Quantum Field Theory II

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Contents

Lecture 1

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Topics. Generalize the formalism for complex scalar fields. Introduce fermions and their functional quantization. Discrete symmetries and PCT theorem. Gauge theories and their functional quantization. Anomalies. Possibly instantons, applications of RG flows, etc.

1 Complex scalar boson fields

[r] sources?

1.1 Generalization

Consider a complex scalar field φ. It has two degrees of freedom: either the fields φ and φ^* or the real and imaginary parts of the field φ. Both are useful for computation

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + \mathrm{i}\varphi_2)\,, \quad \varphi^* = \frac{1}{\sqrt{2}}(\varphi_1 - \mathrm{i}\varphi_2)$$

*https://github.com/M-a-s-o/notes

The action of the $\lambda\varphi^4$ theory is

$$\begin{aligned}
S &= \int \mathrm{d}^4x \, [\partial_\mu\varphi \, \partial^\mu\varphi^* - m^2\varphi\varphi^* - \lambda(\varphi\varphi^*)^2] \\
&= \int \mathrm{d}^4x \, \left[\frac{1}{2}\,\partial_\mu\varphi_1\,\partial^\mu\varphi_1 - \frac{1}{2}m^2\varphi_1^2 + \frac{1}{2}\,\partial_\mu\varphi_2\,\partial^\mu\varphi_2 - \frac{1}{2}m^2\varphi_2^2 - \frac{\lambda}{4}(\varphi_1^2+\varphi_2^2)^2\right]
\end{aligned}$$

Each degree of freedom has an equation of motion. Varying the action with respect to φ gives the equations of motion for φ^* and vice versa. For $\lambda = 0$, the fields components satisfy the Klein–Gordon equation.

Since the interaction term is real, it corresponds to a quartic vertex $\varphi\varphi^*\varphi\varphi^*$. For $\lambda = 0$, one can rewrite the action in terms of real φ_1 and imaginary φ_2 parts and see that the two degrees of freedom decouple since there is no mixed term $\varphi_1^2\varphi_2^2$.

With two degrees of freedom, the Euclidean generating functional depends on two source terms

$$\begin{aligned}
W[J,J^*] &= \int [\mathcal{D}\varphi\,\mathcal{D}\varphi^*] \exp\left[-\int \mathrm{d}^4x\,[\mathcal{L}(\varphi,\varphi^*) - J\varphi - J^*\varphi^*]\right] \\
W[J_1,J_2] &= \int [\mathcal{D}\varphi_1\,\mathcal{D}\varphi_2] \exp\left[-\int \mathrm{d}^4x\,[\mathcal{L}(\varphi_1,\varphi_2) - J_1\varphi_1 - J_2\varphi_2]\right]
\end{aligned}$$

The Euclidean Green's functions are

$$\begin{aligned}
&\langle 0|\, \mathcal{T}\{\varphi(x_1)\cdots\varphi(x_l)\varphi^*(x_{l+1})\cdots\varphi^*(x_n)\}\,|0\rangle = \\
&\qquad = \left.\frac{\delta^n W[J,J^*]}{\delta J(x_1)\cdots\delta J(x_l)\,\delta J^*(x_{l+1})\cdots\delta J^*(x_n)}\right|_{J=J^*=0}
\end{aligned}$$

Similarly

$$\begin{aligned}
&\langle 0|\, \mathcal{T}\{\varphi_1(x_1)\cdots\varphi_1(x_l)\varphi_2(x_{l+1})\cdots\varphi_2(x_n)\}\,|0\rangle = \\
&\qquad = \left.\frac{\delta^n W[J_1,J_2]}{\delta J_1(x_1)\cdots\delta J_1(x_l)\,\delta J_2(x_{l+1})\cdots\delta J_2(x_n)}\right|_{J_1=J_2=0}
\end{aligned}$$

Free theory. For a free real scalar theory, i.e. setting $\lambda = 0$, one can compute the free propagator exactly. In this case the situation is slightly different? [r]. In the complex fields formulation, one has to compute a complex Gaussian integral of the form

$$\int \left[\prod_{j=1}^{n} \mathrm{d}z_j\,\mathrm{d}z_j^*\right] \mathrm{e}^{-z^*Az} = \frac{\pi^n}{\det A}$$

where $z \equiv (z_1,\ldots,z_n)$ and A is a square matrix of dimension n (with hermitian part positive-definite). The Euclidean generating functional is then

$$\begin{aligned}
W[J_1,J_2] &= \int [\mathcal{D}\varphi_1\,\mathcal{D}\varphi_2]\,\mathrm{e}^{-S_0[\varphi_1]-S_0[\varphi_2]} \exp\left[\int \mathrm{d}^4x\,(J_1\varphi_1 + J_2\varphi_2)\right] \\
&= \int [\mathcal{D}\varphi_1]\,\mathrm{e}^{-S[\varphi_1]+\int \mathrm{d}^4x\,J_1\varphi_1} \int [\mathcal{D}\varphi_2]\,\mathrm{e}^{-S[\varphi_2]+\int \mathrm{d}^4x\,J_2\varphi_2} \\
&= W_1[J_1]W_2[J_2]
\end{aligned}$$

The theory factorizes.

The Lagrangian above has a U(1) symmetry of the fields

$$\varphi' = \mathrm{e}^{\mathrm{i}\alpha}\varphi\,, \quad \varphi'^* = \mathrm{e}^{-\mathrm{i}\alpha}\varphi^*\,, \quad \alpha \in \mathbb{R}$$

Equivalently, the Lagrangian is invariant under SO(2) of the components φ_1 and φ_2.

Exercise. Write the most general SO(2) transformation.

Exercise. See Peskin, Problem 12.3, p. 428. The interaction in the action $S[\varphi_1, \varphi_2]$ can be split into self-interaction and coupling between the fields. This form can be generalized

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi_1\,\partial^\mu\varphi_1 + \frac{1}{2}\,\partial_\mu\varphi_2\,\partial^\mu\varphi_2 - \frac{\lambda}{4!}(\varphi_1^4 + \varphi_2^4) - \frac{2\rho}{4!}\varphi_1^2\varphi_2^2$$

Do the following:

- Write the Euclidean Feynman rules.
- Compute all one-loop corrections to the λ-vertex and the ρ-vertex in dimensional regularization $D = 4 - 2\varepsilon$, use massless integrals (discussed in the following).
- Impose the normalization conditions at $s = t = u = \Lambda^2$ such that

  $$\Gamma^{(4)}(s = t = u = \Lambda^2) = -\lambda\,, \quad \Gamma^{(4)}_{\text{mixed}}(s = t = u = \Lambda^2) = -2\rho$$

  [r] minus?
  Notice that this implies that there are no one-loop finite corrections to the couplings.
- Compute the beta functions β_λ, β_ρ and

  $$\beta_{\frac{\lambda}{\rho}} = \mathrm{d}_t\frac{\lambda}{\rho} = \frac{1}{\rho}\,\mathrm{d}_t\lambda - \frac{1}{\rho^2}\lambda\,\mathrm{d}_t\rho = \frac{1}{\rho}\beta_\lambda - \frac{\lambda}{\rho^2}\beta_\rho$$

- Find the fixed points and describe the RG flow.
- Explain what happens for

  $$\frac{1}{3} < \frac{\lambda}{\rho} < 1$$

  and for $\lambda = \rho$.

1.2 Massless integrals in dimensional regularization

Proposition. In dimensional regularization, it holds

$$I_\alpha = \int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{1}{(p^2)^\alpha} = 0\,, \quad \alpha \in \mathbb{C}$$

Notice that it is null also for $\alpha = 0$. This integral describes tadpoles for $\alpha = 1$: the two-point Green's function receives no correction at one-loop if the fields are massless.

The intuitive argument of why this is true can be seen for $D \neq 2\alpha$ where the integral is dimensionful: the result has to be dimensionful as well and must be written in terms of dimensionful parameters, however the integral does not depend on any parameter and so the result must be zero.

Proof. Consider the theorem below and the Euclidean formalism. One may rewrite the integrand

$$\frac{1}{(p^2)^\alpha} = \frac{1}{(p^2)^\alpha}\frac{p^2 + m^2}{p^2 + m^2} = \frac{1}{(p^2)^\alpha}\frac{m^2}{p^2 + m^2} + \frac{1}{(p^2)^{\alpha-1}}\frac{1}{p^2 + m^2}$$

The region of convergence for the integral

$$\int \mathrm{d}^D p\,\frac{1}{(p^2)^\alpha}\frac{m^2}{p^2 + m^2}$$

is given by the following limits. At infinity, one has

$$|p| \to \infty\,, \quad I \sim \frac{1}{p^{2\alpha+2-D}}$$

There is no singularity for $2\alpha + 2 - \operatorname{Re} D > 0$. At the origin, one has

$$|p| \to 0\,, \quad I \sim p^{D-2\alpha}$$

There is no singularity for $\operatorname{Re} D - 2\alpha > 0$. The integral of the first addendum is well-defined for

$$2\alpha < \operatorname{Re} D < 2\alpha + 2$$

Similarly, the region of convergence of the integral of the second addendum is

$$2\alpha - 2 < \operatorname{Re} D < 2\alpha$$

The two regions do not overlap and, by the theorem below, one has

$$I_\alpha = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{m^2}{(p^2)^\alpha (p^2 + m^2)} + \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{(p^2)^{\alpha-1}(p^2 + m^2)} \equiv I_\alpha^{(1)} + I_\alpha^{(2)}$$

Looking at tables of integrals [r], one has a general formula

$$\int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{(p^2)^a (p^2 + m^2)^b} = (m^2)^{\frac{D}{2} - a - b} \frac{\Gamma(D/2 - a)\Gamma(a + b - D/2)}{(4\pi)^{\frac{D}{2}} \Gamma(b) \Gamma(D/2)}$$

For the first integral, one has $a = \alpha$, $b = 1$ and for the second $a = \alpha - 1$ and $b = 1$. Therefore

$$\begin{aligned} I_\alpha &= m^{D-2\alpha} \frac{\Gamma(D/2 - \alpha)\Gamma(1 + \alpha - D/2)}{(4\pi)^{\frac{D}{2}} \Gamma(D/2)} + m^{D-2\alpha} \frac{\Gamma(D/2 + 1 - \alpha)\Gamma(\alpha - D/2)}{(4\pi)^{\frac{D}{2}} \Gamma(D/2)} \\ &= \frac{m^{D-2\alpha}}{(4\pi)^{\frac{D}{2}} \Gamma(D/2)} \Gamma(\alpha - D/2) \Gamma(D/2 - \alpha) \left[\alpha - \frac{D}{2} + \frac{D}{2} - \alpha \right] = 0 \end{aligned}$$

where one applies $\Gamma(z+1) = z\Gamma(z)$ to the Gamma function with three addenda in its argument. □

Theorem. If an integrand can be written as a sum of terms with non-overlapping regions of convergence (for their integrals), then the integral is a sum of their integrals.

One a-posteriori argument to see why the integral is null follows. One may rescale $p = sp'$ to have

$$I_\alpha = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{(p^2)^\alpha} = s^{D-2\alpha} \int \frac{\mathrm{d}^D p'}{(2\pi)^D} \frac{1}{(p'^2)^\alpha} \implies I_\alpha = s^{D-2\alpha} I_\alpha$$

For $D \neq 2\alpha$ then $I_\alpha = 0$. For the case $D = 2\alpha$, one can argue that there is a fine-tuning between the ultraviolet and the infrared divergences. In this case, there divergences are both present and give zero when summed. Consider the particular example of $D = 2 - \varepsilon$ and $\alpha = 1$. Then, the integral becomes

$$\begin{aligned} I_1 &= \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{p^2} = -\int \frac{\mathrm{d}^D p}{(2\pi)^D} \int_0^\infty \mathrm{d}\lambda \, \frac{1}{(p^2 - \lambda)^2} = -\int_0^\infty \mathrm{d}\lambda \int \frac{\mathrm{d}^{2-\varepsilon} p}{(2\pi)^{2-\varepsilon}} \frac{1}{(p^2 - \lambda)^2} \\ &= -\int_0^\infty \mathrm{d}\lambda \, \frac{\Gamma(1 + \varepsilon/2)}{(4\pi)^{1-\frac{\varepsilon}{2}} \Gamma(2)} (-\lambda)^{-\frac{\varepsilon}{2} - 1} = (-1)^{\frac{\varepsilon}{2}} \frac{\Gamma(1 + \varepsilon/2)}{(4\pi)^{1-\frac{\varepsilon}{2}}} \int_0^\infty \frac{\mathrm{d}\lambda}{\lambda^{1+\frac{\varepsilon}{2}}} \end{aligned}$$

Ignoring the coefficients, the integral is

$$\begin{aligned} I_1 &\propto \int_{a_{\mathrm{IR}}}^1 \frac{\mathrm{d}\lambda}{\lambda^{1+\frac{\varepsilon}{2}}} + \int_1^{a_{\mathrm{UV}}} \frac{\mathrm{d}\lambda}{\lambda^{1+\frac{\varepsilon}{2}}} = -\frac{2}{\varepsilon} \lambda^{-\frac{\varepsilon}{2}} \Big|_{a_{\mathrm{IR}}}^1 - \frac{2}{\varepsilon} \lambda^{-\frac{\varepsilon}{2}} \Big|_1^{a_{\mathrm{UV}}} = -\frac{2}{\varepsilon} [-a_{\mathrm{IR}}^{-\frac{\varepsilon}{2}} + a_{\mathrm{UV}}^{-\frac{\varepsilon}{2}}] \\ &= -\frac{2}{\varepsilon} \left[-\frac{\varepsilon}{2} (-\ln a_{\mathrm{IR}} + \ln a_{\mathrm{UV}}) + o(\varepsilon) \right] = \ln a_{\mathrm{UV}} - \ln a_{\mathrm{IR}} + o(\varepsilon^0) = 0 \end{aligned}$$

At the second line, one has integrated up to a cutoff [r]. From this one sees that $a_{\mathrm{IR}} = a_{\mathrm{UV}}$ and $I_\alpha = 0$ is the product of a cancellation between the ultraviolet and infrared divergences.

When dealing with massless theories in two dimensions $D = 2$, the cancellation of the tadpole is due to a balance between infrared and ultraviolet divergences. When interested in either divergence, one has to remove the other divergence in order to renormalize the one of interest. For example, one replaces

$$I_1 = \int \frac{\mathrm{d}^{2-\varepsilon}p}{(2\pi)^{2-\varepsilon}} \frac{1}{p^2} \to \int \frac{\mathrm{d}^{2-\varepsilon}p}{(2\pi)^{2-\varepsilon}} \frac{1}{p^2+\mu^2} = \frac{\Gamma(\varepsilon/2)}{(4\pi)^{1-\frac{\varepsilon}{2}}} \mu^{-\varepsilon} \sim \frac{2}{\varepsilon} \mathrm{e}^{-\varepsilon \ln\mu + \cdots} \sim \frac{2}{\varepsilon} \,, \quad \varepsilon \to 0$$

where μ^2 is an infrared regulator. This is the ultraviolet divergence of the tadpole in two dimensions.

Therefore, in dimensional regularization, massless tadpoles can be ignored when $D \neq 2$. In $D = 2$ the tadpole is dimensionless and gives a contribution when removing one divergence.

Part I
Spin-half fermion fields

2 Introduction

Review – classical fields. See Srednicki. A fermion field is a field describing Dirac spinors. A Dirac spinor in the Weyl basis is comprised of two fixed-chirality Weyl spinors

$$\psi = \begin{bmatrix} \chi_\mathrm{L} \\ \chi_\mathrm{R} \end{bmatrix}$$

where the left-chiral spinor belongs to the $(\frac{1}{2}, 0)$ representation of the Lorentz group $\mathrm{SO}(1,3)$, while the right-chiral spinor belongs to $(0, \frac{1}{2})$.

The equation of motion of the Dirac field is the Dirac equation

$$(\mathrm{i}\,\partial\!\!\!/ - m)\psi(x) = 0$$

where the Dirac matrices are four square matrices of dimension four defining the Dirac algebra $\mathrm{Cl}_{1,3}(\mathbb{C})$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

The equation of motion can be obtained from the action principle with the Lagrangian

$$\mathcal{L} = \mathrm{i}\bar{\psi}\,\partial\!\!\!/\psi - m\bar{\psi}\psi = \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi \,, \quad \bar{\psi} = \psi^\dagger\gamma^0$$

If the field ψ satisfies the Dirac equation of motion, then each of its four components (each two of the Weyl spinors) satisfies the Klein–Gordon equation

$$(\Box + m^2)\psi_j(x) = 0$$

In momentum space, the above is an algebraic equation that gives the dispersion relation

$$(-p^2 + m^2)\psi(p) = 0 \implies p^2 = m^2 \iff p^0 = \sqrt{|\mathbf{p}|^2 + m^2} \equiv \omega$$

The most general solution of the Klein–Gordon equation has the form[1]

$$\psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega} [u(\mathbf{p})\mathrm{e}^{-\mathrm{i}px} + v(\mathbf{p})\mathrm{e}^{\mathrm{i}px}]_{p^0=\omega}$$

Imposing the Dirac equation one finds

$$(p\!\!\!/ - m)u(\mathbf{p}) = 0 \,, \quad (p\!\!\!/ + m)v(\mathbf{p}) = 0$$

One solves these equation by going to the rest frame $\mathbf{p} = 0$ for which

$$p\!\!\!/ = \gamma^0 p_0 = \gamma^0 m \implies (\gamma^0 - 1)u(0) = 0 \,, \quad (\gamma^0 + 1)v(0) = 0$$

[1] Understood as a group of four components that solve the equation.

Each equation has two independent solutions $u_\pm$ and $v_\pm$. To get the solution for an arbitrary momentum one has to perform a boost. Therefore, the general solution to the Dirac equation is

$$\psi(x) = \sum_{s=\pm} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2\omega} [b_s(\mathbf{p}) u_s(\mathbf{p}) \mathrm{e}^{-\mathrm{i}px} + d_s^\dagger v_s(\mathbf{p}) \mathrm{e}^{\mathrm{i}px}]_{p^0=\omega}$$

where b and d are numbers.

The two-spinors u_s and v_s satisfy several identities (see Peskin, p. 48). The normalization is chosen to be

$$\bar{u}_r(\mathbf{p}) u_s(\mathbf{p}) = 2m\delta_{rs}\,, \quad \bar{v}_r(\mathbf{p}) v_s(\mathbf{p}) = -2m\delta_{rs}$$

Equivalent to

$$u_r^\dagger(\mathbf{p}) u_s(\mathbf{p}) = 2p^0 \delta_{rs}\,, \quad v_r^\dagger(\mathbf{p}) v_s(\mathbf{p}) = 2p^0 \delta_{rs}$$

The spinors are orthogonal

$$\bar{u}_r(\mathbf{p}) v_s(\mathbf{p}) = \bar{v}_r(\mathbf{p}) u_s(\mathbf{p}) = 0\,, \quad u_r^\dagger(\mathbf{p}) v_s(-\mathbf{p}) = v_r^\dagger(-\mathbf{p}) u_s(\mathbf{p}) = 0$$

The last two relations are not zero for both momenta being $+\mathbf{p}$. With these, one can obtain

$$b_s(\mathbf{p}) = \int \mathrm{d}^3 x\, \mathrm{e}^{\mathrm{i}px} u_s^\dagger(\mathbf{p}) \psi(x)\,, \quad d_s(\mathbf{p}) = \int \mathrm{d}^3 x\, \mathrm{e}^{\mathrm{i}px} \psi^\dagger(x) v_s(\mathbf{p})$$

To see this, it is sufficient to Fourier transform $u_s^\dagger(\mathbf{p})\psi(x)$, substitute $\psi(x)$ and apply the above rules. Notice that

$$\int \mathrm{d}^3 x\, \mathrm{e}^{\mathrm{i}x(p-q)} = \mathrm{e}^{\mathrm{i}x^0(p^0-q^0)} (2\pi)^3 \delta^3(\mathbf{p}-\mathbf{q})$$

and that

$$|\mathbf{p}| = |\mathbf{q}| \implies p^0 = q^0$$

Lecture 2

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Canonical quantization. See Srednicki, §39. The classical fields are promoted to operator fields by imposing a suitable set of equal-time anti-commutation rules (ACR)

$$\{\psi_\alpha(x), \psi_\beta(y)\} = 0\,, \quad \{\psi_\alpha(x), \psi_\beta^\dagger(y)\} = \delta_{\alpha\beta} \delta^{(3)}(\mathbf{x}-\mathbf{y})$$

From these, one obtains the rules for the Fourier coefficients which are promoted to operators

$$\{b_r(\mathbf{p}), b_s(\mathbf{q})\} = \{d_r(\mathbf{p}), d_s(\mathbf{q})\} = 2\omega(2\pi)^3 \delta_{rs} \delta^{(3)}(\mathbf{p}-\mathbf{q})$$

One defines the vacuum state as the state destroyed by every annihilation operator

$$b_s(\mathbf{p}) |0\rangle = d_s(\mathbf{p}) |0\rangle = 0\,, \quad \forall b, d$$

The excited states are obtained by acting on the vacuum with the creation operators $b_s^\dagger$ and $d_s^\dagger$. One interprets $b_s^\dagger(\mathbf{p}) |0\rangle$ as a single-particle state with momentum $\mathbf{p}$, energy $\omega = \sqrt{|\mathbf{p}|^2 + m^2}$ and projection s (in units) of the spin along the z-direction, $S_z = \frac{1}{2}s$. The difference between the two creation operators is not yet clear.

The use of anti-commutation rules implies the Pauli exclusion principle: there cannot be two fermions with the same quantum numbers. This is reflected in the spectrum of the number operator and the relation

$$\{b_s^\dagger(\mathbf{p}), b_s^\dagger(\mathbf{p})\} = 0 \implies [b_s^\dagger(\mathbf{p})]^2 = 0$$

The Hamiltonian density is obtained by Legendre transforming the Lagrangian density. By substituting the fields ψ and $\bar{\psi}$, one obtains the energy (Hamiltonian) of the system

$$H = \sum_{s=\pm} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2\omega} \omega [N_s^b(\mathbf{p}) + N_s^d(\mathbf{p})]$$

where N_s^a is the number operator of spin s for the ladder operator a

$$N_s^a(\mathbf{p}) \equiv a_s^\dagger(\mathbf{p})a_s(\mathbf{p})$$

The anti-commutation relations imply a finite spectrum

$$[N_s^a(\mathbf{p})]^2 = N_s^a(\mathbf{p}) \implies n = 0, 1$$

The Dirac Lagrangian exhibits a global U(1) symmetry

$$\psi' = \mathrm{e}^{\mathrm{i}\alpha}\psi\,, \quad \bar{\psi} = \mathrm{e}^{-\mathrm{i}\alpha}\bar{\psi}\,, \quad \alpha \in \mathbb{R}$$

Noether's theorem implies a conserved current

$$\partial_\mu J^\mu = 0\,, \quad J^\mu = \bar{\psi}\gamma^\mu\psi$$

The conserved charge is

$$Q = \int \mathrm{d}^3x\, J^0(x) = \int \mathrm{d}^3x\, \bar{\psi}\gamma^0\psi = \sum_{s=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[N_s^b(\mathbf{p}) - N_s^d(\mathbf{p})]$$

The global symmetry gives a minus sign for the operator d: the creation operators have different meanings. The operator d treats particles with opposite U(1) charge of the operator b. The b-particles, with charge $Q = 1$, are simply called particles; while the d-particles, with $Q = -1$, are called anti-particles.

Free fermion propagator. See Srednicki, §42. The free fermion propagator is the inverse of the kinetic term in the Lagrangian

$$S_{\alpha\beta}(x - y) \equiv -\mathrm{i}\,\langle 0|\,\mathcal{T}\{\psi_\alpha(x)\bar{\psi}_\beta(y)\}\,|0\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{(\not{p} + m)_{\alpha\beta}}{p^2 - m^2 + \mathrm{i}\varepsilon}\,\mathrm{e}^{\mathrm{i}p(x-y)}$$

Exercise. Check that the propagator is the Green's function of the Dirac operator

$$(\mathrm{i}\,\not{\partial} - m)_{\alpha\beta}S_{\beta\gamma}(x) = \delta_{\alpha\gamma}\delta^{(4)}(x)$$

3 LSZ reduction formula

See Srednicki, §41. The physical observables are the cross-sections which are expressed in terms of scattering amplitudes $\langle f|i\rangle$. The initial and final states are defined asymptotically in the distant past and future, $t \to \mp\infty$. One utilizes the adiabatic hypothesis: the region where the interaction is non-trivial is finite and, beyond, the theory is essentially free. The hypothesis has to be supported by the ansatz that the free regions are reached in the distant past and future.

One-particle states can be constructed by applying one a creation operator

$$a_s^\dagger(\mathbf{p})\,|0\rangle = |\mathbf{p}, s\rangle$$

Due to the Heisenberg uncertainty principle, a more physical particle is the one comprised of a wave-packet of momenta. The physical creation operator is a Gaussian distribution centered around a central momentum $\mathbf{q}$

$$a_s^\dagger = \int \mathrm{d}^3p\, f(\mathbf{p})a_s^\dagger(\mathbf{p})\,, \quad f(\mathbf{p}) \propto \mathrm{e}^{-\frac{(\mathbf{p}-\mathbf{q})^2}{4\sigma^2}}$$

The operation of applying the creation operator to the vacuum to obtain a particle is valid and meaningful in the asymptotic limit, where the states and theory are free. This holds since the vacuum defined by the annihilation operators is the free vacuum. By introducing an interaction, the vacuum is interacting. One assumes to be able to define a vacuum in the interacting theory and that the particles can be obtained in the same way as the free theory. For instance, a one-particle initial and final state are taken to be

$$|i\rangle = \lim_{t\to-\infty} b_r^\dagger(t)\,|0\rangle\,, \quad |f\rangle = \lim_{t\to\infty} b_s^\dagger(t)\,|0\rangle$$

In the interacting theory, the ladder operators depend on time.

Time dependence. To obtain a useful expression for the scattering amplitude, one needs to study the time dependence of the ladder orpeators. As with the scalar fields, consider

$$\begin{aligned}
b_s(+\infty) - b_s(-\infty) &= \int_{-\infty}^{\infty} \mathrm{d}t\,\partial_0 b_s(t) = \int \mathrm{d}^3p\, f(\mathbf{p}) \int_{-\infty}^{\infty} \mathrm{d}t\,\partial_0 b_s(\mathbf{p}) \\
&= \int \mathrm{d}^3p\, f(\mathbf{p}) \int_{-\infty}^{\infty} \mathrm{d}t \int \mathrm{d}^3x\,\partial_0[\mathrm{e}^{\mathrm{i}px} u_s^\dagger(\mathbf{p})\psi(x)] \\
&= \int \mathrm{d}^3p\, f(\mathbf{p}) \int \mathrm{d}^4x\,\partial_0[\mathrm{e}^{\mathrm{i}px} \bar{u}_s(\mathbf{p})\gamma^0\psi(x)] \\
&= \int \mathrm{d}^3p\, f(\mathbf{p}) \int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}px} \bar{u}_s(\mathbf{p})[\mathrm{i}p_0\gamma^0 + \gamma^0\,\partial_0]\psi(x) \\
&= \int \mathrm{d}^3p\, f(\mathbf{p}) \int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}px} \bar{u}_s(\mathbf{p})[-\mathrm{i}p_j\gamma^j + \mathrm{i}m + \gamma^0\,\partial_0]\psi(x) \\
&= \int \mathrm{d}^3p\, f(\mathbf{p}) \int \mathrm{d}^4x\,\bar{u}_s(\mathbf{p})[-\gamma^j\,\partial_j \mathrm{e}^{\mathrm{i}px} + \mathrm{e}^{\mathrm{i}px}(\mathrm{i}m + \gamma^0\,\partial_0)]\psi(x) \\
&= \int \mathrm{d}^3p\, f(\mathbf{p}) \int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}px} \bar{u}_s(\mathbf{p})[\gamma^j\,\partial_j + \mathrm{i}m + \gamma^0\,\partial_0]\psi(x) \\
&= -\mathrm{i} \int \mathrm{d}^3p\, f(\mathbf{p}) \int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}px} \bar{u}_s(\mathbf{p})[\mathrm{i}\,\not{\partial} - m]\psi(x)
\end{aligned}$$

At the first line, one inserts the expression of the wave-packet operator b_1. At the second line, one inserts the expression of the annihilation operator $b_s(\mathbf{p})$. At the fifth line, one has remembered that

$$\bar{u}(\mathbf{p})(\not{p} - m) = 0 \implies \bar{u}(\mathbf{p})(p_0\gamma^0 + p_j\gamma^j - m) = 0 \implies \bar{u}(\mathbf{p})p_0\gamma^0 = \bar{u}(\mathbf{p})(-p_j\gamma^j + m)$$

At the penultimate line, one has integrated by parts the first addendum.

The integrand is zero if the theory is free, but the theory is interacting and so

$$(\mathrm{i}\,\not{\partial} - m)\psi(x) = -\delta_{\bar{\psi}(x)}\mathcal{L}_{\text{int}} \neq 0$$

and the ladder operators are functions of time. For the creation operators, one has

$$b_s^\dagger(+\infty) - b_s^\dagger(-\infty) = -\mathrm{i} \int \mathrm{d}^3p\, f(\mathbf{p}) \int \mathrm{d}^4x\,\bar{\psi}(x)(\mathrm{i}\overleftarrow{\not{\partial}} + m)u_s(\mathbf{p})\mathrm{e}^{-\mathrm{i}px}$$

[r]

Two-by-two scattering. Consider a two-by-two scattering $p_1p_2 \to p_{1'}p_{2'}$. The scattering amplitude is

$$\langle f|i\rangle = \langle 0|\, b_{2'}(+\infty)b_{1'}(+\infty)b_1^\dagger(-\infty)b_2^\dagger(-\infty)\,|0\rangle$$

where the index of the spin projection is dropped and $1, 2$ or $1', 2'$ indicates the particle. Thanks to the ordering of the operators, the above is equal to

$$\langle f|i\rangle = \langle 0|\,\mathcal{T}\{b_{2'}(+\infty)b_{1'}(+\infty)b_1^\dagger(-\infty)b_2^\dagger(-\infty)\}\,|0\rangle$$

In the limit $\sigma \to 0$, the wave-packet is a Dirac delta giving

$$b_s^\dagger(+\infty) - b_s^\dagger(-\infty) = -\mathrm{i} \int \mathrm{d}^4x\,\mathrm{e}^{-\mathrm{i}px}\bar{\psi}(x)(\mathrm{i}\overleftarrow{\not{\partial}} + m)u_s(\mathbf{p})$$

One would like to replace $b_j^\dagger(-\infty)$ with $b_j^\dagger(+\infty)$ and the integral above. Due to the time-ordered product, the operator $b_j^\dagger(+\infty)$ is moved to the left and gives zero when acting on the bra [r]. Similarly happens for the annihilation operators. Therefore the scattering amplitude is

$$
\begin{aligned}
\langle f|i\rangle = (-1)^2 \mathrm{i}^4 \langle 0| \mathcal{T}\bigg\{ \int & \mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\mathrm{d}^4x_{1'}\,\mathrm{d}^4x_{2'} \\
& \times \mathrm{e}^{\mathrm{i}p_{2'}x_{2'}}\,\bar{u}_{s_{2'}}(\mathbf{p}_{2'})(\mathrm{i}\,\partial\!\!\!/ - m)\psi(x_{2'})\,\mathrm{e}^{\mathrm{i}p_{1'}x_{1'}}\,\bar{u}_{s_{1'}}(\mathbf{p}_{1'})(\mathrm{i}\,\partial\!\!\!/ - m)\psi(x_{1'}) \\
& \times \bar{\psi}(x_1)(\mathrm{i}\,\overleftarrow{\partial\!\!\!/} + m)u_{s_1}(\mathbf{p}_1)\mathrm{e}^{-\mathrm{i}p_1x_1}\,\bar{\psi}(x_2)(\mathrm{i}\,\overleftarrow{\partial\!\!\!/} + m)u_{s_2}(\mathbf{p}_2)\mathrm{e}^{-\mathrm{i}p_2x_2}\bigg\}|0\rangle \\
= \int & \mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\mathrm{d}^4x_{1'}\,\mathrm{d}^4x_{2'}\,\mathrm{e}^{\mathrm{i}p_{2'}x_{2'}}\,\bar{u}_{s_{2'}}(\mathbf{p}_{2'})(\mathrm{i}\,\partial\!\!\!/ - m)_{2'}\mathrm{e}^{\mathrm{i}p_{1'}x_{1'}}\,\bar{u}_{s_{1'}}(\mathbf{p}_{1'})(\mathrm{i}\,\partial\!\!\!/ - m)_{1'} \\
& \times \langle 0|\mathcal{T}\{\psi(x_{2'})\psi(x_{1'})\bar{\psi}(x_1)\bar{\psi}(x_2)\}|0\rangle\,(\mathrm{i}\,\overleftarrow{\partial\!\!\!/} + m)_1 u_{s_1}\mathrm{e}^{\mathrm{i}p_1x_1}(\mathrm{i}\,\overleftarrow{\partial\!\!\!/} + m)_2 u_{s_2}\mathrm{e}^{\mathrm{i}p_2x_2}\}
\end{aligned}
$$

[r] This is the LSZ reduction formula for fermions. In general, the computation of a scattering amplitude can be expressed as a computation of correlation functions

$$G^{(2n)}(x_1,\dots,x_n,x'_1,\dots x'_n) = \langle 0|\mathcal{T}\{\psi(x'_1)\cdots\psi(x'_n)\bar{\psi}(x_1)\cdots\bar{\psi}(x_n)\}|0\rangle$$

[r] One needs to develop a functional approach to compute the above Green's functions.

4 Functional quantization

See Srednicki, §43. Consider an interacting fermionic theory

$$\mathcal{L}(\psi,\bar{\psi}) = \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi + \mathcal{L}_{\mathrm{int}}(\psi,\bar{\psi})$$

One needs to generalize the generating functional for fermions. One introduces the spinorial source terms η and $\bar{\eta}$ to formally write

$$W[\eta,\bar{\eta}] = \int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\exp\left[\mathrm{i}\int \mathrm{d}^4x\,[\mathcal{L}(x) + \bar{\psi}\eta + \bar{\eta}\psi]\right]$$

where the spinorial product is defined using the van der Waerden notation[2]. The integrand must be a real quantity. In fact

$$(\bar{\psi}\eta)^\dagger = (\psi^\dagger\gamma^0\eta)^\dagger = \eta^\dagger\gamma^0\psi = \bar{\eta}\psi$$

which implies that $\bar{\psi}\eta + \bar{\eta}\psi$ is real. Since spinors anti-commute, then

$$\bar{\psi}\eta = -\eta\bar{\psi}$$

Therefore, when differentiating one has to be careful about the signs

$$\delta_{\eta(x)}\int \mathrm{d}^4y\,\bar{\psi}(y)\eta(y) = -\bar{\psi}(x)$$

The Green's function is

$$
\begin{aligned}
G^{(2n)}(x_1,\dots,x_n,y_1,\dots,y_n) &= \langle 0|\mathcal{T}\{\psi_{\alpha_1}(x_1)\cdots\psi_{\alpha_n}(x_n)\bar{\psi}_{\beta_1}(y_1)\cdots\bar{\psi}_{\beta_n}(y_n)\}|0\rangle \\
&= (-1)^n\mathrm{i}^{2n}\left.\frac{\delta^{2n}W[\eta,\bar{\eta}]}{\delta\bar{\eta}_{\alpha_1}(x_1)\cdots\delta\bar{\eta}_{\alpha_n}(x_n)\delta\eta_{\beta_1}(y_1)\cdots\delta\eta_{\beta_n}(y_n)}\right|_{\eta=\bar{\eta}=0} \\
&= \left.\frac{\delta^{2n}W[\eta,\bar{\eta}]}{\delta\bar{\eta}_{\alpha_1}(x_1)\cdots\delta\bar{\eta}_{\alpha_n}(x_n)\delta\eta_{\beta_1}(y_1)\cdots\delta\eta_{\beta_n}(y_n)}\right|_{\eta=\bar{\eta}=0}
\end{aligned}
$$

Each derivative with respect to $\bar{\eta}$ brings a $-\mathrm{i}$ while each derivative with respect to η brings an i.

The Euclidean functional integral is

$$W_{\mathrm{E}}[\eta,\bar{\eta}] = \int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\exp\left[-\int \mathrm{d}^4x\,[\mathcal{L}(x) - \bar{\psi}\eta - \bar{\eta}\psi]\right]$$

[2] See also eigenchris, Spinors for Beginners 9, `https://youtu.be/4NJBvkjpC3E?t=2340`, at minute 39:00 as well as A. Steane, An Introduction to Spinors, `https://arxiv.org/abs/1312.3824`.

and the Euclidean Green's function is [r]

$$G_{\mathrm{E}}^{(2n)}(x_1,\ldots,x_n,y_1,\ldots,y_n) = (-1)^n \left.\frac{\delta^{2n} W_{\mathrm{E}}[\eta,\bar{\eta}]}{\delta\bar{\eta}_{\alpha_1}(x_1)\cdots\delta\bar{\eta}_{\alpha_n}(x_n)\delta\eta_{\beta_1}(y_1)\cdots\delta\eta_{\beta_n}(y_n)}\right|_{\eta=\bar{\eta}=0}$$

The functional integral involves spinorial variables. This kind of integration has to be properly defined.

Exercise. Let $n=1$, then

$$\begin{aligned}
G^{(2)}(x,y) &= (-1)\left.\frac{\delta^2 W_{\mathrm{E}}[\eta,\bar{\eta}]}{\delta\bar{\eta}_\alpha(x)\delta\eta_\beta(y)}\right|_{\eta=\bar{\eta}=0} \\
&= (-1)\frac{\delta}{\delta\bar{\eta}_\alpha(x)}\int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\,\exp\left[-\int \mathrm{d}^4x'\,[\mathcal{L}-\bar{\psi}\eta-\bar{\eta}\psi]\right][-\bar{\psi}_\beta(y)]\Big|_{\eta=\bar{\eta}=0} \\
&= \int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\,\exp\left[-\int \mathrm{d}^4x'\,[\mathcal{L}-\bar{\psi}\eta-\bar{\eta}\psi]\right]\psi_\alpha(x)\bar{\psi}_\beta(y)\Big|_{\eta=\bar{\eta}=0} \\
&= \langle 0|\,\mathcal{T}\{\psi_\alpha(x)\bar{\psi}_\beta(y)\}\,|0\rangle
\end{aligned}$$

Lecture 3

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4.1 Grassmann algebra

See Cheng, §1.3. See also DeWitt, Supermanifolds. The fields ψ and $\bar{\psi}$ are classical spinor fields with four components in Dirac's notation

$$\psi_\alpha\,,\quad \alpha=1,2,3,4$$

Each component is a Grassmann-odd number[3]: it anti-commutes with itself and other Grassmann-odd numbers. For example

$$\psi_1\psi_2 = -\psi_2\psi_1$$

Grassmann numbers are needed when taking the classical limit of the quantized spinor fields which are anti-commuting. When using a functional definition of quantum field theory, the quantities appearing inside the path integral are all classical, not operators, so they have to be Grassmann numbers.

The path integral is the continuum limit of an ordinary integral on a lattice. A field is evaluated only at the sites of the lattice

$$\psi_\alpha(x^i) \equiv \psi_\alpha^i$$

The path integral on the lattice is an ordinary integral but on Grassmann-odd variables[4]

$$\int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}] \leftrightarrow \int \prod_{\alpha,i}\mathrm{d}\psi_\alpha^i \prod_{\alpha,i}\mathrm{d}\bar{\psi}_\alpha^i$$

One needs to define analysis for Grassmann numbers.

Algebra. An n-dimensional Grassmann algebra is generated by n Grassmann variables θ_i that anti-commute

$$\{\theta_i,\theta_j\}=0 \implies \theta_i^2=0$$

A generic element of the algebra can be expanded in a finite Taylor series

$$f(\theta_1,\ldots,\theta_n) = f_0 + f_{1,i}\theta^i + f_{2,ij}\theta^i\theta^j + \cdots + f_{n,i_1\cdots i_n}\theta^{i_1}\cdots\theta^{i_n}$$

[3] Odd or even refers to the number of Grassmann variables θ_i in the expansion of a Grassmann number z in terms of such Grassmann variables θ_i. A Grassmann number may not have definite Grassmann parity, but can be separated into odd and even parts. Odd Grassmann numbers anti-commute between themselves and even Grassmann numbers commute with every Grassmann number.

[4] Grassmann numbers are individual elements of the exterior algebra generated by a set of n Grassmann variables.

This is because if there is any repeated variable, then it is zero $\theta_j^2 = 0$. The coefficients of the expansion are complex numbers that are completely anti-symmetric in the indices i_k.

Consider the expansion of an element of the algebra by writing explicitly the dependence on one particular[5] Grassmann variable θ. In such variable, the expansion is at most linear

$$f(\theta) = f_0 + f_1\theta$$

The coefficients f_0 and f_1 are independent of θ and depend on the other $n-1$ Grassmann variables.

In general, one has to consider a function also of space-time

$$f(x,\theta) = f_0(x) + f_1(x)\theta$$

The coefficient of the θ-expansion are space-time fields. If $f(x,\theta)$ is a Grassmann-even field, then the Grassmann parity must be the same on either side of the equation, so $f_0(x)$ is a Grassmann-even field and $f_1(x)$ is a Grassmann-odd field.

The addition of Grassmann variables to space-time gives superspace[6]. A field on superspace is a superfield and contains different ordinary fields. In supersymmetry, a superfield is a representation of the supersymmetry algebra.

Differentiation. The left and right derivatives are defined as

$$\mathrm{d}_\theta\theta = \theta\overleftarrow{\mathrm{d}}_\theta = 1$$

Therefore, the left derivative of the above element of the algebra is

$$\mathrm{d}_\theta f(\theta) = \begin{cases} +f_1\,, & f \text{ odd} \\ -f_1\,, & f \text{ even} \end{cases}$$

while the right derivative is simpler

$$f(\theta)\overleftarrow{\mathrm{d}}_\theta = f_1$$

Integration. The integration over Grassmann-odd variables is called Berenzin integration[7] which differs from Lebesgue's.

Typically, the integral is the inverse operator of the derivative. Since the partial derivative with respect to a Grassmann variable is a Grassmann-odd operator then the second derivative is zero[8]. The derivative is not an invertible operator. The integral must be defined in another way

$$\int \mathrm{d}\theta\, f(\theta)$$

One imposes two properties: linearity

$$\int \mathrm{d}\theta\,(f+\alpha g) = \int \mathrm{d}\theta\, f + \alpha\int \mathrm{d}\theta\, g\,,\quad \alpha\in\mathbb{C}$$

and translational invariance

$$\int \mathrm{d}\theta\, f(\theta) \equiv \int \mathrm{d}(\theta+\eta)\, f(\theta+\eta)$$

where η is Grassmann-odd number independent of the particular variable θ.

[5]This means that the dependence on the other $n-1$ variables is hidden inside the expansion coefficients which are no longer complex numbers, but Grassmann numbers.

[6]In particular Minkowski superspace: Minkowski space is extended with anti-commuting fermionic degrees of freedom, taken to be anti-commuting Weyl spinors from the Clifford algebra associated to the Lorentz group.

[7]See Berezin, F. A. (1966). The Method of Second Quantization. Pure and Applied Physics. Vol. 24. New York. ISSN 0079-8193. `https://www.sciencedirect.com/bookseries/pure-and-applied-physics/vol/24`.

[8]This is a consequence of $\partial_z z = 1$.

The second property can be expanded to have

$$\int \mathrm{d}\theta\, f(\theta) = \int \mathrm{d}(\theta+\eta)\, f(\theta+\eta) = \int \mathrm{d}\theta\, f(\theta+\eta) \implies \int \mathrm{d}\theta\, (f_0 - \theta f_1) = \int \mathrm{d}\theta\, [f_0 - (\theta+\eta) f_1]$$

Applying linearity, one finds

$$\int \mathrm{d}\theta\, \eta f_1 = 0 \implies \eta f_1 \int \mathrm{d}\theta = 0 \implies \boxed{\int \mathrm{d}\theta = 0}$$

since the above has to hold for every f_1 and η (which do not depend on θ). Using this result, one obtains

$$\int \mathrm{d}\theta\, f(\theta) = \int \mathrm{d}\theta\, f_0 + f_1\theta = \int \mathrm{d}\theta\, f_1\theta = \pm f_1 \int \mathrm{d}\theta\, \theta \equiv \pm f_1 \implies \boxed{\int \mathrm{d}\theta\, \theta = 1}$$

where $+$ is for f odd and $-$ for f even. One may notice that Berenzin integration is equivalent to differentiation

$$\int \mathrm{d}\theta\, f(\theta) = \pm f_1 = \mathrm{d}_\theta f$$

Therefore, the general definition of integration in a Grassmann-odd variable is

$$\boxed{\int \mathrm{d}\theta\, f(\theta) \equiv \mathrm{d}_\theta f|_{\theta=0}}$$

[r] why evaluated at $\theta = 0$? for defining the Berenzin integral also for ordinary functions?

Change of variables. When performing a change of variables, one has to account for the Jacobian. For the Berenzin integral, the inverse Jacobian is produced instead.

For ordinary (Grassmann-even, Riemann or Lebesgue) integration, a change of variable produces

$$\int \mathrm{d}y\, f(y)\,, \quad y = g(x) \implies \int \mathrm{d}y\, f(y) = \int \mathrm{d}x\, |\partial_x y| f(g(x))$$

For Grassmann-odd integration, a change of variables is

$$\theta' = g(\theta) = a + b\theta\,, \quad \mathrm{d}_\theta \theta' = b$$

Notice that a is odd and b is even. Knowing that

$$\int \mathrm{d}\theta'\, f(\theta') = \int \mathrm{d}\theta'\, (f_0 + f_1\theta') = \pm f_1$$

one may consider the integration without Jacobian

$$\int \mathrm{d}\theta\, f(\theta'(\theta)) = \int \mathrm{d}\theta\, f(a+b\theta) = \int \mathrm{d}\theta\, [f_0 + f_1(a+b\theta)] = \pm f_1 b = \pm f_1\, \mathrm{d}_\theta \theta'$$

Therefore, when performing the change of variables, one has to use the inverse Jacobian

$$\int \mathrm{d}\theta'\, f(\theta') = \int \mathrm{d}\theta\, |\mathrm{d}_\theta \theta'|^{-1} f(\theta'(\theta))$$

Generalization to more variables. The differentiation and integration results above can be generalized to more Grassmann variables.

Differentiation. The left and right derivatives are

$$\mathrm{d}_{\theta_i}\theta_j = \theta_j \overleftarrow{\mathrm{d}}_{\theta_i} = \delta_{ij}$$

The derivative are Grassmann-odd operators

$$\{\mathrm{d}_{\theta_i}, \mathrm{d}_{\theta_j}\} = 0\,, \quad \{\mathrm{d}_{\theta_i}, \theta_j\} = 0$$

where, in the second equality, the operator is understood as not acting on θ_j: when computing the derivative of a product, a minus sign appears every time the derivative goes though an odd variable. As such, the derivative of a product is

$$\mathrm{d}_{\theta_i}(\theta_1 \cdots \theta_n) = \delta_{i1}\theta_2 \cdots \theta_n - \theta_1\delta_{i2}\theta_3 \cdots \theta_n + \cdots + (-1)^{n-1}\theta_1 \cdots \theta_{n-1}\delta_{in} = \sum_{j=1}^n (-1)^{j-1}\delta_{ij}\prod_{k\neq j}^n \theta_k$$

while the right derivative is similar

$$(\theta_1 \cdots \theta_n)\overleftarrow{\mathrm{d}}_{\theta_i} = \theta_1 \cdots \theta_{n-1}\delta_{in} - \theta_1 \cdots \delta_{i,n-1}\theta_n + \cdots + (-1)^{n-1}\delta_{i1}\theta_2 \cdots \theta_n = \sum_{j=1}^n (-1)^{n-j}\delta_{ij}\prod_{k\neq j}^n \theta_k$$

Integration. For integration, the measures anti-commute

$$\{\mathrm{d}\theta_i, \mathrm{d}\theta_j\} = 0$$

The fundamental integral properties are

$$\int \mathrm{d}\theta_i\, 1 = 0\,, \quad \int \mathrm{d}\theta_i\, \theta_j = \delta_{ij}$$

The integral in one variable of an element of the algebra is

$$\int \mathrm{d}\theta_i\, f(\theta_1, \ldots, \theta_n) = \mathrm{d}_{\theta_i} f(\theta_1, \ldots, \theta_n)|_{\theta_i=0}$$

The integral in multiple variables is

$$\int \mathrm{d}\theta_n \cdots \mathrm{d}\theta_1\, f(\theta_1, \ldots, \theta_n) = \mathrm{d}_{\theta_n} \cdots \mathrm{d}_{\theta_1} f|_{\theta_1=\cdots=\theta_n=0}$$

Change of variables. For a change of variables $\theta'_i = b_{ij}\theta_j$, one has

$$\int \mathrm{d}\theta'_n \cdots \mathrm{d}\theta'_1 f(\theta'_1, \ldots, \theta'_n) = \int \mathrm{d}\theta_n \cdots \mathrm{d}\theta_1\, (\det b)^{-1} f(\theta'_1(\theta), \ldots, \theta'_n(\theta))$$

Gaussian integrals. The Gaussian integral is

$$G(A) = \int \mathrm{d}\theta_n \cdots \mathrm{d}\theta_1\, \mathrm{e}^{\frac{1}{2}\theta_i A_{ij}\theta_j}$$

where A is an anti-symmetric square matrix of size n, $A_{ij} = -A_{ji}$.

Let $n = 2$. The exponent is

$$\theta_i A_{ij}\theta_j = \begin{bmatrix}\theta_1 & \theta_2\end{bmatrix}\begin{bmatrix}0 & A_{12}\\ -A_{12} & 0\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}$$

The integral is

$$\begin{aligned} G(A) &= \int \mathrm{d}\theta_2\, \mathrm{d}\theta_1\, \exp\left[\frac{1}{2}(\theta_1 A_{12}\theta_2 - \theta_2 A_{12}\theta_1)\right] = \int \mathrm{d}\theta_2\, \mathrm{d}\theta_1 \mathrm{e}^{\theta A_{12}\theta_2} \\ &= \int \mathrm{d}\theta_2\, \mathrm{d}\theta_1\, (1 + \theta_1 A_{12}\theta_2) = \int \mathrm{d}\theta_2\, \mathrm{d}\theta_1\, \theta_1 A_{12}\theta_2 = A_{12}\int \mathrm{d}\theta_2\, \theta_2 = A_{12} = \sqrt{\det A} \end{aligned}$$

This is true for any n. For Berenzin integrals, the determinant is in the numerator of the result and not the denominator.

Table of Gaussian integrals. For real Grassmann-even variables

$$\int \mathrm{d}x_1 \cdots \mathrm{d}x_n \, \mathrm{e}^{-\frac{1}{2}x^\top A x} = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}}$$

For complex Grassmann-even variables

$$\int \left[\prod_{j=1}^{n} \mathrm{d}z_j \, \mathrm{d}\bar{z}_j\right] \mathrm{e}^{-z^\dagger A z} = \frac{\pi^{\frac{n}{2}}}{\det A}$$

[r] For real Grassmann-odd variables

$$\int \mathrm{d}\theta_n \cdots \mathrm{d}\theta_1 \, \mathrm{e}^{\frac{1}{2}\theta A \theta} = \sqrt{\det A}$$

For complex Grassmann-odd variables

$$\int \left[\prod_{j=n}^{1} \mathrm{d}\theta_j \, \mathrm{d}\bar{\theta}_j\right] \mathrm{e}^{\bar{\theta}^\top A \theta} = \det A$$

Functional integrals. The generalization to functional integration can be defined on the lattice. The continuum theory is retrieved in the continuum limit of the lattice.

Example. Consider the free generating functional of a scalar theory

$$W_0[0] = \int [\mathcal{D}\varphi] \, \exp\left[-\int \mathrm{d}^4x \, \frac{1}{2}\varphi(\Box + m^2)\varphi\right] \propto [\det(\Box + m^2)]^{-\frac{1}{2}}$$

where the determinant of an operator is the product of its eigenvalues. For a spinor theory, one has

$$W_0[0] = \int [\mathcal{D}\psi \, \mathcal{D}\bar{\psi}] \, \exp\left[-\int \mathrm{d}^4x \, \frac{1}{2}\bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi\right] \propto \det(\mathrm{i}\,\partial\!\!\!/ - m)$$

4.2 Free field theory

For a free theory, the functional integral can be computed exactly. For a weak interacting theory, one may applie perturbation theory.

Through Grassmann numbers, one may give meaning to the path integral for fermions

$$W[\eta, \bar{\eta}] = \int [\mathcal{D}\psi \, \mathcal{D}\bar{\psi}] \exp\left[\mathrm{i}\int \mathrm{d}^4x \, [\mathcal{L}(x) + \bar{\psi}\eta + \bar{\eta}\psi]\right]$$

Consider the free Lagrangian

$$\mathcal{L}_0(\psi, \bar{\psi}) = \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi$$

The generating functional is

$$W_0[\eta, \bar{\eta}] = N \int [\mathcal{D}\psi \, \mathcal{D}\bar{\psi}] \exp\left[\mathrm{i}\int \mathrm{d}^4x \, [\bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi + \bar{\psi}\eta + \bar{\eta}\psi]\right]$$

The propagator is the inverse of the kinetic term. Using the fact that the propagator is Green's function of the Dirac operator, one gets

$$\int \mathrm{d}^4x \, (\mathrm{i}\,\partial\!\!\!/ - m)_{\alpha\beta} S_{\beta\gamma}(x) = \delta_{\alpha\gamma} \implies S = (\mathrm{i}\,\partial\!\!\!/ - m)^{-1}$$

One may complete the square in the exponent of the generating functional

$$\begin{aligned}
-\mathrm{i}E &= \int \mathrm{d}^4x \, [\bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi + \bar{\psi}\eta + \bar{\eta}\psi] \\
&= \int \mathrm{d}^4x \left\{\left[\bar{\psi} + \int \mathrm{d}^4y \, \bar{\eta}(y)(\mathrm{i}\,\partial\!\!\!/ - m)^{-1}_y\right](\mathrm{i}\,\partial\!\!\!/ - m)_x \left[\psi + \int \mathrm{d}^4y' \, (\mathrm{i}\,\partial\!\!\!/ - m)^{-1}_{y'}\eta(y')\right]\right. \\
&\quad \left. - \int \mathrm{d}^4y \, \mathrm{d}^4y' \, \bar{\eta}(y)(\mathrm{i}\,\partial\!\!\!/ - m)^{-1}_y(\mathrm{i}\,\partial\!\!\!/ - m)_x(\mathrm{i}\,\partial\!\!\!/ - m)^{-1}_{y'}\eta(y')\right\} \\
&= \int \mathrm{d}^4x \, \bar{\chi}(\mathrm{i}\,\partial\!\!\!/ - m)_x\chi - \int \mathrm{d}^4x \, \mathrm{d}^4y' \, \bar{\eta}(x)(\mathrm{i}\,\partial\!\!\!/ - m)^{-1}_{y'}\eta(y')
\end{aligned}$$

[r] where

$$\chi \equiv \psi + \int \mathrm{d}^4y' \,(\mathrm{i}\,\partial\!\!\!/ - m)^{-1}_{y'}\eta(y')\,, \quad (\mathrm{i}\,\partial\!\!\!/ - m)^{-1}_{y}(\mathrm{i}\,\partial\!\!\!/ - m)_x = \delta^{(4)}(x-y)$$

The generating functional becomes

$$\begin{aligned} W_0[\eta,\bar\eta] &= N\int [\mathcal{D}\chi\,\mathcal{D}\bar\chi]\exp\left[\mathrm{i}\int \mathrm{d}^4x\,\bar\chi(\mathrm{i}\,\partial\!\!\!/ - m)_x\chi - \mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y'\,\bar\eta(x)(\mathrm{i}\,\partial\!\!\!/ - m)^{-1}_{y'}\eta(y')\right] \\ &= \exp\left[-\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y'\,\bar\eta(x)(\mathrm{i}\,\partial\!\!\!/ - m)^{-1}_{y'}\eta(y')\right]W_0[0,0] \\ &= \exp\left[-\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y'\,\bar\eta_\alpha(x)S_{\alpha\beta}(x-y')\eta_\beta(y')\right] \end{aligned}$$

At the first line, the path integral of the first addendum of the exponent is a Gaussian integral and can be absorbed into the normalization constant N to give a normalization of $W_0[\eta=\bar\eta=0]=1$ [r].

Remark. The two-point Green's function for the free theory is

$$\begin{aligned} G^{(2)}(x_1,x_2) &= \left.\frac{\delta^2 W_0[\eta,\bar\eta]}{\delta\bar\eta_\gamma(x_1)\delta\eta_\delta(x_2)}\right|_{\eta=\bar\eta=0} = \frac{\delta}{\delta\bar\eta_\gamma(x_1)}\left.\left[\mathrm{i}\int \mathrm{d}^4x\,\bar\eta_\alpha(x)S_{\alpha\delta}(x-x_2)W_0\right]\right|_{\eta=\bar\eta=0} \\ &= \mathrm{i}S_{\gamma\delta}(x_1-x_2) \equiv \langle 0|\,\mathcal{T}\{\psi_\gamma(x_1)\bar\psi_\delta(x_2)\}\,|0\rangle \end{aligned}$$

4.3 Interacting field theory

The Lagrangian is

$$\mathcal{L}(\psi,\bar\psi) = \bar\psi(\mathrm{i}\,\partial\!\!\!/ - m)\psi + \mathcal{L}_{\mathrm{int}}(\psi,\bar\psi)$$

Since the Lagrangian is a scalar, it has to be a function only of bilinears of spinors. The generating functional is

$$W[\eta,\bar\eta] = \int [\mathcal{D}\psi\,\mathcal{D}\bar\psi]\exp\left[\mathrm{i}\int \mathrm{d}^4x\,[\mathcal{L}_0(x) + \mathcal{L}_{\mathrm{int}}(\psi,\bar\psi) + \bar\psi\eta + \bar\eta\psi]\right]$$

Noting that

$$\delta_{\eta(y)}\int \mathrm{d}^4x\,\bar\psi(x)\eta(x) = -\bar\psi(x)\,, \quad \delta_{\bar\eta(x)}\int \mathrm{d}^4x\,\bar\eta(x)\psi(x) = \psi(x)$$

one may apply the useful property of functional integrals[9] and rewrite the fields in terms of derivatives

$$\mathcal{L}_{\mathrm{int}}(\psi,\bar\psi)\exp\left[\mathrm{i}\int \mathrm{d}^4x\,(\bar\psi\eta + \bar\eta\psi)\right] = \mathcal{L}_{\mathrm{int}}(-\mathrm{i}\,\delta_{\bar\eta(x)},\mathrm{i}\,\delta_{\eta(x)})\exp\left[\mathrm{i}\int \mathrm{d}^4x\,(\bar\psi\eta + \bar\eta\psi)\right]$$

Therefore the generating functional is

$$\begin{aligned} W[\eta,\bar\eta] &= \int [\mathcal{D}\psi\,\mathcal{D}\bar\psi]\,\exp\left[\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L}_{\mathrm{int}}(-\mathrm{i}\,\delta_{\bar\eta},\mathrm{i}\,\delta_\eta)\right]\exp\left[\mathrm{i}\int \mathrm{d}^4x\,[\mathcal{L}_0(x) + \bar\psi\eta + \bar\eta\psi]\right] \\ &= \exp\left[\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L}_{\mathrm{int}}(-\mathrm{i}\,\delta_{\bar\eta},\mathrm{i}\,\delta_\eta)\right]W_0[\eta,\bar\eta] \end{aligned}$$

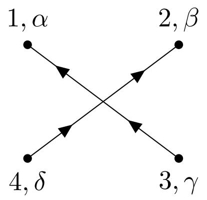
If the coupling constant is weak, then one may expand the exponential of the interaction in a perturbative series in powers of the coupling constant.

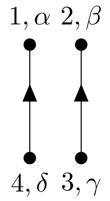
[9] See QFT I.

Wick's theorem. To see the effect of Wick's theorem, one may compute the four-point Green's function in the free theory

$$
\begin{aligned}
G^{(4)}_{\alpha\beta\gamma\delta}(x_1,x_2,x_3,x_4) &= \langle 0|\,\mathcal{T}\{\psi_\alpha(x_1)\psi_\beta(x_2)\bar\psi_\gamma(x_3)\bar\psi_\delta(x_4)\}\,|0\rangle \\
&= \left.\frac{\delta^4 W_0[\eta,\bar\eta]}{\delta\bar\eta_\alpha(x_1)\delta\bar\eta_\beta(x_2)\delta\eta_\gamma(x_3)\delta\eta_\delta(x_4)}\right|_{\eta=\bar\eta=0} \\
&= \frac{\delta^3}{\delta\bar\eta_\alpha(x_1)\delta\bar\eta_\beta(x_2)\delta\eta_\gamma(x_3)}\Bigg\{\exp\left[-\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar\eta_\varepsilon(x)S_{\varepsilon\zeta}(x-y)\eta_\zeta(y)\right] \\
&\quad \times \mathrm{i}\int \mathrm{d}^4x\,\bar\eta_\varepsilon S_{\varepsilon\delta}(x-x_4)\Bigg\}\Bigg|_{\eta=\bar\eta=0} \\
&= \frac{\delta^2}{\delta\bar\eta_\alpha(x_1)\delta\bar\eta_\beta(x_2)}\Bigg[W_0[\eta,\bar\eta]\mathrm{i}\int \mathrm{d}^4y\,\bar\eta_\iota S_{\iota\gamma}(y-x_3) \\
&\quad \times \mathrm{i}\int \mathrm{d}^4x\,\bar\eta_\varepsilon S_{\varepsilon\delta}(x-x_4)\Bigg]\Bigg|_{\eta=\bar\eta=0} \\
&= \frac{\delta}{\delta\bar\eta_\alpha(x_1)}\Bigg[\frac{\delta W_0[\eta,\bar\eta]}{\delta\bar\eta_\beta(x_2)}\,\mathrm{i}\int \mathrm{d}^4y\,\bar\eta_\iota S_{\iota\gamma}(y-x_3)\mathrm{i}\int \mathrm{d}^4x\,\bar\eta_\varepsilon S_{\varepsilon\delta}(x-x_4) \\
&\quad + W_0[\eta,\bar\eta]\mathrm{i}S_{\beta\gamma}(x_2-x_3)\mathrm{i}\int \mathrm{d}^4x\,\bar\eta_\varepsilon S_{\varepsilon\delta}(x-x_4) \\
&\quad - W_0[\eta,\bar\eta]\mathrm{i}\int \mathrm{d}^4y\,\bar\eta_\iota S_{\iota\gamma}(y-x_3)\mathrm{i}S_{\beta\delta}(x_2-x_4)\Bigg]\Bigg|_{\eta=\bar\eta=0} \\
&= \frac{\delta}{\delta\bar\eta_\alpha(x_1)}\Bigg[\frac{\delta W_0[\eta,\bar\eta]}{\delta\bar\eta_\beta(x_2)}\,\mathrm{i}\int \mathrm{d}^4y\,\bar\eta_\iota S_{\iota\gamma}(y-x_3)\mathrm{i}\int \mathrm{d}^4x\,\bar\eta_\varepsilon S_{\varepsilon\delta}(x-x_4)\Bigg]\Bigg|_{\eta=\bar\eta=0} \\
&\quad + \Bigg[\frac{\delta W_0[\eta,\bar\eta]}{\delta\bar\eta_\alpha(x_1)}\mathrm{i}S_{\beta\gamma}(x_2-x_3)\mathrm{i}\int \mathrm{d}^4x\,\bar\eta_\varepsilon S_{\varepsilon\delta}(x-x_4) \\
&\quad - \frac{\delta W_0[\eta,\bar\eta]}{\delta\bar\eta_\alpha(x_1)}\mathrm{i}\int \mathrm{d}^4y\,\bar\eta_\iota S_{\iota\gamma}(y-x_3)\mathrm{i}S_{\beta\delta}(x_2-x_4)\Bigg]\Bigg|_{\eta=\bar\eta=0} \\
&\quad + \mathrm{i}^2\Bigg[W_0[\eta,\bar\eta]S_{\beta\gamma}(x_2-x_3)S_{\alpha\delta}(x_1-x_4) \\
&\quad - W_0[\eta,\bar\eta]S_{\alpha\gamma}(x_1-x_3)S_{\beta\delta}(x_2-x_4)\Bigg]\Bigg|_{\eta=\bar\eta=0} \\
&= S_{\alpha\gamma}(x_1-x_3)S_{\beta\delta}(x_2-x_4) - S_{\beta\gamma}(x_2-x_3)S_{\alpha\delta}(x_1-x_4)
\end{aligned}
$$

In terms of diagrams, one has

$G^{(4)}_{\alpha\beta\gamma\delta}(x_1,x_2,x_3,x_4) =$ 

$-$ 

where every directed line is a propagator. In the Feynman diagrams, scalars are represented as undirected dashed lines, while fermions are represented as directed solid lines.

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The different sign can be interpreted using Wick's theorem: the only non-vanishing contributions are the ones where the fields are contracted completely. The minus sign appears in the second diagram since one has to contract the fields when they are not next to each other

$$
\langle 0|\,\mathcal{T}\{\psi_1\psi_2\bar\psi_3\bar\psi_4\}\,|0\rangle = :\psi_1\psi_2\bar\psi_3\bar\psi_4: + :\psi_1\psi_2\bar\psi_3\bar\psi_4:
$$

This gives a general rule: when performing contractions of spinor fields one has to be careful about the signs.

4.3.1 Yukawa theory

See Srednicki, §§45, 51. The Yukawa theory treats the coupling between a massive real scalar field φ and a massive spinor field ψ. This theory is the analogue of the $\lambda\varphi^4$ theory for scalar fields.

Lagrangian. Let M be the mass of the boson field and m the mass of the fermion field. The Yukawa interaction Lagrangian is

$$\mathcal{L}_{\mathrm{Y}} = g\varphi\bar{\psi}\psi$$

[r] One would like to write the simplest Lagrangian. To this end, one may impose a U(1) symmetry on the spinor field along with Lorentz symmetry and the three discrete symmetries of parity, time reversal and charge conjugation. The scalar terms allowed are all the powers of field with non-negative coupling constant: φ, φ^2, φ^3 and φ^4. While no further spinor terms are allowed other than the kinetic term, the three scalar interaction terms above are too many (notice that φ^2 is a mass term). One may modify the Yukawa interaction

$$\mathcal{L}_{\mathrm{Y}} = \mathrm{i}g\varphi\bar{\psi}\gamma_5\psi$$

The imaginary unit makes the interaction Lagrangian real. In fact, consider

$$(\bar{\psi}\gamma_5\psi)^\dagger = (\psi^\dagger\gamma_0\gamma_5\psi)^\dagger = \psi^\dagger\gamma_5^\dagger\gamma_0^\dagger\psi = \psi^\dagger\gamma_5\gamma_0\psi = -\bar{\psi}\gamma_5\psi$$

This term is anti-hermitian and therefore is purely imaginary. Since the scalar field is real, then an imaginary unit is needed to keep the Lagrangian real.

One notices that under a parity transformation, the interaction term changes sign

$$P(\bar{\psi}\gamma_5\psi) = -\bar{\psi}\gamma_5\psi$$

while the kinetic fermionic term is invariant. For the Lagrangian to conserve parity and in particular $\varphi \to -\varphi$, the scalar field φ must be a pseudo-scalar field and the linear and cubic scalar interactions along with their ultraviolet divergences cannot appear:

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}M^2\varphi^2 + \bar{\psi}(\mathrm{i}\,\not{\partial} - m)\psi - \frac{\lambda}{4!}\varphi^4 + \mathrm{i}g\varphi\bar{\psi}\gamma_5\psi$$

Perturbation theory and Feynman rules. For a weak interacting theory, $\lambda, g \ll 1$, one may apply perturbation theory. Trading

$$\bar{\psi} \to \mathrm{i}\,\delta_\eta\,, \quad \psi \to -\mathrm{i}\,\delta_{\bar{\eta}}\,, \quad \varphi \to -\mathrm{i}\,\delta_J$$

one can write the generating functional

$$W[J,\eta,\bar{\eta}] = \exp\left\{\mathrm{i}\int \mathrm{d}^4x\,\left[-\frac{\lambda}{4!}(-\mathrm{i}\,\delta_{J(x)})^4 + \mathrm{i}g(-\mathrm{i}\,\delta_{J(x)})(\mathrm{i}\,\delta_{\eta(x)})\gamma_5(-\mathrm{i}\,\delta_{\bar{\eta}(x)})\right]\right\}W_0[J,\eta,\bar{\eta}]$$

where the free generating functional is

$$\begin{aligned} W_0[J,\eta,\bar{\eta}] &= \int [\mathcal{D}\varphi]\,\exp\left[\mathrm{i}\int \mathrm{d}^4x\,\frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}M^2\varphi^2 + J\varphi\right] \\ &\quad\times \int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\,\exp\left[\mathrm{i}\int \mathrm{d}^4x\,\bar{\psi}(\mathrm{i}\,\not{\partial} - m)\psi + \bar{\psi}\eta + \bar{\eta}\psi\right] \\ &= \exp\left[\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y\,J(x)\Delta(x-y)J(y)\right] \\ &\quad\times \exp\left[-\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{\eta}_\alpha(x)S_{\alpha\beta}(x-y)\eta_\beta(y)\right] \end{aligned}$$

and the propagators are

$$\Delta(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}k(x-y)}}{k^2 - M^2 + \mathrm{i}\varepsilon}\,, \quad S_{\alpha\beta}(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(\not{k}+m)_{\alpha\beta}}{k^2 - m^2 + \mathrm{i}\varepsilon}\mathrm{e}^{-\mathrm{i}k(x-y)}$$

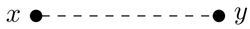
By computing the generating functional at one-loop, one finds

$$W[J,\eta,\bar{\eta}] = W_0[J,\eta,\bar{\eta}] + \mathrm{i}\int \mathrm{d}^4x \left[-\frac{\lambda}{4!}(-\mathrm{i}\,\delta_{J(x)})^4 + \mathrm{i}g(-\mathrm{i}\,\delta_{J(x)})(\mathrm{i}\,\delta_{\eta(x)})\gamma_5(-\mathrm{i}\,\delta_{\bar{\eta}(x)})\right] W_0 + o(\lambda, g)$$

Proceeding as the $\lambda\varphi^4$ theory, the expression above can be expressed in terms of Feynman diagrams according to the following rules:

- Scalar propagator

$$\Delta(x-y) \equiv$$



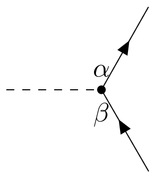
- Fermion propagator

$$S_{\alpha\beta}(x-y) \equiv$$

  Notice the addition of an arrow to denote direction and distinguish between the field ψ and its Dirac adjoint $\bar{\psi}$.

- Internal point

$$\mathrm{i}g\int \mathrm{d}^4x\,(\gamma_5)_{\alpha\beta} \equiv$$



- External scalar point

$$\mathrm{i}\int \mathrm{d}^4y_j\, J(y_j) \to$$

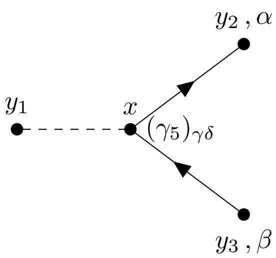
- External fermionic point

$$\mathrm{i}\int \mathrm{d}^4y_j\,\eta_\alpha(y_j) \to$$

,

$$\mathrm{i}\int \mathrm{d}^4y_j\,\bar{\eta}_\alpha(y_j) \to$$

Example. The tree-level diagram for the Yukawa interaction is

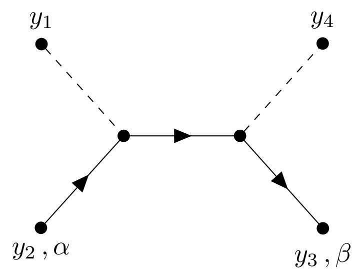


which corresponds to

$$\mathrm{i}^3\int \mathrm{d}^4y_1\,\mathrm{d}^4y_2\,\mathrm{d}^4y_3\,\mathrm{d}^4x\,\bar{\eta}_\alpha(y_2)S_{\alpha\gamma}(y_2-x)(\mathrm{i}g)(\gamma_5)_{\gamma\delta}S_{\delta\beta}(x-y_3)\eta_\beta(y_3)\Delta(x-y_1)J(y_1)$$

[r] Since order matters for spinors, to write the integral without indices one has to go from right to left and follow against the direction of the arrows.

Example. With two Yukawa vertices, one can draw



[r] When writing amputated Green's function, the external lines are cut and only the internal part of the diagram is computed.

Remark. One neglects vacuum diagrams by choosing a suitable normalization

$$W[J = \eta = \bar{\eta} = 0] = 1$$

Remark. One is interested in connected Green's functions because the non-connected diagrams can be written as products of connected ones. The generating functional for connected Green's functions is

$$Z[J, \eta, \bar{\eta}] = \ln W[J, \eta, \bar{\eta}]$$

Remark. In the interest of renormalization, one computes amputated Green's functions $\Gamma^{(n)}$ which are the quantum vertices in the effective action and generated by it.

Remark. Computations are done in momentum space.

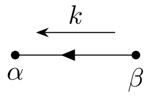
Feynman rules in momentum space. The rules are

- Scalar propagator

$$\frac{1}{k^2 - M^2 + \mathrm{i}\varepsilon} \equiv$$

- Fermion propagator

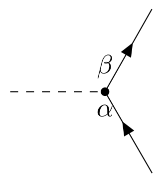
$$\frac{(\not{k} + m)_{\alpha\beta}}{k^2 - m^2 + \mathrm{i}\varepsilon} \equiv$$



  Notice that the momentum arrow follows the line's arrow for particles, but it is in the opposite direction for anti-particles.

- Yukawa vertex

$$\mathrm{i}g(\gamma_5)_{\alpha\beta} \equiv$$



  [r]

Symmetries. The Lagrangian is invariant under a global U(1) transformation of the fermionic fields

$$\psi' = \mathrm{e}^{\mathrm{i}q}\psi\,, \quad \bar{\psi} = \mathrm{e}^{-\mathrm{i}q}\bar{\psi}\,, \quad \varphi' = \varphi$$

Renormalizability. To study renormalizability one has to look at the mass dimension of the coupling constants. One already knows that λ is dimensionless and $\dim\varphi = 1$. From the fermionic mass term, one obtains

$$\dim(\bar{\psi}m\psi) = 4 \implies \dim\psi = \frac{3}{2}$$

The Yukawa coupling is

$$\dim(g\varphi\bar{\psi}\gamma_5\psi) = 4 \implies \dim g = 0$$

Therefore the theory is renormalizable.

By applying the BPHZ renormalization, one finds that the fields are renormalized as

$$\varphi_0 = Z_\varphi^{\frac{1}{2}}\varphi\,, \quad \psi_0 = Z_\psi^{\frac{1}{2}}\psi\,, \quad \bar{\psi}_0 = Z_\psi^{\frac{1}{2}}\bar{\psi}$$

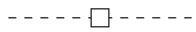
The parameters are renormalized as

$$\lambda_0 = Z_\lambda Z_\varphi^{-2}\lambda\,, \quad g_0 = Z_g Z_\varphi^{-\frac{1}{2}} Z_\psi^{-1} g\,, \quad M_0^2 = Z_M Z_\varphi^{-1} M^2\,, \quad m_0^2 = Z_m Z_\psi^{-1} m^2$$

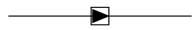
The bare Lagrangian can be split into a renormalized Lagrangian and the counter terms

$$
\begin{aligned}
\mathcal{L}_0 = & \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}M^2\varphi^2 - \frac{\lambda}{4!}\varphi^4 + \bar\psi(\mathrm{i}\,\partial\!\!\!/ - m)\psi + \mathrm{i}g\varphi\bar\psi\gamma_5\psi \\
& + (Z_\varphi - 1)\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}(Z_M - 1)M^2\varphi^2 + (Z_\psi - 1)\bar\psi\mathrm{i}\,\partial\!\!\!/\psi - (Z_m - 1)m\bar\psi\psi \\
& - (Z_\lambda - 1)\frac{\lambda}{4!}\varphi^4 - (Z_g - 1)\mathrm{i}g\varphi\bar\psi\gamma_5\psi
\end{aligned}
$$

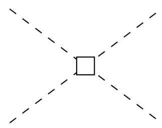
The counter terms (second and third line) produce divergent contributions that cancel the divergent contributions given by the renormalized Lagrangian. Therefore, one has to add the counter term vertices to the one already present. The first two addenda of the second line are a propagator

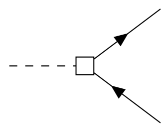


The last two addenda on the second line are another propagator



The third line corresponds to two vertices





The counter terms are fixed in order to cancel the divergences up to a finite part (which is scheme dependent).

5 One-loop contributions

One applies power counting to find the divergent contributions. From the $\lambda\varphi^4$ theory, one already knows that the two-point and four-point scalar Green's function, $\Gamma^{(2)}_\varphi$ and $\Gamma^{(4)}_\varphi$, are divergent. One expects a divergence in the two-point fermionic Green's function $\Gamma^{(2)}_\psi$ and the three-point Yukawa Green's function $\Gamma^{(3)}_{\mathrm{Y}}$.

Exercise. Find the superficial degree of divergence for a generic diagram at L loops with n_φ external scalar lines, $2n_\psi$ external fermion lines, I_φ internal scalar propagators, I_ψ internal fermion propagators, V_{Y} Yukawa vertices, V_4 scalar λ vertices.

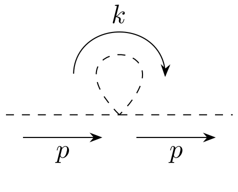
From this one finds that only a particular topology of diagrams is divergent, which are the ones above.

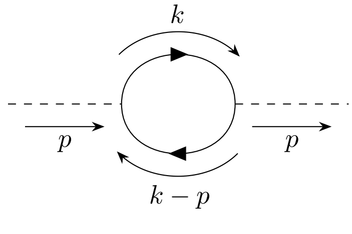
5.1 Two-point scalar Green's function

Recall that the two-point scalar Green's function up to one-loop is

$$
\Gamma^{(2)}_\varphi = p^2 + m^2 - \Sigma_\varphi\,, \quad \Gamma^{(2)}_\varphi|_{1\mathrm{L}} = -\Sigma_\varphi
$$

The one-loop contributions are given by the diagrams





First diagram. For the first diagram, one has found

$$\Sigma_\varphi(\mathrm{I}) = -\frac{\lambda M^2}{32\pi^2}\left[\frac{1}{\varepsilon} + 1 - \gamma - \ln\frac{M^2}{4\pi k^2} + o(\varepsilon^0)\right]$$

[r] where the overall minus is due to working in Minkowski (see Srednicki, eq. 51.21, but remember that their conventions are a spacelike metric, $D = 4 - \varepsilon$ and $4\pi k^2 = \mu^2$).

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Second diagram. From two Yukawa vertices one has to produce two fermionic propagators through contractions. In one of them, the spinor fields have to be swapped and one gets a negative sign. This is a general rule: alla fermionic loops bring a -1. Therefore, the one-loop contribution to the two-point scalar Green's function is

$$\begin{aligned}
\mathrm{i}\Sigma_\varphi(\mathrm{II}) &= \frac{\mathrm{i}^2}{2!}(\mathrm{i}g)^2(-1)\cdot 2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(\not k + m)_{\delta\alpha}}{k^2 - m^2 + \mathrm{i}\varepsilon}(\gamma_5)_{\alpha\beta}\frac{(\not k - \not p)_{\beta\gamma} + m}{(k-p)^2 - m^2 + \mathrm{i}\varepsilon}(\gamma_5)_{\gamma\delta} \\
&= -g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}[(\not k + m)\gamma_5(\not k - \not p + m)\gamma_5]}{(k^2 - m^2 + \mathrm{i}\varepsilon)[(k-p)^2 - m^2 + \mathrm{i}\varepsilon]} \\
&= -g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}[(\not k + m)(\not p - \not k + m)]}{(k^2 - m^2 + \mathrm{i}\varepsilon)[(k-p)^2 - m^2 + \mathrm{i}\varepsilon]} \\
&= -g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[\gamma^\mu\gamma^\nu k_\mu(p-k)_\nu + m^2\right]}{(k^2 - m^2 + \mathrm{i}\varepsilon)[(k-p)^2 - m^2 + \mathrm{i}\varepsilon]} \\
&= -4g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k(p-k) + m^2}{(k^2 - m^2 + \mathrm{i}\varepsilon)[(k-p)^2 - m^2 + \mathrm{i}\varepsilon]}
\end{aligned}$$

[r] why 2 in 1st line?
At the penultimate line, one applies

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4\eta^{\mu\nu}\,, \quad \mathrm{Tr}\, I = 4$$

Since the Dirac matrices disappeared, one can operate a Wick rotation to have

$$k_0 = \mathrm{i}k_0^{\mathrm{E}}\,, \quad k^2 = -k_{\mathrm{E}}^2\,, \quad \mathrm{d}^4k = \mathrm{i}\,\mathrm{d}^4k_{\mathrm{E}}$$

Therefore

$$k(k-p) = k_0(k-p)^0 - \mathbf{k}\cdot(\mathbf{k}-\mathbf{p}) \to -k(k-p)$$

One could begin from Euclidean space instead and rotate the Dirac matrices and the Clifford algebra.

The one-loop contribution is

$$\begin{aligned}
\Sigma_\varphi(\mathrm{II}) &= -4g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-k(p-k) + m^2}{(k^2 + m^2)[(p-k)^2 + m^2]} \\
&= 4g^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{(q + xp)(p - q - xp) - m^2}{(q^2 + D)^2} \\
&= 4g^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{-q^2 + (1-2x)qp + x(1-x)p^2 - m^2}{(q^2 + D)^2} \\
&= 4g^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{-q^2 + x(1-x)p^2 - m^2 + D - D}{(q^2 + D)^2} \\
&= 4g^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{x(1-x)p^2 - m^2 + D}{(q^2 + D)^2} - \frac{1}{q^2 + D} \\
&= 4g^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{2x(1-x)p^2}{(q^2 + D)^2} - \frac{1}{q^2 + D} \\
&\equiv \Sigma_2(\mathrm{II}) + \Sigma_1(\mathrm{II})
\end{aligned}$$

At the second line, one has applied Feynman combining

$$x[(p-k)^2+m^2]+(1-x)(k^2+m^2)=k^2+xp^2-2xpk+m^2=(k-xp)^2+m^2+x(1-x)p^2$$

and one lets

$$q\equiv k-xp\,,\quad D\equiv m^2+x(1-x)p^2$$

At the third line, one notices that linear terms in q vanish due to parity.

The two integral are divergent since their dimensions are 2 and 0. Using dimensional regularization, the first integral is

$$\begin{aligned}\Sigma_1(\mathrm{II})&=-4g^2k^{2\varepsilon}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^nq}{(2\pi)^n}\frac{1}{q^2+D}=-4g^2\int_0^1\mathrm{d}x\,\frac{\Gamma(-1+\varepsilon)}{(4\pi)^{2-\varepsilon}\Gamma(1)}\frac{1}{D^{-1+\varepsilon}}\\&=-\frac{4g^2k^{2\varepsilon}}{(4\pi)^{2-\varepsilon}}\Gamma(-1+\varepsilon)\int_0^1\frac{\mathrm{d}x}{[x(1-x)p^2+m^2]^{-1+\varepsilon}}\\&=-\frac{4g^2}{(4\pi)^2}(4\pi)^\varepsilon\frac{\Gamma(1+\varepsilon)}{(-1+\varepsilon)}\frac{k^{2\varepsilon}}{\varepsilon}\int_0^1\frac{\mathrm{d}x}{[x(1-x)p^2+m^2]^{-1+\varepsilon}}\\&=\frac{4g^2}{(4\pi)^2}(4\pi k^2)^\varepsilon\frac{\Gamma(1+\varepsilon)}{\varepsilon(1-\varepsilon)}\int_0^1\mathrm{d}x\,\frac{x(1-x)p^2+m^2}{[x(1-x)p^2+m^2]^\varepsilon}\\&=\frac{4g^2}{(4\pi)^2}\frac{\Gamma(1+\varepsilon)}{\varepsilon(1-\varepsilon)}\int_0^1\mathrm{d}x\,[x(1-x)p^2+m^2]\exp\left[-\varepsilon\ln\frac{x(1-x)p^2+m^2}{4\pi k^2}\right]\\&=\frac{g^2}{4\pi^2}\frac{1}{\varepsilon}\int_0^1\mathrm{d}x\,[x(1-x)p^2+m^2]+o(\varepsilon^{-1})\\&=\frac{g^2}{4\pi^2}\frac{1}{\varepsilon}\left[\frac{p^2}{6}+m^2\right]+o(\varepsilon^{-1})\end{aligned}$$

At the first line, one has applied

$$\int\frac{\mathrm{d}^nq}{(2\pi)^n}\frac{(q^2)^a}{(q^2+D)^b}=\frac{1}{D^{b-a-\frac{n}{2}}}\frac{\Gamma(b-a-{}^n\!/\!_2)\Gamma(a+{}^n\!/\!_2)}{(4\pi)^{\frac{n}{2}}\Gamma(b)\Gamma({}^n\!/\!_2)}$$

with $a=0$, $b=1$ and $\frac{n}{2}=2-\varepsilon$. At the third line one has applied $\Gamma(z+1)=z\Gamma(z)$.

The second integral is

$$\begin{aligned}\Sigma_2(\mathrm{II})&=8g^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{x(1-x)p^2}{(q^2+D)^2}\\&=8g^2\frac{k^{2\varepsilon}}{(4\pi)^{2-\varepsilon}}\Gamma(\varepsilon)\int_0^1\mathrm{d}x\,\frac{x(1-x)p^2}{[x(1-x)p^2+m^2]^\varepsilon}\\&=\frac{2g^2}{4\pi^2}\frac{1}{\varepsilon}\int_0^1\mathrm{d}x\,x(1-x)p^2+o(\varepsilon^{-1})\\&=\frac{2g^2}{4\pi^2}\frac{1}{\varepsilon}\frac{p^2}{6}+o(\varepsilon^{-1})=\frac{g^2}{4\pi^2}\frac{1}{\varepsilon}\frac{p^2}{3}+o(\varepsilon^{-1})\end{aligned}$$

At the second line, one has applied the integral formula above with $a=1$, $b=2$ and $\frac{n}{2}=2-\varepsilon$.

The total one-loop contribution to the two-point scalar Green's function coming from the second diagram is

$$\begin{aligned}\Sigma_\varphi(\mathrm{II})=\Sigma_1(\mathrm{II})+\Sigma_2(\mathrm{II})&=\frac{g^2}{4\pi^2}\frac{1}{\varepsilon}\left[\frac{p^2}{6}+m^2\right]+\frac{g^2}{4\pi^2}\frac{1}{\varepsilon}\frac{p^2}{3}+\text{finite}\\&=\frac{g^2}{4\pi^2}\frac{1}{\varepsilon}\left[\frac{p^2}{2}+m^2\right]\end{aligned}$$

Notice that, at one-loop, there is a dependence on the momentum as opposed to the $\lambda\varphi^4$ theory. The first addendum of the above contribution is cancelled by a counter term of the kinetic term, while the second addendum is added to the first diagram's contribution and is cancelled by a counter term of the mass term.

Total contribution. The total contribution at one-loop is

$$\Sigma_\varphi|_{1\mathrm{L}} = \Sigma_\varphi(\mathrm{I}) + \Sigma_\varphi(\mathrm{II}) = \frac{g^2}{8\pi^2}\frac{1}{\varepsilon}p^2 + \frac{1}{\varepsilon}\left[\frac{g^2}{4\pi^2}m^2 - \frac{\lambda M^2}{32\pi^2} + \text{finite}\right]$$

This total contributing divergence can be cancelled by a counter term of the kinetic term and mass term that give contributions

$$(Z_\varphi - 1)\,\partial_\mu\varphi\,\partial^\mu\varphi - (Z_M - 1)M^2\varphi^2$$

This counter term is computed in configuration space and not momentum space, like the one-loop contribution above. Integrating by parts, one has

$$(Z_\varphi - 1)\,\partial_\mu\varphi\,\partial^\mu\varphi = -(Z_\varphi - 1)\varphi\,\square\,\varphi$$

Therefore, the counter term contribution in momentum space is

$$(Z_\varphi - 1)p^2 = -(Z_\varphi - 1)p_{\mathrm{E}}^2$$

[r] Wick rotation of the field, do computations? Since one is expanding e^{-S} then the counter term gives

$$(Z_\varphi - 1)\varphi p_{\mathrm{E}}^2\varphi$$

Therefore

$$\frac{g^2}{8\pi^2}\frac{1}{\varepsilon}p_{\mathrm{E}}^2 + (Z_\varphi - 1)p_{\mathrm{E}}^2 = \text{finite} \implies \boxed{Z_\varphi|_{1\mathrm{L}} = 1 - \frac{g^2}{8\pi^2}\frac{1}{\varepsilon} + \text{finite}}$$

The mass term for the scalar field is

$$\left[\frac{g^2}{4\pi^2}m^2 + \frac{\lambda M^2}{32\pi^2} + \text{finite}\right]\frac{1}{\varepsilon} - (Z_M - 1)M^2 \equiv \text{finite}$$

from which one has

$$\boxed{Z_M|_{1\mathrm{L}} = 1 + \left[\frac{\lambda}{32\pi^2} + \frac{g^2}{4\pi^2}\frac{m^2}{M^2}\right]\frac{1}{\varepsilon} + \text{finite}}$$

Lecture 6

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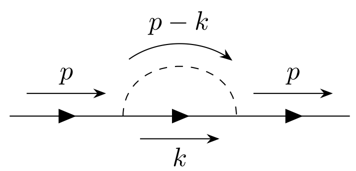
5.2 Two-point spinor Green's function

Consider the diagram



One has to contract to Yukawa vertices

$$\mathrm{i}g\varphi\bar{\psi}_\gamma(\gamma_5)_{\gamma\delta}\psi_\delta\,, \quad \mathrm{i}g\varphi\bar{\psi}_\alpha(\gamma_5)_{\alpha\beta}\psi_\beta$$

to produce a fermion propagator and a scalar propagator (recall that the diagram has to be read in the opposite direction when writing its associated integral). The one-loop contribution to the two-point spinor Green's function is

$$\begin{aligned}
[\mathrm{i}\Sigma_\psi^{(2)}]_{\alpha\delta} &= \frac{\mathrm{i}^2}{2!}(\mathrm{i}g)^2 2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(\gamma_5)_{\alpha\beta}(\not{k}+m)_{\beta\gamma}(\gamma_5)_{\gamma\delta}}{[(p-k)^2 - M^2 + \mathrm{i}\varepsilon](k^2 - m^2 + \mathrm{i}\varepsilon)} \\
&= g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{[\gamma_5(\not{k}+m)\gamma_5]_{\alpha\delta}}{[(p-k)^2 - M^2 + \mathrm{i}\varepsilon](k^2 - m^2 + \mathrm{i}\varepsilon)}
\end{aligned}$$

One applies dimensional regularization to have

$$\begin{aligned}
[\mathrm{i}\Sigma^{(2)}_\psi]_{\alpha\delta} &= g^2 k^{2\varepsilon}\int\frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{(-\not{k}+m)_{\alpha\delta}}{[(p-k)^2-M^2](k^2-m^2)}\\
&= g^2 k^{2\varepsilon}\int_0^1 \mathrm{d}x\int\frac{\mathrm{d}^n q}{(2\pi)^n}\frac{(-\not{q}-x\not{p}+m)_{\alpha\delta}}{(q^2+D)^2}\\
&= g^2 k^{2\varepsilon}\int_0^1 \mathrm{d}x\,(m-x\not{p})_{\alpha\delta}\int\frac{\mathrm{d}^n q}{(2\pi)^n}\frac{1}{(q^2+D)^2}\\
&= g^2 k^{2\varepsilon}\frac{1}{(4\pi)^{2-\varepsilon}}\Gamma(\varepsilon)\int_0^1 \mathrm{d}x\,\frac{(m-x\not{p})_{\alpha\delta}}{[x(1-x)p^2-m^2(1-x)-M^2x]^\varepsilon}\\
&= \frac{g^2}{16\pi^2}\frac{\Gamma(1+\varepsilon)}{\varepsilon}\int_0^1 \mathrm{d}x\,\frac{(m-x\not{p})_{\alpha\delta}}{[\frac{x(1-x)p^2-m^2(1-x)-M^2x}{4\pi k^2}]^\varepsilon}\\
&= \frac{g^2}{16\pi^2}\frac{1}{\varepsilon}\int_0^1 \mathrm{d}x\,(m-x\not{p})_{\alpha\delta}+o(\varepsilon^{-1})\\
&= \frac{g^2}{16\pi^2}\frac{1}{\varepsilon}\left[m-\frac{1}{2}\not{p}\right]_{\alpha\delta}
\end{aligned}$$

[r] check. At the second line, one has applied Feynman combining

$$\mathrm{Den} = k^2-2xkp+xp^2-m^2(1-x)-M^2x = (k-xp)^2+x(1-x)p^2-m^2(1-x)-M^2x$$

and one lets

$$q\equiv k-xp\,,\quad D\equiv x(1-x)p^2-m^2(1-x)-M^2x$$

Renormalization. The mass contribution is [r]

$$[\Sigma^{(2)}_\psi]_m = -\mathrm{i}\frac{g^2}{16\pi^2}\frac{m}{\varepsilon}$$

From the counter term Lagrangian

$$\exp[\mathrm{i}S_{\mathrm{ct}}]\to\exp\left[\mathrm{i}\int\mathrm{d}^4x\,[-(Z_m-1)m\bar{\psi}\psi]\right]$$

one has

$$-\mathrm{i}\frac{g^2}{16\pi^2}\frac{1}{\varepsilon}-\mathrm{i}(Z_m-1)\equiv\text{finite}\implies\boxed{Z_m=1-\frac{g^2}{16\pi^2}\frac{1}{\varepsilon}+\text{finite}}$$

The momentum contribution is

$$[\Sigma^{(2)}_\psi]_p = \mathrm{i}\frac{g^2}{32\pi^2}\frac{1}{\varepsilon}\not{p}$$

Therefore

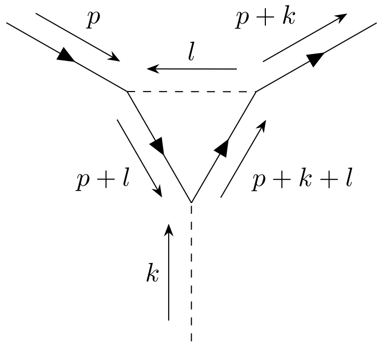
$$\mathrm{e}^{\mathrm{i}S}\to\exp\left[\mathrm{i}\int\mathrm{d}^4x\,\bar{\psi}\mathrm{i}\not{\partial}\psi(Z_\psi-1)\right]\to\exp\left[-\mathrm{i}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\bar{\psi}(-p)\not{p}\psi(p)(Z_\psi-1)\right]$$

[r] $\not{p}=-\mathrm{i}\not{\partial}$?? check Fourier convention
from which

$$\mathrm{i}\frac{g^2}{32\pi^2}\frac{1}{\varepsilon}\not{p}-\mathrm{i}(Z_\psi-1)\not{p}=\text{finite}\implies\boxed{Z_\psi=1+\frac{g^2}{32\pi^2}\frac{1}{\varepsilon}+\text{finite}}$$

5.3 Three-point Yukawa Green's function

One would like to compute the one-loop correction to the Yukawa vertex $\Gamma^{(3)}_{\mathrm{Y}}$. Consider the following diagram



Let $p' = p + k$, the one-loop contribution to the three-point Green's function is

$$
\begin{aligned}
\Gamma^{(3)}_{\mathrm{Y}}|_{1\mathrm{L}} &= \frac{\mathrm{i}^3}{3!}(\mathrm{i}g)^3(3\cdot 2)\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{(\gamma_5)_{\alpha\beta}(\not{p}+\not{l}+m)_{\beta\gamma}(\gamma_5)_{\gamma\delta}(\not{p}'+\not{l}+m)_{\delta\eta}(\gamma_5)_{\eta\rho}}{(l^2-M^2+\mathrm{i}\varepsilon)[(p+l)^2-m^2+\mathrm{i}\varepsilon][(p'+l)^2-m^2+\mathrm{i}\varepsilon]} \\
&= -g^3\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{[(-\not{p}-\not{l}-m)(\not{p}'+\not{l}+m)\gamma_5]_{\alpha\rho}}{(l^2-M^2+\mathrm{i}\varepsilon)[(p+l)^2-m^2+\mathrm{i}\varepsilon][(p'+l)^2-m^2+\mathrm{i}\varepsilon]} \\
&= -g^3k^{3\varepsilon}\int \frac{\mathrm{d}^n l}{(2\pi)^n}\frac{[(-\not{p}-\not{l}-m)(\not{p}'+\not{l}+m)\gamma_5]_{\alpha\rho}}{(l^2-M^2+\mathrm{i}\varepsilon)[(p+l)^2-m^2+\mathrm{i}\varepsilon][(p'+l)^2-m^2]} \\
&= -g^3k^{3\varepsilon}\int_0^1 \mathrm{d}x_1\,\mathrm{d}x_2\int\frac{\mathrm{d}^n q}{(2\pi)^n}\frac{(-q^2\gamma_5+\widetilde{N}+\text{linear in } q)_{\alpha\rho}}{(q^2+D)^3} \\
&= g^3k^{3\varepsilon}\int_0^1 \mathrm{d}x_1\,\mathrm{d}x_2\int\frac{\mathrm{d}^n q}{(2\pi)^n}\frac{q^2}{(q^2+D)^3}(\gamma_5)_{\alpha\rho}+\text{finite} \\
&= g^3k^{3\varepsilon}\frac{1}{(4\pi)^{2-\varepsilon}}\frac{\Gamma(\varepsilon)\Gamma(3-\varepsilon)}{\Gamma(3)\Gamma(2-\varepsilon)}\int_0^1 \mathrm{d}x_1\,\mathrm{d}x_2\,D^{-\varepsilon}(\gamma_5)_{\alpha\rho}+\text{finite} \\
&= \frac{g^3k^{\varepsilon}}{16\pi^2}(4\pi k^2)^{\varepsilon}\frac{1}{\varepsilon}\frac{\Gamma(1-\varepsilon)\Gamma(3-\varepsilon)}{\Gamma(3)\Gamma(2-\varepsilon)}\int_0^1 \mathrm{d}x_1\,\mathrm{d}x_2\,D^{-\varepsilon}(\gamma_5)_{\alpha\rho}+\text{finite} \\
&= \frac{g^3}{16\pi^2}\frac{1}{\varepsilon}(\gamma_5)_{\alpha\rho}+o(\varepsilon^{-1})
\end{aligned}
$$

[r] At the fourth line, one has applied Feynman combining. The denominator is

$$
\mathrm{Den} = (1-x_1x_2)(l^2-M^2)+x_1[(l+p)^2-m^2)]+x_2[(l+p')^2-m^2]\equiv q^2+D
$$

where one has

$$
\begin{aligned}
q &\equiv l+x_1p+x_2p' \\
D &\equiv -(1-x_1-x_2)M^2-(x_1+x_2)m^2+x_1(1-x_1)p^2+x_2(1-x_2)p'^2-2x_1x_2pp'
\end{aligned}
$$

The numerator is

$$
\begin{aligned}
\mathrm{Num} &= (-\not{q}+x_1\not{p}+x_2\not{p}'-\not{p}+m)(\not{q}-x_1\not{p}-x_2\not{p}'+\not{p}'+m)\gamma_5 \\
&= -\not{q}\not{q}\gamma_5+[x_2\not{p}'-(1-x_1)\not{p}+m)][(1-x_2)\not{p}'-x_1\not{p}+m]\gamma_5+(\text{linear in } q) \\
&= -q^2\gamma_5+\widetilde{N}+(\text{linear in } q)
\end{aligned}
$$

where one has

$$
\not{q}\not{q} = \gamma^\mu q_\mu\gamma^\nu q_\nu = \frac{1}{2}q_\mu q_\nu\{\gamma^\mu,\gamma^\nu\} = \frac{1}{2}q_\mu q_\nu 2\eta^{\mu\nu} = q^2
$$

Renormalization. From the exponential, one has gets the counter term

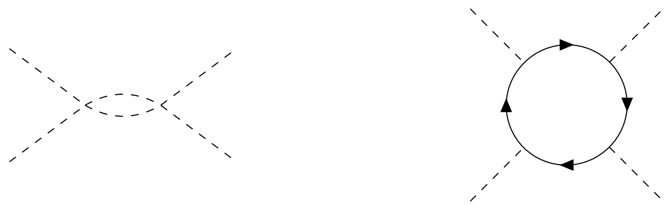
$$
\mathrm{e}^{\mathrm{i}S}\to\exp\left[\mathrm{i}\int\mathrm{d}^4x\,\mathrm{i}g(Z_g-1)\varphi\bar{\psi}\gamma_5\psi\right]\to -g(Z_g-1)\varphi\bar{\psi}\gamma_5\psi
$$

Therefore

$$
\frac{g^3}{16\pi^2}\frac{1}{\varepsilon}-g(Z_g-1)\equiv\text{finite}\implies\boxed{Z_g = 1+\frac{g^2}{16\pi^2}\frac{1}{\varepsilon}+\text{finite}}
$$

5.4 Four-point Green's function

The vertex function for the λ-vertex gets two contributions:



In principle, the Yukawa theory contains the coupling g, but the theory is renormalizable only if one includes the $\lambda\varphi^4$ term. [r] In fact, the second diagram produces a divergent term proportional to λ^4.