MCE 412- Autonomous Robotics

A compact course on linear algebra

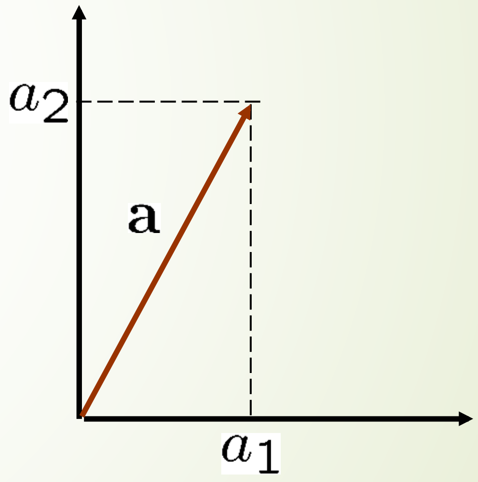
Motivation

- The linear systems are used extensively in the course, perhaps their most critical application in robotics is the representation of position and orientation.

Vectors

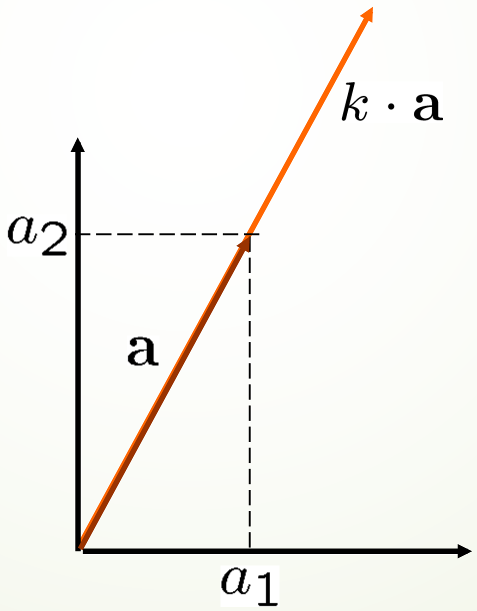
- Arrays of numbers
- Represents a point in a n-dimensional space

$$(a_1) \quad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$



Vectors: Scalar Product

- Scalar-vector product $k \cdot \mathbf{a}$
- Changes the length of the vector, but not its direction

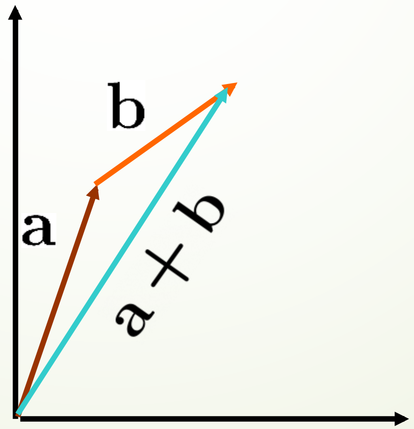


Vectors: Sum

- Sum of vectors (is commutative)

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

- Can be visualized as chaining the vectors

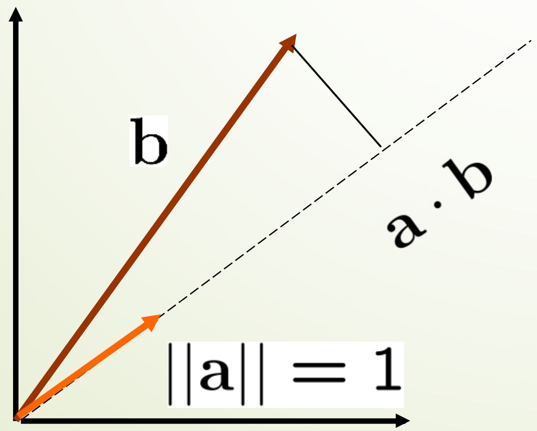


Vectors: Dot Product

- Inner product of vectors (is a scalar)

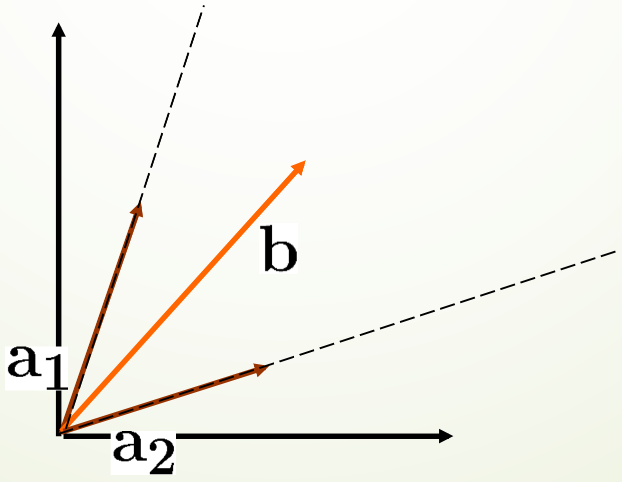
$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = \sum_i a_i \cdot b_i$$

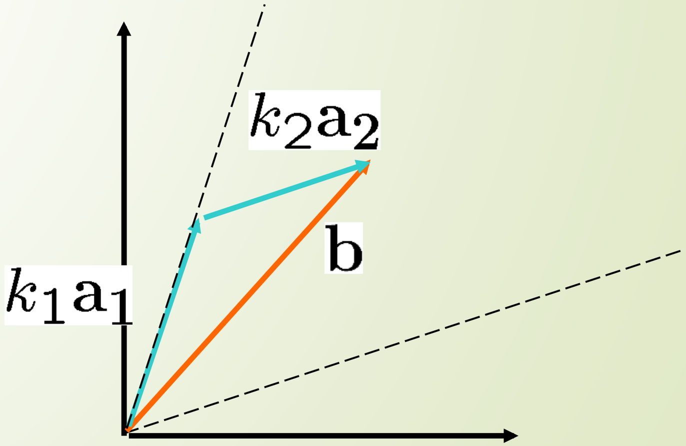
- Length of vector
- If one of the two vectors $\mathbf{a}$ has $||\mathbf{a}|| = 1$ the inner product $\mathbf{a} \cdot \mathbf{b}$ returns the length of the projection of b along the direction of a
- If $\mathbf{a} \cdot \mathbf{b} = 0$ the two vectors are orthogonal



Vectors: Linear (In)Dependence

- A vector $\mathbf{b}$ is **linearly dependent** from $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ if $\mathbf{b} = \sum_i k_i \cdot \mathbf{a}_i$
- In other words if b can be obtained by summing up the $\mathbf{a}_i$ properly scaled.
- If there exists no $\{k_i\}$ such that $\mathbf{b} = \sum_i k_i \cdot \mathbf{a}_i$ then $\mathbf{b}$ is independent from $\{\mathbf{a}_i\}$





Matrices

- A matrix is a collection of vectors and also a matrix is written as a table of values.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

- Can be used in many ways
- Multiplication by a scalar, sum, multiplication by a vector, inversion, transposition are the examples of matrix operation.

Matrices as Collections of Vectors

- Column vectors

$$\begin{pmatrix} \mathbf{a}_{*1} & \mathbf{a}_{*2} & \cdots & \mathbf{a}_{*m} \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

Matrices as Collections of Vectors

- Row Vectors

$$\mathbf{A} = \begin{pmatrix} \boxed{a_{11} \quad a_{12} \quad \cdots \quad a_{1m}} \\ \boxed{a_{21} \quad a_{22} \quad \cdots \quad a_{2m}} \\ \vdots \\ \boxed{a_{n1} \quad a_{n2} \quad \cdots \quad a_{nm}} \end{pmatrix} \begin{pmatrix} \mathbf{a}_{1*}^T \\ \mathbf{a}_{2*}^T \\ \vdots \\ \mathbf{a}_{*n}^T \end{pmatrix}$$

Matrices Operations

- Sum (commutative, associative)
- Product (not commutative)
- Inversion (square, full rank)
- Transposition
- Multiplication by a scalar
- Multiplication by a vector

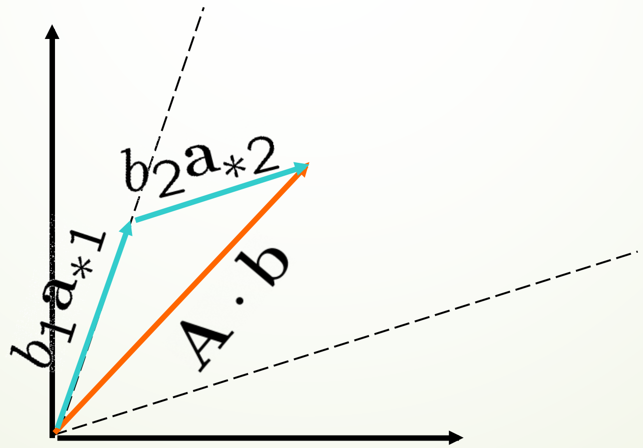
Matrix Vector Product

- The i-th component of $\mathbf{A} \cdot \mathbf{b}$ is the dot product $\mathbf{a}_{i*}^T \cdot \mathbf{b}$
- The vector $\mathbf{A} \cdot \mathbf{b}$ is linearly dependent from $\{\mathbf{a}_{*i}\}$ with coefficients $\{b_i\}$

$$\mathbf{A} \cdot \mathbf{b} = \begin{pmatrix} \mathbf{a}_{1*}^T \\ \mathbf{a}_{2*}^T \\ \vdots \\ \mathbf{a}_{n*}^T \end{pmatrix} \cdot \mathbf{b} = \begin{pmatrix} \mathbf{a}_{1*}^T \cdot \mathbf{b} \\ \mathbf{a}_{2*}^T \cdot \mathbf{b} \\ \vdots \\ \mathbf{a}_{n*}^T \cdot \mathbf{b} \end{pmatrix} = \sum_k \mathbf{a}_{*k} \cdot b_k$$

Matrix Vector Product

- If the column vectors of **A** represents a reference system, the product $\mathbf{A} \cdot \mathbf{b}$ computes the global transformation of the vector $\mathbf{b}$ according to $\{\mathbf{a}_{*i}\}$



Matrix Vector Product

- $a_{i,j}$ can be seen as a linear mixing coefficient that tells how it contributes to $(\mathbf{A} \cdot \mathbf{b})_j$.
- Example: Jacobian of a multi-dimensional function

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} \mathrm{f}_1(\mathbf{x}) \\ \mathrm{f}_2(\mathbf{x}) \\ \vdots \\ \mathrm{f}_n(\mathbf{x}) \end{pmatrix} \quad \mathbf{J}_f = \begin{pmatrix} \frac{d\mathrm{f}_1}{dx_1} & \frac{d\mathrm{f}_1}{dx_2} & \dots & \frac{d\mathrm{f}_1}{dx_m} \\ \frac{d\mathrm{f}_2}{dx_1} & \frac{d\mathrm{f}_2}{dx_2} & \dots & \frac{d\mathrm{f}_2}{dx_m} \\ \vdots & \ddots & & \\ \frac{d\mathrm{f}_n}{dx_1} & \frac{d\mathrm{f}_n}{dx_2} & \dots & \frac{d\mathrm{f}_n}{dx_m} \end{pmatrix}$$

Matrix Matrix Product

- Can be defined through
  - the dot product of row and column vectors
  - the linear combination of the columns of A scaled by the coefficients of the columns of B

$$
\begin{aligned}
\mathbf{C} &= \mathbf{A} \cdot \mathbf{B} \\
&= \begin{pmatrix}
\mathbf{a}_{1*}^T \cdot \mathbf{b}_{*1} & \mathbf{a}_{1*}^T \cdot \mathbf{b}_{*2} & \cdots & \mathbf{a}_{1*}^T \cdot \mathbf{b}_{*m} \\
\mathbf{a}_{2*}^T \cdot \mathbf{b}_{*1} & \mathbf{a}_{2*}^T \cdot \mathbf{b}_{*2} & \cdots & \mathbf{a}_{2*}^T \cdot \mathbf{b}_{*m} \\
\vdots & & & \\
\mathbf{a}_{n*}^T \cdot \mathbf{b}_{*1} & \mathbf{a}_{n*}^T \cdot \mathbf{b}_{*2} & \cdots & \mathbf{a}_{n*}^T \cdot \mathbf{b}_{*m}
\end{pmatrix} \\
&= \begin{pmatrix} \mathbf{A} \cdot \mathbf{b}_{*1} & \mathbf{A} \cdot \mathbf{b}_{*2} & \ldots \mathbf{A} \cdot \mathbf{b}_{*m} \end{pmatrix}
\end{aligned}
$$

Matrix Matrix Product

- If we consider the second interpretation we see that the columns of **C** are the projections of the columns of **B** through **A**.
- All the interpretations made for the matrix vector product hold.

$$
\begin{aligned}
\mathbf{C} &= \mathbf{A} \cdot \mathbf{B} \\
&= \begin{pmatrix} \mathbf{A} \cdot \mathbf{b}_{*1} & \mathbf{A} \cdot \mathbf{b}_{*2} & \ldots \mathbf{A} \cdot \mathbf{b}_{*m} \end{pmatrix} \\
\mathbf{c}_{*i} &= \mathbf{A} \cdot \mathbf{b}_{*i}
\end{aligned}
$$

Linear Systems

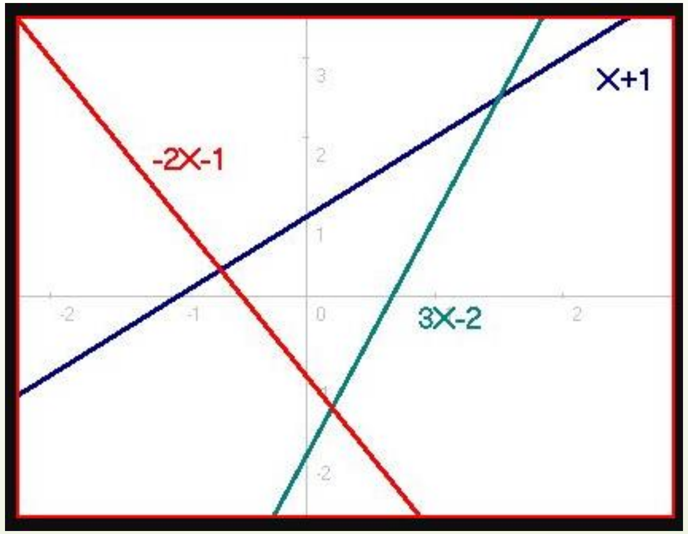
$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

- Interpretations
  - Find the coordinates **x** in the reference system of **A** such that **b** is the result of the transformation of **Ax**.
  - Many efficient solvers
    - Conjugate gradients
    - Sparse Cholesky Decomposition (if SPD)
    - ..
  - The system may be **over** or **under** constrained.
  - One can obtain a reduced system **(A' b')** by considering the matrix **(A,b)** and suppressing all the rows which are linearly dependent.

Linear Systems

- The system is **over-constrained** if the number of linearly independent columns or rows of **A'** is greater than the dimension of **b'.**
- An **over constrained** system does not admit a solution, however one may find a minimum norm solution by pseudo inversion

$$\mathbf{x} = \underset{\mathbf{x}}{\text{argmin}}\, ||\mathbf{A}'\mathbf{x} - \mathbf{b}'|| = (\mathbf{A}'^T\mathbf{A}')^{-1}\mathbf{A}'^T\mathbf{b}'$$



Linear Sytems

- The system is **under-constrained** if the number of linearly independent columns or rows of **A’** is smaller than the dimension of **b’**.
- An under constrained system admits infinite solutions.

Trace

- Only defined for **square matrices**
- **Sum** of the elements on the main diagonal, that is

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}$$

- It is a linear operator with the following properties
  - Additivity: $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$
  - Homogeneity: $\mathrm{tr}(c \cdot A) = c \cdot \mathrm{tr}(A)$
  - Pairwise commutative: $\mathrm{tr}(AB) = \mathrm{tr}(BA), \quad \mathrm{tr}(ABC) \neq \mathrm{tr}(ACB)$
- Trace is similarity invariant $\mathrm{tr}(P^{-1}AP) = \mathrm{tr}((AP^{-1})P) = \mathrm{tr}(A)$
- Trace is transpose invariant $\mathrm{tr}(A) = \mathrm{tr}(A^T)$

Rank

- Maximum number of linearly independent rows (columns)
- Dimension of the image of the transformation $f(\mathbf{x}) = A\mathbf{x}$
- When A is $m \times n$ we have
  - $\operatorname{rank}(A) \geq 0$ and equality holds iff A is the null matrix
  - . $\operatorname{rank}(A) \leq \min(m, n)$
  - If $m = n$ and A is invertible iff $\operatorname{rank}(A) = n$
- Computation of the rank is done by
  - Perform Gaussian elimination on the matrix
  - Count the number of non-zero rows

Identity Matrix

- 3x3 identity matrix

$$I=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Matrix Inversion

$$\mathbf{AB} = \mathbf{I}$$

- If A is a square matrix of full rank, then there is a unique matrix $\mathbf{B}=\mathbf{A}^{-1}$ such that the above equation holds.
- The i^{th} row of A and the j^{th} column of A^{-1} are
  - Orthogonal if $i \neq j$
  - Or their scalar (dot) product is 1 (if i=j)
- The i^{th} column of A^{-1} can be found by solving the following system

$$\mathbf{A}\mathbf{a}^{-1}{}_{*i} = \mathbf{i}_{*i}$$

This is the i^{th} column of the identity matrix

Determinant

- Only defined for square matrices
- $\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{I}$ if and only if $det(\mathbf{A}) \neq 0$
- For 2x2 matrices

  Let $\mathbf{A} = [a_{ij}]$ and $|\mathbf{A}| = det(\mathbf{A})$ then

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

- For 3x3 matrices

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$-a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{11}$$

Determinant

- For general n x n matrices

Let $\mathbf{A}_{ij}$ be the submatrix obtained from A by deleting the i-th row and the j-th column

$$\begin{bmatrix} 1 & 2 & 5 & 0 \\ 2 & 3 & 4 & -1 \\ -5 & 8 & 0 & 0 \\ 0 & 4 & -2 & 0 \end{bmatrix} \Rightarrow \mathbf{A}_{23} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Rewrite the determinant for 3x3 matrices

$$\begin{aligned} det(\mathbf{A}_{3\times 3}) &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad -a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{11} \\ &= a_{11} \cdot det(\mathbf{A}_{11}) - a_{12} \cdot det(\mathbf{A}_{12}) + a_{13} \cdot det(\mathbf{A}_{13}) \end{aligned}$$

Determinant

- For general n x n matrices

$$\begin{aligned} det(\mathbf{A}) &= a_{11}det(\mathbf{A}_{11}) - a_{12}det(\mathbf{A}_{12}) + ... + (-1)^{1+n}a_{1n}det(\mathbf{A}_{1n}) \\ &= \sum_{j=1}^{n}(-1)^{1+j}a_{1j}det(\mathbf{A}_{1j}) \end{aligned}$$

Let $\mathbf{C}_{ij} = (-1)^{i+j}det(\mathbf{A}_{ij})$ be the (i,j)-cofactor, then

$$\begin{aligned} det(\mathbf{A}) &= a_{11}\mathbf{C}_{11} + a_{12}\mathbf{C}_{12} + ... + a_{1n}\mathbf{C}_{1n} \\ &= \sum_{j=1}^{n}a_{1j}\mathbf{C}_{1j} \end{aligned}$$

This is called the **cofactor expansion** across the first row.

Determinant

- Problem: Take a 25x25 matrix (which considered small). The cofactor expansion method requires n! multiplications.

- There are much faster methods, namely using Gaussian elimination to bring the matrix into triangular form. Then

$$\mathbf{A} = \begin{bmatrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ 0 & 0 & d_3 & * \\ 0 & 0 & 0 & d_4 \end{bmatrix} \qquad det(\mathbf{A}) = \prod_{i=1}^{n} d_i$$

- Because for triangular matrices (with A being invertible), the determinant is the product of diagonal elements

Determinant Properties

- Row operations (A still a n x n square matrix)
  - If B results from A by interchanging two rows, then $det(\mathbf{B}) = -det(\mathbf{A})$
  - If B results from A by multiplying one row with a number c, then $det(\mathbf{B}) = c \cdot det(\mathbf{A})$
  - If B results from A by adding a multiple of one row to another row, then $det(\mathbf{B}) = det(\mathbf{A})$
- Transpose: $det(\mathbf{A}^T) = det(\mathbf{A})$
- Multiplication: $det(\mathbf{A} \cdot \mathbf{B}) = det(\mathbf{A}) \cdot det(\mathbf{B})$
- Does not apply to addition $det(\mathbf{A} + \mathbf{B}) \neq det(\mathbf{A}) + det(\mathbf{B})$

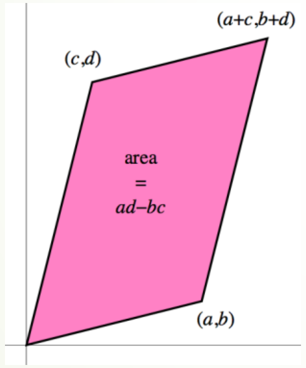
Determinant: Applications

- Compute eigenvalues

  Solve the characteristic polynomial $det(\mathbf{A} - \lambda \cdot \mathbf{I}) = 0$

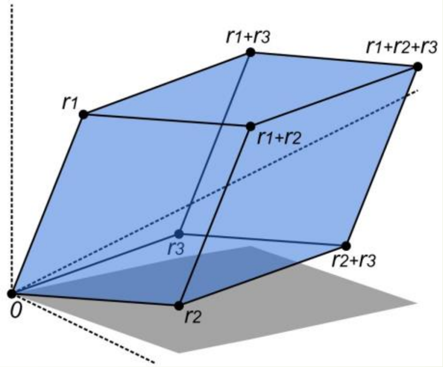
- Area and Volume: $\text{area} = |det(\mathbf{A})|$

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$



$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

(r_i is *i-th* row)



Orthogonal Matrix

- A matrix Q is orthogonal iff its column (row) vectors represent an orthonormal basis

$$q_{*i} \cdot q_{*j} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} , \forall\ i, j$$

- As linear transformation, it is norm preserving, and acts as an isometry in Euclidean space (rotation, reflection)

- Some properties:
  - The transpose is the inverse $QQ^T = Q^TQ = I$
  - Determinant has unity norm $(\pm\ 1)$

$$1 = det(I) = det(Q^TQ) = det(Q)det(Q^T) = det(Q)^2$$

Rotational Matrix

- Important in robotics
  - 2D rotations

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

  - 3D Rotations along the main axes

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \quad R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

- Important: Rotations are not commutative

$$R_x(\frac{\pi}{4}) \cdot R_y(\frac{\pi}{4}) = \begin{bmatrix} 0.707 & 0 & -0.707 \\ -0.5 & 0.707 & -0.5 \\ 0.5 & 0.707 & 0.5 \end{bmatrix}, \ R_x(\frac{\pi}{4}) \cdot R_y(\frac{\pi}{4}) \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1.414 \\ 0.586 \\ 3.414 \end{bmatrix}$$

$$R_y(\frac{\pi}{4}) \cdot R_x(\frac{\pi}{4}) = \begin{bmatrix} 0.707 & -0.5 & -0.5 \\ 0 & 0.707 & -0.707 \\ 0.707 & 0.5 & 0.5 \end{bmatrix}, \ R_y(\frac{\pi}{4}) \cdot R_x(\frac{\pi}{4}) \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1.793 \\ 0.707 \\ 3.207 \end{bmatrix}$$

Matrices as Affine Transformations

- A general and easy way to describe a 3D transformation is via matrices

Translation vector

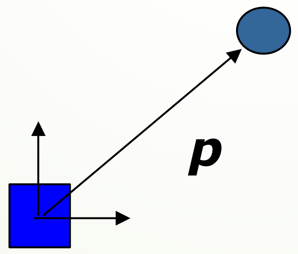
$$\mathbf{A} = \begin{pmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix} \quad \mathbf{A}^{-1} = \begin{pmatrix} \mathbf{R}^T & -\mathbf{R}^T\mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix} \quad \mathbf{p} = \begin{pmatrix} \mathbf{t} \\ 1 \end{pmatrix}$$

Rotation matrix

- Homogeneous behavior in 2D and 3D
- Takes naturally into account the noncommutativity of the transformations

Combining Transformations

- Simple interpretation: chaining of transformations (represented as homogeneous matrices)
  - Matrix A represents the pose of a robot in the space
  - Matrix B represents the position of a sensor on the robot
  - The sensor perceives an object at a given location p, in its own frame (the sensor has no clue on where it is in the world)
  - Where is the object in the global frame?



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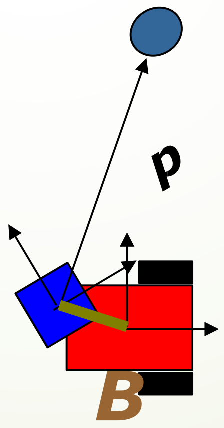

***B*p** gives me the pose of the object wrt the robot

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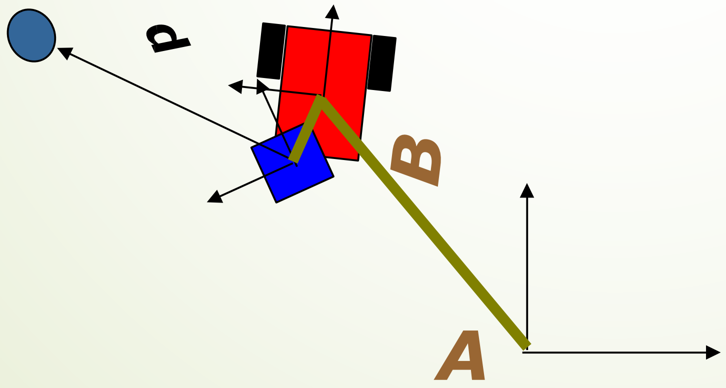


***B*p** gives me the pose of the object wrt the robot

***AB*p** gives me the pose of the object wrt the world

Symmetric matrix

- A matrix A is symmetric if $A = A^T$, e.g. $\begin{bmatrix} 1 & 4 & -2 \\ 4 & -1 & 3 \\ -2 & 3 & 5 \end{bmatrix}$

- A matrix A is anti-symmetric if $A = -A^T$, e.g. $\begin{bmatrix} 0 & 4 & -2 \\ -4 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$

- Every symmetric matrix
  - Can be diagonalizable $D = QAQ^T$ where D is a diagonal matrix of eigenvalues and Q is an orthogonal matrix whose columns are the eigenvectors of A
  - Define a quadratic form

$$q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} = \sum_{i,,j=1}^{n} a_{ij} x_i x_j$$

Positive definite matrix

- The analogous of positive number

- Definition

$$M > 0 \text{ iff } \forall z \neq 0 : z^T M z > 0$$

- Examples

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = z_1^2 + z_2^2 > 0$$

$$M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 2z_1 z_2 < 0, \; z_1 = -z_2$$

Positive definite matrix

- Properties
  - Invertible, with positive definite inverse
  - All eigenvalues >0
  - Trace >0
  - For any positive definite A, $AA^T,\ A^TA$ are positive definite
  - Cholesky decomposition $A = LL^T$

Jacobian Matrix

- It is a non-square matrix nxm in general
- Suppose you have a vector valued function

$$f(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{bmatrix}$$

- Let the gradient operator be the vector of (first order) partial derivatives

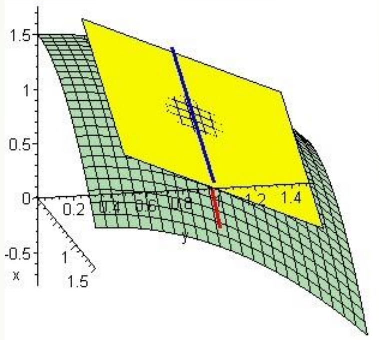
$$\nabla_{\mathbf{x}} = \begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \cdots & \frac{\partial}{\partial x_n} \end{bmatrix}^T$$

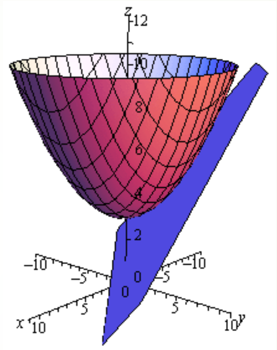
- Then the Jacobian matrix is defined as

$$\mathbf{F_x} = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_n} \end{bmatrix}$$

Jacobian Matrix

- It is the orientation of the tangent plane to the vector valued function at a given point





- Generalizes the gradient of a scalar valued function
- Heavily used for first order error propagation

$$\mathbf{C}_{out} = \mathbf{F} \cdot \mathbf{C}_{in} \cdot \mathbf{F}^T$$

Quadratic Forms

- Many important functions can be locally approximated with a quadratic form

$$\begin{aligned} f(\mathbf{x}) &= \sum_{i,j} a_{ij} x_i x_j + \sum_i b_i x_i + c \\ &= \mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b} \mathbf{x} + c \end{aligned}$$

- Often one is interested in finding the minimum (or maximum) of a quadratic form

$$\hat{\mathbf{x}} = \operatorname*{argmin}_{\mathbf{x}} f(\mathbf{x})$$

Quadratic Forms

- How can we use the matrix properties to quickly compute a solution to this minimization problem?

$$\hat{\mathbf{x}} = \underset{\mathbf{x}}{\text{argmin}}\, f(\mathbf{x})$$

- At the minimum we have $f'(\hat{\mathbf{x}}) = \mathbf{0}$
- By using the definition of matrix product we can compute f'

$$\begin{aligned} f(\mathbf{x}) &= \mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}\mathbf{x} + c \\ f'(\mathbf{x}) &= \mathbf{A}^T\mathbf{x} + \mathbf{A}\mathbf{x} + \mathbf{b} \end{aligned}$$

Quadratic Forms

- The minimum of $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}\mathbf{x} + c$ is where its derivative is set to 0

- Thus we can solve the system

$$0 = \mathbf{A}^T\mathbf{x} + \mathbf{A}\mathbf{x} + \mathbf{b}$$

$$(\mathbf{A}^T + \mathbf{A})\mathbf{x} = -\mathbf{b}$$

- If the matrix is symmetric, the system becomes

$$2\mathbf{A}\mathbf{x} = -\mathbf{b}$$

References

- Matrix Cookbook, http://matrixcookbook.com