Linear Programming

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#Q1

Furnco manufactures desks and chairs. Each desk uses four units of wood, and each chair uses three units of wood. A desk contributes $30 to profit and a chair contributes $20. Marketing restrictions require that the number of chairs produced be at least four times more the number of desks produced. There are 50 units of wood available.

a) Formulate the problem algebraically as a linear program, clearly identifying decision variables, the objective function and constraints.

Set the total numbers of desk and chairs are x_1 and x_2, then we have

$$\begin{aligned} max \quad & 30x_1 + 20x_2 \\ s.t. \quad & 4x_1 + 3x_2 \leq 50 \\ & 4x_1 - x_2 \leq 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

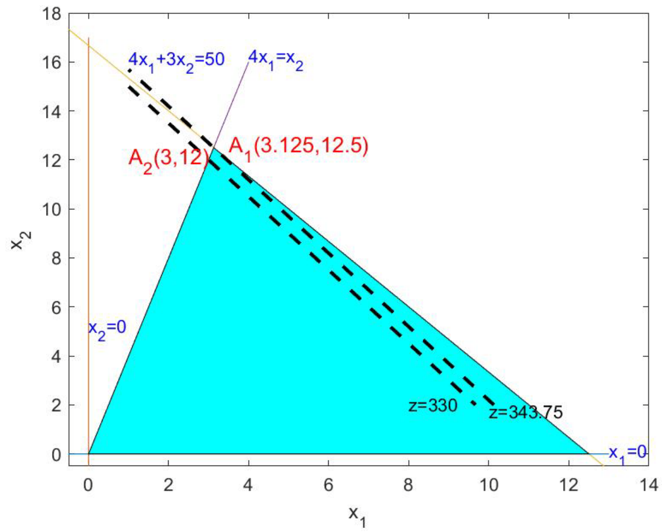
b) Use a spreadsheet solver to maximize Funrnco's profits. Clearly mark the cells containing decision variables, the formula for the objective function, and the cell(s) containing formulas for the constraints. Upload your spreadsheet file separately.

Product	Desk	Chair				
Profit Per Product	30	20		Resources		
Resource Constraints			Usage		Available	Left Over
Wood Usage	4	3	48	<=	50	2

Product	
Desks =	3
Chairs =	12
Profit =	330

c) Confirm graphically that the solution in part a) maximizes Furnco's profit. The simplest way is to use a graph paper and a ruler and draw your graphs, and then scan the image and drag it into your spreadsheet. Make sure your scanned images are not too large (certainly well below 1 Megabyte.) Be neat and clear. Mark each line corresponding to each constraint and shade the feasible region properly. Clearly show the direction of improvement and iso-profit (or iso-cost) lines as appropriate. Mark the optimal solution in your graph.

A_1 is the global optimum when considering x_1, x_2 as continuous variables, while A_2 is solution when considering x_1, x_2 as integers.



d) Formulate your program from part a) in the standard form, i.e. rewrite it as

$$min\, c^T x \quad subject\ to \quad Ax = b,\ x \geq 0 \qquad (1)$$

for appropriately chosen vector c, matrix A and a vector b. Write the dual linear program. (Hint: you can use the slides about duality). Maximize the dual program by hand using a graph paper a ruler, drawing your graphs. What are the dual variables y_* and s_* at the optimum?

Standard form:

$$\begin{aligned} min \quad & -30x_1 - 20x_2 \\ s.t. \quad & 4x_1 + 3x_2 + x_3 = 50 \\ & 4x_1 - x_2 + x_4 = 0 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Dual program:

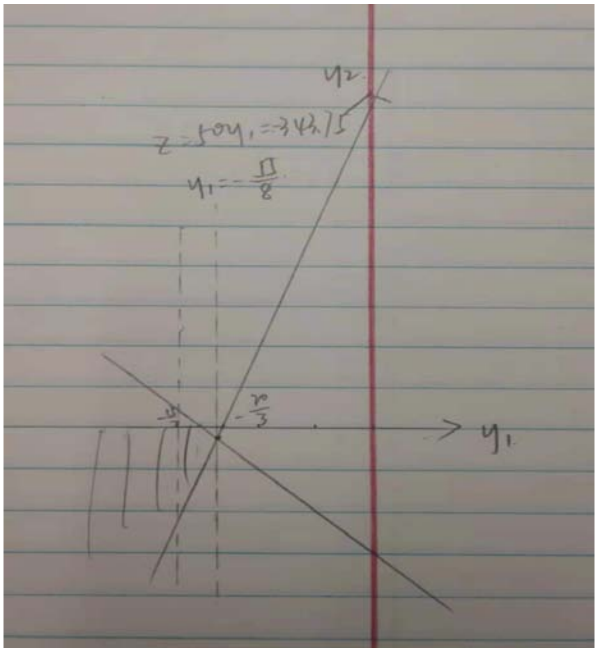
1) from standard form of prime problem:

$$\begin{aligned} max \quad & 50y_1 \\ s.t. \quad & 4y_1 + 4y_2 + s_1 = -30 \\ & 3y_1 - y_2 + s_2 = -20 \\ & y_1 + s_3 = 0 \\ & y_2 + s_4 = 0 \\ & s_1, s_2, s_3, s_4 \geq 0 \end{aligned}$$

2) from table:

$$\begin{aligned} min \quad & 50y_1 \\ s.t. \quad & 4y_1 + 4y_2 \geq 30 \\ & 3y_1 - y_2 \geq 20 \\ & y_1, y_2 \geq 0 \end{aligned}$$

Choose to maximize first dual program with graph:



$$y_* = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -\frac{55}{8} \\ -\frac{5}{8} \end{bmatrix}, \quad \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{55}{8} \\ \frac{5}{8} \end{bmatrix}$$

e) Confirm that complementary slackness conditions, i.e. $x_* s_* = 0$ where x_* is your solution you computed in part c).

In this problem, complementary slackness conditions is that if $x_*^T = (x_1, x_2, x_3, x_4)$ is primal optimal solution, and (y_*, s_*) is dual optimal solution, where $y_*^T = (y_1, y_2)$, $s_*^T = (s_1, s_2, s_3, s_4)$, then we have $x_*^T s_* = 0$, i.e. , $x_1 s_1 = 0,\ x_2 s_2 = 0,\ x_3 s_3 = 0,\ x_4 s_4 = 0$, and $x_i > 0, s_i = 0; s_i > 0, x_i = 0$.

$$x_* = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{25}{8} \\ \frac{25}{2} \\ 0 \\ 0 \end{bmatrix}, \quad s_* = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{55}{8} \\ \frac{5}{8} \end{bmatrix}, \text{ so we have } \begin{cases} x_1 s_1 = 0 \\ x_2 s_2 = 0 \\ x_3 s_3 = 0 \\ x_4 s_4 = 0 \end{cases}$$

f) (Extra credit, 5 points, medium difficulty) Find a basic feasible solution to (1) and show the iterations of the simplex method. How many iterations do you need to compute the optimal solution?

g) (Extra credit, 5 points, medium difficulty) Show the iterations of the dual simplex method.

When solving primary problem, the first iteration is from simplex method, and the second iteration is from dual simplex method.

x_1	x_2	x_3	x_4	
4	3	1	0	50 (x_3 *out*)
4	-1	0	1	0
-30 (x_1 *in*)	-20	0	0	

x_1	x_2	x_3	x_4	
1	3/4	1/4	0	25/2
0	-4	-1	1	-50 (x_4 *out*)
0	5/2 (x_2 *in*)	15/2	0	375

x_1	x_2	x_3	x_4	
1	0	1/16	3/16	25/8
0	1	1/4	-1/4	25/2
0	0	55/8	5/8	343.75

h) (Extra credit, 10 points, difficulty hard) Consider the following optimization problem

$$min \max_{i=1,2,\ldots,k} |a_i^T x - b_i|$$

where $a_i \in \mathbb{R}^n$ are vectors, b_i is a real scalar and the decision variable $x \in \mathbb{R}^n$. Formulate this problem as a linear problem.

Suppose there are k points in the n-dimensional coordinate axis, and linear regression is performed on these points. $|a_i^T x - b_i|$ is the residual at i-th point.

#Q2

Portfolio optimization

Code:

```
exp_ret = [0.2, 0.1, 0.15;
           0.3, 0.2, 0.1;
           0.15, 0.2, 0.17;
           0.13, 0.1, 0.19;
           0.09, 0.02, 0.09;
           0.2, 0.16, 0.15;
           0.18, 0.17, 0.11;
           0.06, 0.11, 0.11;
           0.05, 0.12, 0.1;
           0.07, 0.11, 0.1;
           0.1, 0.1, 0.08];
weight = [0.1, 0.05, 0.13, 0.13, 0.1, 0.05, 0.04, 0.2, 0.1, 0.05, 0.05];
rf = [0.045, 0.05, 0.055];
%calculate r_M respect to each expert
rm = zeros(1,3);
for i = 1:1:3
    for j = 1:1:11
        rm(1,i) = rm(1,i) + weight(1,j) * exp_ret(j,i);
    end
end
%calculate beta_{i,j} respect to each company and expert
b = zeros(12,3);
for i = 1:1:11
    for j = 1:1:3
        b(i,j) = (exp_ret(i,j) - rf(1,j)) / (rm(1,j) - rf(1,j));
    end
end
%max R which is min -R
f = [0,0,0,0,0,0,0,0,0,0,0,-1];
a = zeros(6,12);
%constraint sigma(proportion*beta)+0*R < B with respect to each expert
for i = 1:1:3
    for j = 1:1:11
        a(i,j) = b(j,i);
    end
    a(i,12) = 0;
end
%constraint -sigma(proportion*exp_ret)+R < 0 respect to each expert
for i = 4:1:6
    for j = 1:1:11
        a(i,j) = -exp_ret(j, i-3);
    end
    a(i,12) = 1;
end
b = [1.15; 1.15; 1.15; 0; 0; 0];
%constraint sum of proportion should be 1
aeq = [1,1,1,1,1,1,1,1,1,1,1,0];
beq = 1;
%assume there's no short selling, hence all proportion must be positive
lb = [0,0,0,0,0,0,0,0,0,0,0,-Inf];
ub = [];
opt = linprog(f,a,b,aeq,beq,lb,ub);
fprintf('Optimal proportions:\n');
fprintf('IBM %.4f \t\t', opt(1,1));
fprintf('Microsoft %.4f \t', opt(2,1));
```

```
fprintf('GE %.4f \t', opt(3,1));
fprintf('Exxon %.4f \t', opt(4,1));
fprintf('ATT %.4f \n', opt(5,1));
fprintf('Intel %.4f \t', opt(6,1));
fprintf('Merck %.4f \t', opt(7,1));
fprintf('GM %.4f \t', opt(8,1));
fprintf('Ford %.4f \t', opt(9,1));
fprintf('Texaco %.4f \n', opt(10,1));
fprintf('Citibank %.4f \n\n', opt(11,1));
fprintf('Optimal expected return: %.4f\n', opt(12,1));
```

Output:

>> Problem2

Optimal solution found.

Optimal proportions:
IBM 0.0000 Microsoft 0.0000 GE 0.2774 Exxon 0.0000 ATT 0.1943
Intel 0.2675 Merck 0.0000 GM 0.0000 Ford 0.0000 Texaco 0.2608
Citibank 0.0000

Optimal expected return: 0.1308

#Q3

Diet problem

Code:

```
f = [0.18, 0.22, 0.1, 0.12, 0.1, 0.09, 0.4, 0.16, 0.5, 0.07]; %minimum total cost
intcon = [1,2,3,4,5,6,7,8,9,10] %consider only integer
a = [-90, -110, -100, -90, -75, -35, -65, -100, -120, -65; % -(total calories) has to
less than -420
    0, 2, 2, 2, 5, 3, 0, 4, 0, 1; % total fat has to less than 20
    0, 0, 0, 0, 270, 8, 0, 12, 0, 0; % total cholesteral has to less than 30
    -6, -4, -2, -3, -1, 0, -1, 0, 0, -1; % -(total iron) has to less than -5
    -20, -48, -12, -8, -30, 0, -52, -250, -3, -26; % -(total calcium) has to less than -
400
    -3, -4, -5, -6, -7, -2, -1, -9, -1, -3; % -(total protein) has to less than -20
    -5, -2, -3, -4, 0, 0, -1, 0, 0, -3]; % -(total fiber) has to less than -12
b = [-420; 20; 30; -5; -400; -20; -12];
aeq = [];   beq = []; % no equality constraint
lb = [ 0; 0; 0; 0; 0; 0; 0; 0; 0; 0];               % no bran cereal, which is 0 =< bran
cereal =< 0
ub = [ 0; Inf; Inf; Inf; Inf; Inf; 1; 1; Inf; 1];   % wheat toast =< 1, milk =< 1, orange
=< 1, everything  >= 0
% solve the integer linear programming
opt = intlinprog(f,intcon,a,b,aeq,beq,lb,ub);
fprintf('Optimal Solution:\n')
fprintf('Bran cereal %.0f cup.\t', -opt(1,1));
fprintf('Dry cereal is %.0f cups.\t', opt(2,1));
fprintf('Oatmeal is %.0f cups.\t', opt(3,1));
fprintf('Oat bran is %.0f cup.\n', opt(4,1));
fprintf('Egg is %.0f.\t', opt(5,1));
fprintf('Bacon is %.0f slice.\t', -opt(6,1));
fprintf('Orange is %.0f.\t', -opt(7,1));
fprintf('Milk-2%% is %.0f cup.\t', opt(8,1));
fprintf('Orange juice is %.0f cup.\n', -opt(9,1));
fprintf('Wheat toast is %.0f slice.\n\n', opt(10,1));
value = a*opt;
fprintf('Optimal Nutritients:\n')
fprintf('Calories is %.0f calories.\t', -value(1,1));
fprintf('Fat is %.0f g.\t', value(2,1));
fprintf('Cholesterol is %.0f mg.\n', value(3,1));
fprintf('Iron is %.0f mg.\t\t', -value(4,1));
fprintf('Calcium is %.0f mg.\t\t', -value(5,1));
fprintf('Protein is %.0f g.\t', -value(6,1));
fprintf('Fiber is %.0f g.\n\n', -value(7,1));
fprintf('Optimal total cost is %.2f\n', f*opt);
```

Output:

>> Problem3

Optimal solution found.

Optimal Solution:
Bran cereal 0 cup.	Dry cereal is 2 cups.	Oatmeal is 3 cups.	Oat bran is 0 cup.
Egg is 0. Bacon is 0 slice.	Orange is 0.	Milk-2% is 1 cup.Orange juice is 0 cup.
Wheat toast is 1 slice.

Optimal Nutritients:

Calories is 685 calories.	Fat is 15 g.	Cholesterol is 12 mg.	
Iron is 15 mg.	Calcium is 408 mg.	Protein is 35 g.	Fiber is 16 g.

Optimal total cost is 0.97