Quantum Field Theory II

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Contents

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Lecture 1

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Topics. Generalize the formalism for complex scalar fields. Introduce fermions and their functional quantization. Discrete symmetries and PCT theorem. Gauge theories and their functional quantization. Anomalies. Possibly instantons, applications of RG flows, etc.

14 Complex scalar boson fields

[r] sources?

14.1 Generalization

Consider a complex scalar field φ. It has two degrees of freedom: either the fields φ and φ^* or the real and imaginary parts of the field φ. Both are useful for different computations

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + \mathrm{i}\varphi_2)\,, \quad \varphi^* = \frac{1}{\sqrt{2}}(\varphi_1 - \mathrm{i}\varphi_2)$$

The action of the $\lambda\varphi^4$ theory is

$$\begin{aligned} S &= \int \mathrm{d}^4x\,[\partial_\mu\varphi\,\partial^\mu\varphi^* - m^2\varphi\varphi^* - \lambda(\varphi\varphi^*)^2] \\ &= \int \mathrm{d}^4x\,\left[\frac{1}{2}\,\partial_\mu\varphi_1\,\partial^\mu\varphi_1 - \frac{1}{2}m^2\varphi_1^2 + \frac{1}{2}\,\partial_\mu\varphi_2\,\partial^\mu\varphi_2 - \frac{1}{2}m^2\varphi_2^2 - \frac{\lambda}{4}(\varphi_1^2+\varphi_2^2)^2\right] \end{aligned}$$

Each degree of freedom has an equation of motion. Varying the action with respect to φ gives the equations of motion for φ^* and vice versa. For $\lambda = 0$, the field components satisfy the Klein–Gordon equation.

Since the interaction term is real, it corresponds to a quartic vertex $\varphi\varphi^*\varphi\varphi^*$. For $\lambda = 0$, one can rewrite the action in terms of real φ_1 and imaginary φ_2 parts and see that the two degrees of freedom decouple since there is no mixed term $\varphi_1^2\varphi_2^2$.

With two degrees of freedom, the Euclidean generating functional depends on two source terms

$$\begin{aligned} W[J,J^*] &= \int [\mathcal{D}\varphi\,\mathcal{D}\varphi^*]\,\exp\left[-\int \mathrm{d}^4x\,[\mathcal{L}(\varphi,\varphi^*) - J\varphi - J^*\varphi^*]\right] \\ W[J_1,J_2] &= \int [\mathcal{D}\varphi_1\,\mathcal{D}\varphi_2]\,\exp\left[-\int \mathrm{d}^4x\,[\mathcal{L}(\varphi_1,\varphi_2) - J_1\varphi_1 - J_2\varphi_2]\right] \end{aligned}$$

The Euclidean Green's functions are

$$\begin{aligned} &\langle 0|\,\mathcal{T}\{\varphi(x_1)\cdots\varphi(x_l)\varphi^*(x_{l+1})\cdots\varphi^*(x_n)\}\,|0\rangle = \\ &\qquad = \left.\frac{\delta^n W[J,J^*]}{\delta J(x_1)\cdots\delta J(x_l)\,\delta J^*(x_{l+1})\cdots\delta J^*(x_n)}\right|_{J=J^*=0} \end{aligned}$$

Similarly

$$\begin{aligned} &\langle 0|\,\mathcal{T}\{\varphi_1(x_1)\cdots\varphi_1(x_l)\varphi_2(x_{l+1})\cdots\varphi_2(x_n)\}\,|0\rangle = \\ &\qquad = \left.\frac{\delta^n W[J_1,J_2]}{\delta J_1(x_1)\cdots\delta J_1(x_l)\,\delta J_2(x_{l+1})\cdots\delta J_2(x_n)}\right|_{J_1=J_2=0} \end{aligned}$$

Free theory. For a free real scalar theory, i.e. setting $\lambda = 0$, one can compute the free propagator exactly. In this case the situation is slightly different? [r]. In the complex field formulation, one has to compute a complex Gaussian integral of the form

$$\int \left[\prod_{j=1}^n \mathrm{d}z_j\,\mathrm{d}z_j^*\right]\mathrm{e}^{-z^*Az} = \frac{\pi^n}{\det A}$$

where $z \equiv (z_1, \dots, z_n)$ and A is a square matrix of dimension n (with hermitian part positive-definite). In the real fields formulation, the Euclidean generating functional is

$$
\begin{aligned}
W[J_1, J_2] &= \int [\mathcal{D}\varphi_1 \, \mathcal{D}\varphi_2] \, \mathrm{e}^{-S_0[\varphi_1] - S_0[\varphi_2]} \exp\left[\int \mathrm{d}^4 x \, (J_1 \varphi_1 + J_2 \varphi_2)\right] \\
&= \int [\mathcal{D}\varphi_1] \, \mathrm{e}^{-S[\varphi_1] + \int \mathrm{d}^4 x \, J_1 \varphi_1} \int [\mathcal{D}\varphi_2] \, \mathrm{e}^{-S[\varphi_2] + \int \mathrm{d}^4 x \, J_2 \varphi_2} \\
&= W_1[J_1] W_2[J_2]
\end{aligned}
$$

The theory factorizes.

The Lagrangian above has a U(1) symmetry of the fields

$$
\varphi' = \mathrm{e}^{\mathrm{i}\alpha} \varphi \,, \quad \varphi'^* = \mathrm{e}^{-\mathrm{i}\alpha} \varphi^* \,, \quad \alpha \in \mathbb{R}
$$

Equivalently, the Lagrangian is invariant under SO(2) of the components φ_1 and φ_2. Notice that the two groups are isomorphic.

Exercise. Write the most general SO(2) transformation.

Exercise. See Peskin, Problem 12.3, p. 428. The interaction in the action $S[\varphi_1, \varphi_2]$ can be split into self-interaction and coupling between the fields. This form can be generalized

$$
\mathcal{L} = \frac{1}{2} \, \partial_\mu \varphi_1 \, \partial^\mu \varphi_1 + \frac{1}{2} \, \partial_\mu \varphi_2 \, \partial^\mu \varphi_2 - \frac{\lambda}{4!} (\varphi_1^4 + \varphi_2^4) - \frac{2\rho}{4!} \varphi_1^2 \varphi_2^2
$$

Do the following:

- Write the Euclidean Feynman rules.
- Compute all one-loop corrections to the λ-vertex and the ρ-vertex in dimensional regularization $D = 4 - 2\varepsilon$, use massless integrals (discussed in the following).
- Impose the normalization conditions at $s = t = u = \Lambda^2$ such that

  $$
  \Gamma^{(4)}(s = t = u = \Lambda^2) = -\lambda \,, \quad \Gamma^{(4)}_{\text{mixed}}(s = t = u = \Lambda^2) = -2\rho
  $$

  Notice that this implies that there are no one-loop finite corrections to the couplings.
- Compute the beta functions β_λ, β_ρ and

  $$
  \beta_{\frac{\lambda}{\rho}} = \mathrm{d}_t \frac{\lambda}{\rho} = \frac{1}{\rho} \, \mathrm{d}_t \lambda - \frac{1}{\rho^2} \lambda \, \mathrm{d}_t \rho = \frac{1}{\rho} \beta_\lambda - \frac{\lambda}{\rho^2} \beta_\rho
  $$

- Find the fixed points and describe the renormalization group flow.
- Explain what happens for

  $$
  \frac{1}{3} < \frac{\lambda}{\rho} < 1
  $$

  and for $\lambda = \rho$.

14.2 Massless integrals in dimensional regularization

See Anselmi, §2.1.

Proposition. In dimensional regularization, it holds

$$
I_\alpha = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{(p^2)^\alpha} = 0 \,, \quad \alpha \in \mathbb{C}
$$

Notice that it is null also for $\alpha = 0$. This integral describes tadpoles for $\alpha = 1$: the two-point Green's function receives no correction at one-loop if the fields are massless.

The intuitive argument of why this is true can be seen for $D \neq 2\alpha$ where the integral is dimensionful: the result has to be dimensionful as well and must be written in terms of dimensionful parameters, however the integral does not depend on any parameter and so the result must be zero.

Proof. Consider the theorem below and the Euclidean formalism. One may rewrite the integrand

$$\frac{1}{(p^2)^\alpha} = \frac{1}{(p^2)^\alpha}\frac{p^2+m^2}{p^2+m^2} = \frac{1}{(p^2)^\alpha}\frac{m^2}{p^2+m^2} + \frac{1}{(p^2)^{\alpha-1}}\frac{1}{p^2+m^2}$$

The region of convergence for the integral

$$\int \mathrm{d}^D p \, \frac{1}{(p^2)^\alpha}\frac{m^2}{p^2+m^2}$$

is given by the following limits. At infinity, one has

$$|p| \to \infty \, , \quad I \sim \frac{1}{p^{2\alpha+2-D}}$$

There is no singularity for $2\alpha + 2 - \operatorname{Re} D > 0$. At the origin, one has

$$|p| \to 0 \, , \quad I \sim p^{D-2\alpha}$$

There is no singularity for $\operatorname{Re} D - 2\alpha > 0$. The integral of the first addendum is well-defined for

$$2\alpha < \operatorname{Re} D < 2\alpha + 2$$

Similarly, the region of convergence of the integral of the second addendum is

$$2\alpha - 2 < \operatorname{Re} D < 2\alpha$$

The two regions do not overlap and, by the theorem below, one has

$$I_\alpha = \int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{m^2}{(p^2)^\alpha (p^2+m^2)} + \int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{1}{(p^2)^{\alpha-1}(p^2+m^2)} \equiv I_\alpha^{(1)} + I_\alpha^{(2)}$$

Looking at tables of integrals [r], one has a general formula

$$\int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{1}{(p^2)^a (p^2+m^2)^b} = (m^2)^{\frac{D}{2}-a-b}\frac{\Gamma(D/2 - a)\Gamma(a+b-D/2)}{(4\pi)^{\frac{D}{2}}\Gamma(b)\Gamma(D/2)}$$

For the first integral, one has $a = \alpha$, $b = 1$ and for the second $a = \alpha - 1$ and $b = 1$. Therefore

$$\begin{aligned} I_\alpha &= m^{D-2\alpha}\frac{\Gamma(D/2-\alpha)\Gamma(1+\alpha-D/2)}{(4\pi)^{\frac{D}{2}}\Gamma(D/2)} + m^{D-2\alpha}\frac{\Gamma(D/2+1-\alpha)\Gamma(\alpha-D/2)}{(4\pi)^{\frac{D}{2}}\Gamma(D/2)} \\ &= \frac{m^{D-2\alpha}}{(4\pi)^{\frac{D}{2}}\Gamma(D/2)}\Gamma(\alpha-D/2)\Gamma(D/2-\alpha)\left[\alpha - \frac{D}{2} + \frac{D}{2} - \alpha\right] = 0 \end{aligned}$$

where one applies $\Gamma(z+1) = z\Gamma(z)$ to the Gamma function with three addenda in its argument. □

Theorem. If an integrand can be written as a sum of terms with non-overlapping regions of convergence (for their integrals), then the integral is a sum of their integrals.

One a-posteriori argument to see why the integral is null follows. One may rescale $p = sp'$ to have

$$I_\alpha = \int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{1}{(p^2)^\alpha} = s^{D-2\alpha}\int \frac{\mathrm{d}^D p'}{(2\pi)^D}\frac{1}{(p'^2)^\alpha} \implies I_\alpha = s^{D-2\alpha} I_\alpha$$

For $D \neq 2\alpha$ then $I_\alpha = 0$. For the case $D = 2\alpha$, one can argue that there is a fine-tuning between the ultraviolet and the infrared divergences. In this case, the divergences are both present and give zero when summed. Consider the particular example of $D = 2 - \varepsilon$ and $\alpha = 1$. Then, the integral becomes

$$\begin{aligned} I_1 &= \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{p^2} = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \int_0^\infty \mathrm{d}\lambda \, \frac{1}{(p^2+\lambda)^2} = \int_0^\infty \mathrm{d}\lambda \int \frac{\mathrm{d}^{2-\varepsilon} p}{(2\pi)^{2-\varepsilon}} \frac{1}{(p^2+\lambda)^2} \\ &= \int_0^\infty \mathrm{d}\lambda \, \frac{\Gamma(1+\varepsilon/2)}{(4\pi)^{1-\frac{\varepsilon}{2}}\Gamma(2)} (\lambda)^{-\frac{\varepsilon}{2}-1} = \frac{\Gamma(1+\varepsilon/2)}{(4\pi)^{1-\frac{\varepsilon}{2}}} \int_0^\infty \frac{\mathrm{d}\lambda}{\lambda^{1+\frac{\varepsilon}{2}}} \end{aligned}$$

Ignoring the coefficients, the integral is

$$\begin{aligned} I_1 &\propto \int_{a_{\mathrm{IR}}}^1 \frac{\mathrm{d}\lambda}{\lambda^{1+\frac{\varepsilon}{2}}} + \int_1^{a_{\mathrm{UV}}} \frac{\mathrm{d}\lambda}{\lambda^{1+\frac{\varepsilon}{2}}} = -\frac{2}{\varepsilon}\lambda^{-\frac{\varepsilon}{2}}\bigg|_{a_{\mathrm{IR}}}^1 - \frac{2}{\varepsilon}\lambda^{-\frac{\varepsilon}{2}}\bigg|_1^{a_{\mathrm{UV}}} = -\frac{2}{\varepsilon}[-a_{\mathrm{IR}}^{-\frac{\varepsilon}{2}} + a_{\mathrm{UV}}^{-\frac{\varepsilon}{2}}] \\ &= -\frac{2}{\varepsilon}\left[-\frac{\varepsilon}{2}(-\ln a_{\mathrm{IR}} + \ln a_{\mathrm{UV}}) + o(\varepsilon)\right] = \ln a_{\mathrm{UV}} - \ln a_{\mathrm{IR}} + o(\varepsilon^0) = 0 \end{aligned}$$

At the second line, one has integrated up to a cutoff [r]. From this one sees that $a_{\mathrm{IR}} = a_{\mathrm{UV}}$ and $I_\alpha = 0$ is the product of a cancellation between the ultraviolet and infrared divergences.

When dealing with massless theories in two dimensions $D = 2$, the cancellation of the tadpole is due to a balance between infrared and ultraviolet divergences. When interested in either divergence, one has to remove the other divergence in order to renormalize the one of interest. For example, one replaces

$$I_1 = \int \frac{\mathrm{d}^{2-\varepsilon} p}{(2\pi)^{2-\varepsilon}} \frac{1}{p^2} \to \int \frac{\mathrm{d}^{2-\varepsilon} p}{(2\pi)^{2-\varepsilon}} \frac{1}{p^2+\mu^2} = \frac{\Gamma(\varepsilon/2)}{(4\pi)^{1-\frac{\varepsilon}{2}}}\mu^{-\varepsilon} \sim \frac{2}{\varepsilon}\mathrm{e}^{-\varepsilon \ln\mu + \cdots} \sim \frac{2}{\varepsilon}\,, \quad \varepsilon \to 0$$

where μ^2 is an infrared regulator. This is the ultraviolet divergence of the tadpole in two dimensions.

Therefore, in dimensional regularization, massless tadpoles can be ignored when $D \neq 2$. In $D = 2$ the tadpole is dimensionless and gives a contribution when removing one divergence.

Part II
Spin-half fermion fields

15 Introduction

Review – classical fields. See Srednicki, §§36, 38, 39. A fermion field is a field describing Dirac spinors. A Dirac spinor in the Weyl basis is comprised of two fixed-chirality Weyl spinors

$$\psi = \begin{bmatrix} \chi_{\mathrm{L}} \\ \chi_{\mathrm{R}} \end{bmatrix}$$

where the left-chiral spinor belongs to the $(\frac{1}{2}, 0)$ representation of the Lorentz group $\mathrm{SO}(1,3)$, while the right-chiral spinor belongs to $(0, \frac{1}{2})$.

The equation of motion of the Dirac field is the Dirac equation

$$(\mathrm{i}\,\partial\!\!\!/ - m)\psi(x) = 0$$

where the Dirac matrices are four square matrices of dimension four defining the Dirac algebra $\mathrm{Cl}_{1,3}(\mathbb{C})$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

The equation of motion can be obtained from the action principle with the Lagrangian

$$\mathcal{L} = \mathrm{i}\bar\psi\,\partial\!\!\!/\psi - m\bar\psi\psi = \bar\psi(\mathrm{i}\,\partial\!\!\!/ - m)\psi\,, \quad \bar\psi = \psi^\dagger\gamma^0$$

If the field ψ satisfies the Dirac equation of motion, then each of its four components (each two of the Weyl spinors) satisfies the Klein–Gordon equation

$$(\Box + m^2)\psi_j(x) = 0$$

In momentum space, the above is an algebraic equation that gives the dispersion relation

$$(-p^2+m^2)\psi(p)=0 \implies p^2=m^2 \iff p^0=\sqrt{|\mathbf{p}|^2+m^2}\equiv\omega$$

The most general solution of the Klein–Gordon equation has the form[1]

$$\psi(x)=\int\frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[u(\mathbf{p})\mathrm{e}^{-\mathrm{i}px}+v(\mathbf{p})\mathrm{e}^{\mathrm{i}px}]_{p^0=\omega}$$

Imposing the Dirac equation one finds

$$(\not{p}-m)u(\mathbf{p})=0\,,\quad(\not{p}+m)v(\mathbf{p})=0$$

One solves these equations by going to the rest frame $\mathbf{p}=0$ for which

$$\not{p}=\gamma^0p_0=\gamma^0m \implies (\gamma^0-1)u(0)=0\,,\quad(\gamma^0+1)v(0)=0$$

Each equation has two independent solutions $u_\pm$ and $v_\pm$. To get the solution for an arbitrary momentum one has to perform a boost. Therefore, the general solution to the Dirac equation is

$$\psi(x)=\sum_{s=\pm}\int\frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[b_s(\mathbf{p})u_s(\mathbf{p})\mathrm{e}^{-\mathrm{i}px}+d_s^\dagger v_s(\mathbf{p})\mathrm{e}^{\mathrm{i}px}]_{p^0=\omega}$$

where b and d are numbers.

The two-spinors u_s and v_s satisfy several identities (see Peskin, p. 48). The normalization is chosen to be

$$\bar{u}_r(\mathbf{p})u_s(\mathbf{p})=2m\delta_{rs}\,,\quad\bar{v}_r(\mathbf{p})v_s(\mathbf{p})=-2m\delta_{rs}$$

equivalent to

$$u_r^\dagger(\mathbf{p})u_s(\mathbf{p})=2p^0\delta_{rs}\,,\quad v_r^\dagger(\mathbf{p})v_s(\mathbf{p})=2p^0\delta_{rs}$$

The spinors are orthogonal

$$\bar{u}_r(\mathbf{p})v_s(\mathbf{p})=\bar{v}_r(\mathbf{p})u_s(\mathbf{p})=0\,,\quad u_r^\dagger(\mathbf{p})v_s(-\mathbf{p})=v_r^\dagger(-\mathbf{p})u_s(\mathbf{p})=0$$

The last two relations are not zero for both momenta being $+\mathbf{p}$. With these, one can obtain

$$b_s(\mathbf{p})=\int\mathrm{d}^3x\,\mathrm{e}^{\mathrm{i}px}u_s^\dagger(\mathbf{p})\psi(x)\,,\quad d_s(\mathbf{p})=\int\mathrm{d}^3x\,\mathrm{e}^{\mathrm{i}px}\psi^\dagger(x)v_s(\mathbf{p})$$

To see this, it is sufficient to compute the right-hand sides, substitute $\psi(x)$ and apply the above rules for the spinors. Notice that

$$\int\mathrm{d}^3x\,\mathrm{e}^{\mathrm{i}x(p-q)}=\mathrm{e}^{\mathrm{i}x^0(p^0-q^0)}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})$$

and that

$$|\mathbf{p}|=|\mathbf{q}| \implies p^0=q^0$$

Lecture 2

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15.1 Canonical quantization

See Srednicki, §39. The classical fields are promoted to operator fields by imposing a suitable set of equal-time anti-commutation rules (ACR)

$$\{\psi_\alpha(\mathbf{x},t),\psi_\beta(\mathbf{y},t)\}=0\,,\quad\{\psi_\alpha(\mathbf{x},t),\psi_\beta^\dagger(\mathbf{y},t)\}=\delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{y})$$

From these, one obtains the rules for the Fourier coefficients which are promoted to operators

$$\{b_r(\mathbf{p}),b_s^\dagger(\mathbf{q})\}=\{d_r(\mathbf{p}),d_s^\dagger(\mathbf{q})\}=2\omega(2\pi)^3\delta_{rs}\delta^{(3)}(\mathbf{p}-\mathbf{q})$$

[1] Understood as a group of four components that solve the equation.

One defines the vacuum state as the state destroyed by every annihilation operator

$$b_s(\mathbf{p})\left|0\right\rangle = d_s(\mathbf{p})\left|0\right\rangle = 0\,, \quad \forall b, d$$

The excited states are obtained by acting on the vacuum with the creation operators $b_s^\dagger$ and $d_s^\dagger$. One interprets $b_s^\dagger(\mathbf{p})\left|0\right\rangle$ as a single-particle state with momentum $\mathbf{p}$, energy $\omega = \sqrt{|\mathbf{p}|^2 + m^2}$ and projection s (in units) of the spin along the z-direction, $S_z = \frac{1}{2}s$. The difference between the two creation operators is not yet clear.

The use of anti-commutation rules implies the Pauli exclusion principle: there cannot be two fermions with the same quantum numbers. This is reflected in the spectrum of the number operator and the relation

$$\{b_s^\dagger(\mathbf{p}), b_s^\dagger(\mathbf{p})\} = 0 \implies [b_s^\dagger(\mathbf{p})]^2 = 0$$

The Hamiltonian density is obtained by Legendre transforming the Lagrangian density. By substituting the fields ψ and $\bar{\psi}$, one obtains the energy (Hamiltonian) of the system

$$H = \sum_{s=\pm} \int \mathrm{d}^3 p\, \omega [N_s^b(\mathbf{p}) + N_s^d(\mathbf{p})]$$

where N_s^a is the number operator of spin s for the ladder operator a

$$N_s^a(\mathbf{p}) \equiv \frac{1}{(2\pi)^3 2\omega} a_s^\dagger(\mathbf{p}) a_s(\mathbf{p})$$

The anti-commutation relations imply a finite spectrum

$$[N_s^a(\mathbf{p})]^2 = N_s^a(\mathbf{p}) \implies n = 0, 1$$

The Dirac Lagrangian exhibits a global U(1) symmetry

$$\psi' = \mathrm{e}^{\mathrm{i}\alpha}\psi\,, \quad \bar{\psi} = \mathrm{e}^{-\mathrm{i}\alpha}\bar{\psi}\,, \quad \alpha \in \mathbb{R}$$

Noether's theorem implies a conserved current

$$\partial_\mu J^\mu = 0\,, \quad J^\mu = \bar{\psi}\gamma^\mu\psi$$

The conserved charge is

$$Q = \int \mathrm{d}^3 x\, J^0(x) = \int \mathrm{d}^3 x\, \bar{\psi}\gamma^0\psi = \sum_{s=\pm} \int \mathrm{d}^3 p\, [N_s^b(\mathbf{p}) - N_s^d(\mathbf{p})]$$

The global symmetry gives a minus sign for the operator d: the creation operators have different meanings. The operator d treats particles with opposite U(1) charge of the operator b. The b-particles, with charge $Q = 1$, are simply called particles; while the d-particles, with $Q = -1$, are called anti-particles.

Free fermion propagator. See Srednicki, §42. The free fermion propagator is the inverse of the kinetic term in the Lagrangian

$$S_{\alpha\beta}(x - y) \equiv -\mathrm{i}\left\langle 0\right| \mathcal{T}\{\psi_\alpha(x)\bar{\psi}_\beta(y)\}\left|0\right\rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{(\not{p} + m)_{\alpha\beta}}{p^2 - m^2 + \mathrm{i}\varepsilon}\, \mathrm{e}^{-\mathrm{i}p(x-y)}$$

Exercise. Check that the propagator is the Green's function of the Dirac operator

$$(\mathrm{i}\,\not{\partial} - m)_{\alpha\beta} S_{\beta\gamma}(x) = \delta_{\alpha\gamma}\delta^{(4)}(x)$$

15.2 LSZ reduction formula

See Srednicki, §41. The physical observables are the cross-sections which are expressed in terms of scattering amplitudes $\langle f|i\rangle$. The initial and final states are defined asymptotically in the distant past and future, $t \to \mp\infty$. One utilizes the adiabatic hypothesis: the region where the interaction is non-trivial is finite and, beyond, the theory is essentially free. The hypothesis has to be supported by the ansatz that the free regions are reached in the distant past and future.

One-particle states can be constructed by applying one creation operator

$$a_s^\dagger(\mathbf{p})\,|0\rangle = |\mathbf{p}, s\rangle$$

Due to the Heisenberg uncertainty principle, a more physical particle is the one comprised of a wave-packet of momenta. The physical creation operator is a Gaussian distribution centered around a central momentum $\mathbf{q}$

$$a_s^\dagger = \int \mathrm{d}^3p\, f(\mathbf{p}) a_s^\dagger(\mathbf{p})\,, \quad f(\mathbf{p}) \propto \mathrm{e}^{-\frac{(\mathbf{p}-\mathbf{q})^2}{4\sigma^2}}$$

The operation of applying the creation operator to the vacuum to obtain a particle is valid and meaningful in the asymptotic limit, where the states and theory are free. This holds since the vacuum defined by the annihilation operators is the free vacuum. By introducing an interaction, the vacuum is interacting. One assumes to be able to define a vacuum in the interacting theory and that the particles can be obtained in the same way as the free theory. For instance, a one-particle initial and final state are taken to be

$$|i\rangle = \lim_{t\to-\infty} b_r^\dagger(t)\,|0\rangle\ , \quad |f\rangle = \lim_{t\to\infty} b_s^\dagger(t)\,|0\rangle$$

In the interacting theory, the ladder operators depend on time.

Time dependence. To obtain a useful expression for the scattering amplitude, one needs to study the time dependence of the ladder orpeators. As with the scalar fields, consider

$$\begin{aligned}
b_s(+\infty) - b_s(-\infty) &= \int_{-\infty}^{\infty} \mathrm{d}t\, \partial_0 b_s(t) = \int \mathrm{d}^3p\, f(\mathbf{p}) \int_{-\infty}^{\infty} \mathrm{d}t\, \partial_0 b_s(\mathbf{p}) \\
&= \int \mathrm{d}^3p\, f(\mathbf{p}) \int_{-\infty}^{\infty} \mathrm{d}t \int \mathrm{d}^3x\, \partial_0[\mathrm{e}^{\mathrm{i}px} u_s^\dagger(\mathbf{p})\psi(x)] \\
&= \int \mathrm{d}^3p\, f(\mathbf{p}) \int \mathrm{d}^4x\, \partial_0[\mathrm{e}^{\mathrm{i}px} \bar{u}_s(\mathbf{p})\gamma^0\psi(x)] \\
&= \int \mathrm{d}^3p\, f(\mathbf{p}) \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px} \bar{u}_s(\mathbf{p})[\mathrm{i}p_0\gamma^0 + \gamma^0\,\partial_0]\psi(x) \\
&= \int \mathrm{d}^3p\, f(\mathbf{p}) \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px} \bar{u}_s(\mathbf{p})[-\mathrm{i}p_j\gamma^j + \mathrm{i}m + \gamma^0\,\partial_0]\psi(x) \\
&= \int \mathrm{d}^3p\, f(\mathbf{p}) \int \mathrm{d}^4x\, \bar{u}_s(\mathbf{p})[-\gamma^j\,\partial_j \mathrm{e}^{\mathrm{i}px} + \mathrm{e}^{\mathrm{i}px}(\mathrm{i}m + \gamma^0\,\partial_0)]\psi(x) \\
&= \int \mathrm{d}^3p\, f(\mathbf{p}) \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px} \bar{u}_s(\mathbf{p})[\gamma^j\,\partial_j + \mathrm{i}m + \gamma^0\,\partial_0]\psi(x) \\
&= -\mathrm{i} \int \mathrm{d}^3p\, f(\mathbf{p}) \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px} \bar{u}_s(\mathbf{p})[\mathrm{i}\,\not{\partial} - m]\psi(x)
\end{aligned}$$

At the first line, one inserts the expression of the wave-packet operator b_1. At the second line, one inserts the expression of the annihilation operator $b_s(\mathbf{p})$. At the fifth line, one has remembered that

$$\bar{u}(\mathbf{p})(\not{p} - m) = 0 \implies \bar{u}(\mathbf{p})(p_0\gamma^0 + p_j\gamma^j - m) = 0 \implies \bar{u}(\mathbf{p})p_0\gamma^0 = \bar{u}(\mathbf{p})(-p_j\gamma^j + m)$$

At the penultimate line, one has integrated by parts the first addendum.

The integrand is zero if the theory is free, but the theory is interacting and so

$$(\mathrm{i}\,\not{\partial} - m)\psi(x) = -\delta_{\bar{\psi}(x)}\mathcal{L}_{\text{int}} \neq 0$$

and the ladder operators are functions of time. For the creation operators, one has

$$b_s^\dagger(+\infty) - b_s^\dagger(-\infty) = -\mathrm{i}\int \mathrm{d}^3p\, f(\mathbf{p}) \int \mathrm{d}^4x\, \bar\psi(x)(\mathrm{i}\overleftarrow{\not\partial} + m)u_s(\mathbf{p})\mathrm{e}^{-\mathrm{i}px}$$

[r]

Two-by-two scattering. Consider a two-by-two scattering $p_1p_2 \to p_{1'}p_{2'}$. The scattering amplitude is

$$\langle f|i\rangle = \langle 0|\, b_{2'}(+\infty)b_{1'}(+\infty)b_1^\dagger(-\infty)b_2^\dagger(-\infty)\,|0\rangle$$

where the index of the spin projection is dropped and $1, 2$ or $1', 2'$ indicates the particle. Thanks to the ordering of the operators, the above is equal to

$$\langle f|i\rangle = \langle 0|\, \mathcal{T}\{b_{2'}(+\infty)b_{1'}(+\infty)b_1^\dagger(-\infty)b_2^\dagger(-\infty)\}\,|0\rangle$$

In the limit $\sigma \to 0$, the wave-packet is a Dirac delta giving

$$b_s^\dagger(+\infty) - b_s^\dagger(-\infty) = -\mathrm{i}\int \mathrm{d}^4x\, \mathrm{e}^{-\mathrm{i}px}\bar\psi(x)(\mathrm{i}\overleftarrow{\not\partial} + m)u_s(\mathbf{p})$$

One would like to replace $b_j^\dagger(-\infty)$ with $b_j^\dagger(+\infty)$ and the integral above. Due to the time-ordered product, the operator $b_j^\dagger(+\infty)$ is moved to the left and gives zero when acting on the bra [r]. Similarly happens for the annihilation operators. Therefore the scattering amplitude is

$$\begin{aligned}
\langle f|i\rangle &= (-1)^2\mathrm{i}^4\, \langle 0|\, \mathcal{T}\bigg\{ \int \mathrm{d}^4x_1\, \mathrm{d}^4x_2\, \mathrm{d}^4x_{1'}\, \mathrm{d}^4x_{2'} \\
&\quad \times \mathrm{e}^{\mathrm{i}p_{2'}x_{2'}}\bar u_{s_{2'}}(\mathbf{p}_{2'})(\mathrm{i}\not\partial - m)\psi(x_{2'})\, \mathrm{e}^{\mathrm{i}p_{1'}x_{1'}}\bar u_{s_{1'}}(\mathbf{p}_{1'})(\mathrm{i}\not\partial - m)\psi(x_{1'}) \\
&\quad \times \bar\psi(x_1)(\mathrm{i}\overleftarrow{\not\partial} + m)u_{s_1}(\mathbf{p}_1)\mathrm{e}^{-\mathrm{i}p_1x_1}\, \bar\psi(x_2)(\mathrm{i}\overleftarrow{\not\partial} + m)u_{s_2}(\mathbf{p}_2)\mathrm{e}^{-\mathrm{i}p_2x_2}\bigg\}\,|0\rangle \\
&= \int \mathrm{d}^4x_1\, \mathrm{d}^4x_2\, \mathrm{d}^4x_{1'}\, \mathrm{d}^4x_{2'}\, \mathrm{e}^{\mathrm{i}p_{2'}x_{2'}}\bar u_{s_{2'}}(\mathbf{p}_{2'})(\mathrm{i}\not\partial - m)_{2'}\mathrm{e}^{\mathrm{i}p_{1'}x_{1'}}\bar u_{s_{1'}}(\mathbf{p}_{1'})(\mathrm{i}\not\partial - m)_{1'} \\
&\quad \times \langle 0|\, \mathcal{T}\{\psi(x_{2'})\psi(x_{1'})\bar\psi(x_1)\bar\psi(x_2)\}\,|0\rangle\, (\mathrm{i}\overleftarrow{\not\partial} + m)_1 u_{s_1}\mathrm{e}^{-\mathrm{i}p_1x_1}(\mathrm{i}\overleftarrow{\not\partial} + m)_2 u_{s_2}\mathrm{e}^{-\mathrm{i}p_2x_2}\}
\end{aligned}$$

[r] This is the LSZ reduction formula for fermions. In general, the computation of a scattering amplitude can be expressed as a computation of correlation functions

$$G^{(2n)}(x_1, \dots, x_n, x_1', \dots x_n') = \langle 0|\, \mathcal{T}\{\psi(x_1')\cdots\psi(x_n')\bar\psi(x_1)\cdots\bar\psi(x_n)\}\,|0\rangle$$

[r] One needs to develop a functional approach to compute the above Green's functions.

16 Functional quantization

See Srednicki, §43. Consider an interacting fermionic theory

$$\mathcal{L}(\psi, \bar\psi) = \bar\psi(\mathrm{i}\not\partial - m)\psi + \mathcal{L}_{\text{int}}(\psi, \bar\psi)$$

One needs to generalize the generating functional for fermions. One introduces the spinorial source terms η and $\bar\eta$ to formally write

$$W[\eta, \bar\eta] = \int [\mathcal{D}\psi\, \mathcal{D}\bar\psi] \exp\left[\mathrm{i}\int \mathrm{d}^4x\, [\mathcal{L}(x) + \bar\psi\eta + \bar\eta\psi]\right]$$

where the spinorial product is defined using the van der Waerden notation[2]. The integrand must be a real quantity. In fact

$$(\bar\psi\eta)^\dagger = (\psi^\dagger\gamma^0\eta)^\dagger = \eta^\dagger\gamma^0\psi = \bar\eta\psi$$

[2]See also eigenchris, Spinors for Beginners 9, `https://youtu.be/4NJBvkjpC3E?t=2340`, at minute 39:00 as well as A. Steane, An Introduction to Spinors, `https://arxiv.org/abs/1312.3824`.

which implies that $\bar{\psi}\eta + \bar{\eta}\psi$ is real. Since spinors anti-commute, then

$$\bar{\psi}\eta = -\eta\bar{\psi}$$

Therefore, when differentiating one has to be careful about the signs

$$\delta_{\eta(x)} \int \mathrm{d}^4 y\, \bar{\psi}(y)\eta(y) = -\bar{\psi}(x)$$

The Green's function is

$$\begin{aligned} G^{(2n)}(x_1, \dots, x_n, y_1, \dots, y_n) &= \langle 0| \mathcal{T}\{\psi_{\alpha_1}(x_1)\cdots\psi_{\alpha_n}(x_n)\bar{\psi}_{\beta_1}(y_1)\cdots\bar{\psi}_{\beta_n}(y_n)\} |0\rangle \\ &= (-1)^n \mathrm{i}^{2n} \frac{\delta^{2n} W[\eta, \bar{\eta}]}{\delta\bar{\eta}_{\alpha_1}(x_1)\cdots\delta\bar{\eta}_{\alpha_n}(x_n)\delta\eta_{\beta_1}(y_1)\cdots\delta\eta_{\beta_n}(y_n)}\bigg|_{\eta=\bar{\eta}=0} \\ &= \frac{\delta^{2n} W[\eta, \bar{\eta}]}{\delta\bar{\eta}_{\alpha_1}(x_1)\cdots\delta\bar{\eta}_{\alpha_n}(x_n)\delta\eta_{\beta_1}(y_1)\cdots\delta\eta_{\beta_n}(y_n)}\bigg|_{\eta=\bar{\eta}=0} \end{aligned}$$

Each derivative with respect to $\bar{\eta}$ brings a $-\mathrm{i}$ while each derivative with respect to η brings an i.

The Euclidean functional integral is

$$W_{\mathrm{E}}[\eta, \bar{\eta}] = \int [\mathcal{D}\psi\, \mathcal{D}\bar{\psi}] \exp\left[- \int \mathrm{d}^4 x \,[\mathcal{L}(x) - \bar{\psi}\eta - \bar{\eta}\psi]\right]$$

and the Euclidean Green's function is [r]

$$G_{\mathrm{E}}^{(2n)}(x_1, \dots, x_n, y_1, \dots, y_n) = (-1)^n \frac{\delta^{2n} W_{\mathrm{E}}[\eta, \bar{\eta}]}{\delta\bar{\eta}_{\alpha_1}(x_1)\cdots\delta\bar{\eta}_{\alpha_n}(x_n)\delta\eta_{\beta_1}(y_1)\cdots\delta\eta_{\beta_n}(y_n)}\bigg|_{\eta=\bar{\eta}=0}$$

The functional integral involves spinorial variables. This kind of integration has to be properly defined.

Exercise. Let $n = 1$, then

$$\begin{aligned} G^{(2)}(x, y) &= (-1) \frac{\delta^2 W_{\mathrm{E}}[\eta, \bar{\eta}]}{\delta\bar{\eta}_\alpha(x)\delta\eta_\beta(y)}\bigg|_{\eta=\bar{\eta}=0} \\ &= (-1) \frac{\delta}{\delta\bar{\eta}_\alpha(x)} \int [\mathcal{D}\psi\, \mathcal{D}\bar{\psi}] \,\exp\left[- \int \mathrm{d}^4 x' \,[\mathcal{L} - \bar{\psi}\eta - \bar{\eta}\psi]\right] [-\bar{\psi}_\beta(y)]\bigg|_{\eta=\bar{\eta}=0} \\ &= \int [\mathcal{D}\psi\, \mathcal{D}\bar{\psi}] \,\exp\left[- \int \mathrm{d}^4 x' \,[\mathcal{L} - \bar{\psi}\eta - \bar{\eta}\psi]\right] \psi_\alpha(x)\bar{\psi}_\beta(y)\bigg|_{\eta=\bar{\eta}=0} \\ &= \langle 0| \mathcal{T}\{\psi_\alpha(x)\bar{\psi}_\beta(y)\} |0\rangle \end{aligned}$$

Lecture 3

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16.1 Grassmann algebra

See Cheng, §1.3. See also DeWitt, Supermanifolds[3]. The fields ψ and $\bar{\psi}$ are classical spinor fields with four components in Dirac's notation

$$\psi_\alpha\,, \quad \alpha = 1, 2, 3, 4$$

Each component is a field made of Grassmann-odd numbers[4]: it anti-commutes with itself and other Grassmann-odd numbers. For example

$$\psi_1\psi_2 = -\psi_2\psi_1$$

[3] B. DeWitt, Supermanifolds, 2nd ed. Cambridge: Cambridge University Press, 1992.

[4] Odd or even refers to the number of Grassmann variables θ_i in the expansion of a Grassmann number z in terms of such Grassmann variables θ_i. Notice the careful use of the words "number" and "variable". A Grassmann number may not have definite Grassmann parity, but can be separated into odd and even parts. Odd Grassmann numbers anti-commute between themselves and even Grassmann numbers commute with every Grassmann number.

Grassmann numbers are needed when taking the classical limit of the quantized spinor fields which are anti-commuting. When using a functional definition of quantum field theory, the quantities appearing inside the path integral are all classical, not operators, so they have to be Grassmann numbers.

The path integral is the continuum limit of an ordinary integral on a lattice. A field is evaluated only at the sites of the lattice

$$\psi_\alpha(x^i) \equiv \psi_\alpha^i$$

The path integral on the lattice is an ordinary integral but on Grassmann-odd variables[5]

$$\int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}] \leftrightarrow \int \prod_{\alpha,i} \mathrm{d}\psi_\alpha^i \prod_{\alpha,i} \mathrm{d}\bar{\psi}_\alpha^i$$

One needs to define analysis for Grassmann numbers.

Algebra. An n-dimensional Grassmann algebra is generated by n Grassmann variables θ_i that anti-commute

$$\{\theta_i, \theta_j\} = 0 \implies \theta_i^2 = 0$$

A generic element of the algebra can be expanded in a finite Taylor series

$$f(\theta_1, \dots, \theta_n) = f_0 + f_{1,i}\theta^i + f_{2,ij}\theta^i\theta^j + \cdots + f_{n,i_1\cdots i_n}\theta^{i_1}\cdots\theta^{i_n}$$

This is because if there is any repeated variable, then it is zero $\theta_j^2 = 0$. The coefficients of the expansion are complex numbers that are completely anti-symmetric in the indices i_k.

Consider the expansion of an element of the algebra by writing explicitly the dependence on one particular[6] Grassmann variable θ. In such variable, the expansion is at most linear

$$f(\theta) = f_0 + f_1\theta$$

The coefficients f_0 and f_1 are independent of θ and depend on the other $n-1$ Grassmann variables.

In general, one has to consider a function also of space-time

$$f(x, \theta) = f_0(x) + f_1(x)\theta$$

The coefficient of the θ-expansion are space-time fields. If $f(x,\theta)$ is a Grassmann-even field, then the Grassmann parity must be the same on either side of the equation, so $f_0(x)$ is a Grassmann-even field and $f_1(x)$ is a Grassmann-odd field.

The addition of Grassmann variables to space-time gives superspace[7]. A field on superspace is a superfield and contains different ordinary fields. In supersymmetry, a superfield is a representation of the supersymmetry algebra.

Differentiation. The left and right derivatives are defined as

$$\mathrm{d}_\theta\theta = \theta\,\overleftarrow{\mathrm{d}}_\theta = 1$$

Therefore, the left derivative of the above element of the algebra is

$$\mathrm{d}_\theta f(\theta) = \begin{cases} +f_1\,, & f \text{ odd} \\ -f_1\,, & f \text{ even} \end{cases}$$

while the right derivative is simpler

$$f(\theta)\,\overleftarrow{\mathrm{d}}_\theta = f_1$$

[5] Grassmann numbers are individual elements of the exterior algebra generated by a set of n Grassmann variables.

[6] This means that the dependence on the other $n-1$ variables is hidden inside the expansion coefficients which are no longer complex numbers, but Grassmann numbers.

[7] In particular Minkowski superspace: Minkowski space is extended with anti-commuting fermionic degrees of freedom, taken to be anti-commuting Weyl spinors from the Clifford algebra associated to the Lorentz group.

Integration. The integration over Grassmann-odd variables is called Berenzin integration[8] which differs from Lebesgue's.

Typically, the integral is the inverse operator of the derivative. Since the partial derivative with respect to a Grassmann variable is a Grassmann-odd operator then the second derivative is zero[9]. The derivative is not an invertible operator. The integral must be defined in another way

$$\int \mathrm{d}\theta\, f(\theta)$$

One imposes two properties: linearity

$$\int \mathrm{d}\theta\,(f + \alpha g) = \int \mathrm{d}\theta\, f + \alpha \int \mathrm{d}\theta\, g\,, \quad \alpha \in \mathbb{C}$$

and translational invariance

$$\int \mathrm{d}\theta\, f(\theta) \equiv \int \mathrm{d}\theta\, f(\theta + \eta)$$

where η is Grassmann-odd number independent of the particular variable θ.

The second property can be expanded to have

$$\int \mathrm{d}\theta\, f(\theta) = \int \mathrm{d}(\theta+\eta)\, f(\theta+\eta) = \int \mathrm{d}\theta\, f(\theta+\eta) \implies \int \mathrm{d}\theta\,(f_0 + f_1\theta) = \int \mathrm{d}\theta\,[f_0 + f_1(\theta+\eta)]$$

Applying linearity, one finds

$$\int \mathrm{d}\theta\, \eta f_1 = 0 \implies \eta f_1 \int \mathrm{d}\theta = 0 \implies \boxed{\int \mathrm{d}\theta = 0}$$

since the above has to hold for every f_1 and η (which do not depend on θ). Using this result, one obtains

$$\int \mathrm{d}\theta\, f(\theta) = \int \mathrm{d}\theta\, f_0 + f_1\theta = \int \mathrm{d}\theta\, f_1\theta = \pm f_1 \int \mathrm{d}\theta\, \theta \equiv \pm f_1 \implies \boxed{\int \mathrm{d}\theta\, \theta = 1}$$

where $+$ is for f odd and $-$ for f even. This choice is a convention, the integral could be any other constant. One may notice that Berenzin integration is equivalent to differentiation

$$\int \mathrm{d}\theta\, f(\theta) = \pm f_1 = \mathrm{d}_\theta f$$

Therefore, the general definition of integration in a Grassmann-odd variable is

$$\boxed{\int \mathrm{d}\theta\, f(\theta) \equiv \mathrm{d}_\theta f|_{\theta=0}}$$

[r] why evaluated at $\theta = 0$? for defining the Berenzin integral also for ordinary functions?

Change of variables. When performing a change of variables, one has to account for the Jacobian. For the Berenzin integral, the inverse Jacobian is produced instead.

For ordinary (Grassmann-even, Riemann or Lebesgue) integration, a change of variable produces

$$\int \mathrm{d}y\, f(y)\,, \quad y = g(x) \implies \int \mathrm{d}y\, f(y) = \int \mathrm{d}x\, |\partial_x y| f(g(x))$$

For Grassmann-odd integration, a change of variables is

$$\theta' = g(\theta) = a + b\theta\,, \quad \mathrm{d}_\theta \theta' = b$$

[8]See Berezin, F. A. (1966). The Method of Second Quantization. Pure and Applied Physics. Vol. 24. New York. ISSN 0079-8193. `https://www.sciencedirect.com/bookseries/pure-and-applied-physics/vol/24`.

[9]This is a consequence of $\partial_z z = 1$.

Notice that a is odd and b is even. Knowing that

$$\int \mathrm{d}\theta' \, f(\theta') = \int \mathrm{d}\theta' \, (f_0 + f_1\theta') = \pm f_1$$

one may consider the integration without Jacobian

$$\int \mathrm{d}\theta \, f(\theta'(\theta)) = \int \mathrm{d}\theta \, f(a + b\theta) = \int \mathrm{d}\theta \, [f_0 + f_1(a + b\theta)] = \pm f_1 b = \pm f_1 \, \mathrm{d}_\theta \theta'$$

Therefore, when performing the change of variables, one has to use the inverse Jacobian

$$\int \mathrm{d}\theta' \, f(\theta') = \int \mathrm{d}\theta \, |\mathrm{d}_\theta \theta'|^{-1} f(\theta'(\theta))$$

Generalization to more variables. The differentiation and integration results above can be generalized to more Grassmann variables.

Differentiation. The left and right derivatives are

$$\mathrm{d}_{\theta_i} \theta_j = \theta_j \overleftarrow{\mathrm{d}}_{\theta_i} = \delta_{ij}$$

The derivative are Grassmann-odd operators

$$\{\mathrm{d}_{\theta_i}, \mathrm{d}_{\theta_j}\} = 0 \, , \quad \{\mathrm{d}_{\theta_i}, \theta_j\} = \delta_{ij}$$

the second equality is just the Leibniz product rule for Grassmann-odd variables: when computing the derivative of a product, a minus sign appears every time the derivative goes though an odd variable

$$\{\mathrm{d}_{\theta_i}, \theta_j\} f = \delta_{ij} f \implies \mathrm{d}_{\theta_i}(\theta_j f) + \theta_j \, \mathrm{d}_{\theta_i} f = \delta_{ij} f \implies \mathrm{d}_{\theta_i}(\theta_j f) = \delta_{ij} f - \theta_j \, \mathrm{d}_{\theta_i} f$$

As such, the derivative of a product is

$$\mathrm{d}_{\theta_i}(\theta_1 \cdots \theta_n) = \delta_{i1}\theta_2 \cdots \theta_n - \theta_1 \delta_{i2} \theta_3 \cdots \theta_n + \cdots + (-1)^{n-1} \theta_1 \cdots \theta_{n-1} \delta_{in} = \sum_{j=1}^n (-1)^{j-1} \delta_{ij} \prod_{k \neq j}^n \theta_k$$

while the right derivative is similar

$$(\theta_1 \cdots \theta_n) \overleftarrow{\mathrm{d}}_{\theta_i} = \theta_1 \cdots \theta_{n-1} \delta_{in} - \theta_1 \cdots \delta_{i,n-1} \theta_n + \cdots + (-1)^{n-1} \delta_{i1} \theta_2 \cdots \theta_n = \sum_{j=1}^n (-1)^{n-j} \delta_{ij} \prod_{k \neq j}^n \theta_k$$

Integration. For integration, the measures anti-commute

$$\{\mathrm{d}\theta_i, \mathrm{d}\theta_j\} = 0$$

The fundamental integral properties are

$$\int \mathrm{d}\theta_i \, 1 = 0 \, , \quad \int \mathrm{d}\theta_i \, \theta_j = \delta_{ij}$$

The integral in one variable of an element of the algebra is

$$\int \mathrm{d}\theta_i \, f(\theta_1, \ldots, \theta_n) = \mathrm{d}_{\theta_i} f(\theta_1, \ldots, \theta_n)|_{\theta_i = 0}$$

The integral in multiple variables is

$$\int \mathrm{d}\theta_n \, \cdots \, \mathrm{d}\theta_1 \, f(\theta_1, \ldots, \theta_n) = \mathrm{d}_{\theta_n} \cdots \, \mathrm{d}_{\theta_1} f|_{\theta_1 = \cdots = \theta_n = 0}$$

Change of variables. For a change of variables $\theta'_i = b_{ij}\theta_j$, one has

$$\int \mathrm{d}\theta'_n \cdots \mathrm{d}\theta'_1 f(\theta'_1, \ldots, \theta'_n) = \int \mathrm{d}\theta_n \cdots \mathrm{d}\theta_1 \, (\det b)^{-1} f(\theta'_1(\theta), \ldots, \theta'_n(\theta))$$

Gaussian integrals. The Gaussian integral is

$$G(A) = \int \mathrm{d}\theta_n \cdots \mathrm{d}\theta_1 \, \mathrm{e}^{\frac{1}{2}\theta_i A_{ij}\theta_j}$$

where A is an anti-symmetric square matrix of size n, $A_{ij} = -A_{ji}$.

Let $n = 2$. The exponent is

$$\theta_i A_{ij}\theta_j = \begin{bmatrix}\theta_1 & \theta_2\end{bmatrix} \begin{bmatrix} 0 & A_{12} \\ -A_{12} & 0 \end{bmatrix} \begin{bmatrix}\theta_1 \\ \theta_2\end{bmatrix}$$

The integral is

$$\begin{aligned} G(A) &= \int \mathrm{d}\theta_2 \, \mathrm{d}\theta_1 \, \exp\left[\frac{1}{2}(\theta_1 A_{12}\theta_2 - \theta_2 A_{12}\theta_1)\right] = \int \mathrm{d}\theta_2 \, \mathrm{d}\theta_1 \mathrm{e}^{\theta A_{12}\theta_2} \\ &= \int \mathrm{d}\theta_2 \, \mathrm{d}\theta_1 \, (1 + \theta_1 A_{12}\theta_2) = \int \mathrm{d}\theta_2 \, \mathrm{d}\theta_1 \, \theta_1 A_{12}\theta_2 = A_{12} \int \mathrm{d}\theta_2 \, \theta_2 = A_{12} = \sqrt{\det A} \end{aligned}$$

This is true for any n. For Berenzin integrals, the determinant is in the numerator of the result and not the denominator.

Table of Gaussian integrals. For real Grassmann-even variables

$$\int \mathrm{d}x_1 \cdots \mathrm{d}x_n \, \mathrm{e}^{-\frac{1}{2}x^\top A x} = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}}$$

For complex Grassmann-even variables

$$\int \left[\prod_{j=1}^{n} \mathrm{d}z_j \, \mathrm{d}\bar{z}_j\right] \mathrm{e}^{-z^\dagger A z} = \frac{\pi^{\frac{n}{2}}}{\det A}$$

[r] For real Grassmann-odd variables

$$\int \mathrm{d}\theta_n \cdots \mathrm{d}\theta_1 \, \mathrm{e}^{\frac{1}{2}\theta A \theta} = \sqrt{\det A}$$

For complex Grassmann-odd variables

$$\int \left[\prod_{j=n}^{1} \mathrm{d}\theta_j \, \mathrm{d}\bar{\theta}_j\right] \mathrm{e}^{\bar{\theta}^\top A \theta} = \det A$$

Functional integrals. The generalization to functional integration can be defined on the lattice. The continuum theory is retrieved in the continuum limit of the lattice.

Example. Consider the free generating functional of a scalar theory

$$W_0[0] = \int [\mathcal{D}\varphi] \, \exp\left[-\int \mathrm{d}^4 x \, \frac{1}{2}\varphi(\Box + m^2)\varphi\right] \propto [\det(\Box + m^2)]^{-\frac{1}{2}}$$

where the determinant of an operator is the product of its eigenvalues. For a spinor theory, one has

$$W_0[0] = \int [\mathcal{D}\psi \, \mathcal{D}\bar{\psi}] \, \exp\left[-\int \mathrm{d}^4 x \, \frac{1}{2}\bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi\right] \propto \det(\mathrm{i}\,\partial\!\!\!/ - m)$$

16.2 Free field theory

See Srednicki, §43?, sources? [r]. For a free theory, the functional integral can be computed exactly. For a weak interacting theory, one may apply perturbation theory.

Through Grassmann numbers, one may give meaning to the path integral for fermions

$$W[\eta,\bar{\eta}] = \int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}] \exp\left[\mathrm{i}\int \mathrm{d}^4x\,[\mathcal{L}(x) + \bar{\psi}\eta + \bar{\eta}\psi]\right]$$

Consider the free Lagrangian

$$\mathcal{L}_0(\psi,\bar{\psi}) = \bar{\psi}(\mathrm{i}\,\not{\partial} - m)\psi$$

The generating functional is

$$W_0[\eta,\bar{\eta}] = N\int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}] \exp\left[\mathrm{i}\int \mathrm{d}^4x\,[\bar{\psi}(\mathrm{i}\,\not{\partial} - m)\psi + \bar{\psi}\eta + \bar{\eta}\psi]\right]$$

The propagator is the inverse of the kinetic term. Using the fact that the propagator is the Green's function of the Dirac operator, one gets

$$\int \mathrm{d}^4x\,(\mathrm{i}\,\not{\partial} - m)_{\alpha\beta}S_{\beta\gamma}(x) = \delta_{\alpha\gamma} \implies S = (\mathrm{i}\,\not{\partial} - m)^{-1}$$

One may complete the square in the exponent of the generating functional

$$\begin{aligned}
-\mathrm{i}E &= \int \mathrm{d}^4x\,[\bar{\psi}(\mathrm{i}\,\not{\partial} - m)\psi + \bar{\psi}\eta + \bar{\eta}\psi] \\
&= \int \mathrm{d}^4x\left\{\left[\bar{\psi} + \int \mathrm{d}^4y\,\bar{\eta}(y)(\mathrm{i}\,\not{\partial} - m)_y^{-1}\right](\mathrm{i}\,\not{\partial} - m)_x\left[\psi + \int \mathrm{d}^4y'\,(\mathrm{i}\,\not{\partial} - m)_{y'}^{-1}\eta(y')\right]\right. \\
&\quad \left. - \int \mathrm{d}^4y\,\mathrm{d}^4y'\,\bar{\eta}(y)(\mathrm{i}\,\not{\partial} - m)_y^{-1}(\mathrm{i}\,\not{\partial} - m)_x(\mathrm{i}\,\not{\partial} - m)_{y'}^{-1}\eta(y')\right\} \\
&= \int \mathrm{d}^4x\,\bar{\chi}(\mathrm{i}\,\not{\partial} - m)_x\chi - \int \mathrm{d}^4x\,\mathrm{d}^4y'\,\bar{\eta}(x)(\mathrm{i}\,\not{\partial} - m)_{y'}^{-1}\eta(y')
\end{aligned}$$

[r] where

$$\chi \equiv \psi + \int \mathrm{d}^4y'\,(\mathrm{i}\,\not{\partial} - m)_{y'}^{-1}\eta(y')\,, \quad (\mathrm{i}\,\not{\partial} - m)_y^{-1}(\mathrm{i}\,\not{\partial} - m)_x = \delta^{(4)}(x-y)$$

The generating functional becomes

$$\begin{aligned}
W_0[\eta,\bar{\eta}] &= N\int [\mathcal{D}\chi\,\mathcal{D}\bar{\chi}] \exp\left[\mathrm{i}\int \mathrm{d}^4x\,\bar{\chi}(\mathrm{i}\,\not{\partial} - m)_x\chi - \mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y'\,\bar{\eta}(x)(\mathrm{i}\,\not{\partial} - m)_{y'}^{-1}\eta(y')\right] \\
&= \exp\left[-\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y'\,\bar{\eta}(x)(\mathrm{i}\,\not{\partial} - m)_{y'}^{-1}\eta(y')\right]W_0[0,0] \\
&= \exp\left[-\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y'\,\bar{\eta}_\alpha(x)S_{\alpha\beta}(x-y')\eta_\beta(y')\right]
\end{aligned}$$

At the first line, the path integral of the first addendum of the exponent is a Gaussian integral and can be absorbed into the normalization constant N to give a normalization of $W_0[\eta = \bar{\eta} = 0] = 1$ [r].

Remark. The two-point Green's function for the free theory is

$$\begin{aligned}
G^{(2)}(x_1,x_2) &= \left.\frac{\delta^2 W_0[\eta,\bar{\eta}]}{\delta\bar{\eta}_\gamma(x_1)\delta\eta_\delta(x_2)}\right|_{\eta=\bar{\eta}=0} = \frac{\delta}{\delta\bar{\eta}_\gamma(x_1)}\left.\left[\mathrm{i}\int \mathrm{d}^4x\,\bar{\eta}_\alpha(x)S_{\alpha\delta}(x-x_2)W_0\right]\right|_{\eta=\bar{\eta}=0} \\
&= \mathrm{i}S_{\gamma\delta}(x_1-x_2) \equiv \langle 0|\,\mathcal{T}\{\psi_\gamma(x_1)\bar{\psi}_\delta(x_2)\}\,|0\rangle
\end{aligned}$$

16.3 Interacting field theory

The Lagrangian is

$$\mathcal{L}(\psi,\bar\psi)=\bar\psi(\mathrm{i}\,\partial\!\!\!/-m)\psi+\mathcal{L}_{\text{int}}(\psi,\bar\psi)$$

Since the Lagrangian is a scalar, it has to be a function only of bilinears of spinors. The generating functional is

$$W[\eta,\bar\eta]=\int[\mathcal{D}\psi\,\mathcal{D}\bar\psi]\exp\left[\mathrm{i}\int\mathrm{d}^4x\,[\mathcal{L}_0(x)+\mathcal{L}_{\text{int}}(\psi,\bar\psi)+\bar\psi\eta+\bar\eta\psi]\right]$$

Noting that

$$\delta_{\eta(y)}\int\mathrm{d}^4x\,\bar\psi(x)\eta(x)=-\bar\psi(y)\,,\quad \delta_{\bar\eta(y)}\int\mathrm{d}^4x\,\bar\eta(x)\psi(x)=\psi(y)$$

one may apply the useful property of functional integrals[10] and rewrite the fields in terms of derivatives

$$\mathcal{L}_{\text{int}}(\psi,\bar\psi)\exp\left[\mathrm{i}\int\mathrm{d}^4x\,(\bar\psi\eta+\bar\eta\psi)\right]=\mathcal{L}_{\text{int}}(-\mathrm{i}\,\delta_{\bar\eta(x)},\mathrm{i}\,\delta_{\eta(x)})\exp\left[\mathrm{i}\int\mathrm{d}^4x\,(\bar\psi\eta+\bar\eta\psi)\right]$$

Therefore the generating functional is

$$\begin{aligned}W[\eta,\bar\eta]&=\int[\mathcal{D}\psi\,\mathcal{D}\bar\psi]\,\exp\left[\mathrm{i}\int\mathrm{d}^4x\,\mathcal{L}_{\text{int}}(-\mathrm{i}\,\delta_{\bar\eta},\mathrm{i}\,\delta_\eta)\right]\exp\left[\mathrm{i}\int\mathrm{d}^4x\,[\mathcal{L}_0(x)+\bar\psi\eta+\bar\eta\psi]\right]\\&=\exp\left[\mathrm{i}\int\mathrm{d}^4x\,\mathcal{L}_{\text{int}}(-\mathrm{i}\,\delta_{\bar\eta},\mathrm{i}\,\delta_\eta)\right]W_0[\eta,\bar\eta]\end{aligned}$$

If the coupling constant is weak, then one may expand the exponential of the interaction in a perturbative series in powers of the coupling constant.

Wick's theorem. To see the effect of Wick's theorem, one may compute the four-point Green's function in the free theory

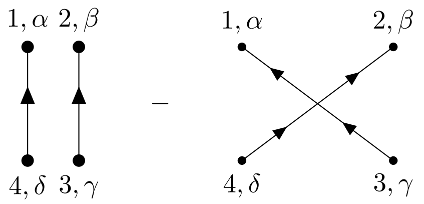
$$\begin{aligned}G^{(4)}_{\alpha\beta\gamma\delta}(x_1,x_2,x_3,x_4)&=\langle 0|\,\mathcal{T}\{\psi_\alpha(x_1)\psi_\beta(x_2)\bar\psi_\gamma(x_3)\bar\psi_\delta(x_4)\}\,|0\rangle\\
&=\left.\frac{\delta^4W_0[\eta,\bar\eta]}{\delta\bar\eta_\alpha(x_1)\delta\bar\eta_\beta(x_2)\delta\eta_\gamma(x_3)\delta\eta_\delta(x_4)}\right|_{\eta=\bar\eta=0}\\
&=\frac{\delta^3}{\delta\bar\eta_\alpha(x_1)\delta\bar\eta_\beta(x_2)\delta\eta_\gamma(x_3)}\left\{\exp\left[-\mathrm{i}\int\mathrm{d}^4x\,\mathrm{d}^4y\,\bar\eta_\varepsilon(x)S_{\varepsilon\zeta}(x-y)\eta_\zeta(y)\right]\right.\\
&\qquad\left.\left.\times\,\mathrm{i}\int\mathrm{d}^4x\,\bar\eta_\varepsilon S_{\varepsilon\delta}(x-x_4)\right\}\right|_{\eta=\bar\eta=0}\\
&=\frac{\delta^2}{\delta\bar\eta_\alpha(x_1)\delta\bar\eta_\beta(x_2)}\left[W_0[\eta,\bar\eta]\mathrm{i}\int\mathrm{d}^4y\,\bar\eta_\iota S_{\iota\gamma}(y-x_3)\right.\\
&\qquad\left.\left.\times\,\mathrm{i}\int\mathrm{d}^4x\,\bar\eta_\varepsilon S_{\varepsilon\delta}(x-x_4)\right]\right|_{\eta=\bar\eta=0}\\
&=\frac{\delta}{\delta\bar\eta_\alpha(x_1)}\left[\frac{\delta W_0[\eta,\bar\eta]}{\delta\bar\eta_\beta(x_2)}\,\mathrm{i}\int\mathrm{d}^4y\,\bar\eta_\iota S_{\iota\gamma}(y-x_3)\mathrm{i}\int\mathrm{d}^4x\,\bar\eta_\varepsilon S_{\varepsilon\delta}(x-x_4)\right.\\
&\qquad+W_0[\eta,\bar\eta]\mathrm{i}S_{\beta\gamma}(x_2-x_3)\mathrm{i}\int\mathrm{d}^4x\,\bar\eta_\varepsilon S_{\varepsilon\delta}(x-x_4)\\
&\qquad\left.\left.-\,W_0[\eta,\bar\eta]\mathrm{i}\int\mathrm{d}^4y\,\bar\eta_\iota S_{\iota\gamma}(y-x_3)\mathrm{i}S_{\beta\delta}(x_2-x_4)\right]\right|_{\eta=\bar\eta=0}\\
&=\frac{\delta}{\delta\bar\eta_\alpha(x_1)}\left.\left[\frac{\delta W_0[\eta,\bar\eta]}{\delta\bar\eta_\beta(x_2)}\,\mathrm{i}\int\mathrm{d}^4y\,\bar\eta_\iota S_{\iota\gamma}(y-x_3)\mathrm{i}\int\mathrm{d}^4x\,\bar\eta_\varepsilon S_{\varepsilon\delta}(x-x_4)\right]\right|_{\eta=\bar\eta=0}\\
&\qquad+\left[\frac{\delta W_0[\eta,\bar\eta]}{\delta\bar\eta_\alpha(x_1)}\mathrm{i}S_{\beta\gamma}(x_2-x_3)\mathrm{i}\int\mathrm{d}^4x\,\bar\eta_\varepsilon S_{\varepsilon\delta}(x-x_4)\right.\end{aligned}$$

[10]See QFT I.

$$
\begin{aligned}
&- \frac{\delta W_0[\eta,\bar\eta]}{\delta\bar\eta_\alpha(x_1)} \mathrm{i} \int \mathrm{d}^4 y\, \bar\eta_\iota S_{\iota\gamma}(y - x_3) \mathrm{i} S_{\beta\delta}(x_2 - x_4)\Bigg]\Bigg|_{\eta=\bar\eta=0} \\
&+ \mathrm{i}^2 \Big[W_0[\eta,\bar\eta] S_{\beta\gamma}(x_2 - x_3) S_{\alpha\delta}(x_1 - x_4) \\
&- W_0[\eta,\bar\eta] S_{\alpha\gamma}(x_1 - x_3) S_{\beta\delta}(x_2 - x_4)\Big]\Big|_{\eta=\bar\eta=0} \\
&= \mathrm{i} S_{\beta\gamma}(x_2 - x_3) \mathrm{i} S_{\alpha\delta}(x_1 - x_4) - \mathrm{i} S_{\alpha\gamma}(x_1 - x_3) \mathrm{i} S_{\beta\delta}(x_2 - x_4)
\end{aligned}
$$

In terms of diagrams, one has

$G^{(4)}_{\alpha\beta\gamma\delta}(x_1, x_2, x_3, x_4) =$



where every directed line is a propagator. In the Feynman diagrams, scalars are represented as undirected dashed lines, while fermions are represented as directed solid lines.

Lecture 4

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The different sign can be interpreted using Wick's theorem: the only non-vanishing contributions are the ones where the fields are contracted completely. The minus sign appears in the second diagram since one has to contract the fields when they are not next to each other

$$
\langle 0| \mathcal{T}\{\psi_1\psi_2\bar\psi_3\bar\psi_4\} |0\rangle = :\psi_1\psi_2\bar\psi_3\bar\psi_4: + :\psi_1\psi_2\bar\psi_3\bar\psi_4:
$$

This gives a general rule: when performing contractions of spinor fields one has to be careful about the signs.

16.3.1 Yukawa theory

See Srednicki, §§45, 51. The Yukawa theory treats the coupling between a massive real scalar field φ and a massive spinor field ψ. This theory is the analogue of the $\lambda\varphi^4$ theory for scalar fields.

Lagrangian. Let M be the mass of the boson field and m the mass of the fermion field. The Yukawa interaction Lagrangian is

$$
\mathcal{L}_{\mathrm{Y}} = g\varphi\bar\psi\psi
$$

[r] One would like to write the simplest Lagrangian. To this end, one may impose a U(1) symmetry on the spinor field along with Lorentz symmetry and the three discrete symmetries of parity, time reversal and charge conjugation. The scalar terms allowed are all the powers of field with non-negative coupling constant: φ, φ^2, φ^3 and φ^4. While no further spinor terms are allowed other than the kinetic term, the three scalar interaction terms above are too many (notice that φ^2 is a mass term) if one desires the simplest theory. One may modify the Yukawa interaction

$$
\mathcal{L}_{\mathrm{Y}} = \mathrm{i} g\varphi\bar\psi\gamma_5\psi
$$

The imaginary unit makes the interaction Lagrangian real. In fact, consider

$$
(\bar\psi\gamma_5\psi)^\dagger = (\psi^\dagger\gamma_0\gamma_5\psi)^\dagger = \psi^\dagger\gamma_5^\dagger\gamma_0^\dagger\psi = \psi^\dagger\gamma_5\gamma_0\psi = -\bar\psi\gamma_5\psi
$$

This term is anti-hermitian and therefore is purely imaginary. Since the scalar field is real, then an imaginary unit is needed to keep the Lagrangian real.

One notices that under a parity transformation, the interaction term changes sign

$$
P(\bar\psi\gamma_5\psi) = -\bar\psi\gamma_5\psi
$$

while the kinetic fermionic term is invariant. For the Lagrangian to conserve parity, the scalar field φ must be a pseudo-scalar field, $P(\varphi) = -\varphi$, and the linear and cubic scalar interactions along with their ultraviolet divergences cannot appear:

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}M^2\varphi^2 + \bar{\psi}(\mathrm{i}\,\not{\partial} - m)\psi - \frac{\lambda}{4!}\varphi^4 + \mathrm{i}g\varphi\bar{\psi}\gamma_5\psi$$

Perturbation theory and Feynman rules. For a weak interacting theory, $\lambda, g \ll 1$, one may apply perturbation theory. Trading

$$\bar{\psi} \to \mathrm{i}\,\delta_\eta\,, \quad \psi \to -\mathrm{i}\,\delta_{\bar{\eta}}\,, \quad \varphi \to -\mathrm{i}\,\delta_J$$

one can write the generating functional

$$W[J,\eta,\bar{\eta}] = \exp\left\{\mathrm{i}\int \mathrm{d}^4x\,\left[-\frac{\lambda}{4!}(-\mathrm{i}\,\delta_{J(x)})^4 + \mathrm{i}g(-\mathrm{i}\,\delta_{J(x)})(\mathrm{i}\,\delta_{\eta(x)})\gamma_5(-\mathrm{i}\,\delta_{\bar{\eta}(x)})\right]\right\}W_0[J,\eta,\bar{\eta}]$$

where the free generating functional is

$$\begin{aligned} W_0[J,\eta,\bar{\eta}] &= \int [\mathcal{D}\varphi]\,\exp\left[\mathrm{i}\int \mathrm{d}^4x\,\frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}M^2\varphi^2 + J\varphi\right] \\ &\quad\times \int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\,\exp\left[\mathrm{i}\int \mathrm{d}^4x\,\bar{\psi}(\mathrm{i}\,\not{\partial} - m)\psi + \bar{\psi}\eta + \bar{\eta}\psi\right] \\ &= \exp\left[-\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,J(x)\Delta(x-y)J(y)\right] \\ &\quad\times \exp\left[-\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{\eta}_\alpha(x)S_{\alpha\beta}(x-y)\eta_\beta(y)\right] \end{aligned}$$

and the propagators are

$$\Delta(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}k(x-y)}}{k^2 - M^2 + \mathrm{i}\varepsilon}\,, \quad S_{\alpha\beta}(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(\not{k} + m)_{\alpha\beta}}{k^2 - m^2 + \mathrm{i}\varepsilon}\mathrm{e}^{-\mathrm{i}k(x-y)}$$

By computing the generating functional at one-loop, one finds

$$W[J,\eta,\bar{\eta}] = W_0[J,\eta,\bar{\eta}] + \mathrm{i}\int \mathrm{d}^4x\,\left[-\frac{\lambda}{4!}(-\mathrm{i}\,\delta_{J(x)})^4 + \mathrm{i}g(-\mathrm{i}\,\delta_{J(x)})(\mathrm{i}\,\delta_{\eta(x)})\gamma_5(-\mathrm{i}\,\delta_{\bar{\eta}(x)})\right]W_0 + o(\lambda, g)$$

Proceeding as the $\lambda\varphi^4$ theory, the expression above can be expressed in terms of Feynman diagrams according to the following rules:

- Scalar propagator

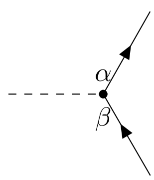
  $\mathrm{i}\Delta(x-y) \equiv \quad x$ y

- Fermion propagator

  $\mathrm{i}S_{\alpha\beta}(x-y) \equiv \quad \psi_\alpha(x)$  $\bar{\psi}_\beta(y)$

  Notice the addition of an arrow to denote direction and distinguish between the field ψ and its Dirac adjoint $\bar{\psi}$.

- Internal point

  $-g\int \mathrm{d}^4x\,(\gamma_5)_{\alpha\beta} \equiv$ 

- External scalar point

  $\mathrm{i}\int \mathrm{d}^4y_j\,J(y_j) \to$

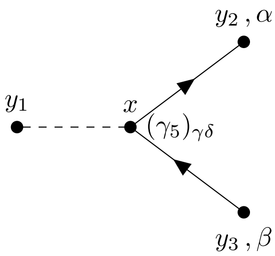
- External fermionic point

$$\mathrm{i}\int \mathrm{d}^4y_j\,\eta_\alpha(y_j) \to$$, $$\mathrm{i}\int \mathrm{d}^4y_j\,\bar\eta_\alpha(y_j) \to$$

The factors of i can be immediately seen by rewriting $W_0[J,\eta,\bar\eta]$ and giving each factor in the exponent its own unit i. For example

$$-\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,J(x)\Delta(x-y)J(y) = \frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{i}J(x)\mathrm{i}\Delta(x-y)\mathrm{i}J(y)$$

Example. The tree-level diagram for the Yukawa interaction is

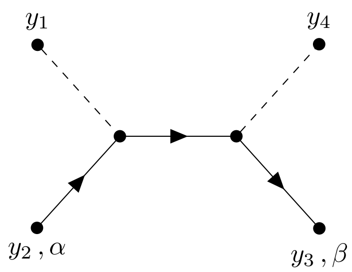


which corresponds to

$$\mathrm{i}^3\mathrm{i}^3\int \mathrm{d}^4y_1\,\mathrm{d}^4y_2\,\mathrm{d}^4y_3\,\mathrm{d}^4x\,\bar\eta_\alpha(y_2)S_{\alpha\gamma}(y_2-x)(-g\gamma_5)_{\gamma\delta}S_{\delta\beta}(x-y_3)\eta_\beta(y_3)\Delta(x-y_1)J(y_1)$$

[r] Since order matters for spinors, to write the integral without indices one has to go from right to left and follow against the direction of the arrows.

Example. With two Yukawa vertices, one can draw



[r] When writing amputated Green's functions, the external lines are cut and only the internal part of the diagram is computed.

Remark. One neglects vacuum diagrams by choosing a suitable normalization

$$W[J=\eta=\bar\eta=0]=1$$

Remark. One is interested in connected Green's functions because the non-connected diagrams can be written as products of connected ones. The generating functional for connected Green's functions is

$$Z[J,\eta,\bar\eta]=\ln W[J,\eta,\bar\eta]$$

Remark. In the interest of renormalization, one computes amputated Green's functions $\Gamma^{(n)}$ which are the quantum vertices in the effective action and are generated by it.

Remark. Computations are done in momentum space.

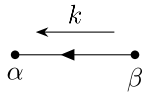
Feynman rules in momentum space. The rules are

- Scalar propagator

$$\frac{\mathrm{i}}{k^2 - M^2 + \mathrm{i}\varepsilon} \equiv$$

- Fermion propagator

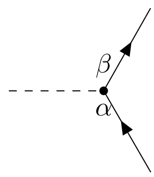
$$\frac{\mathrm{i}(\not{k} + m)_{\alpha\beta}}{k^2 - m^2 + \mathrm{i}\varepsilon} \equiv$$



Notice that the momentum arrow follows the line's arrow for particles, but it is in the opposite direction for anti-particles.

- Yukawa vertex

$$-g(\gamma_5)_{\alpha\beta} \equiv$$



[r]

Symmetries. By construction, the Lagrangian is invariant under a global U(1) transformation of the fermionic fields

$$\psi' = \mathrm{e}^{\mathrm{i}q}\psi\,, \quad \bar{\psi} = \mathrm{e}^{-\mathrm{i}q}\bar{\psi}\,, \quad \varphi' = \varphi$$

Renormalizability. To study renormalizability one has to look at the mass dimension of the coupling constants. One already knows the dimensions of the four-point coupling constant and the scalar field

$$\dim\lambda = 4 - n = 2\varepsilon = 0\,, \quad \dim\varphi = \frac{n-2}{2} = 1$$

where the last equality is for $n = 4$, $\varepsilon = 0$. From the fermionic mass term, one obtains

$$\dim(\bar{\psi}m\psi) = n \implies \dim\psi = \frac{n-1}{2} = \frac{3}{2}$$

The Yukawa coupling constant is

$$\dim(g\varphi\bar{\psi}\gamma_5\psi) = n \implies \dim g = 2 - \frac{n}{2} = \varepsilon = 0$$

Therefore the theory is renormalizable.

By applying the BPHZ renormalization, one finds that the fields are renormalized as

$$\varphi_0 = Z_\varphi^{\frac{1}{2}}\varphi\,, \quad \psi_0 = Z_\psi^{\frac{1}{2}}\psi\,, \quad \bar{\psi}_0 = Z_\psi^{\frac{1}{2}}\bar{\psi}$$

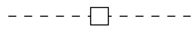
The parameters are renormalized as

$$\lambda_0 = Z_\lambda Z_\varphi^{-2}\lambda\,, \quad g_0 = Z_g Z_\varphi^{-\frac{1}{2}} Z_\psi^{-1} g\,, \quad M_0^2 = Z_M Z_\varphi^{-1} M^2\,, \quad m_0 = Z_m Z_\psi^{-1} m$$

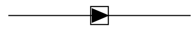
The bare Lagrangian can be split into a renormalized Lagrangian and the counter terms

$$\begin{aligned}
\mathcal{L}_0 = {} & \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}M^2\varphi^2 - \frac{\lambda}{4!}\varphi^4 + \bar{\psi}(\mathrm{i}\,\not{\partial} - m)\psi + \mathrm{i}g\varphi\bar{\psi}\gamma_5\psi \\
& + (Z_\varphi - 1)\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}(Z_M - 1)M^2\varphi^2 + (Z_\psi - 1)\bar{\psi}\mathrm{i}\,\not{\partial}\psi - (Z_m - 1)m\bar{\psi}\psi \\
& - (Z_\lambda - 1)\frac{\lambda}{4!}\varphi^4 + (Z_g - 1)\mathrm{i}g\varphi\bar{\psi}\gamma_5\psi
\end{aligned}$$

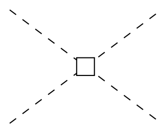
The counter terms (second and third line) produce divergent contributions that cancel the divergent contributions given by the renormalized Lagrangian. Therefore, one has to add the counter term vertices to the one already present. The first two addenda of the second line are a propagator

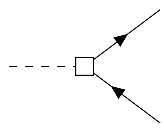


The last two addenda on the second line are another propagator



The third line corresponds to two vertices





The counter terms are fixed in order to cancel the divergences up to a finite part (which is scheme dependent).

17 One-loop contributions

One applies power counting to find the divergent contributions. From the $\lambda\varphi^4$ theory, one already knows that the two-point and four-point scalar Green's functions, $\Gamma_\varphi^{(2)}$ and $\Gamma_\varphi^{(4)}$, are divergent. One expects a divergence in the two-point fermionic Green's function $\Gamma_\psi^{(2)}$ and the three-point Yukawa Green's function $\Gamma_{\mathrm{Y}}^{(3)}$.

Exercise. Find the superficial degree of divergence for a generic diagram at L loops with n_φ external scalar lines, $2n_\psi$ external fermion lines, I_φ internal scalar propagators, I_ψ internal fermion propagators, V_{Y} Yukawa vertices, V_4 scalar λ vertices.

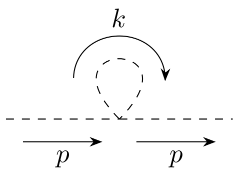
From this one finds that only a particular topology of diagrams is divergent, which are the ones above.

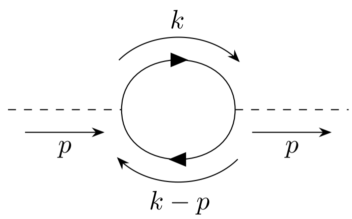
17.1 Two-point scalar Green's function

Recall that in Minkowski, the two-point scalar Green's function up to one-loop is [r]

$$\Gamma_\varphi^{(2)} = p^2 - m^2 - \Sigma_\varphi \,, \quad \Gamma_\varphi^{(2)}|_{1\mathrm{L}} = -\Sigma_\varphi$$

The one-loop contributions are given by the diagrams





First diagram. For the first diagram, one has found (see Cheng, eq. 2.127)

$$-\mathrm{i}\Sigma_\varphi(\mathrm{I}) = \frac{\mathrm{i}\lambda M^2}{32\pi^2}\left[\frac{1}{\varepsilon} + \psi(2) - \ln\frac{M^2}{4\pi k^2} + o(\varepsilon^0)\right]$$

To perform the computation, one Wick rotates only the integral. In this way, the integration contour does not pass over any pole and the value of the integral is unchanged.

Lecture 5

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Second diagram. From two Yukawa vertices one has to produce two fermionic propagators through contractions. In one of them, the spinor fields have to be swapped and one gets a negative sign. This is a general rule: alla fermionic loops bring a -1. Therefore, the one-loop contribution to the two-point scalar Green's function is

$$
\begin{aligned}
-\mathrm{i}\Sigma_\varphi(\mathrm{II}) &= \frac{\mathrm{i}^2}{2!}(-g)^2(-1)\cdot 2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(\not{k}+m)_{\delta\alpha}}{k^2-m^2+\mathrm{i}\varepsilon}(\gamma_5)_{\alpha\beta}\frac{(\not{k}-\not{p})_{\beta\gamma}+m}{(k-p)^2-m^2+\mathrm{i}\varepsilon}(\gamma_5)_{\gamma\delta} \\
&= g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}[(\not{k}+m)\gamma_5(\not{k}-\not{p}+m)\gamma_5]}{(k^2-m^2+\mathrm{i}\varepsilon)[(k-p)^2-m^2+\mathrm{i}\varepsilon]} \\
&= g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}[(\not{k}+m)(\not{p}-\not{k}+m)]}{(k^2-m^2+\mathrm{i}\varepsilon)[(k-p)^2-m^2+\mathrm{i}\varepsilon]} \\
&= g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[\gamma^\mu\gamma^\nu k_\mu(p-k)_\nu+m^2\right]}{(k^2-m^2+\mathrm{i}\varepsilon)[(k-p)^2-m^2+\mathrm{i}\varepsilon]} \\
&= 4g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k(p-k)+m^2}{(k^2-m^2+\mathrm{i}\varepsilon)[(k-p)^2-m^2+\mathrm{i}\varepsilon]}
\end{aligned}
$$

The 2 factor in the first line comes from the combinatorics: the first external point can be connected to either vertex. Differently from the $\lambda\varphi^4$ theory, in this case there is no factor of 2 when drawing the loop since the two (fermionic) lines are not equivalent. At the penultimate line, one has noted that the Dirac matrices are traceless. At the last line, one has applied

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4\eta^{\mu\nu}\,, \quad \mathrm{Tr}\, I = 4$$

Since the Dirac matrices are no longer present, one can operate a Wick rotation to have

$$k_0 = \mathrm{i}k_0^{\mathrm{E}}\,, \quad k^2 = -k_{\mathrm{E}}^2\,, \quad \mathrm{d}^4k = \mathrm{i}\,\mathrm{d}^4k_{\mathrm{E}}$$

Therefore

$$k(p-k) = k_0(p-k)^0 - \mathbf{k}\cdot(\mathbf{p}-\mathbf{k}) \to -k(p-k)$$

This is not the only way to do the computation. One could begin from Euclidean space instead and rotate the Dirac matrices and the Clifford algebra.

The one-loop contribution is

$$
\begin{aligned}
-\Sigma_\varphi(\mathrm{II}) &= 4g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-k(p-k)+m^2}{(k^2+m^2)[(p-k)^2+m^2]} \\
&= -4g^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{(q+xp)(p-q-xp)-m^2}{(q^2+D)^2} \\
&= -4g^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{-q^2+(1-2x)qp+x(1-x)p^2-m^2}{(q^2+D)^2} \\
&= -4g^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{-q^2+x(1-x)p^2-m^2+D-D}{(q^2+D)^2} \\
&= -4g^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{x(1-x)p^2-m^2+D}{(q^2+D)^2}-\frac{1}{q^2+D} \\
&= -4g^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{2x(1-x)p^2}{(q^2+D)^2}-\frac{1}{q^2+D} \\
&\equiv \Sigma_2(\mathrm{II})+\Sigma_1(\mathrm{II})
\end{aligned}
$$

At the second line, one has applied Feynman combining

$$x[(p-k)^2+m^2]+(1-x)(k^2+m^2) = k^2+xp^2-2xpk+m^2 = (k-xp)^2+m^2+x(1-x)p^2$$

and one lets

$$q\equiv k-xp\,, \quad D\equiv m^2+x(1-x)p^2$$

At the third line, one notices that linear terms in q vanish due to parity.

The two integrals are divergent since their dimensions are 0 and 2. Using dimensional regularization, the first integral is

$$\begin{aligned}
\Sigma_1(\mathrm{II}) &= 4g^2k^{2\varepsilon}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^n q}{(2\pi)^n}\frac{1}{q^2+D} = 4g^2k^{2\varepsilon}\int_0^1 \mathrm{d}x\,\frac{\Gamma(-1+\varepsilon)}{(4\pi)^{2-\varepsilon}\Gamma(1)}\frac{1}{D^{-1+\varepsilon}} \\
&= \frac{4g^2k^{2\varepsilon}}{(4\pi)^{2-\varepsilon}}\Gamma(-1+\varepsilon)\int_0^1 \frac{\mathrm{d}x}{[x(1-x)p^2+m^2]^{-1+\varepsilon}} \\
&= \frac{4g^2}{(4\pi)^2}(4\pi)^\varepsilon\frac{\Gamma(1+\varepsilon)}{(-1+\varepsilon)}\frac{k^{2\varepsilon}}{\varepsilon}\int_0^1 \frac{\mathrm{d}x}{[x(1-x)p^2+m^2]^{-1+\varepsilon}} \\
&= -\frac{4g^2}{(4\pi)^2}(4\pi k^2)^\varepsilon\frac{\Gamma(1+\varepsilon)}{\varepsilon(1-\varepsilon)}\int_0^1 \mathrm{d}x\,\frac{x(1-x)p^2+m^2}{[x(1-x)p^2+m^2]^\varepsilon} \\
&= -\frac{4g^2}{(4\pi)^2}\frac{\Gamma(1+\varepsilon)}{\varepsilon(1-\varepsilon)}\int_0^1 \mathrm{d}x\,[x(1-x)p^2+m^2]\exp\left[-\varepsilon\ln\frac{x(1-x)p^2+m^2}{4\pi k^2}\right] \\
&= -\frac{g^2}{4\pi^2}\frac{1}{\varepsilon}\int_0^1 \mathrm{d}x\,[x(1-x)p^2+m^2] + o(\varepsilon^{-1}) \\
&= -\frac{g^2}{4\pi^2}\frac{1}{\varepsilon}\left[\frac{p^2}{6}+m^2\right] + o(\varepsilon^{-1})
\end{aligned}$$

At the first line, one has applied

$$\int \frac{\mathrm{d}^n q}{(2\pi)^n}\frac{(q^2)^a}{(q^2+D)^b} = \frac{1}{D^{b-a-\frac{n}{2}}}\frac{\Gamma(b-a-n/2)\Gamma(a+n/2)}{(4\pi)^{\frac{n}{2}}\Gamma(b)\Gamma(n/2)}$$

with $a=0$, $b=1$ and $\frac{n}{2}=2-\varepsilon$. At the third line one has applied $\Gamma(z+1)=z\Gamma(z)$.

The second integral is

$$\begin{aligned}
\Sigma_2(\mathrm{II}) &= -8g^2\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{x(1-x)p^2}{(q^2+D)^2} \\
&= -8g^2\frac{k^{2\varepsilon}}{(4\pi)^{2-\varepsilon}}\Gamma(\varepsilon)\int_0^1 \mathrm{d}x\,\frac{x(1-x)p^2}{[x(1-x)p^2+m^2]^\varepsilon} \\
&= -\frac{2g^2}{4\pi^2}\frac{1}{\varepsilon}\int_0^1 \mathrm{d}x\,x(1-x)p^2 + o(\varepsilon^{-1}) \\
&= -\frac{2g^2}{4\pi^2}\frac{1}{\varepsilon}\frac{p^2}{6} + o(\varepsilon^{-1}) = -\frac{g^2}{4\pi^2}\frac{1}{\varepsilon}\frac{p^2}{3} + o(\varepsilon^{-1})
\end{aligned}$$

At the second line, one has applied the integral formula above with $a=0$, $b=2$ and $\frac{n}{2}=2-\varepsilon$.

The total one-loop contribution to the two-point scalar Green's function coming from the second diagram is

$$\begin{aligned}
-\Sigma_\varphi(\mathrm{II}) = \Sigma_1(\mathrm{II})+\Sigma_2(\mathrm{II}) &= -\frac{g^2}{4\pi^2}\frac{1}{\varepsilon}\left[\frac{p^2}{6}+m^2\right] - \frac{g^2}{4\pi^2}\frac{1}{\varepsilon}\frac{p^2}{3} + \text{finite} \\
&= -\frac{g^2}{4\pi^2}\frac{1}{\varepsilon}\left[\frac{p_{\mathrm{E}}^2}{2}+m^2\right]
\end{aligned}$$

Notice that, at one-loop, there is a dependence on the momentum as opposed to the $\lambda\varphi^4$ theory. The first addendum of the above contribution is cancelled by a counter term of the kinetic term, while the second addendum is added to the first diagram's contribution and is cancelled by a counter term of the mass term.

Total contribution. The total contribution at one-loop is

$$-\mathrm{i}\Sigma_\varphi|_{1\mathrm{L}} = -\mathrm{i}\Sigma_\varphi(\mathrm{I}) - \mathrm{i}\Sigma_\varphi(\mathrm{II}) = -\frac{\mathrm{i}g^2}{8\pi^2}\frac{1}{\varepsilon}p_{\mathrm{E}}^2 - \frac{\mathrm{i}}{\varepsilon}\left[\frac{g^2}{4\pi^2}m^2 - \frac{\lambda M^2}{32\pi^2}\right] + \text{finite}$$

This total contributing divergence can be cancelled by a counter term of the kinetic term and mass term

$$\mathrm{i}(Z_\varphi - 1)\,\partial_\mu\varphi\,\partial^\mu\varphi - \mathrm{i}(Z_M-1)M^2\varphi^2$$

whose Feynman rules are[11]

$$\mathrm{i}(Z_\varphi - 1)p^2 - \mathrm{i}(Z_M - 1)M^2$$

In fact, the counter term is computed in configuration space and not momentum space, like the one-loop contribution above. Integrating by parts and then going to momentum space gives[12]

$$\mathrm{i}(Z_\varphi - 1)\,\partial_\mu\varphi\,\partial^\mu\varphi \to -\mathrm{i}(Z_\varphi - 1)\varphi\,\Box\,\varphi \to \mathrm{i}(Z_\varphi - 1)p^2$$

Thus the renormalization function of the scalar field is

$$\frac{\mathrm{i}g^2}{8\pi^2}\frac{1}{\varepsilon}p^2 + \mathrm{i}(Z_\varphi - 1)p^2 = \text{finite} \implies \boxed{Z_\varphi|_{1\mathrm{L}} = 1 - \frac{g^2}{8\pi^2}\frac{1}{\varepsilon} + \text{finite}}$$

The mass term for the scalar field is

$$-\frac{\mathrm{i}}{\varepsilon}\left[\frac{g^2}{4\pi^2}m^2 - \frac{\lambda M^2}{32\pi^2}\right] - \mathrm{i}(Z_M - 1)M^2 \equiv \text{finite}$$

from which one has

$$\boxed{Z_M|_{1\mathrm{L}} = 1 + \left[\frac{\lambda}{32\pi^2} - \frac{g^2}{4\pi^2}\frac{m^2}{M^2}\right]\frac{1}{\varepsilon} + \text{finite}}$$

Lecture 6

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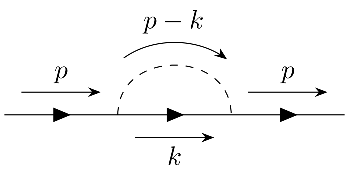
17.2 Two-point spinor Green's function

Consider the diagram



One has to contract two Yukawa vertices

$$-g\varphi\bar{\psi}_\gamma(\gamma_5)_{\gamma\delta}\psi_\delta\,, \quad -g\varphi\bar{\psi}_\alpha(\gamma_5)_{\alpha\beta}\psi_\beta$$

to produce a fermion propagator and a scalar propagator (recall that the diagram has to be read in the opposite direction when writing its associated integral). The one-loop contribution to the two-point spinor Green's function is

$$\begin{aligned}
[-\mathrm{i}\Sigma^{(2)}_\psi]_{\alpha\delta} &= \frac{\mathrm{i}^2}{2!}(-g)^2 2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(\gamma_5)_{\alpha\beta}(\not{k} + m)_{\beta\gamma}(\gamma_5)_{\gamma\delta}}{[(p-k)^2 - M^2 + \mathrm{i}\varepsilon](k^2 - m^2 + \mathrm{i}\varepsilon)} \\
&= -g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{[\gamma_5(\not{k} + m)\gamma_5]_{\alpha\delta}}{[(p-k)^2 - M^2 + \mathrm{i}\varepsilon](k^2 - m^2 + \mathrm{i}\varepsilon)} \\
&= -g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{[(-\not{k} + m)]_{\alpha\delta}}{[(p-k)^2 - M^2 + \mathrm{i}\varepsilon](k^2 - m^2 + \mathrm{i}\varepsilon)}
\end{aligned}$$

One goes to Euclidean[13]

$$\gamma^0 = \gamma_0 \equiv \mathrm{i}\gamma_{0\mathrm{E}} \implies \not{k}_\mathrm{M} = -\not{k}_\mathrm{E}$$

[11] See Peskin, fig. 10.3 on p. 325.

[12] Remember that this is not the formal and correct way to compute the Feynman rules. This procedure fails for the fermion kinetic term, see below.

[13] From the definition, the matrix γ^5 remains the same and $(\gamma^5)^2 = 1$.

and applies dimensional regularization to have

$$
\begin{aligned}
[-\mathrm{i}\Sigma_\psi^{(2)}]_{\alpha\delta} &= -\mathrm{i}g^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{(\not{k}+m)_{\alpha\delta}}{[(p-k)^2+M^2](k^2+m^2)} \\
&= -\mathrm{i}g^2 k^{2\varepsilon} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^n q}{(2\pi)^n} \frac{(\not{q}+x\not{p}+m)_{\alpha\delta}}{(q^2+D)^2} \\
&= -\mathrm{i}g^2 k^{2\varepsilon} \int_0^1 \mathrm{d}x\,(m+x\not{p})_{\alpha\delta} \int \frac{\mathrm{d}^n q}{(2\pi)^n} \frac{1}{(q^2+D)^2} \\
&= -\mathrm{i}g^2 k^{2\varepsilon} \frac{1}{(4\pi)^{2-\varepsilon}} \Gamma(\varepsilon) \int_0^1 \mathrm{d}x \frac{(m+x\not{p})_{\alpha\delta}}{[x(1-x)p^2+m^2(1-x)+M^2x]^\varepsilon} \\
&= -\frac{\mathrm{i}g^2}{16\pi^2} \frac{\Gamma(1+\varepsilon)}{\varepsilon} \int_0^1 \mathrm{d}x \frac{(m+x\not{p})_{\alpha\delta}}{[\frac{x(1-x)p^2+m^2(1-x)+M^2x}{4\pi k^2}]^\varepsilon} \\
&= -\frac{\mathrm{i}g^2}{16\pi^2} \frac{1}{\varepsilon} \int_0^1 \mathrm{d}x\,(m+x\not{p})_{\alpha\delta} + o(\varepsilon^{-1}) \\
&= -\frac{\mathrm{i}g^2}{16\pi^2} \frac{1}{\varepsilon} \left[m + \frac{1}{2}\not{p}_{\mathrm{E}}\right]_{\alpha\delta}
\end{aligned}
$$

At the second line, one has applied Feynman combining

$$\mathrm{Den} = k^2 - 2xkp + xp^2 + m^2(1-x) + M^2x = (k-xp)^2 + x(1-x)p^2 + m^2(1-x) + M^2x$$

and one lets

$$q \equiv k - xp\,, \quad D \equiv x(1-x)p^2 + m^2(1-x) + M^2x$$

At the third line, the contribution from $\not{q}$ is null due to parity.

Renormalization. The mass contribution is [r]

$$[-\mathrm{i}\Sigma_\psi^{(2)}]_m = -\frac{\mathrm{i}g^2}{16\pi^2}\frac{m}{\varepsilon}$$

Therefore, one has

$$-\frac{\mathrm{i}g^2}{16\pi^2}\frac{m}{\varepsilon} - \mathrm{i}(Z_m-1)m \equiv \text{finite} \implies \boxed{Z_m = 1 - \frac{g^2}{16\pi^2}\frac{1}{\varepsilon} + \text{finite}}$$

The momentum contribution is

$$[-\mathrm{i}\Sigma_\psi^{(2)}]_p = -\frac{\mathrm{i}g^2}{32\pi^2}\frac{1}{\varepsilon}\not{p}_{\mathrm{E}} = \frac{\mathrm{i}g^2}{32\pi^2}\frac{1}{\varepsilon}\not{p}$$

The Feynman rule for the kinetic counter term is[14]

$$\mathrm{i}(Z_\psi - 1)\not{p}$$

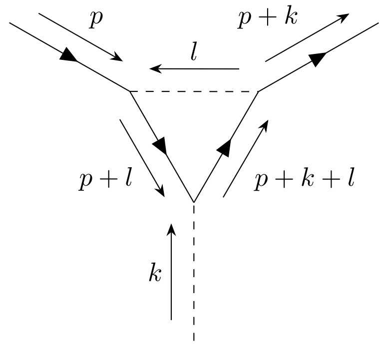
from which

$$\frac{\mathrm{i}g^2}{32\pi^2}\frac{1}{\varepsilon} + \mathrm{i}(Z_\psi - 1) = \text{finite} \implies \boxed{Z_\psi = 1 - \frac{g^2}{32\pi^2}\frac{1}{\varepsilon} + \text{finite}}$$

17.3 Three-point Yukawa Green's function

One would like to compute the one-loop correction to the Yukawa vertex $\Gamma_{\mathrm{Y}}^{(3)}$. Consider the following diagram

[14]See Peskin, fig. 10.4 on p. 332. Notice how the Feynman rule cannot be naively read from the Lagrangian. To understand why, see `https://physics.stackexchange.com/q/335476/`.



Let $p' = p+k$, the one-loop contribution to the three-point Green's function is

$$
\begin{aligned}
\mathrm{i}\Gamma^{(3)}_{\mathrm{Y}}|_{1\mathrm{L}} &= \frac{\mathrm{i}^3}{3!}(-g)^3(3\cdot 2)\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{(\gamma_5)_{\alpha\beta}(\not{p}'+\not{l}+m)_{\beta\gamma}(\gamma_5)_{\gamma\delta}(\not{p}+\not{l}+m)_{\delta\eta}(\gamma_5)_{\eta\rho}}{(l^2-M^2+\mathrm{i}\varepsilon)[(p+l)^2-m^2+\mathrm{i}\varepsilon][(p'+l)^2-m^2+\mathrm{i}\varepsilon]} \\
&= \mathrm{i}g^3\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{[(\not{p}'+\not{l}+m)(-\not{p}-\not{l}+m)\gamma_5]_{\alpha\rho}}{(l^2-M^2+\mathrm{i}\varepsilon)[(p+l)^2-m^2+\mathrm{i}\varepsilon][(p'+l)^2-m^2+\mathrm{i}\varepsilon]} \\
&= g^3\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{[(-\not{p}'-\not{l}+m)(\not{p}+\not{l}+m)\gamma_5]_{\alpha\rho}}{(l^2+M^2)[(p+l)^2+m^2][(p'+l)^2+m^2]} \\
&= g^3 k^{3\varepsilon}\int \frac{\mathrm{d}^n l}{(2\pi)^n}\frac{[(-\not{p}'-\not{l}+m)(\not{p}+\not{l}+m)\gamma_5]_{\alpha\rho}}{(l^2+M^2)[(p+l)^2+m^2][(p'+l)^2+m^2]} \\
&= g^3 k^{3\varepsilon}\int_0^1 \mathrm{d}x_1\,\mathrm{d}x_2 \int \frac{\mathrm{d}^n q}{(2\pi)^n}\frac{(q^2\gamma_5+\widetilde{N}+\text{linear in } q)_{\alpha\rho}}{(q^2+D)^3} \\
&= g^3 k^{3\varepsilon}\int_0^1 \mathrm{d}x_1\,\mathrm{d}x_2 \int \frac{\mathrm{d}^n q}{(2\pi)^n}\frac{q^2}{(q^2+D)^3}(\gamma_5)_{\alpha\rho} + \text{finite} \\
&= \frac{g^3 k^{3\varepsilon}}{16\pi^2}(4\pi)^\varepsilon\Gamma(\varepsilon)(\gamma_5)_{\alpha\rho}\int_0^1 \mathrm{d}x_1\,\mathrm{d}x_2\,\frac{1}{D^\varepsilon} + \text{finite} \\
&= \frac{g^3 k^{\varepsilon}}{16\pi^2}\frac{\Gamma(1+\varepsilon)}{\varepsilon}(\gamma_5)_{\alpha\rho}\int_0^1 \mathrm{d}x_1\,\mathrm{d}x_2\,\frac{(4\pi k^2)^\varepsilon}{D^\varepsilon} + \text{finite} \\
&= \frac{g^3 k^{\varepsilon}}{16\pi^2}\frac{1}{\varepsilon}(\gamma_5)_{\alpha\rho} + o(\varepsilon^{-1})
\end{aligned}
$$

At the third line, one has gone to Euclidean. At the fourth line, one has applied dimensional regularization. At the fifth line, one has applied Feynman combining. The denominator is

$$
\mathrm{Den} = (1-x_1-x_2)(l^2+M^2) + x_1[(l+p)^2+m^2] + x_2[(l+p')^2+m^2] \equiv q^2 + D
$$

where one has

$$
\begin{aligned}
q &\equiv l + x_1 p + x_2 p' \\
D &\equiv (1-x_1-x_2)M^2 + (x_1+x_2)m^2 + x_1(1-x_1)p^2 + x_2(1-x_2)p'^2 - 2x_1x_2 pp'
\end{aligned}
$$

The numerator is

$$
\begin{aligned}
\mathrm{Num} &= (-\not{q} + x_1\not{p} + x_2\not{p}' - \not{p}' + m)(\not{q} - x_1\not{p} - x_2\not{p}' + \not{p} + m)\gamma_5 \\
&= -\not{q}\not{q}\gamma_5 + [x_1\not{p} - (1-x_2)\not{p}' + m][(1-x_1)\not{p} - x_2\not{p}' + m)]\gamma_5 + (\text{linear in } q) \\
&= q^2\gamma_5 + \widetilde{N} + (\text{linear in } q)
\end{aligned}
$$

where one notes that

$$
\not{q}\not{q} = \gamma^\mu q_\mu \gamma^\nu q_\nu = \frac{1}{2}q_\mu q_\nu\{\gamma^\mu,\gamma^\nu\} = \frac{1}{2}q_\mu q_\nu(-2\delta^{\mu\nu}) = -q^2
$$

Renormalization. Recalling the counter term and the Feynman rule to be

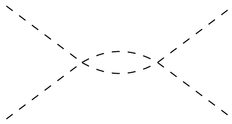
$$
\mathcal{L}_{\mathrm{ct}} \supset \mathrm{i}g(Z_g-1)\varphi\bar{\psi}\gamma_5\psi\,, \quad -g(Z_g-1)\gamma_5
$$

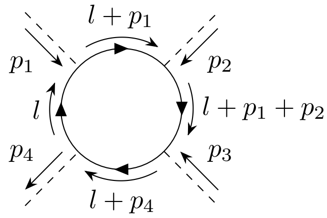
one finds

$$
\frac{g^3}{16\pi^2}\frac{1}{\varepsilon} - g(Z_g-1) \equiv \text{finite} \implies \boxed{Z_g = 1 + \frac{g^2}{16\pi^2}\frac{1}{\varepsilon} + \text{finite}}
$$

17.4 Four-point Green's function

The vertex function for the λ-vertex gets two contributions:





In principle, the Yukawa theory contains the coupling g, but the theory is renormalizable only if one includes the $\lambda\varphi^4$ term. [r] In fact, the second diagram produces a divergent term proportional to λ^4.

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Second diagram. The integral associated with the diagram is

$$\begin{aligned}\mathrm{i}\Gamma^{(4)}_\varphi(\mathrm{II}) &= \frac{\mathrm{i}^4}{4!}(-g)^4 \cdot 4! \cdot (-1) \\ &\quad \times \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{\mathrm{Tr}[(\not{l} + m)\gamma_5(\not{l} + \not{p}_4 + m)\gamma_5(\not{l} + \not{p}_1 + \not{p}_2 + m)\gamma_5(\not{l} + \not{p}_1 + m)\gamma_5]}{(l^2 - m^2)[(l+p_4)^2 - m^2][(l+p_1+p_2)^2 - m^2][(l+p_1)^2 - m^2]} \\ &\quad + 5 \text{ permutations of } (p_2, p_3, p_4)\end{aligned}$$

where $\mathrm{i}\varepsilon$ is understood in all propagators. Since one is interested only in the divergent part, one selects only the l^4 term in the numerator. One may set all external momenta to zero $p_1 = p_2 = p_3 = p_4 = 0$ since the integral has superficial degree of divergence equal to zero and so it does not depend on momentum. The trace is

$$\mathrm{Tr}[\not{l}\not{l}\not{l}\not{l}] = l_\mu l_\nu l_\rho l_\sigma \, \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4[l^4 - l^4 + l^4] = 4l^4$$

Performing the computation in Euclidean and using dimensional regularization, one obtains

$$\mathrm{i}\Gamma^{(4)}_\varphi(\mathrm{II}) = -4\mathrm{i}g^4 \cdot 6 \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{l^4}{(l^2+m^2)^4} + \text{finite} = -\frac{3\mathrm{i}g^4}{2\pi^2}\frac{k^{2\varepsilon}}{\varepsilon} + \text{finite}$$

The coefficient 6 comes from the inequivalent permutations of the momenta of the diagram.

Total contribution. Recalling that the first diagram gives

$$\mathrm{i}\Gamma^{(4)}_\varphi(\mathrm{I}) = \frac{3\mathrm{i}\lambda^2}{32\pi^2}\frac{k^{2\varepsilon}}{\varepsilon} + \text{finite}$$

then one finds

$$\frac{3\mathrm{i}\lambda^2}{32\pi^2} - \frac{3\mathrm{i}g^4}{2\pi^2} - \mathrm{i}\lambda(Z_\lambda - 1) = \text{finite} \implies \boxed{Z_\lambda = 1 + \left[\frac{3\lambda}{32\pi^2} - \frac{3}{2\pi^2}\frac{g^4}{\lambda}\right]\frac{1}{\varepsilon} + \text{finite}}$$

18 Renormalization group

See Srednicki, §52.

Renormalization functions. The renormalization functions at one-loop are

$$\begin{aligned}
Z_\varphi &= 1 - \frac{g^2}{8\pi^2}\frac{1}{\varepsilon} + \text{finite} \\
Z_\psi &= 1 - \frac{g^2}{32\pi^2}\frac{1}{\varepsilon} + \text{finite} \\
Z_g &= 1 + \frac{g^2}{16\pi^2}\frac{1}{\varepsilon} + \text{finite} \\
Z_\lambda &= 1 + \left[\frac{3\lambda}{32\pi^2} - \frac{3}{2\pi^2}\frac{g^4}{\lambda}\right]\frac{1}{\varepsilon} + \text{finite}
\end{aligned}$$

The bare coupling constants are

$$g_0 = Z_g Z_\varphi^{-\frac{1}{2}} Z_\psi^{-1} g k^\varepsilon\,, \quad \lambda_0 = Z_\lambda Z_\varphi^{-2}\lambda k^{2\varepsilon}$$

Using the minimal subtraction scheme to eliminate the dependence of the counter terms on the mass, one writes the renormalization functions as

$$Z = 1 + \sum_n \frac{a_n(g,\lambda)}{\varepsilon^n}$$

Therefore, the coupling constants are

$$g_0 = g k^\varepsilon\left[1 + \sum_{n=1}^\infty \frac{\widetilde{G}_n(g,\lambda)}{\varepsilon^n}\right], \quad \lambda_0 = \lambda k^{2\varepsilon}\left[1 + \sum_{n=1}^\infty \frac{\widetilde{L}_n(g,\lambda)}{\varepsilon^n}\right]$$

Comparing these series with the above expressions for the coupling constants, one sees that the brackets are the products of the renormalization functions

$$\begin{aligned}
1 + \sum_{n=1}^\infty \frac{\widetilde{G}_n(g,\lambda)}{\varepsilon^n} &= Z_g Z_\varphi^{-\frac{1}{2}} Z_\psi^{-1} \\
&= \left[1 + \frac{g^2}{16\pi^2}\frac{1}{\varepsilon} + \cdots\right]\left[1 + \frac{g^2}{16\pi^2}\frac{1}{\varepsilon} + \cdots\right]\left[1 + \frac{g^2}{32\pi^2}\frac{1}{\varepsilon} + \cdots\right] \\
&= 1 + \frac{5g^2}{32\pi^2}\frac{1}{\varepsilon} + \cdots
\end{aligned}$$

where the ellipses contain higher-order corrections in g and λ (and thus higher powers of ε^{-1}). Therefore

$$\widetilde{G}_1(g,\lambda)|_{1\text{L}} = \frac{5g^2}{32\pi^2}$$

Similarly

$$\widetilde{L}_1(g,\lambda)|_{1\text{L}} = \frac{3\lambda}{32\pi^2} + \frac{g^2}{4\pi^2} - \frac{3}{2\pi^2}\frac{g^4}{\lambda}$$

One notices that $\widetilde{G}_1$ depends only on one coupling constant.

Beta functions. Recall that the beta functions are

$$\beta_g = k\,\mathrm{d}_k g\,, \quad \beta_\lambda = k\,\mathrm{d}_k\lambda$$

The bare Yukawa coupling constant can be rewritten as

$$\begin{aligned}
g_0 &= g k^\varepsilon\left[1 + \sum_{n=1}^\infty \frac{\widetilde{G}_n(g,\lambda)}{\varepsilon^n}\right] = g\mathrm{e}^{\varepsilon\ln k}\exp\left[\sum_{n=1}^\infty \frac{G_n(g,\lambda)}{\varepsilon^n}\right] \\
\lambda_0 &= \lambda k^{2\varepsilon}\left[1 + \sum_{n=1}^\infty \frac{\widetilde{L}_n(g,\lambda)}{\varepsilon^n}\right] = \lambda\mathrm{e}^{2\varepsilon\ln k}\exp\left[\sum_{n=1}^\infty \frac{L_n(g,\lambda)}{\varepsilon^n}\right]
\end{aligned}$$

Taking the logarithm, one has

$$\ln g_0 = \ln g + \varepsilon \ln k + \sum_{n=1}^{\infty} \frac{G_n(g,\lambda)}{\varepsilon^n}$$
$$\ln \lambda_0 = \ln \lambda + 2\varepsilon \ln k + \sum_{n=1}^{\infty} \frac{L_n(g,\lambda)}{\varepsilon^n}$$

The right-hand side depends on the mass scale k, but not the left-hand side. Applying $k\,\mathrm{d}_k$, one has

$$0 = k\,\mathrm{d}_k \ln g + \varepsilon + \sum_{n=1}^{\infty} \frac{1}{\varepsilon^n} \left[\partial_g G_n\, k\,\mathrm{d}_k g + \partial_\lambda G_n\, k\,\mathrm{d}_k \lambda\right]$$

Multiplying by the coupling constant g, one has

$$0 = g\varepsilon + k\,\mathrm{d}_k g + \sum_{n=1}^{\infty} \frac{1}{\varepsilon^n}[g\,\partial_g G_n\, k\,\mathrm{d}_k g + g\,\partial_\lambda G_n\, k\,\mathrm{d}_k \lambda]$$

This expression contains the beta functions. Since the beta functions are finite in the limit $\varepsilon \to 0$, one may write

$$k\,\mathrm{d}_k g = \beta_g(g,\lambda) - \varepsilon g\,, \quad k\,\mathrm{d}_k \lambda = \beta_\lambda(g,\lambda) - 2\varepsilon\lambda$$

[r] Substituting this inside the expression above, one obtains

$$\begin{aligned} 0 &= \beta_g + \sum_{n=1}^{\infty} \frac{1}{\varepsilon^n}[g\,\partial_g G_n\,(-g\varepsilon + \beta_g) + g\,\partial_\lambda G_n\,(-2\lambda\varepsilon + \beta_\lambda)] \\ &= \beta_g - g^2\,\partial_g G_1 - 2g\lambda\,\partial_\lambda G_1 + \sum_{n=1}^{\infty} \frac{1}{\varepsilon^n}(\cdots) \end{aligned}$$

For the equality to hold, every coefficient of the poles must be zero. From this one also has

$$\boxed{\beta_g(g,\lambda) = g[g\,\partial_g G_1 + 2\lambda\,\partial_\lambda G_1]}$$

Similarly

$$\boxed{\beta_\lambda = \lambda[g\,\partial_g L_1 + 2\lambda\,\partial_\lambda L_1]}$$

Since $\widetilde{G}_1 = G_1$ and $\widetilde{L}_1 = L_1$, then the one-loop beta functions are

$$\beta_g|_{1\mathrm{L}} = \frac{5}{16\pi^2} g^3\,, \quad \beta_\lambda|_{1\mathrm{L}} = \frac{1}{16\pi^2}(3\lambda^2 + 8g^2\lambda - 48g^4)$$

18.1 Fixed points and renormalization group flow

One introduces a new coupling ρ such that $\lambda = \rho g^2$ and studies the flow of g and ρ. The beta function is

$$\beta_\rho = k\,\mathrm{d}_k \rho = k\,\mathrm{d}_k \frac{\lambda}{g^2} = \frac{1}{g^2}\beta_\lambda - \frac{2\lambda}{g^3}\beta_g$$

At one-loop, one finds

$$\beta_\rho|_{1\mathrm{L}} = \frac{g^2}{16\pi^2}(3\rho^2 - 2\rho - 48)$$

This expression is a product of polynomials of the coupling constants as opposed to the beta function of λ which is a sum of polynomials.

The only fixed point of the flow of g is the trivial one

$$\beta_g = 0 \implies g = 0$$

In order to study the flow, one considers small perturbations away from the fixed point, $|g| \ll 1$. The fixed points for ρ are

$$\beta_\rho = 0 \implies \rho_{1,2} = \frac{1 \pm \sqrt{145}}{3}$$

The first is positive and the second is negative. The small perturbations around the solutions ρ_j can be studied by Taylor expanding

$$k\,\mathrm{d}_k\rho = \beta_\rho = \beta_\rho(\rho_1) + \beta'_\rho(\rho_1)(\rho-\rho_1) + o(\Delta\rho) = \beta'_\rho(\rho_1)(\rho-\rho_1) + o(\Delta\rho)$$

From the sign of the derivative of the beta function, one obtains the flow. If the derivative is positive, then for $\rho > \rho_j$ the derivative of the coupling with respect to the energy is positive, so the coupling strengthens with increasing energy, while for $\rho < \rho_j$, the derivative of the coupling is negative so the coupling weakens; vice versa, if the derivative is negative, then for $\rho > \rho_j$, the coupling weakens, while for $\rho < \rho_j$ it strengthens. The derivative is

$$\beta'_\rho = \frac{g^2}{16\pi^2}(6\rho - 2)$$

from which

$$\beta'(\rho_1) = \frac{g^2}{8\pi^2}\sqrt{145} > 0\,, \quad \beta'(\rho_2) = -\frac{g^2}{8\pi^2}\sqrt{145} < 0$$

[r] The flow is away from ρ_1 and towards ρ_2. The first is an infrared-stable fixed point while the second is an ultraviolet-stable fixed point.

Trajectories. One may study the trajectory of the theory in the (ρ, g) plane. To this end, one needs to find $g(\rho)$. One has to solve

$$\mathrm{d}_\rho g = \mathrm{d}_{\ln k} g\,\mathrm{d}_\rho \ln k = \frac{\beta_g}{\beta_\rho} = \frac{5}{16\pi^2}g^3\left[\frac{g^2}{16\pi^2}3(\rho-\rho_1)(\rho-\rho_2)\right]^{-1} = \frac{5g}{3(\rho-\rho_1)(\rho-\rho_2)}$$

Therefore

$$\frac{1}{g}\,\mathrm{d}_\rho g = \mathrm{d}_\rho \ln g = \frac{5}{3(\rho-\rho_1)(\rho-\rho_2)}$$

after integrating one gets

$$\ln\frac{g(\rho)}{g(0)} = \frac{5}{3}\int_0^\rho \frac{\mathrm{d}x}{(x-\rho_1)(x-\rho_2)} = \frac{5}{3(\rho_1-\rho_2)}\ln\left|\frac{\rho-\rho_1}{\rho-\rho_2}\frac{\rho_2}{\rho_1}\right|$$

from which

$$g(\rho) = g(0)\left|\frac{\rho-\rho_1}{\rho-\rho_2}\right|^{\frac{5}{3(\rho_1-\rho_2)}}$$

Due to the absolute value, the constant-sign domains are

$$\rho < \rho_2\,, \quad \rho_2 < \rho < \rho_1\,, \quad \rho > \rho_1$$

[r] diagr. Let

$$\nu = \frac{5}{3(\rho_1-\rho_2)} > 0\,, \quad g_0 \equiv g(0)$$

then the derivatives are

$$g'(\rho) = g_0\nu\left|\frac{\rho-\rho_1}{\rho-\rho_2}\right|^{\nu-1}\frac{1}{(\rho-\rho_2)^2}\begin{cases}\rho_1-\rho_2 > 0\,, & \rho < \rho_2 \vee \rho > \rho_1\\ \rho_2-\rho_1 < 0\,, & \rho_2 < \rho < \rho_1\end{cases}$$

[r] With two coupling constants, it is interesting to study how they affect each other.

Exercise. Compute the anomalous dimensions. Recall that

$$\gamma_\varphi = \frac{1}{2}k\,\mathrm{d}_k \ln Z_\varphi\,, \quad \gamma_\psi = \frac{1}{2}k\,\mathrm{d}_k \ln Z_\psi$$

Solution. Recalling that up to one-loop one has

$$Z_\varphi = 1 - \frac{g^2}{8\pi^2}\frac{1}{\varepsilon} \implies \ln Z_\varphi = \sum_{n=1}^\infty \frac{F_n(g,\lambda)}{\varepsilon^n} \implies F_1(g,\lambda) = -\frac{g^2}{8\pi^2}$$

for the scalar field follows

$$\begin{aligned}
\gamma_\varphi &= \frac{1}{2}k\,\mathrm{d}_k \ln Z_\varphi = \frac{1}{2}[(\mathrm{d}_g \ln Z_\varphi)(\mathrm{d}_{\ln k} g) + (\mathrm{d}_\lambda \ln Z_\varphi)(\mathrm{d}_{\ln k}\lambda)] \\
&= \frac{1}{2}[(\mathrm{d}_g \ln Z_\varphi)(\beta_g - \varepsilon g) + (\mathrm{d}_\lambda \ln Z_\varphi)(\beta_\lambda - 2\varepsilon\lambda)] \\
&= \frac{1}{2}(\mathrm{d}_g \ln Z_\varphi)(\beta_g - \varepsilon g) + 0 = \frac{1}{2}\,\mathrm{d}_g\left[-\frac{g^2}{8\pi^2}\frac{1}{\varepsilon} + \cdots\right](\beta_g - \varepsilon g) \\
&= \frac{g^2}{8\pi^2} + \cdots
\end{aligned}$$

where the ellipses are higher powers of ε^{-1}.

For the spinor field, one has

$$Z_\psi = 1 - \frac{g^2}{32\pi^2}\frac{1}{\varepsilon} \implies P_1(g,\lambda) = -\frac{g^2}{32\pi^2}$$

The procedure is exactly the same

$$\begin{aligned}
\gamma_\psi &= \frac{1}{2}k\,\mathrm{d}_k \ln Z_\psi = \frac{1}{2}[(\mathrm{d}_g \ln Z_\psi)(\mathrm{d}_{\ln k} g) + (\mathrm{d}_\lambda \ln Z_\psi)(\mathrm{d}_{\ln k}\lambda)] \\
&= \frac{1}{2}[(\mathrm{d}_g \ln Z_\psi)(\beta_g - \varepsilon g) + (\mathrm{d}_\lambda \ln Z_\psi)(\beta_\lambda - 2\varepsilon\lambda)] \\
&= \frac{1}{2}(\mathrm{d}_g \ln Z_\psi)(\beta_g - \varepsilon g) + 0 = \frac{1}{2}\,\mathrm{d}_g\left[-\frac{g^2}{32\pi^2}\frac{1}{\varepsilon} + \cdots\right](\beta_g - \varepsilon g) \\
&= \frac{g^2}{32\pi^2} + \cdots
\end{aligned}$$

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19 Functional determinants

See Srednicki, §53. Functional determinants are useful to motivate the use of an infinite sum of Feynman diagrams.

Example. Consider a complex scalar field χ coupled to a background real scalar field φ

$$\mathcal{L} = \partial_\mu\bar{\chi}\,\partial^\mu\chi - m^2\bar{\chi}\chi + g\varphi\bar{\chi}\chi$$

The generating functional with no source is

$$\mathrm{e}^{\mathrm{i}\Gamma[\varphi]} = \int [\mathcal{D}\chi\,\mathcal{D}\bar{\chi}]\,\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L}}$$

where $\Gamma[\varphi]$ is the effective action for the field φ. The action is quadratic in the complex field and one may rewrite

$$\begin{aligned}
\int \mathrm{d}^4x\,\mathcal{L} &= \int \mathrm{d}^4x\,[-\bar{\chi}(\Box + m^2)\chi + g\varphi\bar{\chi}\chi] \\
&= -\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{\chi}(x)[(\Box + m^2) - g\varphi(x)]\delta^{(4)}(x-y)\chi(y) \\
&= -\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{\chi}(x)M(x,y)\chi(y)
\end{aligned}$$

where the differential operator is

$$M(x,y) \equiv [(\Box_x + m^2) - g\varphi(x)]\delta^{(4)}(x-y)$$

The generating functional is

$$\mathrm{e}^{\mathrm{i}\Gamma[\varphi]} = \int [\mathcal{D}\chi\,\mathcal{D}\bar{\chi}]\,\exp\left[-\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{\chi}(x)M(x,y)\chi(y)\right]$$

One recognizes the functional generalization of the Gaussian integral

$$\int d^n z\, d^n \bar{z}\, e^{-i\bar{z}_i M_{ij} z_j} \propto \frac{1}{\det M}$$

One may rewrite the differential operator as

$$M(x,y) = \int d^4 z\, [(\Box_x + m^2)\delta^{(4)}(x-z)][\delta^{(4)}(z-y) - g\Delta(z-y)\varphi(y)] = \int d^4 z\, M_0(x,z)\tilde{M}(z,y)$$

where

$$M_0(x,z) \equiv (\Box_x + m^2)\delta^{(4)}(x-z)\,, \quad \tilde{M}(z,y) \equiv \delta^{(4)}(z-y) + g\Delta(z-y)\varphi(y)$$

and $\Delta(x)$ is the Klein–Gordon Green's function (i.e. the free propagator)

$$(\Box_x + m^2)\Delta(x-y) = -\delta^{(4)}(x-y)$$

One may notice that the operator M_0 is the operator M with no background, $\varphi = 0$, and the structure of the operator $\tilde{M}$ is $\tilde{M} = I + G$.

In general, the determinant of a linear operator is the product of its eigenvalues. The determinant also satisfies the typical matrix relations [r]

$$\det M = \det(M_0\tilde{M}) = (\det M_0)(\det \tilde{M})$$

Therefore, the generating functional is

$$e^{i\Gamma[\varphi]} \propto \frac{1}{(\det M_0)(\det \tilde{M})} \propto \frac{1}{\det \tilde{M}}$$

Since the operator M_0 does not depend on the background field, then its determinant $\det M_0$ is a constant that can be absorbed into the normalization constant. Knowing that

$$\det \tilde{M} = e^{\operatorname{Tr}\ln \tilde{M}}$$

the generating functional is

$$e^{i\Gamma[\varphi]} \propto e^{-\operatorname{Tr}\ln\tilde{M}} = e^{-\operatorname{Tr}\ln(I+G)} = \exp\left[\operatorname{Tr}\sum_{n=1}^{\infty}(-1)^n\frac{G^n}{n}\right]$$

where one Taylor-expands the logarithm assuming that the theory is weakly interacting, $g \ll 1$. Recalling that $\operatorname{Tr}(A+B) = \operatorname{Tr}A + \operatorname{Tr}B$, then one may write

$$\operatorname{Tr}G^n = g^n \int \left[\prod_{j=1}^{n} d^4 x_j\right] \Delta(x_1 - x_2)\varphi(x_2)\Delta(x_2 - x_3)\varphi(x_3)\cdots\Delta(x_n - x_1)\varphi(x_1)$$

This trace is represented as a loop with n external φ lines and n internal χ propagators. The $(-1)^n$ comes from the factors of i from each propagator and each vertex. Therefore the effective action is

$$i\Gamma[\varphi] \propto \sum_{n=1}^{\infty}(-1)^n\frac{\operatorname{Tr}G^n}{n}$$

This is the typical expansion in powers of the coupling constant g of Feynman diagrams.

Example. Consider a Dirac spinor field ψ with a Yukawa interaction and a background real scalar field φ

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + g\varphi\bar{\psi}\psi$$

The generating functional is

$$e^{i\Gamma[\varphi]} = \int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\, e^{i\int d^4 x\, \mathcal{L}}$$

The action can rewritten as

$$\begin{aligned}\int \mathrm{d}^4x\,\mathcal{L} &= \int \mathrm{d}^4x\,[\bar\psi(\mathrm{i}\,\partial\!\!\!/ - m)\psi + g\varphi\bar\psi\psi] \\ &= -\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar\psi_\alpha(x)[(-\mathrm{i}\,\partial\!\!\!/_x + m)_{\alpha\beta} - g\varphi(x)\delta_{\alpha\beta}]\delta^{(4)}(x-y)\psi_\beta(y) \\ &= -\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar\psi_\alpha(x)M_{\alpha\beta}(x,y)\psi_\beta(y)\end{aligned}$$

where one has

$$M_{\alpha\beta}(x,y) \equiv [(-\mathrm{i}\,\partial\!\!\!/_x + m)_{\alpha\beta} - g\varphi(x)\delta_{\alpha\beta}]\delta^{(4)}(x-y)$$

The generating functional becomes

$$\mathrm{e}^{\mathrm{i}\Gamma[\varphi]} = \int [\mathcal{D}\psi\,\mathcal{D}\bar\psi]\,\exp\left[-\mathrm{i}\int \mathrm{d}^4x\;\mathrm{d}^4y\,\bar\psi(x)M(x,y)\psi(y)\right] \propto \det M$$

recalling that a Grassmann-odd Gaussian integral is proportional to the determinant and not its inverse.

The differential operator may be rewritten as

$$\begin{aligned}M_{\alpha\beta}(x,y) &= \int \mathrm{d}^4z\,[(-\mathrm{i}\,\partial\!\!\!/_x + m)_{\alpha\gamma}\delta^{(4)}(x-y)][\delta_{\gamma\beta}\delta^{(4)}(y-z) + g\varphi(z)S_{\gamma\beta}(y-z)] \\ &= \int \mathrm{d}^4z\,M_0(x,z)_{\alpha\gamma}\tilde M(z,y)_{\gamma\beta}\end{aligned}$$

where one defines

$$\begin{aligned}M_0(x,z)_{\alpha\gamma} &\equiv (-\mathrm{i}\,\partial\!\!\!/_x + m)_{\alpha\gamma}\delta^{(4)}(x-y) \\ \tilde M(z,y)_{\gamma\beta} &\equiv \delta_{\gamma\beta}\delta^{(4)}(y-z) + g\varphi(z)S_{\gamma\beta}(y-z) = I + \tilde G\end{aligned}$$

and S is the Dirac Green's function (i.e. free propagator)

$$(\mathrm{i}\,\partial\!\!\!/ - m)_{\alpha\beta}S_{\beta\gamma}(x-y) = \delta_{\alpha\gamma}\delta^{(4)}(x-y)$$

Like the previous example, the generating functional is

$$\mathrm{e}^{\mathrm{i}\Gamma[\varphi]} \propto \det M_0\,\det\tilde M \propto \det\tilde M = \mathrm{e}^{\mathrm{Tr}\ln\tilde M} = \mathrm{e}^{\mathrm{Tr}\ln(I+\tilde G)} = \exp\left[-\sum_{n=1}^{\infty}\frac{\mathrm{Tr}(-1)^n\tilde G^n}{n}\right]$$

where one has

$$\mathrm{Tr}\,\tilde G^n = g^n\int\left[\prod_{j=1}^{n}\mathrm{d}^n x_j\right]S_{\alpha_1\alpha_2}(x_1-x_2)\varphi(x_2)S_{\alpha_2\alpha_3}(x_2-x_3)\varphi(x_3)\cdots S_{\alpha_n\alpha_1}(x_n-x_1)\varphi(x_1)$$

The corresponding diagram is again a loop with n external scalar lines and n internal spinor propagators. Notice that the negative sign associated with fermion loops is the one present in the exponential.

The effective action is

$$\mathrm{i}\Gamma[\varphi] \propto -\sum_{n=1}^{\infty}(-1)^n\frac{\mathrm{Tr}\,\tilde G^n}{n}$$

Conclusion. The scalar effective action $\Gamma[\varphi]$ is given by an infinite sum of one-loop diagrams with external scalar lines.

20 Discrete symmetries

See Srednicki, §§2, 23, 40. There are many theories forbidden by parity, time reversal and charge conjugation.

Lorentz group. The Lorentz group is

$$\mathrm{O}(1,3) = \{\Lambda \in \mathrm{GL}(4,\mathbb{R}) \mid \Lambda^\top \eta \Lambda = \eta\} = \mathcal{L}$$

where the Minkowski metric η is timelike. The above condition is

$$\Lambda^\mu{}_\nu \eta_{\mu\rho} \Lambda^\rho{}_\sigma = \eta_{\nu\sigma}$$

Relativistic field theories have to be invariant under Lorentz changes of coordinates

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

From the definition of the group, it follows

$$\det \Lambda = \pm 1$$

and

$$\Lambda^\mu{}_0 \eta_{\mu\nu} \Lambda^\nu{}_0 = 1 \implies (\Lambda^0{}_0)^2 \geq 1$$

In this way, the Lorentz group can be split into four disjoint subsets, but only one is a subgroup[15]

$$\begin{aligned}
\mathcal{L}_+^\uparrow &= \{\Lambda \in \mathrm{O}(1,3) \mid \det\Lambda = 1\,, \Lambda^0{}_0 \geq 1\} \\
\mathcal{L}_+^\downarrow &= \{\Lambda \in \mathrm{O}(1,3) \mid \det\Lambda = 1\,, \Lambda^0{}_0 \leq -1\} \\
\mathcal{L}_-^\uparrow &= \{\Lambda \in \mathrm{O}(1,3) \mid \det\Lambda = -1\,, \Lambda^0{}_0 \geq 1\} \\
\mathcal{L}_-^\downarrow &= \{\Lambda \in \mathrm{O}(1,3) \mid \det\Lambda = -1\,, \Lambda^0{}_0 \leq -1\}
\end{aligned}$$

The first one is a group and it is generated by proper rotations and Lorentz boosts. Since it is a subgroup it contains the identity. Furthermore, it is continuously connected to it: transformations can be written infinitesimally close to the identity.

The other subsets contain discrete transformations. One such transformation is parity

$$\Lambda_P = \mathrm{diag}(1,-1,-1,-1) \in \mathcal{L}_-^\uparrow$$

It mirrors all spatial coordinates. Another is time reversal

$$\Lambda_T = \mathrm{diag}(-1,1,1,1) \in \mathcal{L}_-^\downarrow$$

It mirrors time. The combination of these two gives

$$\Lambda_{PT} = \mathrm{diag}(-1,-1,-1,-1) \in \mathcal{L}_+^\downarrow$$

The set of the transformations

$$I = \{1, \Lambda_P, \Lambda_T, \Lambda_{PT}\}$$

is a subgroup of the Lorentz group.

Any transformation in the three non-group subsets of the Lorentz group can be obtained by combining a proper Lorentz transformation with a transformation in group I.

One would like to study how the fields (which are finite-dimensional, non-unitary irreducible representations of the proper Lorentz group) transform under discrete symmetries.

20.1 Representation on scalar fields

Scalar fields are the representation with spin $s = 0$. The transformation of a scalar field under a proper[16] Lorentz transformation is

$$U^{-1}(\Lambda)\varphi(x)U(\Lambda) = \varphi(\Lambda^{-1}x)\,, \quad U(\Lambda) = \mathrm{e}^{\frac{\mathrm{i}}{2}\varepsilon^{\mu\nu}L_{\mu\nu}}$$

where $L_{\mu\nu}$ are the generators Lorentz group.

Similarly, a discrete transformation has a representation in terms of the field.

[15] Called restricted proper Lorentz group, cf. Mathematical Methods for Physics.

[16] For this section, the word "proper" includes the adjective restricted when referring to Lorentz transformations.

Parity. For parity, one has

$$U^{-1}(\Lambda_P)\varphi(x)U(\Lambda_P) = \eta_P \varphi(\Lambda_P x)$$

where η_P is a constant. Applying twice a parity transformation, one gets unity

$$U^{-2}(\Lambda_P)\varphi(x)U^2(\Lambda_P) = \eta_P U^{-1}(\Lambda_P)\varphi(\Lambda_P x)U(\Lambda_P x) = \eta_P^2 \varphi(\Lambda_P^2 x) = \eta_P^2 \varphi(x) \equiv \varphi(x)$$

Therefore, one obtains

$$\eta_P^2 = 1 \implies \eta_P = \pm 1$$

For $\eta_P = 1$, the field is parity-even and is a scalar field[17], while $\eta_P = -1$ the field is parity-odd and is a pseudo-scalar field.

Time reversal. For time reversal, one has

$$U^{-1}(\Lambda_T)\varphi(x)U(\Lambda_T) = \eta_T \varphi(\Lambda_T x)$$

where η_T is a constant. Like before, applying twice the transformation, one gets unity. Thus

$$\eta_T = \pm 1$$

The field can be time reversal-even or time reversal-odd.

Physical constraints. To decide whether the scalar fields are even or odd under a discrete symmetry, one utilizes the physical input. The Lagrangian of a scalar field must be parity-even and time reversal-even. In this way, the action is invariant and the two symmetries are conserved.

20.1.1 Unitarity

The operator associated to parity is unitary

$$U^\dagger(\Lambda_P)U(\Lambda_P) = U(\Lambda_P)U^\dagger(\Lambda_P) = 1$$

while the operator associated to time reversal is anti-unitary

$$\langle Ux|Uy\rangle = \langle x|\, U^\dagger U \,|y\rangle \equiv \langle x|y\rangle^*$$

or equivalently

$$U^{-1}(\Lambda_T)\mathrm{i}U(\Lambda_T) = -\mathrm{i}$$

Motivation of anti-unitarity. Consider a proper Lorentz transformation, $\Lambda \in \mathcal{L}_+^\uparrow$, and a four-momentum p^μ. Under such transformation, one has

$$U^{-1}(\Lambda)p^\mu U(\Lambda) = \Lambda^\mu{}_\nu p^\nu$$

Under a parity transformation and a time reversal transformation, one obtains

$$\begin{aligned} U^{-1}(\Lambda_P)p^\mu U(\Lambda_P) &= (\Lambda_P)^\mu{}_\nu p^\nu \implies p^\mu = (p^0, p^i) \to (p^0, -p^i) \\ U^{-1}(\Lambda_T)p^\mu U(\Lambda_T) &= (\Lambda_T)^\mu{}_\nu p^\nu \implies p^\mu = (p^0, p^i) \to (-p^0, p^i) \end{aligned}$$

Since p^0 is the Hamiltonian density, then time reversal can be a symmetry if and only if $H = 0$, but this is undesirable. One then must have

$$U^{-1}(\Lambda_T)p^\mu U(\Lambda_T) = -(\Lambda_T)^\mu{}_\nu p^\nu$$

and as such one must consider an anti-unitary operator.

[17] Notice that pseudo-tensors transform like tensors under proper rotations, but additionally change sign under improper rotations. For example, under improper rotations, scalars do not change sign, while pseudo-scalars do; vectors change sign, while pseudo-vectors do not.

Proof of anti-unitarity. The need of anti-unitarity is proven as follows. Consider the space-time translation operator

$$T(a) = \mathrm{e}^{-\mathrm{i}a_\mu p^\mu} = 1 - \mathrm{i}a_\mu p^\mu + o(a_\mu)\,, \quad a_\mu \ll 1$$

A translation of the field is

$$T^{-1}(a)\varphi(x)T(a) = \varphi(x-a)$$

Applying a proper Lorentz transformation gives

$$\begin{aligned}
U^{-1}(\Lambda)\varphi(x-a)U(\Lambda) &= U^{-1}(\Lambda)T^{-1}(a)\varphi(x)T(a)U(\Lambda) \\
&= (U^{-1}T^{-1}U)(U^{-1}\varphi U)(U^{-1}TU) \\
\varphi(\Lambda^{-1}(x-a)) &= (U^{-1}T^{-1}U)\varphi(\Lambda^{-1}x)(U^{-1}TU) \\
\varphi(\Lambda^{-1}x - \Lambda^{-1}a) &= (U^{-1}TU)^{-1}\varphi(\Lambda^{-1}x)(U^{-1}TU) \\
\varphi(y - \Lambda^{-1}a) &= (U^{-1}TU)^{-1}\varphi(y)(U^{-1}TU)
\end{aligned}$$

Therefore, the operator that translates by $\Lambda^{-1}a$ is

$$T(\Lambda^{-1}a) = U^{-1}(\Lambda)T(a)U(\Lambda)$$

which can be written as an infinitesimal transformation

$$U^{-1}(\Lambda)(I - \mathrm{i}a_\mu p^\mu)U(\Lambda) = I - \mathrm{i}(\Lambda^{-1}a)_\mu p^\mu = I - \mathrm{i}(\Lambda^{-1})_\mu{}^\nu a_\nu p^\mu$$

Therefore, for a generic Lorentz transformation, one finds

$$\boxed{U^{-1}(\Lambda)(\mathrm{i}a_\mu p^\mu)U(\Lambda) = \mathrm{i}(\Lambda^{-1})_\mu{}^\nu a_\nu p^\mu}$$

Consider a proper Lorentz transformation and the fact that

$$(\Lambda^{-1})_\mu{}^\nu = (\Lambda^\top)_\mu{}^\nu = \Lambda^\nu{}_\mu$$

then one has

$$U^{-1}(\Lambda)(\mathrm{i}a_\mu p^\mu)U(\Lambda) = \mathrm{i}a_\nu \Lambda^\nu{}_\mu p^\mu$$

Knowing that

$$\Lambda^\nu{}_\mu p^\mu = U^{-1}(\Lambda)p^\nu U(\Lambda)$$

one finds

$$U^{-1}(\Lambda)\mathrm{i}U(\Lambda) = \mathrm{i}$$

Consider a time reversal transformation

$$U^{-1}(\Lambda_T)(\mathrm{i}a_\mu p^\mu)U(\Lambda_T) = \mathrm{i}a_\nu (\Lambda_T)^\nu{}_\mu p^\mu$$

Assuming that

$$U^{-1}(\Lambda_T)\mathrm{i}U(\Lambda_T) = \mathrm{i}\eta\,, \quad \eta = \pm 1$$

one has

$$\mathrm{i}\eta a_\nu (\Lambda_T)^\nu{}_\mu p^\mu = U^{-1}(\Lambda_T)(\mathrm{i}a_\nu p^\nu)U(\Lambda_T) = U^{-1}(\Lambda_T)\mathrm{i}U(\Lambda_T)U^{-1}(\Lambda_T)(a_\nu p^\nu)U(\Lambda_T) = \mathrm{i}\eta a_\nu U^{-1}(\Lambda_T)p^\nu U(\Lambda_T)$$

One requires

$$U^{-1}(\Lambda_T)p^\nu U(\Lambda_T) = -(\Lambda_T)^\nu{}_\mu p^\mu$$

and there one obtains

$$\eta = -1$$

from which $U(\Lambda_T)$ is anti-unitary.

Lecture 9

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20.1.2 Charge conjugation

Charge conjugating is a $\mathbb{Z}_2$ symmetry:

$$Z^{-1}\varphi(x)Z = \pm\varphi(x)\,, \quad Z^2 = 1$$

For real scalar fields there is no charge and the discussion ends here. For complex scalar fields, a Lagrangian may be

$$\begin{aligned}\mathcal{L} &= \partial_\mu\varphi^\dagger\,\partial^\mu\varphi - m^2\varphi^\dagger\varphi - \frac{\lambda}{4!}(\varphi^\dagger\varphi)^2 \\ &= \frac{1}{2}\,\partial_\mu\varphi_1\,\partial^\mu\varphi_1 - \frac{1}{2}m^2\varphi_1^2 + \frac{1}{2}\,\partial_\mu\varphi_2\,\partial^\mu\varphi_2 - \frac{1}{2}m^2\varphi_2^2 - \frac{\lambda}{4!}(\varphi_1^2+\varphi_2^2)^2\end{aligned}$$

where one has

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + \mathrm{i}\varphi_2)\,, \quad \varphi^\dagger = \frac{1}{\sqrt{2}}(\varphi_1 - \mathrm{i}\varphi_2)$$

The Lagrangian written with complex fields has a global U(1) symmetry

$$\varphi' = \mathrm{e}^{\mathrm{i}q}\varphi\,, \quad \varphi'^\dagger = \mathrm{e}^{-\mathrm{i}q}\varphi^\dagger$$

The field has charge q while the conjugate field has charge $-q$. They are associated to particles and anti-particles. The Lagrangian written in terms of real fields has a global SO(2) symmetry.

Along the previous symmetry, the complex-field Lagrangian exhibits also a $\mathbb{Z}_2$ symmetry, $\varphi \leftrightarrow \varphi^\dagger$. The generator of this symmetry is given by the operator C:

$$C^{-1}\varphi(x)C = \varphi^\dagger(x)\,, \quad C\varphi^\dagger(x)C^{-1} = \varphi(x) \implies C^{-1}\mathcal{L}C = \mathcal{L}$$

For the real-field Lagrangian, one has

$$C^{-1}\varphi_1(x)C = \varphi_1(x)\,, \quad C^{-1}\varphi_2(x)C = -\varphi_2(x)$$

In matrix form, one has

$$C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\,, \quad \det C = -1$$

Since this operator has negative determinant, one has to enlarge the symmetry group to O(2).

The operator C is responsible for charge conjugation since it exchanges particles of charge q with anti-particles of charge $-q$.

20.2 Representation on spinor fields

Consider spinor fields, that is spin-half fermionic fields, $s = \frac{1}{2}$. Recall that, in canonical quantization, the most general solution to the free Dirac equation is

$$\psi_\alpha(x) = \sum_{s=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[b_s(\mathbf{p})u_{s\alpha}(\mathbf{p})\mathrm{e}^{-\mathrm{i}px} + d_s^\dagger(\mathbf{p})v_{s\alpha}(\mathbf{p})\mathrm{e}^{\mathrm{i}px}]\,, \quad p_0 \equiv \omega = \sqrt{|\mathbf{p}|^2 + m^2}$$

In the Weyl basis, the Dirac matrices are

$$\gamma^0 = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}\,, \quad \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix}\,, \quad \gamma^5 = \begin{bmatrix} -I_2 & 0 \\ 0 & I_2 \end{bmatrix}$$

where σ^k are the Pauli matrices. One introduces the charge conjugation matrix[18]

$$C = \mathrm{i}\gamma^0\gamma^2 = \begin{bmatrix} -\mathrm{i}\sigma^2 & 0 \\ 0 & \mathrm{i}\sigma^2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

[18] Typically the convention is to define $C = \mathrm{i}\gamma^2\gamma_0$. In this convention the following identities change too.

Identities. For the plane-wave spinor solutions (see Peskin, eqs. 3.59, 3.62)

$$u_s(\mathbf{p}) = \begin{bmatrix} \sqrt{\mathbf{p}\cdot\boldsymbol{\sigma}}\xi_s \\ \sqrt{\mathbf{p}\cdot\bar{\boldsymbol{\sigma}}}\xi_s \end{bmatrix}, \quad v_s(\mathbf{p}) = \begin{bmatrix} \sqrt{\mathbf{p}\cdot\boldsymbol{\sigma}}\eta_s \\ -\sqrt{\mathbf{p}\cdot\bar{\boldsymbol{\sigma}}}\eta_s \end{bmatrix}$$

it holds (see Srednicki, eqs. 38.32, 38.40)

$$u_s(-\mathbf{p}) = \gamma^0 u_s(\mathbf{p}), \quad v_s(-\mathbf{p}) = -\gamma^0 v_s(\mathbf{p})$$

also

$$u^*_{-s}(-\mathbf{p}) = -sC\gamma^5 u_s(\mathbf{p}), \quad v^*_{-s}(-\mathbf{p}) = -sC\gamma^5 v_s(\mathbf{p})$$

Proper Lorentz transformation. Under a proper Lorentz transformation, $\Lambda \in \mathcal{L}^\uparrow_+$, a spinor transforms as

$$U^{-1}(\Lambda)\psi(x)U(\Lambda) = D(\Lambda)\psi(\Lambda^{-1}x)$$

with

$$D(\Lambda) = \mathrm{e}^{\frac{\mathrm{i}}{2}\omega_{\mu\nu}S^{\mu\nu}}, \quad S^{\mu\nu} = \frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu]$$

where $S^{\mu\nu}$ is the spin representation of the Lorentz generators.

20.2.1 Parity

One makes the ansatz that

$$U^{-1}(\Lambda_P)\psi(x)U(\Lambda_P) = D(P)\psi(\Lambda_P x)$$

where $D(P)$ is an operator to be determined in the following. By applying the relation above twice, one obtains $-\psi(x)$. One does not require unity because spinors are not observables, only bilinear functions made from them are

$$U^{-2}(\Lambda_P)\psi(x)U^2(\Lambda_P) = D^2(P)\psi(\Lambda_P x) \equiv \pm\psi(x) \implies D^2(P) = \pm 1$$

One requires the ladder operators to transform in a simple way. Since, under parity, the momentum $\mathbf{p}$ changes sign while the angular momentum $\mathbf{J}$ does not, then the parity transformation should reverse the three-momentum while leaving the spin direction unchanged

$$U^{-1}(\Lambda_P)b^\dagger_s(\mathbf{p})U(\Lambda_P) = \eta b^\dagger_s(-\mathbf{p}), \quad U^{-1}(\Lambda_P)d^\dagger_s(\mathbf{p})U(\Lambda_P) = \eta d^\dagger_s(-\mathbf{p})$$

with $\eta^2 = \pm 1$. One introduces the same constant η for b and d to be able to relate the operators when working with Majorana fermions.

[r] Therefore, substituting the wave expansion — and recalling that the dispersion relation holds —, one has

$$\begin{aligned}
U^{-1}(\Lambda_P)\psi(x)U(\Lambda_P) &= \sum_{s=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[\eta^* b_s(-\mathbf{p})u_s(\mathbf{p})\mathrm{e}^{-\mathrm{i}px} + \eta d^\dagger_s(-\mathbf{p})v_s(\mathbf{p})\mathrm{e}^{\mathrm{i}px}]_{p^0=\omega} \\
&= \sum_{s=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[\eta^* b_s(\mathbf{p})u_s(-\mathbf{p})\mathrm{e}^{-\mathrm{i}p\Lambda_P x} + \eta d^\dagger_s(\mathbf{p})v_s(-\mathbf{p})\mathrm{e}^{\mathrm{i}p\Lambda_P x}] \\
&= \sum_{s=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[\eta^* b_s(\mathbf{p})\gamma^0 u_s(\mathbf{p})\mathrm{e}^{-\mathrm{i}p\Lambda_P x} - \eta d^\dagger_s(\mathbf{p})\gamma^0 v_s(\mathbf{p})\mathrm{e}^{\mathrm{i}p\Lambda_P x}] \\
&= \mathrm{i}\gamma^0\sum_{s=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[b_s(\mathbf{p})u_s(\mathbf{p})\mathrm{e}^{-\mathrm{i}p\Lambda_P x} + d^\dagger_s(\mathbf{p})v_s(\mathbf{p})\mathrm{e}^{\mathrm{i}p\Lambda_P x}] \\
&= \mathrm{i}\gamma^0\psi(\Lambda_P x)
\end{aligned}$$

At the second line, one has sent $\mathbf{p} \to -\mathbf{p}$ in the integral. The spatial components of the momentum in the exponential gain an extra minus sign which can be put into the position $\mathbf{x}$ to have

$$p_\mu x^\mu = p^0x^0 - \mathbf{p}\cdot\mathbf{x} \to p^0x^0 - (-\mathbf{p})\cdot\mathbf{x} = p^0x^0 - \mathbf{p}(-\mathbf{x}) = p_\mu(\Lambda_P x)^\mu$$

Notice that there is no overall minus sign since the integration limits are mapped into one another and the sign from the measure is absorbed to swap them. At the third line, one has applied the identities above. At the fourth line, one has required that

$$\eta^* = -\eta \implies \eta \in \mathrm{i}\mathbb{R}$$

so that one obtains a spinor field again. One chooses $\eta = -\mathrm{i}$.

From this, one sees that

$$\boxed{D(P) = \mathrm{i}\gamma^0} \implies D^2(P) = -I$$

Spinor transformation. Consider a spinor[19]

$$\psi = \begin{bmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{bmatrix} = \begin{bmatrix} \chi_\alpha \\ \bar{\xi}^{\dot{\alpha}} \end{bmatrix}$$

The spinor transforms as

$$D(P)\psi = \mathrm{i}\gamma^0\psi = \mathrm{i}\begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}\begin{bmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{bmatrix} = \mathrm{i}\begin{bmatrix} \psi_{\mathrm{R}} \\ \psi_{\mathrm{L}} \end{bmatrix}$$

The chiral components are swapped.

20.2.2 Time reversal

One assumes that under a time reversal transformation, the spinor field changes as

$$U^{-1}(\Lambda_T)\psi(x)U(\Lambda_T) = D(T)\psi(\Lambda_T x)$$

As before, one must have $D^2(T) = \pm 1$.

Noting that, under time reversal, the momentum $\mathbf{p}$ and the angular momentum $\mathbf{J}$ both change sign, one has

$$U^{-1}(\Lambda_T)b_s^\dagger(\mathbf{p})U(\Lambda_T) = \zeta_s b_{-s}^\dagger(-\mathbf{p})\,, \quad U^{-1}(\Lambda_T)d_s^\dagger(\mathbf{p})U(\Lambda_T) = \zeta_s d_{-s}^\dagger(-\mathbf{p})$$

with $\zeta_s^2 = \pm 1$. When one applies the time reversal transformation to the spinor field, one produces the term

$$U^{-1}(\Lambda_T)\psi(x)(\mathbf{p})U(\Lambda_T) \propto U^{-1}(\Lambda_T)b_s(\mathbf{p})u_s(\mathbf{p})\mathrm{e}^{-\mathrm{i}px}U(\Lambda_T) = U^{-1}(\Lambda_T)b_s(\mathbf{p})U(\Lambda_T)u_s^*(\mathbf{p})\mathrm{e}^{\mathrm{i}px}$$

where one inserts UU^{-1} between bu and recalls that $U(\Lambda_T)$ is anti-unitary. Therefore

$$\begin{aligned}
U^{-1}(\Lambda_T)\psi(x)U(\Lambda_T) &= \sum_{s=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[\zeta_s^* b_{-s}(-\mathbf{p})u_s^*(\mathbf{p})\mathrm{e}^{\mathrm{i}px} + \zeta_s d_{-s}^\dagger(-\mathbf{p})v_s^*(\mathbf{p})\mathrm{e}^{-\mathrm{i}px}]_{p^0=\omega} \\
&= \sum_{s=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[\zeta_{-s}^* b_s(\mathbf{p})u_{-s}^*(-\mathbf{p})\mathrm{e}^{-\mathrm{i}p\Lambda_T x} + \zeta_{-s} d_s^\dagger(\mathbf{p})v_{-s}^*(-\mathbf{p})\mathrm{e}^{\mathrm{i}p\Lambda_T x}] \\
&= \sum_{s=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[\zeta_{-s}^* b_s(\mathbf{p})(-sC\gamma^5 u_s(\mathbf{p}))\mathrm{e}^{-\mathrm{i}p\Lambda_T x} \\
&\quad + \zeta_{-s} d_s^\dagger(\mathbf{p})(-sC\gamma^5 v_s(\mathbf{p}))\mathrm{e}^{\mathrm{i}p\Lambda_T x}] \\
&= C\gamma^5\psi(\Lambda_T x)
\end{aligned}$$

At the second line, one has interchanged $\mathbf{p} \to -\mathbf{p}$ and $s \to -s$. Similar to before, the spatial components of the momentum in the exponential gain a minus sign which is put in front along with the minus sign of the time-like component coming from $\Lambda_T x$. At the third line, one has applied the identities above. At the fourth line, to obtain a spinor field, one has to choose

$$\zeta_s = s \implies \zeta_{-s} = -s \implies (-s)^2 = 1$$

Therefore

$$\boxed{D(T) = C\gamma^5} \implies D^2(T) = -I$$

[19] The second equality is in the van der Waerden notation.

Spinor transformation. Noting that

$$C = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -\varepsilon^{\alpha\beta} & 0 \\ 0 & -\varepsilon_{\dot{\alpha}\dot{\beta}} \end{bmatrix} , \quad \gamma^5 = \begin{bmatrix} -I_2 & 0 \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} -\delta_\beta{}^\gamma & 0 \\ 0 & \delta^{\dot{\beta}}{}_{\dot{\gamma}} \end{bmatrix}$$

one finds

$$C\gamma^5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} \varepsilon^{\alpha\gamma} & 0 \\ 0 & -\varepsilon_{\dot{\alpha}\dot{\gamma}} \end{bmatrix}$$

from which

$$C\gamma^5 \begin{bmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{bmatrix} = \begin{bmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{bmatrix} \begin{bmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{bmatrix} = \begin{bmatrix} \varepsilon\psi_{\mathrm{L}} \\ -\varepsilon\psi_{\mathrm{R}} \end{bmatrix}$$

Time reversal does not exchange chiral components, but changes each component into its dual. [r]

20.2.3 Charge conjugation

Let the charge conjugation matrix be C and the charge conjugation generator on the field ψ be $\mathcal{C}$. The operator $\mathcal{C}$ exchanges the Weyl spinors inside a Dirac spinor[20]. On the former, it acts as

$$\mathcal{C}^{-1}\chi_\alpha(x)\mathcal{C} = \xi_\alpha , \quad \mathcal{C}^{-1}\xi_\alpha(x)\mathcal{C} = \chi_\alpha(x)$$

Through some keen observations, one finds that a charge conjugation transformation is

$$\mathcal{C}^{-1}\psi(x)\mathcal{C} = C\bar{\psi}^\top(x) = C(\psi^\dagger\gamma^0)^\top = C\gamma^0\psi^*$$

Similarly

$$\mathcal{C}^{-1}\bar{\psi}(x)\mathcal{C} = \psi^\top(x)C$$

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The charge conjugation matrix satisfies

$$C^\top = C^\dagger = C^{-1} = -C , \quad \{\gamma^0, C\} = 0 , \quad C^{-1}\gamma^\mu C = -(\gamma^\mu)^\top , \quad C^{-1}\gamma^5 C = \gamma^5$$

20.2.4 Majorana spinors

Definition. A Majorana spinor is a bi-spinor satisfying

$$\psi \equiv \psi^c = C\bar{\psi}^\top = C\gamma^0\psi^*$$

Recalling that

$$\gamma^0 = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix} \implies C\gamma^0 = \begin{bmatrix} 0 & -\mathrm{i}\sigma^2 \\ \mathrm{i}\sigma^2 & 0 \end{bmatrix}$$

The Majorana condition is equivalent to

$$C\gamma^0\psi^* = \psi \iff \begin{bmatrix} 0 & -\mathrm{i}\sigma^2 \\ \mathrm{i}\sigma^2 & 0 \end{bmatrix} \begin{bmatrix} \psi_{\mathrm{L}}^* \\ \psi_{\mathrm{R}}^* \end{bmatrix} = \begin{bmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{bmatrix} \implies \begin{cases} -\mathrm{i}\sigma^2\psi_{\mathrm{L}}^* = \psi_{\mathrm{L}} \\ \mathrm{i}\sigma^2\psi_{\mathrm{R}}^* = \psi_{\mathrm{R}} \end{cases}$$

The degrees of freedom are half the ones of a Dirac spinor.

[20]See Srednicki, around eq. 36.31.

Summary. A brief summary of parity, time reversal and charge conjugation follows

$$\begin{aligned}U^{-1}(\Lambda_P)\psi(x)U(\Lambda_P) &= \mathrm{i}\gamma^0\psi(\Lambda_P x)\\ U^{-1}(\Lambda_T)\psi(x)U(\Lambda_T) &= C\gamma^5\psi(\Lambda_T x)\\ \mathcal{C}^{-1}\psi(x)\mathcal{C} &= C\gamma^0\psi^*(x)\end{aligned}$$

One may consider the transformations for the Dirac adjoint field $\bar\psi$. Parity gives

$$\begin{aligned}U^{-1}(\Lambda_P)\bar\psi(x)U(\Lambda_P) &= U^{-1}(\Lambda_P)\psi^\dagger(x)\gamma^0U(\Lambda_P) = U^{-1}(\Lambda_P)\psi^\dagger(x)\gamma^0U(\Lambda_P)\\ &= -\mathrm{i}\psi^\dagger(\Lambda_P x)\gamma^0\gamma^0 = -\mathrm{i}\bar\psi(\Lambda_P x)\gamma^0\end{aligned}$$

For time reversal, one finds

$$U^{-1}(\Lambda_T)\bar\psi(x)U(\Lambda_T) = \bar\psi(\Lambda_T x)\gamma^5C^{-1}$$

Therefore, the summary for the Dirac adjoint is

$$\begin{aligned}U^{-1}(\Lambda_P)\bar\psi(x)U(\Lambda_P) &= -\mathrm{i}\bar\psi(\Lambda_P x)\gamma^0\\ U^{-1}(\Lambda_T)\bar\psi(x)U(\Lambda_T) &= \bar\psi(\Lambda_T x)\gamma^5C^{-1}\\ \mathcal{C}^{-1}\bar\psi(x)\mathcal{C} &= \psi^\top(x)C\end{aligned}$$

20.3 Spinor bilinears

A bilinear in the spinor field is a real expression of the type $\bar\psi A\psi$ where A is a product of Dirac matrices such that

$$(\bar\psi A\psi)^\dagger = \bar\psi A\psi \implies \bar A = \gamma^0A^\dagger\gamma^0 = A$$

where the bar denotes the Dirac adjoint of a matrix.

Parity. A parity transformation of a bilinear is

$$\begin{aligned}U^{-1}(\Lambda_P)(\bar\psi A\psi)U(\Lambda_P) &= [U^{-1}(\Lambda_P)\bar\psi U(\Lambda_P)][U^{-1}(\Lambda_P)AU(\Lambda_P)][U^{-1}(\Lambda_P)\psi U(\Lambda_P)]\\ &= [-\mathrm{i}\bar\psi\gamma^0A\mathrm{i}\gamma^0\psi](\Lambda_P x) = \bar\psi\gamma^0A\gamma^0\psi = \bar\psi A^\dagger\psi\end{aligned}$$

To go further, one needs to specify the bilinear and in particular A.

Let $A = I$, one has a scalar

$$U^{-1}(\Lambda_P)\bar\psi\psi U(\Lambda_P) = \bar\psi\psi$$

Let $A = \mathrm{i}\gamma^5$, one has a pseudo-scalar

$$U^{-1}(\Lambda_P)\mathrm{i}\bar\psi\gamma^5\psi U(\Lambda_P) = -\mathrm{i}\bar\psi\gamma^5\psi$$

Let $A = \gamma^\mu$, one has a vector (also called polar vector)

$$U^{-1}(\Lambda_P)\bar\psi\gamma^\mu\psi U(\Lambda_P) = (\Lambda_P)^\mu{}_\nu(\bar\psi\gamma^\nu\psi)$$

where the parity transformation Λ_P encodes the following relations

$$\langle\psi|\,(\gamma^\mu)^\dagger\psi = \bar\psi\gamma^0\gamma^\mu\gamma^0\psi = \begin{cases}\bar\psi\gamma^0\psi\,, & \mu = 0\\ -\bar\psi\gamma^i\psi\,, & \mu = i\end{cases}$$

Let $A = \gamma^\mu\gamma^5$, one has a pseudo-vector (also called axial vector)

$$U^{-1}(\Lambda_P)\bar\psi\gamma^\mu\gamma^5\psi U(\Lambda_P) = -(\Lambda_P)^\mu{}_\nu(\bar\psi\gamma^\nu\gamma^5\psi)$$

Time reversal. The transformation of a bilinear is

$$\begin{aligned}U^{-1}(\Lambda_T)(\bar{\psi}A\psi)U(\Lambda_T) &= [U^{-1}(\Lambda_T)\bar{\psi}U(\Lambda_T)][U^{-1}(\Lambda_T)AU(\Lambda_T)][U^{-1}(\Lambda_T)\psi U(\Lambda_T)] \\ &= \bar{\psi}\gamma^5 C^{-1}A^* C\gamma^5\psi\end{aligned}$$

The second bracket evaluates to A^* since the operator $U(\Lambda_T)$ is anti-unitary. For the following it is useful to remember that

$$\gamma_{0,5} = \gamma_{0,5}^\dagger\,, \quad \gamma_i = -\gamma_i^\dagger$$

which can be shortened to

$$\gamma_5 = \gamma_5^\dagger\,, \quad (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$$

Let $A = I$, the bilinear is T-even

$$U^{-1}(\Lambda_T)(\bar{\psi}\psi)U(\Lambda_T) = \bar{\psi}\gamma^5 C^{-1}C\gamma^5\psi = \bar{\psi}\psi$$

Let $A = \mathrm{i}\gamma^5$, the bilinear is T-odd

$$U^{-1}(\Lambda_T)(\bar{\psi}\mathrm{i}\gamma^5\psi)U(\Lambda_T) = \bar{\psi}\gamma^5 C^{-1}(-\mathrm{i}\gamma^5)C\gamma^5\psi = -\mathrm{i}\bar{\psi}(\gamma^5)^3\psi = -\mathrm{i}\bar{\psi}\gamma^5\psi$$

where one applies the properties of the charge conjugation matrix. Let $A = \gamma^\mu$, the bilinear is T-odd

$$U^{-1}(\Lambda_T)(\bar{\psi}\gamma^\mu\psi)U(\Lambda_T) = \bar{\psi}\gamma^5 C^{-1}(\gamma^\mu)^* C\gamma^5\psi = -\bar{\psi}\gamma^5(\gamma^\mu)^\dagger\gamma^5\psi = \bar{\psi}(\gamma^\mu)^\dagger\psi = -(\Lambda_T)^\mu{}_\nu\bar{\psi}\gamma^\nu\psi$$

The sign difference is encoded into the time reversal transformation Λ_T.

Exercise. Check that $\bar{\psi}\gamma^\mu\gamma^5\psi$ is also T-odd.

Charge conjugation. The transformation of a bilinear is

$$\begin{aligned}\mathcal{C}^{-1}(\bar{\psi}A\psi)\mathcal{C} &= [\mathcal{C}^{-1}\bar{\psi}\mathcal{C}][\mathcal{C}^{-1}A\mathcal{C}][\mathcal{C}^{-1}\psi\mathcal{C}] = \psi^\top CAC\bar{\psi}^\top \\ &= (\psi^\top CAC\bar{\psi}^\top)^\top = -\bar{\psi}(CAC)^\top\psi = \bar{\psi}C^{-1}A^\top C\psi\end{aligned}$$

At the second line, the first equality holds since all spinor indices are contracted[21]; at the second equality, one has noticed that the transpose of a bilinear gains a minus sign due to interchanging the spinor fields[22].

Let $A = I$, the bilinear is C-even

$$\mathcal{C}^{-1}\bar{\psi}\psi\mathcal{C} = \bar{\psi}\psi$$

Let $A = \mathrm{i}\gamma^5$, the bilinear is C-even

$$\mathcal{C}^{-1}(\bar{\psi}\mathrm{i}\gamma^5\psi)\mathcal{C} = \bar{\psi}\mathrm{i}\gamma^5\psi$$

Let $A = \gamma^\mu$, the bilinear is C-odd

$$\mathcal{C}^{-1}(\bar{\psi}\gamma^\mu\psi)\mathcal{C} = -\bar{\psi}\gamma^\mu\psi$$

Let $A = \gamma^\mu\gamma^5$, the bilinear is C-even

$$\mathcal{C}^{-1}(\bar{\psi}\gamma^\mu\gamma^5\psi)\mathcal{C} = \bar{\psi}\gamma^\mu\gamma^5\psi$$

Remark. If the spinor field is Majorana, then

$$\psi^c = \psi \iff \mathcal{C}^{-1}\psi\mathcal{C} = \psi\,, \quad \mathcal{C}^{-1}\bar{\psi}\mathcal{C} = \bar{\psi}$$

Therefore, one has [r]

$$\mathcal{C}^{-1}\bar{\psi}A\psi\mathcal{C} = \bar{\psi}A\psi$$

This means that the odd bilinears have to be zero. In particular

$$\bar{\psi}\gamma^\mu\psi = 0$$

[21]The product is thought of as being a 1×1 matrix, $x^\top = x$.

[22]See `https://physics.stackexchange.com/q/458451`.

20.4 Representation on vector fields

[r] Source?.

Charge conjugation. Each component of a vector field $V^\mu(x)$ behaves as a scalar field under charge conjugation. [r]

Parity. Under parity, there are vectors and pseudo-vectors

$$U^{-1}(\Lambda_P)\mathbf{V}(x)U(\Lambda_P) = -\mathbf{V}(\Lambda_P x)\,, \quad U^{-1}(\Lambda_P)\mathbf{A}(x)U(\Lambda_P) = \mathbf{A}(\Lambda_P x)$$

[r]

Time reversal. For time reversal one has T-even and T-odd vectors, similar to vector bilinears.

Theorem (Furry's in QED)**.** In QED, the interaction term is

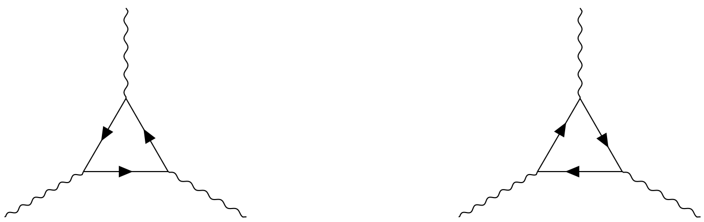
$$\mathcal{L}_{\text{int}} = J^\mu A_\mu\,, \quad J^\mu \propto \bar{\psi}\gamma^\mu\psi$$

A Lagrangian invariant under charge conjugation implies that the four-potential A^μ has to be odd

$$\mathcal{C}^{-1}A^\mu\mathcal{C} = -A^\mu$$

since the current J^μ is odd. This implies that the correlation functions of an odd number of gauge fields A^μ are zero.

Example. For example, the triangle diagram has two configurations depending on the orientation of the internal fermionic propagators. These two configurations sum up to zero.



20.5 CPT theorem

See Weinberg, vol. 1, §5.8, Srednicki, §40.

Theorem. The action of a local and hermitian[23] Lagrangian is symmetric under a simultaneous transformation of charge conjugation, parity and time reversal

$$(CPT)^{-1}\mathcal{L}(x)(CPT) = \mathcal{L}(-x) \implies S = \int \mathrm{d}^4x\,\mathcal{L}(-x) = \int \mathrm{d}^4x\,\mathcal{L}(x)$$

Proof. For scalar fields, one has

$$P^{-1}\varphi P = \pm\varphi = \eta\varphi\,, \quad T^{-1}\varphi T = \pm\varphi = \zeta^*\varphi\,, \quad \mathcal{C}^{-1}\varphi\mathcal{C} = \pm\varphi^\dagger = \xi\varphi^\dagger$$

Therefore

$$(CPT)^{-1}\varphi(CPT) = \eta\zeta^*\xi\varphi^\dagger(-x) = \varphi^\dagger(-x)$$

where one chooses the inversion phases such that

$$\eta\zeta^*\xi = 1$$

This can also be done for vector fields.

[23]Hermitian so that it guarantees unitarity.

For spinor fields, one has

$$\begin{aligned}(CPT)^{-1}\bar{\psi}\psi(CPT) &= \bar{\psi}\psi \\ (CPT)^{-1}\bar{\psi}\mathrm{i}\gamma^5\psi(CPT) &= \bar{\psi}\mathrm{i}\gamma^5\psi \\ (CPT)^{-1}\bar{\psi}\gamma^\mu\psi(CPT) &= -\bar{\psi}\gamma^\mu\psi \\ (CPT)^{-1}\bar{\psi}\gamma^\mu\gamma^5\psi(CPT) &= -\bar{\psi}\gamma^\mu\gamma^5\psi\end{aligned}$$

where the first two are CPT-even and the other two are CPT-odd. This is true for any bilinear. In general, tensors (including the ones made from bilinears) with an even number of vector indices are even under CPT; while for an odd number, they are CPT-odd. This is also includes four-derivatives. In fact

$$(CPT)^{-1}\partial_\mu\varphi(x)(CPT) = -\partial_\mu\varphi^\dagger(-x)$$

On to the actual proof. A hermitian combination of fields and derivatives is even or odd depending on the total number of uncontracted vector indices. Any hermitian combination of any set of fields (scalar, spinor, vector) and their derivatives that is a Lorentz scalar is CPT-even. A Lagrangian must be formed out of such combinations and as such is a Lorentz scalar: it is even under CPT [r]. The action is therefore invariant. □

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Part III
Gauge fields

21 Review

Classical fields — abelian gauge theory. See Cheng, §8.1. Consider QED as an abelian gauge theory. The free Lagrangian is

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi$$

The Lagrangian is invariant under a global U(1) transformation

$$\psi' = e^{\mathrm{i}\alpha}\psi\,, \quad \bar{\psi}' = \bar{\psi}e^{-\mathrm{i}\alpha}$$

The gauge principle involves promoting the global parameter α to a function of space-time $\alpha(x)$. By requiring a local U(1) invariance, one introduces a gauge covariant derivative[24] and a gauge field

$$D_\mu = \partial_\mu + \mathrm{i}qA_\mu\,, \quad A'_\mu = A_\mu - \frac{1}{q}\,\partial_\mu\alpha$$

where q is the charge (with sign) of the bispinor field. The gauge-invariant Lagrangian has an interaction term

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\,D\!\!\!\!/ - m)\psi = \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi - qA_\mu\bar{\psi}\gamma^\mu\psi$$

If there is no kinetic term for the gauge field, then the field is a background field. To make it dynamical, one has to insert such kinetic term

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

where the field strength tensor is defined by

$$\mathrm{i}qF_{\mu\nu} = [D_\mu, D_\nu]$$

and it is invariant under a gauge transformation.

One observes that gauge invariance forbids any mass term for the gauge field. The gauge Lagrangian term is a free Lagrangian — meaning it does not contain any interaction vertex — since the gauge field A_μ is neutral (because it is real) and cannot couple to itself.

[24]Note that Cheng uses $q = e$ where, for an electron, $e < 0$.

Classical fields — non-abelian gauge theory. An example of non-abelian gauge theory is QCD. Consider the gauge group $G = \mathrm{SU}(n)$ (so the theory is a Yang–Mills theory). The gauge group has $n^2 - 1$ generators that satisfy the (fundamental representation of the) $\mathfrak{su}(n)$ Lie algebra

$$[T^a, T^b] = \mathrm{i}c^{abc}T^c$$

where c^{abc} are the structure constants, abc are indices labelling the dimensions of the algebra and the generators are traceless Hermitian complex $n \times n$ matrices[25]. Consider a set of Dirac spinors in some representation of the group $\mathrm{SU}(n)$. The Lagrangian is

$$\mathcal{L} = \bar{\psi}_j(\mathrm{i}\,\partial\!\!\!/ - m)\psi_j$$

It has a global $\mathrm{SU}(n)$ symmetry

$$\psi_j' = U_{jk}(\theta)\psi_k\,, \quad \bar{\psi}_j' = \bar{\psi}_k U^\dagger_{kj}(\theta)$$

where the transformation matrix is

$$U_{jk}(\theta) = [\mathrm{e}^{-\mathrm{i}\theta^a T^a}]_{jk}\,, \quad UU^\dagger = U^\dagger U = I$$

By applying the gauge principle, one obtains $\theta = \theta(x)$. The requirement of local $\mathrm{SU}(n)$ symmetry implies

$$D_\mu = \partial_\mu - \mathrm{i}gT^a A^a_\mu\,, \quad (A^a_\mu)' = A^a_\mu + c^{abc}\theta^b A^c_\mu - \frac{1}{g}\,\partial_\mu\theta^a$$

The second relation is understood infinitesimally. Let $A_\mu = T^a A^a_\mu$, then the infinitesimal transformation is

$$A_\mu' = A_\mu - \mathrm{i}[\theta, A_\mu] - \frac{1}{g}\,\partial_\mu\theta$$

To make the gauge fields dynamical, one introduces the field strengths

$$[D_\mu, D_\nu] = -\mathrm{i}gT^a F^a_{\mu\nu}\,, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gc^{abc}A^b_\mu A^c_\nu$$

equivalently

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g[A_\mu, A_\nu]$$

The field strength is not gauge-invariant, but transforms infinitesimally as

$$F'^a_{\mu\nu} = F^a_{\mu\nu} + c^{abc}\theta^b F^c_{\mu\nu}$$

while for a finite transformation as

$$F'_{\mu\nu} = U^\dagger(\theta)F_{\mu\nu}U(\theta)$$

However the gauge kinetic term is invariant. This term is

$$\begin{aligned}\mathcal{L}_{\text{gauge}} &= -\frac{1}{2}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{2}F^a_{\mu\nu}F^{b\mu\nu}\,\mathrm{Tr}\big(T^aT^b\big) = -\frac{1}{4}F^a_{\mu\nu}F^{b\mu\nu}\delta^{ab}\\ &= -\frac{1}{2}\,\partial_\mu A^a_\nu(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) - gc^{abc}\,(\partial_\mu A^a_\nu)A^{b\mu}A^{c\nu} - \frac{g^2}{4}c^{abc}c^{adf}A^b_\mu A^c_\nu A^{d\mu}A^{f\nu}\end{aligned}$$

When inserting the explicit expression for the field strength one finds the following general structure

$$\text{kinetic} + g\partial A\,A^2 + g^2A^4$$

There is derivative three-point vertex and a four-point vertex. The gauge Lagrangian term is a not free Lagrangian since it contains two vertices.

One observes that the covariant derivative introduces an interaction term

$$gA^a_\mu\bar{\psi}_j T^a_{jk}\gamma^\mu\psi_k = gA^a_\mu J^\mu_a$$

where jk label the bispinor fields of the theory and a is an $\mathfrak{su}(n)$ index. [r]

While in abelian gauge theories, one may have different matter sectors coupled to the same gauge field with different coupling constants, in this case the coupling constant g is fixed when choosing the gauge group and it has to be the same for all matter fields since it appears in the field strength.

For each simple gauge group (e.g. $\mathrm{SU}(n)$), there is a corresponding single coupling constant.

[25] In the physicists' convention. In the mathematicians' convention they are traceless anti-Hermitian matrices. The two conventions differ by a factor of i.

Canonical quantization. Consider QED. Out of the four components of a gauge field, only two are physical degrees of freedom, typically A^1 and A^2. This is due to gauge invariance which can be used to eliminate non-physical degrees of freedom.

One may use gauge invariance to fix the Lorenz gauge. If there are no sources in the free theory, one may perform a further gauge transformation while being in the Lorenz gauge to set $A_3 = 0$. In fact, consider a generic gauge

$$\partial_\mu A^\mu \neq 0$$

One may transform the gauge field to go to the Lorenz gauge

$$A'_\mu(x) = A_\mu(x) - \partial_\mu \Lambda(x)\,, \quad 0 = \partial_\mu A'^\mu = \partial_\mu A^\mu - \Box\,\Lambda \implies \Box\,\Lambda = \partial_\mu A^\mu$$

From the gauge field A', one can always choose another gauge field within the Lorenz gauge

$$A''_\mu(x) = A'_\mu(x) - \partial_\mu \lambda(x)\,, \quad 0 = \partial^\mu A''_\mu = \partial_\mu A'^\mu - \Box\,\lambda \implies \Box\,\lambda = 0$$

Since λ is an arbitrary superposition of plane waves, one may choose

$$A''_3 = A'_3 - \partial_3 \lambda = 0 \implies \partial_3 \lambda = A'_3$$

There are two ways to perform canonical quantization.

1. One eliminates the non-physical degrees of freedom and quantizes only the physical ones. This method is not Lorentz-invariant because one fixes a component, $A_3 = 0$.
2. The other possibility follows the approach by Gupta–Bleuler[26] in quantizing all four degrees of freedom and removing the non-physical ones at the end. This approach produces negative-norm states that can be eliminated from the spectrum of physical states by defining the physical states as
$$(\partial_\mu A^\mu)^{(+)} \left|\text{phys}\right\rangle = 0$$
where the superscript $(+)$ denotes the positive frequencies of the gauge field, those corresponding to the annihilation operators, $a_r(\mathbf{k})$.

22 Functional quantization

See Cheng, §9. The method of functional quantization is an alternative to canonical quantization. The LSZ reduction formula can also be formulated for vector fields. The scattering amplitudes are directly related to the computation of correlation functions.

For a pure gauge theory (i.e. no matter fields), one would like to construct a generating functional that provides all the possible correlation functions

$$W[J] = \int [\mathcal{D}A_\mu]\, \exp\left[\mathrm{i} \int \mathrm{d}^4x\, (\mathcal{L}_\mathrm{g} + J^a_\mu A^\mu_a)\right]$$

The Green's functions are

$$\langle 0|\, \mathcal{T}\{A^{\mu_1}_{a_1}(x) \cdots A^{\mu_n}_{a_n}(x_n)\}\, |0\rangle = (-\mathrm{i})^n \left.\frac{\delta^n W[J]}{\delta J^{a_1}_{\mu_1}(x_1) \cdots \delta J^{a_n}_{\mu_n}(x_n)}\right|_{J=0}$$

22.1 Free theory

The free Lagrangian (which does not contain the self-interactions) is

$$\mathcal{L}_0 = -\frac{1}{2}\, \partial_\mu A^a_\nu (\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu})$$

The associated generating functional is

$$W_0[J] = \int [\mathcal{D}A_\mu]\, \exp\left[\mathrm{i} \int \mathrm{d}^4x\, (\mathcal{L}_0 + J^a_\mu A^\mu_a)\right]$$

[26] See Theoretical Physics II for a covariant quantization of the electromagnetic field.

[r] The free action is

$$S_0 = -\frac{1}{2}\int \mathrm{d}^4x\,\partial_\mu A^a_\nu(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) = \frac{1}{2}\int \mathrm{d}^4x\,A^a_\mu(\eta^{\mu\nu}\,\Box - \partial^\mu\partial^\nu)A^a_\nu = \frac{1}{2}\int \mathrm{d}^4x\,AKA$$

The differential operator is the kinetic operator

$$K^{\mu\nu} \equiv \eta^{\mu\nu}\,\Box - \partial^\mu\partial^\nu$$

This operator is not invertible. If it could be inverted, one would complete the square in the exponent of the generating functional

$$\int \mathrm{d}^4x\,(AKA+2JA) = \int \mathrm{d}^4x\,[(A+JK^{-1})K(A+K^{-1}J) - JK^{-1}J] = \int \mathrm{d}^4x\,(\tilde{A}K\tilde{A} - JK^{-1}J)$$

therefore the generating functional would be

$$\begin{aligned} W_0[J] &= \exp\left[-\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\,JK^{-1}J\right]\int [\mathcal{D}\tilde{A}_\mu]\,\exp\left[\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\,\tilde{A}K\tilde{A}\right] \\ &\propto \frac{1}{\sqrt{\det K}}\exp\left[-\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\,JK^{-1}J\right] \end{aligned}$$

However, the inverse of the kinetic operator does not exist. Consider

$$K_{\mu\nu}K^\nu{}_\rho = (\eta_{\mu\nu}\,\Box - \partial_\mu\partial_\nu)(\delta^\nu_\rho\,\Box - \partial^\nu\partial_\rho) = \eta_{\mu\rho}\,\Box^2 - \Box\,\partial_\mu\partial_\rho = \Box(\eta_{\mu\rho}\,\Box - \partial_\mu\partial_\rho) = \Box\,K_{\mu\rho}$$

From this one sees that the operator K behaves like a projection operator and as such is not invertible (because it is not injective).

One looks for the Green's function G of the kinetic operator

$$K_{\mu\nu}G^{\nu\rho} = \delta^\rho_\mu \iff (\eta_{\mu\nu}\,\Box - \partial_\mu\partial_\nu)G^{\nu\rho}(x-y) = \delta^\rho_\mu\delta^{(4)}(x-y)$$

By Fourier transformation, one has

$$G^{\nu\rho}(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{\mathrm{i}k(x-y)}\widetilde{G}^{\nu\rho}(k)$$

from which the equation above becomes

$$(-k^2\eta_{\mu\nu} + k_\mu k_\nu)\widetilde{G}^{\nu\rho}(k) = \delta^\rho_\mu$$

The most general type-(2,0) tensor field, function of the momentum k has the form

$$\widetilde{G}^{\nu\rho}(k) = A(k^2)\eta^{\mu\nu} + B(k^2)k^\nu k^\rho$$

Inserting this expression above, one finds no possible functions A and B.

Remark. Since the kinetic term is not invertible, then at least one eigenvalue is zero. The operator has a zero mode. Its determinant is zero and the generating functional is ill-defined. This is because the functional integral goes over all configurations of the gauge field. Recalling that the field is gauge invariant, the exponential in the generating functional is the same for each field connected by a gauge transformation. One is summing the same phase.

Remark. If one were to fix the gauge, one could modify the kinetic term to obtain an invertible operator.

Lecture 12

gio 16 mag 2024 10:30

22.1.1 Faddeev–Popov prescription

All the field configurations linked by a gauge transformation constitute a gauge orbit. An orbit has infinite volume which makes the generating functional ill-defined. The problem can be solved by employing the Faddeev–Popov prescription. One rewrites the generating functional as

$$W[J] \sim \mathrm{e}^{\mathrm{i}S} \times (\text{gauge orbit volume})$$

and removes the infinite volume of the gauge orbits. One declares this as the well-defined prescription.

Example. An illustrative example with ordinary integrals is the following. Consider a Lebesgue bidimensional integral

$$W = \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{\mathrm{i}S(x,y)}$$

Going to polar coordinates (r, θ)

$$x = r\cos\theta\,, \quad y = r\sin\theta\,, \quad \mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$$

one has

$$W = \int r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{e}^{\mathrm{i}S(r,\theta)}$$

To each fixed radius corresponds a circle. The prescription is read as integrating along a circle and then integrating over all circles. Let S be a function invariant under rotation $\theta \to \theta+\varphi$. This invariance is the analogue of gauge invariance in QFT. This implies that the function depends only on the radius

$$S(r,\theta) = S(r)$$

The integral is

$$W = \int r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{e}^{\mathrm{i}S(r)} = 2\pi \int r\,\mathrm{d}r\,\mathrm{e}^{\mathrm{i}S(r)}$$

The factor 2π is the volume of the circle, which corresponds to the volume of the orbit. In this case, the volume of the orbit is finite, but in QFT it is infinite.

One may compute the integral in an alternative way. One inserts unity into the initial integral

$$1 = \int_0^{2\pi} \mathrm{d}\varphi\,\delta(\theta-\varphi)$$

therefore

$$W = \int_0^{2\pi} \mathrm{d}\varphi \int r\,\mathrm{d}r\,\mathrm{d}\theta\,\delta(\theta-\varphi)\mathrm{e}^{\mathrm{i}S(r)} \equiv \int_0^{2\pi} \mathrm{d}\varphi\,W_\varphi$$

This corresponds to fixing an angle $\theta = \varphi$, integrating over the radius and then summing over all angles. The integral W_φ is invariant under a rotation by an angle φ (because the integrand is invariant). Therefore W_φ does not depend on such angle φ. Thus, the whole integral may be written as

$$W = \int_0^{2\pi} \mathrm{d}\varphi\,W_\varphi = W_\varphi \int_0^{2\pi} \mathrm{d}\varphi = 2\pi W_\varphi$$

The φ-integral is the integral on the orbit and the original integral W can be normalized by the volume

$$\frac{W}{\text{volume orbit}} = W_\varphi$$

and one considers this normalized function.

The insertion of the delta function means integrating first over the radius and then over the angle. One can generalize the approach by integrating first over an arbitrary line (i.e. not straight) that intersects each orbit only once and then integrate over the angle. This line is given by a gauge function $g(r,\theta)$ and is such that, for any r,

$$g(r, \theta+\varphi) = 0$$

has a unique φ solution. One generalizes unity as

$$1 = \Delta_g(\mathbf{r})\Delta_g^{-1}(\mathbf{r})\,, \quad \mathbf{r}_\varphi = (r, \theta+\varphi)$$

where one has

$$\Delta_g^{-1}(\mathbf{r}) = \int_0^{2\pi} \mathrm{d}\varphi\,\delta(g(\mathbf{r}_\varphi)) = \int \mathrm{d}g\,\partial_g\varphi\,\delta(g) = \partial_g\varphi|_{g=0} \implies \Delta_g(\mathbf{r}) = \partial_\varphi g|_{g=0}$$

A property of this object is gauge invariance under rotations

$$\Delta_g^{-1}(\mathbf{r}_{\varphi'}) = \int_0^{2\pi} \mathrm{d}\varphi\,\delta(g(\mathbf{r}_{\varphi+\varphi'})) = \int_0^{2\pi} \mathrm{d}(\varphi+\varphi')\,\delta(g(\mathbf{r}_{\varphi'+\varphi})) = \int_0^{2\pi} \mathrm{d}\varphi''\,\delta(g(\mathbf{r}_{\varphi''})) = \Delta_g^{-1}(\mathbf{r})$$

where one sets $\varphi'' = \varphi + \varphi'$.

Inserting this tool inside the integral W, one finds

$$\begin{aligned} W &= \int r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{e}^{\mathrm{i}S(r)} = \int r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{e}^{\mathrm{i}S(r)}\Delta_g(\mathbf{r})\Delta_g^{-1}(\mathbf{r}) \\ &= \int_0^{2\pi} \mathrm{d}\varphi \int r\,\mathrm{d}r\,\mathrm{d}\theta\,\Delta_g(\mathbf{r})\delta(g(\mathbf{r}_\varphi))\mathrm{e}^{\mathrm{i}S(r)} \\ &= \int_0^{2\pi} \mathrm{d}\varphi \int r\,\mathrm{d}r\,\mathrm{d}(\theta+\varphi)\,\Delta_g(\mathbf{r}_\varphi)\delta(g(\mathbf{r}_\varphi))\mathrm{e}^{\mathrm{i}S(\mathbf{r}_\varphi)} \\ &= \int_0^{2\pi} \mathrm{d}\varphi \int r\,\mathrm{d}r\,\mathrm{d}\theta'\,\Delta_g(\mathbf{r}')\delta(g(\mathbf{r}'))\mathrm{e}^{\mathrm{i}S(\mathbf{r}')} \\ &\equiv \int_0^{2\pi} \mathrm{d}\varphi\, W_\varphi \end{aligned}$$

At the third line, one has applied

$$\Delta_g(\mathbf{r}) = \Delta_g(\mathbf{r}_\varphi)\,, \quad S(r) = S(\mathbf{r}) = S(\mathbf{r}_\varphi)\,, \quad r\,\mathrm{d}r\,\mathrm{d}\theta = r\,\mathrm{d}r\,\mathrm{d}(\theta+\varphi)$$

At the fourth line one renames

$$\mathbf{r}_\varphi = (r, \theta+\varphi) \to \mathbf{r}' = (r, \theta')$$

The quantity W_φ is independent of the angle φ (due to the integral over θ'?), therefore one can normalize the integral and consider this normalized quantity

$$W \to \frac{W}{\int_0^{2\pi}\,\mathrm{d}\varphi} = W_\varphi = \int r\,\mathrm{d}r\,\mathrm{d}\theta\,\Delta_g(\mathbf{r})\delta(g(\mathbf{r}))\mathrm{e}^{\mathrm{i}S(r)}$$

The reason why one is allowed to normalize by the orbit volume is because the path integral of the generating functional counts infinitely many times the same phase. The normalization removes this over-counting.

Functional quantization. The discussion proceeds by analogy to the previous example. Consider gauge invariance under finite gauge transformations of the group $G = \mathrm{SU}(n)$

$$A_\mu^\theta = U(\theta)A_\mu U^{-1}(\theta) - \frac{\mathrm{i}}{g}\,[\partial_\mu U(\theta)]U^{-1}(\theta)\,, \quad U(\theta) = \mathrm{e}^{-\mathrm{i}\theta^a T^a}$$

The generating functional with no source is

$$W = \int [\mathcal{D}A_\mu]\,\mathrm{e}^{\mathrm{i}S_\mathrm{g}(A_\mu)}\,, \quad S_\mathrm{g}(A_\mu) = S_\mathrm{g}(A_\mu^\theta)\,, \quad \forall\theta$$

where the index g stands for “gauge”. Notice that the action is gauge-invariant.

One chooses a set of $\dim G$ gauge functions and imposes the gauge-fixing condition

$$f_a(A_\mu) = 0$$

These functions are such that, for each A_μ,

$$f_a(A_\mu^\theta) = 0$$

has a unique solution $\theta(x)$. The gauge functions intersect each orbit only once.

One has to define the integration over the group space. In terms of the representation matrices, the multiplication of group elements takes on the form

$$U(\theta)U(\theta') = U(\theta\theta')$$

The integration measure over the group space can be chosen as

$$[\mathrm{d}\theta(x)] = \prod_a \mathrm{d}\theta_a(x)$$

which is invariant in the sense that

$$[\mathrm{d}\theta] = [\mathrm{d}(\theta\theta')]$$

The operator providing unity is

$$\begin{aligned}
\Delta_f^{-1}[A_\mu] &= \int [\mathrm{d}\theta(x)] \prod_a \delta(f_a(A_\mu^\theta(x))) \\
&= \int \prod_a [\mathrm{d}f_a] |\det \partial_{f_a} \theta_b| \prod_a \delta(f_a) \\
&= \left|\det \partial_{\theta_b} f_a(A_\mu^\theta)\right|^{-1}_{f_a=0} = \det M_{ab}^{-1}
\end{aligned}$$

from which

$$\Delta_f = \left|\det \partial_{\theta_b} f_a(A_\mu^\theta)\right|_{f_a=0} = \det M_{ab}$$

An important property is gauge invariance

$$\Delta_f[A_\mu] = \Delta_f[A_\mu^\theta]\,, \quad \forall\theta$$

This can be checked as follows

$$\begin{aligned}
\Delta_f^{-1}[A_\mu^\theta] &= \int [\mathrm{d}\theta'(x)] \prod_a \delta(f_a(A_\mu^{\theta\theta'})) = \int [\mathrm{d}\theta\theta'(x)] \prod_a \delta(f_a(A_\mu^{\theta\theta'})) \\
&= \int [\mathrm{d}\theta''(x)] \prod_a \delta(f_a(A_\mu^{\theta''})) = \Delta_f^{-1}[A_\mu]
\end{aligned}$$

where $\mathrm{d}\theta' = \mathrm{d}(\theta\theta')$ and $\theta'' = \theta\theta'$. Therefore, the generating functional is

$$\begin{aligned}
W &= \int [\mathcal{D}A_\mu]\, \mathrm{e}^{\mathrm{i}S_\mathrm{g}[A_\mu]} \Delta_f[A_\mu] \int [\mathrm{d}\theta(x)] \prod_a \delta(f_a(A_\mu^\theta)) \\
&= \int [\mathrm{d}\theta(x)] \int [\mathcal{D}A_\mu]\, \mathrm{e}^{\mathrm{i}S_\mathrm{g}[A_\mu]} \Delta_f[A_\mu] \prod_a \delta(f_a(A_\mu^\theta(x))) \\
&= \int [\mathrm{d}\theta(x)] \int [\mathcal{D}A_\mu^\theta]\, \mathrm{e}^{\mathrm{i}S_\mathrm{g}[A_\mu^\theta]} \Delta_f[A_\mu^\theta] \prod_a \delta(f_a(A_\mu^\theta(x))) \\
&= \int [\mathrm{d}\theta(x)] \int [\mathcal{D}A'_\mu]\, \mathrm{e}^{\mathrm{i}S_\mathrm{g}[A'_\mu]} \Delta_f[A'_\mu] \prod_a \delta(f_a(A'_\mu(x))) \\
&= \infty \times (\text{smth})
\end{aligned}$$

at the third line, one notices that the action is gauge invariant [r]. At the fourth line, one has renamed the field $A_\mu^\theta = A'_\mu$.

One notices that the second integral does not depend on the parameter θ and the first integral is the infinite volume of the gauge orbits. One normalizes the generating functional by the orbits' volume and considers the result as the well-defined generating functional for gauge theories

$$\boxed{W_f = \int [\mathcal{D}A_\mu]\, \Delta_f[A_\mu] \prod_a \delta(f_a(A(x)))\mathrm{e}^{\mathrm{i}S[A_\mu]} = \int [\mathcal{D}A_\mu]\, (\det M_f) \prod_a \delta(f_a(A(x)))\mathrm{e}^{\mathrm{i}S[A_\mu]}}$$

where

$$(M_f)_{ab} = \left.\frac{\delta f_a(A_\mu^\theta)}{\delta\theta_b}\right|_{f_a=0}$$

[r] where absolute value?
This is the Faddeev–Popov prescription. One is not integrating over every field configuration due to the presence of the delta function: one is summing only one single representative of each orbit.

Determinant. One may rewrite the coefficients of the exponential as phases. Recall

$$\int \left[\prod_{j=n}^{1} \mathrm{d}\theta_j \,\mathrm{d}\bar{\theta}_j\right] \mathrm{e}^{\mathrm{i}\bar{\theta}_a M_{ab}\theta_b} \propto \det M$$

One introduces a set of Grassmann-odd fields

$$c_a(x)\,, \quad \bar{c}_a(x) = c_a^\dagger(x)\,, \quad a = 1, \ldots, \dim G$$

so the determinant is

$$\det M_f \propto \int [\mathrm{d}c\,\mathrm{d}\bar{c}] \,\exp\left[\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{c}_a(x)[M_f(x,y)]_{ab}c_b(y)\right]$$

These new fields are non-physical degrees of freedom called Faddeev–Popov ghosts.

Delta function. One generalizes the gauge-fixing condition from $f_a(A_\mu) = 0$ to

$$f_a(A_\mu(x)) = B_a(x)$$

where $B_a(x)$ is a smooth function independent of the gauge field A_μ. The delta function is then

$$\delta(f_a(A_\mu) - B_a)$$

Considering that the original generating functional W is independent of B_a due to the dependence being inside $\Delta\Delta^{-1}$, one may change the normalization by a constant

$$\text{const} \equiv \int [\mathcal{D}B_a] \,\exp\left[-\frac{\mathrm{i}}{2\xi}\int \mathrm{d}^4x\,B_a(x)B_a(x)\right]$$

Therefore

$$\begin{aligned}
W[J] &= \int [\mathcal{D}B_a]\,[\mathcal{D}A_\mu]\,(\det M_f)\prod_a \delta(f_a(A_\mu) - B_a)\,\exp\left[-\frac{\mathrm{i}}{2\xi}\int \mathrm{d}^4x\,B_a(x)B_a(x)\right]\mathrm{e}^{\mathrm{i}S_\mathrm{g}[A_\mu]} \\
&= \int [\mathcal{D}A_\mu]\,(\det M_f)\,\exp\left[-\frac{\mathrm{i}}{2\xi}\int \mathrm{d}^4x\,[f_a(A_\mu)]^2\right]\mathrm{e}^{\mathrm{i}S_\mathrm{g}[A_\mu]} \\
&= \int [\mathcal{D}A_\mu\,\mathcal{D}c_a\,\mathcal{D}\bar{c}_a]\,\mathrm{e}^{\mathrm{i}S_\mathrm{g}[A_\mu]} \\
&\quad\times \exp\left[-\frac{\mathrm{i}}{2\xi}\int \mathrm{d}^4x\,[f_a(A_\mu)]^2 + \mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{c}_a(x)(M_f)_{ab}(x,y)c_b(y)\right]
\end{aligned}$$

At the second line, one integrates using the delta function. From the third line, one sees that ghosts are propagating fields, but are only quantum: there can be no ghost background fields [r].

The total action is therefore

$$S = S_\text{gauge} + S_\text{gauge-fixing} + S_\text{ghost}$$

where one has

$$\begin{aligned}
S_\mathrm{g}[A^\mu] &= -\frac{1}{4}\int \mathrm{d}^4x\,F^a_{\mu\nu}F^{\mu\nu}_a \\
S_\mathrm{gf}[A^\mu] &= -\frac{1}{2\xi}\int \mathrm{d}^4x\,[f_a(A^\mu)]^2 \\
S_\mathrm{gh}[A^\mu, c, \bar{c}] &= \int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{c}_a(x)(M_f)_{ab}(x,y)c_b(y)
\end{aligned}$$

with ξ an arbitrary gauge parameter and

$$f_a(A^\mu) = B_a(x)\,, \quad (M_f)_{ab}(x,y) = \left.\frac{\delta f_a(A^\theta_\mu(x))}{\delta\theta_b(y)}\right|_{f^a=0}$$

Remark. The total action is not gauge-invariant due to the presence of the gauge-fixing term.

Lecture 13

lun 20 mag 2024 10:30

Example (Axial gauge). Consider the axial gauge, also called Arnowitt–Fickler gauge[27],

$$f_a(A^\mu) = A^3_a$$

An infinitesimal gauge transformation by $\theta_a(x)$ is

$$f^a(A^\theta_\mu) = A^a_3 + c^{abc}\theta^b A^c_3 - \frac{1}{g}\,\partial_3\theta^a = -\frac{1}{g}\,\partial_3\theta_a$$

[r] where one applies the gauge-fixing condition $f_a(A^\mu) = A^3_a = 0$. The matrix M is

$$M_{ab}(x,y) = \left.\frac{\delta f_a(A_\mu(x))}{\delta\theta_b(y)}\right|_{f_a=0} = -\frac{1}{g}\,\partial_3[\delta^{(4)}(x-y)\,\delta_{ab}]$$

Notice how it does not depend on the gauge field A^a_μ. This implies that the ghost Lagrangian depends only on the ghosts: the gauge field decouples from the ghosts. The functional integral of the ghosts in the generating functional can be performed and absorbed into the normalization constant, so that the ghosts do not appear anymore.

From this follows that the ghost action describes non-propagating degrees of freedom[28]

$$\begin{aligned} S_{\text{gh}} &= \int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{c}_a(x)(M_f)_{ab}(x,y)c_b(y) = -\frac{1}{g}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{c}_a\,\partial_3(x)[\delta^{(4)}(x-y)\delta_{ab}]c_b(y) \\ &= \frac{1}{g}\int \mathrm{d}^4x\,\partial_3\bar{c}_a(x)\,c_a(x) \end{aligned}$$

where one applies the properties of the Dirac delta function under differentiation noting that ∂_3 is done with respect to x. Since the ghost fields are non-propagating, they can be ignored. In the axial gauge there are no ghosts. This result may be generalized: a gauge-fixing condition that is linear in the gauge field implies that M_{ab} is independent of the field and the ghosts decouple from it.

Example (Abelian theory). Consider an abelian theory. The gauge transformation is

$$A^\theta_\mu = A_\mu - \frac{1}{g}\,\partial_\mu\theta$$

Taking a gauge-fixing condition linear in the gauge field permits to decouple the ghosts. This is why QED can be a theory without ghosts. An example of non-linear condition is

$$A_\mu A^\mu = 0$$

Example (Abelian Lorenz gauge). Consider the Lorenz gauge

$$f(A_\mu) = \partial_\mu A^\mu = 0$$

The gauge transformation is

$$f_a(A^\theta_\mu) = \partial_\mu A^\mu_\theta = \partial_\mu A^\mu - \frac{1}{g}\,\Box\,\theta = -\frac{1}{g}\,\Box\,\theta$$

The matrix is

$$M(x,y) = -\frac{1}{g}\partial_{\theta(x)}[\Box\,\theta(y)] = -\frac{1}{g}\,\Box[\delta^{(4)}(x-y)]$$

The ghost action is

$$S_{\text{gh}} = -\frac{1}{g}\int \mathrm{d}^4x\,\bar{c}(x)\,\Box\,c(x)$$

This is a free action, describing propagating but non-interacting fields.

[27] Generally it is written as $n^\mu A_\mu$ with n^μ a unit four-vector specifying the direction.

[28] Notice how there is only one derivative. The equations of motion imply no wave equation: $\partial_3\bar{c} = \partial_3 c = 0$. Any function independent of x^3 is a solution, and there is no sense that the past determines future.

Example (Non-abelian Lorenz gauge)**.** Consider the Lorenz gauge like before

$$f_a(A_\mu) = \partial_\mu A^\mu_a = 0$$

The gauge function for the transformed field is

$$\begin{aligned}
f^a(A^\theta_\mu) &= \partial_\mu \left[A^{a\mu} + c^{abc}\theta^b A^{c\mu} - \frac{1}{g}\,\partial^\mu\theta^a\right] = -\frac{1}{g}\,\partial_\mu(\partial^\mu\theta^a - gc^{abc}\theta^b A^{c\mu}) \\
&= -\frac{1}{g}\,\partial_\mu(\partial^\mu\theta^a + gc^{abc}A^{b\mu}\theta^c) = -\frac{1}{g}\,\partial_\mu(D^\mu\theta)^a
\end{aligned}$$

where the covariant derivative D_μ acts on a field $\theta_a(x)$ in the adjoint representation[29]. In fact, recalling that

$$D_\mu = \partial_\mu - \mathrm{i}gA^a_\mu T^a$$

the covariant derivative acts on the adjoint representation of a field φ as

$$[D_\mu, \varphi] = [\partial_\mu - \mathrm{i}gA^a_\mu T^a, \varphi^b T^b] = (\partial_\mu\varphi^a + gc^{abc}A^b_\mu\varphi^c)T^a$$

Notice that

$$[\partial_\mu, f]g = \partial_\mu(fg) - f\,\partial_\mu g = (\partial_\mu f)g$$

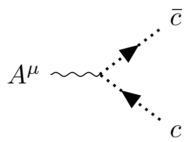
One obtains

$$M_{ab}(x,y) = -\frac{1}{g}\,\partial_{\theta^b(y)}[\partial_\mu(D^\mu\theta(x))_a]|_{f_a=0} = -\frac{1}{g}\,\partial^\mu(D_\mu)_{ab}\delta^{(4)}(x-y)$$

The ghost action is

$$\begin{aligned}
S_{\mathrm{gh}} &= -\frac{1}{g}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{c}_a(x)[\partial^\mu D^{ab}_\mu\delta^{(4)}(x-y)]c_b(y) = \frac{1}{g}\int \mathrm{d}^4x\,\partial^\mu\bar{c}_a(x)\,D^{ab}_\mu\,c_b(x) \\
&= \frac{1}{g}\int \mathrm{d}^4x\,[\partial^\mu\bar{c}_a(x)\,\partial_\mu c_a(x) + gc^{acb}\,\partial_\mu\bar{c}_a(x)A^{c\mu}(x)c^b(x)]
\end{aligned}$$

where one integrates by parts. The second addendum is present only for non-abelian theories: it is a three-point vertex between the gauge field and two ghosts:



The complete action is then

$$S = S_{\mathrm{g}} + S_{\mathrm{gf}} + S_{\mathrm{gh}} = S_0 + S_{\mathrm{I}}$$

where the free and interacting parts are

$$\begin{aligned}
S_0 &= \int \mathrm{d}^4x\,\left[-\frac{1}{2}\,\partial_\mu A^a_\nu(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 - \bar{c}_a\,\Box\,c_a\right] \\
S_{\mathrm{I}} &= \int \mathrm{d}^4x\,\left[-gc^{abc}\,\partial_\mu A^a_\nu\,A^{b\mu}A^{c\nu} - \frac{g^2}{4}c^{abc}c^{ade}A^b_\mu A^c_\nu A^{d\mu}A^{e\nu} - gc^{abc}\,(\partial_\mu\bar{c}^a)c^bA^{c\mu}\right]
\end{aligned}$$

where the fields c and $\bar{c}$ have been redefined to

$$c_a \to \sqrt{g}c_a$$

The interacting part includes a three-point and a four-point self-coupling vertices and a three-point ghost-field vertex.

[29]For the meaning of a field to be in a representation, see `https://physics.stackexchange.com/q/412232`.

22.2 Perturbation theory

[r] The kinetic term found using ghosts is invertible. The generating functional is

$$W[J,\eta,\bar{\eta}] = \int [\mathcal{D}A_\mu\,\mathcal{D}c\,\mathcal{D}\bar{c}]\,\exp\left[\mathrm{i}S + \mathrm{i}\int \mathrm{d}^4x\,(J^a_\mu A^\mu_a + \bar{\eta}^a c^a + \bar{c}^a\eta^a)\right]$$

As with scalar and spinor fields, one may replace

$$A^a_\mu(x) \to -\mathrm{i}\,\frac{\delta}{\delta J^\mu_a(x)}\,,\quad c^a \to -\mathrm{i}\,\frac{\delta}{\delta\bar{\eta}^a}\,,\quad \bar{c}^a \to \mathrm{i}\,\frac{\delta}{\delta\eta^a}$$

to have

$$W[J,\eta,\bar{\eta}] = \exp\left[\mathrm{i}S_\mathrm{I}(-\mathrm{i}\,\partial_J, -\mathrm{i}\,\delta_{\bar{\eta}}, \mathrm{i}\,\delta_\eta)\right]W_0[J,\eta,\bar{\eta}]$$

noting that the free generating functionals split

$$W_0[J,\eta,\bar{\eta}] = W_0[J]W_0[\eta,\bar{\eta}]$$

with

$$\begin{aligned} W_0[J] &= \int [\mathcal{D}A_\mu]\,\exp\left[\mathrm{i}(S^{(0)}_\mathrm{g} + S_\mathrm{gf}) + \mathrm{i}\int \mathrm{d}^4x\,J^\mu_a A^a_\mu\right] \\ W_0[\eta,\bar{\eta}] &= \int [\mathcal{D}c\,\mathcal{D}\bar{c}]\,\exp\left[\mathrm{i}S^{(0)}_\mathrm{gh} + \mathrm{i}\int \mathrm{d}^4x\,(\bar{\eta}c + \bar{c}\eta)\right] \end{aligned}$$

where the superscript (0) indicates the free part.

Gauge field propagator. Consider the free action of the gauge boson field

$$\begin{aligned} S^{(0)}_\mathrm{g} + S_\mathrm{gf} &= \int \mathrm{d}^4x\,\left[-\frac{1}{2}\,\partial_\mu A^a_\nu(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) - \frac{1}{2\xi}(\partial_\mu A^{a\mu})(\partial_\nu A^{a\nu})\right] \\ &= \int \mathrm{d}^4x\,\frac{1}{2}A^a_\nu\left[\eta^{\mu\nu}\,\Box - \left(1-\frac{1}{\xi}\right)\partial^\mu\,\partial^\nu\right]A^a_\mu \\ &= \int \mathrm{d}^4x\,\mathrm{d}^4y\,\frac{1}{2}A^a_\nu(x)\left[\eta^{\mu\nu}\,\Box - \left(1-\frac{1}{\xi}\right)\partial^\mu\,\partial^\nu\right]\delta^{(4)}(x-y)\delta^a_b A^b_\mu(y) \\ &= \int \mathrm{d}^4x\,\mathrm{d}^4y\,\frac{1}{2}A^a_\nu(x)K^{\mu\nu}_{ab}(x,y)A^b_\mu(y) \end{aligned}$$

At the second line one has integrated by parts.

The kinetic term is an invertible operator

$$K^{\mu\nu}_{ab}(x-y) = \left[\eta^{\mu\nu}\,\Box - \left(1-\frac{1}{\xi}\right)\partial^\mu\,\partial^\nu\right]\delta^{(4)}(x-y)\delta_{ab}$$

[r] In fact, its Green's function $G_{\mu\nu}(x-y)$ that satisfies

$$K^{\mu\nu}(x-y)G_{\nu\rho}(y-z) = \delta^\mu_\rho\delta^{(4)}(x-z)$$

Applying the Fourier transform

$$G_{\nu\rho}(y-z) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{\mathrm{i}k(y-z)}G_{\nu\rho}(k)$$

gives

$$\left[-k^2\eta^{\mu\nu} + \left(1-\frac{1}{\xi}\right)k^\mu k^\nu\right]G_{\nu\rho}(k) = \delta^\mu_\rho$$

which has solution

$$G^{ab}_{\nu\rho}(k) = \left[-\eta_{\nu\rho} + (1-\xi)\frac{k_\nu k_\rho}{k^2}\right]\frac{\delta^{ab}}{k^2 + \mathrm{i}\varepsilon}$$

Completing the square of the generating functional $W_0[J]$ gives

$$W_0[J] = \exp\left[-\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,J^a_\mu(x)G^{\mu\nu}_{ab}(x-y)G^b_\nu(y)\right]$$

The two-point Green's function is

$$(G^{\mu\nu}_{ab})^{(2)}(x-y) = \langle 0|\,\mathcal{T}\{A^\mu_a(x)A^\nu_b(y)\}\,|0\rangle = (-\mathrm{i})^2\,\frac{\delta^2 W_0[J]}{\delta J^a_\mu(x)\delta J^b_\nu(y)}\bigg|_{J=0} = \mathrm{i}\Delta^{\mu\nu}_{ab}(x-y) = \mathrm{i}G^{\mu\nu}_{ab}(x-y)$$

and in Fourier space it is

$$(G^{ab}_{\mu\nu})^{(2)}(k) = \langle 0|\,\mathcal{T}\{A^a_\mu(k)A^b_\nu(-k)\}\,|0\rangle = \mathrm{i}\left[-\eta_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]\frac{\delta^{ab}}{k^2+\mathrm{i}\varepsilon} = \mathrm{i}\Delta^{ab}_{\mu\nu}(k)$$

Ghost propagator. The free generating functional is

$$W_0[\eta,\bar\eta] = \int [\mathcal{D}c\,\mathcal{D}\bar c]\,\exp\left[\mathrm{i}\int \mathrm{d}^4x\,[-\bar c_a\,\Box\,c_a + \bar\eta_a c_a + \bar c_a\eta_a\right]$$

Knowing that the kinetic term in momentum space is

$$\delta_{ab}\,\Box \implies -\delta_{ab}k^2$$

one obtains

$$G_{ab}(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{\mathrm{e}^{\mathrm{i}k(x-y)}}{k^2+\mathrm{i}\varepsilon}\delta_{ab}$$

[r] Completing the square in the exponent gives

$$W_0[\eta,\bar\eta] = \exp\left[-\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar\eta_a G_{ab}(x-y)\eta_b(y)\right]$$

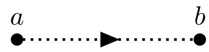
The two-point Green's function is

$$G^{ab}(x-y) = \langle 0|\,\mathcal{T}\{\bar c^a(x)c^b(y)\}\,|0\rangle = \frac{\delta^2 W_0[\eta,\bar\eta]}{\delta\bar\eta_a(x)\delta\eta_b(y)}\bigg|_{\eta=\bar\eta=0} = \mathrm{i}\Delta^{ab}(x-y)$$

and in momentum space, one has

$$G_{ab}(k) = \frac{\mathrm{i}\delta_{ab}}{k^2+\mathrm{i}\varepsilon} = \mathrm{i}\Delta_{ab}(k)$$

which corresponds to the line



23 Becchi–Rouet–Stora–Tyutin quantization

[r] See Cheng, §9.3. Adding a gauge-fixing Lagrangian implies that the action is no longer gauge-invariant. One would like to find another symmetry (and the corresponding transformations) for the gauge-fixed action. This symmetry is the Becchi–Rouet–Stora–Tyutin (BRST) symmetry.

Consider a gauge field minimally coupled to some spinor fields (called matter fields). The most general Yang–Mills Lagrangian (gauge group SU(n)) is

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \bar\psi(\mathrm{i}\,\not{D} - m)\psi - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 + \bar c_a\,\partial^\mu(D_\mu)_{ab}c_b$$

[r] why plus sign in ghost term?
where

$$F^a_{\mu\nu} = \partial_{[\mu}A^a_{\nu]} + gc^{abc}A^b_\mu A^c_\nu\,, \quad D_\mu\psi = (\partial_\mu - \mathrm{i}gA^a_\mu T^a)\psi\,, \quad (D_\mu)_{ab}c_b = \partial_\mu c_a + gc^{abc}A^b_\mu c^c$$

It is convenient to express the complex Grassmann-odd ghost fields as real Grassmann-odd fields

$$c_a = \frac{1}{\sqrt{2}}(\rho_a + \mathrm{i}\sigma_a)\,, \quad \bar{c}_a = \frac{1}{\sqrt{2}}(\rho_a - \mathrm{i}\sigma_a)$$

This implies that the ghost Lagrangian is

$$\mathcal{L}_{\mathrm{gh}} = -\mathrm{i}\,\partial^\mu \rho^a\,(D_\mu)_{ab}\sigma^b$$

which can be obtained noting that

$$\rho^a\,\partial^\mu (D_\mu)_{ab}\rho^b = \sigma^a\,\partial^\mu (D_\mu)_{ab}\sigma^b = 0$$

Proposition. The action is invariant under the BRST transformation

$$\begin{aligned}
\delta A^a_\mu &= \omega D^{ab}_\mu \sigma_b \\
\delta\rho^a &= -\frac{\mathrm{i}}{\xi}\omega\,\partial^\mu A^a_\mu \\
\delta\sigma^a &= -\frac{g}{2}\omega c^{abc}\sigma^b\sigma^c \\
\delta\psi &= \mathrm{i}g\omega(T^a\sigma^a)\psi
\end{aligned}$$

where ω is a constant Grassmann-odd number and ξ is the gauge parameter.

In general, infinitesimal transformations can be written as the commutator of a parameter ε and the generator Q of a symmetry

$$\delta\chi = [\varepsilon Q, \chi]$$

In this case, the parameter $\varepsilon = \omega$ and the generator are Grassmann-odd quantities. This also implies $Q^2 = 0$ and that applying twice the transformations gives zero.

Proof. The action is

$$S = (S_{\mathrm{g}} + S_\psi) + S_{\mathrm{gf}} + S_{\mathrm{gh}}$$

The first two addenda are gauge-invariant. The transformations of the gauge field and the matter fields are the typical (infinitesimal) gauge transformations with parameter $\theta^a = -g\omega\sigma^a$

$$(A^a_\mu)' = A^a_\mu + \delta A^a_\mu = A^a_\mu + c^{abc}\theta^b A^c_\mu - \frac{1}{g}\,\partial_\mu\theta^a\,, \quad \psi' = \psi + \delta\psi = \psi - \mathrm{i}\theta^a T^a\psi$$

Thus one has

$$\delta_{\mathrm{BRST}}(S_{\mathrm{g}} + S_\psi) = 0$$

The gauge-fixing Lagrangian transforms as

$$\delta L_{\mathrm{gf}} = -\frac{1}{2\xi}\,\delta(\partial_\mu A^{a\mu})^2 = -\frac{1}{\xi}\,\partial_\mu A^{a\mu}\,\partial_\nu(\delta A^{a\nu}) = -\frac{1}{\xi}\,\partial_\mu A^{a\mu}\,\partial_\nu[\omega(D^\nu)^{ab}\sigma^b]$$

The ghost Lagrangian transforms as

$$\begin{aligned}
\delta\mathcal{L}_{\mathrm{gh}} &= \delta[-\mathrm{i}\,\partial^\mu\rho_a\,D^{ab}_\mu\sigma_b] = -\mathrm{i}\,\partial^\mu(\delta\rho_a)\,D^{ab}_\mu\sigma_b - \mathrm{i}(\partial^\mu\rho_a)\,\delta[D^{ab}_\mu\sigma_b] \\
&= -\frac{1}{\xi}\omega\,\partial^\mu(\partial^\nu A^a_\nu)\,D^{ab}_\mu\sigma_b - \mathrm{i}(\partial^\mu\rho_a)\,\delta[D^{ab}_\mu\sigma_b]
\end{aligned}$$

At the second line, one may integrate by parts the first addendum and see that it cancels the variation of the gauge-fixing Lagrangian.

The variation of the total Lagrangian (up to four-divergences) is

$$\delta\mathcal{L} = -\mathrm{i}(\partial^\mu\rho_a)\,\delta[D^{ab}_\mu\sigma_b]$$

One needs to show that the last variation is zero

$$\begin{aligned}
\delta[D_\mu^{ab}\sigma_b] &= \delta[\partial_\mu\sigma^a - gc^{abc}\sigma^b A_\mu^c] \\
&= \partial_\mu(\delta\sigma^a) - gc^{abc}(\delta\sigma^b)A_\mu^c - gc^{abc}\sigma^b(\delta A_\mu^c) \\
&= -\frac{g}{2}\omega c^{abc}\,\partial_\mu(\sigma^b\sigma^c) + gc^{abc}\frac{g}{2}\omega c^{bef}\sigma^e\sigma^f A_\mu^c - gc^{abc}\sigma^b\omega D_\mu^{cd}\sigma^d \\
&= -\frac{g}{2}\omega c^{abc}\,\partial_\mu(\sigma^b\sigma^c) + \frac{g^2}{2}\omega c^{abc}c^{bef}\sigma^e\sigma^f A_\mu^c + g\omega c^{abc}\sigma^b(\partial_\mu\sigma^c - gc^{cdf}\sigma^d A_\mu^f) \\
&= \frac{g^2}{2}\omega c^{abc}c^{bef}\sigma^e\sigma^f A_\mu^c - g^2\omega c^{abc}c^{cef}\sigma^b\,\sigma^e A_\mu^f \\
&= \frac{g^2}{2}\omega c^{abe}c^{bcf}\sigma^c\sigma^f A_\mu^e - g^2\omega c^{acb}c^{bfe}\sigma^c\,\sigma^f A_\mu^e \\
&= \frac{g^2}{2}\omega(c^{abe}c^{bcf} + c^{abc}c^{bfe} - c^{abf}c^{bce})\sigma^c\sigma^f A_\mu^e \\
&= 0
\end{aligned}$$

[r] At the fourth line, the last addendum has a positive sign since ω and σ are both Grassmann-odd. At the fifth line, one has considered the two terms without A_μ and has recalled that the structure constants c^{abc} are totally anti-symmetric. At the sixth line, one renames $c \leftrightarrow e$ in the first addendum, and $c \leftrightarrow b$ and $f \leftrightarrow e$ in the second addendum. At the penultimate line, in the third addendum one exchanges the two σs and renames $c \leftrightarrow f$.

The result is zero due to the Jacobi identity. This means that the variation of the Lagrangian is zero under BRST transformations. □

Lecture 14

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Remark. The above variation can be written as

$$\delta_{\text{BRST}}(D_\mu\sigma)^a = 0$$

23.1 Slavnov–Taylor identities

The Slavnov–Taylor identities are the generalization of the Ward–Takahashi identity (originally derived in QED) to non-abelian gauge theories.

Review – Ward identity in QED. Quantum electrodynamics is a renormalizable theory. Its ultraviolet divergences can be taken care of using the renormalization functions

$$\psi_0 = Z_2^{\frac{1}{2}}\psi\,, \quad A_0^\mu = Z_3^{\frac{1}{2}}A^\mu\,, \quad q_0 = k^\varepsilon Z_1 Z_2^{-1}Z_3^{-\frac{1}{3}}q\,, \quad m_0 = Z_m Z_2^{-1}m$$

Gauge invariance imposes the use of covariant derivatives

$$D_\mu = \partial_\mu + \mathrm{i}qA_\mu\,, \quad g = |q|$$

where q is the charge (with sign) of the matter field. Assuming that gauge invariance is preserved after quantization implies that the covariant derivative of a matter field $D_\mu\psi$ has to be protected against quantum corrections

$$D_{\mathrm{R}}^\mu \equiv D_{\text{bare}}^\mu \iff (D_\mu\psi)_{\mathrm{R}} \equiv Z_2^{-\frac{1}{2}}(D_\mu\psi)_{\text{bare}}$$

This implies that

$$(qA_\mu)_{\mathrm{R}} = (qA_\mu)_{\text{bare}} \implies (qA_\mu)_{\text{bare}} = q_0 A_{0\mu} = \frac{Z_1}{Z_2}qA_\mu \equiv qA_\mu$$

from which one obtains the Ward–Takahashi identity in QED

$$\boxed{Z_1 = Z_2}$$

Non-abelian theories. One assumes that the BRST symmetry is preserved after quantization

$$\delta S = 0 \implies \delta W = 0$$

If the classical action S is invariant, then so is the effective action (equivalently the generating functional W). Since the generating functional contains the classical action, then the phase e^{iS} is invariant under BRST transformations. The above statement implies that the functional measure must be invariant.

Consider a generating functional containing the sources for the physical fields, the ghost fields and the composite fields that appear in the BRST transformations

$$W[J,\alpha,\beta,\chi,\bar{\chi},\kappa,\nu,\lambda,\bar{\lambda}] = \int [\mathcal{D}A_\mu\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\mathcal{D}\rho\,\mathcal{D}\sigma]\,\exp\left[\mathrm{i}\int \mathrm{d}^4x\,(\mathcal{L}+\Sigma)\right]$$

where the source term is

$$\Sigma = J^a_\mu A^\mu_a + \alpha_a\rho_a + \beta_a\sigma_a + \bar{\chi}\psi + \bar{\psi}\chi + \kappa^a_\mu\,(D^\mu\sigma)^a + \frac{1}{2}\nu_a c^{abc}\sigma_b\sigma_c + \bar{\lambda}T^a\sigma^a\psi + \bar{\psi}T^a\sigma^a\lambda$$

notice that the last four addenda involve composite fields and, among their sources, only κ is Grassmann-odd [r]. The assumption that the generating functional is invariant implies that

$$\delta W = \int \mathrm{d}^4x \int [\mathcal{D}A_\mu\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\mathcal{D}\rho\,\mathcal{D}\sigma]\,\mathrm{i}\,\delta\Sigma\,\exp\left[\mathrm{i}\int \mathrm{d}^4x\,(\mathcal{L}+\Sigma)\right] = 0 \implies \delta\Sigma = 0$$

If the transformation of the source term Σ is non-trivial, then the equation above gives useful relations between the composite fields. The variation is

$$\begin{aligned}\delta\Sigma &= J^a_\mu\,\delta A^\mu_a + \alpha_a\,\delta\rho_a + \beta_a\,\delta\sigma_a + \bar{\chi}\,\delta\psi + \delta\bar{\psi}\,\chi + \kappa^a_\mu\,\delta(D^\mu\sigma)^a \\ &+ \frac{1}{2}\nu_a\delta(c^{abc}\sigma_b\sigma_c) + \bar{\lambda}T^a\delta(\sigma^a\psi) + \delta(\bar{\psi}T^a\sigma^a)\lambda\end{aligned}$$

From the proof of invariance the gauge-fixed action one has $\delta(D_\mu\sigma)^a = 0$. The variation of all the composite operators is zero (see Cheng, p. 276 for computations)

$$\delta(c^{abc}\sigma_b\sigma_c) = T^a\delta(\sigma^a\psi) = \delta(\bar{\psi}T^a\sigma^a) = 0$$

Inserting the BRST transformations, one finds

$$\delta\Sigma = \omega\left[J^a_\mu\,(D^\mu\sigma)^a + \frac{\mathrm{i}}{\xi}\alpha_a\,\partial_\mu A^\mu_a + \frac{g}{2}\beta_a c^{abc}\sigma_b\sigma_c - \mathrm{i}g\bar{\chi}(T^a\sigma^a)\psi - \mathrm{i}g\bar{\psi}(T^a\sigma^a)\chi\right]$$

In the functional integral, the fields can be replaced with the derivative with respect to the corresponding source. In this way, the source term Σ is no longer function of the integrated fields and can be brought outside the functional integral. The variation of the generating functional is then

$$\delta W = \int \mathrm{d}^4x\,\mathrm{i}\,\delta\Sigma\,W[J,\alpha,\beta,\chi,\bar{\chi},\kappa,\nu,\lambda,\bar{\lambda}] = 0$$

which implies

$$\boxed{\int \mathrm{d}^4x\,\omega\left[J^a_\mu\frac{\delta}{\delta\kappa^a_\mu} + \frac{\mathrm{i}}{\xi}\alpha_a\,\partial_\mu\frac{\delta}{\delta J^a_\mu} + g\beta_a\frac{\delta}{\delta\nu_a} - \mathrm{i}g\bar{\chi}\frac{\delta}{\delta\bar{\lambda}} - \mathrm{i}g\frac{\delta}{\delta\lambda}\chi\right]W[J,\alpha,\beta,\chi,\bar{\chi},\kappa,\nu,\lambda,\bar{\lambda}] = 0}$$

This is the Slavnov–Taylor identity.

From the identity one may obtain relations between different types of Green's functions by applying derivatives with respect to the sources and setting them to zero afterwards. This procedure is equivalent to the implication that, since the generating functional W is invariant under BRST transformations, so are the Green's function obtained from it

$$\delta W = 0 \implies \delta G^{(n)} = 0$$

Example. Consider a two-point Green's function for the gauge field

$$(G^{ab}_{\mu\nu})^{(2)}(x) = \langle 0| \mathcal{T}\{A^a_\mu(x) A^b_\nu(0)\} |0\rangle$$

If the BRST symmetry holds after quantization, one has to find

$$\delta G^{(2)}(x) = 0$$

Explicitly calculating the variation gives

$$\begin{aligned}\delta(G^{ab}_{\mu\nu})^{(2)}(x) &= \delta \langle 0| \mathcal{T}\{A^a_\mu(x) A^b_\nu(0)\} |0\rangle \\ &= \langle 0| \mathcal{T}\{\delta A^a_\mu(x)\, A^b_\nu(0)\} |0\rangle + \langle 0| \mathcal{T}\{A^a_\mu(x)\, \delta A^b_\nu(0)\} |0\rangle \\ &= \langle 0| \mathcal{T}\{\omega (D_\mu \sigma)^a(x)\, A^b_\nu(0)\} |0\rangle + \langle 0| \mathcal{T}\{A^a_\mu(x) \omega (D_\nu \sigma)^b(0)\} |0\rangle \\ &= 0\end{aligned}$$

This implies that the two addenda are related.

Example. Consider a four-point Green's function

$$G^{(4)} = \langle 0| \mathcal{T}\{\rho^a A^a_\mu \bar{\psi}\psi\} |0\rangle$$

The variation is

$$\begin{aligned}\delta G^{(4)} &= -\frac{\mathrm{i}}{\xi}\omega \big\langle \mathcal{T}\{\partial_\nu A^\nu_a\, A^a_\mu \bar{\psi}\psi\}\big\rangle - \omega \big\langle \mathcal{T}\{\rho^a (D_\mu\sigma)^a \bar{\psi}\psi\}\big\rangle \\ &\quad + \mathrm{i}g\omega \big\langle \mathcal{T}\{\rho^a A^a_\mu \bar{\psi}(T^b\sigma^b)\psi\}\big\rangle + \mathrm{i}g\omega \big\langle \mathcal{T}\{\rho^a A^a_\mu \bar{\psi}(T^b\sigma^b)\psi\}\big\rangle \\ &= 0\end{aligned}$$

This linear combination of different Green's function has to be zero.

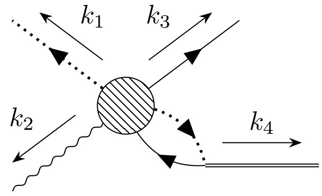
23.2 Unitarity

From the relation in the above example, one may derive physically important identities[30]. Thanks to the LSZ reduction formula, one may compute scattering amplitudes with an integral expression of the kinetic operators applied to a Green's function (which can be a linear combination of different kinds of Green's functions). The LSZ formula gives a non-trivial expression for a scattering amplitude when the Green's function contains propagators that simplify the kinetic terms. This is necessary since external particles are on-shell: the kinetic terms are zero, while the propagators diverge. One then would like to understand which terms in a Green's function contribute.

Consider the above example. When setting the external particles on-shell, the terms in the second line do not contribute. The reasoning is the following. Consider the Green's function

$$\big\langle \mathcal{T}\{\rho(x_1) A_\mu(x_2) \bar{\psi}(x_3) T^a (\sigma^a \psi)(x_4)\big\rangle$$

Notice that $\sigma^a\psi$ is composite field. In momentum space, there are four momenta associated to the above Green's function



[30]Since they come from the Slavnov–Taylor identity they are referred to as (being part of the) Slavnov–Taylor identities.

Since the momentum k_4 splits into the sum of two momenta, one never gets a propagator like

$$\frac{1}{\not{k}_4 - m}$$

which cancels the associated kinetic term of the form $\not{k}_4 - m$ in the LSZ reduction formula. Since, on-shell it (formally) holds $\not{k}_4 - m = 0$, then this Green's function does not contribute on-shell to the scattering amplitude. In general, only the terms linear in each field survive.

Therefore, on-shell the four-point Green's function above gives

$$\frac{\mathrm{i}}{\xi}\langle\mathcal{T}\{\partial_\nu A^\nu_a\, A^a_\mu \bar{\psi}\psi\}\rangle + \langle\mathcal{T}\{\rho_a (D_\mu\sigma)_a \bar{\psi}\psi\}\rangle = 0$$

Since the above discussion also applies for the composite operator σA_μ, then the covariant derivative can be traded for the ordinary derivative

$$\frac{\mathrm{i}}{\xi}\langle\mathcal{T}\{\partial_\nu A^\nu_a\, A^a_\mu \bar{\psi}\psi\}\rangle + \langle\mathcal{T}\{\rho_a\, \partial_\mu\sigma_a\, \bar{\psi}\psi\}\rangle = 0$$

In momentum space, the derivatives become momenta. In the 't Hooft–Feynman gauge, $\xi = 1$, one may define

$$\mathrm{i}T^{ab}_{\nu\mu} \equiv \langle 0|\, \mathcal{T}\{A^a_\nu A^b_\mu \bar{\psi}\psi\}\, |0\rangle \ , \quad \mathrm{i}S^{ab} \equiv \langle 0|\, \mathcal{T}\{\rho^a\sigma^b\bar{\psi}\psi\}\, |0\rangle$$

[r] source?
With these definitions, the above Slavnov–Taylor identity in momentum space becomes

$$\boxed{k_1^\nu T^{ab}_{\nu\mu} = \mathrm{i}S^{ab}(k_2)_\mu}$$

Multiplying by k_2^μ gives

$$\boxed{k_1^\nu T^{ab}_{\nu\mu} k_2^\mu = 0}$$

because $(k_2)^2 = 0$ on-shell.

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Optical theorem. From these two identities, one may show that unitarity is preserved. Unitarity is equivalent to the conservation of probability which implies that the S-matrix is unitary. The scattering matrix enables one to write

$$|f\rangle = S\,|i\rangle$$

The unitarity of the S-matrix can be written as

$$SS^\dagger = I \implies S_{ac}S^*_{bc} = \delta_{ab}$$

The scattering matrix can be written as

$$S_{ab} = \delta_{ab} + \mathrm{i}(2\pi)^4\delta^{(4)}(p_a - p_b)T_{ab}$$

where T_{ab} is the transition amplitude. Inserting this expression into the unitarity constraint, one obtains the optical theorem

$$\boxed{\operatorname{Im} T_{ab} = \frac{1}{2}T_{ac}T^*_{bc}(2\pi)^4\delta^{(4)}(p_a - p_c)}$$

The imaginary part of a transition amplitude is the product of the transition amplitudes associated to the transitions to all possible intermediate physical states.

The optical theorem is satisfied through the two Slavnov–Taylor identities above. Consider a fermion-fermion $\bar{\psi}\psi \to \bar{\psi}\psi$ scattering in a Yang–Mills theory with intermediate states being two gauge bosons (and also ghosts). To compute the imaginary part of the transition amplitude, one considers the spinor, vector and ghost propagators. These last two are

$$\Delta^{ab}_{\mu\nu}(k) = -\frac{\delta^{ab}\eta_{\mu\nu}}{k^2 + \mathrm{i}\varepsilon}\,, \quad \Delta^{ab}(k) = \frac{\delta^{ab}}{k^2 + \mathrm{i}\varepsilon}$$

One employs the Sokhotski–Plemelj theorem

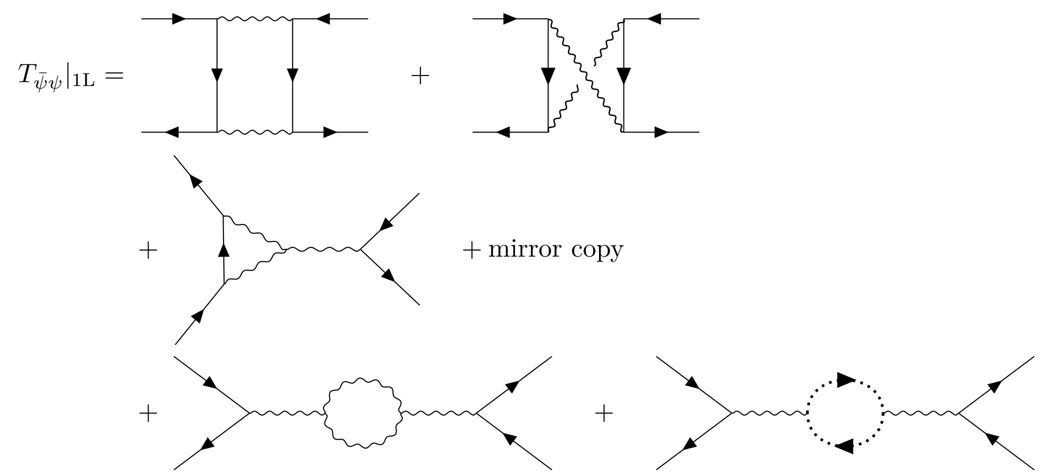
$$\lim_{\varepsilon \to 0^+} \frac{1}{k^2 + \mathrm{i}\varepsilon} = P\frac{1}{k^2} - \mathrm{i}\pi\delta(k^2)$$

where P is the Cauchy principal value. Close to the k^0 singularity, one has

$$\operatorname{Im}\Delta^{ab}_{\mu\nu} = \pi\eta_{\mu\nu}\delta^{ab}\delta(k^2)\theta(k^0)\,, \quad \operatorname{Im}\Delta^{ab} = -\pi\delta^{ab}\delta(k^2)\theta(k^0)$$

To compute the imaginary part of a scattering amplitude (i.e. the left-hand side of the optical theorem), one replaces the propagators in the intermediate states by their imaginary parts (i.e. the theorem above) and multiplies them by the on-shell (intermediate) scattering amplitudes. This is the Cutkosky rule[31]. The right-hand side of the optical theorem is obtained by cutting the internal propagators of a diagram and then multiplying together the on-shell amplitudes obtained from the cuts (notice that amplitudes with ghosts in the final state do not contribute since they are not physical). For more, see Cheng, p. 269, last paragraph.

Example. The one-loop contributions to the fermion-fermion scattering are



To compute only the imaginary part, one has to cut the internal propagators. The imaginary part is the product of tree-level amplitudes (see Peskin for proof). In the first five diagrams, there are products of amplitudes of the type $\bar\psi\psi AA$ which corresponds to the transition amplitude T, while the last diagram involves the amplitude $\bar\psi\psi\rho\sigma$ which corresponds to the scattering amplitude S. Therefore, the left-hand side of the optical theorem is

$$\operatorname{Im} T_{\bar\psi\psi} = \int \mathrm{d}\phi_2 \left[\frac{1}{2}T^{ab}_{\mu\nu}(T^*)^{ab}_{\rho\sigma}\eta^{\mu\rho}\eta^{\nu\sigma} - S^{ab}(S^*)^{ab}\right]$$

where the measure $\mathrm{d}\phi_2$ denotes the massless two-particle phase space. The minus comes from cutting a fermionic loop.

For the right-hand side of the optical theorem, the intermediate states cannot be ghosts, but only vector bosons, since the sum is over physical states. One builds the diagrams with these amplitudes

[31] See also Peskin, §7.3, p236, and §16.3.

and sums over all the intermediate physical states. The two diagrams correspond both to the transition amplitude T. In the sum, one has to consider the polarization of the vector bosons. The general classical solution to the massless Klein–Gordon equation for vector bosons is

$$A^\mu(x) = \sum_{r=0}^{3} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} [\varepsilon_r^\mu(\mathbf{k}) a_r(\mathbf{k}) \mathrm{e}^{-\mathrm{i}kx} + \varepsilon_r^\mu(\mathbf{k}) a_r^\dagger(\mathbf{k}) \mathrm{e}^{\mathrm{i}kx}]_{k^0=\omega}$$

where ε_r are the polarization four-vectors. The physical polarization are ε_1 and ε_2. Therefore, the sum over the intermediate physical states involves only the physical polarizations. In the sum there is a product of the polarization vectors. In the end, one obtains a right-hand side equal to

$$\mathrm{RHS} = \frac{1}{2} \int \mathrm{d}\phi_2 \, T_{\mu\nu}^{ab} (T^*)_{\rho\sigma}^{ab} P^{\mu\rho}(k_1) P^{\nu\sigma}(k_2)$$

where one has

$$P^{\mu\rho}(k_j) = \sum_{r=1,2} \varepsilon_r^\mu(\mathbf{k}_j) \varepsilon_r^\rho(\mathbf{k}_j)$$

One may check the validity of the optical theorem for this particular example

$$\mathrm{Im}\, T \stackrel{?}{=} \mathrm{RHS}$$

A useful choice of polarization basis is

$$\varepsilon_0^\mu = (1, 0, 0, 0) = n^\mu \,, \quad \varepsilon_i^\mu = (0, \boldsymbol{\varepsilon}_i)$$

where

$$\boldsymbol{\varepsilon}_i \cdot \boldsymbol{\varepsilon}_j = \delta_{ij} \,, \quad \boldsymbol{\varepsilon}_{1,2} \cdot \mathbf{k} = 0$$

One sets the longitudinal polarization vector to be

$$\varepsilon_3^\mu = \frac{k^\mu - (nk) n^\mu}{[(nk)^2 - k^2]^{\frac{1}{2}}}$$

The first vector ε_0 is the scalar polarization, while the transverse polarizations are $\varepsilon_{1,2}$. The polarization vectors satisfy

$$\sum_{r=0}^{3} \zeta_r \varepsilon_r^\mu(\mathbf{k}) \varepsilon_r^\nu(\mathbf{k}) = -\eta^{\mu\nu} \,, \quad \zeta_r = \begin{cases} -1 \,, & r = 0 \\ 1 \,, & r = 1, 2, 3 \end{cases}$$

Therefore, one has

$$\begin{aligned} P^{\mu\rho}(k_1) &= \sum_{r=1,2} \varepsilon_r^\mu(\mathbf{k}_1) \varepsilon_r^\rho(\mathbf{k}_1) = \sum_{r=0}^{3} \zeta_r \varepsilon_r^\mu(\mathbf{k}_1) \varepsilon_r^\rho(\mathbf{k}_1) + \varepsilon_0^\mu(\mathbf{k}_1) \varepsilon_0^\rho(\mathbf{k}_1) - \varepsilon_3^\mu(\mathbf{k}_1) \varepsilon_3^\rho(\mathbf{k}_1) \\ &= -\eta^{\mu\rho} + \varepsilon_0^\mu(\mathbf{k}_1) \varepsilon_0^\rho(\mathbf{k}_1) - \varepsilon_3^\mu(\mathbf{k}_1) \varepsilon_3^\rho(\mathbf{k}_1) = -\eta^{\mu\rho} + \frac{(nk)(k^\mu n^\rho + n^\mu k^\rho) - k^\mu k^\rho}{(kn)^2} \end{aligned}$$

The second addendum is the contribution of non-physical polarizations. Inserting the above expression into the right-hand side of the optical theorem, and applying the Slavnov–Taylor identities, one finds the left-hand side, $\mathrm{Im}\, T$.

Remark. The optical theorem holds thanks to the Slavnov–Taylor identities which arise from the BRST symmetry being preserved in the quantum theory.

Remark. The non-physical degrees of freedom are exactly compensated by the ghosts. The term SS present in the imaginary part $\mathrm{Im}\, T$ can be rewritten on the right-hand side to see that it balances the non-physical polarizations.

24 Quantum electrodynamics

See Ramond, §8.2 and consider the Euclidean formalism.

24.1 One-loop structure

Consider an abelian gauge theory based on U(1) with one massive Dirac spinor field ψ minimally coupled to the gauge field A^μ. The covariant derivative is[32]

$$D_\mu = \partial_\mu + \mathrm{i}qA_\mu$$

The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(\mathrm{i}\,\not{D} - m)\psi - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

The generating functional is

$$W[J,\eta,\bar{\eta}] = \int [\mathcal{D}A_\mu\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\,\exp\left[\mathrm{i}\int \mathrm{d}^4x\,(\mathcal{L} + J_\mu A^\mu + \bar{\psi}\eta + \bar{\eta}\psi)\right]$$

Wick rotation. See Ramond, §5.2. When going to Euclidean space, one would like to keep the covariant derivative

$$x^0 \equiv -\mathrm{i}x^0_{\mathrm{E}}\,, \quad \partial_0 \equiv \mathrm{i}\,\partial_{0\mathrm{E}}\,, \quad A_0 \equiv \mathrm{i}A_{0\mathrm{E}}$$

The Dirac matrices are rotated as

$$\gamma_0 \equiv \mathrm{i}\gamma_{0\mathrm{E}} \implies \{\gamma^\mu_{\mathrm{E}}, \gamma^\nu_{\mathrm{E}}\} = -2\delta^{\mu\nu}$$

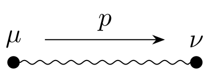
The Euclidean generating functional becomes

$$W_{\mathrm{E}}[J,\eta,\bar{\eta}] = \int [\mathcal{D}A_\mu\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\,\exp\left[-\int \mathrm{d}^4x\,(\mathcal{L}_{\mathrm{E}} - J_\mu A^\mu - \mathrm{i}\bar{\eta}\psi - \mathrm{i}\bar{\psi}\eta)\right]$$

where the Euclidean Lagrangian is[33]

$$\mathcal{L}_{\mathrm{E}} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \bar{\psi}(\not{\partial} + \mathrm{i}m)\psi - \mathrm{i}qA_\mu(\bar{\psi}\gamma^\mu\psi)$$

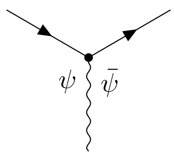
Euclidean Feynman rules. The gauge field propagator in the 't Hooft–Feynman gauge, $\xi = 1$, is

$$\frac{\delta_{\mu\nu}}{p^2} =$$ 

The spinor propagator is

$$-\frac{\mathrm{i}}{\not{p} + m} =$$ 

The vertex is

$$\mathrm{i}qk^\varepsilon\gamma^\mu =$$ 

Each fermion loop acquires a minus sign. One may use Furry's theorem: all odd-fermion loops are zero.

Power counting. The superficial degree of divergence obtained through power counting is

$$D = 4 - E_{\mathrm{B}} - \frac{3}{2}E_{\mathrm{F}}$$

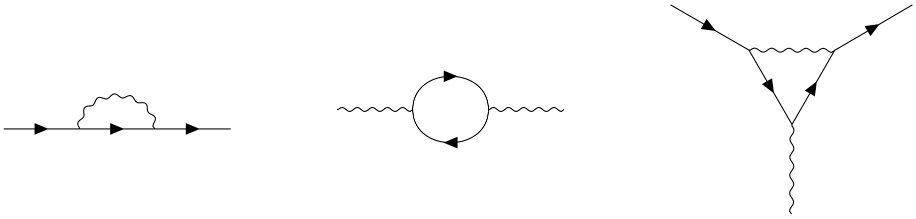
where E_{B} is the number of external boson lines while E_{F} is the number of external fermion lines. Pure vector diagrams, $E_{\mathrm{F}} = 0$, are divergent for $E_{\mathrm{B}} = 2$ and $E_{\mathrm{B}} = 4$: the self-energy of the gauge field and the light-light diagram. For two external fermion lines, $E_{\mathrm{F}} = 2$, the only divergent diagrams have $E_{\mathrm{B}} = 1$ and $E_{\mathrm{B}} = 0$: the vertex correction and the self-energy of the fermion field.

[32] Note that Ramond uses $q = -e$ for electrons, with $e > 0$.

[33] Notice that the partial derivative and the interaction term have different signs. The Euclidean Lagrangian is defined starting from the Euclidean action where one sends $x_{\mathrm{E}} \to -x_{\mathrm{E}}$ in the integral for the derivative. See notes last year, Lecture 14, p. 18.

24.1.1 Regularization

At one-loop, one has to evaluate the diagrams



The light-light diagram is not divergent thanks to gauge invariance (see Peskin, eq10.9). To perform the computations, one has to

- compute the combinatorial factor,
- apply Feynman combining,
- move to spherical coordinates,
- evaluate momentum integrals of the form

$$\int \frac{\mathrm{d}^n l}{(2\pi)^n} \frac{(l^2)^a}{(l^2+D)^b} = \frac{1}{D^{b-a-\frac{n}{2}}} \frac{\Gamma(b-a-{}^n\!/_2)\Gamma(a+{}^n\!/_2)}{(4\pi)^{\frac{n}{2}}\Gamma(b)\Gamma({}^n\!/_2)}$$

  See Ramond, appendix B.

- consider the simple poles of the regularization parameter ε.

One-loop contributions. The fermion self-energy is

$$\Sigma(p) = -\frac{\mathrm{i}q^2}{16\pi^2}\frac{\not{p}+4m}{\varepsilon} + \text{finite}$$

The gauge field self-energy is

$$\Pi_{\mu\nu}(p) = \frac{q^2}{12\pi^2}(p_\mu p_\nu - p^2\delta_{\mu\nu})\frac{1}{\varepsilon} + \text{finite}$$

The vertex function is

$$\Gamma_\mu(p,q) = \frac{\mathrm{i}q^3 k^\varepsilon}{16\pi^2}\gamma_\mu\frac{1}{\varepsilon} + \text{finite}$$

24.1.2 Renormalization

One may apply the BPHZ renormalization. Let the bare quantities be

$$\psi_0 = Z_2^{\frac{1}{2}}\psi\,, \quad A_0^\mu = Z_3^{\frac{1}{2}}A^\mu\,, \quad q_0 = Z_1 Z_2^{-1} Z_3^{-\frac{1}{2}} q k^\varepsilon\,, \quad m_0 = Z_m Z_2^{-1} m\,, \quad \xi_0^{-1} = Z_\xi Z_3^{-1}\xi^{-1}$$

The bare Lagrangian is

$$\begin{aligned}
\mathcal{L}_0 &= \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \bar\psi\gamma^\mu\,\partial_\mu\psi + \mathrm{i}m\bar\psi\psi - \mathrm{i}qA_\mu\bar\psi\gamma^\mu\psi + \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \\
&\quad + (Z_3-1)(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + (Z_2-1)\bar\psi\gamma^\mu\,\partial_\mu\psi + (Z_m-1)\mathrm{i}m\bar\psi\psi \\
&\quad - (Z_1-1)\mathrm{i}qA_\mu\bar\psi\gamma^\mu\psi + (Z_\xi-1)\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \\
&= \mathcal{L}_\mathrm{R} + \mathcal{L}_\mathrm{ct}
\end{aligned}$$

In the following, let $K_j = Z_j - 1$.

Fermion field: mass term. The divergence of the mass term from the fermion self-energy is cancelled by the Feynman rule of the mass counter term $\mathrm{i}mK_m$ to have

$$-\frac{\mathrm{i}}{\varepsilon}\frac{q^2}{4\pi^2}m - \mathrm{i}mK_m = \text{finite} \implies K_m = -\frac{q^2}{4\pi^2}\frac{1}{\varepsilon} + \text{finite} \implies \boxed{Z_m = 1 - \frac{q^2}{4\pi^2}\frac{1}{\varepsilon} + \text{finite}}$$

Remember that the minus sign of the counter term comes from the minus sign of the action in the exponential of the generating functional.

Fermion field: kinetic term. From the self-energy one has

$$-\frac{\mathrm{i}}{\varepsilon}\frac{q^2}{16\pi^2}\not{p} - \mathrm{i}K_2\not{p} = \text{finite} \implies K_2 = -\frac{q^2}{16\pi^2}\frac{1}{\varepsilon} + \text{finite} \implies \boxed{Z_2 = 1 - \frac{q^2}{16\pi^2}\frac{1}{\varepsilon} + \text{finite}}$$

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Gauge field. [r] From the counter term Lagrangian, one reads

$$\begin{aligned}
\mathcal{L}_{\text{ct}} &\supset (Z_3 - 1)\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{1}{2\xi}(Z_\xi - 1)(\partial_\mu A^\mu)^2 \\
&= \frac{1}{2}K_3\,\partial_\mu A_\nu\,\partial^\mu A^\nu - \frac{1}{2}K_3\,\partial_\mu A_\nu\,\partial^\nu A^\mu + \frac{1}{2\xi}K_\xi\,\partial_\mu A^\mu\,\partial_\nu A^\nu \\
&= -\frac{1}{2}K_3 A^\nu\,\Box\,A_\nu - \frac{1}{2}\left[\frac{K_\xi}{\xi} - K_3\right]A^\nu\,\partial_\mu\partial_\nu A^\mu + \text{4-div}
\end{aligned}$$

At the third line, one has integrated all terms by parts. Remembering the minus sign coming from the action in the exponential of the generating functional, one finds

$$-S \supset \frac{1}{2}K_3 A^\nu\,\Box\,A_\nu + \frac{1}{2}\left[\frac{K_\xi}{\xi} - K_3\right]A^\nu\,\partial_\mu\partial_\nu A^\mu$$

In momentum space, this becomes

$$\frac{1}{2}K_3 A_\nu\,(-p^2)A^\nu + \frac{1}{2}\left[\frac{K_\xi}{\xi} - K_3\right]A^\nu(-p_\mu p_\nu)A^\mu$$

The counter term contribution is then

$$-\frac{1}{2}2\left[\delta_{\mu\nu}K_3 p^2 + \left(\frac{K_\xi}{\xi} - K_3\right)p_\mu p_\nu\right]$$

where the 2 is a symmetry factor [r]. Therefore, the total contribution is

$$\frac{q^2}{12\pi^2}(p_\mu p_\nu - p^2\delta_{\mu\nu})\frac{1}{\varepsilon} - \left[\delta_{\mu\nu}K_3 p^2 + \left(\frac{K_\xi}{\xi} - K_3\right)p_\mu p_\nu\right] = \text{finite}$$

The terms proportional to p^2 give

$$-p^2\delta_{\mu\nu}\left[\frac{q^2}{12\pi^2}\frac{1}{\varepsilon} + K_3\right] = \text{fin} \implies K_3 = -\frac{q^2}{12\pi^2}\frac{1}{\varepsilon} + \text{fin} \implies \boxed{Z_3 = 1 - \frac{q^2}{12\pi^2}\frac{1}{\varepsilon} + \text{finite}}$$

while the ones proportional to $p_\mu p_\nu$ give

$$\left[\frac{q^2}{12\pi^2}\frac{1}{\varepsilon} - \left(\frac{K_\xi}{\xi} - K_3\right)\right]p_\mu p_\nu = \text{finite} \implies \frac{q^2}{12\pi^2}\frac{1}{\varepsilon} + K_3 - K_\xi = \text{finite}\,, \quad \xi = 1$$

In the 't Hooft–Feynman gauge, the above relation implies no gauge parameter renormalization

$$K_\xi = \text{finite} = 0 \implies Z_\xi = 1$$

In this gauge, the divergences of the two terms of self-energy $\Pi_{\mu\nu}$ are both cancelled by K_3.

Vertex function. The vertex counter term is

$$-S \supset K_1 \mathrm{i} q k^\varepsilon A_\mu \bar{\psi}\gamma^\mu\psi$$

which leads to the Feynman rule

$$K_1 \mathrm{i} q k^\varepsilon \gamma^\mu$$

From this one finds

$$K_1 = -\frac{q^2}{16\pi^2}\frac{1}{\varepsilon} + \text{finite} \implies \boxed{Z_1 = 1 - \frac{q^2}{16\pi^2}\frac{1}{\varepsilon} + \text{finite}}$$

Remark. Notice that at one-loop, the Ward identity of QED holds: $Z_1 = Z_2$.

24.1.3 Beta function

The coupling constant is the elementary charge $g = |q|$. In the minimal subtraction scheme (i.e. no finite parts), one has

$$g_0 = Z_1 Z_2^{-1} Z_3^{-\frac{1}{2}} g k^\varepsilon = Z_3^{-\frac{1}{2}} g k^\varepsilon = \left[1 - \frac{g^2}{12\pi^2}\frac{1}{\varepsilon}\right]^{-\frac{1}{2}} g k^\varepsilon = \left[1 + \frac{g^2}{24\pi^2}\frac{1}{\varepsilon} + o(g^2)\right] g k^\varepsilon$$

From this, one reads the coefficient of the simple pole

$$a_1 = \frac{g^3}{24\pi^2}$$

Recall that for the $\lambda\varphi^4$ theory it holds

$$\lambda_0 = k^{2\varepsilon}\left[\lambda + \sum_{r=1}^\infty \frac{a_r(\lambda)}{\varepsilon^r}\right] \implies \beta(\lambda) = 2\left[\lambda\,\mathrm{d}_\lambda - 1\right] a_1$$

where the factor of 2 comes from the 2 in the exponent of the energy scale. For QED, one has

$$\beta(g) = \left[g\,\mathrm{d}_g - 1\right] a_1 = 2a_1 = \frac{g^3}{12\pi^2}$$

The beta function enters in the renormalization group flow equation

$$k\,\mathrm{d}_k g = \beta(g)$$

Since the coupling constant is positive, so is the beta function and as such at higher energies the coupling constant gets stronger.

Landau pole. One may solve the above equation. One considers $g^2(k)$. The equation is

$$k\,\mathrm{d}_k g^2 = 2gk\,\mathrm{d}_k g = \frac{g^4}{6\pi^2} \implies \frac{1}{g^4}\,\mathrm{d}_k g^2 = \frac{1}{6\pi^2 k}$$

Integrating from $g(k = k_0) = g_0$ to g gives

$$\boxed{g^2(k) = \frac{g_0^2}{1 - \frac{g_0^2}{6\pi^2}\ln\frac{k}{k_0}}}$$

The running coupling constant of QED has a Landau pole at

$$k = k_0 \exp\left[\frac{6\pi^2}{g_0^2}\right]$$

When getting close to the pole, the coupling constant gets bigger than 1 and perturbation theory cannot be applied, so this result should not be trusted in such region.

There is no asymptotic freedom in QED because the coupling constant grows stronger with the energy scale. This lack of asymptotic freedom implies a well-defined notion of asymptotic states (of the S-matrix): at large distances, which correspond to low energies, the coupling constant is weak [r].

24.2 Ward identities

See Ramond, §8.3. Since QED is an abelian theory, one may use the results of non-abelian theories and set the structure constants to zero, $c^{abc} = 0$. The generating functional is

$$W[J,\eta,\bar\eta,\alpha,\beta] = \int [\mathcal{D}A_\mu\,\mathcal{D}\psi\,\mathcal{D}\bar\psi]\,\exp\left[-\int \mathrm{d}^4x\,(\mathcal{L} - J_\mu A^\mu - \mathrm{i}\bar\eta\psi - \mathrm{i}\bar\psi\eta)\right]$$
$$\times \int [\mathcal{D}\rho\,\mathcal{D}\sigma]\,\exp\left[-\int \mathrm{d}^4x\,(\mathcal{L}_{\mathrm{gh}} - \alpha\rho - \beta\sigma)\right]$$

Since ghosts decouple, one introduces the ghost fields with Lagrangian[34]

$$\mathcal{L}_{\mathrm{gh}} = \rho\,\Box\,\sigma$$

The total Lagrangian

$$\mathcal{L} = \mathcal{L}_{\mathrm{g}} + \mathcal{L}_{\mathrm{gf}} + \mathcal{L}_{\mathrm{gh}} + \mathcal{L}_\psi$$

is invariant under BRST transformations[35]

$$\begin{aligned}
\delta A_\mu &= \omega\,\partial_\mu\sigma \\
\delta\rho &= -\frac{\omega}{\xi}(\partial_\mu A^\mu) \\
\delta\sigma &= 0 \\
\delta\psi &= \mathrm{i}g\omega\sigma\psi \\
\delta\bar\psi &= -\mathrm{i}g\omega\sigma\bar\psi
\end{aligned}$$

Recall that $\mathcal{L}_{\mathrm{g}} + \mathcal{L}_\psi$ is invariant thanks to gauge invariance. The remaining terms are also invariant

$$\delta(\mathcal{L}_{\mathrm{gf}} + \mathcal{L}_{\mathrm{gh}}) = 0$$

With these transformations, one may obtain the Ward identity imposing that

$$\delta W[J,\eta,\bar\eta,\alpha,\beta] = 0$$

where one has

$$W[J,\eta,\bar\eta,\alpha,\beta] = \int [\mathcal{D}A_\mu\,\mathcal{D}\psi\,\mathcal{D}\bar\psi\,\mathcal{D}\rho\,\mathcal{D}\sigma]\mathrm{e}^{-S}\exp\left[\int \mathrm{d}^4x\,(J_\mu A^\mu + \mathrm{i}\bar\eta\psi + \mathrm{i}\bar\psi\eta + \alpha\rho + \beta\sigma)\right]$$

The variation of the generating functional gives

$$\begin{aligned}
0 &= J_\mu\omega\,\partial_\mu\sigma + \mathrm{i}\bar\eta(\mathrm{i}g\omega\sigma\psi) + \mathrm{i}(-\mathrm{i}g\omega\sigma\bar\psi)\eta - \alpha\frac{\omega}{\xi}(\partial_\mu A^\mu) \\
&= J_\mu\omega\,\partial_\mu\sigma - g\omega\sigma\bar\eta\psi + g\omega\sigma\bar\psi\eta + \omega\frac{\alpha}{\xi}(\partial_\mu A^\mu)
\end{aligned}$$

It is useful to rewrite this Ward identity in terms of the effective action Γ. In the $\lambda\varphi^4$ theory, one has

$$\Gamma[\varphi_c] = -Z[J] + \int \mathrm{d}^4x\,J\varphi_c\,, \quad J(x) = \delta_{\varphi(x)}\Gamma[\varphi]$$

Doing the same for QED, one has

$$\Gamma[A_\mu,\psi,\bar\psi,\rho,\sigma] = -Z[J,\eta,\bar\eta,\alpha,\beta] + \int \mathrm{d}^4x\,[J_\mu A^\mu + \mathrm{i}\bar\eta\psi + \mathrm{i}\bar\psi\eta + \alpha\rho + \beta\sigma]$$

where all the fields are classical and a subscript c is understood. One has

$$J_\mu = \delta_{A^\mu}\Gamma\,, \quad \mathrm{i}\eta = \delta_{\bar\psi}\Gamma\,, \quad \mathrm{i}\bar\eta = -\delta_\psi\Gamma\,, \quad \alpha = -\delta_\rho\Gamma\,, \quad \beta = -\delta_\sigma\Gamma$$

[r] Imposing the invariance under the BRST transformations of the effective action Γ above gives the Ward identity

$$\boxed{\delta_{A_\mu}\Gamma\,\partial_\mu\sigma + \mathrm{i}g\,\delta_\psi\Gamma\,\sigma\psi - \mathrm{i}g\sigma\bar\psi\,\delta_{\bar\psi}\Gamma - \frac{1}{\xi}(\partial_\mu A^\mu)\,\delta_\rho\Gamma = 0}$$

where one recalls that $\delta Z = 0$.

[34]The original minus sign is the one in the exponential. One then renames $\mathrm{i}\rho \to \rho$.

[35]See previous note. Notice a missing factor of i in the transformation of ρ and that $(\partial_\mu A^\mu)_{\mathrm{M}} \to -(\partial_\mu A_\mu)_{\mathrm{E}}$ which preserves the minus sign when performing a Wick rotation.

Effective action of QED. Recall that

$$\mathrm{e}^{-\Gamma} = \int [\mathcal{D}A_\mu\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \mathcal{D}\rho\, \mathcal{D}\sigma]\, \exp\left[-\int \mathrm{d}^4x\, (\mathcal{L}_{\mathrm{QED}} + \mathcal{L}_{\mathrm{gh}})\right]$$

The effective action is

$$\begin{aligned}\Gamma[A_\mu, \psi, \bar{\psi}, \rho, \sigma] = \int \mathrm{d}^4x\, \mathrm{d}^4y \bigg[& \rho(x)\Delta^{-1}(x-y)\sigma(y) \\ &+ \frac{1}{2}A_\mu(x)\Delta^{-1}_{\mu\nu}(x-y)A_\nu(y) + \bar{\psi}(x)S^{-1}(x-y)\psi(y) \\ &+ g\int \mathrm{d}^4z\, \bar{\psi}(x)A^\rho(y)\Gamma_\rho(x,y,z)\psi(z) + o(g)\bigg]\end{aligned}$$

Since the ghosts do not interact with the fields, then their inverse propagator $\Delta^{-1} = \Box$ gets no loop corrections. The other propagators and the vertex function are all corrected.

Kinetic term of the gauge field. One may use the Ward identity to find the general expression of the kinetic term of the gauge field $\Delta^{-1}_{\mu\nu}$. One may set the fermion fields to zero. After an integration by parts of the first term, the Ward identity becomes

$$0 = \left[\partial_\mu\, \delta_{A_\mu}\Gamma\right]\sigma + \frac{1}{\xi}(\partial_\mu A^\mu)\, \delta_\rho\Gamma = \partial_\mu(\Delta^{-1}_{\mu\nu}A^\nu)\sigma + \frac{1}{\xi}(\partial_\mu A^\mu)\, \Delta^{-1}\sigma$$

Recalling that $\Delta^{-1} = \Box$, in momentum space one has

$$\mathrm{i}k_\mu \Delta^{-1}_{\mu\nu}A^\nu\sigma + \frac{1}{\xi}\mathrm{i}k_\mu A^\mu(-k^2)\sigma = 0 \implies \boxed{k^\mu\Delta^{-1}_{\mu\nu}(k) - \frac{1}{\xi}k_\nu k^2 = 0}$$

This constraint must hold at every loop order.

One may obtain the general form of the kinetic term. Starting from the ansatz

$$\Delta^{-1}_{\mu\nu}(k) = A(k^2)\delta_{\mu\nu} + B(k^2)k_\mu k_\nu$$

From the constraint above, one obtains

$$A(k^2) = [1 - B(k^2)]k^2\,, \quad \xi = 1$$

Therefore

$$\boxed{\Delta^{-1}_{\mu\nu}(k) = k^2\delta_{\mu\nu} - (\delta_{\mu\nu}k^2 - k_\mu k_\nu)B(k^2)}$$

When computing quantum corrections to the kinetic term, one has to determine $B(k^2)$ which is always multiplied by the projection operator

$$\Pi_{\mu\nu} = \delta_{\mu\nu}k^2 - k_\mu k_\nu$$

that satisfies

$$\Pi_{\mu\nu}\Pi_{\nu\rho} = \Pi_{\mu\rho}k^2$$

The projector is not invertible, but the first addendum in the kinetic term Δ^{-1} comes from gauge-fixing and makes the kinetic term invertible.

Renormalization functions. Another application of the Ward identity is the following. Let $A_\mu = 0$. Then the Ward identity is

$$\delta_{A_\mu}\Gamma|_{A_\mu=0}\, \partial_\mu\sigma + \mathrm{i}q\, \delta_\psi\Gamma\, \sigma\psi + \mathrm{i}q\bar{\psi}\sigma\, \delta_{\bar{\psi}}\Gamma = 0$$

One gets a non-trivial contribution from the vertex function. Inserting the effective action gives

$$\begin{aligned}0 = g\int \mathrm{d}^4x\, \mathrm{d}^4z\, \bar{\psi}(x)\Gamma_\rho(x,w,z)\psi(z)\, \partial_\rho\sigma(w) - \mathrm{i}g\int \mathrm{d}^4x\, \bar{\psi}(x)S^{-1}(x-w)\sigma(w)\psi(w) \\ + \mathrm{i}g\int \mathrm{d}^4y\, \bar{\psi}(w)\sigma(w)S^{-1}(w-y)\psi(y)\end{aligned}$$

In momentum space, one has

$$\boxed{S^{-1}(p) - S^{-1}(q) = (q-p)^\rho \Gamma_\rho(p, q-p, q)}$$

[r] check. Knowing that S^{-1} is renormalized by Z_2 and Γ is renormalized by Z_1, this Ward identity gives

$$\boxed{Z_1 = Z_2}$$

Lecture 17

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25 Yang–Mills gauge theories

See Ramond, §§8.5, 8.6. The Euclidean Lagrangian is

$$\begin{aligned}\mathcal{L} = \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \frac{1}{2\xi}(\partial_\mu A^\mu_a)^2 + \mathrm{i}\,\partial_\mu \bar{c}^a\,\partial_\mu c^a - \frac{\mathrm{i}}{2} g k^\varepsilon c^{abc} \bar{c}^a \overleftrightarrow{\partial}_\mu c^b\, A^c_\mu \\ - \frac{\mathrm{i}}{2} g k^\varepsilon c^{abc} \bar{c}^a c^b\, \partial_\mu A^c_\mu + \bar{\psi}(\not{\partial} + \mathrm{i}m)\psi + \mathrm{i} g k^\varepsilon A^a_\mu \bar{\psi}_i \gamma^\mu T^a_{ij} \psi_j\end{aligned}$$

[r] check

Ghost Lagrangian. The ghost Lagrangian is

$$\mathcal{L}_{\mathrm{gh}} = \mathrm{i}\partial_\mu \bar{c}^a\, D_\mu c^a\,, \quad D_\mu = \partial_\mu + \mathrm{i} g T^a A^a_\mu$$

From the second addendum in the covariant derivative — recalling that it acts in the adjoint representation —, one obtains

$$\begin{aligned}-\partial_\mu \bar{c}^a\, c^{abc} A^b_\mu c^c &= \partial_\mu \bar{c}^a\, c^{abc} c^b A^c_\mu \\ &= \frac{1}{2}\partial_\mu \bar{c}^a\, c^{abc} c^b A^c_\mu + \frac{1}{2}\partial_\mu \bar{c}^a\, c^{abc} c^b A^c_\mu \\ &= \frac{1}{2} c^{abc}(\partial_\mu \bar{c}^a\, c^b - \bar{c}^a\, \partial_\mu c^b) A^c_\mu - \frac{1}{2} c^{abc} \bar{c}^a c^b\, \partial_\mu A^c_\mu + \text{4-div} \\ &= -\frac{1}{2} c^{abc} \bar{c}^a \overleftrightarrow{\partial}_\mu c^b\, A^c_\mu - \frac{1}{2} c^{abc} \bar{c}^a c^b\, \partial_\mu A^c_\mu + \text{4-div}\end{aligned}$$

At the second line, one has integrated by parts the second addendum.

One defines

$$\bar{c}^a \overleftrightarrow{\partial}_\mu c^b \equiv \bar{c}^a\, \partial_\mu c^b - \partial_\mu \bar{c}^a\, c^b\,, \quad A \equiv -\frac{\mathrm{i}}{2} g k^\varepsilon c^{abc} \bar{c}^a c^b\, \partial_\mu A^c_\mu$$

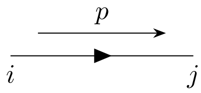
One notices that the remaining term A is proportional to the gauge term [r] and is zero when in the Lorenz gauge.

25.1 Feynman rules

See also §8.1 or appendix C. The Euclidean Feynman rules are as follows.

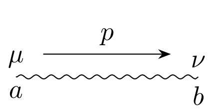
Propagators. The fermion propagator is

$$-\mathrm{i}\frac{\delta_{ij}}{\not{p} + m} =$$



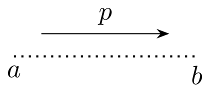
The gauge field propagator is

$$\frac{\delta_{ab}}{p^2}\left[\delta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right] =$$



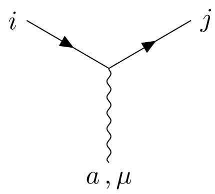
where the a, b indices are the indices in the adjoint representation $a, b = 1, \ldots, n^2 - 1$. The ghost propagator is

$$-\mathrm{i}\frac{\delta_{ab}}{p^2} =$$

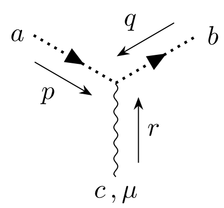


Vertices. Consider the vertices from the expansion of the exponential e^{-S}. The fermion-boson vertex is

$$-\mathrm{i}gk^{\varepsilon}\gamma_\mu (T^a)_{ji} =$$



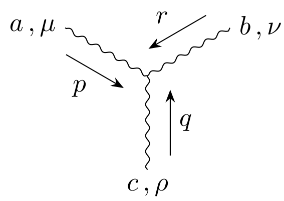
The ghost-boson vertex is



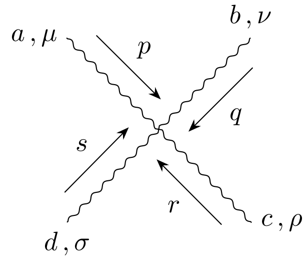
$$= \begin{cases} -gk^{\varepsilon} c^{abc} q_\mu \,, & \text{with } A \\ \dfrac{1}{2} gk^{\varepsilon} c^{abc} (p-q)_\mu \,, & \text{without } A \end{cases}$$

with all momenta entering. The adjoint indices have to be put in the same order every time: in this case clockwise. The three-point self-coupling of the gauge field is

$$-\mathrm{i}gk^{\varepsilon} c^{abc} [(r-q)_\mu \delta_{\nu\rho} + (q-p)_\rho \delta_{\mu\nu} + (p-r)_\nu \delta_{\rho\mu}] =$$



The four-point vertex is



$$\begin{aligned} = -g^2 k^{2\varepsilon} [& c^{abe} c^{cde} (\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\nu\rho}\delta_{\mu\sigma}) \\ & + c^{cbe} c^{ade} (\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\nu\mu}\delta_{\rho\sigma}) \\ & + c^{dbe} c^{cae} (\delta_{\sigma\rho}\delta_{\nu\mu} - \delta_{\nu\rho}\delta_{\mu\sigma})] \end{aligned}$$

25.2 Renormalizability

Power counting. Consider a diagram with

- L loops,
- V_1 first-kind vertices,
- V_2 second-kind vertices,
- V_3 third-kind vertices,
- V_4 fourth-kind vertices,

- I internal gauge propagators A^μ,
- i internal fermion propagators ψ,
- I_g internal ghost propagators c,
- E external gauge lines A_μ,
- e external fermion lines ψ,
- E_g external ghost lines c.

The number of loops is given by

$$L = I + i + I_g - (V_1 + V_2 + V_3 + V_4) + 1$$

There are various constraints

$$\begin{aligned} 2V_1 &= 2i + e \implies i = V_1 - \frac{1}{2}e \\ 2V_2 &= 2I_g + E_g \\ V_1 + V_2 + 3V_3 + 4V_4 &= 2I + E \end{aligned}$$

The superficial degree of divergence is

$$\begin{aligned} D &= 4L + V_2 + V_3 - 2I - i - 2I_g \\ &= 4(I + i + I_g - V_1 - V_2 - V_3 - V_4 + 1) - 2I - i - 2I_g + V_2 + V_3 \\ &= 4 + 2I + 3i + 2I_g - 4V_1 - 3V_2 - 3V_3 - 4V_4 \\ &= 4 - E - \frac{3}{2}e - E_g \end{aligned}$$

Notice that it does not depend on the internal lines, but only on the external lines: there is a finite number of topologies of Green's functions that diverge.

Topologies. There are five different topologies:

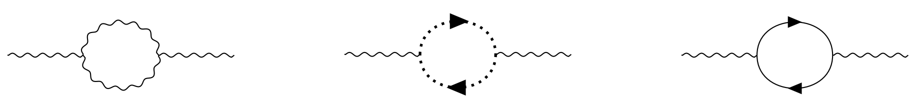
1. Consider pure gauge, $e = E_g = 0$. Then $d = 4 - E \geq 0$ for $E = 1, 2, 3, 4$. Tadpoles, $E = 1$, are not considered [r].
2. Consider pure fermions, $E = E_g = 0$. Then $D = 4 - \frac{3}{2}e \geq 0$. Since fermions must appear as bilinears, then $e = 2$.
3. Let $E = e = 0$. Then $D = 4 - E_g \geq 0$ for $E_g = 2, 4$ since ghosts are also Grassmann-odd fields and must come in pairs. There is no four-point ghost vertex, so if the corresponding diagram diverges, it is a problem; however it is not due to the Ward identity.
4. Let $e = 2$ and $E_g = 0$. Then $D = 1 - E \geq 0$ for $E = 1$.
5. Let $e = 0$ and $E_g = 2$. Then $D = 2 - E \geq 0$ for $E = 1$.

The topologies resemble terms in the Lagrangian so they can be renormalized through counter terms. Yang–Mills theories are renormalizable. These theories describe three of the four fundamental interactions.

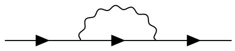
25.3 One-loop regularization

The renormalization is done after using dimensional regularization. This implies that massless tadpoles are zero. The one-loop diagrams are the following.

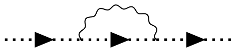
Gauge propagator. The diagrams contributing to the gauge propagator are



Fermion propagator. The diagram contributing to the fermion propagator is

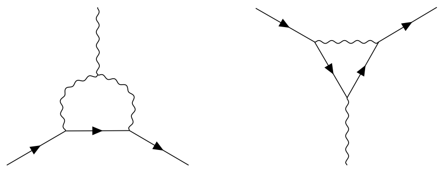


Ghost propagator. The diagram contributing to the ghost propagator is

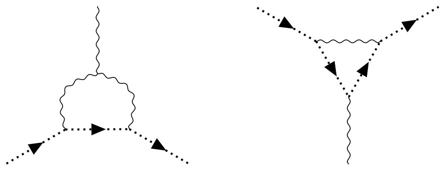


Three-point vertex. There are eight diagrams

$\bar\psi\psi A$



$\bar c c A$



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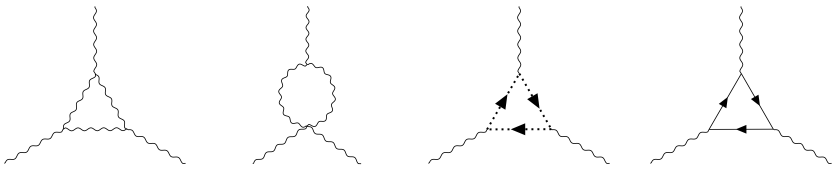
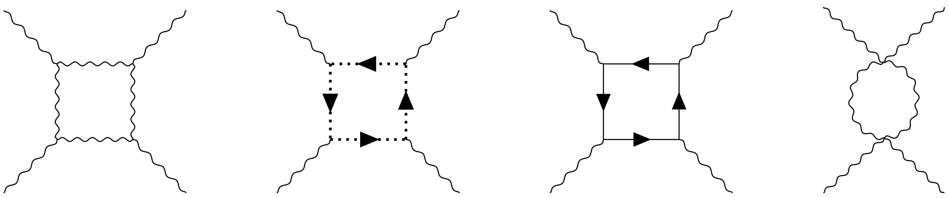


Four-point vertex. There are four diagrams



BPHZ renormalization. Starting from the bare Lagrangian, one has

$$\begin{aligned}
\mathcal{L}_0 &= \frac{1}{4}(\partial_\mu A^a_{0\nu} - \partial_\nu A^a_{0\mu})^2 - g_0' c^{abc}\,\partial_\mu A^a_{0\nu}\,A^b_{0\mu}A^c_{0\nu} + \frac{1}{4}g_0''^2 c^{abc}c^{ade}A^b_{0\mu}A^c_{0\nu}A^d_{0\mu}A^e_{0\nu} \\
&\quad + \frac{1}{2\xi_0}(\partial_\mu A^a_{0\mu})^2 + \mathrm{i}\,\partial_\mu \bar c^a_0\,\partial_\mu c^a_0 - \frac{\mathrm{i}}{2}g_0''' c^{abc}\bar c^a_0 \overleftrightarrow{\partial}_\mu c^b_0\,A^c_{0\mu} - \frac{\mathrm{i}}{2}g_0^{(4)}k^\varepsilon c^{abc}\bar c^a_0 c^b_0\,\partial_\mu A^c_{0\mu} \\
&\quad + \bar\psi_0\,\not{\partial}\psi_0 + \mathrm{i}m_0\bar\psi_0\psi_0 + \mathrm{i}g_0 A^a_{0\mu}\bar\psi_0\gamma_\mu T^a\psi_0 \\
&= \mathcal{L}_{\mathrm{R}} + \mathcal{L}_{\mathrm{ct}}
\end{aligned}$$

The most general Lagrangian has different bare couplings for the interactions. The renormalized quantities are given by

$$\begin{aligned}
\psi_0 &= Z_2^{\frac{1}{2}}\psi \\
A^a_{0\mu} &= Z_3^{\frac{1}{2}}A^a_\mu \\
c^a_0 &= Z_6^{\frac{1}{2}}c^a \\
m_0 &= Z_m Z_2^{-1} m \\
\frac{1}{\xi_0} &= \frac{1}{\xi}Z_\xi Z_3^{-1} \\
g_0 &= Z_1 Z_2^{-1} Z_3^{-\frac{1}{2}} g k^\varepsilon \\
g_0' &= Z_4 Z_3^{-\frac{3}{2}} g k^\varepsilon \\
g_0'' &= Z_5^{\frac{1}{2}} Z_3^{-1} g k^\varepsilon \\
g_0''' &= Z_7 Z_6^{-1} Z_3^{-\frac{1}{2}} g k^\varepsilon \\
g_0^{(4)} &= Z_8 Z_6^{-1} Z_3^{-\frac{1}{2}} g k^\varepsilon
\end{aligned}$$

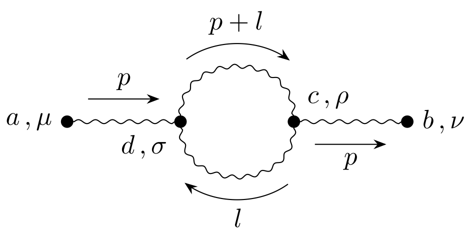
The counter term Lagrangian has a similar form to the bare and renormalized Lagrangians.

25.3.1 Two-point gauge Green's function

The two-point gauge Green's function gets contributions from three diagrams, one for each kind of loop.

Gauge loop. Consider the two-point gauge Green's function with a gauge loop

$$\Pi^{ab}_{\mu\nu}(p^2)|^{\text{gauge}}_{1\text{L}} =$$



One has to multiply two three-point vertices. The colour indices of the two are taken both clockwise[36]. Recall that the formula given previously is for entering momenta. Therefore

$$\begin{aligned}
\Pi^{ab}_{\mu\nu}(p^2)|^{\text{gauge}}_{1\text{L}} &= \frac{1}{2!}(-\mathrm{i}gk^\varepsilon)^2 c^{acd}c^{bdc}[(-p-l-p)_\sigma\delta_{\mu\rho} + (l-(-p-l))_\mu\delta_{\rho\sigma} + (p-l)_\rho\delta_{\mu\sigma}] \\
&\quad\times[(-l+p)_\rho\delta_{\nu\sigma} + (-p-l-p)_\sigma\delta_{\rho\nu} + (l+p+l)_\nu\delta_{\rho\sigma}] \\
&\quad\times\int\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{1}{l^2(p+l)^2} \\
&= -\frac{1}{2}g^2k^{2\varepsilon}c^{acd}c^{bdc}\int\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{N_{\mu\nu}(l,p)}{l^2(p+l)^2}
\end{aligned}$$

with

$$N_{\mu\nu}(l,p) = (10-8\varepsilon)l_\mu l_\nu + (5-4\varepsilon)(l_\mu p_\nu + l_\nu p_\mu) - 2(1+\varepsilon)p_\mu p_\nu + [(p-l)^2 + (2p+l)^2]\delta_{\mu\nu}$$

Notice that the combinatorial factor is inside the three-point vertex. Applying Feynman combining and using a table of integrals gives

$$\begin{aligned}
\Pi^{ab}_{\mu\nu}(p)|^{\text{gauge}}_{1\text{L}} &= \int_0^1 \mathrm{d}x\left\{\delta_{\mu\nu}(9-6\varepsilon)\frac{\Gamma(-1+\varepsilon)}{[p^2x(1-x)]^{-1+\varepsilon}}\right. \\
&\quad\left.+\frac{\Gamma(\varepsilon)}{[p^2x(1-x)]^\varepsilon}[\delta_{\mu\nu}p^2(5-2x(1-x)) + p_\mu p_\nu(-2(1+\varepsilon)-(10-8\varepsilon)x(1-x))]\right\}
\end{aligned}$$

[36] When writing the structure constants. Notice how the pairs of momenta are taken counter-clockwise.

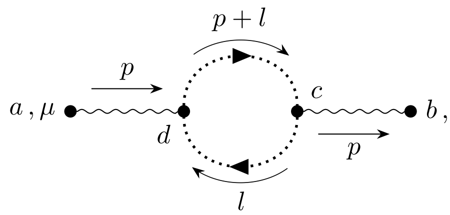
[r] Noting that

$$\Gamma(-1+\varepsilon) = \frac{\Gamma(1+\varepsilon)}{\varepsilon(-1+\varepsilon)} \sim -\frac{1}{\varepsilon}\,, \quad \Gamma(\varepsilon) \sim \frac{1}{\varepsilon}$$

one may find

$$\boxed{\Pi^{ab}_{\mu\nu}(p^2)|^{\text{gauge}}_{\text{1L}} = \frac{g^2}{32\pi^2} c^{acd} c^{bcd} \frac{1}{\varepsilon}\left[\frac{19}{6}\delta_{\mu\nu}p^2 - \frac{11}{3}p_\mu p_\nu\right] + o(\varepsilon^{-1})}$$

Ghost loop. Consider the two-point gauge Green's function with a ghost loop

$\Pi^{ab}_{\mu\nu}(p^2)|^{\text{ghost}}_{\text{1L}} =$ 

This diagram is computed including the A term. The left and right vertices give

$$-gk^\varepsilon c^{dca}(-p-l)_\mu\,, \quad -gk^\varepsilon c^{cdb}(-l)_\nu$$

Therefore

$$\begin{aligned}\Pi^{ab}_{\mu\nu}(p^2)|^{\text{ghost}}_{\text{1L}} &= g^2 k^{2\varepsilon} c^{dca} c^{cdb} (-\mathrm{i})^2(-1) \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{(p+l)_\mu l_\nu}{l^2(p+l)^2} \\ &= \boxed{\frac{g^2}{32\pi^2} c^{acd} c^{bcd} \left[\frac{1}{6}p^2\delta_{\mu\nu} + \frac{1}{3}p_\mu p_\nu\right]\frac{1}{\varepsilon} + o(\varepsilon^{-1})}\end{aligned}$$

The $-\mathrm{i}$ comes from the ghost propagators, while -1 comes from the Grassmann-odd loop integral.

The computation without the A term is slightly different. The two vertices contribute with

$$\frac{1}{2}gk^\varepsilon c^{dca}(2l+p)_\mu\,, \quad \frac{1}{2}gk^\varepsilon c^{cdb}(2l+p)_\nu$$

Thus

$$\begin{aligned}\Pi^{ab}_{\mu\nu}(p^2)|^{\text{ghost}}_{\text{1L}} &= \frac{g^2}{4} k^{2\varepsilon} c^{dca} c^{cdb} \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{(2l+p)_\mu (2l+p)_\nu}{l^2(l+p)^2} \\ &= \frac{g^2}{32\pi^2} c^{acd} c^{bcd} \left[\frac{1}{6}p^2\delta_{\mu\nu} - \frac{2}{3}p_\mu p_\nu\right]\frac{1}{\varepsilon} + o(\varepsilon^{-1})\end{aligned}$$

The only a difference is in the coefficient of the longitudinal part: there is one term $p_\mu p_\nu$ less. This part is taken care of by the gauge-fixing part. Therefore, the fact that it is different does not matter.

Lecture 18

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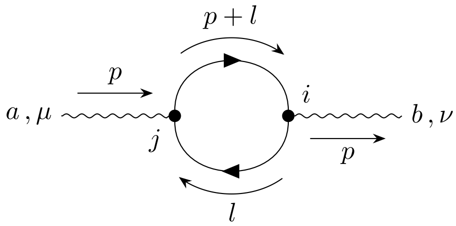
Sum of contributions. The sum of the two above contributions is

$$\Pi^{ab}_{\mu\nu}(p^2)|^{\text{gauge}}_{\text{1L}} + \Pi^{ab}_{\mu\nu}(p^2)|^{\text{ghost}}_{\text{1L}} = \frac{g^2}{32\pi^2} c^{acd} c^{bcd} \frac{10}{3}(p^2\delta_{\mu\nu} - p_\mu p_\nu)\frac{1}{\varepsilon} + o(\varepsilon^{-1})$$

where one has included the A term. Notice that the parenthesis is the projection operator on transverse degrees of freedom.

Fermion loop. Consider the two-point gauge Green's function with a fermion loop

$$\Pi^{ab}_{\mu\nu}(p^2)|^{\text{fermion}}_{1\text{L}} =$$



The two vertices correspond to

$$-\mathrm{i}gk^\varepsilon\gamma_\mu(T^a)_{ji}\,, \quad -\mathrm{i}gk^\varepsilon\gamma_\nu(T^b)_{ij}$$

They are similar to the ones of QED up to a colour term T. Therefore

$$\begin{aligned}\Pi^{ab}_{\mu\nu}(p^2)|^{\text{fermion}}_{1\text{L}} &= (T^a_f)_{ji}(T^b_f)_{ij}\Pi^{\text{QED}}_{\mu\nu} \\ &= -\operatorname{Tr}\big(T^a_fT^b_f\big)\frac{g^2}{16\pi^2}(\delta_{\mu\nu}p^2 - p_\mu p_\nu)\frac{4}{3}\frac{1}{\varepsilon} + o(\varepsilon^{-1})\end{aligned}$$

where the generators T_f are in a generic representation f of the fermions.

Total contribution. The total contribution is

$$\begin{aligned}\Pi^{ab}_{\mu\nu}(p^2)|_{1\text{L}} &= \frac{g^2}{16\pi^2}(p^2\delta_{\mu\nu} - p_\mu p_\nu)\left[\frac{5}{3}c^{acd}c^{bcd} - \frac{4}{3}\operatorname{Tr}\big(T^a_fT^b_f\big)\right]\frac{1}{\varepsilon} + o(\varepsilon^{-1}) \\ &= \frac{g^2}{16\pi^2}(p^2\delta_{\mu\nu} - p_\mu p_\nu)\left[\frac{5}{3}c(\text{adj}) - \frac{4}{3}c(f)\right]\delta^{ab}\frac{1}{\varepsilon} + o(\varepsilon^{-1})\end{aligned}$$

At the second line, one has used the colour structure of the $\mathrm{SU}(n)$ group.

25.3.2 Color structure

See Peskin, §15.4.

Proposition (Dynkin index)**.** [r] The Dynkin index is $c(f)$ defined through

$$\operatorname{Tr}\big(T^a_fT^b_f\big) = c(f)\delta^{ab}$$

Proposition. It holds

$$c^{acd}c^{bcd} = c(\text{adj})\delta^{ab}$$

Proof. In the adjoint representation, it holds

$$(T^a)_{bc} = -\mathrm{i}c^{abc}$$

therefore one has

$$\operatorname{Tr}\big(T^aT^b\big) = (T^a)_{cd}(T^b)_{dc} = -\mathrm{i}c^{acd}(-\mathrm{i}c^{bdc}) \implies c(\text{adj})\delta^{ab} = c^{acd}c^{bcd}$$

□

Proposition (Casimir operator)**.** Knowing

$$(T^2)_{ij} = (T^aT^a)_{ij} = T^a_{ik}T^a_{kj}$$

the Casimir operator T^2 is such that

$$[T^2, T^b] = 0\,, \quad \forall b$$

The Casimir index c_2 is defined through

$$T^2_f = c_2(f)I$$

Proposition. For the adjoint representation, the Casimir index is the Dynkin index

$$c_2(\text{adj}) = c(\text{adj})$$

Proposition. For a representation f it holds

$$c_2(f)\dim f = c(f)\dim G$$

Proof. From the Dynkin index, one traces (also) over the group indices

$$\mathrm{Tr}\big(T_f^a T_f^a\big) = c(f)\dim G$$

Tracing over the definition of Casimir index, one gets

$$\mathrm{Tr}\big(T_f^2\big) = \mathrm{Tr}\big(T_f^a T_f^a\big) = c_2(f)\dim f$$

Thus the thesis follows. □

Example. Consider $G = \mathrm{SU}(n)$ and its fundamental representation f. The dimensions are

$$\dim G = n^2 - 1\,, \quad \dim f = n$$

From the identity above, one has

$$c_2(\text{fond})n = c(\text{fond})(n^2 - 1)$$

It is convention to consider $c(\text{fond}) = \frac{1}{2}$ and

$$\mathrm{Tr}\big(T^a_{\text{fond}} T^b_{\text{fond}}\big) = \frac{1}{2}\delta^{ab}$$

Therefore

$$c_2(\text{fond}) = \frac{n^2 - 1}{2n}$$

Proposition. In the adjoint representation of $\mathrm{SU}(n)$, it holds

$$c_2(\text{adj}) = c(\text{adj}) = n$$

Proof. The adjoint representation can be built from the fundamental representation n and the anti-fundamental representation $\bar{n}$ recalling that the product of such two representations can be decomposed as the direct sum of irreducible representations

$$n \otimes \bar{n} = \sum_i r_i = \text{trivial} \oplus \text{adj}$$

The object that transform under $n \otimes \bar{n}$ carry an index for each $A_{a\bar{a}}$. Therefore, a generator can be written as

$$T^a_{n\otimes\bar{n}} = T^a_n \otimes I_{\bar{n}} + I_n \otimes T^a_{\bar{n}}$$

Therefore, the Casimir operator is

$$T^2_{n\otimes\bar{n}} = T^a_{n\otimes\bar{n}} T^a_{n\otimes\bar{n}} = T^2_n \otimes I + I \otimes T^2_{\bar{n}} + 2T^2_n \otimes T^2_{\bar{n}}$$

Taking the trace gives

$$\mathrm{Tr}\big(T^2_{n\otimes\bar{n}}\big) = [c_2(n) + c_2(\bar{n})]\dim n\,\dim\bar{n} = [c_2(n) + c_2(\bar{n})]n^2 = 2c_2(n)n^2$$

The trace can also be written differently

$$\mathrm{Tr}\big(T^2_{n\otimes\bar{n}}\big) = \sum_i c_2(r_i)\dim r_i = c_2(1)\dim 1 + c_2(\text{adj})\dim\text{Adj} = 0 + c_2(\text{adj})(n^2 - 1)$$

Comparing these two, one finds

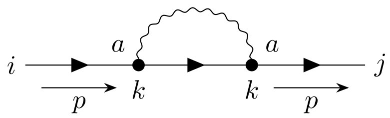
$$2c_2(n)n^2 = c_2(\text{adj})(n^2 - 1) \implies \frac{n^2 - 1}{n}n^2 = c_2(\text{adj})(n^2 - 1) \implies c_2(\text{adj}) = n$$

Noting that $c_2(\text{adj}) = c(\text{adj})$ due to a previous proposition, one obtains the thesis. □

25.3.3 Two-point fermion Green's function

Consider the two-point fermion Green's function

$$\Sigma_{ji}(p^2)|_{1\mathrm{L}} =$$

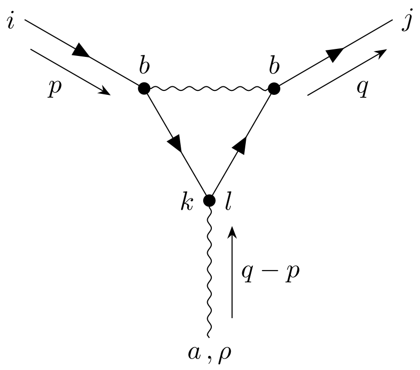


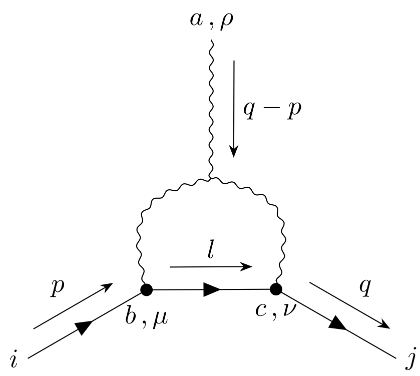
Apart from the colour factors, the computation is the same as in QED. Therefore

$$\begin{aligned}\Sigma_{ji}(p^2)|_{1\mathrm{L}} &= (T_f^a)_{jk}(T_f^a)_{ki}\Sigma^{\mathrm{QED}}(p^2) = -(T_f^aT_f^a)_{ji}\frac{\mathrm{i}}{\varepsilon}\frac{g^2}{16\pi^2}(\not{p}+4m)+o(\varepsilon^{-1})\\ &= \boxed{-\frac{\mathrm{i}g^2}{16\pi^2}c_2(f)\delta_{ji}(\not{p}+4m)\frac{1}{\varepsilon}+o(\varepsilon^{-1})}\end{aligned}$$

25.3.4 Three-point vertices

Spinor-vector vertex. The spinor-vector vertex gets contributions from two diagrams





The first diagram is

$$\Gamma^a_{\rho ji}(p,q)|^{\mathrm{I}}_{1\mathrm{L}} = (T_f^b)_{jl}(T_f^a)_{lk}(T_f^b)_{ki}\Gamma_\rho^{\mathrm{QED}} = (T_f^bT_f^aT_f^b)_{ji}\Gamma_\rho^{\mathrm{QED}}$$

The colour structure is

$$\begin{aligned}T_f^bT_f^aT_f^b &= [T_f^b,T_f^a]T_f^b + T_f^aT_f^bT_f^b = \mathrm{i}c^{bad}T_f^dT_f^b + T_f^ac_2(f)\\ &= \frac{\mathrm{i}}{2}c^{bad}[T_f^d,T_f^b] + c_2(f)T_f^a = -\frac{1}{2}c^{bad}c^{dbg}T_f^g + c_2(f)T_f^a\\ &= -\frac{1}{2}c(\mathrm{adj})T_f^a + c_2(f)T_f^a\end{aligned}$$

Therefore

$$\Gamma^a_{\rho ij}(p,q)|^{\mathrm{I}}_{1\mathrm{L}} = -\mathrm{i}gk^\varepsilon\gamma_\rho T_f^a\left[c_2(f)-\frac{1}{2}c(\mathrm{adj})\right]\frac{g^2}{16\pi^2}\frac{1}{\varepsilon}+o(\varepsilon^{-1})$$

The second diagram uses the vertices

$$-\mathrm{i}gk^\varepsilon\gamma_\mu(T^b)_{ji}\,,\quad -\mathrm{i}gk^\varepsilon c^{abc}[(r-q)_\mu\delta_{\nu\rho}+(p-r)_\nu\delta_{\rho\mu}+(q-p)_\rho\delta_{\mu\nu}]$$

Therefore

$$\Gamma^a_{\rho ji}(p,q)|^{\mathrm{II}}_{1\mathrm{L}} = -g^3k^{3\varepsilon}c^{acb}(T^c)_{jk}(T^b)_{ki}\int\frac{\mathrm{d}^dl}{(2\pi)^d}\gamma^\nu\frac{1}{\not{l}+m}\gamma^\mu\frac{N_{\mu\nu\rho}(p,q)}{(p-l)^2(q-l)^2}$$

check [r] where minus from?
where one has

$$N_{\mu\nu\rho}(p,q) = (q+p-2l)_\rho\delta_{\mu\nu} + (l-2p+q)_\nu\delta_{\mu\rho} + (l+p-2q)_\mu\delta_{\nu\rho}$$

The colour structure is

$$c^{acb}T_f^cT_f^b = \frac{1}{2}c^{acb}[T_f^c, T_f^b] = \frac{\mathrm{i}}{2}c^{acb}c^{cbd}T_f^d = \frac{\mathrm{i}}{2}c(\mathrm{adj})T_f^a$$

See Peskin also. One obtains

$$\Gamma^a_{\rho ji}(p,q)|^{\mathrm{II}}_{1\mathrm{L}} = -\mathrm{i}gk^\varepsilon\gamma_\rho T_f^a\frac{3}{2}c(\mathrm{adj})\frac{g^2}{16\pi^2}\frac{1}{\varepsilon} + o(\varepsilon^{-1})$$

The sum of the two diagrams is

$$\boxed{\Gamma^a_{\rho ji}(p,q)|_{1\mathrm{L}} = -\mathrm{i}gk^\varepsilon\gamma_\rho(T_f^a)_{ji}\frac{g^2}{16\pi^2}[c(\mathrm{adj}) + c_2(f)]\frac{1}{\varepsilon} + o(\varepsilon^{-1})}$$

Other vertices. The other vertices produce divergences that are already taken care of by the counter terms of the above divergences. [r]

25.4 One-loop renormalization

The bare Lagrangian is split

$$\mathcal{L}_0 = \mathcal{L}_\mathrm{R} + \mathcal{L}_\mathrm{ct}$$

where the counter term Lagrangian contains

$$\mathcal{L}_\mathrm{ct} \supset K_3\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + K_2\bar\psi\,\partial\!\!\!/\psi + K_m\mathrm{i}m\bar\psi\psi + K_\xi\frac{1}{2}(\partial_\mu A^a_\mu)^2 + K_1\mathrm{i}gk^\varepsilon A^a_\mu\bar\psi\gamma_\mu T_f^a\psi$$

Let $K_j \equiv Z_j - 1$.

Fermion mass. The mass gets a contribution from the fermion Green's function at one-loop

$$\Sigma_{ji}(p^2)|_{1\mathrm{L}} \implies -\frac{\mathrm{i}g^2}{16\pi^2}c_2(f)4m\frac{1}{\varepsilon} = -\frac{\mathrm{i}g^2}{4\pi^2}c_2(f)m\frac{1}{\varepsilon}$$

Recalling the minus sign from the exponential of the counter term action, one obtains

$$-\mathrm{i}m\left[K_m + \frac{g^2}{4\pi^2}c_2(f)\frac{1}{\varepsilon}\right] = \text{finite} \implies \boxed{Z_m = 1 - \frac{g^2}{4\pi^2}c_2(f)\frac{1}{\varepsilon} + \text{finite}}$$

Fermion field. The kinetic term gets a contribution from the fermion Green's function

$$\Sigma_{ji}(p^2)|_{1\mathrm{L}} \implies -\frac{\mathrm{i}g^2}{16\pi^2}c_2(f)\frac{1}{\varepsilon}p\!\!\!/$$

[r] Recalling that $\partial_\mu \leftrightarrow \mathrm{i}p_\mu$ when doing Fourier transformations, the counter term contribution is $-\mathrm{i}K_2 p\!\!\!/$, already including the minus sign from the exponential. Therefore

$$-\mathrm{i}p\!\!\!/\left[K_2 + \frac{g^2}{16\pi^2}c_2(f)\frac{1}{\varepsilon}\right] = \text{finite} \implies \boxed{Z_2 = 1 - \frac{g^2}{16\pi^2}c_2(f)\frac{1}{\varepsilon} + \text{finite}}$$

Gauge field. The divergent contribution is

$$\Pi^{ab}_{\mu\nu}(p^2)|_{1\mathrm{L}} = \frac{g^2}{16\pi^2}(p^2\delta_{\mu\nu} - p_\mu p_\nu)\left[\frac{5}{3}c(\mathrm{adj}) - \frac{4}{3}c(f)\right]\delta^{ab}\frac{1}{\varepsilon} + o(\varepsilon^{-1})$$

From the counter term Lagrangian

$$\mathcal{L}_\mathrm{ct} \supset K_3\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2$$

One integrates by parts and goes to momentum space

$$\mathcal{L}_{\text{ct}} \rightsquigarrow \frac{1}{2}K_3 A^a_\mu (p^2\delta_{\mu\nu} - p_\mu p_\nu) A^a_\nu$$

The counter term renormalization gives

$$(p^2\delta_{\mu\nu} - p_\mu p_\nu)\delta^{ab}\left[-\frac{1}{2}K_3 \cdot 2 + \frac{g^2}{16\pi^2}\left(\frac{5}{3}c(\text{adj}) - \frac{4}{3}c(f)\right)\frac{1}{\varepsilon}\right] = \text{finite}$$

from which one has

$$\boxed{Z_3 = 1 + \frac{g^2}{16\pi^2}\left[\frac{5}{3}c(\text{adj}) - \frac{4}{3}c(f)\right]\frac{1}{\varepsilon} + \text{finite}}$$

The gauge parameter ξ does not need to be renormalized when considering the A term. Without it, one obtains

$$\Pi^{ab}_{\mu\nu}(p^2)|_{1\text{L}} = \frac{g^2}{16\pi^2}\left[\frac{5}{3}c(\text{adj}){+}?c(f)\right]\frac{1}{\varepsilon}(p^2\delta_{\mu\nu} - 2p_\mu p_\nu)\delta^{ab}$$

check [r]
The extra factor of 2 in the projection operator (the parenthesis) means renormalizing the gauge parameter. In fact, the counter term gives

$$K_\xi \frac{1}{2}(\partial_\mu A^a_\mu)^2 \to -\frac{1}{2}K_\xi A^a_\nu\, \partial_\mu\partial_\nu A^a_\mu \to \frac{1}{2}K_\xi A^a_\nu p_\mu p_\nu A^a_\mu$$

One may separate the extra term of the projection operator from the actual projection operator and renormalize the gauge parameter

$$\boxed{Z_\xi = 1 - \frac{g^2}{16\pi^2}\left[\frac{5}{3}c(\text{adj}){+}?c(f)\right]\frac{1}{\varepsilon} + \text{finite}}$$

check [r]

Vertex. The divergent contribution is

$$\Gamma^a_{\rho ij}(p,q)|_{1\text{L}} = -\mathrm{i}gk^\varepsilon \gamma_\rho (T^a_f)_{ij}\frac{g^2}{16\pi^2}[c(\text{adj}) + c_2(f)]\frac{1}{\varepsilon} + o(\varepsilon^{-1})$$

The counter term gives a contribution of

$$-\mathrm{i}K_1 g k^\varepsilon \gamma_\rho T^a_f$$

Therefore one obtains

$$-\mathrm{i}gk^\varepsilon\gamma_\rho T^a_f\left[K_1 + \frac{g^2}{16\pi^2}(c(\text{adj}) + c_2(f))\frac{1}{\varepsilon}\right] = \text{finite} \implies \boxed{Z_1 = 1 - \frac{g^2}{16\pi^2}[c(\text{adj}) + c_2(f)]\frac{1}{\varepsilon} + \text{finite}}$$

Remark. The ward identity of QED, $Z_1 = Z_2$, does not hold due to $c(\text{adj})$.

Lecture 19

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25.5 Slavnov–Taylor identities

See Ramond, §8.7.

Remark. One may not renormalize the primed coupling constants because, for non-abelian gauge theories, there is only one coupling constant. The gauge-fixing term breaks the gauge invariance of the Lagrangian, but keeps BRST invariance. Gauge invariance implies the Slavnov–Taylor identities, which state that the coupling constants renormalize in the same way. One need not compute Z_4, Z_5, Z_7 and Z_8.

Remark. The renormalization function Z_6 appears in

$$c_0^a = Z_6^{\frac{1}{2}} c^a$$

To be renormalized, one has to compute the diagram contributing to the ghost propagator. The integral reduces to one already computed.

Exercise. Find Z_6 at one-loop.

Exercise. Find the explicit form of the Slavnov–Taylor identity for non-abelian gauge theories. See Ramond, eq. 8.7.40.

Remark. The renormalization functions Z_1, Z_2, Z_3 and Z_6 are gauge dependent. The computations differ based on the gauge. The structure of the ghost action depends on the gauge-fixing condition used. For example, in the axial gauge, $A_3^a = 0$, ghosts decouple and one does not have a ghost-loop gauge field diagram.

25.6 Asymptotic freedom

See Ramond, §8.8. Consider the one-loop renormalization functions

$$\begin{aligned}
Z_1 &= 1 - \frac{g^2}{16\pi^2}[c(\mathrm{adj}) + c_2(f)]\frac{1}{\varepsilon} + \text{finite} \\
Z_2 &= 1 - \frac{g^2}{16\pi^2}c_2(f)\frac{1}{\varepsilon} + \text{finite} \\
Z_3 &= 1 + \frac{g^2}{16\pi^2}\left(\frac{5}{3}c(\mathrm{adj}) - \frac{4}{3}c(f)\right)\frac{1}{\varepsilon} + \text{finite}
\end{aligned}$$

The bare coupling constant is

$$\begin{aligned}
g_0 &= Z_1 Z_2^{-1} Z_3^{-\frac{1}{2}} g k^\varepsilon \\
&= g k^\varepsilon \left[1 - \frac{g^2}{16\pi^2}\left(c(\mathrm{adj}) + \frac{N}{d_f}c(f)\right)\frac{1}{\varepsilon} + o(\varepsilon^{-1})\right]\left[1 + \frac{g^2}{16\pi^2}\frac{N}{d_f}c(f)\frac{1}{\varepsilon} + o(\varepsilon^{-1})\right] \\
&\quad \times \left[1 - \frac{1}{2}\frac{g^2}{16\pi^2}\left(\frac{5}{3}c(\mathrm{adj}) - \frac{4}{3}c(f)\right)\frac{1}{\varepsilon} + o(\varepsilon^{-1})\right] \\
&= k^\varepsilon \left[g - \frac{g^3}{16\pi^2}\left(\frac{11}{6}c(\mathrm{adj}) - \frac{2}{3}c(f)\right)\frac{1}{\varepsilon} + \text{finite}\right]
\end{aligned}$$

where N is the dimension of the group and d_f is the dimension of the representation. At the second line, one has used

$$c_2(f)\dim f = c(f)\dim G \implies c_2(f)d_f = c(f)N \implies c_2(f) = \frac{N}{d_f}c(f)$$

In dimensional regularization, the beta function is obtained from the residue of the simple pole in ε. In general

$$g_0 = k^{n\varepsilon}\left[g + \frac{a_1}{\varepsilon} + \cdots\right]$$

from which the beta function is

$$\beta(g) = k\,\mathrm{d}_k g = -n\left[a_1 - g\,\mathrm{d}_g a_1\right] = n\left[g\,\mathrm{d}_g - 1\right]a_1$$

In this case, $n = 1$ and

$$a_1 = -\frac{g^3}{16\pi^2}\left(\frac{11}{6}c(\mathrm{adj}) - \frac{2}{3}c(f)\right)$$

The beta function is

$$\beta(g)|_{1\mathrm{L}} = 2a_1 = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}c(\mathrm{adj}) - \frac{4}{3}c(f)\right)$$

Abelian gauge theory. For an abelian gauge theory, like QED, one sets $c(\text{adj}) = 0$ and obtains
$$\beta(g)|_{1\text{L}} = \frac{g^3}{12\pi^2} c(f)$$
For QED, one takes the fundamental representation and $c(f) = 1$. One notices that the beta function is non-negative. Since the beta function is related to the running of the coupling constant
$$k\, \mathrm{d}_k g = \beta(g)$$
At higher energies, the coupling constant gets stronger.

Non-abelian gauge theory. For non-abelian gauge theories, a positive beta function means that the behaviour of the theory is similar to QED. Though, one may obtain a negative beta function for
$$\frac{11}{3} c(\text{adj}) > \frac{4}{3} c(f)$$
At higher energies, the coupling weakens. This is the asymptotic freedom: perturbation theory is only applicable in the ultraviolet.

Example. Consider $G = \mathrm{SU}(n)$ with the fundamental and adjoint representations giving
$$c(\text{fund}) = \frac{1}{2}\,, \quad c(\text{adj}) = n$$
Then
$$\beta(g) = -\frac{g^3}{16\pi^2} \left[\frac{11}{3} n - \frac{2}{3}\right]$$
The bracket is positive for $n > \frac{2}{11}$. In general, for more than one flavour of fermions
$$\mathcal{L} \propto \bar{\psi}^A \not{\partial} \psi^A + \mathrm{i} m_A \bar{\psi}^A \psi^A + \mathrm{i} g A^a_\mu \bar{\psi}^A T^a \psi^A$$
where $A = 1, \dots, n_F$ is the flavour index, one has a n_F multiplying $c(\text{fund})$. One may recall that the gauge fields live in the adjoint representation of the group, while the fermions live in the fundamental representation [r]
$$\beta(g)|_{1\text{L}} = 2a_1 = -\frac{g^3}{16\pi^2} \left(\frac{11}{3} c(\text{adj}) - \frac{4}{3} c(f) n_F\right)$$
In order to make the bracket positive, one may play with the choice of the gauge group or the number of flavours n_F.

For QCD, one has $G = \mathrm{SU}(3)$ and $n_F = 6$. Therefore
$$\frac{11}{3} c(\text{adj}) - \frac{4}{3} c(\text{fund}) n_F = \frac{11}{3} \cdot 3 - \frac{4}{3} \cdot \frac{1}{2} \cdot 6 = 11 - 4 = 7 > 0$$
The theory exhibits asymptotic freedom. It is sufficient to look at one-loop to determine the presence of asymptotic freedom, since higher loops give smaller contributions [r].

Absence of a Landau pole. Instead of integrating the beta function of g, it is better to integrate the beta function of g^2
$$\beta(g^2) = \mathrm{d}_{\ln k} g^2 = 2g\, \mathrm{d}_{\ln k} g = 2g\beta(g) = -\frac{g^4}{8\pi^2} \left[\frac{11}{3} c(\text{adj}) - \frac{4}{3} c(f) n_F\right]$$
since it gives a simpler result
$$g^2(k) = g^2(k_0) \left[1 + \frac{g^2(k_0)}{8\pi^2} \left(\frac{11}{3} c(\text{adj}) - \frac{4}{3} c(f) n_F\right) \ln \frac{k}{k_0}\right]^{-1}$$
If the parenthesis positive, then there is no Landau pole due to the presence also of a plus sign. This means that asymptotic freedom and a Landau pole are mutually exclusive.

For small values of the coupling constant, one obtains

$$g^2(k) = g^2(k_0) - \frac{g^4(k_0)}{8\pi^2}\left(\frac{11}{3}c(\mathrm{adj}) - \frac{4}{3}c(f)n_F\right)\ln\frac{k}{k_0} + o(\ln {}^{k}/_{k_0})$$

The first term is the leading contribution, the second term is the next-to-leading contribution and so on.

The non-pertubative region can be described either with computations on the lattice or with the AdS/CFT correspondence. The computations can be done in supersymmetric theories giving a superconformal boundary theory. Recent work has been done on theories that break both supersymmetry and conformal symmetries.

Confinement. At low energies, meaning great distances, one may define free asymptotic states. For a theory with asymptotic freedom, far away particles can only be composite particles. One cannot define free asymptotic states for elementary particles. The free asymptotic states can only be defined for composite particles that are neutral with respect to the gauge group (singlets).

For example in QCD, through the quark model and eightfold way, one may classify bound states with an approximate flavour symmetry: light quarks are in the fundamental representation of $\mathrm{SU}(3)_{\mathrm{F}}$ while light anti-quarks are in the anti-fundamental representation. For two-quark mesons, one has $\mathrm{SU}(2)_{\mathrm{F}}$ with

$$3 \otimes \bar{3} = 1 \oplus 8 = \text{trivial} + \text{adjoint}$$

For three-quark baryons, one has $\mathrm{SU}(3)_{\mathrm{F}}$ and

$$3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$$

See Cheng, §4.4, from p. 117.

Intuitive interpretation of asymptotic freedom. See Peskin, §16.7. In QED, the coupling constant gets weaker for greater distances because the charges get screened by the self-energy. A lone charge self-interacts generating virtual particle–anti-particle pairs.

From Maxwell's equations

$$\partial_\mu F^{\mu\nu} = J^\nu$$

one finds

$$F^{i0} = E^i \implies \partial_i E^i = q\,\delta^{(3)}(\mathbf{x})\,, \quad E^i = \frac{q}{4\pi}\frac{x^i}{r^3} \implies E \sim \frac{1}{r^2}$$

For non-abelian gauge theories, one has

$$D_\mu F^{\mu\nu a} = J^{\nu a}$$

Consider the Coulomb potential of a point particle of magnitude +1 and orientation $a = 1$

$$\partial_i E^{ai} = g\,\delta^{(3)}(\mathbf{x})\delta^{a1} + gc^{abc}A^b_i E^{ci}$$

The extra term matters: it can contribute positively or negatively to the first term. This term acts as a source. [r] This may act as a counter-screening: a charge may get stronger. For example, $a = 1$ gives

$$\partial_i E^{1i} = g\,\delta^{(3)}(\mathbf{x}) + gc^{1bc}A^b_i E^{ci}$$

26 Spontaneous symmetry breaking

See Cheng, §8.3. Gauge theories with spontaneous symmetry breaking are renormalizable as proved by 't Hooft[37].

[37]See G. 't Hooft, Renormalization of massless Yang-Mills fields, Nuclear Physics B, Volume 33, Issue 1, 1971, Pages 173-199, ISSN 0550-3213, `https://doi.org/10.1016/0550-3213(71)90395-6` and G. 't Hooft, Renormalizable Lagrangians for massive Yang-Mills fields, Nuclear Physics B, Volume 35, Issue 1, 1971, Pages 167-188, ISSN 0550-3213, `https://doi.org/10.1016/0550-3213(71)90139-8`.

26.1 Review of abelian gauge theories

Consider an abelian gauge theory, in particular a complex scalar theory with U(1) gauge invariance. The fields trasform as

$$\varphi' = \mathrm{e}^{-\mathrm{i}\alpha(x)}\varphi\,, \quad A'_\mu = A_\mu - \frac{1}{g}\,\partial_\mu\alpha$$

The covariant derivative is

$$D_\mu = \partial_\mu - \mathrm{i}gA_\mu$$

The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D_\mu\varphi)^\dagger(D_\mu\varphi) + m^2\varphi^\dagger\varphi - \lambda(\varphi^\dagger\varphi)^2$$

where potential is

$$V(\varphi) = m^2\varphi^\dagger\varphi - \lambda(\varphi^\dagger\varphi)^2$$

Notice the plus sign for the mass term in order to induce a spontaneous symmetry breaking[38]. Its extrema are given by

$$0 = \frac{\partial V}{\partial|\varphi|^2} = -m^2 + 2\lambda|\varphi|^2 \implies |\varphi|^2 = \frac{m^2}{2\lambda} \implies |\varphi| = \frac{1}{\sqrt{2}}\sqrt{\frac{m^2}{\lambda}} = \frac{v}{\sqrt{2}} \neq 0$$

The potential resembles a Mexican hat. The minima are related by a U(1) transformation. One chooses a particular minimum by fixing the expectation value

$$\langle 0|\,\varphi\,|0\rangle = \frac{v}{2}$$

One may rewrite the field in terms of its real and imaginary components

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + \mathrm{i}\varphi_2)$$

so that one chooses

$$\langle 0|\,\varphi_1\,|0\rangle = v\,, \quad \langle 0|\,\varphi_2\,|0\rangle = 0$$

The gauge transformation of the real fields is an SO(2) transformation

$$\begin{bmatrix}\varphi'_1\\ \varphi'_2\end{bmatrix} = \begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}$$

which relates the vacua as well.

The kinetic term of the field becomes

$$(D_\mu\varphi)^\dagger(D_\mu\varphi) = \frac{1}{2}(\partial_\mu\varphi_1 + gA_\mu\varphi_2)^2 + \frac{1}{2}(\partial_\mu\varphi_2 - gA_\mu\varphi_1)^2$$

The potential is

$$V(\varphi) = m^2\varphi^\dagger\varphi - \lambda(\varphi^\dagger\varphi)^2 = \frac{1}{2}m^2(\varphi_1^2 + \varphi_2^2) - \frac{\lambda}{4}(\varphi_1^2 + \varphi_2^2)^2$$

Since it is easier to work with a field with zero vacuum expectation value, one parameterizes the fluctuations around the vacuum as

$$\varphi'_1 = \varphi_1 - v \quad \varphi'_2 = \varphi_2$$

Substituting this into the Lagrangian, one finds a massless real scalar boson φ'_2. For a global symmetry this is a Goldstone boson, while for a gauge symmetry it is a would-be Goldstone boson. Also, the gauge field becomes massive

$$\frac{1}{2}g^2v^2A_\mu A^\mu \implies m = gv$$

[38]Typically, the mass term is $-m^2\varphi^\dagger\varphi$ and to obtain the Mexican hat potential one would take $m^2 < 0$. In this case one uses $m^2 > 0$ and writes explicitly the minus sign.

Counting the degrees of freedom, one notices that the would-be Goldstone boson should become the third degree of freedom of the gauge field.

The Lagrangian after spontaneous symmetry breaking is still gauge invariant under

$$\begin{bmatrix}\varphi_1' + v \\ \varphi_2'\end{bmatrix} \to \begin{bmatrix}\cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\varphi_1' + v \\ \varphi_2'\end{bmatrix}$$

but gauge invariance is broken in the equations of motion.

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Unitary gauge. A better way to count physical degrees of freedom is to parametrize[39] the field in polar coordinates

$$\varphi(x) = \frac{1}{\sqrt{2}}\eta(x)\mathrm{e}^{\frac{\mathrm{i}}{v}\xi(x)}$$

with η and ξ real scalar fields. The vacuum expectation values of the fields are

$$\langle 0|\,\eta\,|0\rangle = v\,, \quad \langle 0|\,\xi\,|0\rangle = 0$$

One parametrizes the fluctuations around the vacuum to have

$$\varphi(x) = \frac{1}{\sqrt{2}}[\eta(x) + v]\mathrm{e}^{\frac{\mathrm{i}}{v}\xi(x)}$$

For small fluctuations, one may expand the above to get

$$\varphi = \frac{1}{\sqrt{2}}(\eta + v)\left[1 + \frac{\mathrm{i}}{v}\xi + o(\xi)\right] = \frac{1}{\sqrt{2}}[v + \eta + \mathrm{i}\xi + o(\eta,\xi)]$$

At the lowest order, the fields correspond to the previous ones

$$\eta \sim \varphi_1'\,, \quad \xi \sim \varphi_2'$$

The free Lagrangian with these fields (for small fluctuations) is

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\eta\,\partial^\mu\eta + \partial_\mu\xi\,\partial^\mu\xi)$$

while the interaction Lagrangian is

$$\mathcal{L}_{\text{int}} = \frac{1}{2}m^2(\eta^2 + \xi^2) - \frac{\lambda}{4}(\eta^2 + 2v\eta + v^2 + \xi^2)^2 + \cdots$$

One may combine quadratic terms in ξ to see that they give zero: the field ξ has no mass term. In fact

$$\mathcal{L}_{\text{int}} \supset \xi^2\left[\frac{1}{2}m^2 - \frac{\lambda}{4}2v^2\right] = \xi^2\left[\frac{1}{2}m^2 - \frac{\lambda}{2}\frac{m^2}{\lambda}\right] = 0$$

Therefore, the field ξ is a would-be Goldstone mode. This mode can be removed by performing a gauge transformation that removes the exponential in the parametrization of the field φ:

$$\varphi'(x) = \mathrm{e}^{-\frac{\mathrm{i}}{v}\xi(x)}\varphi(x) = \frac{1}{\sqrt{2}}[\eta(x) + v]$$

This particular choice of the gauge parameter of the U(1) transformation is called unitary gauge. This gauge transformation has to be balanced by a transformation of the gauge field

$$A_\mu'(x) = A_\mu(x) - \frac{1}{gv}\,\partial_\mu\xi(x)$$

The gauge Lagrangian in terms of these fields is invariant

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}F_{\mu\nu}'F'^{\mu\nu}$$

[39] Another reason this is done is due to the unclear interpretation of mixed terms, like $-gvA^\mu\,\partial_\mu\varphi_2'$.

The Lagrangian of the field, which has the role of the Higgs boson, is

$$\begin{aligned}\mathcal{L}_{\text{Higgs}} &= (D_\mu\varphi)^\dagger(D_\mu\varphi) + m^2\varphi^\dagger\varphi - \lambda(\varphi^\dagger\varphi)^2 = (D_\mu\varphi)'^\dagger(D_\mu\varphi)' + m^2\varphi'^\dagger\varphi' - \lambda(\varphi'^\dagger\varphi')^2 \\ &= \frac{1}{2}[\partial_\mu\eta - \mathrm{i}gA'_\mu(v+\eta)][\partial^\mu\eta + \mathrm{i}gA'^\mu(v+\eta)] + \frac{1}{2}m^2(v+\eta)^2 - \frac{\lambda}{4}(v+\eta)^4\end{aligned}$$

The field $\xi(x)$ disappears and the gauge field gets a mass term. One loses the degree of freedom of the field $\xi(x)$, but gains one degree from the mass term

$$\frac{1}{2}g^2v^2A'_\mu A'^\mu \implies M = gv$$

The real boson field $\eta(x)$ (the Higgs boson) has mass $m_\eta = \sqrt{2}m$.

The full Lagrangian in the unitary gauge is

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{2}(gv)^2A'_\mu A'^\mu + \frac{1}{2}\,\partial_\mu\eta\,\partial^\mu\eta - \frac{1}{2}(2m^2)\eta^2 \\ &+ \frac{1}{2}g^2A'_\mu A'^\mu\eta(2v+\eta) - \lambda v^2\eta^3 - \frac{\lambda}{4}\eta^4\end{aligned}$$

26.2 Renormalizability

See Cheng, §9.2, "R_ξ gauges in spontaneously broken gauge theories". In the unitary gauge, unitarity is manifest, but renormalizability is not manifest. One may change gauge where the latter becomes manifest.

Consider the kinetic term of the gauge field (rename $A' \to A$)

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{1}{2}g^2v^2A_\mu A^\mu = \cdots = \frac{1}{2}A_\mu[(\Box + M^2)\eta^{\mu\nu} - \partial^\mu\partial^\nu]A_\nu$$

up to a four-divergence since one integrates by parts. In momentum space, the kinetic term is

$$\frac{1}{2}A_\mu[(-p^2+M^2)\eta^{\mu\nu} + p^\mu p^\nu)]A_\nu = -\frac{1}{2}A_\mu[(p^2-M^2)\eta^{\mu\nu} - p^\mu p^\nu)]A_\nu = -\frac{1}{2}A_\mu K^{\mu\nu}A_\nu$$

where

$$K^{\mu\nu} = (p^2-M^2)\eta^{\mu\nu} - p^\mu p^\nu$$

This kinetic term is invertible and one obtains the propagator

$$\Delta_{\mu\nu}(p) = \frac{1}{p^2 - M^2 + \mathrm{i}\varepsilon}\left[-\eta_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}\right]$$

Exercise. Check that it is the inverse

$$K^{\mu\nu}\Delta_{\mu\nu} = \delta^\mu_\rho$$

R_ξ gauges. One has found that gauge theories present a finite number of topologies of divergent diagrams [r]. While in the ordinary massless case, the propagator is damped in the ultraviolet, in this case the propagator is constant in such limit. By revisiting of the superficial degree of divergence of a diagram, one finds that the diagram appears not to be renormalizable: there are an infinite number of topologies of apparently diverging diagrams. The symmetry-broken theory seems to be non-renormalizable.

This cannot be the end. One has only chosen the classical solution of the Lagrangian which is still gauge invariant. One still expects renormalizability to be realized due to non-trivial cancellations of ultraviolet-divergent diagrams. Power counting only gives the worst possible divergence of a diagram, but a diagram may still converge.

As mentioned, 't Hooft proved the renormalizability of Yang–Mills gauge theories with spontaneous symmetry breaking. One has to choose a different group of gauges, called R_ξ gauges.

Consider the parametrization of the field with real and imaginary components

$$\varphi(x) = \frac{1}{\sqrt{2}}(\varphi_1 + \mathrm{i}\varphi_2)$$

and recall the kinetic term

$$(D_\mu\varphi)^\dagger(D_\mu\varphi) = \frac{1}{2}(\partial_\mu\varphi_1 + gA_\mu\varphi_2)^2 + \frac{1}{2}(\partial_\mu\varphi_2 - gA_\mu\varphi_1)^2$$

In the above expression, one has to substitute the following to parametrize the small fluctuations

$$\varphi_1 \to \varphi_1 + v$$

In this way one finds a mixed term with an unclear interpretation

$$-gvA^\mu\,\partial_\mu\varphi_2$$

It looks like a two point vertex of a mass term.

One observers that, before the spontaneous symmetry breaking, the Lagrangian is renormalizable. The gauge field is massless and one has to include the gauge-fixing term. One may choose a gauge condition different from Lorenz's. The original gauge-fixing term

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

is replaced by a general gauge-fixing function

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2\xi}(\partial_\mu A^\mu + \xi M\varphi_2)^2 = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 - M\,\partial_\mu A^\mu\,\varphi_2 - \frac{1}{2}\xi M^2\varphi_2^2$$

The second addendum cancels the mixed term when integrating by parts. One also notices that the field φ_2 gets a mass term.

Propagators. The free Lagrangian without the Faddeev–Popov ghost term becomes

$$\begin{aligned}\mathcal{L}_0 &= \frac{1}{2}(\partial_\mu\varphi_1\,\partial^\mu\varphi_1 - 2m^2\varphi_1^2) + \frac{1}{2}(\partial_\mu\varphi_2\,\partial^\mu\varphi_2 - \xi M^2\varphi_2^2)\\ &\quad - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \frac{1}{2}M^2 A_\mu A^\mu\end{aligned}$$

From this term one reads the propagators. The two scalar propagators

$$\Delta_1(p) = \frac{1}{p^2 - 2m^2 + \mathrm{i}\varepsilon}\,, \quad \Delta_2(p) = \frac{1}{p^2 - \xi M^2 + \mathrm{i}\varepsilon}$$

One should be careful about the second expression, because the pole of a propagator is the physical mass of a particle: one is propagating some particles. The gauge field propagator is

$$\Delta_{\mu\nu}(p) = \frac{1}{p^2 - M^2 + \mathrm{i}\varepsilon}\left[-\eta_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2 - \xi M^2}\right] \sim \frac{1}{p^2}\,, \quad p^2 \to \infty$$

In the ultraviolet, the gauge field propagator is damped and power counting works correctly: spontaneous-symmetry-broken gauge theories are renormalizable.

One must notice that this is just the Higgs sector: one should also add matter fields.

Exercise. Check that $\Delta_{\mu\nu}$ is the inverse of the kinetic term.

Vertices. The interaction Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\text{int}} &= gA_\mu(\varphi_2\,\partial^\mu\varphi_1 - \varphi_1\,\partial^\mu\varphi_2) + \frac{1}{2}g^2 A_\mu A^\mu(\varphi_1^2 + \varphi_2^2)\\ &\quad - \frac{\lambda}{4}(\varphi_1^2 + \varphi_2^2)^2 + g^2 v A_\mu A^\mu\varphi_1 - \lambda v\varphi_1(\varphi_1^2 + \varphi_2^2)\end{aligned}$$

Exercise. Derive this expression from the original Lagrangian.

Remark. In the R_ξ gauges, the field φ_2 does not decouple and describes peculiar particles which are also found in the gauge field propagator. One expects that unitarity is restored due to the balance between non-physical particles of the field φ_2 and the non-physical contributions in the gauge field A_μ.

Renormalizability is manifest, but not unitarity. There is a propagation of spurious degrees of freedom. Unitarity is restored if the contributions coming from spurious degrees of freedom cancel.

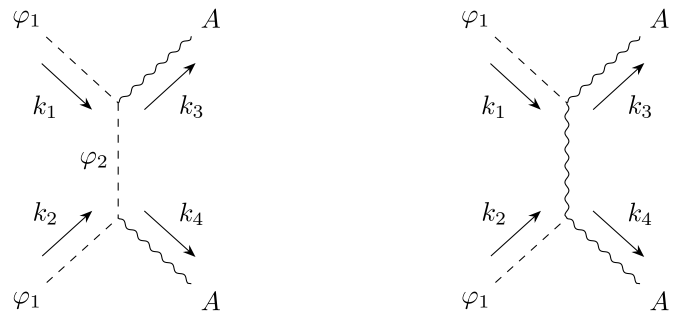
Example. To check that the cancellation happens, one may rewrite the gauge field propagator in a different way

$$\Delta_{\mu\nu}(p) = \frac{-\eta_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}}{p^2 - M^2 + \mathrm{i}\varepsilon} - \frac{p_\mu p_\nu}{M^2}\frac{1}{p^2 - \xi M^2 + \mathrm{i}\varepsilon}$$

One see a division from the typical massive propagator and the unphysical part. Consider the tree-level contributions to the scattering process[40]

$$\varphi_1(k_1) + \varphi_1(k_2) \to A(k_3) + A(k_4)$$

the relevant contributing diagrams are



Looking at the vertices of the theory and computing the Feynman amplitudes, one finds that the gauge field propagator cancels the spurious contribution (i.e. pole $p^2 = \xi M^2$) from the φ_2 field propagator.

Remark. In the unitary gauge, the spurious scalar field disappears and the gauge field contains only the physical massive propagator. One may notice that the unitary gauge is recovered from the R_ξ gauge in the limit $\xi \to \infty$.

Exercise. Check. See Cheng, eqs. 9.97 and 9.98.

Remark. The whole discussion also applies for non-abelian gauge theories.

Remark. This implies that the Standard Model is a renormalizable theory.

27 Anomalies

See Srednicki, Cheng (triangle alternative way, not seen), Ramond (triangle), Serone[41], Bilal[42], Tong[43], and Weinberg, vol. 2, §22.2. [r] chapters.

Intuition. Source?. Consider a classical theory with a global symmetry. Let ϕ be the set of fields. The symmetry formally corresponds to the fact that the classical action is invariant under the global transformation of the symmetry

$$S[\phi] = S[\phi']$$

[40]Taken from Fujikawa, K., Lee, B., & Sanda, A. (1972). Generalized Renormalizable Gauge Formulation of Spontaneously Broken Gauge Theories. Phys. Rev. D, 6, 2923–2943. https://doi.org/10.1103/PhysRevD.6.2923.

[41]Notes on Quantum Field Theory, SISSA.

[42]Lectures on Anomalies, arXiv: 0802.0634.

[43]Gauge Theory?.

For a quantum theory, one considers the effective action (Serone, eq. 9.4.5)

$$e^{-\Gamma[\varphi_c]} = \int [\mathcal{D}\phi]\, e^{-S[\phi]}$$

Under the symmetry transformation, the classical action is invariant, but the path integral measure may not be. If the measure is not invariant, then the symmetry is not preserved in the quantum theory and it exhibits an anomaly.

If the measure is invariant, the effective action $\Gamma[\phi]$ is invariant: the symmetry is preserved in the quantum theory. If the measure is not invariant, then the symmetry is broken in the quantum theory: the symmetry is anomalous.

The breaking of the symmetry is due to the regularization procedure of the theory. For example, massless QED is a scale-invariant field theory. In the regularization, one introduces an energy scale and the theory is no longer scale-invariant. This is the reason why the coupling constant runs. At the fixed points of the RG flow, one recovers scale-invariance by also seeing that the stress-energy tensor is traceless since

$$\langle T^\mu_\mu \rangle \sim \beta(g)$$

Thus, at the fixed points, the parameters no longer run.

Example. A few examples of anomalous symmetries are the following.

- For space-time [r] symmetries, there are scale symmetry and conformal symmetry. Poincaré symmetry is always preserved (except when putting a cutoff).
- For internal symmetries, in four space-time dimensions there is only chiral symmetry acting on fermions.

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27.1 Classical chiral symmetry

See Ramond, §8.9. Consider Euclidean formalism. Consider the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(\not{D} + \mathrm{i}m)\psi = \bar{\psi}(\not{\partial} + \mathrm{i}m)\psi + \mathrm{i}q\bar{\psi}\not{A}\psi\,, \quad D_\mu = \partial_\mu + \mathrm{i}qA_\mu$$

Phase transformation. The gauge U(1) transformations of the fields are

$$\psi' = e^{\mathrm{i}\alpha(x)}\psi\,, \quad \bar{\psi}' = e^{-\mathrm{i}\alpha(x)}\bar{\psi}\,, \quad A'_\mu = A_\mu - \frac{1}{q}\partial_\mu\alpha$$

The global symmetry corresponds to a conserved Noether current

$$J^\mu = \mathrm{i}\bar{\psi}\gamma^\mu\psi$$

The conserved current can be found in a simple way. Consider the free action

$$S = \int d^4x\, \bar{\psi}(\not{\partial} + \mathrm{i}m)\psi$$

The variation of the action under the gauge transformation is

$$\begin{aligned}\delta S &= \int d^4x\, [\delta\bar{\psi}\,(\not{\partial} + \mathrm{i}m)\psi + \bar{\psi}(\not{\partial} + \mathrm{i}m)\,\delta\psi] \\ &= \int d^4x\, \mathrm{i}\,\partial_\mu\alpha(x)\,\bar{\psi}\gamma^\mu\psi = -\int d^4x\,\alpha(x)\,\partial_\mu(\mathrm{i}\bar{\psi}\gamma^\mu\psi)\end{aligned}$$

The symmetry of the action under any function $\alpha(x)$ implies

$$\delta S = 0 \implies \partial_\mu(\mathrm{i}\bar{\psi}\gamma^\mu\psi) = 0 \implies \partial_\mu J^\mu = 0$$

Chiral transformation. Consider chiral transformations of the fields

$$\psi' = \mathrm{e}^{\mathrm{i}\beta\gamma_5}\psi \,, \quad \bar{\psi}' = \bar{\psi}\mathrm{e}^{\mathrm{i}\beta\gamma_5}$$

The kinetic term transforms as

$$\begin{aligned}\bar{\psi}' \,\partial\!\!\!/ \psi' &= \bar{\psi}\mathrm{e}^{\mathrm{i}\beta\gamma_5} \,\partial\!\!\!/ (\mathrm{e}^{\mathrm{i}\beta\gamma_5}\psi) = \bar{\psi}\,\partial\!\!\!/ \psi + \bar{\psi}\mathrm{e}^{\mathrm{i}\beta\gamma_5}\,\partial_\mu(\mathrm{e}^{-\mathrm{i}\beta\gamma_5})\gamma^\mu\psi \\ &= \bar{\psi}\,\partial\!\!\!/ \psi - \mathrm{i}\,\partial_\mu\beta(x)\,\bar{\psi}\gamma_5\gamma^\mu\psi = \bar{\psi}\,\partial\!\!\!/ \psi + \mathrm{i}\,\partial_\mu\beta(x)\,\bar{\psi}\gamma^\mu\gamma_5\psi\end{aligned}$$

The mass term transforms as

$$\bar{\psi}'\psi' = \bar{\psi}\mathrm{e}^{\mathrm{i}\beta\gamma_5}\mathrm{e}^{\mathrm{i}\beta\gamma_5}\psi = \bar{\psi}\mathrm{e}^{2\mathrm{i}\beta\gamma_5}\psi = \bar{\psi}\psi + 2\mathrm{i}\beta\bar{\psi}\gamma_5\psi + o(\beta)$$

Therefore, the variation of the action is

$$\begin{aligned}\delta S &= \int \mathrm{d}^4x\,[\mathrm{i}\,\partial_\mu\beta(x)\,\bar{\psi}\gamma^\mu\gamma_5\psi - 2m\beta(x)\bar{\psi}\gamma_5\psi] \\ &= -\int \mathrm{d}^4x\,\beta(x)[\partial_\mu(\mathrm{i}\bar{\psi}\gamma^\mu\gamma_5\psi) + 2m\bar{\psi}\gamma_5\psi] = 0\end{aligned}$$

This implies that

$$\boxed{\partial_\mu(\mathrm{i}\bar{\psi}\gamma^\mu\gamma_5\psi) = -2m\bar{\psi}\gamma_5\psi}$$

One defines the chiral current to be

$$J^5_\mu = \mathrm{i}\bar{\psi}\gamma_\mu\gamma_5\psi$$

For massive fields, the chiral current is not conserved

$$\partial^\mu J^5_\mu = -2m\bar{\psi}\gamma_5\psi$$

and chiral transformations are not a symmetry of the chiral theory. For massless fields, the theory possesses chiral symmetry

$$\partial^\mu J^5_\mu = 0$$

27.2 Quantum chiral symmetry — ABJ anomaly in massless QED

One would like to study whether chiral symmetry is maintained also in the massless quantum theory (i.e. it survives loops corrections). One finds that the symmetry is not kept. In massless QED, it is called ABJ (Adler–Bell–Jackiw) chiral anomaly.

One may study the anomaly (in particular the conservation equations) in two ways.

- The first is perturbative: at one-loop, one finds that the current is not conserved. However, this result also holds non-perturbatively, because anomalies typically do not get higher order corrections, like in this case.
- The second is studying how the path integral measure is not invariant under chiral transformations.

27.2.1 One-loop contribution

Consider the action

$$S = \int \mathrm{d}^4x \left[\bar{\psi}(\partial\!\!\!/ + \mathrm{i}qA\!\!\!/)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]$$

The matter field term contains the global U(1) current

$$\mathrm{i}\bar{\psi}A\!\!\!/\psi = A_\mu J^\mu$$

If one defines a functional integral where one integrates only the spinor fields, then the gauge field is a background field that acts a source for the current. [r] To recover the original functional integral, the gauge field is integrated at a later time. Applying the same reasoning to the chiral current, one defines

$$\mathrm{e}^{-Z[A_\mu, S^5_\mu]} = \int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\,\mathrm{e}^{-S[\psi,\bar{\psi},A_\mu,S^5_\mu]}$$

where one has

$$S[\psi,\bar{\psi},A_\mu,S^5_\mu]=\int \mathrm{d}^4x\,[\bar{\psi}\,\partial\!\!\!/\psi+\mathrm{i}qA_\mu\bar{\psi}\gamma^\mu\psi+\mathrm{i}S^5_\mu\bar{\psi}\gamma^\mu\gamma_5\psi]$$

The last two terms are currents and they are composite operators. This is the generating functional of such composite operators.

One would like to see whether the chiral current is still conserved

$$\partial^\mu J^5_\mu=0$$

in the quantum theory. One computes the three-point correlation function

$$\langle J_\mu(x)J_\nu(y)J^5_\rho(z)\rangle$$

and, if the chiral symmetry is preserved, one should find

$$\frac{\partial}{\partial z^\rho}\langle J_\mu(x)J_\nu(y)J^5_\rho(z)\rangle=\langle J_\mu(x)J_\nu(y)\,\partial_\rho J^5_\rho(z)\rangle=0$$

Remark. Since the chiral current J^5 is even under charge conjugation, the only non-zero n-point correlation functions are the ones with an even number of currents J_μ or fields A_μ (which are odd under charge conjugation).

Remark. Integrating in four dimensions, one finds integrals with γ_5. Typically, one brings the Dirac matrices outside the integral and computes a pure momentum integral by analytically continuing the integral in $D=4-2\varepsilon$ dimensions. The fifth gamma matrix is not defined outside of four dimensions. One writes the matrix as product of the other matrices [r].

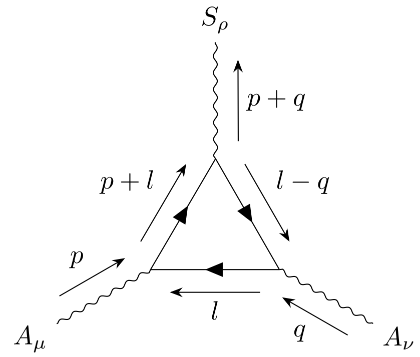
One may combine dimensional regularization with dimensional reduction [r] source?: one brings the external momenta outside the integral and analytically continues only the integration variable; when computing the gamma structure, the Dirac matrices related to the external momenta are in four dimensions, while the ones coming from the integrated momentum are in different dimensions. In this way one also keeps γ_5 in four dimensions. One has to first compute the integral then apply the identities of the gamma matrices.

This problem was already mention in an article[44] by 't Hooft and it is solved with the HVBM scheme[45].

The three-point correlation function is given by

$$\langle J_\mu(x)J_\nu(y)J^5_\rho(z)\rangle=-\left.\frac{\delta^3\mathrm{e}^{-Z[A_\mu,S^5_\mu]}}{\delta A_\mu(x)\,\delta A_\nu(y)\,\delta S^5_\rho(z)}\right|_{A=S^5=0}$$

One needs to compute the diagram

$T_{\mu\nu\rho}(p,q)\equiv$ 

In momentum space, the conservation of the chiral current means

$$(p+q)_\rho T_{\mu\nu\rho}(p,q)=0$$

[44] G. 't Hooft, M. Veltman, Regularization and renormalization of gauge fields, Nuclear Physics B, Volume 44, Issue 1, 1972, Pages 189-213, ISSN 0550-3213, `https://doi.org/10.1016/0550-3213(72)90279-9`.

[45] Breitenlohner, P., Maison, D. Dimensional renormalization and the action principle. Commun. Math. Phys. 52, 11–38 (1977). `https://doi.org/10.1007/BF01609069`. For a clear description of this regularization procedure see Jäger, Barbara (2004), Studies of hadronic spin structure in hard scattering processes at the next-to-leading order of QCD. PhD, Universität Regensburg. `https://doi.org/10.5283/epub.10206`. Section §2.2.

Feynman rules. The fermion propagator, and the vector and axial vertices are

$$-\frac{\mathrm{i}}{\not p}\,, \quad \mathrm{i}qk^\varepsilon\gamma_\mu\,, \quad \mathrm{i}k^\varepsilon\gamma_\mu\gamma_5$$

Triangle diagram. The integral associated to the diagram is

$$\begin{aligned}
T_{\mu\nu\rho}(p,q) &= -\mathrm{i}q^2k^{3\varepsilon}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\operatorname{Tr}\left[\gamma_\rho\gamma_5\frac{-\mathrm{i}}{\not l-\not q}\gamma_\nu\frac{-\mathrm{i}}{\not l}\gamma_\mu\frac{-\mathrm{i}}{\not l+\not p}\right]\\
&= q^2k^{3\varepsilon}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{\operatorname{Tr}[\gamma_5\gamma_\rho(\not l-\not q)\gamma_\nu\not l\gamma_\mu(\not l+\not p)]}{(l-q)^2l^2(l+p)^2}
\end{aligned}$$

At the second line, one has applied[46]

$$-\frac{\mathrm{i}}{\not l} = \mathrm{i}\frac{\not l}{l^2}$$

One has to symmetrize the external photon lines $\mu\leftrightarrow\nu$, $p\leftrightarrow q$ (and send $l\to -l$ in the second term) to find

$$\begin{aligned}
T &= T_{\mu\nu\rho}(p,q)+T_{\nu\mu\rho}(q,p)\\
&= q^2k^{3\varepsilon}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{1}{(l-q)^2l^2(l+q)^2}\\
&\quad\times\left\{\operatorname{Tr}[\gamma_5\gamma_\rho(\not l-\not q)\gamma_\nu\not l\gamma_\mu(\not l+\not p)]-\operatorname{Tr}[\gamma_5\gamma_\rho(\not l+\not p)\gamma_\mu\not l\gamma_\nu(\not l-\not q)]\right\}\\
&= q^2k^{3\varepsilon}[\operatorname{Tr}(\gamma_5\gamma_\rho\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\mu\gamma_\gamma)-\operatorname{Tr}(\gamma_5\gamma_\rho\gamma_\gamma\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\alpha)]\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{(l-q)_\alpha l_\beta(l+p)_\gamma}{(l-q)^2l^2(l+p)^2}\\
&= \text{Traces}\times 2\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^dl}{(2\pi)^d}\left[\frac{l_\alpha l_\beta l_\gamma}{(\text{Den})^3}+\frac{p_\gamma l_\alpha l_\beta - q_\alpha l_\beta l_\gamma}{(\text{Den})^3}-q_\alpha p_\gamma\frac{l_\beta}{(\text{Den})^3}\right]\\
&= \text{Traces}\times I = q^2k^{3\varepsilon}(T_1+T_2+T_3)
\end{aligned}$$

At the third line, one uses dimensional regularization with dimensional reduction. At the fifth line, one has applied Feynman combining with

$$\text{Den} = x(l-q)^2+y(l+p)^2+(1-x-y)l^2 = l^2+2lP+M^2$$

where

$$P_\mu = yp_\mu - xq_\mu\,, \quad M^2 = xq^2+yp^2$$

and integrals

$$\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y$$

To evaluate the integral, one applies eqs. B.17, B.18, B.19 of Ramond. Consider the first integral

$$I_1 = 2\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{(-1)}{(4\pi)^{2-\varepsilon}\Gamma(3)}\left[P_\alpha P_\beta P_\gamma\frac{\Gamma(1+\varepsilon)}{D^{1+\varepsilon}}+\frac{1}{2}(\delta_{\alpha\gamma}P_\beta+\delta_{\beta\gamma}P_\alpha+\delta_{\alpha\beta}P_\gamma)\frac{\Gamma(\varepsilon)}{D^\varepsilon}\right]$$

where

$$D\equiv M^2-P^2$$

One has to compute the integral with the traces. Consider the first addendum

$$\begin{aligned}
\operatorname{Tr}_{1\mathrm{a}} &= [\operatorname{Tr}(\gamma_5\gamma_\rho\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\mu\gamma_\gamma)-\operatorname{Tr}(\gamma_5\gamma_\rho\gamma_\gamma\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\alpha)]P_\alpha P_\beta P_\gamma\\
&= \operatorname{Tr}(\gamma_5\gamma_\rho\not P\gamma_\nu\not P\gamma_\mu\not P)-(\mu\leftrightarrow\nu)\\
&= P^2\operatorname{Tr}(\gamma_5\gamma_\rho\gamma_\nu\gamma_\mu\not P)-2\operatorname{Tr}(\gamma_5\gamma_\rho\not P\gamma_\mu\not P)P_\nu-(\mu\leftrightarrow\nu)\\
&= P^2\operatorname{Tr}(\gamma_5\gamma_\rho\gamma_\nu\gamma_\mu\not P)+0-(\mu\leftrightarrow\nu)\\
&= 2P^2\operatorname{Tr}(\gamma_5\gamma_\rho\gamma_\nu\gamma_\mu\not P)
\end{aligned}$$

[46] Notice that the gamma matrices in Euclidean are not the same as in Minkowski. See Ramond, p. 169, in particular eq. 5.2.42. The change of sign comes from the definition of the Clifford algebra, see eq. 5.2.35.

At the third line and beyond, one has applied the following gamma technology

$$\begin{gathered}\not{P}\gamma_\mu\not{P} = P^2\gamma_\mu - 2P_\mu\not{P} \\ \mathrm{Tr}[\gamma_5\gamma_\mu\gamma_\nu] = 0\,, \qquad \text{in 4D} \\ \gamma_\alpha\gamma_\mu\gamma^\alpha = 2(1-\varepsilon)\gamma_\mu\end{gathered}$$

The second addendum is

$$\begin{aligned}\mathrm{Tr}_{1\mathrm{b}} &= [\mathrm{Tr}(\gamma_5\gamma_\rho\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\mu\gamma_\gamma) - \mathrm{Tr}(\gamma_5\gamma_\rho\gamma_\gamma\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\alpha)](\delta_{\alpha\gamma}P_\beta + \delta_{\beta\gamma}P_\alpha + \delta_{\alpha\beta}P_\gamma) \\ &= [8(1-\varepsilon) - 4(1-\varepsilon) + 8]\,\mathrm{Tr}(\gamma_5\gamma_\rho\gamma_\nu\gamma_\mu\not{P})\end{aligned}$$

Therefore, the total contribution is

$$\mathrm{Tr}_1 = \mathrm{Tr}_{1\mathrm{a}} + \mathrm{Tr}_{1\mathrm{b}} = 4(3-\varepsilon)\,\mathrm{Tr}(\gamma_5\gamma_\rho\gamma_\nu\gamma_\mu\not{P})$$

The contribution to the triangle diagram is then

$$T_1 = -2\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, \frac{\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}D^{1+\varepsilon}} \left[P^2 + \frac{3-\varepsilon}{\varepsilon}D\right] \mathrm{Tr}(\gamma_5\gamma_\rho\gamma_\nu\gamma_\mu\not{P})$$

Similarly, the second contribution is

$$\begin{aligned}T_2 = \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, \frac{\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}D^{1+\varepsilon}} \bigg[&\left(2P^2 + 2\frac{1-\varepsilon}{\varepsilon}D\right) \mathrm{Tr}[\gamma_5\gamma_\rho\gamma_\nu\gamma_\mu(\not{p}-\not{q})] \\ &- 2P_\nu\,\mathrm{Tr}[\gamma_5\gamma_\rho\not{P}\gamma_\mu(\not{p}-\not{q})] + 2P_\mu\,\mathrm{Tr}[\gamma_5\gamma_\rho\not{P}\gamma_\nu(\not{p}-\not{q})]\bigg]\end{aligned}$$

[r] check, see Ramond, eq. 8.9.73
while the third contribution is

$$T_3 = -\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, \frac{\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}D^{1+\varepsilon}} [\mathrm{Tr}(\gamma_5\gamma_\rho\not{p}\gamma_\mu\not{P}\gamma_\nu\not{q}) - \mathrm{Tr}(\gamma_5\gamma_\rho\not{q}\gamma_\nu\not{P}\gamma_\mu\not{p})]$$

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The first two integrals contain diverging contributions. The divergent terms are proportional to D which is simplified by one factor at the denominator. The expansion of the integral in ε gives a constant term.

The divergent term from the first integral becomes

$$J_1 = -2\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \frac{\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}D^\varepsilon}\frac{1}{2}\frac{2-\varepsilon}{\varepsilon}\,\mathrm{Tr}[\gamma_5\gamma_\rho\gamma_\nu\gamma_\mu(\not{p}-\not{q})]$$

The divergent term from the second integral becomes

$$J_2 = 2\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, \frac{\Gamma(1+\varepsilon)}{(2\pi)^{2+\varepsilon}D^\varepsilon}\frac{1-\varepsilon}{\varepsilon}\,\mathrm{Tr}[\gamma_5\gamma_\rho\gamma_\nu\gamma_\mu(\not{p}-\not{q})]$$

[r] The sum of the two is

$$\begin{aligned}J_1 + J_2 &= \left[-2\frac{1}{2}\frac{2-\varepsilon}{\varepsilon} + 2\frac{1-\varepsilon}{\varepsilon}\right]\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, \frac{\Gamma(1+\varepsilon)}{(2\pi)^{2+\varepsilon}D^\varepsilon}\,\mathrm{Tr}[\gamma_5\gamma_\rho\gamma_\nu\gamma_\mu(\not{p}-\not{q})] \\ &= -\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, \frac{\Gamma(1+\varepsilon)}{(2\pi)^{2+\varepsilon}D^\varepsilon}\,\mathrm{Tr}[\gamma_5\gamma_\rho\gamma_\nu\gamma_\mu(\not{p}-\not{q})]\end{aligned}$$

Therefore, the triangle diagram gives

$$\begin{aligned}T_{\mu\nu\rho}(p,q) &= -q^2\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, \frac{\Gamma(1+\varepsilon)}{(2\pi)^{2-\varepsilon}D^\varepsilon}\,\mathrm{Tr}[\gamma_5\gamma_\rho\gamma_\nu\gamma_\mu(\not{p}-\not{q})] + \text{finite} \\ &= -\frac{q^2}{16\pi^2}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\ \mathrm{Tr}[\gamma_5\gamma_\rho\gamma_\nu\gamma_\mu(\not{p}-\not{q})] + \text{finite}\,, \quad \varepsilon \to 0 \\ &= -\frac{q^2}{32\pi^2}\,\mathrm{Tr}[\gamma_5\gamma_\rho\gamma_\nu\gamma_\mu(\not{p}-\not{q})] + \text{finite}\end{aligned}$$

At the second line, one may take the limit $\varepsilon \to 0$ and perform the integrals.

One may then find

$$(p+q)_\rho T_{\mu\nu\rho}(p,q) = -\frac{q^2}{32\pi^2}\,\mathrm{Tr}[\gamma_5(\not{p}+\not{q})\gamma_\nu\gamma_\mu(\not{p}-\not{q})]$$

The finite terms do not contribute when contracting with $p+q$. Finally, one may evaluate the trace

$$\begin{aligned}
\mathrm{Tr} = \mathrm{Tr}[\gamma_5(\not{p}+\not{q})\gamma_\nu\gamma_\mu(\not{p}-\not{q})] &= (p+q)_\rho(p-q)_\sigma\,\mathrm{Tr}(\gamma_5\gamma_\rho\gamma_\nu\gamma_\mu\gamma_\sigma) \\
&= (p_\rho p_\sigma - q_\rho q_\sigma - p_\rho q_\sigma + q_\rho p_\sigma)\,\mathrm{Tr}(\gamma_5\gamma_\rho\gamma_\nu\gamma_\mu\gamma_\sigma) \\
&= (p_\rho p_\sigma - q_\rho q_\sigma - p_\rho q_\sigma + q_\rho p_\sigma)\,\mathrm{Tr}(\gamma_\sigma\gamma_5\gamma_\rho\gamma_\nu\gamma_\mu) \\
&= -(p_\rho p_\sigma - q_\rho q_\sigma + q_\rho p_\sigma - p_\rho q_\sigma)\,\mathrm{Tr}(\gamma_5\gamma_\sigma\gamma_\rho\gamma_\nu\gamma_\mu) \\
&= (p_\rho q_\sigma - q_\rho p_\sigma)\,\mathrm{Tr}(\gamma_5\gamma_\sigma\gamma_\rho\gamma_\nu\gamma_\mu) \\
&= p_\rho q_\sigma\,\mathrm{Tr}(\gamma_5[\gamma_\sigma,\gamma_\rho]\gamma_\nu\gamma_\mu) \\
&= 2p_\rho q_\sigma\,\mathrm{Tr}(\gamma_5\gamma_\sigma\gamma_\rho\gamma_\nu\gamma_\mu) - p_\rho q_\sigma\,\mathrm{Tr}(\gamma_5\{\gamma_\sigma,\gamma_\rho\}\gamma_\nu\gamma_\mu) \\
&= 2p_\rho q_\sigma\,\mathrm{Tr}(\gamma_5\gamma_\sigma\gamma_\rho\gamma_\nu\gamma_\mu) \\
&= 2p_\rho q_\sigma(-4\varepsilon_{\sigma\rho\nu\mu}) = -8\varepsilon_{\mu\nu\rho\sigma}p_\rho q_\sigma \neq 0
\end{aligned}$$

At the fifth line, by the symmetry $p \leftrightarrow q$, one has noted $p_\rho p_\sigma = q_\rho q_\sigma$. At the sixth line, one has noticed that the expression is anti-symmetric in $\rho\sigma$ and one may introduce the commutator of gamma matrices in the trace

$$(p_\rho q_\sigma - q_\rho p_\sigma)\gamma_\sigma\gamma_\rho = p_\rho q_\sigma\gamma_\sigma\gamma_\rho - q_\rho p_\sigma\gamma_\sigma\gamma_\rho = p_\rho q_\sigma\gamma_\sigma\gamma_\rho - q_\sigma p_\rho\gamma_\rho\gamma_\sigma = p_\rho q_\sigma[\gamma_\sigma,\gamma_\rho]$$

where at the second equality one renames $\rho \leftrightarrow \sigma$ in the second addendum, since they are dummy indices. At the seventh line, one has traded the commutator for the anti-commutator

$$2\gamma_\sigma\gamma_\rho = \{\gamma_\sigma,\gamma_\rho\} + [\gamma_\sigma,\gamma_\rho] \implies [\gamma_\sigma,\gamma_\rho] = 2\gamma_\sigma\gamma_\rho - \{\gamma_\sigma,\gamma_\rho\}$$

At the penultimate line, the second term is zero since the anti-commutator gives the metric and the trace of two matrices and the fifth matrix is identically zero.

In the end, one finds the ABJ anomaly

$$\boxed{(p+q)_\rho T_{\mu\nu\rho}(p,q) = \frac{q^2}{4\pi^2}\varepsilon_{\mu\nu\rho\sigma}p_\rho q_\sigma}$$

27.2.2 Non-perturbative contribution

See also Srednicki, §77, and Weinberg, vol. 2, §22.2. One may wonder if the anomaly gets contributions from higher orders. To this end, one may compute the anomaly using Fujikawa's non-perturbative method.

The relevant functional integral is

$$\mathrm{e}^{-Z[A]} = \int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\,\mathrm{e}^{-S[\psi,\bar{\psi},A]}$$

where the gauge field is a background field which acts as a source for the electromagnetic current. The action is

$$S[\psi,\bar{\psi},A] = \bar{\psi}\,\not{D}\psi = \bar{\psi}\,\not{\partial}\psi + \mathrm{i}gA_\mu\bar{\psi}\gamma^\mu\psi$$

A local chiral transformation of the spinor fields is

$$\psi' = \mathrm{e}^{\mathrm{i}\beta(x)\gamma_5}\psi \sim \psi + \mathrm{i}\beta(x)\gamma_5\psi\,, \quad \bar{\psi}' = \bar{\psi}\mathrm{e}^{\mathrm{i}\beta(x)\gamma_5} \sim \bar{\psi} + \mathrm{i}\beta(x)\bar{\psi}\gamma_5$$

For an infinitesimal transformation, the action becomes

$$S'[\psi',\bar{\psi}',A] = S[\psi,\bar{\psi},A] - \int \mathrm{d}^4x\,\beta(x)\,\partial_\mu(\mathrm{i}\bar{\psi}\gamma_\mu\gamma_5\psi)$$

One would like to understand whether the consistency of the procedure requires the conservation of the chiral current

$$\partial_\mu J^5_\mu = 0$$

One has to impose the chiral transformation also on the measure. If the Jacobian coming from the measure is non-trivial then the current is not conserved and an anomaly appears.

Jacobian. The spinor fields are Grassmann-odd fields and the Jacobian of the change of variables is the reciprocal of typical integration

$$[\mathcal{D}\psi'\,\mathcal{D}\bar{\psi}'] = (\det J)^{-2}[\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]$$

The trick to compute the Jacobian is the following: the fields ψ and ψ' are integration variables differing by an infinitesimal local chiral transformation, the functional integral can be written with either. Therefore, one obtains

$$\begin{aligned}\mathrm{e}^{-Z[A]} &= \int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\,\mathrm{e}^{-S[\psi,\bar{\psi},A]} = \int [\mathcal{D}\psi'\,\mathcal{D}\bar{\psi}']\,\mathrm{e}^{-S[\psi',\bar{\psi}',A]} \\ &= \int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\,(\det J)^{-2}\mathrm{e}^{-S[\psi,\bar{\psi},A]}\exp\left[\int \mathrm{d}^4x\,\beta(x)\,\partial_\mu J^5_\mu\right]\end{aligned}$$

[r] The chiral transformation is implemented by the functional matrix $J(x,y)$ (i.e. the Jacobian matrix)

$$\psi'(x) = \mathrm{e}^{\mathrm{i}\beta(x)\gamma_5}\psi(x) = \int \mathrm{d}^4y\,\mathrm{e}^{\mathrm{i}\beta(x)\gamma_5}\delta^{(4)}(x-y)\psi(y) = \int \mathrm{d}^4y\,J(x,y)\psi(y)$$

Consider the infinitesimal transformation (Weinberg, vol. 2, eq. 22.2.9)

$$J(x,y) = 1 + \mathrm{i}\beta(x)\gamma_5\delta^{(4)}(x-y)$$

The Jacobian is then

$$\begin{aligned}(\det J)^{-2} &= \exp[-2\ln\det J] = \exp[-2\,\mathrm{Tr}\ln J] \\ &= \exp\left[-2\,\mathrm{Tr}\Big(\mathrm{i}\beta(x)\gamma_5\delta^{(4)}(x-y)\Big)\right] \\ &= \exp\left[-2\mathrm{i}\int \mathrm{d}^4x\,\beta(x)\delta^{(4)}(x-x)\,\mathrm{Tr}\,\gamma_5\right]\end{aligned}$$

At the second line, one has applied $\ln(1+x)\sim x$; the trace is done on both the spinor discrete indices (and eventually flavour indices) and the continuous position indices. At the third line, the trace is over only spinor (and flavour) indices: the integral over $\mathrm{d}x$ is the trace over continuous indices.

The obtained expression is only formal. To make sense of it, one has to regularize (and substitute) the delta function (with a Gaussian)

$$\delta^{(4)}(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{\mathrm{i}k(x-y)} \to \int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-\frac{k^2}{M^2}}\mathrm{e}^{\mathrm{i}k(x-y)}$$

where M is a regularization parameter that is removed in the limit $M\to\infty$. The momentum in the exponent can be replaced by the derivative in the position

$$\delta^{(4)}(x-y) \to \int \frac{\mathrm{d}^4k}{(2\pi)^4}\exp\left[\frac{\partial_x^2}{M^2}\right]\mathrm{e}^{\mathrm{i}k(x-y)} = \exp\left[\frac{\partial_x^2}{M^2}\right]\delta^{(4)}(x-y)$$

However, one would like to keep gauge invariance even during regularization: one has to use the covariant derivative. Therefore

$$\partial_x \to \mathrm{i}\not{D}_x$$

where the imaginary unit is needed to keep the self-adjoint nature of the operator. Thus

$$\begin{aligned}\delta^{(4)}(x-y)|_{\mathrm{reg}} &= \exp\left[\frac{(\mathrm{i}\not{D}_x)^2}{M^2}\right]\delta^{(4)}(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\exp\left[-\frac{(\not{D}_x)^2}{M^2}\right]\mathrm{e}^{\mathrm{i}k(x-y)} \\ &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{\mathrm{i}k(x-y)}\exp\left[-\frac{(\not{D}_x+\mathrm{i}\not{k})^2}{M^2}\right]\end{aligned}$$

At the second line, one has used the following general identity

$$f(\partial_x)\mathrm{e}^{\mathrm{i}kx} = \mathrm{e}^{\mathrm{i}kx}f(\partial_x+\mathrm{i}k)$$

which holds also for the covariant derivative. The exponent is

$$\begin{aligned}(\not{D}+\mathrm{i}\not{k})^2 &= \not{D}\not{D}-\not{k}\not{k}+\mathrm{i}\not{D}\not{k}+\mathrm{i}\not{k}\not{D}\\ &=\gamma_\mu\gamma_\nu D_\mu D_\nu-\gamma_\mu\gamma_\nu k_\mu k_\nu+\mathrm{i}\gamma_\mu\gamma_\nu(D_\mu k_\nu+k_\mu D_\nu)\\ &=\frac{1}{2}(\{\gamma_\mu,\gamma_\nu\}+[\gamma_\mu,\gamma_\nu])D_\mu D_\nu-\frac{1}{2}\{\gamma_\mu,\gamma_\nu\}k_\mu k_\nu+\mathrm{i}\{\gamma_\mu,\gamma_\nu\}D_\mu k_\nu\\ &=-D^2+\frac{1}{2}[\gamma_\mu,\gamma_\nu]D_\mu D_\nu+k^2-2\mathrm{i}D_\mu k_\mu=-D^2-2\mathrm{i}\frac{\mathrm{i}}{4}[\gamma_\mu,\gamma_\nu]D_\mu D_\nu+k^2-2\mathrm{i}D_\mu k_\mu\\ &=-D^2-2\mathrm{i}\gamma_{\mu\nu}D_\mu D_\nu+k^2-2\mathrm{i}D_\mu k_\mu=-D^2+k^2-2\mathrm{i}D_\mu k_\mu-2\mathrm{i}\gamma_{\mu\nu}D_\mu D_\nu\\ &=-D^2+k^2-2\mathrm{i}D_\mu k_\mu-\mathrm{i}\gamma_{\mu\nu}[D_\mu,D_\nu]\\ &=-D^2+k^2-2\mathrm{i}D_\mu k_\mu-\mathrm{i}\gamma_{\mu\nu}\mathrm{i}gF_{\mu\nu}=-D^2+k^2-2\mathrm{i}D_\mu k_\mu+\gamma_{\mu\nu}gF_{\mu\nu}\end{aligned}$$

At the third line, one has rewritten $\gamma_\mu\gamma_\nu$ in the first two addenda as the sum of symmetric and anti-symmetric parts. In the second addendum, the second is zero due to $k_\mu k_\nu$ being symmetric. At the fourth line, one has applied the Clifford algebra

$$\{\gamma_\mu,\gamma_\nu\}=-2\delta_{\mu\nu}$$

At the fifth line, one recognizes the generators of the Lorentz group in the spinor representation

$$\gamma_{\mu\nu}\equiv\frac{\mathrm{i}}{4}[\gamma_\mu,\gamma_\nu]$$

At the sixth line, one rewrites the two covariants derivatives in terms of symmetric and anti-symmetric parts. At the ultimate line, one recalls (alternatively see[47] Ramond, eq. 8.9.39)

$$\mathrm{i}gF_{\mu\nu}=[D_\mu,D_\nu]$$

The regularized delta function is

$$\begin{aligned}\delta^{(4)}(x-y)|_{\mathrm{reg}} &= \int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{\mathrm{i}k(x-y)}\exp\left[-\frac{1}{M^2}(-D^2+k^2-2\mathrm{i}D_\mu k_\mu+\gamma_{\mu\nu}gF_{\mu\nu})\right]\\ &=M^4\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\mathrm{e}^{\mathrm{i}Mk'(x-y)}\mathrm{e}^{-k'^2}\exp\left[\frac{D^2}{M^2}+\frac{2\mathrm{i}}{M}k'_\mu D_\mu-\frac{g}{M^2}\gamma_{\mu\nu}F_{\mu\nu}\right]\end{aligned}$$

At the second line, one changes variables

$$k_\mu=Mk'_\mu$$

One may compute the trace (over spinor indices) with the regularized Delta function

$$\begin{aligned}\mathrm{Tr}\Big[\delta^{(4)}_{\mathrm{reg}}(x-x)\gamma_5\Big] &= M^4\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-k^2}\,\mathrm{Tr}\left[\exp\left(\frac{D^2}{M^2}+\frac{2\mathrm{i}}{M}k_\mu D_\mu-\frac{g}{M^2}\gamma_{\mu\nu}F_{\mu\nu}\right)\gamma_5\right]\\ &=M^4\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-k^2}\,\mathrm{Tr}\left[\gamma_5+\left(\frac{D^2}{M^2}+\frac{2\mathrm{i}}{M}k_\mu D_\mu-\frac{g}{M^2}\gamma_{\mu\nu}F_{\mu\nu}\right)\gamma_5+\cdots\right]\\ &=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-k^2}\frac{1}{2}g^2\,\mathrm{Tr}[\gamma_{\mu\nu}F_{\mu\nu}\gamma_{\rho\sigma}F_{\rho\sigma}\gamma_5]\,,\quad M\to\infty\\ &=\frac{g^2}{2}\left[\frac{\mathrm{i}}{4}\right]^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-k^2}F_{\mu\nu}F_{\rho\sigma}\,\mathrm{Tr}([\gamma_\mu,\gamma_\nu][\gamma_\rho,\gamma_\sigma]\gamma_5)\\ &=-\frac{g^2}{2}\frac{1}{16}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-k^2}F_{\mu\nu}F_{\rho\sigma}4\,\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5)\\ &=-\frac{g^2}{2}\frac{\pi^2}{4(2\pi)^4}F_{\mu\nu}F_{\rho\sigma}\,\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5)=-\frac{g^2}{8}\frac{\pi^2}{(2\pi)^4}F_{\mu\nu}F_{\rho\sigma}(-4)\varepsilon_{\mu\nu\rho\sigma}\\ &=\frac{g^2}{32\pi^2}\varepsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\end{aligned}$$

[47] Also recall that Srednicki is in Minkowski, while this computation is being done in Euclidean.

At the second line, one has Taylor expanded the exponential in the limit $M \to \infty$. At the third line, one has applied

$$\operatorname{Tr}\gamma_5 = 0\,, \quad \operatorname{Tr}(\gamma_5\gamma_\alpha\gamma_\beta) = 0\,, \quad \operatorname{Tr}(\gamma_5\gamma_\alpha\gamma_\beta\gamma_\gamma) = 0$$

[r] and noted that only one term survives which comes for the quadratic term in the expansion. At the fifth line, one may evaluate the integral by going to spherical coordinates and recalling the definition of the Euler's Gamma function

$$\int \mathrm{d}^4k\, \mathrm{e}^{-k^2} = \pi^2 \int_0^\infty \mathrm{d}t\, t\mathrm{e}^{-t} = \pi^2\Gamma(2) = \pi^2$$

The Jacobian is thus

$$\begin{aligned}(\det J)^{-2} &= \exp\left[-2\mathrm{i}\int \mathrm{d}^4x\,\beta(x)\delta^{(4)}(x-x)\operatorname{Tr}\gamma_5\right] \\ &= \lim_{M\to\infty}\exp\left[-2\mathrm{i}\int \mathrm{d}^4x\,\beta(x)\operatorname{Tr}\left(\delta^{(4)}_{\mathrm{reg}}(x-x)\gamma_5\right)\right] \\ &= \exp\left[-\mathrm{i}\frac{g^2}{16\pi^2}\int \mathrm{d}^4x\,\beta(x)\varepsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right]\end{aligned}$$

Anomaly. Finally, the transformation of the measure together with the transformation of the action gives the Adler–Bardeen theorem

$$-\frac{\mathrm{i}g^2}{16\pi^2}\int \mathrm{d}^4x\,\beta(x)\varepsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + \int \mathrm{d}^4x\,\beta(x)\,\partial_\mu J^5_\mu = 0$$

From which the chiral anomaly

$$\boxed{\partial_\mu J^5_\mu = \frac{\mathrm{i}g^2}{16\pi^2}\varepsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}} = \frac{\mathrm{i}g^2}{8\pi^2}F_{\mu\nu}\widetilde{F}_{\mu\nu} \equiv -\mathcal{A}$$

where the Hodge dual tensor is

$$\widetilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$$

The generalization to N fermions is

$$\partial_\mu J^5_\mu = \frac{\mathrm{i}g^2}{8\pi^2}NF_{\mu\nu}\widetilde{F}_{\mu\nu}$$

Comparison with the triangle diagram. Consider the anomaly above written differently

$$\partial_\rho J^5_\rho = \frac{\mathrm{i}g^2}{16\pi^2}\varepsilon_{\alpha\mu\sigma\nu}F_{\alpha\mu}F_{\sigma\nu} = \frac{\mathrm{i}g^2}{16\pi^2}\,\partial_{[\alpha}A_{\mu]}\,\partial_{[\sigma}A_{\nu]} = \frac{\mathrm{i}g^2}{4\pi^2}\varepsilon_{\alpha\mu\sigma\nu}\,\partial_\alpha A_\mu\,\partial_\sigma A_\nu$$

Going to momentum space and setting the momenta like in the triangle diagram

$$\partial_\rho \to \mathrm{i}(p+q)_\rho\,, \quad \partial_\alpha \to -\mathrm{i}p_\alpha\,, \quad \partial_\sigma \to -\mathrm{i}q_\sigma$$

Therefore

$$\mathrm{i}(p+q)_\rho T_{\mu\nu\rho}(p,q) = \frac{\mathrm{i}g^2}{4\pi^2}(-\mathrm{i})^2 p_\alpha q_\sigma\varepsilon_{\alpha\mu\sigma\nu} \implies (p+q)_\rho T_{\mu\nu\rho}(p,q) = \frac{g^2}{4\pi^2}\varepsilon_{\mu\nu\alpha\sigma}p_\alpha q_\sigma$$

This is the same result as the one-loop computation. The chiral anomaly is one-loop exact: since the computation does not rely on an expansion in power of the coupling constant, there are no higher-order corrections.

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27.3 Abelian gauge theories

Sources? [r], Bilal, §4.1.2?. The Fujikawa procedure can be generalized to any kind of classical symmetry, not only chiral: one is interested in how the functional measure changes under such symmetry. One wonders if it is possible to have an anomaly corresponding to a classical conserved current of a gauge symmetry. In the previous Fujikawa method, one has momentarily assumed that the transformation parameter is local. For gauge theories, the parameter is already local and one may apply the method immediately.

Consider a U(1) gauge theory (i.e. massless QED) with one fermion. Starting from the functional integral

$$\mathrm{e}^{-Z[A]} = \int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\,\exp\left[-\int \mathrm{d}^4x\,\bar{\psi}\,\not{D}\psi\right], \quad D_\mu = \partial_\mu + \mathrm{i}A_\mu\,, \quad g = 1$$

the quantum fields are only the fermions and the gauge field is a background field. If one would like to consider all of the content of the theory, one has to also consider the kinetic term of the gauge field. The generating functional is

$$W = \int [\mathcal{D}A_\mu]\,\exp\left[\int \mathrm{d}^4x\,\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]\mathrm{e}^{-Z[A]} \equiv \int [\mathcal{D}A_\mu]\,\mathrm{e}^{-\widetilde{S}[A]}$$

where

$$\widetilde{S}[A] = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + Z[A]$$

One may study if the generating functional W is gauge invariant. Assuming no Jacobian for the measure (which typically happens only for fermions), then the generating functional is gauge invariant if $\widetilde{S}[A]$ is invariant.

Gauge invariance. Following the same steps as the previous section, $\widetilde{S}$ is gauge invariant only if $Z[A]$ is gauge invariant, which is equivalent to the absence of anomalies. Consider the variation

$$A'_\mu = A_\mu + \delta A_\mu\,, \quad \delta A_\mu = \partial_\mu\lambda(x)$$

Therefore

$$\mathrm{e}^{-Z[A+\delta A]} = \int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\,\mathrm{e}^{-S[\psi,\bar{\psi},A+\delta A]}$$

Performing a change of variables $\psi \to \psi'$ where ψ' are the gauge-transformed fields, one finds

$$\mathrm{e}^{-Z[A+\delta A]} = \int [\mathcal{D}\psi'\,\mathcal{D}\bar{\psi}']\,\mathrm{e}^{-S[\psi',\bar{\psi}',A+\delta A]}$$

Because of gauge invariance, the action does not change

$$S[\psi',\bar{\psi}',A+\delta A] = S[\psi,\bar{\psi},A]$$

while the measure changes as

$$[\mathcal{D}\psi'\,\mathcal{D}\bar{\psi}'] = [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\,(\det J)^{-2} = [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\,\exp\left[-\int \mathrm{d}^4x\,\lambda(x)\mathcal{A}(x)\right]$$

Therefore

$$\begin{aligned}\mathrm{e}^{-Z[A+\delta A]} &= \int [\mathcal{D}\psi'\,\mathcal{D}\bar{\psi}']\,\mathrm{e}^{-S[\psi',\bar{\psi}',A+\delta A]} = \int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\,\mathrm{e}^{-S[\psi,\bar{\psi},A]}\,\exp\left[-\int \mathrm{d}^4x\,\lambda(x)\mathcal{A}(x)\right]\\ &= \exp\left[-\int \mathrm{d}^4x\,\lambda(x)\mathcal{A}(x)\right]\mathrm{e}^{-Z[A]}\end{aligned}$$

The gauge transformation of the generating functional is

$$\delta_{\text{gauge}}Z[A] = Z[A+\delta A] - Z[A] = -\int \mathrm{d}^4x\,\lambda(x)\mathcal{A}(x)$$

If the gauge symmetry is anomalous, then $Z[A]$ is not gauge invariant. This implies that $\widetilde{S}[A]$ and W are not gauge invariant

$$\boxed{\delta_{\text{gauge}} W \neq 0}$$

This result is undesirable for two reasons:

- Gauge invariance is required in order to cancel non-physical degrees of freedom (i.e. longitudinal) of the quantized gauge field. No gauge invariance means that the longitudinal degrees of freedom do not decouple from physical scattering amplitudes.
- A non-invariant theory under gauge transformations is also not Lorentz-invariant. The most general Lorentz transformation of the gauge field is (see Weinberg, eq. 5.9.31)

  $$U^{-1}(\Lambda) A_\mu(x) U(\Lambda) = \Lambda_\mu{}^\nu A_\nu(\Lambda^{-1} x) + \partial_\mu \lambda(x)$$

  In the action, the relevant term transforms as

  $$\begin{aligned}\int \mathrm{d}^4x\, A_\mu(x) J^\mu(x) \to \int &\mathrm{d}^4x\, (U^{-1} A_\mu U + \partial_\mu \lambda) U^{-1} J^\mu U \\ &= \int \mathrm{d}^4x\, U^{-1} A_\mu J^\mu U + \int \mathrm{d}^4x\, \partial_\mu \lambda\, U^{-1} J^\mu U \\ &= \int \mathrm{d}^4x\, U^{-1} A_\mu J^\mu U - \int \mathrm{d}^4x\, \lambda U^{-1}\, \partial_\mu J^\mu U \\ &= \int \mathrm{d}^4x\, A_\mu J^\mu + \int \mathrm{d}^4x\, \lambda U^{-1}\, \mathcal{A} U\end{aligned}$$

  which is not Lorentz-invariant due to the second integral.

One concludes that gauge theories cannot exhibit anomalies. Equivalently, gauge currents cannot develop an anomaly term $\mathcal{A}(x)$.

Example. See Ramond, §6.1?. An example of anomalous theory is massless chiral QED. A Dirac field ψ can be split into chiral components

$$\psi_{\mathrm{L}} = \frac{1-\gamma_5}{2} \psi$$

Consider the Lagrangian

$$\mathcal{L}_{\text{chiral}} = \bar{\psi}_{\mathrm{L}}\, \not{D} \psi_{\mathrm{L}} \,, \quad D_\mu = \partial_\mu + \mathrm{i} A_\mu$$

This Lagrangian is classically invariant under the gauge transformations of $\mathrm{U}(1)_{\mathrm{L}}$

$$\psi'_{\mathrm{L}} = \mathrm{e}^{\mathrm{i}\alpha(x)} \psi_{\mathrm{L}} \,, \quad \bar{\psi}'_{\mathrm{L}} = \bar{\psi}_{\mathrm{L}} \mathrm{e}^{-\mathrm{i}\alpha(x)} \,, \quad A'_\mu = A_\mu - \partial_\mu \alpha(x)$$

In fact, the transformation of the Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{chiral}} &= \bar{\psi}_{\mathrm{L}}\, \not{\partial} \psi_{\mathrm{L}} + \mathrm{i} A_\mu \bar{\psi}_{\mathrm{L}} \gamma^\mu \psi_{\mathrm{L}} = \bar{\psi}_{\mathrm{L}}\, \not{\partial} \psi_{\mathrm{L}} + \mathrm{i} A_\mu \bar{\psi} \gamma^\mu \frac{1}{2}(1-\gamma_5)\psi \\ &= \bar{\psi}_{\mathrm{L}}\, \not{\partial} \psi_{\mathrm{L}} + \frac{1}{2} A_\mu J^\mu - \frac{1}{2} A^\mu J^5_\mu\end{aligned}$$

where one has

$$J_\mu = \mathrm{i} \bar{\psi}\, \gamma_\mu \psi \,, \quad J^5_\mu = \mathrm{i} \bar{\psi} \gamma_\mu \gamma_5 \psi$$

Computing the four-divergence of the chiral current, one finds

$$\partial^\mu J^5_\mu = -\mathcal{A}(x) = \frac{\mathrm{i} g^2}{16\pi^2} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

where $\mathcal{A}(x)$ is the chiral anomaly. This current is not associated to a global chiral transformation, but it is a current associated to a gauge transformation. Therefore gauge invariance is broken and the theory is inconsistent.

27.4 Non-abelian gauge theories

See Srednicki, §75 (at the very end), Peskin, § 19.4, Tong, §3.4.2, others? [r]. One looks for possible anomalies in non-abelian global symmetries. One may relax the global condition keeping in mind that if the symmetry is global to begin with, then the anomaly can be accepted, but not for a local symmetry.

Recall the triangle diagrams with two axial currents and one chiral current (and include the contribution from swapping the two gauge field lines). In a non-abelian theory, the external currents possess color indices

$$J^a_\mu = \mathrm{i}\bar{\psi}^j (T^a)_{jk}\gamma_\mu \psi^k \,, \quad J^{5a}_\mu = \mathrm{i}\bar{\psi}^j (T^a)_{jk}\gamma_\mu\gamma_5 \psi^j$$

In general, one considers matter fields to be in a generic representation R of the gauge group. Following the computations of the triangle diagrams and keeping in mind the color structure, the expression of the anomaly has to be adjusted. Due to the fermionic loop, one sums over all the color indices of the representation and one produces a trace of the gauge group generators. Since the expression is symmetrized under the exchange of the two axial currents, one finds (see Peskin, eq. 19.139)

$$\mathrm{Tr}\big(\{T^a_R, T^b_R\} T^c_R\big) \equiv \frac{1}{2} A(R) d^{abc}$$

where $A(R)$ is called the anomaly coefficient[48] of the representation R and d^{abc} is a completely symmetric tensor independent of the representation[49]. The triangle diagram corresponds to the three-point function

$$\big\langle J^a_\mu(x) J^b_\nu(y)\, \partial_\rho J^{5c}_\rho(z)\big\rangle = -\mathcal{A}\, d^{abc}(R)$$

In non-abelian gauge theories, there is a color factor d^{abc} that may give no anomaly if it is zero.

In a given representation R, one has (see Tong)

$$d^{abc}(R) = A(R) d^{abc}(\text{fund})\,, \quad A(\text{fund}) = 1\,, \quad d^{abc}(\text{fund}) \equiv d^{abc}$$

The second factor comes from group theory, while $A(R)$ is the charges of the particles in the representation R [r] ??. From group theory, one knows that

$$A(\bar{R}) = -A(R)$$

This is valid for fermionic particles in the representation $R \oplus \bar{R}$. The matter particles in $\bar{R}$ give the opposite contributions of the matter particles in R. For more families of matter, one has to sum multiple triangle diagrams. The total contribution is then

$$\begin{aligned} d^{abc}(R \oplus \bar{R}) &= A(R \oplus \bar{R}) d^{abc}(\text{fund}) = [A(R) + A(\bar{R})] d^{abc}(\text{fund}) \\ &= [A(R) - A(R)] d^{abc}(\text{fund}) = 0 \end{aligned}$$

One concludes that, in vector theories (i.e. particles come in pairs in the representation $R \oplus \bar{R}$), there are no anomalies. In chiral theories, some fermions are in a representation R while the paired fermions are not in the conjugate representation $\bar{R}$ [r] (for example a Lagrangian with left-chiral Weyl spinors that transform, while right-chiral Weyl spinors that do not, see Peskin, eq. 19.121). Anomalies may arise only in chiral theories.

For real and pseudo-real representations, one has $A(R) = 0$. Example of anomaly-free groups are SU(2), SO(n) and the exceptional groups. The anomalies may come from U(1) and SU(n) with $n \geq 3$.

In general, the total anomaly coefficient is given by

$$A = \sum_R n(R) A(R)$$

where the sum is over the representations R of the matter fields and $n(R)$ is the number of fermions in such representation.

The presence of an anomaly depends on the matter content of the theory: how many fermions and how many in which representation.

[48] Sometimes called instead anomaly of the representation.

[49] It appears in the fundamental representation of SU(n) with $n \geq 3$, $\{T^a_n, T^b_n\} = n^{-1}\delta^{ab} + d^{abc} T^c_n$.

Composite group. See Tong and others? [r]. Recall that

$$d^{abc} \propto \operatorname{Tr}\big(\{T^a, T^b\}T^c\big)$$

The symmetry group may not be a simple group, but a product of subgroups

$$G = G_1 \times G_2$$

In this case, the generators of the group are of the form

$$T = T_1^a \otimes I_2 + I_1 \otimes T_2^i$$

One may produce any kind of d factor. One may have all generators of one subgroup, or the other subgroup or any combination

$$d^{abc}\,, \quad d^{abi}\,, \quad d^{aij}\,, \quad d^{ijk}$$

When taking the trace, one traces over the representations of both subgroups

$$\begin{aligned}
d^{abc}(R) &= d^{abc}(R_1)\dim R_2 \\
d^{abi}(R) &\propto 2\operatorname{Tr}_2(T^i)C(R_1)\delta^{ab} \\
d^{aij}(R) &\propto 2\operatorname{Tr}_1(T^a)C(R_2)\delta^{ij} \\
d^{ijk}(R) &= d^{ijk}(R_2)\dim R_1
\end{aligned}$$

where $C(f)$ is the Dynkin index [r] and $R = R_1 \otimes R_2$.

Example. Consider the singlet anomaly $G_2 = \mathrm{U}(1)$. The generator T^i of the subgroup is the identity. Therefore

$$d^{abi} = \operatorname{Tr}\big(\{T^a, T^b\}\big) = \operatorname{Tr}_1(\{T^a, T^b\})\operatorname{Tr}_2 I$$

From the current associated to the subgroup G_2, one finds

$$\partial_\mu J^{\mu i} \propto d^{abi}\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$$

where $F_{\mu\nu}$ is the abelian field strength tensor. This corresponds to the triangle diagram with one abelian current and two non-abelian currents.

27.4.1 Standard Model

See Weinberg, §22.4, Bilal, §7.3, Tong, §3.4.4. The gauge group of the Standard Model is $\mathrm{SU}(3)_\mathrm{C}\times\mathrm{SU}(2)_\mathrm{L}\times\mathrm{U}(1)_\mathrm{Y}$. Consider a single generation of leptons and quarks. The representations and the U(1) charges[50] are

	SU(3)	SU(2)	U(1)
q_L	3	2	$1/6$
l_L	1	2	$-1/2$
u_R	$\bar{3}$	1	$-2/3$
d_R	$\bar{3}$	1	$1/3$
e_R	1	1	1

The upper part corresponds to $A(R)$ and the lower to $A(\bar{R})$ [r]. Notice that the right-chiral fermions are intended as left-chiral anti-fermions (by charge conjugation, see Bilal).

Pure color group. [r] Considering the anomaly diagram $G = [\mathrm{SU}(3)]^3$ with

$$\operatorname{Tr}\big(\{T^a, T^b\}T^c\big)\,, \quad T^a \in \mathfrak{su}(3)$$

Since all particles come in pairs of $3 \oplus \bar{3}$, then there is no pure SU(3) anomaly.

[50] Recall that the (weak version of the) Gell-Mann–Nishijima formula is $Q = T_3 + \frac{1}{2}Y$. Here one uses the alternative half-scale $Q = T_3 + Y$. Note that Weinberg has $Q = T_3 - Y$.

Pure isospin group. Consider $G = [\mathrm{SU}(2)]^3$ and $T^a \in \mathfrak{su}(2)$. The particles either come in doublets (corresponding to the fundamental representation) or in singlets (trivial representation). They are pseudo-real and real representations respectively (see Weinberg, Tong). Therefore $A(R) = 0$ (see Bilal, eq. 7.15) and there is no SU(2) anomaly.

Pure hypercharge group. Consider $G = [\mathrm{U}(1)]^3$. One has $T^a = IQ$, where Q is the U(1) charge. The anomaly the sum of the hypercharges multiplied by the dimensions of the non-abelian factors

$$A_{\mathrm{Y}} = \left[(3 \cdot 2) \cdot \left(\frac{1}{6}\right)^3 + 2 \cdot \left(-\frac{1}{2}\right)^3\right] + \left[3 \cdot \left(-\frac{2}{3}\right)^3 + 3 \cdot \left(\frac{1}{3}\right)^3 + 1\right] = 0$$

Remark. The matter content of the Standard Model and the hypercharge assignment are consistent with a theory free of anomaly.

One has also to consider the mixed anomalies.

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Mixed anomalies. The only non-trivial mixed anomalies that may appear are $\mathrm{U}(1) \times [\mathrm{SU}(3)]^2$ and $\mathrm{U}(1) \times [\mathrm{SU}(2)]^2$. For a single factor of the special unitary group, the d-coefficient is proportional to the trace of the generators, but these are traceless.

Let the index of the unitary group be Latin and the index of the special unitary group be Greek. Recall that

$$d^{a\beta\gamma} \propto \mathrm{Tr}_1(T^a) C(R_2) \delta^{\beta\gamma}$$

If R_2 is the fundamental representation, then the Dynkin index is $C(R_2) = \frac{1}{2}$. If R_2 is the singlet representation (i.e. trivial), then $C(R_2) = 0$.

For $\mathrm{U}(1) \times [\mathrm{SU}(2)]^2$, there is no contribution from the right-chiral sector, only from the left-chiral. The SU(3) group is a spectator, but contributes with a factor of 3 due to the number of colors. Therefore

$$d^{a\beta\gamma} \propto \frac{1}{2}\left[3 \cdot \frac{1}{6} + 1 \cdot \left(-\frac{1}{2}\right)\right] + 0 \cdot \left[3 \cdot \left(-\frac{2}{3}\right) + 3 \cdot \frac{1}{3} + 1 \cdot 1\right] = 0$$

For $\mathrm{U}(1) \times [\mathrm{SU}(3)]^2$, one has [r]

$$d^{a\beta\gamma} \propto \frac{1}{2}\left[2 \cdot \frac{1}{6} + 1 \cdot \left(-\frac{2}{3}\right) + 1 \cdot \frac{1}{3}\right] + 0 \cdot \left[1 \cdot \left(-\frac{1}{2}\right) + 1 \cdot 1\right] = 0$$

Conclusion. This concludes the study of anomalies in the Standard Model. If the gauge symmetry is anomalous, one loses gauge invariance and Lorentz invariance. The gauge group of the Standard Model is not anomalous. This is why the quantum numbers of the fields are fixed as in the table above.

Remark. Notice that requiring that fermions in $\mathrm{SU}(2)_{\mathrm{L}}$ couple to electromagnetism in an anomaly-free gauge theory implies that there is an equal number of families of quarks and leptons (see Peskin, eq. 19.136).

27.4.2 Quantum chromodynamics

See Serone, §9.2 (towards the end). Consider QCD with n_f flavours of massless fermions in the fundamental representation of the (global) flavour group

$$\mathrm{SU}(n_f)_{\mathrm{V}} \times \mathrm{SU}(n_f)_{\mathrm{A}} \times \mathrm{U}(1)_{\mathrm{V}} \times \mathrm{U}(1)_{\mathrm{A}}$$

where V and A stand for vector and axial.

The anomalies may be:

- There is no pure gauge group $\mathrm{SU}(3)_{\mathrm{C}}$ anomaly, because fermions come in pairs.

- Consider an anomaly diagram with only one $\mathrm{SU}(n_f)_{\mathrm{V/A}}$ factor and recall the formula for the factor $d^{ab\gamma}$. This coefficient is proportional to the trace of a generator, but the generators of this group are traceless.

- The only possible anomalies come from $\mathrm{U}(1)_{\mathrm{V}} \times [\mathrm{SU}(3)]^2$ and $\mathrm{U}(1)_{\mathrm{A}} \times [\mathrm{SU}(3)]^2$. In QCD, quarks and anti-quarks have opposite $\mathrm{U}(1)_{\mathrm{V}}$ charges, so the first anomaly is zero; though, they have the same $\mathrm{U}(1)_{\mathrm{A}}$ charge and such group is anomalous.

27.5 Wess–Zumino consistency condition

See Bilal, §9.1, Serone, §9.5, Weinberg, §22.6, others? [r]. One would like to find a general structure of anomalies [r]. One would like to use the existence of global anomalies and 't Hooft anomalies to constrain the theory, in particular in the non-perturbative regime.

Consider the functional integral

$$\mathrm{e}^{-Z[A]} = \int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\,\exp\left[-\int \mathrm{d}^4x\,\bar{\psi}\,\not{D}\psi\right]$$

In the presence of an anomaly, for a gauge transformation of the form

$$\delta A_\mu = \partial_\mu \lambda$$

one has

$$\delta Z[A] = Z[A+\delta A] - Z[A] = -\int \mathrm{d}^4x\,\lambda(x)\,\mathcal{A}(x)$$

One may also have

$$\begin{aligned}\delta Z[A] &= \int \mathrm{d}^4x\,\delta A_\mu(x)\,\frac{\delta Z[A]}{\delta A_\mu(x)} = \int \mathrm{d}^4x\,\partial_\mu\lambda(x)\,\frac{\delta Z[A]}{\delta A_\mu(x)} \\ &= -\int \mathrm{d}^4x\,\lambda(x)\,\partial_\mu\frac{\delta Z[A]}{\delta A_\mu(x)} = -\int \mathrm{d}^4x\,\lambda(x)X(x)Z[A]\end{aligned}$$

where one defines

$$X(x) \equiv \partial_\mu \frac{\delta}{\delta A_\mu(x)}$$

Comparing these two expressions for any λ, one finds

$$\boxed{X(x)Z[A] = \mathcal{A}(x)}$$

For non-abelian gauge theories, the gauge field also carries a colour index. The procedure is the same, but with the covariant derivative. The gauge transformation is

$$A^a \to A^a + \delta A^a\,, \quad \delta A^a = (D_\mu)^{ab}\lambda^b(x)\,, \quad (D_\mu)^{ab}\lambda^b(x) = \partial_\mu\lambda^a + f^{abc}A^b_\mu\lambda^c\,, \quad g=1$$

The variation of the generating functional is

$$\begin{aligned}\delta Z[A] &= \int \mathrm{d}^4x\,\delta A^a_\mu(x)\,\frac{\delta Z[A]}{\delta A^a_\mu(x)} = \int \mathrm{d}^4x\,(D_\mu)^{ab}\lambda^b(x)\,\frac{\delta Z[A]}{\delta A^a_\mu(x)} \\ &= -\int \mathrm{d}^4x\,\lambda^b(x)\,(D_\mu)^{ba}\,\frac{\delta Z[A]}{\delta A^a_\mu(x)} = -\int \mathrm{d}^4x\,\lambda^a(x)X^a(x)Z[A]\end{aligned}$$

where one defines

$$X^b(x) \equiv (D_\mu)^{ba}\,\frac{\delta}{\delta A^a_\mu(x)}$$

Comparing for any λ, one finds

$$\boxed{X^a(x)Z[A] = \mathcal{A}^a(x)}$$

Algebra. The object $X^a(x)$ satisfies the algebra (Bilal, eq. 9.7)

$$[X^a(x), X^b(y)] = -f^{abc} X^c(x) \delta^{(4)}(x-y)$$

To check this, one may write X explicitly

$$[X^a(x), X^b(y)] = \left[\partial_{x\mu} \frac{\delta}{\delta A^a_\mu(x)} + f^{acd} A^c_\mu(x) \frac{\delta}{\delta A^d_\mu(x)}, \partial_{y\nu} \frac{\delta}{\delta A^b_\nu(y)} + f^{bfg} A^f_\nu(y) \frac{\delta}{\delta A^g_\nu(y)} \right]$$

where one knows that

$$[\partial_\mu, f] g = \partial_\mu (fg) - f\, \partial_\mu g = (\partial_\mu f) g$$

and apply the commutator to a dummy object. For the complete computation[51] see Bilal, eq. 9.5. [r]

Consider applying the algebra of X to the generating functional $Z[A]$ keeping in mind the relation with the anomaly:

$$\begin{aligned} [X^a(x), X^b(y)] Z[A] &= -f^{abc} X^c(x) Z[A] \delta^{(4)}(x-y) \\ X^a(x)(X^b(y) Z[A]) - X^b(y)(X^a(x) Z[A]) &= -f^{abc} X^c(x) Z[A] \delta^{(4)}(x-y) \\ \Aboxed{X^a(x) \mathcal{A}^b(y) - X^b(y) \mathcal{A}^a(x) &= -f^{abc} \mathcal{A}^c(x) \delta^{(4)}(x-y)} \end{aligned}$$

This is the Wess–Zumino consistency condition.

27.6 BRST transformations

See Srednicki, §74?, Bilal, §9.2, Weinberg, §22.6, from p. 398. One may reformulate the consistency condition of an anomaly using BRST transformations. In a gauge theory, one has to introduce a gauge-fixing term which adds ghost fields. The Lagrangian is not gauge invariant, but is BRST invariant. The BRST transformations are

$$\delta A^a_\mu = \omega (D_\mu)^{ab} \sigma^b \,, \quad \delta \rho^a = -\frac{\mathrm{i}}{\xi} \omega\, \partial^\mu A^a_\mu \,, \quad \delta \sigma^a = -\frac{\omega}{2} f^{abc} \sigma^b \sigma^c \,, \quad \delta \psi = \mathrm{i} \omega (T^a \sigma^a) \psi$$

where ω is a Grassmann-odd constant and $g = 1$. The transformations may be rewritten as the commutator of the field with the BRST charge (i.e. the operator that enacts the transformation)

$$\delta A^a_\mu = [\omega Q_{\mathrm{BRST}}, A^a_\mu] \implies Q A^a_\mu = (D_\mu)^{ab} \sigma^b$$

One has found that $Q^2 = 0$: applying twice the BRST transformations gives zero. This is the reason why the total Lagrangian is invariant. The total Lagrangian is

$$\mathcal{L} = (\mathcal{L}_\psi + \mathcal{L}_{\mathrm{g}}) + (\mathcal{L}_{\mathrm{gf}} + \mathcal{L}_{\mathrm{gh}})$$

The last parenthesis can be rewritten as a product with the charge. For this reason, one finds

$$Q\mathcal{L} = Q(\mathcal{L}_\psi + \mathcal{L}_{\mathrm{g}}) + QQ(\cdots) = Q(\mathcal{L}_\psi + \mathcal{L}_{\mathrm{g}}) + 0 = 0$$

The first addendum is zero because of classical gauge invariance.

In order to study the connection between the BRST charge and anomalies, it is better to use the complex ghost fields. The ghost Lagrangian (in the non-abelian Lorenz gauge) is[52]

$$\mathcal{L}_{\mathrm{gh}} = \bar{c}^a\, \partial_\mu (D^\mu)^{ab} c^b$$

The gauge-fixing Lagrangian may be rewritten as

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2\xi} (\partial_\mu A^a_\mu)^2 = b^a (\partial_\mu A^a_\mu) + \frac{1}{2} \xi b^a b^a$$

[51] And a few considerations like Schwarz's theorem for the symmetry of second derivatives and the derivative of the Dirac delta function.

[52] In this case, there is no need to identify $\bar{c}$ with the Hermitian conjugate of c. It is more convenient to treat them as separate real Grassmann-odd fields. See Srednicki, after eq. 74.16.

where b^a is the Lautrup–Nakanishi auxiliary field, it is not dynamical. This can be done because, when integrating out (i.e. replacing) the field b with its equations of motion, one gets back the original term: the equality holds on-shell. In fact

$$\frac{\delta\mathcal{L}_{\mathrm{gf}}}{\delta b^a} = (\partial_\mu A^a_\mu) + \xi b^a = 0 \implies b^a = -\frac{1}{\xi}(\partial_\mu A^a_\mu)$$

The BRST transformations have to be changed to accomodate also this new field. One may check that the total Lagrangian is invariant under

$$\delta A^a_\mu = \omega(D_\mu)^{ab}c^b\,, \quad \delta c^a = -\frac{\omega}{2}f^{abc}c^b c^c\,, \quad \delta\bar{c}^a = b^a\,, \quad \delta b^a = 0\,, \quad \delta\psi = \mathrm{i}\omega(T^a c^a)\psi$$

Consider the following functional

$$G(c, A) \equiv \int \mathrm{d}^4x\, c^a(x)\mathcal{A}^a(x)$$

Since the BRST transformations of the gauge field and the matter field are the same as gauge transformations with $\lambda^a = -\omega c^a$, one may wonder what happens when considering

$$\delta_{\mathrm{BRST}}Z[A] = [\omega Q, Z[A]] = \int \mathrm{d}^4x\,\omega c^a(x)\mathcal{A}^a(x)$$

From this, one finds

$$QZ[A] = \int \mathrm{d}^4x\, c^a(x)\mathcal{A}^a(x) = G(c, A)$$

It follows that, by applying again a BRST transformation, one must have

$$0 = Q^2 Z[A] = QG(c, A)$$

One may write explicitly the transformation to find

$$\begin{aligned}
0 = QG(c, A) &= \int \mathrm{d}^4x\,[Qc^a(x)\,\mathcal{A}^a(x) - c^a(x)Q\mathcal{A}^a(x)] \\
&= \int \mathrm{d}^4x \left[-\frac{1}{2}f^{abc}c^b(x)c^c(x)\mathcal{A}^a(x) - c^a(x)\int \mathrm{d}^4y\,(D_\mu)^{bc}c^c(y)\,\frac{\delta\mathcal{A}^a(x)}{\delta A^b_\mu(y)}\right] \\
&= \int \mathrm{d}^4x\,\mathrm{d}^4y \left[-\frac{1}{2}c^a(x)c^b(y)\right]\left[f^{abc}\mathcal{A}^c(x)\delta^{(4)}(x-y) - 2(D_\mu)^{bc}\,\frac{\delta}{\delta A^c_\mu(y)}\mathcal{A}^a(x)\right] \\
&= \int \mathrm{d}^4x\,\mathrm{d}^4y \left[-\frac{1}{2}c^a(x)c^b(y)\right]\left[f^{abc}\mathcal{A}^c(x)\delta^{(4)}(x-y) - 2X^b(y)\mathcal{A}^a(x)\right] \\
&= \int \mathrm{d}^4x\,\mathrm{d}^4y \left[-\frac{1}{2}c^a(x)c^b(y)\right]\left[f^{abc}\mathcal{A}^c(x)\delta^{(4)}(x-y) - X^b(y)\mathcal{A}^a(x) + X^a(x)\mathcal{A}^b(y)\right]
\end{aligned}$$

At the first line, the minus sign is due to the Grassmann-odd nature of the field c. At the second line, one has used the formula for the variation of a functional

$$\delta F[A] = \int \mathrm{d}^4x\,\delta A^a_\mu(x)\,\frac{\delta F[A]}{\delta A^a_\mu(x)} = \int \mathrm{d}^4x\,\omega(D_\mu)^{ab}c^b(x)\,\frac{\delta F[A]}{\delta A^a_\mu(x)}$$

recalling that $\mathcal{A}^a = X(x)^a Z[A]$ and $\delta A = [\omega Q, A]$. At the third line, one has integrated by parts the covariant derivative in the second addendum. At the last line, one has rewritten the last addendum as symmetric and anti-symmetric parts while noting that $c^a(x)c^b(y)$ is anti-symmetric in the exchange $(a, x) \leftrightarrow (b, y)$.

This implies that the second bracket must be zero. One obtains the Wess–Zumino consistency condition. The condition is a different way to state that $Q^2 = 0$ in the BRST transformations.

Consider the two relations

$$QG(c, A) = 0\,, \quad QZ[A] = G(c, A)$$

for the two following observations.

Remark. One observers that $G(c, A)$ is defined up to Q-exact terms. In fact consider

$$\widetilde{G}(c, A) = G(c, A) + Q\hat{G}[A]$$

The first relation above holds for $\widetilde{G}$. The second the relation is

$$QZ[A] = \widetilde{G}(c, A) - Q\hat{G} \implies Q(Z[A] + \hat{G}) = \widetilde{G}(c, A)$$

By changing the functional integral, one may change the anomaly.

Remark. The first relation has solutions

$$G(c, A) = Q\hat{G}[A]$$

The second relation becomes

$$QZ[A] = Q\hat{G}[A] \implies Q(Z[A] - \hat{G}[A]) = 0$$

One may cancel the anomaly with the addition of the term $\hat{G}$ to the generating functional. This term is interpreted as a local counter term.

Conclusions. When looking for non-trivial (i.e. non-removable) anomalies, one has to consider

$$G \in {}^{\ker(Q)}\!/\!_{\mathrm{Im}(Q)}$$

This means that G is Q-closed, but not Q-exact. The above set is the cohomology of Q.

In other words, anomalies are defined up to equivalence classes

$$G(c, A) \sim G(c, A) + Q\hat{G}[A]$$

where $\hat{G}[A]$ is a functional and does not depend on the ghost field, while G does. The equivalence classes form the cohomology of Q.

27.7 Stora–Zumino descent equations

See Bilal, §§8.3.2?, 9.4?, Weinberg, §22.6, from p. 402, others? [r]. One may find solutions to the consistency condition through the Stora–Zumino descent equations. The functional G in d dimensions can be expressed as a d-form

$$G(c, A) \equiv \int_M \mathrm{d}^d x \, c^a(x) \mathcal{A}^a(x) = \int G_1^{(d)}$$

where the integral is over the space-time manifold M. One introduces the ghost number that counts the ghost fields. The object $G_g^{(k)}$ is a k-form

$$G^{(k)} = G_{\mu_1 \cdots \mu_k} \, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k}$$

with g ghosts. The BRST operator Q has ghost number $g = 1$ and acts as

$$QG_g^{(k)} = G_{g+1}^{(k)}$$

The differential (i.e. exterior derivative) has ghost number $g = 0$ and acts as

$$\mathrm{d}G^{(k)} = \partial_\mu G_{\mu_1 \cdots \mu_k} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k} = G^{(k+1)} \, , \quad \mathrm{d}^2 G^{(k)} = 0$$

Therefore, one has $Q^2 = d^2 = 0$ [r]. The one-forms $\mathrm{d}x^\mu$ are understood to anti-commute with fermionic fields like the ghost field c^a, so the operator d anti-commutes with the operator Q:

$$\{\mathrm{d}, Q\} = 0$$

The set of descent equations can be constructed as follows. From the consistency condition one finds

$$0 = QG(c, A) = \int QG_1^{(d)} = -\int \mathrm{d}G_2^{(d-1)}$$

At the third equality, one considers manifolds without boundary conditions (or trivial conditions). One does not assume that the integrand is zero, but that it is the exterior derivative of a local function so that its integral vanishes. From this, it follows

$$QG_1^{(d)} + \mathrm{d}G_2^{(d-1)} = 0$$

Applying Q to the above gives

$$0 = Q(\mathrm{d}G_2^{(d-1)}) = -\mathrm{d}(QG_2^{(d-1)})$$

Since the last term has to be zero, the parenthesis is a closed form and one applies Poincaré's theorem[53] (with proper hypotheses) to have

$$QG_2^{(d-1)} + \mathrm{d}G_3^{(d-2)} = 0$$

Iterating the procedure, one finds the Stora–Zumino descent equations

$$\begin{aligned} QG_1^{(d)} + \mathrm{d}G_2^{(d-1)} &= 0 \\ QG_2^{(d-1)} + \mathrm{d}G_3^{(d-2)} &= 0 \\ &\vdots \\ QG_{d+1}^{(0)} &= 0 \end{aligned}$$

These equations may be written in another way. Consider a set of generalized forms Ω_{k+g}: the set of forms with the same value of $k+g$. One notices

$$\mathrm{d} : \Omega_{k+g} \to \Omega_{k+1+g}\,, \quad Q : \Omega_{k+g} \to \Omega_{k+g+1}$$

The two operators are endomorphisms, they map the set into itself. It is useful to consider the operator

$$\delta \equiv \mathrm{d} + Q\,, \quad \delta^2 = 0$$

Consider a form in Ω_{d+1}:

$$G_{d+1} = G_0^{(d+1)} + G_1^{(d)} + G_2^{(d-1)} + \cdots + G_{d+1}^{(0)}$$

From

$$\boxed{\delta G_{d+1} = 0}$$

one may collect terms with the same ghost numbers. All the terms have to be set to zero and these conditions are the descent equations.

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27.8 't Hooft anomaly

See Serone, §9.6. In (3+1)D quantum field theory, there are two kinds of anomalies: ABJ (chiral) anomaly and 't Hooft anomaly.

Consider a non-abelian gauge theory with gauge group G_s containing n_f massless fermions minimally coupled to the gauge field which realize a given representation of the flavour symmetry $(G_f)_\mathrm{V} \times (G_f)_\mathrm{A}$.

Non-abelian anomalies are proportional to the trace of the generators d^{abc}. One may decide to have the external currents in the triangle diagrams to be of the group G_s or G_f. For three local currents one has d^{sss}, for two local and one global d^{ssf}, for one local and two global d^{sff} and finally for three global d^{fff}.

For G_f being a non-abelian group, one finds

$$d^{ssf} \propto \mathrm{Tr}\, T^f = 0$$

[53] In a simply connected region of the manifold, any closed k-form is exact. See Weinberg, vol. 1, after eq. 8.8.2. On Wikipedia it is called Poincaré's lemma.

If the group is abelian, then $d^{ssf} \neq 0$. In both cases, one considers the gauge group to be non-abelian so one has
$$d^{sff} \propto \operatorname{Tr} T^s = 0$$
The relevant coefficients that appear in the computations are all but d^{sff}.

Example. For QCD one has

$$G_s = \mathrm{SU}(3)_{\mathrm{C}}\,, \quad G_f = \mathrm{SU}(n_f)_{\mathrm{V}} \times \mathrm{SU}(n_f)_{\mathrm{A}} \times \mathrm{U}(1)_{\mathrm{V}} \times \mathrm{U}(1)_{\mathrm{A}}$$

Scenarios. One finds three possible scenarios. The anomaly makes the functional integral

$$\mathrm{e}^{-Z[A]} = \int [\mathcal{D}\psi\, \mathcal{D}\bar{\psi}]\, \mathrm{e}^{-S[\psi,\bar{\psi},A]}$$

not invariant under gauge transformations

$$\delta Z[A] = -\int \mathrm{d}^4x\, \lambda^a(x) \mathcal{A}^a(x)$$

This means that, under a symmetry transformation, one has

$$\mathrm{e}^{-Z[A]} \to \mathrm{e}^{-Z[A+\delta A]} = \mathrm{e}^{-Z[A]} \exp\left[\int \mathrm{d}^4x\, \lambda^a(x)\mathcal{A}^a(x)\right] \neq \mathrm{e}^{-Z[A]}$$

noting that the anomaly is a functional of the gauge field A_μ [r].

First case. Consider the case with only $d^{sss} \neq 0$. It corresponds to a pure gauge anomaly. If one has gauged the symmetry, then the functional integral $\mathrm{e}^{-Z[A]}$ is not the quantity of interest, but
$$W = \int [\mathcal{D}A_\mu]\, \mathrm{e}^{-Z[A]}$$
This generating functional is not invariant because of the anomaly and the theory is not consistent.

Second case. Consider the case with only $d^{ssf} \neq 0$. The theory is consistent. The global symmetry is anomalous and it is broken.

Third case. Consider the case with only $d^{fff} \neq 0$. The theory is consistent. The global current (i.e. the one of G_f) is anomalous, but the symmetry is not broken. The gauge fields, that are sources to the global current, are not path integrated, so that $\mathrm{e}^{-Z[A]}$ is the generating functional considered. This functional transforms with a constant (i.e. the anomaly) but it can be reabsorbed into the normalization.

This case is a disaster when attempting to gauge the group G_f. One has to path integrate the gauge field. This case is the 't Hooft anomaly. The global group G_f of a theory with a 't Hooft anomaly cannot be gauged.

27.8.1 't Hooft anomaly matching condition

Consider asymptotically free Yang–Mills theories. These can be studied perturbatively in the ultraviolet region, but become strongly coupled in the infrared. The matching condition is the solution to the following. Suppose that the theory has a 't Hooft anomaly, $d^{fff} \neq 0$, in the ultraviolet, the matching condition describes the fate of the global anomaly when the theory flows to the infrared. The 't Hooft anomaly d^{fff} keeps the same value in both the ultraviolet and the infrared.

The evaluation of the coefficients d^{abc} deeply depends on the matter content of the theory. The matter content in the ultraviolet is given by the elementary degrees of freedom. When the theory flows to the infrared, it exhibits confinement and the degrees of freedom are no longer elementary, but bound states and composite fields of the original fields. In the two regions the degrees of freedom are different. The 't Hooft matching condition implies that the effective degrees of freedom in the infrared organize themselves in such a way that the 't Hooft anomaly is the same as in the ultraviolet.

In an ultraviolet theory that is asymptotically free, one may know everything about it with a Lagrangian and perturbation theory. In the infrared, one does not know anything, not even how the elementary degrees of freedom are organized to give the effective degrees of freedom. The matching condition gives a clue as to how [r].

Proof. A proof of the matching condition is the following. For an asymptotically free Yang–Mills theory, there is a gauge symmetry group G_s which confines and one extra global symmetry G_f that gives $d^{fff} \neq 0$. One may not gauge the global symmetry group in the ultraviolet (due to the presence of an anomaly). To do so anyway, one is forced to introduce extra massless fermions in some representation of the group G_f in such a way that they produce another coefficient $d^{fff}_{\text{spectator}}$ for which

$$d^{fff}_{\text{spectator}} + d^{fff} = 0$$

In this way the aforementioned anomaly is cancelled. At this point one may gauge the global group in the ultraviolet. The theory is consistent and has to remain such at any value of the coupling constant. In particular, in the infrared, $|g| \gg 1$. [r] One may choose the spectators to be singlets of the gauge group G_s, i.e. they belong to its trivial representation. This means that in the infrared, the particles charged under G_s confine, while the singlets do not confine (hence they are spectators). This implies that the spectators flow to the infrared as if they are still in the ultraviolet. In the infrared, the total set of degrees of freedom are made of confined states (i.e. bound states) and spectators (the same as in the ultraviolet). Since the theory is consistent in the infrared, the total coefficient d^{fff}_{tot} is still zero:

$$d^{fff}_{\text{tot}} = d^{fff}_{\text{IR}} + d^{fff}_{\text{spect}} = 0 \implies d^{fff}_{\text{IR}} = -d^{fff}_{\text{spect}} = d^{fff}_{\text{UV}}$$

This argument works for a weakly gauge group G_f, but also for global symmetries ($g \to 0$?). $\square$

27.9 Conformal anomaly

Conformal anomalies are due to the introduction of a scale during quantization that breaks conformal invariance in a controlled way. [r]

The trace of the stress-energy tensor is related to the beta function.

Part IV
Addenda

Monopoles, solitons and instantons. See Srednicki, Cheng, Coleman, Rajaraman.

28 Monopoles

This section deals with the Dirac theory of monopoles. Consider Maxwell equations

$$\partial_\mu F^{\mu\nu} = J^\nu \,, \quad \partial_{[\mu} F_{\nu\rho]} = 0$$

The second can be rewritten

$$\partial_{[\mu} F_{\nu\rho]} = 0 = \varepsilon_{\sigma\mu\nu\rho} \partial^\mu F^{\nu\rho} \implies \partial_\mu \widetilde{F}^{\mu\nu} = 0$$

In the vacuum, Maxwell equations are symmetric

$$\partial_\mu F^{\mu\nu} = 0 \,, \quad \partial_\mu \widetilde{F}^{\mu\nu} = 0$$

They are invariant under duality transformation

$$F^{\mu\nu} \to \widetilde{F}^{\mu\nu} \,, \quad \widetilde{F}^{\mu\nu} \to -F^{\mu\nu}$$

The physical fields are

$$E^i = F^{0i}\,,\quad B^i = \varepsilon^{ijk} F^k$$

The transformation maps them into

$$\mathbf{E} \to \mathbf{B}\,,\quad \mathbf{B} \to -\mathbf{E}$$

If one assumes that there is a current for the magnetic field, one may still have Maxwell equations to be symmetric. The current must be such that

$$J^\nu \to K^\nu\,,\quad K^\nu \to -J^\nu$$

from which the equations become

$$\partial_\mu F^{\mu\nu} = J^\nu\,,\quad \partial_\mu \widetilde{J}^{\mu\nu} = K^\nu$$

The electric current for point-like charges is defined as

$$J^\nu(x) = \sum q_i \int \mathrm{d}x_i^\nu\, \delta^{(4)}(x - x_i)$$

Similarly, one may write

$$K^\nu(x) = \sum_i g_i \int \mathrm{d}x_i^\nu\, \delta^{(4)}(x - x_i)$$

These charges are magnetic monopoles of charge g_i.

Quantization condition. The Dirac quantization condition is the following. Consider

$$\frac{q_i g_i}{4\pi} \equiv \frac{1}{2} n_{ij}\,,\quad n_{ij} \in \mathbb{Z}$$

For a single monopole and electric charge, one has

$$\frac{qg}{4\pi} = \frac{1}{2} n$$

This implies that the charges have to come in pair and have to satisfy this condition.

The simplest way to prove this quantization condition is to consider the motion of an electric charge q in a magnetic field generated by the monopole g. For a point-like monopole, the magnetic field is

$$\mathbf{B} = \frac{g}{4\pi r^2} \hat{\mathbf{r}}$$

The electric charge of mass m is subject to the Lorentz force

$$m\ddot{\mathbf{r}} = q\mathbf{v} \wedge \mathbf{B} = \frac{qg}{4\pi} \dot{\mathbf{r}} \wedge \frac{1}{r^2} \hat{\mathbf{r}}$$

The orbital angular momentum is

$$\mathbf{L} = \mathbf{r} \wedge m\dot{\mathbf{r}}$$

The torque is

$$\tau = \mathrm{d}_t \mathbf{L} = \mathrm{d}_t(\mathbf{r} \wedge m\dot{\mathbf{r}}) = \mathbf{r} \wedge m\ddot{\mathbf{r}} = \mathbf{r} \wedge m(q\mathbf{v} \wedge \mathbf{B}) = \mathrm{d}_t \left[\frac{qg}{4\pi} \hat{\mathbf{r}}\right]$$

from which

$$\mathrm{d}_t \left[\mathbf{L} - \frac{qg}{4\pi} \hat{\mathbf{r}}\right] = 0$$

The second addendum is the angular momentum of the field. Let the total conserved momentum be

$$\mathbf{J} \equiv \mathbf{L} - \frac{qg}{4\pi} \hat{\mathbf{r}}$$

After quantization [r], the only possible eigenvalues of the momentum are proportional to half-integers. The orbital momentum is quantized with integers, therefore

$$\frac{qg}{4\pi} \hat{\mathbf{r}}$$

has to be quantized with half-integers, from which

$$\frac{qg}{4\pi} = \frac{1}{2} n$$

29 Solitons

This section deals with solitons in $(1+1)$ dimensions. Consider a Lagrangian for one real scalar field subject to a quartic potential

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - V(\varphi)\,, \quad V(\varphi) = \frac{\lambda}{8}(\varphi^2 - v^2)^2$$

The Lagrangian can be rewritten as

$$\mathcal{L} = \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\partial_x\varphi)^2 - V(\varphi)$$

The Hamiltonian density is

$$\mathcal{H} = \int \mathrm{d}x \left[\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\partial_x\varphi)^2 + V(\varphi)\right]$$

29.1 Vacuum solutions

Vacuum solutions are constant field configurations that solve

$$\frac{\delta V}{\delta\varphi^2} = 0 \implies \varphi = \pm v$$

[r] Vacuum states correspond to zero energy $\mathcal{H} = 0 \implies E = 0$.

One may wonder if there are solutions that

- are static $\dot{\varphi} = 0$,
- correspond to finite energy.

The action is

$$S = \int \mathrm{d}^2x\,\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\partial_x\varphi)^2 - V(\varphi) = -\int \mathrm{d}^2x\,\frac{1}{2}(\partial_x\varphi)^2 + V(\varphi)$$

The equations of motions give

$$\delta S = 0 \implies \delta\left[\frac{1}{2}(\partial_x\varphi)^2 + V(\varphi)\right] = 0$$

These correspond to the study of a one-dimensional motion in the potential $-V(\varphi)$ [r]. The vacua of the manifold is $\{-v, v\}$. The spatial boundary is $\{-\infty, +\infty\}$. The two sets have the same topology. One may always find mappings between the two.

The trivial map is

$$\varphi(+\infty) = \pm v\,, \quad \varphi(-\infty) = \pm v$$

The identity map is

$$\varphi(+\infty) = \pm v\,, \quad \varphi(-\infty) = \mp v$$

Consider the identity map. One considers solutions to the equations of motion with boundary conditions

$$\lim_{x\to\pm\infty} \varphi(x) = \pm v$$

The static solutions interpolate the two vacua [r]. One has to check if the energy is finite when going from one vacuum to the other.

Consider the energy density

$$\begin{aligned}
\mathcal{H} &= \int \mathrm{d}x \, \left[\frac{1}{2}(\partial_x\varphi)^2 + V(\varphi)\right] \\
&= \int \mathrm{d}x \, \left[\frac{1}{2}(\partial_x\varphi - \sqrt{2V(\varphi)})^2 + \sqrt{2V(\varphi)}\,\partial_x\varphi\right] \\
&= \int \mathrm{d}x \, \frac{1}{2}(\partial_x\varphi - \sqrt{2V(\varphi)})^2 + \int \mathrm{d}x \, \sqrt{2V(\varphi)}\,\partial_x\varphi \\
&= \int \mathrm{d}x \, \frac{1}{2}(\partial_x\varphi - \sqrt{2V(\varphi)})^2 + \int_{-v}^{+v} \mathrm{d}\varphi \, \sqrt{2V(\varphi)} \\
&= \int \mathrm{d}x \, \frac{1}{2}(\partial_x\varphi - \sqrt{2V(\varphi)})^2 + \frac{2}{3}\sqrt{\lambda}v^3 \\
&= M + \int \mathrm{d}x \, \frac{1}{2}(\partial_x\varphi - \sqrt{2V(\varphi)})^2 \geq M
\end{aligned}$$

At the third line, one notices that the first integral is non-negative. At the last line, one sets

$$M \equiv \frac{2}{3}\sqrt{\lambda}v^3$$

The energy is bounded by M. The solutions corresponding to $E = M$ are

$$\partial_x\varphi = \sqrt{2V(\varphi)} \implies \varphi(x) = v\tanh\left[\frac{\sqrt{\lambda}}{2}v(x - x_0)\right]$$

where x_0 is an integration constant. This is the soliton solution. It interpolates between the two vacua with a finite amount of energy.

One may generate further solutions by applying Lorentz boosts.

29.2 Topological number

The solutions can be classified in terms of their boundary conditions. A good number may be the topological number

$$n \equiv \varphi(+\infty) - \varphi(-\infty) = \begin{cases} 0\,, & \text{trivial map} \\ \pm 2v\,, & \text{identity map} \\ 2kv\,, & k \in \mathbb{Z} \end{cases}$$

The first is the vacuum, while the second is the kink map (plus) and anti-kink map (minus). The quantum number n may be written as

$$n = \int \mathrm{d}x \, \partial_x\varphi$$

The topological current is

$$J_\mu \equiv \varepsilon_{\mu\nu}\,\partial^\nu\varphi$$

This current is conserved since ε is anti-symmetric. This is the off-shell conservation. The quantum number can be written as

$$n = \int \mathrm{d}x \, J_0 \equiv Q$$

where Q si the topological charge. The solitons solutions can be classified according to the topological number.

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29.3 Higher dimensions

See Srednicki. In higher dimensions there are no solitons. Consider a theory in (2+1) dimensions of a complex scalar interacting with a quartic potential

$$\mathcal{L} = \partial_\mu\varphi^\dagger\,\partial^\mu\varphi - V(\varphi)\,, \quad V(\varphi) = \frac{\lambda}{4}(\varphi^\dagger\varphi - v^2)^2$$

Vacuum configurations are constant field configurations solving

$$\frac{\delta V}{\delta\left|\varphi\right|^2} = 0 \implies \left|\varphi\right|^2 = v^2 \implies \varphi = v\mathrm{e}^{\mathrm{i}\alpha}$$

The topology of the manifold of the solutions is a circle S^1 described by the real parameter α. The boundary of the spatial volume is a circle too

$$x^1 = r\cos\theta\,, \quad x^2 = r\sin\theta\,, \quad \theta \in [0, 2\pi]$$

One may build mappings between the spatial boundary described by a fixed radius r and an angle θ, and the vacua manifold described by the parameters v and α:

$$U(\theta) = \mathrm{e}^{\mathrm{i}\alpha(\theta)}$$

One looks for solutions with classical equations of motion satisfying the boundary conditions

$$\lim_{r\to\infty}\varphi(r,\theta) = vU(\theta)$$

One would like to select solutions corresponding to finite energy density

$$\mathcal{H} \to \int_{S^1} \mathrm{d}^2x\,\left|\nabla\varphi\right|^2 + \cdots\,, \quad \left|\nabla\varphi\right|^2 = (\nabla\varphi^\dagger)\cdot(\nabla\varphi)$$

Taking the ansatz for the solution to the equations of motion to be

$$\varphi(r,\theta) = vf(r)\mathrm{e}^{\mathrm{i}n\theta}\,, \quad \lim_{r\to\infty} f(r) = 1\,, \quad \lim_{r\to 0} f(r) = 0$$

Therefore, one finds

$$\left|\nabla\varphi\right|^2 = v^2\left[f'^2 + \frac{n^2}{r^2}f^2\right]$$

Therefore, the energy is

$$E \to \int \mathrm{d}^2x\,\left|\nabla\varphi\right|^2 \implies \int_0^\infty \mathrm{d}r\,rv^2\left[f'^2 + \frac{n^2}{r^2}f^2\right] \implies v^2n^2\int_0^\infty \mathrm{d}r\,\frac{1}{r}f^2 \sim \frac{1}{r}\,, \quad r\to\infty$$

The integral is divergente: there are no finite energy solutions.

Theorem (Derrick)**.** Only with scalars, one may not construct solitons solutions in more than $(1+1)$ dimensions.

The way to solve the problem is to add a gauge group. One may gauge the global U(1) of the Lagrangian

$$\mathcal{L} = (D_\mu\varphi)^\dagger(D^\mu\varphi) - V(\varphi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

The gradient now becomes the covariant gradient. The energy becomes

$$E \to \int \mathrm{d}^2x\,(\partial_j - \mathrm{i}eA_j)\varphi^\dagger(\partial_j + \mathrm{i}eA_j)\varphi$$

The gauge field contributes ease the divergence.

Example ('t Hooft–Polyakov monopole)**.** An example of theory is the 't Hooft–Polyakov monopole. One may consider a $(3+1)$ dimensional theory. The spatial boundary is

$$S^2 \simeq \mathrm{SO}(3) \simeq \mathrm{SU}(2)$$

Therefore, one introduces a triplet of real scalar fields φ^a in the adjoint representation of the SO(2) group

$$\mathcal{L} = \frac{1}{2}(D_\mu\varphi)^a(D^\mu\varphi)^a - V(\varphi) - \frac{1}{4}\operatorname{Tr}(F_{\mu\nu}F^{\mu\nu})$$

where one has

$$(D_\mu\varphi)^a = \partial_\mu\varphi^a + e\varepsilon^{abc}A^b_\mu\varphi^c\,, \quad F_\mu\nu^a = \partial_{[\mu}A^a_{\nu]} + e\varepsilon^{abc}A^b_\mu A^c_\nu$$

The potential is

$$V(\varphi) = \frac{\lambda}{8}(\varphi^a\varphi^b - v^2)^2$$

This is the Georgi–Glashow model. The vacuum configurations correspond to

$$A_\mu = 0$$

and constant field φ solutions of

$$\frac{\delta V}{\delta|\varphi|^2} = 0$$

Therefore

$$\varphi^a\varphi^a = v^2 \implies \varphi^a = v\hat{\varphi}^a$$

where $\hat{\varphi}^a$ is the unit vector in the field space

$$\hat{\varphi}^a\hat{\varphi}^a = 1$$

The classical equations of motion are

$$(D_\nu F^{\nu\mu})^a = e\varepsilon^{abc}\varphi^b(D_\mu\varphi)^c\,, \quad (D^\mu D_\mu\varphi)^a = -\frac{\lambda}{2}\varphi^a(\varphi^b\varphi^c - v^2)$$

with boundary conditions

$$\lim_{r\to\infty}\varphi^a(\mathbf{x}) = v\hat{x}^a\,, \quad r = |\mathbf{x}|\,, \quad \hat{x}^a = \frac{1}{r}x^a$$

One constructs a static solution which is a map between the spatial sphere and the vacua manifold sphere. The boundary conditions on the gauge field A_μ are

$$\lim_{r\to\infty}(D_\mu\varphi)^a = 0 \implies x_b A^b_i = 0\,, \quad r\to\infty$$

See p. 567 for explicit solutions. One ma focus on the structure of the energy density

$$\mathcal{H} = \int \mathrm{d}^3x\,\left[\frac{1}{2}|(\nabla - \mathrm{i}e\mathbf{A})\varphi|^2 + V(\varphi) + \frac{1}{2}B^a_iB^a_i\right]$$

where one has

$$B^a_i = \varepsilon_{ijk}F^a_{jk}$$

The energy gives

$$E_i \to F^a_{0i} = \partial_0A^a_i - \partial_iA^a_0 = -\partial_iA^a_0 = 0$$

The first term is zero due to the solution being static and the second is set to zero in the axial gauge $A^a_0 = 0$.

Considering the first and last terms of the Hamiltonian, one finds

$$\frac{1}{2}[(D_i\varphi)^a(D_i\varphi)^a + \frac{1}{2}B^a_iB^a_i = \frac{1}{2}[(D_i\varphi)^a + B^a_i]^2 - (D_i\varphi)^aB^a_i$$

Consider the last term

$$(D_i\varphi)^a B_i^a = \partial_i(\varphi^a B_i^a) - \varphi^a\, D_i B_i^a = \partial_i(\varphi^a B_i^a)$$

At the first equality one integrates by parts. The Hamiltonian becomes

$$\mathcal{H}\int \mathrm{d}^3x\, \partial_i(\varphi^a B_i^a) = \int \mathrm{d}S_i\, v B_i^a \hat{x}^a = v\Phi(B) = v\left[-\frac{4\pi}{e}n\right] = -vg$$

At the second equality, one applies Stokes's theorem. The coefficient n is a topological number which is an integer. At the last equality one has applied Dirac's quantization condition.

The energy density can be written as

$$\mathcal{H} = \frac{4\pi}{e}|v|n + \int \mathrm{d}^3x\, \left[\frac{1}{2}[(D_i\varphi)^a + B_i^a]^2 + V(\varphi)\right] \geq \frac{4\pi}{e}|v|n$$

The bound i called Bogonolny bound. The minimum energy corresponds to the solution

$$[(D_i\varphi)^a + B_i^a]^2 = -2V(\varphi)$$

The explicit solution is called 't Hooft–Poliakov monopole[54]. The solutions that saturate the bound are called BPS solutions. [r]

In four dimensions one has to abandon pure scalar theories, but has to include gauge fields. This is an example of theory that admits monopoles. This model was proposed as a model for the weak interactions but was ruled out due to the presence of solutions.

30 Instantons

In Minkowski space-time, one may construct theories with solitons. In Euclidean space, there is no longer a notion of energy, since it is replaced by the action. One looks for solutions with finite action. Instantons (in Euclidean) are the equivalent of solitons in Minkowski. One is interested in instantons because in the path integral one is summing over trajectories weighted by the exponential of the action and the instantons corresponds to finite action so are counted in the integral.

Consider 4 dimensional Euclidean space. Let $r = |x|$, the boundary is a three-sphere S^3. There is a natural mapping between S^3 and SU(2). An element of the latter group can be written as

$$g = \mathrm{e}^{\mathrm{i}\boldsymbol{\varepsilon}\cdot\boldsymbol{\tau}}\,, \quad \boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$$

For an element near the identity

$$g = u_0 I + \mathrm{i}\mathbf{u}\cdot\boldsymbol{\tau}$$

From unitarity one finds

$$gg^\dagger = I \implies u_0^2 + |\mathbf{u}| = 1$$

This is the equation of S^3 embedded in $\mathbb{R}^4$. One may always define mappings between S^3 and SU(2)

$$f(x^0, \mathbf{x}) = x^0 + \mathrm{i}\mathbf{x}\cdot\boldsymbol{\tau} \in \mathrm{SU}(2)$$

Recall homotopy: two curves f_0 and f_1 are homotopic if one can define a family $F(x^0, \mathbf{x}, t)$ with $t \in [0, 1]$ such that

$$F(x^0, \mathbf{x}, t = 0) = f_0(x^0, \mathbf{x})\,, \quad F(x^0, \mathbf{x}, t = 1) = f_1(x^0, \mathbf{x})$$

One may build group mappings according to the homotopy equivalence relations. Each homotopy class is identified by a topological number called winding number defined as

$$n = -\frac{1}{24\pi^2}\int_{S^3} \mathrm{d}\theta_1\, \mathrm{d}\theta_2\, \mathrm{d}\theta_3\, \mathrm{Tr}(\varepsilon_{ijk}A_iA_jA_k)\,, \quad A_i = \frac{1}{f}\,\partial_i f$$

[54] It is a soliton solution, but is called monopole because of the Dirac's quantization condition used above.

One considers a pure gauge SU(2) theory. The generators of the group are the Pauli matrices

$$\frac{1}{2}t^a$$

One lets

$$A_\mu = \frac{1}{2}\tau^a A^a_\mu\,, \quad F_{\mu\nu} = F^a_{\mu\nu}\frac{\tau^a}{2} = \partial_{[\mu}A_{\nu]} + [A_\mu, A_\nu]$$

The above are mappings between an SO(3) object and SU(2) one [r]. The Lagrangian is

$$\mathcal{L} = \frac{1}{2g^2}\operatorname{Tr}(F_{\mu\nu}F^{\mu\nu})$$

The Lagrangian is invariant under the gauge transformation

$$A'_\mu = U^{-1}A_\mu U + U^{-1}\,\partial_\mu U$$

One looks for solutions that satisfy the following boundary conditions

$$\lim_{r\to\infty}\operatorname{Tr}(F_{\mu\nu}F_{\mu\nu}) = 0$$

This implies

$$\lim_{r\to\infty}F_{\mu\nu}(x) = 0 \implies A_\mu = U^{-1}\,\partial_\mu U$$

This is the pure gauge [r]. The map between S^3 and SU(2).

$$U = \exp\left[\mathrm{i}\alpha^a\frac{\tau^a}{2}\right]$$

One can classify all the maps in homotopy classes, each one ? by the topological number

$$\begin{aligned} n &= -\frac{1}{24\pi^2}\int_{S^3}\operatorname{Tr}\big(U^{-1}\,\partial_i U\,U^{-1}\,\partial_j U\,U^{-1}\partial_k U\big)\varepsilon_{ijk} \\ &= \cdots \\ &= \frac{1}{16\pi^2}\int \mathrm{d}^4x\ \operatorname{Tr}(F_{\mu\nu}F_{\mu\nu}) \end{aligned}$$

The last trace is reminiscent of the anomaly.

One looks for solutions with constant action

$$S = \frac{1}{2g^2}\int \mathrm{d}^4x\ \operatorname{Tr}(F_{\mu\nu}F_{\mu\nu})$$

Consider the quantity

$$\operatorname{Tr}\Big[(F_{\mu\nu}\pm\widetilde{F}_{\mu\nu})^2\Big] = 2\operatorname{Tr}(F_{\mu\nu}F_{\mu\nu}) \pm 2\operatorname{Tr}\left(F_{\mu\nu}\widetilde{F}_{\mu\nu}\right) \geq 0$$

Integrating both sides, one finds

$$\int \mathrm{d}^4x\ \operatorname{Tr}(F_{\mu\nu}F_{\mu\nu}) \geq \left|\int \mathrm{d}^4x\ \operatorname{Tr}\left(F_{\mu\nu}\widetilde{F}_{\mu\nu}\right)\right| = 16\pi^2|n|$$

Therefore, one finds a bound for the action

$$S_\mathrm{E} \geq \frac{16\pi^2|n|}{2g^2} = \frac{8\pi^2}{g^2}n$$

When the bound is saturated, the action is at a minimum. For this to happen, the trace above has to be zero. The boundary is saturated by solutions satisfying

$$F_{\mu\nu} = \widetilde{F}_{\mu\nu}\,, \quad F_{\mu\nu} = -\widetilde{F}_{\mu\nu}$$

The first is are self-dual solutions while the second are the anti-self-dual solutions. These two solutions are the instantons. The instantons with winding number n has finite action

$$S_\mathrm{E} = \frac{8\pi^2}{g^2}n$$

The action depends on the inverse of the coupling constant. For weak coupling, the action is great and contributes very little to the path integral. Viceversa, at strong coupling, the action is small and the instantons contribute to the path integral.

Perturbation theory. In perturbation theory, one considers the integral

$$\int [\mathcal{D}A_\mu]\, \mathrm{e}^{-S[A]} \equiv \langle 0|0\rangle_{\mathrm{pert}}$$

where the boundary conditions are trivial and $n = 0$ solutions (i.e. the perturbative vacuum solution). Away from the perturbative regime, there is a different configuration at the boundary identified by the winding number. There is an infinite set of vacua labelled by the topological number n. It makes sense to consider transitions between vacua

$$\langle m|n\rangle = \int [\mathcal{D}A_\mu]_{m-n}\, \mathrm{e}^{-S[A]}$$

Generally, the θ-vacua are

$$|\theta\rangle = \sum_n \mathrm{e}^{-\mathrm{i}n\theta}\, |n\rangle$$

The transition from one vacuum to another is

$$\langle \theta'|\theta\rangle = \int [\mathcal{D}A_\mu]_\nu \mathrm{e}^{-\widetilde{S}[A]}$$

where

$$\widetilde{S}[A] = S[A] + \frac{\mathrm{i}\theta}{16\pi^2}\operatorname{Tr}\Big(F_{\mu\nu}\widetilde{F}_{\mu\nu}\Big)$$

See Cheng for computation.

Conclusion. Forgetting about instantons contribution, one may work with the ordinary action. Once the theory is an instantons configuration, one has also to consider an extra trace.