FAST

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Assignment #02

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MVC

Q1 $C(x,y) = e^x + \sin y - 4x - y$

$f_x = e^x - 4$ $\quad$ $f_y = \cos y - 1$

$f_{xx} = e^x$ $\quad$ $f_{yy} = -\sin y$

$f_{xy} = 0$

$e^x - 4 = 0$ $\quad$ $\cos y - 1 = 0$

$x = \ln 4$ $\quad$ $y = 0 \;,\; y = 2\pi$

critical points are $(\ln 4, 2\pi k)$ $k \in \mathbb{Z}$

$D = f_{xx} \cdot f_{yy} - (f_{xy})^2 \Rightarrow e^x \cdot (-\sin y) - 0^2$

$D(\ln 4, 0) = e^{\ln 4} \cdot (-\sin 0) = 0$

As $D=0$, second derivative test is inconclusive!

Q2 $M(a,b) = 100a + 150b - 2a^2 - 3b^2 - ab$

$f_a = 100 - 4a - b$ $\quad$ $f_b = 150 - 6b - a$

$f_{aa} = -4$ $\quad$ $f_{bb} = -6$

$f_{ab} = -1$

$100 - 4a - b = 0$ $\quad$ $150 - 6b - a = 0$

$b = 100 - 4a$

$150 - 600 + 24a = a \Rightarrow 23a = 450$

$\left(a = \frac{450}{23} \;,\; b = \frac{500}{23}\right) \rightarrow$ critical points

$D = f_{aa} \cdot f_{bb} - (f_{ab})^2 \Rightarrow (-4)(-6) - (-1)^2 = 23$

$23 > 0$ and ~~$f_{ab} = -1 <$~~ $f_{aa} = -4 < 0$ so it's a local maxima.

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Q3

$C(m,n) = 3m^2 + 2mn + 4n^2 \quad m^2 + n^2 + mn = 400$

$\vec{\nabla} C = \lambda \vec{\nabla} g$

$6m + 2n = \lambda(2m+n), \quad 2m + 8n = \lambda(2n+m)$

$\lambda = \dfrac{6m+2n}{2m+n}, \quad \lambda = \dfrac{2m+8n}{2n+m}$

$\dfrac{6m+2n}{2m+n} = \dfrac{2m+8n}{2n+m}$

$12mn + 6m^2 + 4n^2 + 2mn = 4m^2 + 16mn + 2mn + 8n^2$

$6m^2 + 4n^2 + 14mn = 8n^2 + 4m^2 + 18mn$

$4n^2 - 2m^2 + 4mn = 0 \Rightarrow m^2 - 2mn - 2n^2 = 0$

* By treating n as a constant & solving for m:

$\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow \dfrac{-2n \pm \sqrt{(2n)^2 - 4(1)(2n^2)}}{2}$

$m = n(1 + \sqrt{3})$

* Subsituting back: $n + n\sqrt{3} + n^2 + n(n + \sqrt{3}n) = 400$

$2n^2 + n + n\sqrt{3} + n + n\sqrt{3} = 400$

$n = \sqrt{200(3\sqrt{3} - 5)} \quad m = \sqrt{200(3\sqrt{3}-5)} + \sqrt{3}(\sqrt{200(3\sqrt{3}-5)})$

$\boxed{n = 6.26}, \boxed{m = 17.11}$

Q4

$E(p,q,r) = p^2 + 4q^2 + 2r^2 \qquad p^2 + q^2 + r^2 = 250{,}000$

$\vec{\nabla} E = \lambda \vec{\nabla} g$

$2p = \lambda 2p, \quad 8q = \lambda 2q, \quad 4r = \lambda 2r$

for $p \neq 0: \lambda = 1$, $q \neq 0: \lambda = 4$, $r \neq 0: \lambda = 2$

$\lambda = 1: \; 8q = 2q \Rightarrow q = 0, \quad 4r = 2r \Rightarrow r = 0 \qquad p^2 = 250{,}000$

$\lambda = 4: \; 2p = 8p \Rightarrow p = 0, \quad 4r = 8r \Rightarrow r = 0 \qquad q^2 = 250{,}000$

$\lambda = 2: \; 2p = 4p \Rightarrow p = 0, \quad 8q = 4q \Rightarrow q = 0 \qquad r^2 = 250{,}000$

Optimal power distibution:

$(\pm 500, 0, 0), \; (0, \pm 500, 0), \; (0, 0, \pm 500)$

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Q5 $J(\theta_1, \theta_2) = \sin(\theta_1) + \theta_2^2 + \theta_1\theta_2$

initial values : $\theta_1^0 = 1$, $\theta_2^0 = 1$ rate : $\alpha = 0.1$

$\frac{\partial J}{\partial \theta_1} = \cos(\theta_1) + \theta_2$, $\frac{\partial J}{\partial \theta_2} = 2\theta_2 + \theta_1$

Initial:

$\frac{\partial J}{\partial \theta_1} : \cos(1) + 1 = 1.9998$, $\frac{\partial J}{\partial \theta_2} : 2 + 1 = 3$

$x_1 = x_0 - \alpha \vec{\nabla} f$

$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 0.1 \begin{bmatrix} 1.9998 \\ 3 \end{bmatrix}$ $x_1 = \begin{bmatrix} 0.8 \\ 0.7 \end{bmatrix} \Rightarrow f(x_1) = 1.064$

1st iteration: $x_2 = x_1 - \alpha \vec{\nabla} J$

$x_2 = \begin{bmatrix} 0.8 \\ 0.7 \end{bmatrix} - 0.1 \begin{bmatrix} \cos(0.8) + 0.7 \\ 2(0.7) + 0.8 \end{bmatrix}$

$x_2 = \begin{bmatrix} 0.63 \\ 0.48 \end{bmatrix} \Rightarrow f(x_2) = 0.544$

2nd iteration: $x_3 = x_2 - \alpha \vec{\nabla} J$

$x_3 = \begin{bmatrix} 0.63 \\ 0.48 \end{bmatrix} - 0.1 \begin{bmatrix} \cos(0.63) + 0.48 \\ 2(0.48) + 0.63 \end{bmatrix}$

$x_3 = \begin{bmatrix} 0.48 \\ 0.32 \end{bmatrix} \Rightarrow f(x_3) = 0.266$

Q6 $R(f, g, h) = f^2 + 2g^2 + 3h^2 + fg - 5f - 7g - 9h + 30$ $\alpha = 0.05$ $(f_0, g_0, h_0) = (1, 1, 1)$

$\vec{\nabla} R = (2f + g - 5)\hat{i} + (4g + f - 7)\hat{j} + (6h - 9)\hat{k}$

Initial:

$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 0.05 \begin{bmatrix} 2(1) + 1 - 5 \\ 4(1) + 1 - 7 \\ 6(1) - 9 \end{bmatrix}$

$x_1 = \begin{bmatrix} 1.1 \\ 1.1 \\ 1.15 \end{bmatrix}$ $f(x_1) = 15.26$

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1^{st} iteration:

$$X_2 = \begin{bmatrix} 1.1 \\ 1.1 \\ 1.15 \end{bmatrix} - 0.05 \begin{bmatrix} 2(1.1) + 1.1 - 5 \\ 4(1.1) + 1.1 - 7 \\ 6(1.1) - 9 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 1.185 \\ 1.175 \\ 1.255 \end{bmatrix} \qquad f(X_2) = 14.84$$

2^{nd} iteration:

$$X_3 = \begin{bmatrix} 1.185 \\ 1.175 \\ 1.255 \end{bmatrix} - 0.05 \begin{bmatrix} 2(1.185) + 1.175 - 5 \\ 4(1.175) + 1.185 - 7 \\ 6(1.255) - 9 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 1.258 \\ 1.231 \\ 1.329 \end{bmatrix} \qquad f(X_3) = 14.59$$

Q7 $J(\theta_1, \theta_2, \theta_3) = e^{\theta_1} + \theta_2^2 + 4\theta_3^2 + \theta_1\theta_2$

$\frac{\partial J}{\partial \theta_1} = e^{\theta_1} + \theta_2 \qquad \frac{\partial^2 J}{\partial \theta_1^2} = e^{\theta_1}$

$\frac{\partial J}{\partial \theta_2} = 2\theta_2 + \theta_1 \qquad \frac{\partial^2 J}{\partial \theta_2^2} = 2 \qquad H = \begin{bmatrix} e^{\theta_1} & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

$\frac{\partial J}{\partial \theta_3} = 8\theta_3 \qquad \frac{\partial^2 J}{\partial \theta_3^2} = 8 \qquad \det(H_2) = 2e^{\theta_1} > 1 \Rightarrow e^{\theta_1} > \frac{1}{2}$

$\det(H_3) = 4e^{\theta_1} - 2 > 0 \Rightarrow e^{\theta_1} > \frac{1}{2}$

$e^{\theta_1} + \theta_2 = 0, \quad 2\theta_2 + \theta_1 = 0, \quad 8\theta_3 = 0$

$-2\theta_2 = \theta_1 \qquad \theta_3 = 0$

$e^{-2\theta_2} + \theta_2 = 0 \Rightarrow e^{-2\theta_2} = -\theta_2 \Rightarrow -2\theta_2 = \ln(-\theta_2)$

$\ln(-\theta_2)$ is only defined when $-\theta_2 > 0$ so $\theta_2 < 0$.

As $\theta_2 < 0$ then e^{θ_1} is always positive

$\det(H_1) = e^{\theta_1} > 0$ so Hessian matrix is a local minimum

~~and the function has a saddle point. Thus no optimal value at this point.~~ So optimal value exist when $e^{\theta_1} > \frac{1}{2}$

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Q8 $T(x,y) = xy - x^2 \quad 0 \le x \le 1 \quad 1 \le y \le 3$

$$\int_1^3\int_0^1 xy - x^2 \, dx \Rightarrow \int_1^3 \left[\frac{x^2y}{2} - \frac{x^3}{3}\right]_0^1 dy \Rightarrow \int_1^3 \frac{y}{2} - \frac{1}{3}\, dy$$

$$\left[\frac{y^2}{4} - \frac{y}{3}\right]_1^3 \Rightarrow \left(\frac{9}{4} - 1\right) - \left(\frac{1}{4} - \frac{1}{3}\right) \Rightarrow \boxed{\frac{4}{3}}$$

Q9 $z = 10 - x^2 - y^2 \quad (0,0), (2,0), (2,3)$

$$\frac{3-0}{2-0} = \frac{3}{2}$$

$$\int_0^2\int_0^{1.5x} 10 - x^2 - y^2 \, dy \Rightarrow \int_0^2 \left[10y - x^2y - \frac{y^3}{3}\right]_0^{1.5x}$$

$$\int_0^2 15x - 1.5x^3 - \frac{9}{8}x^3 \Rightarrow \int_0^2 15x - \frac{21}{8}x^3 \Rightarrow \left[\frac{15x^2}{2} - \frac{21}{32}x^4\right]_0^2$$

$$\frac{15(2)^2}{2} - \frac{21}{32}(2)^4 - 0 \Rightarrow \boxed{\frac{39}{2} = 19.5}$$

Q10 a) $f(x,y,z) = x^2 + y^2 - z = 0 \quad (1,2,5)$

$$\vec\nabla f(1,2,5) = 2x\hat{i} + 2y\hat{j} - \hat{k} \Rightarrow 2(1) + 2(2) + 1 \Rightarrow \boxed{2i + 4j - k}$$

Normal vector

Tangent: $2(x-1) + 4(y-2) (z-5) = 0$

$$2x - 2 + 4y - 8 - z + 5 = 0$$

$$\boxed{2x + 4y - z = 5}$$

b) $f_1(x,y,z) = x^2 + y^2 + z^2 - 9 = 0 \quad f_2(x,y,z) = x + 2y - 3z + 4 = 0 \quad (2,-1,1)$

$$\vec\nabla f_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k} \Rightarrow \vec\nabla f_1(2,-1,1) = 4i - 2j + 2k$$

$$\vec\nabla f_2 = \hat{i} + 2\hat{j} - 3\hat{k}$$

$$\cos\theta = \frac{\vec\nabla f_1 \cdot \vec\nabla f_2}{|\vec\nabla f_1||\vec\nabla f_2|} \Rightarrow \cos\theta = \frac{4 - 4 - 6}{2\sqrt{6} \times \sqrt{14}}$$

$$\boxed{\theta = 109.11^\circ}$$