
The first graph above, "Case 1", shows two distinct non-parallel lines that cross at exactly one point. This is called an "independent" system of equations, and the solution is always some x,y-point.



The second graph above, "Case 2", shows two distinct lines that are parallel. Since parallel lines never cross, then there can be no intersection; that is, for a system of equations that graphs as parallel lines, there can be no solution. This is called an "inconsistent" system of equations, and it has no solution.



The third graph above, "Case 3", appears to show only one line. Actually, it's the same line drawn twice. These "two" lines, really being the same line, "intersect" at every point along their length. This is called a "dependent" system, and the "solution" is the whole line.



This shows that a system of equations may have one solution (a specific x,y-point), no solution at all, or an infinite solution (being all the solutions to the equation). You will never have a system with two or three solutions; it will always be one, none, or infinitely-many.

Example 0.1.3 $\left\{\begin{smallmatrix} y=\frac{4}{3}-\frac{2}{3}x \\ y=-6+4x \end{smallmatrix}\right\}$

$m_1 = -2/3, m_2 = 4$

Since the slopes of the lines are different, then they are intersect in one point. Therefore, the system is ***consistent*** *and has a* ***unique solution****.*

Example 0.1.4 $\left\{\begin{smallmatrix} 2x+3y=4 \\ 4x+6y=2 \end{smallmatrix}\right\} = \left\{\begin{smallmatrix} y=\frac{4}{3}-\frac{2}{3}x \\ y=\frac{1}{3}-\frac{2}{3}x \end{smallmatrix}\right\}$

$m_1 = -\frac{2}{3}, m_2 = -\frac{2}{3}, b1 = 4/3, b2 = 1/3$

Since the slopes of the lines are similar, and the intercepts are different then there is no intersection point (i.e. Parallel Lines). Therefore, the system is ***inconsistent*** *and* ***has no solution****.*

Example 0.1.5 $\left\{\begin{smallmatrix} 2x+3y=4 \\ 4x+6y=8 \end{smallmatrix}\right\} = \left\{\begin{smallmatrix} y=\frac{4}{3}-\frac{2}{3}x \\ y=\frac{8}{6}-\frac{4}{6}x \end{smallmatrix}\right\}$

$m1 = -\frac{2}{3} = -\frac{4}{6} = m2, b1 = \frac{4}{3} = \frac{8}{6} = b2$

One of the equations is multiple of the other (i.e. ***Dependent system****). So, these tow lines are identical and have the same graph. Therefore, the system is* ***consistent*** *and there are* ***infinitely many solutions****.*

0.2 Solve Systems of Linear Equations by Substitution.

The method of solving "by substitution" works by solving one of the equations (**you choose which one**) for one of the variables (**you choose which one**), and then plugging this back into the other equation, "**substituting**" for the chosen variable and solving for the other. Then you back-solve for the first variable.

Example 0.2.1 *Solve the system*

$$\begin{cases} 2x - 3y = -2 \\ 4x + y = 24 \end{cases}$$

Step 1: *Solve one of the equations (**you choose which one**) for one of the variables (**you choose which one**).*

Remark 0.2.2 *The idea here is to solve one of the equations for one of the variables, and plug this into the other equation. It does not matter which equation or which variable you pick. There is no right or wrong choice; the answer will be the same, regardless. But — some choices may be better than others.*

For instance, in this case, can you see that it would probably be simplest to solve the second equation for "$y =$", since there is already a y floating around loose in the middle there? I could solve the first equation for either variable, but I'd get fractions, and solving the second equation for x would also give me fractions. It wouldn't be "wrong" to make a different choice, but it would probably be more difficult. Being lazy, I'll solve the second equation for y:

$$4x + y = 24$$

$$y = -4x + 24 \quad \text{........ Eq*}$$

Step 2 : Substitute that expression for y in the other linear equation. You'll get an equation in x .

$2x - 3y = -2$

$2x - 3(-4x + 24) = -2$

Step 3 : Solve this, and you have the x -coordinate of the intersection.

$2x - 3(-4x + 24) = -2$

$2x + 12x - 72 = -2$

$14x = 70$

$x = 5$

Step 4 : plug this $x$value back into either equation, and solve for y. But since I already have an expression for "$y =$" in Eq*, it will be simplest to just plug into this:

$y = -4x + 24 \quad$ .... Eq*

$y = -4(5) + 24 = -20 + 24 = 4$

Then the solution is $(x, y) = (5, 4)$.

Example 0.2.3 *Solve the system*

$$\begin{cases} 3x + 2y = 16 \\ 7x + y = 19 \end{cases}$$

Example 0.2.4 *Solve the second equation for y.*

$y = 19 - 7x$ .... *Eq**

Substitute $19 - 7x$ *for y in the first equation and solve for x.*

$$\begin{aligned} 3x + 2y &= 16 \\ \Rightarrow 3x + 2(\mathbf{19} - \mathbf{7x}) &= 16 \\ \Rightarrow 3x + 38 - 14x &= 16 \\ \Rightarrow -11x &= -22 \\ \Rightarrow x &= 2 \end{aligned}$$

Substitute 2 *for x in $y = 19 - 7x$ and solve for y .*

$$\begin{aligned} y &= 19 - 7(2) \\ y &= 5 \end{aligned}$$

The solution is $(2, 5)$.

Example 0.2.5 *A box containing 3 dictionaries and 8 atlases weighs 35 pounds. Each dictionary weighs twice as much as an atlas. How much does each type of book weigh?*

Solution 0.2.6 *Assume $x =$the weigh of dictionary, $y =$the weigh of atlas*

$$\begin{aligned} 3x + 8y &= 35 \\ x &= 2y \end{aligned}$$

Substitute $2y$ *for x in the first equation and solve for y.*

$$\begin{aligned} 3(2y) + 8y &= 35 \\ 14y &= 35 \\ y &= 2.5 \end{aligned}$$

Substitute 2.5 *for y in $x = 2y$ and solve for x.*

$x =$

Therefore, the solution of the linear system is $(5, 2.5)$.

0.3 The elimination method for solving linear systems

Another way of solving a linear system is to use the elimination method. In the elimination method you either add or subtract the equations to get an equation in one variable.

When the coefficients of one variable are **opposites you add the equations to eliminate** a variable and when the coefficients of one variable are equal you **subtract the equations to eliminate a variable**.

Example 0.3.1
$$\begin{aligned} y &= 9 - 3x \\ 5x &= 4y - 22 \end{aligned}$$

Step1: *Rewrite the equations in standard form if they are not.*

$$\begin{aligned} 3x + y &= 9 \\ 5x + 4y &= 22 \end{aligned}$$

Step 2: *Multiply each equation by a suitable number to make the coefficients of one of the variables are the same or opposites.*

Remember that: *If you make them same you need to subtract and if you make them opposites you need to add systems of equations to eliminate that variable.*

Therefore, Multiply the first equation by -4 *so that the coefficients of* y *are opposites*

$$\begin{aligned} -12x + -4y &= -36 \\ 5x + 4y &= 22 \end{aligned}$$

Step 3: Add the second equation to the first we get.

$$-7x = -14$$

Step 4: Solve this new equation for x.

$$-7x = -14 \Longrightarrow x = 2$$

Step 5: Substitute $x = 2$ into either first equation or second equation above and solve for x. We'll use first equation.

$3x + y = 9$
$3(2) + y = 9$
$y = 3$
Finally, the solution of the linear system is $(2, 3)$.

0.4 Solve Equivalent Systems of Linear Equations

- **Equivalent equations** are algebraic equations that have identical solutions or roots.

- Adding or subtracting the same number or expression to both sides of an equation produces an equivalent equation.
- Multiplying or dividing both sides of an equation by the same non-zero number produces an equivalent equation.

`Try these questions`

1. Solve each system of equations by using substitution or elimination.

$$\begin{cases} 3x+6y=3 \\ 4x=8y-5 \end{cases}$$

$$\begin{cases} 3x=3-6y \\ 4x=8y-5 \end{cases} \Longrightarrow \begin{cases} x=1-2y \\ 4x=8y-5 \end{cases}$$

Substitute the first "$x=1-2y$" equation in the second "$4x=8y-5$"

$4x=8y-5$

$4(1-2y)=8y-5$

$4-8y=8y-5$

$4+5=8y+8y$

$9=16y$

$y=\frac{9}{16}$

Find the value of x by substituting $y=\frac{9}{16}$ in " $x=1-2y$"

$x=1-2(\frac{9}{16})=-\frac{1}{8}$

2 $$\begin{cases} 4a-2b=10 \\ 3a+5b=-18\frac{1}{2} \end{cases}$$

$$\begin{cases} 4a-2b=10 \\ 3a+5b=-\frac{37}{2} \end{cases} \xrightarrow{\text{Multiply he second equation by 2}} \begin{cases} 4a-2b=10 \\ 6a+10b=-37 \end{cases}$$

Now, Multiply the first equation by 5 to make the coefficients of y are **opposite**

$$\begin{cases} 20a-10b=50 \\ 6a+10b=-37 \end{cases}$$

Add the equations of system

$26a=13$

$a=\frac{1}{2}$

To find the value of b substitute $a=\frac{1}{2}$ in any equation you want (I will substitute in second one)

$6a+10b=-37$

$6(\frac{1}{2})+10b=-37$

$10b=-37-6(\frac{1}{2})$

$10b=-40$

$b = -4$

Therefore, the solution is the point (1/2,-4).

5 How would you solve the following system:

$$\begin{cases} 3x + 2y = 7 \\ 2y - 1 = 5x \end{cases}$$

1. Would you use the substitution or the elimination method? Show how you

   would set up the system, and explain why you chose that method.

By Elimination: $\begin{cases} 3x + 2y = 7 \\ 2y - 1 = 5x \end{cases}$

$\Longrightarrow \begin{cases} 3x + 2y = 7 \\ -5x + 2y = 1 \end{cases} \xrightarrow{\text{Subtract}} 8x = 6 \Longrightarrow x = \frac{3}{4}$

By Substitution: If I want to solve by substitution method. Then I will solve the second equation for $2y$ because it is common term between the tow equations of the system, and no need to solve for y in this case.

$2y - 1 = 5x$

$2y = 5x + 1$

Now, substitute $2y = 5x + 1$ in the first equation directly.

$3x + 2y = 7$

$3x + (5x + 1) = 7$

$8x + 1 = 7$

$8x = 6$

$x = \frac{6}{8} = \frac{3}{4}$

In Conclusion, Solving by elimination method takes less time.

0.5 Apply Systems of Linear Equations

Remark 0.5.1 *In general, there are five steps for problem solving in algebra.*

1. *Familiarize yourself with the problem situation.*
2. *Translate the problem to an equation.*
3. *Solve the equation.*
4. *Check the answer in the original problem.*
5. *State the answer to the problem clearly.*

Example 0.5.2 *Alan bought 5 pens, some* ***blue pens*** *for $0.70 each and some* ***red pens*** *for $.65 each. If he received $1.65 change from $5.00, how many red pens did he buy?*

Step 1: Determine your variables. (Look to the last phrase of your question how many red pens did he buy?)

r= The number red pens

b=The number blue pens

Step 2: Read carefully, then construct your equations from the given pieces of information by translating the problem into equations.

First equation will related to number pens

Red pens+Blue pens=5

$r + b = 5$........ Eq 1

The second equation is related to price of pens

Firstly, the price of all pens

$5\% - \$1.65 = \$3.35$

The price of **blue pens** + The price of **red pens** =$3.35

$0.70b + 0.65r = 3.35$........ Eq 2

Step 3: *Solve the system*

$$\begin{cases} r + b = 5 \\ 0.70b + 0.65r = 3.35 \end{cases}$$

Multiply the Eq 2 by 100$\Longrightarrow \begin{cases} r + b = 5 \\ 70b + 65r = 335 \end{cases}$

Solving for r in the first equation$\Longrightarrow \begin{cases} r = 5 - b \\ 70b + 65r = 335 \end{cases}$

$$\Longrightarrow \begin{cases} r = 5 - b \\ 70b + 65r = 335 \end{cases}$$

$\Longrightarrow 70b + 65(5 - b) = 335$

$70b + 65(5 - b) = 335$

$\Longrightarrow 5b + 325 = 335$

$\Longrightarrow 5b = 10$

$\Longrightarrow b = 2$

$\Longrightarrow r = 3$

Example 0.5.3 *Brianna has \$3.30 in dimes and quarters. The number of dimes she has is 3 fewer than twice the number of quarters. How many of each coin does she have?*
d= Number of dimes
q=Number of quarters
$d + q = 3.3$
$d = 2q - 3$

0.6 Graph Systems of Linear Inequalities

Definition 0.6.1 *A system of linear inequalities is a set of two or more linear inequalities with the same variables.*

Example 0.6.2 *Sylvia's profit on her handmade jewelry is \$8 on a pair of earrings and \$12 on a necklace. She has the materials to make at most 20 pairs of earrings and 16 necklaces. From these, Sylvia wants to make a profit of at least \$240. Write a system of inequalities to represent the situation. Graph the system to show possible solutions.*
$8e + 12n \geq 240$
$n \leq 16$
$e \leq 20$
$e > 0$
$n > 0$

Example 0.6.3 *For her birthday, Djana received a \$75 gift card to an online bookstore. She wants to buy at least 6 books. If all books at this store cost either \$8 or \$5, write a system of inequalities that describe the situation.*

Then graph the system to show all possible solutions.

$8x + 5y \leq 75$

$x + y \geq 6$
$x \geq 0$
$y \geq 0$

Example 0.6.4 *The Band Boosters hope to make at least \$400 at a fund-raiser. They have 40 date books to sell at a profit of \$8 each, and 50 calendars to sell at a profit of \$5 each. How many of each could they sell to make at least \$400?*

$d =$ *date books*
$c =$ *calendars*
$8d + 5c \geq 400$
$d \leq 40$
$d \geq 0$
$c \leq 50$
$c \geq 0$