

# Mathematical Methods for Physics

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November 26, 2023

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## Lecture 1

### 1 Introduction

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The course is about group theory and explaining where it appears in theoretical physics. Group theory is motivated by the fact that the current understanding of Nature is based on symmetries. Symmetries are properties that are invariant under some transformations. Utilizing Noether's theorem, one can link the conservation of charges to transformations and invariants. Symmetries play an important role in physics and are described by groups. There are several classes of symmetries.

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\*<https://github.com/M-a-s-o/notes>

**Continuous symmetries.** Symmetries can be parameterized by continuous variables. Some examples of continuous symmetries are:

- (space) translation, equivalent to the assumption that space is homogeneous: one can obtain the conservation of momentum;
- time translation, equivalent to the assumption that time is homogeneous: conservation of energy;
- space rotation, equivalent to the assumption of the isotropy of space: conservation of angular momentum;
- Lorentz and Poincaré transformations: conservation of the energy–momentum tensor.
- Local gauge symmetry: it is a symmetry of an internal degree of freedom (i.e. not related to temporal nor spacial degrees of freedom) and the transformation depends on the space-time coordinates

$$V_{ab}(\mathbf{x}, t)\phi(\mathbf{x}, t)$$

These kinds of symmetries describe the dynamics of quantum field theory and the properties of fundamental interactions.

- Internal, global symmetry, for example isospin: swapping the up and down quarks does not change the physics.

**Discrete symmetries.** A few examples of discrete symmetries are:

- parity  $\mathbf{x} \rightarrow -\mathbf{x}$ : parity is conserved in quantum electrodynamics and in quantum chromodynamics, but not in electroweak theory.
- Time reversal  $t \rightarrow -t$ ;
- charge conjugation;
- discrete lattice: no more continuous translation symmetries, because there is only a subset of them. Lattices are used for the study of QCD in the non-perturbative regime.

This course will be dedicated to the classification and study of groups and their representations.

**Definition 1.1.** A group  $G$  is a set endowed with a binary operation  $\circ$  (called group multiplication) that has the following properties:

- closure:  $\forall g_1, g_2 \in G, \exists! g_3 \in G$  such that  $g_3 = g_1 \circ g_2$ .
- associativity:  $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3), \forall g_1, g_2, g_3 \in G$ .
- identity element:  $\exists e \in G$  such that  $g \circ e = e \circ g = g, \forall g \in G$ .
- inverse element:  $\forall g \in G, \exists g^{-1} \in G$  such that  $g \circ g^{-1} = g^{-1} \circ g = e$ .

**Example 1.2.** A few examples:

- The set  $\mathbb{R}$  of the real numbers endowed with summation is a group  $(\mathbb{R}, +)$  with identity element 0 and inverse  $-g$ .
- The whole numbers with summation  $(\mathbb{Z}, +)$  is a group.
- The positive integers with summation  $(\mathbb{N}, +)$  are not a group: there does not exist the inverse.
- The real numbers with multiplication  $(\mathbb{R}, \cdot)$  has identity element 1, but is not a group because 0 does not have an inverse. To obtain a group, one can consider  $\mathbb{R} \setminus \{0\}$ .
- Consider the set of square matrices of size  $n$  with either real entries  $\text{Mat}(n, \mathbb{R})$  or complex entries  $\text{Mat}(n, \mathbb{C})$ . Taking the operation to be matrix multiplication, the identity to be  $I = \text{diag}(1, 1, 1, \dots)$ , the set is not a group because the inverse element, inverse matrices, does not exist for matrices with null determinant.

**Definition 1.3.** The general linear group is defined as  $\text{GL}(n, \mathbb{R}) \equiv \{M \in \text{Mat}(n, \mathbb{R}) \mid \det M \neq 0\}$  and is the largest and most general group of matrices. The definition is equivalent for matrices with complex entries.

**Definition 1.4.** The cyclic group  $\mathbb{Z}_n$  is discrete and defined by

$$\mathbb{Z}_n \equiv \{e, a, a^2, a^3, \dots, a^{n-1}\}, \quad a^n \equiv e$$

with generic operation  $\circ$  where the notation implies  $a^2 \equiv a \circ a$ .

**Example 1.5.** Some examples are:

- $\mathbb{Z}_2 = \{0, 1\}$  with summation mod 2,
- the group  $G = \{1, -1\}$  with multiplication,
- endowing the integers  $\mathbb{Z}$  with summation mod  $n$ , one can split the set into equivalence classes that are the elements of the cyclic group. For example, if  $n = 2$  then the sets

$$\{0\} = \{0, 2, 4, \dots\}, \quad \{1\} = \{1, 3, 5, \dots\}$$

are the elements of the cyclic group  $\mathbb{Z}_2$ . This group is important in particle physics.

**Definition 1.6.** The order of a group  $G$ , denoted by  $|G|$ , is given by the number of elements of the group.

**Definition 1.7.** A group  $G$  is finite if its order is finite. Otherwise it is infinite.

**Definition 1.8.** A group  $G$  is discrete if it is a countable set.

**Definition 1.9.** A group  $G$  is abelian if its operation is commutative  $g_1 \circ g_2 = g_2 \circ g_1, \forall g_1, g_2$ .

**Example 1.10.** A few examples of abelian and non-abelian groups:

- Some abelian groups are  $(\mathbb{R}, +)$ ,  $(\mathbb{Z}, +)$  which are also infinite. Though, the first is not countable, while the second is and as such it is a discrete group.
- All cyclic groups  $\mathbb{Z}_n$  are abelian:  $a^2 \circ a^3 = a^3 \circ a^2$ .
- The general linear group  $\text{GL}(n, \mathbb{R})$  is not abelian: the order of matrix multiplication matters.

**Definition 1.11.** A ring  $R$  is an abelian group with composition  $+$ , endowed with a second operation  $\cdot : R \times R \rightarrow R$  with the properties:

- associativity  $(r_1 \cdot r_2) \cdot r_3 = r_1 \cdot (r_2 \cdot r_3), \forall r_1, r_2, r_3 \in R$ ;
- distributivity with respect to the first composition:  $r_1 \cdot (r_2 + r_3) = r_1 \cdot r_2 + r_1 \cdot r_3$  and equivalently for right multiplication.

**Remark 1.12.** The identity element of the relation  $+$  is 0, while the inverse is  $-r$ . The second relation need not an identity nor an inverse.

**Definition 1.13.** A field  $\mathbb{F}$  is a ring such that  $\mathbb{F} \setminus \{0\}$  is an abelian group with respect to the ring's second operation.

**Example 1.14.** Some examples and counterexamples of fields are the following.

- Consider the integers with summation  $(\mathbb{Z}, +)$ : it is an abelian group. It is also a ring with the multiplication  $n_1 \cdot n_2 \in \mathbb{Z}$ . However,  $(\mathbb{Z}, \cdot)$  is not a group because  $n^{-1} \notin \mathbb{Z}$ , therefore the set of integers is not a field.
- The real numbers with summation and multiplication is a field.
- The complex numbers are also a field.

- The cyclic group  $\mathbb{Z}_n$  with summation is an abelian group. Adding the multiplication  $\{n_1\} \cdot \{n_2\} \equiv \{n_1 \cdot n_2\}$ , the cyclic group is a field only for  $n$  prime. See the following theorem.

The fact that real numbers and complex numbers are fields implies one can define sets of matrices with entries from those fields.

**Theorem 1.15.** If  $n$  is prime, then the cyclic group  $\mathbb{Z}_n$  is a field (and vice versa).

**Example 1.16.** Some more examples:

- Consider the cyclic group  $\mathbb{Z}_5$ . It is an abelian group with respect to summation. One can study if  $(\mathbb{Z}_5 \setminus \{0\}, \cdot)$  is an abelian group:

$$\{2\} \cdot \{3\} = \{6\} = \{1\}$$

This means that  $\{3\}$  is the inverse of  $\{2\}$ . Continuing, one gets

$$\{4\} \cdot \{4\} = \{1 + 5 \cdot 3\} = \{1\}$$

So  $\{4\}$  is its own inverse. Since the remaining elements are the additive identity  $\{0\}$  and multiplicative identity  $\{1\}$ , the set  $\mathbb{Z}_5$  is a field.

- The group  $(\mathbb{Z}_4 \setminus \{0\}, \cdot)$  is not abelian because

$$\{2\} \cdot \{2\} = \{4\} = \{0\}$$

the set is not closed under multiplication. Therefore  $\mathbb{Z}_4$  is not a field.

**Definition 1.17.** A **module**  $M$  with respect to a **ring**  $R$  is an abelian group endowed with an action  $\cdot : R \times M \rightarrow M$  such that the following properties hold for every  $r \in R$  and  $m \in M$ :

- associativity  $(r_1 \cdot r_2) \cdot m = r_1 \cdot (r_2 \cdot m)$
- distributivity  $r_1 \cdot (m_1 + m_2) = r_1 \cdot m_1 + r_1 \cdot m_2$  and  $(r_1 + r_2) \cdot m = r_1 \cdot m + r_2 \cdot m$

Pay attention to whom the operations apply: operations in different sets have the same notation.

**Definition 1.18.** A vector space  $V$  with respect to a field  $\mathbb{F}$  is a module over  $\mathbb{F}$  such that the multiplicative identity of  $\mathbb{F}$  acts as the identity on  $V$ :  $1 \cdot v = v \cdot 1 = v$ ,  $\forall v \in V$ .

**Definition 1.19.** An associative algebra  $A$  over a ring  $R$  is a ring and also a module with respect to  $R$  such that the action of  $R$  on  $A$  is bilinear

$$r \cdot (a_1 \cdot a_2) = (r \cdot a_1) \cdot a_2 = a_1 \cdot (r \cdot a_2)$$

Again, pay attention to whom the operations apply.

[r] image of recap

**Example 1.20.** The set of square matrices of size  $n$ ,  $\text{Mat}(n, \mathbb{F})$ , with summation, matrix multiplication and scalar product forms an algebra over the field  $\mathbb{F}$ .

## Lecture 2

**Definition 1.21.** The set of vectors  $\{|e_i\rangle\}$  is a basis of a vector space  $V$  if and only if  $|v\rangle = v^i |e_i\rangle$ ,  $v^i \in \mathbb{F}$  for all  $|v\rangle \in V$ .

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**Definition 1.22.** The action of an inner product on a basis defines a metric  $g^i_j = \langle e^i | e_j \rangle$ . The metric  $g$  is hermitian and if  $g^i_j = \delta^i_j$  then the basis is orthonormal.

**Lemma 1.23.** Linear operators over  $V$  form an algebra called  $\text{End}(V)$ :

$$A : |v\rangle \mapsto |Av\rangle \in V, \quad A : |w\rangle = \alpha |v_1\rangle + \beta |v_2\rangle \mapsto |Aw\rangle = \alpha |Av_1\rangle + \beta |Av_2\rangle$$

**Proposition 1.24.** Linear operators follow these properties:

- composition:  $(A \circ B)|v\rangle = A|Bv\rangle = |ABv\rangle$ ;
- given a basis  $\{|e_i\rangle\}$ , the operator  $A$  is a matrix  $n \times n$  over a field  $\mathbb{F}$ ;
- a basis rotation  $S$  is a linear operator which is invertible ( $\det S \neq 0$ ):

$$|e'_i\rangle = S^j_i |e_j\rangle \implies \langle e'^i | e'_j \rangle = (S^\dagger)^i_k \langle e^k | e_l \rangle S^l_j \implies g' = S^\dagger g S$$

**Definition 1.25.** A linear operator  $A$  is idempotent if  $A^2 = A$  (and  $A \neq I$ ). Given a basis  $\{|e_i\rangle\}$ , an example of idempotent linear operator the projector  $A = |e_i\rangle\langle e^i|$ :

$$A|v\rangle = |e_i\rangle\langle e^i|v^k|e_k\rangle = v^i|e_i\rangle$$

**Definition 1.26.** A linear operator  $A$  is nilpotent if  $\exists n \in \mathbb{N}$  such that  $A^n = 0$  where 0 is the null operator. For example

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

One wants to categorize type of transformations to be able to study symmetries.

## 1.1 Classical groups of matrices

**Linear group.** The general linear group  $\text{GL}(n, \mathbb{C})$  is the largest group in the endomorphism algebra  $\text{End}(V)$ . The group is parametrized by  $2n^2$  real elements. The dimension of the group is given by the number of free parameters.

The special linear group is defined by

$$\text{SL}(n, \mathbb{C}) \equiv \{M \in \text{GL}(n, \mathbb{C}) \mid \det M = 1\} \subset \text{GL}(n, \mathbb{C})$$

This set is also a group, not just a subset of matrices. The number of free parameters is  $2n^2 - 1$  because of the condition on the determinant. The definition is analogous for  $\text{SL}(n, \mathbb{R})$ . The set  $\text{SL}(n, \mathbb{Z})$  is also a group.

Consider a change of coordinates from  $x^i$  to  $\bar{x}^j = S^j_i x^i$  with the intention to perform an integral over space-time. The measures of the two coordinate systems are related by the Jacobian

$$d^n \bar{x} = |\partial_x \bar{x}| d^n x = \det S d^n x$$

Matrices in the special linear group preserve the volume.

**Unitary group.** Consider the groups that preserve the metric of  $V$  over  $\mathbb{C}$ :

$$|\bar{e}_i\rangle = U^j_i |e_j\rangle \implies \delta^i_j = \langle \bar{e}^i | \bar{e}_j \rangle = \langle e^k | (U^\dagger)^i_k U^l_j |e_l\rangle = (U^\dagger)^i_k U^l_j \delta^k_l \implies U^\dagger U = I$$

Unitary matrices belong to the unitary group  $\text{U}(n, \mathbb{C})$  which is defined as

$$\text{U}(n, \mathbb{C}) \equiv \{M \in \text{GL}(n, \mathbb{C}) \mid M^\dagger M = M M^\dagger = I\}$$

The number of free parameters is  $2n^2 - n - n(n-1) = n^2$ :

- the unitary condition  $U^\dagger U$  imposes  $n$  conditions corresponding to the elements on the diagonal,  $[U^\dagger U]^i_i = 1$  (no sum);
- the upper and lower triangular parts of the matrix provide  $\frac{n(n-1)}{2}$  conditions each.

The unitary group  $\text{U}(1)$  describes quantum electrodynamics. The determinant of a unitary matrix is

$$\det(U^\dagger U) = |\det U|^2 = 1 \implies \det U = e^{i\varphi}, \quad \varphi \in \mathbb{R}$$

One can define the special unitary group as

$$\text{SU}(n, \mathbb{C}) \equiv \{M \in \text{U}(n, \mathbb{C}) \mid \det M = 1\}$$

The number of free parameters is  $n^2 - 1$ . The strong and weak interactions are described by special unitary groups.

**Orthogonal group.** Consider the groups that preserve the metric of  $V$  over  $\mathbb{R}$ . The orthogonal group is

$$O(n, \mathbb{R}) \equiv \{M \in GL(n, \mathbb{R}) \mid M^\top M = MM^\top = I\}$$

The number of free parameters is  $n^2 - n - \frac{n(n-1)}{2} = \frac{n(n-1)}{2}$ . The determinant of the orthogonal matrices is

$$\det(O^\top O) = (\det O)^2 = 1 \implies \det O = \pm 1$$

The special orthogonal group is

$$SO(n, \mathbb{R}) \equiv \{M \in O(n, \mathbb{R}) \mid \det M = 1\}$$

The number of free parameters is the same as above. For the unitary group and orthogonal group, the field is not usually specified, but rather understood to be  $\mathbb{C}$  and  $\mathbb{R}$  respectively.

**Indefinite orthogonal group.** Consider the groups that preserve the metrics of  $V$  over  $\mathbb{R}$  with signature  $(p, q)$ . Consider the metric to be

$$g^i_j = \eta^i_j = \text{diag}(\underbrace{1, \dots, 1}_p, \underbrace{-1, \dots, -1}_q)$$

Then

$$|\bar{e}'_i\rangle = R^j_i |e_j\rangle \implies \bar{\eta} = R^\top \eta R$$

The indefinite orthogonal group is

$$O(p, q, \mathbb{R}) \equiv \{R \in GL(n, \mathbb{R}) \mid \eta = R^\top \eta R\}$$

The indefinite special orthogonal group is

$$SO(p, q, \mathbb{R}) \equiv \{R \in O(p, q, \mathbb{R}) \mid \det R = 1\}$$

The number of free parameters is the same as the orthogonal group. The Lorentz group is  $SO(1, 3)$ . The group  $SO(p, q)$  contains both ordinary rotations  $SO(p)$  and  $SO(q)$  subgroups.

### 1.1.1 Symplectic forms

**Definition 1.27.** A symplectic vector space  $V$  is a vector space over a field  $\mathbb{F}$  equipped with a bilinear form  $\Omega : V \times V \rightarrow \mathbb{F}$ . The bilinear form is:

- anti-symmetric:  $\Omega(\mathbf{v}, \mathbf{w}) = -\Omega(\mathbf{w}, \mathbf{v})$ ;
- non-singular:  $\Omega(\mathbf{v}, \mathbf{w}) = 0, \forall \mathbf{w} \implies \mathbf{v} = \mathbf{0}$ ,

**Theorem 1.28.** Given a square matrix  $A$  of size  $n$  and a constant  $c$ , then

$$\det(cA) = c^n \det A$$

*Proof.* From the hypotheses, it follows

$$\det(cA) = \det(cI) \det A = c^n \det A$$

□

Given a vector space  $V$  over  $\mathbb{R}$ , one of its bases  $\{|e_i\rangle\}$  and a bilinear form  $\Omega_{ij} = \Omega(|e_i\rangle, |e_j\rangle)$ , the aim is to represent the form  $\Omega$  as a linear operator acting on the basis of  $V$ . Using anti-symmetry and the theorem above, it follows

$$\det \Omega = \det(-\Omega^\top) = (-1)^n \det \Omega$$

Because of non-singularity, the dimension  $n$  can only be even. Using anti-symmetry and setting  $\mathbf{w} = \mathbf{v}$  then

$$\Omega(\mathbf{v}, \mathbf{v}) = -\Omega(\mathbf{v}, \mathbf{v}) \implies \Omega_{ii} = 0, \quad \Omega_{ij} = -\Omega_{ji}$$

Fixing  $i$ , if  $\Omega_{ij} = 0$  for all  $j$  then  $|e_i\rangle = 0$ . Negating this statement means if all basis' vectors are non-zero (which must hold for a vector to be part of a basis) then, for at least one  $j$ , it must be true that  $\Omega_{ij} \neq 0$ . One can choose  $\Omega_{ij} = 0$  for all  $j \neq i+1$  and  $\Omega_{i,i+1} = \lambda_i$ . With this choice, one can obtain the canonical form

$$\Omega = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & 1 & \\ & -1 & 0 & 1 \\ & & -1 & 0 & 1 \end{pmatrix}$$

Another canonical form obtained by change of basis from the previous is

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

given by the conditions

$$\Omega_{i, \frac{n}{2}+i} = 1, \quad \Omega_{ij} = 0, \quad \forall j \neq \frac{n}{2} + i, \quad i < \frac{n}{2}$$

These two forms amount to a decomposition of the vector space  $V = V_1 \oplus V_1^*$  — where  $V_1^*$  is the dual space of  $V_1$  — such that a basis of  $V$  can be written as

$$\{|e_1\rangle, \dots, |e_m\rangle, |f_1\rangle, \dots, |f_m\rangle\}, \quad |e_i\rangle \leftrightarrow V_1, \quad |f_i\rangle \leftrightarrow V_1^*$$

As such the dimension is always even

$$n = \dim V = \dim V_1 + \dim V_1^* = 2 \dim V_1 = 2m$$

A generic transformation of a symplectic form

$$\Omega' = A\Omega B$$

needs to preserve the form's properties:

$$\begin{aligned} (\Omega')^\top &= B^\top \Omega^\top A^\top = -B^\top \Omega A^\top \\ &= -\Omega' = -A\Omega B \implies A = B^\top \end{aligned}$$

**Symplectic group.** The symplectic group is

$$\mathrm{Sp}(2n, \mathbb{F}) \equiv \{M \in \mathrm{Mat}(2n, \mathbb{F}) \mid M^\top \Omega M = \Omega\}, \quad (\det M)^2 = 1$$

This group also preserves the pfaffian

$$[\mathrm{Pf}(N)]^2 = \det N$$

## 1.2 Morphisms

**Definition 1.29.** A morphism  $\phi$  is a map from a group  $G$  to another group  $G'$ ,  $\phi : G \rightarrow G'$ ,  $\phi(g) = g'$  with  $g \in G$  and  $g' \in G'$ .

**Definition 1.30.** A homomorphism is a morphism that respects the composition law of  $G$  and  $G'$ :  $\forall g_1, g_2 \in G$ ,  $\phi(g_1 \cdot g_2) = \phi(g_1) \cdot \phi(g_2)$ . The first composition is in  $G$ , the second is in  $G'$ .

**Definition 1.31.** An isomorphism is an invertible homomorphism. It is a bijective map:  $\forall g \in G$ ,  $\phi(g) = g' \in G'$  and  $\exists \phi^{-1} : G' \rightarrow G$  such that  $\phi^{-1}(g') = \phi^{-1}(\phi(g)) = g$ .

**Definition 1.32.** An automorphism is an isomorphism with  $G' = G$ .

### Lecture 3

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**Definition 1.33.** A representation  $\rho$  is a homomorphism where  $G'$  is a group of matrices:

$$\rho : G \rightarrow \text{GL}(n, \mathbb{C})$$

The number  $n$  is the dimension of the representation.

**Example 1.34.** A few examples:

- the trivial homomorphism exists: it maps all elements into the identity,  $\phi(g) = e', \forall g \in G, e' \in G'$ ; this is used in the context of representations because, if it exists, there is always a group element  $g$  mapped to the identity of the general linear group.
- Take a function between two cyclic groups  $\phi : \mathbb{Z}_4 \rightarrow \mathbb{Z}_2$ . To create a homomorphism one can take  $\phi(e) = \phi(a^2) = e'$  and  $\phi(a) = \phi(a^3) = a'$ . One then has

$$\phi(ea) = \phi(a) = a', \quad \phi(ea) = \phi(e)\phi(a) = e'a' = a'$$

- It's possible to construct an isomorphism between the definitions of  $\mathbb{Z}_n$ : the set  $\{e, a, \dots, a^{n-1}\}$  with multiplication and the set  $\{\{0\}, \{1\}, \dots, \{n-1\}\}$  with addition.
- One can also construct an isomorphism between  $(\mathbb{R}, +)$  and  $(\mathbb{R}^+, \cdot)$ . The following should hold  $\phi(g_1 + g_2) = \phi(g_1)\phi(g_2)$ . Such an isomorphism is the exponential function  $e^{g_1+g_2} = e^{g_1}e^{g_2}$ . Its inverse is the natural logarithm.
- An example of non trivial automorphism considers  $\mathbb{Z}_3$ . One can define

$$\phi(e) = e', \quad \phi(a) = a'^2, \quad \phi(a^2) = a'$$

and one should check that the morphism respects the composition law

$$\phi(a)\phi(a^2) = \phi(a^3) = e', \quad \phi(a)\phi(a^2) = a'^2a = e'$$

However, this is not true for  $\mathbb{Z}_4$ . Consider the mapping

$$\phi(e) = e', \quad \phi(a) = a'^2, \quad \phi(a^2) = a'$$

The fourth element is given by

$$\phi(a)\phi(a^3) = b^3$$

but

$$\phi(a)\phi(a^3) = \phi(a^4) = \phi(e) = e', \quad \phi(a)\phi(a^3) = a'^2e' = a'^2$$

**Definition 1.35.** Two elements  $h_1$  and  $h_2$  of a group  $G$  are conjugate of one another if there exists a third element  $g$  of the group such that  $h_2 \equiv g^{-1}h_1g$ .

One can then build equivalence classes.

**Definition 1.36.** An internal automorphism is the mapping given by conjugating every element of a group while fixing  $g$ :

$$\phi_g : G \rightarrow G, \quad \phi_g : h \mapsto g^{-1}hg$$

The inverse mapping is

$$\phi_g^{-1} : h \mapsto ghg^{-1}$$

**Lemma 1.37.** The set of automorphisms of a group is itself a group  $\text{Aut}(G)$  with the operation of composition of functions.

**Definition 1.38.** Given a set  $S = \{s_1, \dots, s_k\}$  of  $k$  elements, the set of all words  $S'$  created by the concatenation of the elements of the set  $S$ , together with an equivalence relation, forms a free group. A word can be of any length, with as many repeated elements as one wants. The composition is given by concatenating words. The identity is the null word.



**Definition 1.39.** Given a subset of words  $R \subset S'$ , a presentation is a group whose elements are given by the equivalence classes in  $S$  of the relation  $r = e$  for all  $r \in R$ . The set  $R$  is the set of relations.

**Remark 1.40.** In terms of groups, the set  $S'$  is a group and one wants to identify the elements of  $S$  that give all the elements of the group utilizing equivalence classes. The set  $S$  is the set of generators: the minimal set that gives all the group elements.

**Definition 1.41.** The number of generators is called rank.

**Example 1.42.** The set of generators of the cyclic group  $\mathbb{Z}_n$  has rank one because it only contains the element  $a$  with relation  $a^n = e$ : all other elements can be generated from  $a$ .

**Theorem 1.43.** Every finite group  $G$  admits an homomorphism of the form  $\phi : G \rightarrow G'$  where  $G'$  is a free group. Therefore, a presentation always exists.

**Example 1.44.** A presentation for  $\mathbb{Z}_n$  is  $\{a, a^n = e\}$ .

## 2 Dihedral groups

**Definition 2.1.** A dihedral group  $D_k$  is a group of symmetries of a  $k$ -gon.

**Remark 2.2.** The course considers the Euclidean plane  $\mathbb{R}^2$  and its isometries, transformations that keep invariant the distance between points. Such isometries are translation, rotation and reflection.

**Definition 2.3.** A figure is a subset of  $\mathbb{R}^2$  delimited by a closed curve.

**Definition 2.4.** In  $\mathbb{R}^2$ , a  $k$ -gon is a regular polygon figure with  $k$  side.

**Three points.** A 3-gon is a regular triangle. One wants to find all transformations that keep the triangle in the same situation as initially. Such transformations are the identity, rotations by  $e^{i\frac{2}{3}\pi}$  or  $e^{i\frac{4}{3}\pi}$ , and reflections  $R_i$  by each side's axis of the triangle. The number of elements of the dihedral group is 6. [r] image

One wants to find the minimal set of transformations as to find the generators. Let  $r = e^{i\frac{2}{3}\pi}$  and  $\sigma = R_1$ . Combining some transformations one gets

$$\sigma r = R_3, \quad r\sigma = R_2, \quad r^2 = e^{i\frac{4}{3}\pi}, \quad r^2\sigma = R_3$$

Note that the order of operation is from right to left. The presentation of the dihedral group is

$$D_3 = \{\{r, \sigma\}, r^3 = e, \sigma^2 = e, (\sigma r)^2 = e\}$$

The rank is then 2.

**Two points.** Since two sides are not enough for a planar figure, one can consider two points connected by two overlapping segments. The transformations include rotations by  $r = e^{i\pi}$  and reflections along the  $x$  and  $y$  axes. Let  $\sigma = R_x$ , then  $\sigma r = R_y$ . The presentation is

$$D_2 = \{\{r, \sigma\}, \sigma^2 = e, r^2 = e, (\sigma r)^2 = e\}$$

This group is abelian.

**One point.** There is only one transformation to be made and that is a reflection about an arbitrary axis. Given  $\sigma = R$  then the presentation is

$$D_1 = \{\{\sigma\}, \sigma^2 = e\}$$

This group is abelian and it's isomorphic to the cyclic group:  $D_1 \simeq \mathbb{Z}_2$ .

**Four points.** The transformations are rotations every  $e^{i\frac{\pi}{2}}$  and four reflections, two along the diagonals, two splitting the sides. The presentation is

$$D_4 = \{\{\sigma, r\}, \sigma^2 = e, r^4 = e, (\sigma r)^2 = e\}$$

**Arbitrary points.** The  $n$  dihedral group has  $n+n$  elements given by rotations and reflections. The rank is 2 for  $n > 1$ . The group is not abelian for  $n > 2$ . The presentation is

$$D_n = \{\{\sigma, r\}, r^n = e, \sigma^2 = e, (\sigma r)^2 = e\}$$

**Subgroups.** The rotations are a subgroup  $\mathbb{Z}_n = \{r, r^n = e\}$  and so is each reflection  $\mathbb{Z}_2 = \{\sigma, \sigma^2 = e\}$  along some axes: there are  $n$  copies of  $\mathbb{Z}_2$ .

**Definition 2.5.** A subgroup is a subset  $H$  of a group  $G$  which is a group itself under the same composition of  $G$ . It suffices to check

$$\forall h_1, h_2 \in H, h_1 \cdot h_2 \in H, \quad \forall h \in H, \exists h^{-1} \in H \mid hh^{-1} = h^{-1}h = e$$

**Example 2.6.** Some trivial examples are  $H = G$  and  $H = \{e\}$ .

## Lecture 4

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### 3 Finite groups of small order

Groups can be represented through a multiplication table, like the following

	$e$	$g_1$	$g_2$
$e$	$g_1$	$g_2$	$g_2$
$g_1$	$g_1$	$g_1^2$	$g_1g_2$

**Order one.** A group of order one contains only the identity.

**Order two.** For a group of order two,  $G = \{e, a\}$ , the table is

	$e$	$a$
$e$	$e$	$a$
$a$	$a$	$a^2$

To be a group, then  $a^2$  must be an element of  $G$ . If  $a^2 = a$  then

$$a^{-1}a^2 = a, \quad a^{-1}a^2 = a^{-1}a = e$$

which is a contradiction. So if  $a^2 = e$  then one has  $\mathbb{Z}_2 = \{e, a\}$  with  $a^2 = e$ . It has rank one, it is abelian, it has no subgroups and it is the only group of order two.

**Order three.** For a group of order three, the table is

	$e$	$a$	$b$
$e$	$e$	$a$	$b$
$a$	$a$	$a^2$	$ab$
$b$	$b$	$ba$	$b^2$

One can follow a branching logic as before. Setting  $a^2 = a$  leads to  $a = e$  which means that the group is no longer of order three.

So if  $a^2 = e$  then  $a^{-1} = a$ . The product  $ab$  must be an element of the group. Therefore

- if  $ab = e$  then  $b = a$ ;

- if  $ab = b$  then  $a = e$ ;
- if  $ab = a$  then  $b = e$ .

These are not acceptable conditions since the order would no longer be three. So one must modify the previous condition to  $a^2 = b$ :

- if  $ab = b$  then  $a = e$ , or if  $ab = a$  then  $b = e$ , which are not acceptable;
- if  $ab = e$  then  $b^2$  must be an element of the group also:
  - ◊ if  $b^2 = b$  then  $b = e$ , not acceptable;
  - ◊ if  $b^2 = e$  then  $b = a$ , not acceptable;
  - ◊ if  $b^2 = a$  then it is a valid condition.

The only such group is  $\mathbb{Z}_3$ :

	$e$	$a$	$b$
$e$	$e$	$a$	$b$
$a$	$a$	$b$	$e$
$b$	$b$	$e$	$a$

When the multiplication table is symmetric, the group is abelian. From the multiplication table one can identify subgroups by looking at smaller multiplication tables already known (not necessarily with entries one next to each other). In this case,  $\mathbb{Z}_3$  has rank one, it is the only group of order three and has no subgroups.

**Order four.** For a group of order four, the table is

	$e$	$a$	$b$	$c$
$e$	$e$	$a$	$b$	$c$
$a$	$a$	$a^2$	$ab$	$ac$
$b$	$b$	$ba$	$b^2$	$bc$
$c$	$c$	$ca$	$cb$	$c^2$

Repeating the same branching procedure gives two ways of filling the multiplication table. The first one leads to  $\mathbb{Z}_4$ . It is abelian and has rank 1 with  $a^4 = e$ :

	$e$	$a$	$b$	$c$
$e$	$e$	$a$	$b$	$c$
$a$	$a$	$b$	$c$	$e$
$b$	$b$	$c$	$e$	$a$
$c$	$c$	$e$	$a$	$b$

Looking at the table one can see the sub-table of  $\mathbb{Z}_2 = \{e, a^2\} = \{e, b\}$  in the upper-left corner.

The second way of filling the multiplication table gives the dihedral group  $D_2$ . It is abelian and has rank 2:

	$e$	$a$	$b$	$c$
$e$	$e$	$a$	$b$	$c$
$a$	$a$	$e$	$c$	$b$
$b$	$b$	$c$	$e$	$a$
$c$	$c$	$b$	$a$	$e$

One can identify three subgroups  $\mathbb{Z}_2$ , two the corners and one at the center. There's also a fourth hidden  $\mathbb{Z}_2$  subgroup that is found by combining the corners in  $2 \times 2$  blocks.

These two groups constructed with the branching procedure are not isomorphic: one can look at the subgroups, in this case how many there are. Also by looking at their subgroups, it holds

$$D_2 = \mathbb{Z}_2 \otimes \mathbb{Z}_2$$

where  $\otimes$  is the Cartesian product later called direct product.

**Higher orders.** At order five there is only  $\mathbb{Z}_5$ . At order six there are  $\mathbb{Z}_6$  and the group of permutations  $S_3$ . At order seven there is  $\mathbb{Z}_7$ . One may notice that for prime-number orders, there are only cyclic groups.

### 3.1 Groups of permutations

**Definition 3.1.** The set of  $n!$  permutations of  $n$  objects forms a group  $S_n$  (also called the symmetric group).

**Remark 3.2.** A permutation is an automorphism over the set of  $n$  objects. The group of permutations  $S_n$  is a group of automorphisms.

**Notation.** The permutation of  $n$  elements is represented as

$$p = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ p_1 & p_2 & p_3 & \cdots & p_n \end{pmatrix}$$

In the product  $PQ$  the permutation that acts first is  $Q$  followed by  $P$ .

**Representation in matrix notation.** Given a permutation  $p \in S_n$ , its  $n \times n$  matrix representation is  $[P]_{ij} = \delta_{jp_i}$ . [r]

A representation is a homomorphism so it must preserve the group structure. In fact the matrix representation of a product of permutations must represent the product of the permutations:

$$[PQ]_{ij} = \sum_k P_{ik} Q_{kj} = \sum_k \delta_{ip_k} \delta_{kq_j} = \delta_{i,(pq)_j}$$

[r]

Permutations are used for rotations on the lattice.

**Cycle notation.** The cycle notation is compact way to write permutations. It has the properties

- the sum of all cycle lengths is equal to  $n$ ,
- the order of the cycles is irrelevant,
- the cyclic order within a cycle is irrelevant.

For example

$$(134)(2)$$

means that 2 stays put, while 1 and 3 interchange then 3 and 4 interchange. The last property means  $(134) = (341) = (413)$ . It is possible rewrite a cycle of length  $n$  in terms of two cycles that pivot around the same element:  $(134) = (14)(13)$ .

**Lemma 3.3.** Every cycle can be written as the product of two cycles with repeated elements. For convention, one always chooses 1 to be the pivotal, repeated number.

**Proposition 3.4.** It holds

$$(1\ 2\ \cdots\ m) = (1\ m)(1\ m-1)\cdots(13)(12)$$

It follows that

$$(ij) = (1i)(1j)(1i)$$

From the cycle notation, one can represent the entire group. By combining two-cycles one can get arbitrary permutations. So, the symmetric group  $S_n$  has the following generators

$$\{(12), (13), \dots, (1n)\}, \quad \text{rank } S_n = n - 1$$

The identity is given by  $(12)^2(13)^2(14)^2$  but is not part of the generators.

**Definition 3.5.** The parity of an element  $g \in S_n$  is the number of two-cycles in any two-cycle decomposition.

**Definition 3.6.** Since the identity  $e$  is even under parity, the set of even permutations forms a subgroup called alternating group  $A_n$ . Its order is  $2^{-1}n!$ .

**Example 3.7.** Consider  $S_3 = \{e, (12), (13), (23), (123), (132)\}$ . Let  $g_1 = (12)$  and  $g_2 = (23)$ . It is not abelian

$$abc \xrightarrow{g_1} bac \xrightarrow{g_2} bca, \quad abc \xrightarrow{g_2} acb \xrightarrow{g_1} cab$$

The identity is given by  $e = (12)^2(13)^2$ . There are also three copies of  $\mathbb{Z}_2$ :

$$\{e, (12)\}, \quad \{e, (13)\}, \quad \{e, (23)\}$$

The alternating group is given by the even permutations

$$A_3 = \{e, (123), (132)\}$$

At order three, there is only one group, so  $A_3 \simeq \mathbb{Z}_3$ . At order six, one has  $D_3 \simeq S_3$ . For general groups,  $S_n \not\simeq D_n$  because  $|D_n| = 2n$  and  $|S_n| = n!$ . The dihedral group is a subset of the symmetric group: there are permutations that cannot be achieved from rotations and reflections.

**Lemma 3.8** (Rearrangement). If  $g, a, b$  are elements of a group  $G$  and  $g \circ a = g \circ b$ , then  $a = b$ .

*Proof.* Multiplying  $g \circ a = g \circ b$  by  $g^{-1}$  on the left one finds the thesis.  $\square$

**Corollary 3.9.** Consider the group  $G$  of  $n$  elements. Multiplying every element by  $h \in G$  gives a new group, mapping the group  $G$  to itself. It is an automorphism.

**Definition 3.10.** A morphism  $\varphi_h : G_n \rightarrow S_{|G|}$ ,  $h \mapsto p_h$  such the above gives the permutations of the set  $G$ . It is a homomorphism:  $\varphi_{h_1} \varphi_{h_2} = \varphi_{h_1 h_2}$ . The range of  $\varphi_h$  is a subset of  $S_{|G|}$  and it is a subgroup  $P_h$ . [r]

**Theorem 3.11** (Cayley's). Every group  $G$  of order  $n$  is isomorphic to a subgroup of  $S_n$  called Cayley's group.

## Lecture 5

**Corollary 3.12.** Two consequences are

- given a permutation  $P_h$  in Cayley's group, it cannot contain 1-cycles except for the identity  $h = e$ ;
- a permutation  $P_h$  in Cayley's group can be decomposed in cycles of equal length. If  $n$  is prime, the corresponding permutations in Cayley's group can only contain non-factorized  $n$ -cycles. This means that one 1-cycle is the identity as it is one  $n$ -cycle: the entire Cayley's group is built from them and as such has rank one and order  $n$ . It is  $\mathbb{Z}_n$ .

**Remark 3.13.** Applying  $n$  times a cycle of length  $n$ , one gets the identity.

**Theorem 3.14.** If the order  $n$  of a group  $G$  is prime, then it must be isomorphic to  $\mathbb{Z}_n$ .

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## 4 Structural properties of groups

**Definition 4.1.** Two elements are said to be conjugate  $g_1 \sim g_2$  if and only if  $\exists h \in G$  such that  $hg_1h^{-1} = g_2$ . The relation induces a partitioning of the group into equivalence classes.

**Definition 4.2.** If  $H$  is a subgroup of  $G$  and  $a \in G$ , then its conjugate group is

$$H' = \{aha^{-1}, \forall h \in H\} \equiv aHa^{-1}$$

which is also a subgroup.

**Lemma 4.3.** If  $H$  and  $H'$  are conjugate subgroups, then either  $H \cap H' = \{e\}$  or  $H = H'$ .

**Definition 4.4.** An invariant subgroup  $H$  of  $G$  is identical to all its conjugate subgroups. It is also called normal. The set  $H \subseteq G$  is invariant if and only if  $gHg^{-1} = H$  for all  $g \in G$ .

**Example 4.5.** A few examples:

- Consider the group of permutations

$$S_3 = \{e, (12), (13), (23), (123), (132)\}$$

It has three copies of  $\mathbb{Z}_2$  given by the 2-cycles

$$\{e, (12)\}, \quad \{e, (13)\}, \quad \{e, (23)\}$$

The conjugate subgroup of the first  $\mathbb{Z}_2$  is

$$(23)^{-1}e(23) = e, \quad (23)^{-1}(12)(23) = (23)(12)(23) = (13)$$

but this does not remain within the group. Remember that two-cycles are their own inverse. So  $H' \neq H$  but only share the identity. Since the two sets do not coincide, by the lemma  $\mathbb{Z}_2$  is not invariant. Consider instead the subgroup

$$H_2 = \{e, (123), (132)\} \simeq \mathbb{Z}_3$$

Its conjugate group is

$$(12)H_2(12) = \{e, (132), (123)\} = H_2$$

Therefore  $\mathbb{Z}_3$  is invariant.

- Consider

$$\mathbb{Z}_4 = \{e, a, a^2, a^3\}$$

It has one subgroup

$$\mathbb{Z}_2 = \{e, a^2\}$$

Its conjugate is

$$a^{-1}\{e, a^2\}a = \{e, a^2\}, \quad (a^3)^{-1}\{e, a^2\}a^3 = \{e, a^2\}$$

Remember that  $\mathbb{Z}_2$  is abelian. It is also abelian.

- Consider the dihedral group  $D_n$ . The relations used  $r^n = e$  and  $\sigma^2 = e$  corresponding to a  $\mathbb{Z}_n$  and  $\mathbb{Z}_2$ . In general,  $\mathbb{Z}_n \subset D_n$  is normal, while  $\mathbb{Z}_2 \subset D_n$  is not invariant. On the other hand, in general  $\mathbb{Z}_n \subset S_n$  is not invariant.

**Lemma 4.6.** All subgroups of abelian groups are invariant subgroups.

**Definition 4.7.** A group is simple if it does not contain any non-trivial invariant subgroup.

**Definition 4.8.** A group is semi-simple if it does not contain any abelian invariant subgroup.

**Example 4.9.** Some examples:

- The group  $\mathbb{Z}_4$  has a  $\mathbb{Z}_2$  subgroup which is invariant and abelian: the group  $\mathbb{Z}_4$  is not simple nor semi-simple.

- The group  $S_3$  has a  $\mathbb{Z}_3$  subgroup which is invariant and abelian.
- The cyclic group  $\mathbb{Z}_n$  with  $n$  prime is a simple group (because it does not admit subgroups).

**Remark 4.10.** The simple property describes whether a group is fundamental or has substructures.

**Definition 4.11.** Consider a set  $H$  subgroup of  $G$ , an element  $g \in G$  such that  $g \notin H$ . The set of elements

$$gH = \{gh_1, gh_2, \dots\}$$

is a left coset. Similarly  $Hg$  is a right coset.

**Remark 4.12.** Cosets have the same order of their parent group.

**Lemma 4.13.** Cosets are not subgroups.

*Proof.* Cosets do not have the identity element and  $g$  can't be the identity if it is not part of the subgroup  $H$ .  $\square$

**Lemma 4.14.** Two cosets either coincide completely or have no common elements.

*Proof.* Consider two cosets  $pH$  and  $qH$ . If  $ph_i = qh_j$  then  $q^{-1}p = h_jh_i^{-1} \in H$ . If it holds for all elements, then

$$q^{-1}pH = H \implies pH = qH$$

[r]  $\square$

**Theorem 4.15** (Lagrange). The order of a finite group must be an integer multiple of the order of any of its subgroups.

**Example 4.16.** Consider the symmetric group  $S_3$  and one of its subgroup

$$H_1 = A_3 = \{e, (123), (132)\}$$

The only coset that can be built is

$$(12)H_1 = (13)H_1 = (23)H_1 = \{(12), (13), (23)\}$$

Cosets induce a partitioning of the group. Consider one  $\mathbb{Z}_2$  subgroup

$$\mathbb{Z}_2 \simeq H_2 = \{e, (12)\}$$

It has two cosets

$$(23)H_2 = (132)H_2 = \{(23), (132)\}, \quad (13)H_2 = (123)H_2 = \{(13), (123)\}$$

**Theorem 4.17.** If  $H$  is an invariant subgroup of  $G$ , then its left and right cosets coincide. The set of cosets endowed with the composition

$$pH \cdot qH = (p \cdot q)H$$

forms a group called quotient group of  $G$  denoted by  $G/H$ , [r] with order  $|G|/|H|$ .

*Proof.* It holds

$$gH = gH(g^{-1}g) = (gHg^{-1})g = Hg$$

Also

$$ph_iqh_j = p(qq^{-1})h_iqh_j = (pq)h_k$$

where  $q^{-1}h_iq \in H$  and  $h_j \in H$ .  $\square$

**Example 4.18.** Two examples are:

- Consider the group  $\mathbb{Z}_4$ . It has an invariant subgroup

$$\mathbb{Z}_2 \simeq H = \{e, a^2\}$$

which has only one coset

$$M = aH = \{a, a^3\} = a\mathbb{Z}_2 = a^3\mathbb{Z}_2$$

In the sense of sets, that is multiplying all the elements together, one has

$$HH = H, \quad HM = MH = M, \quad MM = H$$

Therefore  $\mathbb{Z}_4/\mathbb{Z}_2 \simeq \mathbb{Z}_2$  the partitioning sets behave as elements of  $\mathbb{Z}_2$ .

- The set  $S_3$  has three elements as it does  $A_3$ . The quotient has only two elements. Since the only group of order two is  $\mathbb{Z}_2$  then

$$S_3/A_3 \simeq \mathbb{Z}_2$$

## 4.1 Multiplication of groups

**Definition 4.19.** Given two groups  $H_1$  and  $H_2$ , their direct product is the set of pairs

$$\{(h_1, h_2) \mid h_1 \in H_1, h_2 \in H_2\}$$

with the composition law

$$(h_1, h_2) \cdot (h'_1, h'_2) = (h_1 h'_1, h_2 h'_2) \in H_1 \otimes H_2$$

The order of the direct product is  $|H_1 \otimes H_2| = |H_1||H_2|$ . The sets  $H_1$  and  $H_2$  are invariant subgroups of  $H_1 \otimes H_2$ .

**Remark 4.20.** Consider

$$(h'_1, h'_2)^{-1}(e_1, h_2)(h'_1, h'_2) = (h'^{-1}_1, h'^{-1}_2)(e_1, h_2)(h'_1, h'_2) = (h'^{-1}_1 h'_1, h'^{-1}_2 h_2 h'_2) = (e_1, h'^{-1}_2 h_2 h'_2)$$

[r]

**Definition 4.21.** Given a group  $G = H_1 \otimes H_2$ , it is the direct product group of the subgroups  $H_1$  and  $H_2$  if

- $h_1 h_2 = h_2 h_1$  for all  $h_1 \in H_1$  and  $h_2 \in H_2$ ;
- the intersection is  $H_1 \cap H_2 = \{e\}$ ;
- for all  $g \in G$ ,  $\exists h_1 \in H_1, h_2 \in H_2$  such that  $g = h_1 h_2$ .

**Remark 4.22.** If  $G = H_1 \otimes H_2$  with  $H_1$  and  $H_2$  invariant subgroups then

$$G/H_1 \simeq H_2, \quad G/H_2 \simeq H_1$$

[r]

## Lecture 6

**Example 4.23.** Consider

$$\mathbb{Z}_6 = \{e, a, a^2, a^3, a^4, a^5\}$$

Two subgroups are

$$H_1 = \{e, a^3\} \simeq \mathbb{Z}_2, \quad H_2 = \{e, a^2, a^4\} \simeq \mathbb{Z}_3$$

The subgroups are abelian because the group is. They are also invariant  $ghg^{-1} = hgg^{-1} = h$ . Therefore one can write

$$\mathbb{Z}_6 = \mathbb{Z}_2 \otimes \mathbb{Z}_3 \implies \mathbb{Z}_6/\mathbb{Z}_3 \simeq \mathbb{Z}_2$$

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One may think that one can factor a cyclic group in terms of smaller cyclic groups by using factorization.

**Theorem 4.24** (Chinese remainder). Let  $p$  and  $q$  be coprime integers. Then

$$\mathbb{Z}_{pq} \simeq \mathbb{Z}_p \otimes \mathbb{Z}_q$$

**Theorem 4.25.** Let  $\phi : G \rightarrow G'$  be a homomorphism. The kernel of  $\phi$

$$\text{Ker } \phi = \{g \in G \mid \phi(g) = e'\} \equiv K$$

is an invariant subgroup of  $G$ . The quotient  $G/\text{Ker } \phi$  is isomorphic to  $G'$ .

**Example 4.26.** Consider again  $\mathbb{Z}_6$  and assume not knowing its structure. Then

$$H_1 = \{e, a^3\}, \quad aH_1 = \{a, a^4\}, \quad a^2H_1 = \{a^2, a^5\}$$

There is a natural mapping  $\phi$  using this structure. One can map every coset to the elements that multiply  $H_1$ :

$$\{e', b, b^2\}$$

The set  $H_1 \simeq \mathbb{Z}_2$  is the kernel of  $\phi$ . Therefore

$$G/H_1 \simeq G' = \mathbb{Z}_3$$

The direct product is restrictive because it requires subgroups to be invariant. One may want a more loose definition of product.

**Definition 4.27.** Given two groups  $G$  and  $K$  with different operations  $g_i \cdot g_j, k_i \circ k_j$ , assuming  $G$  acts as a group of transformations over  $K$

$$\forall g \in G, \quad g : K \rightarrow K, \quad k_i \mapsto g(k_i) = k_j$$

the semi-direct product,  $G \ltimes K$ , is the group given by the direct product endowed with the composition law

$$(g_1, k_1)(g_2, k_2) = (g_1 \cdot g_2, k_1 \circ g_1(k_2))$$

**Proposition 4.28.** A few properties of this definition:

- the definition is the composition law of Poincaré transformations,
- the inverse element of  $(g, k)$  is

$$(e, e) = (g, k)^{-1}(g, k) \implies (g, k)^{-1} = (g^{-1}, [g^{-1}(k)]^{-1})$$

- the order is

$$|G \ltimes K| = |G||K|$$

- the set  $K$  is an invariant subgroup of  $G \ltimes K$  but not of  $G$

$$(e, k) \in K \subset G \ltimes K \implies (g, h)^{-1}(e, k)(g, h) = (g^{-1}eg, \dots) \in K$$

- if  $G$  acts trivially on  $K$

$$g(k) = k, \forall g \in G, \forall k \in K$$

then

$$G \ltimes K = G \otimes K$$

**Proposition 4.29.** Given the subgroups  $K$  and  $H$  of  $G$ , then  $G = K \ltimes H$  if

- $H$  is invariant,
- the intersection is  $K \cap H = \{e\}$ ,

- given  $g \in G$ , there exists  $k \in K$  and  $h \in H$  such that

$$g = (k, e)(e, h)$$

If only one subgroup is invariant, then one uses a semi-direct product. If both subgroups are invariant, one uses the direct product.

**Example 4.30.** In the dihedral groups, one can identify two subgroups:  $\mathbb{Z}_k$  and  $\mathbb{Z}_2$ . Only the first is invariant. Therefore

$$D_k/\mathbb{Z}_k \simeq \mathbb{Z}_2 \implies D_k \simeq \mathbb{Z}_2 \ltimes \mathbb{Z}_k$$

**Example 4.31.** Consider

$$\mathbb{Z}_4/\mathbb{Z}_2 \simeq \mathbb{Z}_2, \quad \mathbb{Z}_4 \not\simeq \mathbb{Z}_2 \ltimes \mathbb{Z}_2$$

Because one cannot identify a second  $\mathbb{Z}_2$  subgroup inside  $\mathbb{Z}_4$ .

## 5 Classification of finite simple groups

From simple groups one can build any other group.

There are 18 infinite series of finite simple groups:

- $\mathbb{Z}_p$  with  $p$  prime,
- the alternating subgroup  $A_n$  subgroup with the permutation group with  $n \geq 5$ ,
- classical groups of matrices over finite fields.

There are also 26 exceptional groups (simple and finite):

- the smallest such group is Mathieu group  $M_{11}$  of order  $7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$ .
- the biggest such group is the monster group  $F_1$ , it contains 20 out of the 26 exceptional groups. Its order is about  $8.08 \times 10^{53}$ . This group is associated with Galois Theory.

## 6 Groups of space-time symmetries

**Definition 6.1.** The Euclidean group  $ISO(n)$  (inhomogeneous special orthogonal group?) consists of all continuous linear transformations of the  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  which leave the norm of all vectors invariant. The norm is defined as

$$\|x\|^2 = \sum_{i=0}^{n-1} (x^i)^2$$

A generic element of the Euclidean group  $ISO(n)$  maps

$$x^i \mapsto x'^i = R^i_j x^j + b^i$$

where  $R$  is a rotation and  $b$  is a translation.

### 6.1 Two dimensions

**Rotations.** An active transformation rotates a point  $P$  and keeps the axes fixed, while a passive rotation is the opposite, the axes are rotated. The latter will be used. The rotation is then

$$\begin{bmatrix} x'_p \\ y'_p \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_p \\ y_p \end{bmatrix} = R(\theta) \begin{bmatrix} x_p \\ y_p \end{bmatrix}$$

All rotation matrices  $R(\theta)$  form a group. In fact

$$R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2), \quad RR^\top = I$$

The matrices are orthogonal and have determinant equal to  $\det R = 1$ . In two dimensions, rotations form the special orthogonal group  $SO(2)$ . It is abelian, its identity is  $R(\theta = 0) = I$ , the inverse is  $R^{-1}(\theta) = R(-\theta)$ .

**Translations.** A translation is a transformation such that

$$T(\mathbf{b}) : x^i \mapsto x'^i = T(\mathbf{b}_1 + \mathbf{b}_2)$$

It holds

$$T(\mathbf{b}_1)T(\mathbf{b}_2) = T(\mathbf{b}_1 + \mathbf{b}_2), \quad T(\mathbf{0}) = e, \quad T^{-1}(\mathbf{b}) = T(-\mathbf{b})$$

The translations form an abelian group.

**Euclidean group.** An element of the Euclidean group  $\text{ISO}(2)$  is

$$g(\theta, \mathbf{b}) : \begin{bmatrix} x' \\ y' \end{bmatrix} = R \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix}$$

One may want to understand if elements  $g$  with both compositions form a group. It holds

$$g(\theta_1, \mathbf{b}_1)g(\theta_2, \mathbf{b}_2) = g(\theta_1 + \theta_2, \mathbf{b}_1 + R(\theta_1) \cdot \mathbf{b}_2)$$

In fact, applying one element  $g$  after the other, one gets

$$R_1(R_2\mathbf{x} + \mathbf{b}_2) + \mathbf{b}_1$$

The inverse is

$$g^{-1}(\theta, \mathbf{b}) = g(-\theta, -[R(-\theta) \cdot \mathbf{b}])$$

The identity is

$$g(0, \mathbf{0}) = e$$

The Euclidean group  $\text{ISO}(2)$  is a group and it is a semi-direct product

$$\text{ISO}(2) = \text{SO}(2) \ltimes \text{T}_2$$

**Exercise.** Prove that  $\text{T}_2$  is invariant and  $\text{SO}(2)$  is not.

**Reflections.** Consider the reflection with respect to the  $x$  axis

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

One may want a reflection about an arbitrary line. If the line has an angle  $\theta$  from the  $x$  axis, then the reflected point  $P''$  forms an angle with the origin and  $P'$  of  $2\theta$ . So a reflection about an arbitrary axis is a reflection about the  $x$  axis and then a reflection:

$$I(\theta) = R(2\theta)I(0)$$

For a passive transformation one has

$$I(\theta) = I(0)R(-2\theta)$$

Since one should get the same result for both active and passive transformations then

$$R(2\theta)I(0) = I(0)R(-2\theta) \implies I^{-1}(0)R(2\theta)I(0) = R(-2\theta)$$

Since  $\theta$  is arbitrary, so is  $2\theta = \varphi$ :

$$I^{-1}(0)R(\varphi)I(0) = R(-\varphi) = R^{-1}(\varphi)$$

Substituting this relation, one gets

$$I(\theta) = I(0)R(-2\theta) = I(0)[I^{-1}(0)R(\theta)I(0)]R(-\theta) = R(\theta)I(0)R(-\theta)$$

[r] A reflection through the origin is

$$\begin{bmatrix} x''' \\ y''' \end{bmatrix} = -I \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

Noting that  $-I = R(\pi)$ . A reflection about the  $y$  axis is

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = R(\pi)I(0)$$

Therefore

- all reflections are built from a single reflection  $I(0)$  and reflections;
- it holds

$$\det[I(\theta)] = \det R \det I(0) = \det I(0) = -1$$

- it holds

$$I^2(0) = I, \quad I^{-1}(0) = I(0)$$

The element  $I(0)$  represents a generator of  $\mathbb{Z}_2$ .

- the metric transforms as

$$g' = I(0)gI(0)$$

If  $g = \delta$  then

$$g' = I^2(0) = I$$

The matrix  $I(0)$  preserves the  $2 \times 2$  metric and as such it is an element of the orthogonal group  $O(2)$ .

- Elements  $M$  in the orthogonal group have  $\det M = \pm 1$ ; this group has two disjoint sets

$$SO(2) = \{M \in O(2) \mid \det M = 1\}, \quad I(0)SO(2) = \{M \in O(2) \mid \det M = -1\}$$

Therefore

$$O(2) \simeq \mathbb{Z}_2 \ltimes SO(2)$$

Because only  $SO(2)$  is invariant.

## Lecture 7

### 6.2 Arbitrary dimensions

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**Definition 6.2.** The set of all transformations, including reflections, of  $n$ -dimensional Euclidean space which preserve the norm of a vector is called the Euclidean group  $E_n$ . Therefore

$$T_n \subset ISO(n) \subset E_n, \quad SO(n) \subset ISO(n), \quad SO(n) \subset O(n) \subset E_n, \quad D_n \subset O(n)$$

The relation above can be generalized to any dimension

$$O(n) \simeq \mathbb{Z}_2 \ltimes SO(n)$$

The special orthogonal group is always an invariant group

$$O(n)/SO(n) \simeq \mathbb{Z}_2$$

[r] By adding translations?

$$E_n \simeq O(n) \ltimes T_n, \quad E_n/T_n \simeq O(n)$$

## 7 Representations of groups

A presentation  $\rho$  of  $G$  is a homomorphism  $\rho : G \rightarrow GL(V)$ ,  $g \mapsto \rho(g)$ . One defines

- the dimension of the vector space  $V$  over a field  $\mathbb{F}$  is the dimension  $n$  of the representation;
- a representation is faithful if the homomorphism is also an isomorphism, otherwise the representation is degenerate;
- given a degenerate representation, the set  $G/\text{Ker } \rho$  determines a faithful representation;
- it holds

$$\rho(e) = I, \quad \rho(g^{-1}) = \rho^{-1}(g), \quad \rho(g_1)\rho(g_2) = \rho(g_1g_2)$$

**Example 7.1.** A few examples:

- every group  $G$  admits a trivial representation

$$\rho(g) = I, \quad \forall g \in G, \quad V = \mathbb{C}$$

- All groups of matrices admit a one dimensional representation

$$V = \mathbb{C}, \quad \forall M \in G \subset \text{GL}(n, \mathbb{C}), \quad \rho(M) = \det M$$

The representation is degenerate

$$\rho(M_1)\rho(M_2) = \rho(M_1M_2) = \det(M_1M_2) = \det M_1 \det M_2$$

- The cyclic group  $\mathbb{Z}_2 = \{e, a\}$ ,  $a^2 = e$  admits two representations with dimension two with  $V = \mathbb{R}^2$ :

◊ first:  $\rho(e) = I$ ,  $\rho(a) = -I$ , this can be generalized to arbitrary dimensions;

◊ second:  $\rho(e) = I$ ,  $\rho(a) = \text{diag}(1, -1)$ .

There does not exist a change of basis which maps one into the other.

- The dihedral group  $D_2 = \{e, R_x, R_y, r\}$  has two representations in two dimensions:

$$e = I, \quad \rho(R_x) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \rho(R_y) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho(r) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

From the representation one can also see that  $\mathbb{Z}_2 \subset D_2$ . The group structure is preserved.

- A one dimensional representation of

$$\mathbb{Z}_n = \{e, a, \dots, a^{n-1}\}, \quad a^n = e, \quad \rho(a^n) = \rho^n(a) = \rho(e) = 1$$

and then

$$\rho(a) = e^{\frac{2\pi i}{n}}, \quad \rho(a^k) = e^{\frac{2\pi i}{n}k}, \quad k = 0, 1, \dots, n-1$$

- A representation gives a way to multiply the elements  $g \in G$  by a vector  $\mathbf{v} \in V$ . One can see the representation  $\rho$  as a map

$$\rho : G \times V \rightarrow V, \quad (g, \mathbf{v}) \mapsto \rho(g)\mathbf{v}$$

Therefore  $\rho$  is a module.

**Definition 7.2.** The representation of a group of matrices coinciding with the group itself is a fundamental representation.

**Example 7.3.** Two examples:

- For the special linear group  $\text{SO}(2)$ , one can use

$$R(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

The matrix  $R$  is two dimensional acting on a vector space over  $\mathbb{R}$ .

- Using orthogonal matrices to represent  $\text{O}(n)$  defines the orthogonal group's fundamental representation.

**Theorem 7.4.** If a group  $G$  admits an invariant subgroup  $H$ , then every representation of  $G/H$  induces a degenerate representation of  $G$ .

**Example 7.5.** Consider the alternating group

$$A_3 = \{e, (123), (321)\}$$

it is an invariant subgroup of  $S_3$ . Therefore

$$S_3/A_3 \simeq \mathbb{Z}_2, \quad \rho(\mathbb{Z}_2) = \{1, -1\}$$

The representation  $\rho$  induces a one dimensional representation of  $S_3$ : the parity of the elements. For every element, there is an associated cycle which has an even or odd number of two-cycles.

**Definition 7.6.** A similarity transformation between two matrices is defined by an invertible matrix  $S$  such that

$$M_1 = S^{-1}M_2S$$

Notice that it is one matrix for all group elements.

**Definition 7.7.** Two representations  $\rho_1$  and  $\rho_2$  of  $G$  are equivalent if and only if

$$\rho_1(G) \sim \rho_2(G) \iff \exists s \mid \forall g \in G, \rho_1(g) = S^{-1}\rho_2(g)S$$

**Definition 7.8.** The character  $\chi_\rho(g)$  for an element  $g \in G$  of a representation  $\rho$  is defined as

$$\chi_\rho(g) = \text{Tr}[\rho(g)]$$

The character is a function of the equivalence classes, not of the representation

$$\text{Tr}[\rho_1(g)] = \text{Tr}[S^{-1}\rho_2(g)S] = \text{Tr}[\rho_2(g)SS^{-1}] = \text{Tr}[\rho_2(g)]$$

## 7.1 Irreducible representations

**Definition 7.9.** Let  $\rho$  be a representation of  $G$  on a vector space  $V$ . Let  $V_1$  be a subspace of  $V$  such that

$$\rho(g)|v_1\rangle \in V_1, \quad \forall |v_1\rangle \in V_1$$

The set  $V_1$  is an invariant subspace of  $V$  with respect to  $\rho(G)$ . If  $V_1$  does not contain any non-trivial invariant subspace, then  $V_1$  is said to be minimal or proper.

**Definition 7.10.** A representation  $\rho$  of  $G$  on  $V$  is

- irreducible (irrep) if there is no invariant subspace of  $V$ ,
- reducible if there is at least one invariant subspace,
- fully reducible or decomposable if the orthogonal complement

$$V_1^\perp = \{|v\rangle \in V \mid \langle v_1|v\rangle = 0, \forall |v_1\rangle \in V_1\}$$

is also invariant.

**Remark 7.11.** One may note:

- the representation  $\rho$  is reducible if there exists a basis for all  $g \in G$  such that

$$\rho(g) = \begin{bmatrix} \rho_1(g) & \rho_2(g) \\ 0 & \rho_3(g) \end{bmatrix}$$

the representation is block diagonal, but the subspaces are mixed.

- the representation  $\rho$  is fully reducible if there exists a basis such that

$$\rho(g) = \begin{bmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{bmatrix}$$

the representation is block diagonal.

**Definition 7.12.** The direct sum of two representations is

$$\rho = \rho_1 \oplus \rho_2 = \begin{bmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{bmatrix}$$

acting on the vector space  $V = V_1 \oplus V_2$  with  $\dim V = \dim V_1 + \dim V_2$ .

**Definition 7.13.** The direct product of two representations  $\rho = \rho_1 \otimes \rho_2$  is defined on the vector space  $V = V_1 \otimes V_2$  with  $\dim V = \dim V_1 \dim V_2$  and vectors

$$|v\rangle = v^{ij} |e_i^{(1)}\rangle \otimes |e_j^{(2)}\rangle \in V$$

for a basis  $\{|e_i^{(j)}\rangle\}$  of  $V_j$ ; such that

$$\rho(g) |v\rangle = v^{ij} (\rho_1(g) |e_i^{(1)}\rangle) \otimes (\rho_2(g) |e_j^{(2)}\rangle) = v^{ij} \rho_1(g)^k{}_i [\rho_2(g)]^m{}_j |e_k^{(1)}\rangle \otimes |e_m^{(2)}\rangle = |v'\rangle$$

In matrix notation one has

$$v' = \rho_1(g) v \rho_2(g)^\top$$

**Remark 7.14.** The central problem in representation theory and physics is classification. One wants to write any representation as sums or products of irreps  $\rho$ :

$$D = \bigotimes_{\nu=1}^m [\rho_\nu(G)]^{b_\nu} = \bigoplus_{\mu=1}^n a_\mu \rho_\mu(G)$$

**Example 7.15.** Some examples:

- Consider the cyclic group  $\mathbb{Z}_2$ . The trivial representation and another representation both one dimensional are

$$\rho_0(\mathbb{Z}_2) = \{1, 1\}, \quad \rho_1(\mathbb{Z}_2) = \{1, -1\}$$

One can build a two dimensional representation

$$\rho_2(\mathbb{Z}_2) = \rho_0(\mathbb{Z}_2) \oplus \rho_1(\mathbb{Z}_2) = \begin{bmatrix} \rho_0(g) & 0 \\ 0 & \rho_1(g) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Consider the representation

$$\tilde{\rho}_2(\mathbb{Z}_2) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

Letting  $V = \mathbb{R}^2$ , one basis is

$$\{|v_1\rangle, |v_2\rangle\}, \quad |v_i\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix}$$

and one gets

$$\tilde{\rho}_2(a) |v_1\rangle = |v_1\rangle, \quad \tilde{\rho}_2(a) |v_2\rangle = -|v_2\rangle$$

The subspaces generated by  $\tilde{\rho}_2 |v_i\rangle$  are separate. One can introduce a similarity matrix

$$S = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \implies S^{-1} \tilde{\rho}_2(a) S = \rho_2(a)$$

So the representations  $\tilde{\rho}_2$  and  $\rho_2$  are equivalent. The property of equivalence can be stated by other means.

## Lecture 8

- The matrix

$$\begin{bmatrix} 1 & -a \\ a & 1 \end{bmatrix}, \quad a \neq 0$$

does not admit an invariant subspace. In fact, imposing

$$\begin{bmatrix} 1 & -a \\ a & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \implies \begin{cases} x - ay = x \\ ax + y = y \end{cases} \implies x = y = 0$$

[r] The group  $S_3$  admit a two dimensional representation with

$$\rho_2((123)) = -\frac{1}{2} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

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- The fundamental representation of  $\text{SO}(2)$

$$R = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

is an irreducible representation over  $V = \mathbb{R}^2$ . However, consider now  $V = \mathbb{C}^2$  and

$$|v_{\pm}\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm i \end{bmatrix}$$

Applying the rotation matrix, one gets

$$R(\theta) |v_{\pm}\rangle = \frac{e^{\pm i\theta}}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm i \end{bmatrix} = e^{\pm i\theta} |v_{\pm}\rangle$$

So the vectors  $|v_{\pm}\rangle$  define an invariant subspace. Defining a unitary transformation to be

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}, \quad S^{\dagger} = S^{-1}$$

one gets a similarity transformation to diagonalize the rotation matrix

$$S^{-1} R(\theta) S = \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}$$

Thus, the representation above is not an irreducible representation over  $V = \mathbb{C}^2$  and is fully reducible written as a direct sum

$$\rho_2(\theta) = \rho_1(\theta) \oplus \rho_{-1}(\theta)$$

The single representations are also representations of  $\text{U}(1)$ . The fundamental representation of  $\text{SO}(2)$  over  $\mathbb{C}^2$  can be written as the sum of  $\text{U}(1)$  representations.

## 7.2 Unitary representations

**Definition 7.16.** A vector space is hermitian if it admits an inner product which is sesquilinear  $\varphi : V \otimes V \rightarrow \mathbb{F}$ , typically  $\mathbb{F} = \mathbb{C}$ :

- Linear in the arguments

$$\varphi(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}) = \varphi(\mathbf{v}_1, \mathbf{w}) + \varphi(\mathbf{v}_2, \mathbf{w}), \quad \varphi(\mathbf{v}, \mathbf{w}_1 + \mathbf{w}_2) = \dots$$

- It holds

$$\varphi(\mathbf{v}, \lambda \mathbf{w}) = \lambda \varphi(\mathbf{v}, \mathbf{w})$$

- then [r]

$$\varphi(\mathbf{v}, \mathbf{w}) = [\varphi(\mathbf{w}, \mathbf{v})]^*$$

- it holds

$$\varphi(\mathbf{v}, \mathbf{v}) \geq 0, \quad \varphi(\mathbf{v}, \mathbf{v}) = 0 \iff \mathbf{v} = \mathbf{0}$$

**Definition 7.17.** If the operators  $\rho(g)$  of a representation of a group  $G$  over a hermitian vector space are unitary

$$[\rho(g)]^{-1} = \rho(g^{-1}) = [\rho(g)]^{\dagger}, \quad \forall g \in G$$

then the representation is called unitary. That is, the representation  $\rho$  is unitary if  $\rho(g)$  is a unitary matrix.

**Theorem 7.18.** Any representation of a finite group on a hermitian vector space  $V$  is equivalent to a unitary representation.



*Proof.* Since the vector space is hermitian one can build the conjugate representation  $\rho^\dagger(g)$  and construct

$$M = \sum_{g \in G} \rho^\dagger(g) \rho(g)$$

but  $\rho$  is not unitary. Therefore

$$\rho^\dagger(g) M \rho(g) = \rho^\dagger(g) \sum_{g'} \rho^\dagger(g') \rho(g') \rho(g) = \sum_{g'} \rho^\dagger(g'g) \rho(g'g) = \sum_{g''} \rho^\dagger(g'') \rho(g'') = M$$

Since  $M = M^\dagger$  is hermitian then it is diagonalizable

$$M = U^\dagger \Lambda U, \quad \Lambda = \text{diag}(\Lambda_1, \dots, \Lambda_n)$$

The matrix  $M$  is positive definite:

$$\langle v^\dagger M | v \rangle = \langle v^\dagger \sum_g \rho^\dagger(g) \rho(g) | v \rangle = \sum_g \|\rho(g) | v \rangle\|^2 \geq \|\rho(e) | v \rangle\|^2 = \| | v \rangle \|^2 > 0$$

Therefore its eigenvalues are real and positive. Taking the squaring root of the eigenvalues then multiply by  $U^\dagger U$  one gets the square root of the matrix

$$\sqrt{M} = U^\dagger \sqrt{\Lambda} U, \quad \sqrt{M} \sqrt{M} = M, \quad \sqrt{M} = (\sqrt{M})^\dagger$$

Starting from the beginning expression of  $M$  one multiplies by  $M^{-\frac{1}{2}}$ :

$$M = \rho^\dagger(g) M \rho(g) = [M^{-\frac{1}{2}} \rho^\dagger(g) M^{\frac{1}{2}}] [M^{\frac{1}{2}} \rho(g) M^{-\frac{1}{2}}] = I$$

Therefore, the matrix

$$M^{\frac{1}{2}} \rho(g) M^{-\frac{1}{2}}$$

is unitary. □

**Theorem 7.19.** If the representation  $\rho$  is reducible and unitary, then it is fully reducible.

*Proof.* Consider an invariant subspace  $V_1$  and its orthogonal complement  $V_1^\perp$ . One needs to prove that the orthogonal complement is an invariant subspace:

$$\rho(g) | v \rangle \in V_1, \quad \rho(g^{-1}) | v \rangle \in V_1, \quad | v \rangle \in V_1$$

Then, using  $w \in V_1^\perp$ , one gets

$$\langle w | \rho^\dagger(g) | v \rangle = \langle w | \rho^{-1}(g) | v_1 \rangle = \langle w | \rho(g^{-1}) | v_1 \rangle = 0 \implies \rho(g) | w \rangle \in V_1^\perp$$

So the orthogonal complement is invariant. The representation  $\rho$  is thus fully reducible. □

**Corollary 7.20** (both theorems). All reducible representations of finite groups are fully reducible.

**Lemma 7.21** (Schur's I). Let  $\rho$  be an irreducible representation of a group  $G$  on a vector space  $V$ , and  $A$  an operator on  $V$ . If  $A$  commutes with all operators

$$A \rho(g) = \rho(g) A, \quad \forall g \in G$$

then  $A = \lambda I$ .

*Proof.* With no loss of generality, consider  $A$  to be a hermitian operator since one can always have

$$A_\pm = \frac{A \pm A^\dagger}{2}$$

Consider an orthonormal basis formed by the eigenvectors of  $A$

$$A | u_{i,\alpha} \rangle = \lambda_i | u_{i,\alpha} \rangle$$

where  $\alpha$  is the index of degeneracy of an eigenvalue. Applying the hypothesis:

$$A\rho(g)|u_{i,\alpha}\rangle = \rho(g)A|u_{i,\alpha}\rangle = \rho(g)\lambda_i|u_{i,\alpha}\rangle$$

so the vector  $\rho(g)|u_{i,\alpha}\rangle$  is also an eigenvector of  $A$ , so at most  $\rho$  changes  $\alpha$ . Defining the subspace

$$V_i = \{|u_{i,\alpha}\rangle\}$$

then

$$\rho(g)|u_{i,\alpha}\rangle \in V_i$$

so the vector space  $V_i$  is an invariant subspace. However, if  $\rho$  is an irreducible representation, then it does not admit an invariant (proper) subspace: there is only one subspace, the trivial one  $V_i = V$ . This means that  $A$  has only one eigenvalue

$$A = \lambda I$$

□

**Lemma 7.22** (Schur's II). Let  $\rho$  and  $\rho'$  be two irreducible representation of a group  $G$  on the vector space  $V$  and  $V'$ . Let  $A : V \rightarrow V'$  be a linear operator satisfying

$$A\rho'(g) = \rho(g)A, \quad g \in G$$

It follows either  $A = 0$ , or  $V = V'$  (so  $A$  is an isomorphism) and  $\rho \sim \rho'$ .

*Proof.* The image (or range) of  $A$  in  $V'$  is

$$\text{im } A = \{|x'\rangle \in V' \mid |x'\rangle = A|x\rangle, \quad |x\rangle \in V\}$$

It holds

$$\forall g \in G, \rho'(g)|x'\rangle = \rho'(g)A|x\rangle = A\rho(g)|x\rangle \in \text{im } A$$

So the image is an invariant subspace of  $V'$  with respect to  $\rho'(g)$ . But  $\rho'$  is an irreducible representation, therefore either  $\text{im } A = 0$  implies  $A = 0$ , or  $\text{im } A = V$  so  $A$  maps on the entire  $V'$ .

One can apply the same reasoning on the kernel

$$\text{Ker } A = \{|x\rangle \in V \mid A|x\rangle = |0\rangle \in V'\}$$

Considering  $|x\rangle \in \text{Ker } A$ , then

$$A\rho(g)|x\rangle = \rho'(g)A|x\rangle = \rho'(g)|0\rangle = |0\rangle \implies \rho(g)|x\rangle \in \text{Ker } A$$

Therefore, the kernel is an invariant subspace of  $V$  with respect to  $\rho$ . But  $\rho$  is an irrep so either  $\text{Ker } A = V$  implies  $A = 0$ , or  $\text{Ker } A = 0$  implies

$$A|x'\rangle = A|y'\rangle \implies |x'\rangle = |y'\rangle$$

This last case, combined with the last above, the map  $A$  is a bijective map. Therefore it is invertible and is an isomorphism

$$V = V', \quad \dim V = \dim V'$$

Since  $A$  is invertible, then the two representation are equivalent

$$A\rho' = \rho A \implies \rho' = A^{-1}\rho A \implies \rho' \sim \rho$$

□

**Notation.** The labels of irreps are  $\mu, \nu$ . The dimension of irrep  $\mu$  is  $d_\mu$ . The character  $\chi_i^\mu$  has the index  $i$  that labels equivalence classes. The number of equivalence classes is  $n_c$ . The number of elements in an equivalence class is  $n_i$ .

**Theorem 7.23 (A).** Irreducible representations are orthonormal to each other, valid only for finite groups:

$$\frac{d_\mu}{|G|} \sum_{g \in G} [\rho_\mu^\dagger(g)]_j^i [\rho_\nu(g)]_l^k = \delta_{\mu\nu} \delta_l^i \delta_j^k$$

The reason it describes orthonormality can be seen by the following

$$\langle g|u, i, j\rangle = \sqrt{\frac{d_\mu}{|G|}} [\rho_\mu(g)]_j^i \implies \sum_g \langle u, i, j|g\rangle \langle g|\nu, k, l\rangle = \delta_{\mu\nu} \delta_l^i \delta_j^k$$

*Proof.* Consider two irreps  $\rho_\mu$  and  $\rho_\nu$  with dimensions  $d_\mu$  and  $d_\nu$ . Let  $X$  be a  $d_\mu \times d_\nu$  matrix

$$M_x = \sum_g \rho_\mu^\dagger(g) X \rho_\nu(g)$$

From the rearrangement lemma

$$\rho_\mu^{-1}(g) M_x \rho_\nu(g) = M_x, \quad \forall g \in G$$

Considering

$$M_x \rho_\nu(g) = \rho_\mu(g) M_x, \quad \forall g \in G$$

from the second Schur's lemma one has either  $\mu \neq \nu$  and  $M_x = 0$ , or  $\mu = \nu$  and  $M_x = \lambda I$  (from the first Schur's lemma). This is the first step towards  $\delta_{\mu\nu}$ .

Consider the following matrices  $X_l^k$  indexed by  $k, l$

$$[X_l^k]_j^i = \delta_j^k \delta_l^i$$

Therefore

$$[M_l^k]^m_n = \sum_g [\rho_\mu^\dagger(g)]^m_i [X_l^k]_j^i [\rho_\mu(g)]_n^j = \sum_g [\rho_\mu^\dagger(g)]^m_l [\rho_\mu(g)]_n^k = [\lambda_l^k] I_{d_\mu}$$

Taking the trace one gets

$$\text{Tr}[M_l^k] = \sum_g [\rho_\mu^\dagger(g)]^m_l [\rho_\mu(g)]_m^k = \sum_g [\rho_\mu(g) \rho_{\mu u}^\dagger]_l^k = \sum_g I_l^k = |G| \delta_l^k \equiv \lambda_l^k d_\mu \implies \lambda_l^k = \frac{|G|}{d_\mu} \delta_l^k$$

from which the thesis follows.  $\square$

**Example 7.24.** Consider the cyclic group  $\mathbb{Z}_2 = \{e, a\}$ . One wants to construct all possible irreps using the theorem above. Starting from an irrep, one can build perpendicular irreps so independent. Starting from the trivial irrep

$$\rho(e) = \rho(a) = 1$$

One can write it as a vector  $(1, 1)$  which has only one orthogonal vector  $(1, -1)$ . So the cyclic group  $\mathbb{Z}_2$  admits only two irreps.

**Remark 7.25.** Abelian groups have all irreps with dimension  $d = 1$ . The theorem above is then

$$\frac{1}{|G|} \sum_g \rho_\mu^\dagger(g) \rho_\mu(g) = \delta_{\mu\nu}$$

**Corollary 7.26.** The number of inequivalent irreps of a finite group is

$$\sum_\mu d_\mu^2 \leq |G|$$

where the sum over  $\mu$  is the sum over the irreps.

*Proof.* Consider the vector notation  $\langle g|u, i, j\rangle$  which means the  $g$ -th component of the vector  $|\mu, i, j\rangle$ . This vector has two indices, so it is a matrix and as such has  $d_\mu^2$  components. The number of linearly independent vectors less than  $\dim V$  is  $|G|$ .  $\square$

## Lecture 9

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**Theorem 7.27** (B). Irreducible representations are complete. It holds

$$\sum_{\mu} d_{\mu}^2 = |G|, \quad \frac{1}{|G|} \sum_{\mu} d_{\mu} \sum_{i,j=1}^{d_{\mu}} [\rho_{\mu}(g_1)]_j^i [\rho_{\mu}^{\dagger}(g_2)]_i^j = \delta_{g_1 g_2}$$

The argument of the second relation is the trace of  $\rho \rho^{\dagger}$ . The proof of the first relation can be found in the proof of Theorem 7.37.

**Proposition 7.28** (Properties of characters). A few properties of characters:

- the character  $\chi$  labels equivalence classes of irreps; in fact, two irreps are equivalent  $\rho \sim \rho'$  if it exists a matrix  $S$  such that  $\rho' = S^{-1} \rho S$ , therefore

$$\chi_{\rho'}(g) = \text{Tr}[\rho'(g)] = \text{Tr}[S^{-1} \rho(g) S] = \chi_{\rho}(g)$$

- The character  $\chi$  also depends on conjugacy classes; two elements are conjugate if

$$g \sim g' \iff h g h^{-1} = g', \quad \forall h$$

then

$$\chi_{\rho}(g') = \text{Tr}[\rho(g')] = \text{Tr}[\rho(h) \rho(g) \rho(h^{-1})] = \text{Tr}[\rho(h) \rho(g) \rho(h)^{-1}] = \chi_{\rho}(g)$$

- One defines

$$\chi_{\mu}(g) = \chi_{\mu}^i = \text{Tr}[\rho_{\mu}(g)], \quad g \in \zeta_i$$

where  $\zeta_i$  is a conjugacy class.

- The character of the identity is

$$\chi_{\rho}(e) = \text{Tr}[\rho(e)] = \dim \rho = \dim V$$

For an irrep one has  $\chi_{\mu}(e) = d_{\mu}$ .

**Theorem 7.29** (C.1). The characters of irreducible representations are orthonormal

$$\frac{1}{|G|} \sum_{i=1}^{n_c} (\chi_{\mu}^i)^* \chi_{\nu}^i n_i = \delta_{\mu\nu}$$

*Proof.* Consider Theorem 7.23

$$\frac{d_{\mu}}{|G|} \sum_g [\rho_{\mu}^{\dagger}(g)]_i^k [\rho_{\nu}(g)]_l^j = \delta_{\mu\nu} \delta_l^k \delta_i^j$$

Setting  $i = k$  and  $j = l$  then

$$\frac{1}{|G|} \sum_g \chi_{\mu}^*(g) \chi_{\nu}(g) = \delta_{\mu\nu}$$

The sum can be decomposed in a sum over classes

$$\sum_g = \sum_i \sum_{h \in \zeta_i}$$

where  $\zeta_i$  is a class. However, the character  $\chi$  is invariant of classes so

$$\sum_g = \sum_{i=1}^{n_c} \sum_{h \in \zeta_i} \implies \sum_g = \sum_{i=1}^{n_c} n_i$$

therefore the thesis.  $\square$

**Lemma 7.30.** It holds

$$\sum_{h \in \zeta_i} \rho_\mu(h) = \frac{n_i}{d_\mu} \chi_\mu^i I_{d_\mu}$$

*Proof.* Defining

$$A \equiv \sum_{h \in \zeta_i} \rho_\mu(h)$$

it follows

$$\rho_\mu(g) A \rho_\mu^{-1}(g) = \sum_{h \in \zeta_i} \rho_\mu(ghg^{-1}) = A$$

From the first Schur's lemma one has  $A = \lambda I$  and its trace is

$$\text{Tr } A = \sum_{h \in \zeta_i} \text{Tr}[\rho_\mu(h)] = \sum_h \chi_\mu^i = n_i \chi_\mu^i = \lambda \text{Tr } I = \lambda d_\mu \implies \lambda = \frac{n_i}{d_\mu} \chi_\mu^i$$

from which follows the thesis.  $\square$

**Theorem 7.31** (C.2). The characters obey a completeness relation

$$\frac{n_i}{|G|} \sum_\mu \chi_\mu^i (\chi_\mu^j)^* = \delta^{ij}$$

*Proof.* Consider the second thesis of Theorem 7.27

$$\delta_{g_1 g_2} = \frac{1}{|G|} \sum_\mu d_\mu \sum_{kl} [\rho(g_1)]_l^k [\rho_\mu^\dagger(g_2)]_k^l$$

Summing over all elements  $g_1, g_2$  in two classes, one gets

$$\begin{aligned} n_i \delta^{ij} &= \frac{1}{|G|} \sum_\mu d_\mu \sum_{kl} \left[ \sum_{g_1 \in \zeta_i} [\rho(g_1)]_l^k \right] \left[ \sum_{g_2 \in \zeta_j} [\rho(g_2)]_k^l \right] \\ &= \frac{1}{|G|} \sum_\mu d_\mu \sum_{kl} \left( \frac{n_i}{d_\mu} \chi_\mu^i \delta_l^k \right) \left( \frac{n_j}{d_\mu} (\chi_\mu^j)^* \delta_k^l \right) \\ &= \frac{1}{|G|} \sum_\mu \sum_{kl} (n_i \chi_\mu^i \delta_l^k) \left( \frac{n_j}{d_\mu} (\chi_\mu^j)^* \delta_k^l \right) \\ &= \frac{1}{|G|} \sum_\mu \frac{n_i n_j}{d_\mu} (\chi_\mu^i) (\chi_\mu^j)^* d_\mu \end{aligned}$$

from which the thesis

$$\frac{1}{|G|} = \sum_\mu n_j (\chi_\mu^i) (\chi_\mu^j)^* = \delta^{ij}$$

$\square$

**Corollary 7.32.** The number of inequivalent irreducible representations for any finite group is equal to the number  $n_c$  of conjugacy classes. The characters  $\chi_\mu^i$  can be put in an  $n_c \times n_c$  matrix called character table. Sometimes they are normalized as

$$\chi_\mu^i \rightarrow \sqrt{\frac{n_i}{|G|}} \chi_\mu^i$$

**Theorem 7.33.** One can reduce a generic representation into irreducible representations

$$\rho(G) = \bigoplus_{\mu=1}^{n_c} a_\mu \rho_\mu(G)$$

then the number  $a_\mu$  of times an irreducible representation  $\rho_\mu$  occurs is

$$a_\mu = \sum_i \frac{(\chi_\mu^i)^* \chi^i n_i}{|G|}$$

*Proof.* A generic representation for a finite group is similar to a unitary representation which is fully reducible, therefore it can be written as a direct sum. Since the number of irreducible representations is the number of conjugacy class, the sum runs from 1 to  $n_c$ .

Defining

$$\chi_i = \text{Tr}[\rho(G)] = \sum_{\mu} a_{\mu} \chi_{\mu}^i$$

and using Theorem 7.29

$$\frac{1}{|G|} \sum_i n_i (\chi_{\mu}^i)^* \chi^i = \frac{1}{|G|} \sum_{i\nu} n_i (\chi_{\mu}^i)^* a_{\nu} \chi_{\nu}^i = \sum_{\nu} a_{\nu} \delta_{\mu\nu} = a_{\nu}$$

□

**Theorem 7.34** (Irreducibility condition). A necessary and sufficient condition for a representation  $\rho$  with characters  $\chi_i$  to be irreducible is

$$\sum_{i=1}^{n_c} n_i |\chi_i|^2 = |G|$$

*Proof.* Using Theorem 7.29

$$\sum_i \frac{n_i}{|G|} (\chi^i)^* \chi^i = \sum_i \frac{n_i}{|G|} \sum_{\mu\nu} a_{\mu} (\chi_{\mu}^i)^* a_{\nu} \chi_{\nu}^i = \sum_{\mu} |a_{\mu}|^2$$

If  $\rho$  is an irreducible representation, then for one  $\mu$  one has  $a_{\mu} = 1$ , while for all others  $a_{\mu} = 0$ . Conversely, if

$$\sum_i \frac{n_i}{|G|} |\chi_i|^2 = 1$$

then

$$\sum_{\mu} |a_{\mu}|^2 = 1$$

Since  $a_{\mu} \in \mathbb{N}_0$ , this is possible only if  $a_{\mu} = 1$  for one  $\mu$  and  $a_{\mu} = 0$  for all others. □

### 7.3 Regular representations

**Theorem 7.35.** Every finite group admits a representation with  $|G| \times |G|$  matrices called regular representation. For all  $g_i, g_j \in G$  it holds

$$g_i g_j = g_k = g_m [\rho^R(g_i)]^m_j$$

where

$$[\rho^R(g_i)]^m_j = \begin{cases} 1, & m = k \\ 0, & m \neq k \end{cases}, \quad i, j, m = 1, 2, \dots, |G|$$

**Remark 7.36.** One may notice:

- The matrices  $\rho^R$  have one 1 and all zeros in each row. Consider the cyclic group  $\mathbb{Z}_3 = \{e, a, a^2\}$ . The relations between the elements can be written as a linear system of equations

$$\begin{cases} ea = a \\ ea^2 = a^2 \\ ee = e \end{cases} \implies X \begin{bmatrix} e \\ a \\ a^2 \end{bmatrix} = \begin{bmatrix} e \\ a \\ a^2 \end{bmatrix} \implies X = \rho^R(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Consider now the relations with  $a$ :

$$X \begin{bmatrix} e \\ a \\ a^2 \end{bmatrix} = \begin{bmatrix} a \\ a^2 \\ e \end{bmatrix} \implies X = \rho^R(a) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Note that  $\rho^R(a)$  permutes the elements.

- The matrices  $\rho^R$  are representations of  $S_{|G|}$  group elements. This is again Cayley's theorem.
- Since the matrices are permutations then

$$\text{Tr}[\rho^R(e)] = |G|, \quad \text{Tr}[\rho^R(g)] = 0, \quad g \neq e$$

**Theorem 7.37.** The regular representation contains every inequivalent irreducible representation  $d_\mu$  times

$$\rho^R(G) = \bigoplus_{\mu=1}^{n_c} d_\mu \rho_\mu(G), \quad \sum_{\mu} d_\mu^2 = |G|$$

*Proof.* Using Theorem 7.33, since any representation can be written as sum of irreducible representations it holds

$$\rho^R(G) = \bigoplus_{\mu=1}^{n_c} a_\mu \rho_\mu(G), \quad a_\mu = \sum_i \frac{(\chi_\mu^i)^* \chi^i n_i}{|G|}, \quad \chi_i = \text{Tr}[\rho^R(G)]$$

Using the last property above one gets

$$\chi^i = \text{Tr}[\rho^R(G)] = \text{Tr}[\rho^R(e)] = \chi^0 = |G|$$

therefore

$$a_\mu = \frac{(\chi_\mu^0)^* \chi^0 n_0}{|G|} = (\chi_\mu^0)^* n_0 = (\chi_\mu^0)^* = \text{Tr}[\rho_\mu(e)] = \text{Tr}[I_{d_\mu}] = d_\mu$$

in the third equality one considers that the conjugacy class of the identity has only one element and that is the identity

$$geg^{-1} = e, \quad \forall g$$

so  $n_0 = 1$ . So every irreducible representation appears always  $d_\mu$  times.

The second statement follows from

$$\text{Tr}[\rho^R(e)] = \sum_{\mu} d_\mu \text{Tr}[\rho_\mu(e)] = \sum_{\mu} d_\mu^2 = |G|$$

the first equality is a consequence of the first thesis of the theorem.  $\square$

## Lecture 10

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**Example 7.38.** One would like to construct all irreducible representations. The symmetric group  $S_3$  has order  $3! = 6$ . One can count the irreducible representations. The trivial one always exists,  $d = 1$  and  $\rho_0 : G \rightarrow 1$ . Knowing that abelian groups have only representations of dimension one, since  $S_3$  is not abelian, it must have a representation with dimension greater than one.

The number of irreducible representations is equal to the number of conjugacy classes  $\chi_\mu^i$ . The symmetric group has a subgroup  $A_3 = \{e, (123), (132)\} \simeq \mathbb{Z}_3$  which is invariant: the effect of conjugation keeps all elements in the group  $[r]$ . The equivalence classes are

$$\{e\}, \quad \{(12), (13), (23)\}, \quad \{(123), (321)\}$$

One looks for the next irreducible representations  $\rho_1$ . Knowing that the quotient  $S_3/A_3 \simeq \mathbb{Z}_2$  induces a representation of the entire group, one gets the parity  $\sigma(p) = \pm 1, p \in S_3$ .

The last irreducible representation can be built from orthogonality. Looking at the character table

	1	2	3
0	1	1	1
1	1	-1	1
2	2	0	-1

where the last row has been constructed considering the entries as vectors and finding the vector perpendicular to both first and second vector. In fact, one has seen that

$$\rho_2(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_2(r) = \frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \quad \rho_2(\sigma) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

for  $r$  and  $\sigma$  in  $D_3$ , but  $S_3 \simeq D_3$ .

One can build the regular representation of the symmetric group  $S_3$ :

$$\rho^R(S_3) = \rho_0(S_3) \oplus \rho_1(S_3) \oplus 2\rho_2(S_3)$$

**Example 7.39.** Consider the dihedral group  $D_2$  with order  $|D_2| = 4$ . Knowing the trivial representation to exist and that the group is abelian, it must have four irreducible representations of dimension one. The strategy is to find invariant subgroups, consider the quotient and induce the representations.

The subgroup  $\{e, r, \sigma_1, \sigma_2\}$  is an invariant and has a subgroup  $\mathbb{Z}_2 \simeq \{e, r\}$ . The quotient is isomorphic to another  $\mathbb{Z}_2$

$$\rho_1(e) = 1, \quad \rho_1(r) = 1, \quad \rho_1(\sigma_1) = -1, \quad \rho_1(\sigma_2) = -1$$

The group has another subgroup  $\{e, \sigma_1\} \simeq \mathbb{Z}_2$  whose quotient is another  $\mathbb{Z}_2$ . One can then build

$$\rho_2 \begin{pmatrix} e \\ r \\ \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

The character table is then

	$e$	$r$	$\sigma_1$	$\sigma_2$
0	1	1	1	1
1	1	1	-1	-1
2	1	-1	1	-1
3	1	-1	-1	1

[r] One has four conjugacy classes  $\{e\}$ ,  $\{r\}$ ,  $\{\sigma_1\}$  and  $\{\sigma_2\}$ . In fact, one can note that  $ghg^{-1} = h$  therefore every element is its own conjugacy class.

**Example 7.40.** Consider the cyclic group  $\mathbb{Z}_n$ . It is abelian, so one expects  $n$  one-dimensional irreps. Previously, a representation has been constructed as

$$\rho_1(a^k) = e^{\frac{2\pi i}{n}k} \implies \rho_\mu(a^k) = e^{\frac{2\pi i}{n}\mu k}, \quad \mu, k = 0, \dots, n-1$$

## 7.4 Representations of the symmetric group

The cycle decomposition is invariant under conjugation. This means that the classes are given by cycle structures.

Suppose to have  $\nu_n$   $n$ -cycles for  $n \in \mathbb{N}$ , then for each class of  $S_n$ , one has

$$n = \sum_{k=1}^n k\nu_k$$

[r]

**Definition 7.41.** A Young tableau of order  $k$  is a way of stacking  $k$  squares in columns ordered such that the  $(i+1)$ -th column is not longer than the  $i$ -th. [r] for  $S_4$  remember that  $2, 1, 1$  exists.

**Theorem 7.42.** The number of Young tableau is equal to the number of classes, which itself is equal to the number of irreps.



**Example 7.43.** The cyclic group appears in physics as a parity symmetry:

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi + m\phi^2 + \lambda\phi^4, \quad \phi \rightarrow -\phi$$

For a vector field, one has instead

$$m\phi^\top \phi \implies \phi \rightarrow \rho(g)\phi \implies \rho(g) \in O(n)$$

[r] In quantum field theory, the tame divergences, one discretizes space-time into a lattice: the continuous symmetries become finite group symmetries.

## 8 Continuous groups

**Definition 8.1.** A topological space is a set  $X$  where a topology is defined, which is a set  $U$  of subsets  $U_i \subseteq X$  such that

- the empty set and  $X$  belong both to  $U$ ,
- every union of  $U_i$  is a subset of  $U$ ,
- every intersection of  $U_i$  is a subset of  $U$ .

**Definition 8.2.** The subsets  $U_i$  are called open neighbourhoods.

**Remark 8.3.** A topological space is the most general structure where continuity is defined.

**Definition 8.4.** A topological space has a Hausdorff topology if all pairs of elements  $(x, y) \in X$  admit a disjoint neighbourhood. [r] disjoint?

**Definition 8.5.** A homeomorphism  $\phi$  between two topological spaces  $A$  and  $B$  with Hausdorff topology is a continuous bijective (therefore invertible) map, with the inverse  $\phi^{-1}$  also continuous. The spaces  $A$  and  $B$  are said to be homeomorphic.

**Definition 8.6.** A manifold  $M$  of dimension  $N = \dim M$  is a space locally homeomorphic to  $\mathbb{R}^n$ : for every point  $p \in M$  there exists an open neighbourhood  $U_p$  homeomorphic to an open neighbourhood of  $\mathbb{R}^n$ .

**Definition 8.7.** A chart  $(U, \varphi)$  is an open set  $U \subseteq M$  with  $\varphi : U \rightarrow V \subseteq \mathbb{R}^n$ ,  $x \mapsto \{x^i\}$  an homeomorphism, where  $x^i$  are the coordinates on the chart.

**Definition 8.8.** An atlas is a set of charts  $\{(U_\alpha, \varphi_\alpha)\}$  such that

- the union of all charts is  $\bigcup_\alpha U_\alpha = M$ ,
- the transition functions  $f_{\beta\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1}$  are differentiable  $C^k$ , then the atlas is  $C^k$ .

**Remark 8.9.** The transition functions are functions  $f_{\beta\alpha} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  so the notion of derivative is well-defined. In an intersection  $U_\alpha \cap U_\beta$  there are two charts, so two coordinate systems  $x$  and  $y$ . A change of coordinates is given by

$$y^i(p) = \varphi_\beta \circ \varphi_\alpha^{-1}(x^i(p)) = \varphi_\beta(\varphi_\alpha^{-1}(x^i(p))) = f_{\beta\alpha}(x^i(p))$$

**Definition 8.10.** A smooth manifold is a manifold  $M$  with an atlas  $C^p$  (typically  $C^\infty$ ).

**Example 8.11.** Consider the two-sphere  $S^2$  embedded in  $\mathbb{R}^3$ :  $x^2 + y^2 + z^2 = 1$ . One can have two charts

$$U_S = S^2 - (0, 0, 1), \quad U_N = S^2 - (0, 0, -1)$$

which are the sphere without one of the poles, with the homeomorphism  $\varphi_{S,N}$  a stereographic projection. The stereographic projection is singular at the poles, so one must remove one of them. The dimension of the manifold is 2 because

$$\varphi_S : U_S \rightarrow \mathbb{R}^2, \quad \varphi_S = \begin{cases} \frac{x}{1-z} \\ \frac{y}{1-z} \end{cases}$$

so a point  $p \in S^2$  is given by the embedding  $(x, y, z)$  in  $\mathbb{R}^3$ . There is only one transition function  $f_{SN} = \varphi_S \circ \varphi_N^{-1}$  and it is  $C^\infty$ , so one has a smooth manifold.

**Definition 8.12.** A scalar function  $g : M \rightarrow \mathbb{R}$  is smooth,  $C^\infty$ , if given a chart  $(U, \varphi)$  then  $g \circ \varphi^{-1}$  is also smooth  $C^\infty$ . This definition considers just one point.

**Remark 8.13.** The homeomorphism is  $\varphi^{-1} : \mathbb{R}^n \rightarrow M$  and the function is  $g : M \rightarrow \mathbb{R}$  therefore one has a notion of derivative through  $f \circ \varphi^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}$ .

[r]

**Definition 8.14.** Consider two smooth manifolds  $M_1$  and  $M_2$ . A map  $g : M_1 \rightarrow M_2$  smooth  $C^\infty$  at  $p \in M_1$  if for a chart  $(U, \varphi) \in M_1$  and  $(V, \psi) \in M_2$  then  $\psi \circ g \circ \varphi^{-1}$  is smooth  $C^\infty$ . If  $g$  is bijective (therefore invertible) and  $g, g^{-1}$  are smooth  $C^\infty$  everywhere, then  $g$  is a diffeomorphism.

**Definition 8.15.** A Lie group  $G$  is a smooth manifold  $C^\infty$  with composition  $\varphi : G \times G \rightarrow G$  of class  $C^\infty$  and smooth inverse such that it holds

- closure  $\forall x, y \in G, \exists! z \in G \mid z = \varphi(x, y)$ ,
- associativity  $\forall x, y, z \in G, \varphi(x, \varphi(y, z)) = \varphi(\varphi(x, y), z)$ ,
- identity  $\exists e \in G, \forall x \in G$  such that  $\varphi(e, x) = \varphi(x, e) = x$ ,
- inverse  $\forall x \in G, \exists x^{-1} \in G$  such that  $\varphi(x, x^{-1}) = \varphi(x^{-1}, x) = e$ .

**Example 8.16.** Examples of Lie groups already known are

$$\begin{aligned} \text{GL}(n, \mathbb{R}), \quad \dim[\text{GL}(n, \mathbb{R})] &= n^2 \\ \text{SL}(n, \mathbb{R}), \quad \dim[\text{SL}(n, \mathbb{R})] &= n^2 - 1 \\ \text{O}(n), \quad \dim \text{O}(n) &= \frac{n(n+1)}{2} \\ \text{SO}(n), \quad \dim \text{SO}(n) &= \frac{n(n-1)}{2} \end{aligned}$$

## Lecture 11

**Definition 8.17.** The center of a group is a subgroup of all commuting elements

$$Z(G) = \{a \in G \mid g^{-1}ag = a, \forall g \in G\}$$

It is an invariant abelian subgroup of  $G$ , so it admits a quotient  $G/Z(G) \simeq H$ .

**Remark 8.18.** Matrices that commute with unitary matrices are of the form  $cI$ .

**Example 8.19.** One can study the center of the unitary group  $U(n)$  for which  $M^\dagger = M^{-1}$ . Let  $M = cI$ , then

$$M^\dagger M = I \implies \det(M^\dagger M) = c^* c = |c|^2 = 1 \implies c = e^{i\theta}$$

therefore, the center and the quotient group are

$$Z(U(n)) = U(1), \quad U(n)/U(1) \simeq \text{SU}(n)$$

**Example 8.20.** Consider  $\text{SU}(n)$  and take  $M = cI$  since it is a subgroup of the above. Therefore

$$1 = \det M = c^n, \quad |c|^2 = 1 \implies c = e^{\frac{2\pi i k}{n}}$$

The center and the quotient are

$$Z(\text{SU}(n)) = \mathbb{Z}_n, \quad \text{SU}(n)/\mathbb{Z}_n \otimes U(1) \simeq U(n)$$

**Definition 8.21.** A curve  $\gamma$  passing through a point  $p$  is a smooth function

$$\gamma : I \rightarrow M, \quad t \mapsto \gamma(t)$$

where  $\gamma(0) = p$ ,  $I = [0, 1] \subset \mathbb{R}$  and  $M$  is manifold.

**Remark 8.22.** Considering a scalar function  $g : M \rightarrow \mathbb{R}$ , one can define the derivative of  $g$  along a curve  $\gamma$ . One evaluates  $g$  over  $\gamma$

$$g_\gamma : I \subset \mathbb{R} \rightarrow \mathbb{R}, \quad t \mapsto g \circ \gamma(t) = g(\gamma(t))$$

Taking a chart  $(U_\alpha, \varphi_\alpha) \in M$ , one assigns the coordinates such that

$$x_\alpha^\mu(t) = \varphi_\alpha(\gamma(t))$$

and one has

$$g \circ \varphi_\alpha^{-1} \circ \varphi_\alpha \circ \gamma(t) = g(\gamma(t))$$

**Definition 8.23.** Given a curve  $\gamma$  passing through a point  $p$ , and a scalar function  $g$ , the tangent vector  $t_p^{(\alpha)}$  in the chart  $(U_\alpha, \varphi_\alpha)$  is

$$d_t g_\gamma(t)|_{t=0} = \partial_{x_\alpha^\mu} g \, d_t x_\alpha^\mu|_{t=0} = \partial_{x_\alpha^\mu} g \, b_\alpha^\mu \equiv t_p^{(\alpha)} g$$

**Definition 8.24.** Consider the space of curves  $\gamma$  passing through a point  $p$

$$\Gamma_p = \{\gamma : \mathbb{R} \rightarrow M \mid \gamma(0) = p\}$$

Two curves are equivalent if the tangent vectors coincide for a given chart  $(U_\alpha, \varphi_\alpha)$  to which  $p$  belongs:

$$d_t(\varphi_\alpha \circ \gamma_1)|_{t=0} = d_t(\varphi_\alpha \circ \gamma_2)|_{t=0} \implies \gamma_1 \sim \gamma_2$$

The equivalence class

$$T_p M = \{\gamma_1, \gamma_2 \in \Gamma_p \mid \gamma_1 \sim \gamma_2\}$$

is called tangent space. It is a vector space with dimension  $\dim M = n$ . [r]

**Remark 8.25.** The tangent vector  $t_p^{(\alpha)}$  is a map from the algebra of smooth functions at  $p$  into itself

$$t_p^{(\alpha)} : C^\infty(U^\alpha) \rightarrow C^\infty(U^\alpha), \quad g_\gamma \mapsto t_p^{(\alpha)} g_\gamma$$

**Definition 8.26.** A vector field  $v$  over a manifold  $M$  is a linear operator from the space of smooth functions  $C^\infty(M)$  into itself, such that on each chart  $(U_\alpha, \varphi_\alpha)$  is a directional derivative

$$v = b_\alpha^\mu = \partial_{x_\alpha^\mu}$$

**Remark 8.27.** A change of coordinates, that is a change of charts

$$(U_\alpha, \varphi_\alpha) \rightarrow (U_\beta, \varphi_\beta), \quad p \in U_\alpha \cap U_\beta$$

is given by

$$d_t g_\gamma(0) = \partial_{x_\alpha^\mu} g \, d_t x_\alpha^\mu(0) = \partial_{x_\beta^\mu} g \, d_t x_\beta^\mu(0)$$

and one can rewrite

$$b_\alpha^\mu = d_t x_\alpha^\mu(0) = \partial_{x_\beta^\nu} x_\alpha^\mu \, d_t x_\beta^\nu(0) = \partial_{x_\beta^\nu} x_\alpha^\mu \, b_\beta^\nu$$

A vector field transforms with the Jacobian.

**Definition 8.28.** The Lie bracket of two vector fields  $v$  and  $w$  is the commutator

$$[v, w](f) = v(w(f)) - w(v(f))$$

where one has

$$[v, w] = v^\mu \partial_\mu (w^\nu \partial_\nu) - w^\mu \partial_\mu (v^\nu \partial_\nu)$$

**Remark 8.29.** The Lie bracket is invariant under change of coordinates

$$u = [v, w], \quad (U_\alpha, \varphi_\alpha) \rightarrow (U_\beta, \varphi_\beta), \quad p \in U_\alpha \cap U_\beta$$

Given

$$u = u^\nu \partial_\nu = v^\mu \partial_\mu (w^\nu \partial_\nu) - w^\mu \partial_\mu (v^\nu \partial_\nu), \quad u_\alpha^\nu = u_\beta^\sigma \partial_{x_\beta^\sigma} x_\alpha^\nu$$

one has

$$\begin{aligned} v_\alpha^\mu \partial_\mu w_\alpha^\nu &= v_\alpha^\mu \partial_\mu (w_\beta^\sigma \partial_{x_\beta^\sigma} x_\alpha^\nu) = v_\beta^\delta \partial_{x_\beta^\delta} x_\alpha^\mu \partial_{x_\alpha^\mu} (w_\beta^\sigma \partial_{x_\beta^\sigma} x_\alpha^\nu) \\ &= v_\beta^\delta \partial_{x_\beta^\delta} (w_\beta^\sigma \partial_{x_\beta^\sigma} x_\alpha^\nu) + v_\beta^\sigma w_\beta^\delta \partial_{x_\beta^\sigma} \partial_{x_\beta^\delta} x_\alpha^\nu \end{aligned}$$

doing the same for the second addendum, one finds that the double derivatives cancel.

**Example 8.30.** The orthogonal group  $O(n)$  is a Lie group so a manifold. To build the tangent space one needs a curve

$$O : \mathbb{R} \rightarrow O(n), \quad t \mapsto O(t), \quad O(0) = I$$

Starting from the group defining relation

$$O(t)O^\top(t) = I$$

taking the derivative in the parameter and then evaluating at  $t = 0$  one gets

$$0 = (\partial_t O)O^\top + O \partial_t O^\top = \partial_t O + \partial_t O^\top = A + A^\top \implies A = -A^\top$$

so the matrix  $A$  is anti-symmetric over  $\mathbb{R}$ . Therefore, the tangent space at the identity is

$$\mathfrak{o}(n) \equiv T_e O(n) = \{A \in M(n, \mathbb{R}) \mid A + A^\top = 0\}$$

**Example 8.31.** The unitary group  $U(n)$  is a Lie group. Taking the curve  $U(t)$  such that  $U(0) = I$ . From the group relation one has

$$U^\dagger U = I \implies 0 = \partial_t U^\dagger U + U^\dagger \partial_t U = \partial_t U^\dagger + \partial_t U$$

[r] The tangent space is composed of anti-hermitian matrices

$$\mathfrak{u}(n) = T_e U(n) = \{A \in M(n, \mathbb{C}) \mid A + A^\dagger = 0\}$$

**Example 8.32.** Consider  $SU(n)$  and take a curve  $U(t)$  such that  $U(0) = I$ . Since this is a subgroup of  $U(n)$  then the tangent space is made of anti-hermitian matrices. At an infinitesimal time  $t$  one can write the curve as

$$U(t) = I + tA + o(t)$$

From the determinant one has

$$\det U(t) = 1, \quad \det U(t) = 1 + t \operatorname{Tr}(A) + o(t)$$

Using the derivative, one has

$$\partial_t \det U(0) = 0 \implies \operatorname{Tr} A = 0$$

Therefore the tangent space is

$$\mathfrak{su}(n) = T_e SU(n) = \{A \in \mathfrak{u}(n) \mid \operatorname{Tr}(A) = 0\}$$

**Example 8.33.** The same can be done for  $SO(n)$  which has traceless anti-symmetric matrices.

**Example 8.34.** Considering  $GL(n, \mathbb{F})$  and curve  $M(t)$  such that  $M(0) = I$ , one can expand

$$M(t) = I + tA + o(t), \quad A \in M(n, \mathbb{F})$$

and then the tangent space is

$$\mathfrak{gl}(n, \mathbb{F}) = T_e GL(n, \mathbb{F}) = M(n, \mathbb{F})$$

Similarly for  $SL(n, \mathbb{F})$ , one has

$$\mathfrak{sl}(n, \mathbb{F}) = T_e SL(n, \mathbb{F}) = \{A \in M(n, \mathbb{F}) \mid \operatorname{Tr} A = 0\}$$

**Remark 8.35.** In all the cases above, the elements of the tangent space are not elements of the manifold.

**Definition 8.36.** A Lie algebra is a vector space  $\mathcal{G}$  over a field  $\mathbb{F}$  (typically  $\mathbb{R}$  or  $\mathbb{C}$ ) with a bracket (called Lie product)  $[\cdot, \cdot] : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$  such that

- linearity

$$[x, \alpha y + \beta z] = \alpha[x, y] + \beta[x, z], \quad [\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z], \quad \alpha, \beta \in \mathbb{F}, \quad x, y, z \in \mathcal{G}$$

- anti-symmetry

$$[x, y] = -[y, x], \quad \forall x, y \in \mathcal{G}$$

- Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \quad \forall x, y, z \in \mathcal{G}$$

**Example 8.37.** A vector space  $M(n, \mathbb{F})$  is a Lie algebra with

$$[M_1, M_2] = M_1 M_2 - M_2 M_1$$

**Remark 8.38.** Any associative algebra can be turned into a Lie algebra by imposing

$$[a_1, a_2] \equiv a_1 a_2 - a_2 a_1$$

**Definition 8.39** (Structure constants). Let  $\mathcal{G}$  be a Lie algebra and  $\{t_i\}$  a basis, that is  $x = x^i t_i$  with  $x^i \in \mathbb{F}$  for all  $x \in \mathcal{G}$ . The bracket of two elements is closed in  $\mathcal{G}$

$$[t_i, t_j] = c_{ij}^k t_k$$

where  $c_{ij}^k$  are called structure constants. Their properties are

- anti-symmetry  $c_{ij}^k = -c_{ji}^k$ ,

- Jacobi identity

$$c_{ij}^m c_{km}^l + c_{ki}^m c_{jm}^l + c_{jk}^m c_{im}^l = 0$$

- change of basis  $t'_i = S_i^j t_j$

$$(c')_{ij}^k = S_i^m S_j^n c_{mn}^l (S^{-1})_l^k$$

The first two follow from the properties of the Lie bracket.

## Lecture 12

[r] change notation of Lie algebra to Fraktur.

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**Definition 8.40.** A morphism of Lie algebras is a mapping  $\phi : \mathcal{G}_1 \rightarrow \mathcal{G}_2$  such that

$$\phi([x, y]_{\mathcal{G}_1}) = [\phi(x), \phi(y)]_{\mathcal{G}_2}, \quad \forall x, y \in \mathcal{G}_1$$

**Definition 8.41.** An isomorphism of Lie algebras is an invertible morphism.

**Definition 8.42.** A representation of a Lie algebra  $\mathcal{G}$  is a morphism  $\rho : \mathcal{G} \rightarrow \mathfrak{gl}(n, \mathbb{F})$ . An injective representation  $\rho$  is called faithful.

**Definition 8.43.** The adjoint representation,  $\text{ad}$ , of a Lie algebra  $\mathcal{G}$  is defined by

$$\text{ad}_x(y) \equiv [x, y], \quad \forall x, y \in \mathcal{G}$$

This representation has dimension  $\dim \mathcal{G}$ . The Jacobi identity guarantees that this is a representation.

**Remark 8.44.** Fixing a basis  $x = x^i t_i$ , from the regular representation one has

$$[t_i, t_j] = c_{ij}^k t_k$$

One can see that the structure constants give the adjoint representation

$$t_i \rightarrow [T_i]_j^k = -c_{ij}^k$$

**Example 8.45.** An example of a Lie algebra is  $\mathbb{R}^3$  treated as a vector space with the outer product. Its properties are

- $\mathbf{v}_1 \wedge \mathbf{v}_2 = -\mathbf{v}_2 \wedge \mathbf{v}_1$
- $\mathbf{v}_1 \wedge (\mathbf{v}_2 \wedge \mathbf{v}_3) + \mathbf{v}_2 \wedge (\mathbf{v}_3 \wedge \mathbf{v}_1) + \mathbf{v}_3 \wedge (\mathbf{v}_1 \wedge \mathbf{v}_2) = 0$
- the basis satisfies  $\mathbf{e}_i \wedge \mathbf{e}_j = \varepsilon_{ij}^k \mathbf{e}_k$ , where  $\varepsilon$  is the Levi-Civita tensor, it is the structure constants of this Lie algebra.

**Example 8.46.** Consider the Pauli matrices  $\sigma_i$ . Defining

$$t_i = -\frac{i}{2}\sigma_i, \quad \sigma_i \sigma_j = \delta_{ij} + i\varepsilon_{ij}^k \sigma_k, \quad \sigma_i \sigma_j - \sigma_j \sigma_i = 2i\varepsilon_{ij}^k \sigma_k$$

Then

$$[t_i, t_j] = \varepsilon_{ij}^k t_k$$

form an algebra. Note that the properties

$$\text{Tr } \sigma_i = 0, \quad \sigma_i + \sigma_i^\dagger = 0, \quad d = 2$$

suggest that the Pauli matrices are a basis for  $\mathfrak{su}(2)$ . Since  $(t_i)_j^k$  for  $i = 1, 2, 3$  and  $j, k = 1, 2$  then the elements  $t_i$  are not the adjoint representation because it should have dimension 3.

**Example 8.47.** Given the Levi-Civita tensor, the elements  $L_i$  such that

$$[L_i]_j^k = -\varepsilon_{ij}^k$$

form an algebra. In fact one has

$$[L_i, L_j] = \varepsilon_{ij}^k L_k, \quad i, j, k = 1, 2, 3$$

A generic vector  $A = a^i L_i$  is such that

$$A^\top = a_i L_i^\top \implies [L_i^\top]_j^k = [L_i]_k^j = -[L_i]_j^k$$

So they are anti-symmetric matrices. Moreover

$$\varepsilon_{ij}^k = 0 \implies \text{Tr } A = 0$$

This is  $\mathfrak{so}(3)$  algebra and  $L_i$  is the adjoint representation because  $\varepsilon_{ij}^k$  are structure constants.

**Remark 8.48.** The algebra  $\mathfrak{su}(2)$  has dimension  $n^2 - 1 = 3$  and the algebra has dimension  $\frac{1}{2}n(n-1) = 3$ . There is an isomorphism between the two algebras. The two have the same structure constants.

## 8.1 Lie groups as groups of transformations

Consider the circle  $S^1$ . It is a manifold. Take two charts, the first

$$(U, \varphi), \quad \varphi : S^1 \rightarrow \mathbb{R}^2, \quad \rho \mapsto x^2 + y^2 = 1$$

the second

$$(U', \varphi'), \quad \varphi' : S^1 \rightarrow \mathbb{C}, \quad \rho \mapsto e^{i\theta}$$

A generic rotation  $R(\theta)$  is a translation along the circle, so

$$R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$$

so rotations constitute a group. An infinitesimal shift is

$$R(\theta + \delta\theta) = R(\theta) + \delta\theta \partial_\theta R(0) + o(\delta\theta), \quad J \equiv \partial_\theta R(0)$$

Setting  $\theta = 0$  one gets

$$R(\delta\theta) = 1 + \delta\theta J + o(\delta\theta)$$

The expression above is also

$$R(\theta + \delta\theta) = R(\theta)R(\delta\theta) = R(\theta)[1 + \delta\theta J + o(\delta\theta)]$$

From these two relations one has

$$d_\theta R(\theta)|_{\delta\theta=0} = JR(\theta), \quad R(0) = 1 \implies R(\theta) = e^{J\theta}$$

One obtains an exponential map from group properties together with smoothness conditions.

For the chart  $(U', \varphi')$ , a generic element of the circle is  $e^{i\theta}$ . A rotation along the circle is  $e^{i\theta'}$ : therefore  $J = i$  and the rotation group is  $U(1)$ . For the chart  $(U, \varphi)$ , a generic rotation

$$R(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

So, letting  $\theta = 0$ , one gets

$$J = \partial_\theta R(0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

To prove that  $R(\theta) = e^{J\theta}$  one can see that  $J(-J) = I$  from which

$$R(\theta) = e^{J\theta} = \sum_{n=0}^{\infty} \frac{(J\theta)^n}{n!} = I \cos \theta + J \sin \theta$$

note that reordering the terms in an infinite series is allowed only if the series absolutely converges (see Riemann series theorem).

**Remark 8.49.** One can construct  $SO(2)$  and  $U(1)$  from the same exponential map and  $J$ . Both groups are manifolds and are  $S^1$  so they are isomorphic. Notice that these are groups, the example before involved Lie algebras.

**Definition 8.50.** The left and right translations over a group

$$Lg : h \mapsto Lg(h) = gh, \quad Rg : h \mapsto Rg(h) = hg$$

maps the group into itself. Consider some charts  $(U_\alpha, \varphi_\alpha)$  and assign some coordinates  $\alpha^\mu$  to  $h$ , one has

$$\varphi_\alpha : G \rightarrow \mathbb{R}^d, \quad \alpha^\mu = [\varphi_\alpha(g)]^\mu, \quad d = \dim G$$

Consider a second set of charts  $(U_\beta, \varphi_\beta)$  and assign  $\beta^\mu$  to  $g$ . The left translation is

$$Lg : \alpha^\mu \mapsto \varphi^\mu(\beta, \alpha)$$

Since the group is closed, then there is a set of charts  $(U_\gamma, \varphi_\gamma)$  where

$$\gamma^\rho = \varphi^\rho(\beta, \alpha)$$

This is the group product  $gh$  but with coordinates.

**Definition 8.51** (Generators of a Lie group). Consider  $g = e$  and consider a char  $(U, \varphi)$  such that  $\beta^\mu = 0$ . Performing a translation of  $h$  with  $g$  infinitesimally close to  $e$ , one has

$$\alpha^\mu \xrightarrow{Lg} \varphi^\mu(\delta\beta, \alpha) = \varphi^\mu(0, \alpha) + \partial_{\beta^\nu} \varphi^\mu|_{\beta=0} \delta\beta^\nu + o(\delta\beta^\nu) = \alpha^\mu + \partial_{\beta^\nu} \varphi^\mu|_{\beta=0} \delta\beta^\nu + o(\delta\beta^\nu)$$

Noting that

$$\partial_{\alpha^\nu} \alpha^\mu = \delta_\nu^\mu, \quad \partial_{\beta^\nu} \varphi^\mu = \partial_{\beta^\nu} \varphi^\rho \delta_\rho^\mu$$

one gets

$$\varphi^\mu(\delta\beta, \alpha) = \alpha^\mu + [\delta\beta^\nu \partial_{\beta^\nu} \varphi^\rho \partial_{\alpha^\rho}] \alpha^\mu + o(\delta\beta^\nu) = [1 + \delta\beta^\nu J_\nu(\alpha)] \alpha^\mu + o(\delta\beta^\nu)$$

Therefore

$$J_\nu(\alpha) = J_\nu^\rho \partial_{\alpha^\rho}, \quad J_\nu^\rho(\alpha) = \partial_{\beta^\nu} \varphi^\rho(\beta, \alpha)|_{\beta=0}$$

The generators  $J$  are vector fields over the group manifold, but are not elements of the group.

**Example 8.52.** Consider the general linear group  $GL(n, \mathbb{R})$  with

$$\varphi(\alpha, \lambda) \leftrightarrow N_i^k M_k^j$$

The generators are

$$J_{ij} = \partial_{M_{ij}}(NM)|_{M=0}$$

Setting  $N = I$  one gets

$$[J_{kl}]_{ij} = \partial_{M_{ij}} M_{kl} = \delta_i^k \delta_j^l$$

One can see that  $\det J = 0$  so  $J \notin GL(n, \mathbb{R})$ , but  $J \in \mathfrak{gl}(n, \mathbb{R})$ .

**Definition 8.53.** Let  $Lg$  be a left translation and  $v$  a vector field

$$v = v^\mu \partial_{\alpha^\mu}$$

The left translation of  $v$  is given by

$$dLg : v^\mu(\alpha) \mapsto w^\mu(\varphi(\beta, \alpha)) = \partial_{\alpha^\nu} \varphi^\mu(\beta, \alpha) v^\nu(\alpha)$$

**Definition 8.54.** The left-invariant vector fields are given by the relation

$$v^\mu(\varphi(\beta, \alpha)) = \partial_{\alpha^\nu} \varphi^\mu(\beta, \alpha) v^\nu(\alpha)$$

meaning  $v^\mu = w^\mu$ .

**Theorem 8.55** (First Lie's). The generators of a Lie group  $G$  are left and right invariant.

*Proof.* Starting from  $\alpha^\mu$  on the manifold, one uses  $\beta$  to translate to  $\gamma^\mu$ . One considers an infinitesimal shift by  $\varepsilon$  along  $\gamma^\mu$  to get to  $\gamma^\mu + \delta\gamma^\mu$ . This point can be reached from a different group element  $\beta + \delta\beta$ . So

$$\gamma^\mu = \varphi^\mu(\beta, \alpha)$$

Shifting both  $\gamma \rightarrow \gamma + \delta\gamma$  and  $\beta \rightarrow \beta + \delta\beta$  one gets

$$\varphi^\mu(\beta + \delta\beta, \alpha) = \gamma^\mu + \delta\gamma^\mu = \varphi^\mu(\varepsilon, \gamma), \quad \beta^\mu + \delta\beta^\mu = \varphi^\mu(\varepsilon, \beta)$$

When  $\varepsilon$  is infinitesimally close to the identity, then  $g = e$  from which

$$\begin{aligned} \beta^\mu + \delta\beta^\mu &= \varphi^\mu(0, \beta) + \varepsilon^\nu \partial_{\varepsilon^\nu} \varphi^\mu(0, \beta) + o(\varepsilon) \geq \delta\beta^\mu = \varepsilon^\nu \partial_{\varepsilon^\nu} \varphi^\mu(0, \beta) + o(\varepsilon) \\ \gamma^\mu + \delta\gamma^\mu &= \varphi^\mu(0, \gamma) + \varepsilon^\nu \partial_{\varepsilon^\nu} \varphi^\mu(0, \gamma) + o(\varepsilon) \implies \delta\gamma^\mu = \varepsilon^\nu \partial_{\varepsilon^\nu} \varphi^\mu(0, \gamma) + o(\varepsilon) \end{aligned}$$

These become

$$\delta\beta^\mu = J^\mu_\nu(\beta) \varepsilon^\nu, \quad \delta\gamma^\mu = J^\mu_\nu(\gamma) \varepsilon^\nu$$

From which (not rigorously follows)

$$\delta\gamma = \delta\beta J^{-1} J \implies J^\sigma_\nu(\beta) \partial_{\beta^\sigma} \gamma^\mu = J^\mu_\nu(\gamma)$$

and thus the generators are left invariant. In this way one can calculate the generators along the manifold.  $\square$

## Lecture 13

**Theorem 8.56** (Second Lie's). Given a Lie group  $G$ , the generators form a Lie algebra (which is a sub-algebra of the algebra of vector fields over  $M$ )

$$[J_\mu, J_\nu](\gamma) = c_{\mu\nu}^\rho J_\rho(\gamma)$$

*Proof.* Consider the definition

$$\psi = J^{-1}$$

and take

$$\partial_{\beta^\sigma} \gamma^\mu = J^\mu_\nu(\gamma) \psi^\nu_\sigma(\beta)$$

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from the first Lie's theorem. Its second derivative is

$$\partial_{\beta\rho}^2 \gamma^\mu = \partial_{\beta\rho} [J^\mu_\nu(\gamma) \psi^\nu_\sigma(\beta)] = \partial_{\beta\sigma} [J^\mu_\nu(\gamma) \psi^\nu_\rho(\beta)]$$

The second equality comes from the symmetry of the second derivative. Therefore

$$\begin{aligned} J^\mu_\nu(\gamma) [\partial_{\beta\rho} \psi^\nu_\sigma(\beta) - \partial_{\beta\sigma} \psi^\nu_\rho(\beta)] &= -[\partial_{\beta\rho} J^\mu_\nu(\gamma)] \psi^\nu_\sigma(\beta) + [\partial_{\beta\sigma} J^\mu_\nu(\gamma)] \psi^\nu_\rho(\beta) \\ &= -[\partial_{\beta\rho} \gamma^\lambda \partial_{\gamma\lambda} J^\mu_\nu(\gamma)] \psi^\nu_\sigma(\beta) + [\partial_{\beta\sigma} \gamma^\lambda \partial_{\gamma\lambda} J^\mu_\nu(\gamma)] \psi^\nu_\rho(\beta) \\ &= -[J^\lambda_\alpha(\gamma) \psi^\alpha_\rho(\beta) \partial_{\gamma\lambda} J^\mu_\nu(\gamma)] \psi^\nu_\sigma(\beta) \\ &\quad + [J^\lambda_\alpha(\gamma) \psi^\alpha_\sigma(\beta) \partial_{\gamma\lambda} J^\mu_\nu(\gamma)] \psi^\nu_\rho(\beta) \\ &= \psi^\nu_\sigma(\beta) \psi^\alpha_\rho(\beta) [-(J^\lambda_\alpha \partial_{\gamma\lambda} J^\mu_\nu) + (J^\lambda_\nu \partial_{\gamma\lambda} J^\mu_\alpha)](\gamma) \\ &= \psi^\nu_\sigma(\beta) \psi^\alpha_\rho(\beta) [J_{\alpha\nu}, J_\sigma] \end{aligned}$$

Multiply from the left by  $JJ$  to cancel  $\psi\psi$  of  $\beta$  and multiplying by  $\psi$  to cancel  $J(\gamma)$  one gets

$$\psi(\gamma) [J \partial J - J \partial J](\gamma) = J(\beta) J(\beta) [\partial\psi - \partial\psi](\beta)$$

There is a complete separation of coordinates. The only way this is satisfied is that the two expressions are the same constant. This means that the left-hand side is

$$c_{\mu\nu}{}^\rho = [J^{-1}]^\rho_\lambda [J^\sigma_\mu \partial_{\gamma\sigma} J^\lambda_\nu - (\mu \leftrightarrow \nu)]$$

□

**Remark 8.57.** The generators  $J$  form a basis among left and right invariant vector fields. The theorem states that the structure constants do not depend on the point  $p \in G$ , but are global. The generators are defined everywhere on a group. The vector field from  $J$  is non singular because the Jacobian must exist. [r]

**Definition 8.58** (Lie algebra of a Lie group). The Lie algebra  $\text{Lie}(G)$  of a Lie group  $G$  is the tangent space  $T_e G$  considered as a vector space.

**Proposition 8.59.** Taking a left-invariant field to translate the tangent space  $T_e G$  to another point

$$v^\mu(\beta) = \partial_{\alpha\nu} \varphi^\mu(\beta, 0) v^\nu(0) \equiv \bar{J}^\mu_\nu(\beta) v^\nu(0)$$

The generators  $\bar{J}$  are the base for left-invariant fields, while  $J$  for right-invariant fields. The new vector space  $T_p G$  generated from the vector  $v^\mu(\beta)$  with  $\beta^\mu$  coordinates of  $p$  is defined everywhere because the Jacobian must exist and be non-singular. The Lie bracket

$$[v_1(\beta), v_2(\beta)]$$

is the same as the Lie bracket

$$[v_1(0), v_2(0)] \in T_e G$$

The vector spaces are different, but the Lie brackets are the same.

**Example 8.60.** The special orthogonal group  $\text{SO}(2)$  has elements

$$R(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

the function that couples two group elements is

$$\varphi(\alpha, \beta) = R(\alpha)R(\beta) = R(\alpha + \beta)$$

The generator is

$$J = \partial_\alpha \varphi(0, \beta)$$

Starting from the identity and  $T_e G$  one imposes  $\beta = 0$  and gets

$$J = \partial_\alpha R(0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The algebra is one-dimensional because there is only one generator. The commutator is null  $[J, J] = 0$  because the group is abelian.

For this group, one also has

$$J^\top + J = 0$$

Taking  $O \in O(2)$  then

$$(I + \delta\theta J)^\top (I + \delta\theta J) = I \iff I + \delta\theta(J^\top J) + o(\delta\theta) = I \implies 0 + o(\delta\theta) = 0$$

So the generator  $J$  is also the generator of  $O(2)$ . [r]

**Example 8.61.** Consider the Lie group  $(\mathbb{R}, +)$ . It is an abelian group, so the left and right translations are the same. A translation of  $x \in G$  by  $a \in G$  is

$$x \mapsto x + a = \varphi(\alpha, \beta), \quad \alpha = x, \quad \beta = a$$

One can infer the generators from left-invariant vector fields. Consider a generic vector field

$$v = v^0(x) \partial_x$$

Imposing the left-invariant condition, one gets

$$v^0(\varphi(\beta, \alpha)) = \partial_{\alpha^0} \varphi^0(\beta, \alpha) v^0(\alpha) \implies v^0(x + a) = 1 v^0(x) \implies v^0(x) = k = \text{const.}$$

So the vector field is

$$v = k \partial_x$$

Translations in one dimension admit a one-dimensional Lie algebra given by one generator: the momentum operator above.

**Example 8.62.** Consider  $G = (R^+, \cdot)$  and  $x, a \in G$ . The translation is

$$x \mapsto ax = \varphi(a, x)$$

To build a left-invariant vector field one has

$$v^0(x + a) = \partial_x \varphi(a, x) v^0(x) = a v^0(x) \implies v^0(x) = kx$$

the solution is a first-degree polynomial. Therefore the generator is

$$v = kx \partial_x$$

This group is a dilation over  $\mathbb{R}^+$ .

## 8.2 Exponential map

One has seen that generator  $J_\mu \in \mathcal{G}$  is an element of the tangent space, but not of the group  $J_\mu \notin G$ . Close at the identity one has

$$(e + \varepsilon \alpha^\mu J_\mu) \in G$$

Taking

$$g = e + \frac{\varepsilon}{k} \alpha^\mu J_\mu$$

then  $gg \in G$  and  $ggg \in G$  etc. Also

$$\lim_{k \rightarrow \infty} g^k = \lim_{k \rightarrow \infty} \left(1 + \frac{\varepsilon}{k} \alpha^\mu J_\mu\right)^k = e^{\varepsilon \alpha^\mu J_\mu} \in G$$

The mapping between the algebra and the group is valid for any matrix group because the exponential map admits a representation in terms of a Taylor expansion

$$M = \exp(m) = \sum_{n=0}^{\infty} \frac{m^n}{n!}$$

**Proposition 8.63** (Properties). A few properties

- the exponential map is smooth  $C^\infty$  and invertible,
- it holds

$$e^{n+m} = e^m e^n \iff [m, n] = 0$$

and in general one uses the Baker–Campbell–Hausdorff formula

$$e^m e^n = \exp \left[ m + n + \frac{1}{2}[m, n] + \frac{1}{12}([m, [m, n]] + [[m, n], n]) + \dots \right]$$

The importance of  $m, n$  being in a Lie algebra, they are generators, then the commutators are given by the structure constants: the same structure repeats itself.

- For a change of basis  $S$  in a vector space then

$$e^{S^{-1}mS} = S^{-1}e^m S$$

- It holds

$$\det e^m = e^{\text{Tr } m}$$

This implies that a group with elements that have determinants one has traceless generators in its algebra.

**Theorem 8.64** (Taylor). Given a Lie group  $G$  and its Lie algebra  $\mathcal{G}$  of generators  $J_\mu$ , and given a parametrization such that  $\alpha^\mu(e) = 0$ , then the elements  $\alpha^\mu J_\mu \in \mathcal{G}$  define one-dimensional abelian subgroups

$$g = e^{\lambda \alpha^\mu J_\mu}, \quad \lambda \in \mathbb{R}$$

**Theorem 8.65.** Every element of the group  $G$  connected with the identity  $e$  through a continuous curve belong to a single abelian subgroup as defined above.

**Remark 8.66.** From the generators and the exponential map one can infer if the group is compact or not.

## Lecture 14

**Example 8.67.** One can build an entire group through the exponential map. Consider  $U \in \text{SU}(2)$ ,  $U^\dagger U = I$ ,  $\det U = 1$  and  $u \in \mathfrak{su}(2)$ . Then

$$U = e^u$$

It follows

$$I = \det U = \det e^u = e^{\text{Tr } u} \implies \text{Tr } u = 0$$

The unitary condition on  $U$  implies that  $u$  is anti-hermitian. A generic parametrization of  $2 \times 2$  complex matrices is

$$u(\alpha_i) = \frac{i}{2}(\alpha_0 I + \alpha_i \sigma^i), \quad i = 1, 2, 3$$

where  $\sigma^i$  are the Pauli matrices. Taking the trace, one gets

$$\text{Tr}[u(\alpha_i)] = i\alpha_0$$

Imposing the trace, one gets  $\alpha_i = 0$ . One has a larger and a smaller algebra [r]. So

$$I(\alpha_i) = \exp \left[ \frac{i}{2}\alpha_0 I + \frac{i}{2}\alpha_i \sigma^i \right] = e^{\frac{i}{2}\alpha_i} e^{\frac{i}{2}\alpha_i \sigma^i}$$

The first factor is a phase, so an element of  $U(1)$ , while the second is an element of  $\text{SU}(2)$ . [r] One can expand the second term to get

$$e^{\frac{i}{2}\alpha_i \sigma^i} = \left[ I + \frac{i}{2}\alpha_i \sigma^i - \frac{1}{8}\alpha_i \sigma^i \alpha_j \sigma^j + \dots \right]$$

using the property  $\sigma^i \sigma^j = \delta^{ij} + i\varepsilon^{ij}_k \sigma^k$  one gets

$$e^{\frac{i}{2}\alpha_i \sigma^i} = \left[ I + \frac{i}{2}\alpha_i \sigma^i - \frac{1}{8}|\alpha|^2 - \frac{1}{8}\alpha_i \alpha_j \varepsilon^{ij}_k \sigma^k + \dots \right]$$

Whenever one encounters at least two Pauli matrices, one can use the identity above to get either the identity or one matrix fewer. Therefore

$$e^{\frac{i}{2}\alpha_i \sigma^i} = \left[ I \cos \frac{|\alpha|}{2} + i \hat{\alpha}_i \sigma^i \sin \frac{|\alpha|}{2} \right] = U$$

This matrix can be easily calculated by a computer. This is the fundamental representation of  $SU(2)$ . One can study the underlying manifold. From the unit vector  $\hat{\alpha}_i$  one gets

$$\hat{\alpha}_1^2 + \hat{\alpha}_2^2 + \hat{\alpha}_3^2 = 1$$

that is the two-sphere  $S^2$ , which has dimension 2. The group  $SU(2)$  has dimension three, so one has to study  $|\alpha|$  also. When  $|\alpha| = 0$  then  $U = I$ . Similarly, when  $|\alpha| = 2\pi$  one has  $U = -I$ . When  $|\alpha| = 4\pi$  one has  $U = I$ . The parameter  $|\alpha|$  is an angle so the whole manifold is  $S^3$ . One can also see this by reparametrizing the parameter to see that it is another colatitude.

One can do the construction of the elements of  $SO(3)$ . The algebra has anti-symmetric matrices. Consider

$$M \in SO(3), \quad M = e^{\alpha_i L^i} = \exp \begin{bmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{bmatrix}$$

Using the property

$$[\alpha_i L^i]^3 = -|\alpha|^2 \alpha_i L^i$$

one has

$$\begin{aligned} M &= I + |\alpha| \hat{\alpha}_i L^i + \frac{1}{2} |\alpha|^2 (\hat{\alpha}_i L^i)^2 - \frac{1}{6} |\alpha|^3 \hat{\alpha}_i L^i + \dots \\ &= I + \sin |\alpha| \hat{\alpha}_i L^i + (1 - \cos |\alpha|) (\hat{\alpha}_i L^i)^2 \end{aligned}$$

The odd powers are rearranged into the sine, while the even powers are arranged into  $L^i$ , not the identity. The parameters  $\hat{\alpha}_i$  define again a two-sphere  $S^2$ . When  $|\alpha| = 0$ , then  $M = I$ . When  $|\alpha| = 2\pi$ , then  $M = I$ . One has

$$M(\hat{\alpha}, |\alpha| = \pi) = I + 2(\hat{\alpha}_i L^i)^2 = M(-\hat{\alpha}, |\alpha| = \pi)$$

One can identify anti-podal points. At  $|\alpha| = \pi$  one has  $S^2/\mathbb{Z}_2$  and not  $S^2$ . The global manifold is  $S^3/\mathbb{Z}_2$ .

Therefore

$$SU(2) \simeq S^3, \quad SO(3) \simeq S^3/\mathbb{Z}_2, \quad SU(3) \simeq SU(2)/\mathbb{Z}_2$$

The algebra of both is built at the identity and it is the same. The exponential map gives different groups.

**Example 8.68.** Consider  $SU(3)$ , the symmetry group of the strong force. Its generators are proportional to the Gell-Mann matrices  $\lambda_a$

$$t_a = -\frac{i}{2} \lambda_a, \quad \lambda_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

also

$$\lambda_6 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Conventions for the generators  $t_a$  may vary. The generators are eight as it is the dimension of the group.

**Example 8.69.** One can build the generators for the pseudo-orthogonal group  $O(p, q)$ ,  $SO(p, q)$ . The metric is

$$\eta_{\mu\nu} = \text{diag}(-1, \dots, -1, 1, \dots, 1)$$

each  $p$  and  $q$  times. For  $\Lambda \in O(p, q)$  one has

$$\Lambda^\top \eta \Lambda = \eta$$

To find the generators one has to look at the Lie algebra which is the tangent space at the identity. So

$$(I + \lambda^\top) \eta (I + \lambda) = \eta, \quad \Lambda = e^\lambda$$

Using  $\eta = \eta^{-1}$  one has

$$\lambda^\top \eta + \eta \lambda = 0 \implies \eta^{-1} \lambda^\top = -\lambda \eta^{-1}$$

so  $\lambda \eta^{-1}$  is anti-symmetric. To find the generators one can use this most generic anti-symmetric matrix

$$(J_{\rho\sigma})^\mu{}_\nu = (\delta_\rho{}^\mu \delta_\sigma{}^\lambda - \delta_\rho{}^\lambda \delta_\sigma{}^\mu) \eta_{\lambda\nu}$$

The indices  $\rho\sigma$  parametrize the tangent space of the generators. The commutator of two generators defines the structure constants giving the algebra

$$[J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\mu\rho} J_{\nu\sigma} + \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\nu\sigma} J_{\mu\rho}$$

This is the algebra of the Lorentz group.

**Example 8.70.** The group  $G = (\mathbb{R}, +)$  of translations over  $\mathbb{R}$  has generators  $J = \partial_x$  which is an infinitesimal translation. The exponential map leads to a finite translation

$$\exp(aJ)f(x) = \exp(a\partial_x)f(x) = \left[1 + a\partial_x + \frac{1}{2}a^2\partial_x^2 + \dots\right]f(x) = f(x+a)$$

In this way one can see how  $J$  generates momentum.

### 8.3 Global properties of Lie groups

**Definition 8.71** (Connectedness). A topological space is arc-wise connected if any two points on a manifold  $M$  can be connected by a continuous curve.

**Definition 8.72.** A Lie group  $G$  is simply connected if all loops drawn on the manifold  $G$  are contractible.

**Definition 8.73** (Compactness). If a manifold  $M$  can be embedded in a higher-dimensional flat space (like  $\mathbb{R}^d$ ), then  $M$  is compact if there exists a ball of finite radius containing such manifold.

For groups of matrices, one can infer compactness if the range of parameters is finite.

**Example 8.74.** Consider  $U(n)$ ,  $U^\dagger U = I$ , then

$$(U_j^i)^* U_k^j = \delta_k^i \implies \delta_i^i = n = \sum_{ij} |U_j^i|^2$$

The group is compact because the sum is finite.

**Example 8.75.** The group  $SU(2) \simeq S^3$  is compact and simply connected. The group  $SO(3) \simeq S^3/\mathbb{Z}_2$  is compact but not simply connected; the algebras are isomorphic  $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$ , they see the local structure, not the global properties.

**Definition 8.76.** Two groups  $G_1$  and  $G_2$  are locally isomorphic if there exists an isomorphic mapping  $w : \mathcal{U}_{e_1} \rightarrow \mathcal{U}_{e_2}$  among open neighbourhoods of the identity.

**Theorem 8.77** (Third Lie's). All groups  $G_i$  whose algebras  $\mathcal{G}_i$  are isomorphic, are locally isomorphic. Among all the groups, there is only one simply connected group  $G_0$  called universal cover. Given an isomorphism  $w_i : \mathcal{U}_{e_0} \rightarrow \mathcal{U}_{e_i}$ , then the kernel

$$\ker w_i = D_i$$

is a discrete group and  $G_i = G_0/D_i$ .

*Proof.* Proof of the second statement. If both groups  $G_1$  and  $G_2$  are simply connected and locally isomorphic, then they are globally isomorphic.  $\square$

**Example 8.78.** Consider  $\mathrm{SO}(3) \simeq \mathrm{SO}(2)/\mathbb{Z}_2$ , then  $\mathrm{SO}(3) = G_i$ ,  $\mathrm{SO}(2)$  is the simply connected universal cover  $G_0$ , and  $\mathbb{Z}_2 = D_i$ .

**Example 8.79.** The group of translations  $(\mathbb{R}, +)$  is simply connected because loops are just points. The group  $(\mathbb{Z}, +)$  is an invariant subgroup also called  $\mathbb{Z}$ :

$$(\mathbb{R}, +)/\mathbb{Z} \simeq S^1 \simeq \mathrm{U}(1)$$

So a circle which is not simply connected.

**Example 8.80.** Consider the group  $\mathrm{GL}(n, \mathbb{R})$ . The set of determinants of these matrices is discontinuous because  $\det M \neq 0$ . The set is not connected. There are two disjoint sets. The identity is the identity matrix and determinant 1. Exponentiating the algebra, one only gets  $\mathrm{SL}(n, \mathbb{R})$  (that is the set with  $\det M > 0$ ), because it is connected with the identity. To recover the other half, one considers

$$Mg, \quad g \in \mathrm{SL}(n, \mathbb{R}), \quad \det M < 0, \quad M^2 = I$$

Therefore the relation one obtains is

$$\mathrm{GL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{R}) \simeq \mathbb{Z}_2$$

not related to the previous theorem. This is similar to

$$\mathrm{O}(n)/\mathrm{SO}(n) \simeq \mathbb{Z}_2$$

**Lemma 8.81.** If  $G_1 \simeq G_2$  then also the Lie algebras  $\mathcal{G}_1 \simeq \mathcal{G}_2$  are isomorphic (but not viceversa).

## Lecture 15

**Definition 8.82** (Haar measure). The Haar measure is a left, or right, invariant integration measure

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$$\int_G d\mu_R(g) f(gh) = \int_G d\mu_R(gh) f(gh) = \int_G d\mu_R(g) f(g)$$

**Remark 8.83.** If a group is abelian then left and right measure are the same  $d\mu_L = d\mu_R$ .

If a group is compact, then the volume is

$$\mathrm{vol}(G) = \int_G d\mu_R(g) < \infty, \quad d\mu_R = d\mu_L = d\mu_H$$

The measure can be defined from a parametrization

$$d\mu_L(g) = \mu_L(g) d^d \alpha = \mu_L(\varphi(\beta, \alpha)) d^d \varphi = \mu_L(\varphi) \det(\partial_\alpha \varphi) d^d \alpha$$

Taking  $g = e$  one defines

$$d\mu_L = [\det \partial_\alpha \varphi(\beta, 0)]^{-1} d^d \alpha$$

One may notice that  $\mu_L(\alpha = 0)$  are constants.

**Example 8.84.** Consider  $\mathrm{SO}(2)$  which is parametrized by a single angle

$$\varphi(\theta_1, \theta_2) = \theta_1 + \theta_2$$

The jacobian and the measure are

$$\partial_{\theta_1} \varphi = 1, \quad d\mu_R = d\mu_L(\theta) = \frac{d\theta}{2\pi}$$

**Example 8.85.** Consider  $GL(n, \mathbb{R})$ . One has

$$\varphi(M_1, M_2) = M_1 M_2, \quad \det \partial_M \varphi(M, N) = \det(\partial_{M_{ij}} M^{nl} N_{lm}) = \det(IN) = \det N$$

the measure is

$$d\mu_L(M) = \frac{1}{\det M} \prod_{ij} dM_{ij}$$

**Example 8.86.** Consider  $SL(n, \mathbb{R})$ ,  $\det M = 1$ , then

$$d\mu_L(M) = \frac{1}{\det M} \prod_{ij} dg_{ij} \delta(1 - \det M)$$

**Theorem 8.87** (Orthogonality and completeness). The following relations are called Peter–Weyl’s. It holds

$$\int_G d\mu_H(g) [\rho_\mu(g)]^i_j [\rho_\nu(g^{-1})]^k_l = \frac{1}{d_\mu} \delta_{\mu\nu} \delta^k_l$$

Also

$$\int_G d\mu_H(g) \chi_\mu(g) \chi_\nu^*(g) = \delta_{\mu\nu}$$

Similarly, given

$$\int_G d\mu_H(g) f(g) \delta(g, h) = f(h)$$

then

$$\sum_\mu d_\mu \operatorname{Tr}[\rho_\mu(g_1) \rho_\mu(g_2^{-1})] = \delta(g_1, g_2)$$

Likewise, given

$$\hat{\delta}(g_1, g_2) = \delta(hg_1h^{-1}, hg_2h^{-1})$$

one has

$$\sum_\mu \chi_\mu(g_1) \chi_\mu^*(g_2) = \hat{\delta}(g_1, g_2)$$

## 8.4 Structure of Lie algebras

**Definition 8.88.** A Lie subalgebra  $\mathcal{H} \subseteq \mathcal{G}$  is a subspace which is closed under the Lie bracket, namely

$$[h_1, h_2] \in \mathcal{H}, \quad \forall h_1, h_2 \in \mathcal{H}$$

**Definition 8.89.** A Lie subspace  $\mathcal{H} \subseteq \mathcal{G}$  is an invariant (also said ideal) if

$$[\mathcal{H}, \mathcal{G}] = \mathcal{H}$$

**Definition 8.90.** A Lie subalgebra is the direct sum  $\mathcal{G}_1 \oplus \mathcal{G}_2$  if it is a direct sum as a vector space  $V_1 \oplus V_2$  and  $[\mathcal{G}_1, \mathcal{G}_2] = 0$  (so  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are ideals).

**Example 8.91.** Consider euclidean four-space’s algebra  $\mathfrak{so}(4)$  (or equivalently Minkowski space after a Wick’s rotation, that is with the imaginary time formalism). The generators obey

$$[L_{\mu\nu}, L_{\rho\sigma}] = \delta_{\mu\sigma} L_{\nu\rho}$$

[r] remembering that the two indices are motivated by  $\Lambda = w^{\mu\nu} \lambda_{\mu\nu}$  where  $w$  is the infinitesimal parameter and  $\lambda$  is a generator. Inside there must be a  $\mathfrak{so}(3)$  subalgebra

$$M_i = \frac{1}{2} \varepsilon_{ijk} L^{jk}, \quad [M_i, M_j] = \varepsilon_{ij}^{\quad k} M_k, \quad i, j, k = 1, 2, 3$$

The algebra has 6 dimensions

$$\dim \mathfrak{so}(4) = \frac{n(n-1)}{2}$$

The boosts are

$$N_i = L_{i4}, \quad [N_i, N_j] = \varepsilon_{ijk} M^k$$

meaning that they do not form a subalgebra. Considering the change of basis

$$J_i = \frac{M_i + N_i}{2}, \quad K_i = \frac{M_i - N_i}{2}$$

one has

$$[J_i, J_j] = \varepsilon_{ijk} J^k, \quad [K_i, K_j] = \varepsilon_{ijk} K^k, \quad [J_i, K_j] = 0$$

In this way, one has found two ideals, both being  $\mathfrak{so}(3)$ . Therefore

$$\mathfrak{so}(4) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3)$$

**Theorem 8.92.** Subalgebras exponentiate into subgroups. Ideals exponentiate into invariant subgroups.

**Remark 8.93.** Consider the algebra  $\mathfrak{u}(n)$ . A matrix in this algebra is anti-hermitian  $A^\dagger = -A$ . Consider

$$B = aA, \quad a \in \mathbb{C}$$

one has

$$B^\dagger = a^* A^\dagger = -a^* A \neq -B$$

So the algebra  $\mathfrak{u}(n)$  is a real Lie algebra:

$$a \in \mathbb{R} \implies aA \in \mathfrak{u}(n)$$

**Definition 8.94.** Given two vector spaces  $V_1$  and  $V_2$ , their tensor product  $V_1 \otimes V_2$  is defined from the Cartesian product  $V_1 \times V_2$  up to an equivalence relation

$$(\lambda v_1) \otimes v_2 = v_1 \otimes (\lambda v_2) = \lambda(v_1 \otimes v_2), \quad \forall v_1, v_2$$

**Definition 8.95.** Given a real vector space  $V$ , its complexification is  $V \otimes \mathbb{C}$ , that is

$$v \in V, \quad \lambda \in \mathbb{C} \implies \lambda v \in V \otimes \mathbb{C}$$

In this way one can complexify real Lie algebras.

**Example 8.96.** Consider  $\mathrm{SL}(2, \mathbb{R})$ . Its algebra  $\mathfrak{sl}(2, \mathbb{R})$  is made up of  $2 \times 2$  traceless matrices over  $\mathbb{R}$ . The parametrization is

$$m(\alpha) = \begin{pmatrix} \alpha_0 & \alpha_1 \\ \alpha_2 & -\alpha_0 \end{pmatrix}$$

From this parametrization one can construct the generators as

$$L_i = \partial_{\alpha_i} m(\alpha), \quad L_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The group can be recovered with the exponential map

$$M(\alpha) = e^{\alpha_i L_i} \in \mathrm{SL}(2, \mathbb{R}), \quad \alpha_i \in \mathbb{R}$$

To complexify, one can introduce three new generators for a total of six

$$J_i = L_i, \quad J_k = iL_{k-3}, \quad k = 4, 5, 6$$

One can either consider  $\alpha_i J_i$  with  $\alpha_i \in \mathbb{R}$  or  $\alpha_i L_i$  with  $\alpha_i \in \mathbb{C}$ . Both ways give the Lie algebra  $\mathfrak{sl}(2, \mathbb{C}) \simeq \mathfrak{sl}(2, \mathbb{R})_{\mathbb{C}}$ .

One may note that

$$t_3 = iL_0, \quad t_1 = \frac{i}{2}(L_1 + L_2), \quad t_2 = -\frac{i}{2}(L_1 - L_2)$$

corresponding to the generators  $t_i = \frac{i}{2}\sigma_i$  of  $\mathfrak{su}(2)$ . Therefore, one has found

$$\mathfrak{sl}(2, \mathbb{R}) \simeq \mathfrak{su}(2), \quad \mathfrak{sl}(2, \mathbb{R})_{\mathbb{C}} \simeq \mathfrak{sl}(2, \mathbb{C}) \simeq \mathfrak{su}(2)_{\mathbb{C}}$$



**Definition 8.97.** A simple algebra does not admit ideal. The simple algebra exponentiate into a simple group connected to the identity.

**Definition 8.98.** A semi-simple algebra does not admit abelian ideals (since the ideal are  $\{0\}$  and  $\mathcal{G}$  itself).

**Definition 8.99.** An abelian algebra has  $[x, y] = 0$  for all  $x, y \in \mathcal{G}$ .

**Definition 8.100.** The derived algebra  $\mathcal{G}^1$  or  $D\mathcal{G}$  is a subalgebra of  $\mathcal{G}$  whose elements are all commutators  $[x, y]$  for all  $x, y \in \mathcal{G}$

$$D\mathcal{G} = [\mathcal{G}, \mathcal{G}] = \{z \in \mathcal{G} \mid \exists x, y \in \mathcal{G}, z = [x, y]\}$$

The derived algebra is an ideal of  $\mathcal{G}$  and if  $\mathcal{G}$  is abelian, then  $D\mathcal{G} = \mathbf{0}$ .

**Example 8.101.** One has  $D[\text{su}(2)] = \text{su}(2)$ .

Consider the group  $\text{ISO}(2)$ . A group element is

$$g(\theta, \mathbf{b}) = \begin{bmatrix} \cos \theta & \sin \theta & b_1 \\ -\sin \theta & \cos \theta & b_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

The generators are

$$x_R = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The commutators are

$$[x_1, x_2] = 0, \quad [x_R, x_1] = x_2, \quad [x_R, x_2] = -x_1$$

The subalgebra of translations is

$$t_2 = \{x_1, x_2\}$$

it is abelian and an ideal. Also

$$D[\text{iso}(2)] = t_2$$

Its derived algebra contains only zero.

**Definition 8.102.** A Lie algebra is solvable if the  $k$ -th derived algebra  $G^k = \{0\}$  for some  $k \in \mathbb{N}$ .

**Definition 8.103** (Semi-direct sum). Let  $\mathcal{H}$  and  $\mathcal{J}$  be two Lie algebras such that  $\mathcal{J}$  admits a representation on  $\mathcal{H}$ , that is a homomorphism from  $\mathcal{J}$  to the space of linear operators on  $\mathcal{H}$

$$\forall s \in \mathcal{J}, \exists \hat{s} : \mathcal{H} \rightarrow \mathcal{H}, \quad h \mapsto \hat{\Delta}(h)$$

The semi-direct sum algebra  $\mathcal{G} = \mathcal{J} +_s \mathcal{H}$  is  $\mathcal{J} \oplus \mathcal{H}$  as a vector space and it is equipped with a Lie bracket:

$$\forall g \in \mathcal{G}, \quad \exists s \in \mathcal{J}, h \in \mathcal{H} \mid g = s + h$$

and

$$\forall g_1, g_2, \quad [g_1, g_2]_{\mathcal{G}} = [s_1 + h_1, s_2 + h_2] = [s_2, s_2]_{\mathcal{J}} + [h_1, h_2]_{\mathcal{H}} + \hat{s}_1(h_2) - \hat{s}_2(h_1)$$

The condition  $g = s + h$  is important. The vector  $s$  and  $h$  are matrices that obey the previous vector linear combination, but also the Lie bracket must fulfilled: in particular the last two terms are the action of the algebra  $\mathcal{J}$  on elements of the algebra  $\mathcal{H}$ .