Theory and Phenomenology of Fundamental Interactions

Maso*

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Contents

Lecture 1

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The course will make more sense after taking QFT 1 and 2.

Exam. Computation of tree-level amplitude. [r] The oral part is comprised of 3 questions discussing theoretical aspects and fundamental aspects of the theory.

1 Introduction

The first part of the course is about the completion of the theoretical description of the electroweak sector of the Standard Model. Then Yukawa interaction linked to the [r] matrix: weak interactions are not diagonal with respect to flavor families. [r] Then discussion of phenomenology of the electroweak and Higgs sectors at the LHC.

The second part treats QCD, its gauge invariance; perturbative regime, the subtleties, universal divergences, properties of non-abelian gauge theories.

The last part deals with hadronic collision and its non-trivial description.

Further topics are neutrino masses, etc.

*https://github.com/M-a-s-o/notes

1.1 The bigger picture

The course deals with the Standard Model. One needs to understand how it must be viewed from a historical perspective and a modern perspective. To study fundamental interactions (excluding gravity) the popolar choice is quantum field theory: quantum electrodynamics, electroweak theory, quantum chromodynamics. The first is an abelian gauge theory, while the last two are non-abelian.

In these theories, the Lagrangian density is the fundamental object: it is Lorentz invariant, invariant under Poincaré transformation, and describes quantized fields. The particles are excitations of the fields. Every Lagrangian can be divided in the free term and the interaction term

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\mathrm{I}}$$

The free Lagrangian contains kinetic terms, the propagators [r]. The interaction Lagrangian contains the interaction terms, which are represented as vertices in Feynman diagrams. The quantum electrodynamics Lagrangian is

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(\mathrm{i}\not{\partial} - m)\psi - eQ\bar{\psi}\gamma^\mu \psi A_\mu$$

where e is the elementary charge and Q is the quantum number of charge.

If one assumes that fields and interactions to be truly fundamental:

- then the theory has to be unitary and predictive at all energies (so predict final results and amplitudes that do not exceed 1).
- The non-trivial link to quantum field theory is that the theory must be renormalizable. The lagrangian must be constrained: the coupling constants cannot have negative mass dimension. Since fundamental interactions are well described by gauge theories, gauge invariance is a fundamental requirement of the renormalizability of the theory. [r] Once one picks a gauge group, it must produce observable phenomena and one must keep abiding its rules to keep gauge invariance.

Effective Standard Model. The Standard Model [r] but it is not a complete theory. There is no natural candidate for dark matter, gravity is not accounted for, neutrino masses are not explained. The Model has to fail at some point, at some energy scale where a phenomenon cannot be described with the field content of the Model. Therefore, the Standard Model must be an effective theory (as opposed to a fundamental theory). Therefore, it is allowed to add non-renormalizable operators: terms in the Lagrangian with negative mass dimension. If one wants to understand the physics beyond the Standard Model from a bottom-up approach, this is a middle ground: one modifies the theory enough to compute phenomena [r]. A similar story happened when going from Fermi's four-interaction theory to the intermedia vector boson theory. The interaction term for Fermi theory is $\bar{\psi}\psi\bar{\psi}\psi$ with dimension six, so the coupling constant must have mass dimension -2. [r] One may add non-renormalizable operators built from Standard Model objects that respect its symmetry group. In this paradigm one may not use renormalizability and unitarity, so the predictions are valid up to some energies.

1.2 Weyl spinors

A massless Dirac field is made of two Weyl field. A massive Dirac spinor is made of left- and right-chiral components. A term like

$$\bar{\psi}_{\mathrm{L}} \psi_{\mathrm{R}}$$

is not invariant under $\mathrm{SU}(2)_{\mathrm{L}}$.

Lorentz group. The proper Lorentz group [r] has six generators. [r] through the exponential map as

$$R(\hat{e}, \theta) = \exp(-\mathrm{i}\theta \hat{e} \cdot \mathbf{J}) \,, \quad B(\hat{u}, \eta) = \exp(-\mathrm{i}\eta \hat{u} \cdot \mathbf{K})$$

where $\mathbf{J}$ are the generators of rotations and $\mathbf{K}$ are the generators of boosts. The explicit form of the generators can be obtained from infinitesimal transformations. For example

$$J_z = \mathrm{i}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_x = \mathrm{i}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

These are a fundamental representation of the Lorentz algebra SO(1, 3)

$$[J_i, J_j] = \mathrm{i}\varepsilon_{ijk}J_k\,, \quad [K_i, K_j] = -\mathrm{i}\varepsilon_{ijk}J_k\,, \quad [J_i, K_j] = \mathrm{i}\varepsilon_{ijk}K_k$$

The above algebra can be rewritten as

$$[M_{\mu\nu}, M_{\rho\sigma}] = \mathrm{i}(\eta_{\nu\rho}M_{\mu\sigma} + \cdots)$$

where $M^{\mu\nu}$ an anti-symmetric tensor such that

$$M^{0i} = K_i\,, \quad M^{ij} = \varepsilon_{ijk}J_k$$

[r] In order to label the representations one has to use mass and spin.

The Lorentz algebra can be decomposed in two other algebras

$$\mathrm{so}(1,3) \simeq \mathrm{su}(2) \oplus \mathrm{su}(2)$$

[r] In fact, one combines

$$J_k^{\pm} = \frac{1}{2}(J_k \pm \mathrm{i}K_k) \implies [J_i^+, J_j^-] = 0\,, \quad [J_i^{\pm}, J_j^{\pm}] = \mathrm{i}\varepsilon_{ijk}J_k^{\pm}$$

In order to label the possible elementary fields associated to Lorentz group (in general the Poincaré group) one needs two non-negative half-integers (s_1, s_2). For $(0,0)$ the transformation under each su(2) is trivial, so they do not transform, they are a singlet: it is a scalar field. The next representations are $(1/2, 0)$ and $(0, 1/2)$ for right-chiral Weyl spinor and left-chiral Weyl spinor. For a vector field one has $(1/2, 1/2)$. P 146 Schwartz

For right-chiral Weyl spinors u_{R}, the generators are

$$J_i^+ = \frac{1}{2}\sigma_i\,, \quad J_i^- = 0 \implies J_i = \frac{1}{2}\sigma_i\,, \quad \mathrm{i}K_i = \frac{1}{2}\sigma_i$$

Therefore, for a rotation, one has

$$R = \exp(-\mathrm{i}\boldsymbol{\theta}\cdot\mathbf{J}) = \exp\left(-\frac{1}{2}\mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}\right)$$

and for a boost

$$B = \exp(-\mathrm{i}\boldsymbol{\eta}\cdot\mathbf{K}) = \exp\left(-\frac{1}{2}\boldsymbol{\eta}\cdot\sigma\right)$$

Similarly for a left-chiral Weyl spinor. For a four-vector one has

$$J_i = J_i^+ + J_i^-$$

One may realize that there are two states that transforms as [r] The singlet component under rotation is A^0, while the triplet is A^i.

Parity. Under parity, the generators of rotations do not transform $J \to J$, while boosts do $K \to -K$. Also parity maps

$$(s_1, s_2) \to (s_2, s_1)$$

Therefore, left-chiral spinor becomes a right-chiral spinor, while a vector is still a vector.

Weyl spinors. Weyl spinors can be combined into a vector. Considering

$$\sigma_{\pm}^{\mu} = (I, \pm\boldsymbol{\sigma})$$

which is equivalent to the notation

$$\sigma^{\mu} = (I, \sigma^{i})\,, \quad \bar{\sigma}^{\mu} = (I, -\sigma^{i})$$

Therefore a vector is given by

$$u_{\mathrm{R}}^{\dagger}\sigma^{\mu}u_{\mathrm{R}}\,, \quad u_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}u_{\mathrm{L}}$$

These are bilinear objects in the spinor fields. One may use them to construct Lagrangians.

Lagrangian. One can build a Lagrangian from these fields by requiring that

$$u_{\mathrm{R,L}} \to \mathrm{e}^{\mathrm{i}\theta}u_{\mathrm{R,L}}$$

In fact, one may have

$$\mathcal{L}_{\mathrm{Weyl}} = \mathrm{i}u_{\mathrm{R,L}}^{\dagger}\sigma_{\pm}^{\mu}\,\partial_{\mu}u_{\mathrm{R,L}}$$

[r] The equations of motion are

$$\sigma^{\mu}\,\partial_{\mu}\psi_{\mathrm{R}} = 0\,, \quad \bar{\psi}^{\mu}\,\partial_{\mu}\psi_{\mathrm{L}} = 0 \implies (\partial_0 \pm \boldsymbol{\sigma}\cdot\nabla)\psi_{\mathrm{R,L}} = 0$$

Acting on the last equation with $(\partial_0 \mp \boldsymbol{\sigma}\cdot\nabla)$ on the left side, one gets a massless Klein–Gordon equation

$$\Box\,\psi_{\mathrm{R,L}} = 0$$

In momentum space one has [r]

$$\psi_{\mathrm{R,L}} = \hat{\psi}_{\mathrm{R,L}}(\mathbf{k})\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\,, \quad k^0 = |\mathbf{k}|$$

where the hat indicates the Fourier transform. In momentum space, the Weyl equations are

$$[k^0 \mp (\mathbf{k}\cdot\sigma)]\hat{\psi}_{\mathrm{R,L}} = 0$$

So the spinor is an eigenvector of the helicity operator

$$\frac{1}{2}\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{|\mathbf{k}|}$$

[r] from this, right-chiral spinors have positive helicity and similar.

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Weyl equations are important because electroweak theory violates parity: left- and right-chiralities transform differently.

If one relaxes the global U(1) symmetry, one can add a Lorentz scalar to the Lagrangian. Composing two left-chiral representations

$$\left(\frac{1}{2}, 0\right) \otimes \left(\frac{1}{2}, 0\right) = (0, 0) \oplus (1, 0)$$

One has a part that transforms as a vector under SU(2) and one as a scalar. The singlet combination can be extracted as

$$\varepsilon_{ab}u_{\pm}^{a}u_{\pm}^{b}\,, \quad \varepsilon_{ab} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

where each spinor has two complex entries. The above combination is Lorentz invariant. When checking that it is Lorentz invariant, the Levi–Civita tensor gives the determinant of some matrix, which is the exponential of the Pauli matrices, so it is 1. The combination is not zero because the spinor product is not symmetric: the spinor components are Grassmann odd fields

$$\{u^a_\pm, u^b_\pm\} = 0$$

Such a term in a Lagrangian is allowed

$$\mathcal{L}^\pm_{\text{Weyl}} = \mathrm{i} u^\dagger_\pm \sigma^\mu_\pm \partial_\mu u_\pm - \frac{1}{2} m[\varepsilon_{ab} u^a_\pm u^b_\pm + \text{h.c.}]$$

This is a Majorana mass term, it is bilinear in the fields. This term is not invariant under U(1) of the spinors. If one would like to implement a global or local transformation such the previous cannot have a mass term like the one above. [r] For charged (under some symmetry group) chiral fermions, one cannot have a Majorana mass term. In QED a charged fermion transforms non trivially under $\mathrm{U}(1)_{\mathrm{EM}}$.

For a non charged particle of any symmetry of the Standard Model, such a term is allowed, like right-handed neutrinos.

1.3 Dirac spinors

Under parity, the right- and left-chiral representations are mapped into one another. Parity invariance means the presence of both chiralities. A Dirac spinor is a combination of Weyl spinors (in the Weyl basis)

$$\psi = \begin{bmatrix} u_- \\ u_+ \end{bmatrix} = \begin{bmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{bmatrix}, \quad \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$$

This is a direct sum of representations because the two chiral representations do not mix. In the Dirac basis, the Dirac spinor is

$$\psi = \frac{1}{\sqrt{2}} \begin{bmatrix} u_+ + u_- \\ u_+ - u_- \end{bmatrix}$$

The course uses Weyl basis (also called chiral basis). In this representation, the Dirac matrices are

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}, \quad \gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} -I_2 & 0 \\ 0 & I_2 \end{bmatrix}$$

The projection operators are then

$$P_{\mathrm{L,R}} = \frac{1 \mp \gamma^5}{2}, \quad P_{\mathrm{L}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_{\mathrm{R}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad P^2_{\mathrm{L,R}} = P_{\mathrm{L,R}}, \quad P_{\mathrm{L}} P_{\mathrm{R}} = P_{\mathrm{R}} P_{\mathrm{L}} = 0$$

The Dirac conjugate spinors are

$$\bar{\psi} = \psi^\dagger \gamma^0, \quad \bar{\psi}_{\mathrm{L,R}} = \gamma^\dagger_{\mathrm{L,R}} \gamma^0 = \bar{\psi} \frac{1 \pm \gamma^5}{2}$$

notice how they have the opposite chirality of their non conjugate part. Some properties of the fifth gamma matrix are

$$\gamma_5 = \gamma_5^\dagger, \quad \gamma_5^2 = I, \quad \{\gamma_\mu, \gamma_5\} = 0, \quad \{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$$

The free Dirac lagrangian in terms of Weyl spinors is

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi = \bar{\psi}_{\mathrm{L}} \mathrm{i}\,\partial\!\!\!/ \psi_{\mathrm{L}} + \bar{\psi}_{\mathrm{R}} \mathrm{i}\,\partial\!\!\!/ \psi_{\mathrm{R}} - m(\bar{\psi}_{\mathrm{R}} \psi_{\mathrm{L}} + \bar{\psi}_{\mathrm{L}} \psi_{\mathrm{R}})$$

This mass term is different from Majorana's. This term couples the two chiral fields. It is invariant under $\mathrm{U}(1)_{\mathrm{EM}}$

$$\psi \to \mathrm{e}^{\mathrm{i}\alpha} \psi$$

but it is not invariant under different transformations of the chiral fields (like $\mathrm{SU}(2)_{\mathrm{L}}$). In electroweak theory, one needs to [r].

Vector theory. A vector theory does not distinguish the chiral parts of a field. As a consequence, parity is conserved. This is the reason why the fermionic current is a vector

$$\bar{\psi}\gamma^\mu\psi = \bar{\psi}_\mathrm{L}\gamma^\mu\psi_\mathrm{L} + \bar{\psi}_\mathrm{R}\gamma^\mu\psi_\mathrm{R}$$

which is invariant under

$$\psi_\mathrm{L,R} \to U(x)\psi_\mathrm{L,R}$$

Chiral theory. A chiral theory treats fields differently based on their chirality. It is parity violating. A Dirac mass term is not gauge invariant for chiral theories.

For QED and QCD a Dirac mass term is allowed; only the weak sector of the Standard Model creates problems. Under $\mathrm{SU}(2)_\mathrm{L}$ a left-chiral field transforms non trivially

$$\psi_\mathrm{L} \to U(x)\psi_\mathrm{L}$$

while a right-chiral field remains the same.

1.4 Conventions

A Dirac field may be written in a Fourier series as

$$\psi(t,\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\sum_s [u_s(\mathbf{k})a_s(\mathbf{k})\mathrm{e}^{-\mathrm{i}kx} + v_s(\mathbf{k})b^\dagger(\mathbf{k})\mathrm{e}^{ikx}]$$

where u and v are wave functions? [r]. The normalization of the free spinors is

$$\sum_s u_s(\mathbf{p})\bar{u}_s(\mathbf{p}) = \not{p} + m\,, \quad \sum_s v_s(\mathbf{p})\bar{v}_s(\mathbf{p}) = \not{p} - m$$

which solve

$$(\not{p}-m)u_s = 0\,, \quad \bar{u}_s(\not{p}-m) = 0\,, \quad (\not{p}+m)v_s = 0\,, \quad \bar{v}_s(\not{p}+m) = 0$$

[r] signs
A fermion propagator is

$$\frac{\mathrm{i}(\not{p}+m)}{p^2-m^2+\mathrm{i}\varepsilon}$$

The Feynman amplitude is

$$\mathrm{i}\mathcal{M} = \sum \text{Feynman diagrams}$$

The sum over quantum numbers of external particles is

$$\sum|\mathcal{M}|^2$$

The sum and average is instead

$$\overline{\sum}|\mathcal{M}|^2$$

[r] The tree-level cross-section is obtained

$$\mathrm{d}\sigma = \mathcal{F}\overline{\sum}|\mathcal{M}|^2\mathrm{d}\phi_n$$

where $\mathcal{F}$ is the flux factor and $\mathrm{d}\phi_n$ is the phase space

$$\mathrm{d}\phi_n = (2\pi)^4\delta^{(4)}(\Delta p^\mu)\prod_{i=1}^n[\mathrm{d}k_i]$$

where the Lorentz invariant phase space measure is

$$[\mathrm{d}k_i] = \frac{\mathrm{d}^3k_i}{(2\pi)^3 2E_i}\,, \quad E_i^2 = m_i^2 + |\mathbf{k}_i|^2$$

In general, the phase space contains also symmetry factors $\frac{1}{n!}$ if the final state particles are identical bosons.

The decay width of a particle M decaying is

$$\mathrm{d}\Gamma = \frac{1}{2M}\overline{\sum}|\mathcal{M}|^2 \,\mathrm{d}\phi_n$$

The total width is

$$\Gamma = \int \mathrm{d}\Gamma = \frac{1}{2M}\int \overline{\sum}|\mathcal{M}|^2 \,\mathrm{d}\phi_n$$

The Dirac traces is not explicitly written but a bracket is present [r] for Bhabha scattering one has [r] wrong. For computing the cross section one needs $\mathcal{M}^*$ and therefore

$$\sum|\mathcal{M}|^2 = [\cdots]$$

Four products of momenta get shortened

$$p_1^\mu p_{2\mu} = (12) = (p_1 \cdot p_2)$$

1.5 Quantum electrodynamics

One applies the gauge principle to go from a global symmetry to a local symmetry and make the lagrangian invariant. The interactions appear in the covariant derivative.

The QED lagrangian is

$$\mathcal{L}_{\mathrm{QED}} = \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - q\bar{\psi}\gamma^\mu\psi A_\mu$$

The free lagrangian

$$\mathcal{L}_0 = \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi$$

is invariant under global U(1). Making it local and requiring invariance, the fields transform as

$$\psi'(x) = \mathrm{e}^{\mathrm{i}\alpha(x)}\psi(x)\,, \quad A'_\mu = A_\mu - \frac{1}{q}\,\partial_\mu\alpha(x)$$

The covariant derivative is then

$$D_\mu = \partial_\mu + \mathrm{i}qA_\mu$$

So that the invariant lagrangian is

$$\mathcal{L} = \bar{\psi}(\mathrm{i}D\!\!\!\!/ - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

The derivative of the field does not transform the same way as the field, however the covariant derivative does

$$(\partial_\mu\psi)' \neq \mathrm{e}^{\mathrm{i}\alpha(x)}\partial_\mu\psi\,, \quad (D_\mu\psi)' = \mathrm{e}^{\mathrm{i}\alpha(x)}D_\mu\psi$$

Therefore the term $\bar{\psi}D_\mu\psi$ is gauge invariant. The field strength tensor is also gauge invariant.

Remark. The term

$$(F_{\mu\nu}F^{\mu\nu})^2$$

is not included because its mass dimension is 8 and the theory is not renormalizable.

Remark. The term $A^\mu A_\mu$ is not gauge invariant and it corresponds to a mass term, but the photon is massless.

Lecture 3

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2 Non-abelian gauge groups

The Standard Model is based in part on SU(n) symmetry groups. Consider $U \in \mathrm{SU}(n)$, it can be expressed as

$$U = \exp[\mathrm{i}\theta^a t^a]\,, \quad a = 1, \cdots, n^2 - 1$$

where t^a are the generators of the group and θ^a are real parameters. Elements of such group have the properties

$$UU^\dagger = U^\dagger U = I\,, \quad \det U = 1$$

The generators, which belong to the algebra, obey

$$t^a = (t^a)^\dagger\,, \quad 1 = \det e^{t^a} = e^{\mathrm{Tr}\, t^a} \implies \mathrm{Tr}\, t^a = 0$$

So the generators are hermitian traceless matrices. The generators are normalized to give

$$\mathrm{Tr}\big(t^a t^b\big) = T_R \delta^{ab}\,, \quad T_R = \frac{1}{2}$$

The commutation relations of the Lie algebra su(n) are

$$[t^a, t^b] = \mathrm{i} f^{abc} t^c$$

The coefficient f^{abc} are the structure constants of the Lie algebra. For non-abelian gauge groups, the commutator is not identically zero.

Exercise. Defining the matrix

$$\tau^{ab} \equiv \mathrm{i}[t^a, t^b]$$

where a, b do not label the components. Show that

$$\mathrm{Tr}\, \tau^{ab} = 0\,, \quad (\tau^{ab})^\dagger = \tau^{ab}$$

and that f is totally anti-symmetric and real.

In general, a d-dimensional representation of an algebra is a set of $d \times d$ matrices that satisfy the commutation relation

$$[T^a, T^b] = \mathrm{i} f^{abc} T^c$$

The number of T^a is the dimension of the Lie group.

The important representations are the fundamental, anti-fundamental and the adjoint representations. The fundamental representation is an N-dimensional representation. It acts on N-dimensional objects. The adjoint representation is given by the structure constants

$$(T^a)_{bc} = \mathrm{i} f^{bac}$$

where bc are the components of the matrix T^a. It is a $N^2 - 1$ dimensional representation.

Gauge symmetry. The matrix

$$U(x) = \mathrm{e}^{\mathrm{i}\theta^a(x) t^a}$$

depends on space-time coordinates. In the fundamental representation, the matrix acts on N dimensional objects. A spinor does not transform trivially, but as

$$\psi'(x) = U(x)\psi(x)$$

The derivative transforms as

$$(\partial_\mu \psi)' = \partial_\mu[U(x)\psi(x)] = (\partial_\mu U)\psi + U\, \partial_\mu \psi$$

but this is not favorable. One uses the covariant derivative

$$(D_\mu)_{ij} = \partial_\mu \delta_{ij} + \mathrm{i} g t^a_{ij} A^a_\mu$$

where the ij indices treats gauge group components. Therefore, there are $N^2 - 1$ gauge fields A^a_μ. The covariant transforms in the same way as the field

$$(D_\mu \psi)' = D'_\mu \psi' = U(x) D_\mu \psi$$

This implies that the gauge fields transform as

$$(t^a A^a_\mu)' = t^a A'^a_\mu = U(t^a A^a_\mu) U^{-1} + \frac{\mathrm{i}}{g} (\partial_\mu U) U^{-1}$$

To define the field strength tensor one goes by analogy with quantum electrodynamics

$$\mathrm{i} q F^{\mathrm{QED}}_\mu = [D^{\mathrm{QED}}_\mu, D^{\mathrm{QED}}_\nu] \,, \quad D^{\mathrm{QED}}_\mu = \partial_\mu + \mathrm{i} q A_\mu$$

In general, one defines

$$\mathrm{i} g t^a F^a_{\mu\nu} = [D_\mu, D_\nu]$$

This gives

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$

Under gauge transformation one has

$$(t^a F^a_{\mu\nu})' = t^a F'^a_{\mu\nu} = U(t^a F^a_{\mu\nu}) U^{-1}$$

The kinetic term for gauges boson is

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = -\frac{1}{2} \operatorname{Tr}(\mathbf{F}^{\mu\nu} \mathbf{F}_{\mu\nu}) \,, \quad \mathbf{F}_{\mu\nu} = t^a F^a_{\mu\nu} \,, \quad F^a_{\mu\nu} F^{a\mu\nu} = 2 \operatorname{Tr}(\mathbf{F}^{\mu\nu} \mathbf{F}_{\mu\nu})$$

One sees that

$$\mathbf{F}'^{\mu\nu} = U \mathbf{F}^{\mu\nu} U^{-1}$$

For a non-abelian gauge group the Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\psi}_j(x) [\mathrm{i} \not{D}_{jk} - m \delta_{jk}] \psi_k(x)$$

The indices jk are the one carried by the generators t^a_{jk}, not the four components of Dirac spinors. In the fundamental representation one has

$$a = 1, \dots, N^2 - 1 \,, \quad j, k = 1, \dots, N$$

The covariant derivative is

$$D^\mu_{jk} = \partial^\mu \delta_{jk} + \mathrm{i} g t^a_{jk} A^a_\mu$$

There are $N^2 - 1$ gauge boson fields A^a_μ and they are massless because massive terms $m^2 A^a_\mu A^{a\mu}$ are not guage invariant.

A term $(F^a_{\mu\nu} F^{a\mu\nu})^n$ is not renormalizable for $n > 1$.

The covariant derivative for the electroweak sector

$$D_\mu = \partial_\mu - \mathrm{i} q t^a A^a_\mu$$

while for the quantum chromodynamics sector is

$$D_\mu = \partial_\mu + \mathrm{i} g t^a A^a_\mu$$

[r] DC

The interaction term

$$\bar{\psi} \not{D} \psi \rightsquigarrow -g (\bar{\psi} t^a \gamma^\mu \psi) A^a_\mu$$

generates a vertex equal to

$$-\mathrm{i}gt^a_{ji}\gamma^\mu$$

In the kinetic part FF there are three-vertices

$$FF \rightsquigarrow gf^{abc}(\partial_\mu A^a_\nu)A^b_\mu A^c_\nu$$

which is called derivative vertex. In momentum space, the derivative is a momentum [r]

$$-gf^{abc}[(p_a - p_b)^\gamma \eta^{\alpha\beta} + (p_b - p_c)^\alpha \eta^{\beta\gamma} + (p_c - p_a)^\beta \eta^{\alpha\gamma}]$$

The charged boson have to interact with the carrier of the force, which is another boson.

In the kinetic part there are also four-vertices

$$FF \rightsquigarrow g^2 ffAAAA$$

2.1 History of the Standard Model

Fermi theory. Fermi theory is a theory of the electroweak sector. The Lagrangian is

$$\mathcal{L} = -\frac{G_\mathrm{F}}{\sqrt{2}} J^\mu J^\dagger_\mu \,, \quad J_\mu = \bar\psi_l \gamma_\mu(1-\gamma_5)\psi_{\nu_l} + \bar\psi_d\gamma_\mu(1-\gamma_5)\psi_u = L_\mu + H_\mu$$

The current contains both the leptonic part and the hadronic part. Examples of weak decays are the muon beta decay and the neutron beta decay.

One can compute tree-level total decay widths Γ and differential widths $\mathrm{d}_\Omega\Gamma$ from which one obtains the Fermi constant $G_\mathrm{F} = 1.16 \times 10^{-5}\,\mathrm{GeV}^{-2}$. The Lagrangian correctly describes these two processes [r].

Fermi theory is a V-A theory because bilinears of the types

$$V^\mu = \bar\psi_1 \gamma^\mu \psi_2$$

transforms like a polar vector while bilinears like

$$A^\mu \bar\psi_1 \gamma^\mu \gamma^5 \psi_2$$

transforms like an axial vector, or pseudo-vector. A theory of this type is in accordance with experiments. A V-A theory is maximally parity violating. Check that $V'^\mu = \Lambda^\mu{}_\nu V^\mu$ and $A'^\mu = \bar\Lambda^\mu{}_\nu A^\nu$ [r]. To see how it is maximally violating, one can see that from the Lagrangian one has terms like

$$A^\mu V_\mu \to -A^\mu V_\mu$$

Processes like neutrino deep-inelastic scattering. For example $\bar\nu_e u \to de^+$ one has a differential cross section

$$\mathrm{d}_\Omega \sigma = \frac{G_\mathrm{F}^2}{8\pi^2}\frac{s}{4}(1+\cos\theta)^2 \,, \quad s = (p_1+p_2)^2$$

but this cross section implies that the scattering matrix is not unitary.

Therefore Fermi theory is not renormalizable because the coupling constant has negative mass dimensions and it is not unitary. To fix the second problem one introduces a vector gauge boson.

Lecture 4

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Intermediate vector boson theory. A neutrino scattering is an exchange in the t-channel of a vector boson [r] diagr. The diagram gives an amplitude

$$\mathcal{M} \sim g_W^2 J^\mu_{e\nu}\left[-\eta_{\mu\nu} + \frac{Q_\mu Q_\nu}{M^2}\right]\frac{1}{Q^2 - M_W^2} J^2_{ud}$$

In the limit of the momentum going to zero

$$Q^\mu = P^\mu_1 - P^\mu_3 \to 0$$

the amplitude is asymptotic to

$$M \sim g_W^2 J_\mu^{e\nu} J_\nu^{ud} \frac{\eta^{\mu\nu}}{Q^2 - M_W^2}$$

Confronting this result with Fermi theory, one obtains a coupling constant of

$$-\frac{G_\mathrm{F}}{\sqrt{2}} = \frac{1}{8} \frac{g_W^2}{Q^2 - M_W^2}$$

The intermediate vector boson is an advanced theory respect to Fermi's. Also, in the low momentum limit $Q \to 0$ (equivalent to $Q^2 \ll M_W^2$) one obtains the Fermi theory

$$\frac{G_\mathrm{F}}{\sqrt{2}} = \frac{g_W^2}{8M_W^2}$$

In the high energy limit, $Q^2 \gg M^2$, the differential cross section does not diverge

$$\mathrm{d}_\Omega \sigma = \frac{g_W^2}{4\pi^2} \frac{s}{(s - M_W^2)^2} \,, \quad s = (p_1 + p_2)^2$$

An explicit mass term in the Lagrangian breaks the gauge invariance

$$\mathcal{L} \sim M_W^2 W_\mu W^\mu$$

The theory presents a problem: the need of a theory with massive vector boson spoils the gauge invariance. Also a problem is that the theory is not renormalizable. The propagator of a massive vector boson is

$$G_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M^2} \left[-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{M^2} \right]$$

If one wants to understand the renormalizabilty through power counting arguments, one sees that the propagator is constant

$$G_{\mu\nu}(\mathbf{k}) \sim \frac{k_\mu k_\nu}{k^2} \,, \quad k \to \infty$$

The renormalizable nature of a theory is given by the convergence of the propagators in a Feynman diagram.

Another problem is the loss of unitary. When scattering two vector bosons in two vector bosons, one has to build the amplitude by summing all diagrams. One can check how the amplitude scales with the energy. The behaviour of E^4 cancels and remains E^2: the amplitude grows indefinitely.

Electroweak sector. One has to associate currents to the left-chiral part of the Dirac field. A leptonic current is

$$J_\mu^\mathrm{lept} = \frac{1}{2} \bar{\nu} \gamma_\nu (1 - \gamma_5) e$$

The current must come from a covariant derivative. The current must be Noether current. Therefore

$$\partial_\mu - \mathrm{i} g T^a A_\mu^a$$

This term is always between two fermionic fields, so the current is

$$\bar{\psi}_i \gamma_\mu T_{ij}^a \psi_j$$

where the indices ij are associated to the transformation of the gauge group.

One defines a left-chiral leptonic doublet

$$L(x) = P_\mathrm{L} \begin{bmatrix} \nu_e \\ e \end{bmatrix}$$

for which the current is

$$J_\mu^{\text{lept}} = \bar{L}\gamma_\mu \tau^+ L$$

[r] Confronting with the current above one has

$$\tau^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}(\tau_1 + \mathrm{i}\tau_2)$$

where τ_i are the Pauli matrices. So the gauge group is SU(2). Moreover, there is

$$(J_\mu^{\text{lept}})^\dagger = \bar{L}\gamma_\mu \tau^- L\,, \quad \tau^- = \frac{1}{2}(\tau_1 - \mathrm{i}t_2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

There is bijective correspondence between currents and generators. Since the algebra is close, there is a third current given by

$$(J_\mu^{\text{lept}})^3 = \bar{L}\gamma_\mu [\tau^+, \tau^-] L = \bar{L}\gamma_\mu \tau^3 L$$

This current implies a bilinear combination of spinors of the type

$$e\gamma_\mu e\,, \quad \nu\gamma_\mu \nu$$

If one searches for more generators, one may compute the commutator. However, in this case the algebra is closed

$$[\tau^3, \tau^\pm] = 2\tau^\pm$$

So there are no other currents.

The leptonic electroweak sector has $\mathrm{SU}(2)_\mathrm{L} \times U(1)_Y$ gauge group. The Lagrangian respecting this symmetry is given by

$$\mathcal{L}_\mathrm{f} = \mathrm{i}\bar{L}\not{D}L + \mathrm{i}\bar{e}_\mathrm{R}\not{D} + \mathrm{i}\bar{\nu}_{e\mathrm{R}}\not{D}\nu_{e\mathrm{R}}$$

where

$$D_\mu = \partial_\mu - \mathrm{i}gW_\mu^j T^j - \frac{1}{2}\mathrm{i}g' Y(\psi) B_\mu$$

where $j = 1, 2, 3$ and W_μ is the field associated to the local SU(2) symmetry, while B_μ is associated to U(1). This last gauge group is not the one of electromagnetism.

The generators T^i are the generators of the Lie algebra in the representation in which the fields transform

$$T^i = \begin{cases} \frac{1}{2}\tau^i\,, & \text{left-chiral fields} \\ 0\,, & \text{right-chiral fields} \end{cases}$$

For the hypercharge symmetry, the hypercharge is an abelian group and the associated charge can depend on the field itself.

The Lagrangian $\mathcal{L}_f$ is invariant under

$$U(x) = \exp\left[\frac{1}{2}\mathrm{i}g\theta(x)\tau^i\right]\,, \quad U(x) = \exp\left[\frac{1}{2}\mathrm{i}g'\alpha(x)Y\right]$$

The kinetic part of the Lagrangian is

$$\mathcal{L}_f^{\text{kin}} = \mathrm{i}\bar{L}\,\not{\partial}L + \mathrm{i}\bar{\nu}_{e\mathrm{R}}\,\not{\partial}\nu_{e\mathrm{R}} + \mathrm{i}\bar{e}_\mathrm{R}\,\not{\partial}e_\mathrm{R}$$

One may notice that the electrons and neutrinos are both massless, but when measured they are not massless.

The charge current interaction is

$$\begin{aligned}\mathcal{L}_{\text{cc}} &= \frac{1}{2}gW_\mu^1\bar{L}\gamma^\mu\tau^1 L + \frac{1}{2}gW_\mu^2\bar{L}\gamma^\mu\tau^2 L = \frac{g}{\sqrt{2}}[W_\mu^+\bar{L}\gamma^\mu\tau^+ L + W_\mu^-\gamma^\mu\tau^- L] \\ &= \frac{g}{\sqrt{2}}[W_\mu^+\bar{\nu}_\mathrm{L}\gamma^\mu e_\mathrm{L} + W_\mu^-\bar{e}_\mathrm{L}\gamma^\mu\nu_\mathrm{L}]\end{aligned}$$

where one defines

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp \mathrm{i}W_\mu^2)$$

The Feynman rule is [r] diagr
$$\mathrm{i}\frac{g}{\sqrt{2}}\gamma^\mu\frac{1-\gamma_5}{2}$$
These are the charged currents because the carrier boson is charged.

The neutral current part is
$$\mathcal{L}_{\mathrm{nc}}=\frac{1}{2}gW_3^\mu[\bar{\nu}_{e\mathrm{L}}\gamma_\mu\nu_{e\mathrm{L}}-\bar{e}_\mathrm{L}\gamma_\mu e_\mathrm{L}]+\frac{1}{2}g'B^\mu[Y(L)(\bar{\nu}_{e\mathrm{L}}\gamma_\mu\nu_{e\mathrm{L}}+\bar{e}_\mathrm{L}\gamma_\mu e_\mathrm{L})+Y(e_\mathrm{R})(\bar{e}_\mathrm{R}\gamma_\mu e_\mathrm{R})+Y(\nu_{e\mathrm{R}})(\bar{\nu}_{e\mathrm{R}}\gamma_\mu v_{e\mathrm{R}})]$$
Introducing
$$\Psi=\begin{bmatrix}\nu_{e\mathrm{L}} & e_\mathrm{L} & \nu_{e\mathrm{R}} & e_\mathrm{R}\end{bmatrix}^\top$$
the third value of the isospin is
$$T_3=\frac{1}{2}\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$
while the hypercharge is
$$Y=\mathrm{diag}[Y(L),Y(L),Y(\nu_{e\mathrm{R}}),Y(e_\mathrm{R})]$$
The Lagrangian becomes
$$\mathcal{L}_{\mathrm{nc}}=g(\bar{\Psi}\gamma^\mu T_3\Psi)W_\mu^3+\frac{1}{2}g'(\bar{\Psi}\gamma^\mu Y\Psi)B_\mu$$
From these fields one would like to recognize quantum electrodynamics. One considers linear combinations of the two fields through an orthogonal rotation: the kinetic term [r]. Therefore
$$\begin{bmatrix}B_\mu\\W_\mu^3\end{bmatrix}=\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}\begin{bmatrix}A_\mu\\Z_{mu}\end{bmatrix}$$
The angle is called Weinberg's angle. One needs to look at the kinetic terms and how the interactions change. The interaction Lagrangian is
$$L_{\mathrm{YM,int}}=-\frac{1}{4}B_{\mu\nu}B^{\mu\nu}-\frac{1}{4}W_{\mu\nu}^3W_3^{\mu\nu}\to-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-\frac{1}{4}Z_{\mu\nu}Z^{\mu\nu}$$
where $F^{\mu\nu}$ is the field strength tensor of the electromagnetic field A^μ, and $Z^{\mu\nu}$ for the Z^μ field. There are no mixed kinetic terms $F_{\mu\nu}Z^{\mu\nu}$.

The neutral current part of the Lagrangian is
$$\mathcal{L}_{\mathrm{nc}}=\bar{\Psi}\gamma_\mu[g\sin\theta\,T_3+\frac{1}{2}g'\cos\theta\,Y]\Psi A^\mu+\bar{\Psi}\gamma_\mu[g\cos\theta\,T_3-\frac{1}{2}g'\sin\theta\,Y]\Psi Z^\mu$$
Noting that the interaction Lagrangian of quantum electrodynamics is
$$\mathcal{L}_{\mathrm{QED}}^{\mathrm{int}}=\bar{\Psi}\gamma^\mu eQ\Psi A_\mu$$
where $e>0$ is the elementary charge and
$$Q=\mathrm{diag}(Q_\nu,Q_e,Q_\nu,Q_e)=\mathrm{diag}(0,-1,0,-1)$$
Therefore, one has
$$g\sin\theta\,T_3+\frac{1}{2}g'\cos\theta\,Y\equiv eQ$$
Since Y is always with g', then one can fix a value for Y and the other follow
$$Y(L)\equiv-1$$
therefore, for the left-chiral neutrino field $\nu_{e\mathrm{L}}$ one has $T_3=\frac{1}{2}$ and the left-chiral electron field one has $T_3=-\frac{1}{2}$. For the neutrino and electron one has
$$\frac{1}{2}g\sin\theta-\frac{1}{2}g\cos\theta=0\,,\quad-\frac{1}{2}g\sin\theta-\frac{1}{2}g'\cos\theta$$

This implies

$$g\sin\theta + g'\cos\theta = e$$

Therefore one has the Gell-Mann–Nijishima equation

$$Q = T_3 + \frac{1}{2}Y$$

For the right-chiral fields, one has $T_3 = 0$ and

$$Y(\nu_\mathrm{R}) = 0\,, \quad Y(e_\mathrm{R}) = -2$$

The right-chiral neutrino behaves like it does not exist (apart from gravity).

The Feynman rules for quantum electrodynamics $e\bar{\Psi}\gamma_\mu Q\Psi A^\mu$ and neutral currents $e\bar{\Psi}\gamma_\mu Q_Z\Psi Z^\mu$ are [r] diagr

$$\mathrm{i}eQ_f\gamma^\mu\,, \quad \mathrm{i}e\gamma_\mu(c_\mathrm{L}P_\mathrm{L} + c_\mathrm{R}P_\mathrm{R}) = \mathrm{i}e\gamma^\mu(v_f - a_f\gamma_5)$$

where

$$Q_Z = \frac{1}{\sin\theta\cos\theta}[T_3 - Q\sin^2\theta]\,, \quad \Psi = \Psi_\mathrm{L} + \Psi_\mathrm{R}$$

and

$$c_\mathrm{L} = \frac{1}{\sin\theta\cos\theta}(T_f^3 - Q_f\sin^2\theta)\,, \quad c_\mathrm{R} = -\tan\theta\, Q_f$$

likewise

$$v_f = \frac{1}{2}(c_\mathrm{L} - c_\mathrm{R}) = \frac{T_f^3 - 2Q_f^2\sin^2\theta}{2\sin\theta\cos\theta}\,, \quad a_f = \frac{1}{2}(c_\mathrm{L} + c_\mathrm{R}) = \frac{T_f^3}{2\sin\theta\cos\theta}$$

Lecture 5

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The pure Yang–Mills part is

$$\mathcal{L}_\mathrm{YM} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}W_{\mu\nu}W_i^{\mu\nu}\,, \quad i = 1,2,3$$

where one has

$$W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + gf^{abc}W_\mu^b W_\nu^c$$

Inserting the physical fields through the relations

$$W_\mu^1 = \frac{1}{\sqrt{2}}(W_\mu^+ + W_\mu^-)\,, \quad W_\mu^2 = \frac{\mathrm{i}}{\sqrt{2}}(W_\mu^+ - W_\mu^-)\,, \quad W_\mu^3 = A_\mu\sin\theta + Z_\mu\cos\theta\,, \quad B_\mu = A_\mu\cos\theta - Z_\mu\sin\theta$$

one obtains

$$\begin{gathered}\mathcal{L}_\mathrm{YM} = \mathcal{L}_\mathrm{kin} + \mathcal{L}_\mathrm{int} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{2}W_{\mu\nu}^+W_-^{\mu\nu} \\ + \text{3-point interactions} + \text{4-point interactions}\end{gathered}$$

in the first line there are no mixed terms: the fields are mass eigenstates. The three-point vertex contains derivative so they scale with the energy of the momentum, while the four-point have no dependence on momentum.

Isospin and hypercharge Hadronic sector. See Ridolfi. This sector has a symmetry

$$\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_Y$$

Hadrons are not elementary particles, but are made of quarks. A neutron decays into

$$n \to pe^-\bar{\nu}_e\,, \quad |udd\rangle \to |uud\rangle + e^- + \bar{\nu}_e$$

In the low-energy limit, the decay can be explained using Fermi's four-point interaction. The interaction has to be generated from a term

$$J_\mu^{\text{lept}} J^\mu_{\text{had}}$$

where one has the charged currents

$$J_\mu^{\text{had}} = \frac{1}{2}\bar{u}\gamma_\mu(1-\gamma_5)d$$

The up and down quarks are not the only ones. Experiments showed the presence of strange hadrons $K^\pm$, K^0, Λ^0, etc. These particles decay slowly, they have a short decay width so the interaction has a weak coupling constant: they decay weakly. One assumes that a kaon is made of another quark, the strange quark

$$\left|K^+\right\rangle = |u\bar{s}\rangle$$

One of its decay chains is

$$\left|K^+\right\rangle \to \left|\pi^0\right\rangle e^+\nu$$

It starts from a kaon and has leptons in the final state: the interaction is the weak. The strange quark is postulated to have electromagnetic charge of

$$Q_s = -\frac{1}{3}$$

The strangeness quantum number of the strange quark is $Q_s = -1$.

The natural phenomenological hypothesis is considering that the hadronic current has two parts

$$J_\mu^{\text{had}} = \cos\theta\frac{1}{2}\bar{u}\gamma_\mu(1-\gamma_5)d + \sin\theta\frac{1}{2}\bar{u}\gamma_\mu(1-\gamma_5)s$$

where $\theta \approx 12^\circ$ is the Cabibbo angle.

One may extend the model supposing the current is Noether's and one may find new interactions. The hadronic current is

$$J_\mu^{\text{had}} = \begin{bmatrix}\bar{u}_\text{L} & \bar{d}_\text{L} & \bar{s}_\text{L}\end{bmatrix}\gamma_\mu T^+ \begin{bmatrix}u_\text{L}\\ d_\text{L}\\ s_\text{L}\end{bmatrix}, \quad T^+ = \begin{bmatrix}0 & \cos\theta & \sin\theta\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

The hermitian conjugate current is

$$(J_\mu^{\text{had}})^\dagger = \cdots, \quad T^- = \begin{bmatrix}0 & 0 & 0\\ \cos\theta & 0 & 0\\ \sin\theta & 0 & 0\end{bmatrix}$$

One imagines that the matrices $T^\pm$ are elements of a Lie group ([r] still Cabibbo angle? or which one is it?)

$$[T^+, T^-] = T^3 = \begin{bmatrix}1 & 0 & 0\\ 0 & -\cos^2\theta & -\sin\theta\cos\theta\\ 0 & -\sin\theta\cos\theta & -\sin^2\theta\end{bmatrix}$$

This matrix is not diagonal. The current associated with this element is

$$J_\mu^{\text{had},3} = \begin{bmatrix}\bar{u}_\text{L} & \bar{d}_\text{L} & \bar{s}_\text{L}\end{bmatrix}\gamma_\mu T^3 \begin{bmatrix}u_\text{L}\\ d_\text{L}\\ s_\text{L}\end{bmatrix} = \bar{u}_\text{L}\gamma_\mu u_\text{L} - \cos^2\theta\bar{d}_\text{L}\gamma_\mu d_\text{L} - \sin^2\theta\bar{s}_\text{L}\gamma_\mu s_\text{L} - \sin\theta\cos\theta[\bar{d}_\text{L}\gamma_\mu s_\text{L} + \bar{s}_\text{L}\gamma_\mu d_\text{L}]$$

The last parenthesis is the flavor-changing neutral current (FCNC). These currents exist in nature but are extremely suppressed. However, this theory does not predict a suppressed FCNC.

The flavor-changing charge current can be seen from

$$\left|K^+\right\rangle \to \left|\pi^0\right\rangle e^+\nu\,, \quad |u\bar{s}\rangle \to |u\bar{u}\rangle$$

mediated by

$$\sin\theta_c\,\bar{s}_\text{L}\gamma^\mu u_\text{L}$$

[r] Instead, for the mediating FCNC is

$$\sin\theta_c \cos\theta_c \bar{s}_{\mathrm{L}}\gamma^\mu d_{\mathrm{L}}$$

The ratio of the experimental widths is

$$\frac{\Gamma(K^+ \to \pi^+ e^+ e^-)}{\Gamma(K^+ \to \pi^0 e^+ \nu)} \approx 10^{-5}$$

while in theory one has

$$\sim \frac{(\sin\theta_c \cos\theta_c)^2}{\sin^2\theta_c} \sim 0.97$$

To resolve the issue, one may postulate the existence of a fourth quark, called charm quark. Its electric charge is

$$Q = \frac{2}{3}$$

It has to be heavy and coupled to the down and strange quarks through charged currents. The current is

$$\begin{aligned} J_\mu^{\mathrm{had}} &= \cos\theta_c \bar{u}_{\mathrm{L}}\gamma_\mu d_{\mathrm{L}} + \sin\theta_c \bar{u}_{\mathrm{L}}\gamma_\mu s_{\mathrm{L}} - \sin\theta_c \bar{c}_{\mathrm{L}}\gamma_\mu d_{\mathrm{L}} + \cos\theta_c \bar{c}_{\mathrm{L}}\gamma_\mu s_{\mathrm{L}} \\ &= \bar{u}_{\mathrm{L}}\gamma_\mu d'_{\mathrm{L}} + \bar{c}_{\mathrm{L}}\gamma_\mu s'_{\mathrm{L}} \end{aligned}$$

where one has

$$\begin{bmatrix} d' \\ s' \end{bmatrix} = \begin{bmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{bmatrix} \begin{bmatrix} d \\ s \end{bmatrix}$$

The current is then

$$J_\mu^{\mathrm{had}} = \begin{bmatrix} \bar{u}_{\mathrm{L}} & \bar{d}'_{\mathrm{L}} \end{bmatrix} \gamma_\mu \tau^+ \begin{bmatrix} u_{\mathrm{L}} \\ d'_{\mathrm{L}} \end{bmatrix} + \begin{bmatrix} \bar{c}_{\mathrm{L}} & \bar{s}'_{\mathrm{L}} \end{bmatrix} \gamma_\mu \tau^- \begin{bmatrix} c_{\mathrm{L}} \\ s'_{\mathrm{L}} \end{bmatrix}$$

The third current is

$$\begin{aligned} J_\mu^{\mathrm{had},3} &= \begin{bmatrix} \bar{u}_{\mathrm{L}} & \bar{d}'_{\mathrm{L}} \end{bmatrix} \gamma_\mu \tau^3 \begin{bmatrix} u_{\mathrm{L}} \\ d'_{\mathrm{L}} \end{bmatrix} + \begin{bmatrix} \bar{c}_{\mathrm{L}} & \bar{s}'_{\mathrm{L}} \end{bmatrix} \gamma_\mu \tau^3 \begin{bmatrix} c_{\mathrm{L}} \\ s'_{\mathrm{L}} \end{bmatrix} \\ &= \bar{u}_{\mathrm{L}}\gamma_\mu u_{\mathrm{L}} + \bar{c}_{\mathrm{L}}\gamma_\mu c_{\mathrm{L}} - \bar{d}_{\mathrm{L}}\gamma_\mu d_{\mathrm{L}} - \bar{s}_{\mathrm{L}}\gamma_\mu s_{\mathrm{L}} \end{aligned}$$

There are no flavour-changing neutral currents.

The existence of the charm quark was discovered four years after postulating its existence. It was discovered the bound state $|c\bar{c}\rangle$ which is the J/ψ particle. The mass of the quark is $m_c \approx 1.5\,\mathrm{GeV}$.

Those who postulated its existence worked also on the GIM mechanics: there are no FCNC at tree-level, the FCNC are suppressed at 1 loop if $\Delta s \neq 0$, explains the mass difference of K_L and K_S.

At present time, the mechanism involves the mixing between three quark families: there are three angles and one complex phase; this phase implies CP violation.

Summary. The Standard Model has the following fields. The quarks are

$$\begin{bmatrix} u_{\mathrm{L}} \\ d'_{\mathrm{L}} \end{bmatrix}, \quad \begin{bmatrix} c_{\mathrm{L}} \\ s'_{\mathrm{L}} \end{bmatrix}, \quad \begin{bmatrix} t_{\mathrm{L}} \\ b'_{\mathrm{L}} \end{bmatrix}, \quad u_{\mathrm{R}} d'_{\mathrm{R}} c_{\mathrm{R}} s'_{\mathrm{R}} t_{\mathrm{R}} b'_{\mathrm{R}}$$

The leptons are

$$\begin{bmatrix} \nu_{el} \\ e_L \end{bmatrix}, \quad \begin{bmatrix} \nu_{\mu L} \\ \mu_L \end{bmatrix}, \quad \begin{bmatrix} \nu_{\tau L} \\ \tau_L \end{bmatrix}, \quad e_R \mu_R \tau_R \nu_{eR} \nu_{\mu R} \nu_{\tau R}$$

The left-chiral leptons are organized in doublets of $\mathrm{SU}(2)_L$ with $T_3 = \pm\frac{1}{2}$, while the right-chiral are singlets with $T_j = 0$. One also has

$$Q = T_3 + \frac{1}{2}Y$$

where one has

$$Y(u_R) = \frac{4}{3}\,, \quad Y(d_R) = -\frac{2}{3}\,, \quad Y(Q_L) = \frac{1}{3}\,, \quad Q_u = \frac{2}{3}\,, \quad Q_d = -\frac{1}{3}\,, \quad Q_L = \begin{bmatrix} u_L \\ d'_L \end{bmatrix}\,, \cdots$$

The charged-current Lagrangian

$$\mathcal{L}_{cc} = \frac{g}{\sqrt{2}} \sum_f [\bar{L}_f \gamma_\mu \tau_+ L_f + \bar{Q}_f \gamma_\mu \tau_+ Q_f] W^+_\mu + \text{h.c.}$$

where $f = 1, 2, 3$ is the index associated with the three quark and leptonic families and

$$L_f \in \{ \begin{bmatrix} v_{eL} \\ e_L \end{bmatrix}\,, \begin{bmatrix} v_{\mu L} \\ \mu_L \end{bmatrix}\,, \cdots \}\,, \quad Q_f = \{ \begin{bmatrix} u_L \\ d'_L \end{bmatrix}\,, \cdots$$

the Lagrangian is

$$\mathcal{L}_{cc} = \mathcal{L}_{\text{lept}} + \frac{g}{\sqrt{2}} \left[\sum_{f,g} (\bar{u}^f_L \gamma_\mu V^{fg} d^g_L) W^+_\mu + \text{h.c.} \right]$$

where V_{fg} is a 3×3 non-diagonal matrix called CKM matrix. [r] one has the Feynman

$$\sim V^{12} [\gamma_\mu \frac{1}{2}(1 - \gamma_5) \frac{g}{\sqrt{2}}]$$

The neutral current Lagrangian has no FCNC but is diagonal

$$\mathcal{L}_{\text{nc}} = e(\bar{\Psi} \gamma_\mu Q \Psi) A_\mu + (\bar{\Psi} \gamma^\mu Q_Z \Psi) Z_\mu$$

Higgs mechanism. None of the fields have any masses because gauge symmetry must be respected. Masses are given through the spontaneous symmetry breaking of gauge theories: the Higgs mechanism. This gives masses to the gauge boson.

A Yukawa interaction is allowed and a Higgs field with vacuum expectation value gives masses to fermions.

One may study the number of degrees of freedom needed. In the electroweak sector, the gauge fields are massless each with two transverse polarization; though in nature three are massive, so one more degree for each is needed.

One needs a scalar field charge under the gauge group $\mathrm{SU}(2)_L \times U(1)_Y$ with non zero vacuum expectation value. The gauge group must be broken, but there must remain a gauge subgroup of $U(1)_{\text{EM}}$. The simplest and minimal choice is using a doublet of scalar complex fields

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

It has four real degrees of freedom. Three of them make up the masses of three gauge boson, while the last is the Higgs field. The doublet transforms as

$$T_3 \phi = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \phi\,, \quad Y(\phi) = \cdots$$

the hypercharge is not yet fixed.

To realize the spontaneous symmetry breaking one needs a potential [r], to respect gauge invariance, and the theory to be normalizable. The potential can only be

$$V(\phi) = m^2 |\phi|^2 + \lambda |\phi|^4\,, \quad |\phi|^2 = |\phi_1|^2 + |\phi_2|^2$$

The mass dimension of λ is zero, so the theory is renormalizable. A cubic term is not gauge invariant.

A spontaneous symmetry breaking implies a non zero vacuum expectation value. The minimum of the potential is

$$|\phi_{\min}|^2 = -\frac{m^2}{2\lambda} = \frac{1}{2} v^2 \implies m^2 < 0$$

The minimum configuration must be invariant under $U(1)$ so that this gauge group is unbroken and the photon does not gain mass. One parametrizes the minimum field as

$$\phi_{\min} = \frac{1}{\sqrt{2}} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} , \quad |v_1|^2 + |v_2|^2 = v^2$$

The residual invariance implies

$$e^{i\alpha(x)Q} \phi_{\min} = \phi_{\min} \implies Q\phi_{\min} = 0$$

Lecture 6

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One needs to have

$$\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \frac{1}{2} \begin{bmatrix} 1+Y & 0 \\ 0 & -1+Y \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

There are two choices $v_1 = 0$ and $v_2 = |v|$ or $v_1 = |v|$ and $v_2 = 0$. For the first one has $Y(\phi) = 1$ while for the second $Y(\phi) = -1$. One picks the first choice

$$Q_{\mathrm{EM}}(\phi_1) = 1 , \quad Q_{\mathrm{EM}}(\phi_2) = 0$$

The doublet can rewritten as

$$\phi = \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix}$$

To obtain the content of the theory, one parametrizes the field around the vacuum expectation value. This can be done through Cartesian coordinates

$$\phi = \begin{bmatrix} \xi_1 + i\xi_2 \\ \frac{1}{\sqrt{2}}(v + H(x) + i\chi) \end{bmatrix}$$

but it is more useful to use complex coordiantes

$$\phi = \frac{1}{2} \exp\left[\frac{1}{2} i\tau^a \theta^a(x)\right] \begin{bmatrix} 0 \\ v + H(x) \end{bmatrix}$$

where $H(x)$ is the Higgs field. The four fields above have zero expectation value. One imposes the unitary gauge

$$\phi(x)' = U(x)\phi(x) , \quad U(x) = \exp\left[-\frac{1}{2} i\tau^a \theta^a(x)\right]$$

This gauge is allowed because the Lagrangian is gauge invariant. The θ^a fields are absorbed into the $W^\pm$ and Z fields as longitudinal polarization, so the associated bosons are massive. The field $H(x)$ is associated with a physical particle. The spontaneous symmetry patterns is such that three of the five degrees of freedom [r]

$$\tau_1 \varphi_{\min} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{bmatrix} \neq 0 , \quad \tau_2 \phi_{\min} \neq 0$$

whereas

$$\frac{1}{2}(\tau_3 + Y)\phi_{\min} = 0$$

Physical consequences. The Lagrangian of the field is

$$\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) - V(\phi) , \quad \phi = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ v + H(x) \end{bmatrix}$$

The potential is

$$
\begin{aligned}
V = m^2|\phi|^2 + \lambda|\phi|^4 &= \frac{1}{2}m^2[H^2 + 2vH] + \frac{1}{4}\lambda[H^4 + 4H^3v + 6H^2v^2 + 4Hv^3] + \text{const.} \\
&= \frac{1}{2}(2\lambda v^2)H^2 + \lambda vH^3 + \frac{1}{4}\lambda H^4
\end{aligned}
$$

at the second line one uses the fact that

$$-m^2 = \lambda v^2$$

The first term is a mass term $m_{\mathrm{H}}^2 = 2\lambda v^2$. The other two terms are a three-point and four-point self-couplings of the Higgs field. The two vertices are proportional to the free parameter λ.

Within the covariant derivative there is the Higgs kinetic term and the interactions with the vector boson HVV, HHVV. The covariant derivative is

$$
\begin{aligned}
D_\mu\phi &= \left[\partial_\mu - \frac{1}{2}\mathrm{i}gW_\mu^i\tau^i - \frac{1}{2}\mathrm{i}g'YB_\mu\right]\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ v + H(x)\end{bmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ \partial_\mu H(x)\end{bmatrix} - \frac{\mathrm{i}}{2}\left[1 + \frac{M}{v}\right]\begin{bmatrix} gvW_\mu^+ \\ -v\sqrt{\frac{g^2+g'^2}{2}}Z_\mu\end{bmatrix}
\end{aligned}
$$

where $Y = 1$ for the Higgs field. Therefore, one gets

$$
(D_\mu\phi)^\dagger(D^\mu\phi) = \cdots = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \left(1 + \frac{M}{v}\right)^2\left[\left(\frac{gv}{2}\right)^2 W_\mu^+W_\mu^- + \frac{v^2}{2}\frac{g^2+g'^2}{4}Z_\mu Z^\mu\right]
$$

The masses of the gauge bosons are then

$$
m_W^2 = \frac{1}{2}gv\,, \quad m_Z^2 = v^2\frac{g^2+g'^2}{4} = \frac{m_W^2}{\cos^2\theta}
$$

while the mass of the electromagnetic field is zero because a massa term does not appear. [r]

The vacuum expectation value must be

$$
\frac{G_\mathrm{F}}{\sqrt{2}} = \frac{g^2}{8M_W^2} \implies v = (\sqrt{2}G_\mathrm{F})^{-\frac{1}{2}} \approx 246\,\mathrm{GeV}
$$

This energy is the energy scale associated to a symmetry breaking.

Interactions. The three-point interaction is

$$
\begin{aligned}
\mathrm{HVV} &= \frac{2m_W^2}{v}W_\mu^+W_-^\mu H + \frac{m_Z^2}{v}Z_\mu Z^\mu H \\
&= gm_W\mathrm{WWH} + \frac{1}{2}g\frac{m_Z}{\cos\theta}\mathrm{ZZH}
\end{aligned}
$$

at the second equality one uses the relations above for the masses. The Higgs boson couples in a way proportional to the mass of the vector bosons. The vertex is [r]

$$
i\eta_{\mu\nu}\begin{cases} m_W\,, & \mathrm{HWW} \\ \frac{m_W}{\cos\theta}\,, & \mathrm{HZZ}\end{cases}
$$

Another interaction is the HHVV.

Remark. SSB produces a Lagrangian with massive vector boson in a gauge invariant way. This is done through a scalar field with non zero vev.

However, there are heavy fermions which have

$$\langle\bar{\psi}\psi\rangle \neq 0$$

If they were charged properly for SU(2) left [r] If such field exists, its energy is higher than the current probed energy. A model like this is called composite Higgs.

The Higgs mass is

$$m_H \approx 125\,\text{GeV}$$

The mass is known from experiment and one has

$$m_H^2 = 2\lambda v^2$$

To measure directly the parameter λ one has to obtain a three-point Higgs self-coupling. The Higgs can be obtain from gluon fusion. [r] however there is a background. The dependence on λ comes from the destructive interference between the two diagrams.

Yukawa interaction. [r] One has to recover a Dirac mass. [r] The kinetic term for fermions, one can compactly write

$$\mathcal{L}_{\text{fermions}} = \sum_{f=1}^{n}\sum_{k=1}^{5} \bar{\psi}_{\text{R}}^{(f)} \mathrm{i}\not{D}\psi_{\text{R}}^{(f)}$$

where

$$k \in \{ \begin{bmatrix} u_{\text{L}} \\ d_{\text{L}} \end{bmatrix}, u_{\text{R}}, d_{\text{R}}, \begin{bmatrix} \nu_{eL} \\ e_{\text{L}} \end{bmatrix}, e_{\text{R}} \}$$

and $f = 1, 2, 3$ so n is the number of fermionic families.

The Lagrangian is invariant under the transformation U(n)

$$\psi_k^{(f)} \to U^{fg}\psi_k^{(g)}$$

This is an accidental global symmetry.

One has to mix different families, so the notation used is the following: the primed denotes the interaction eigenstates, the interactions with gauge bosons are diagonal; the unprimed denotes the mass eigenstates. A diagonal matrix is useful because one can read out the propagators. One would like to express everything in terms of the mass eigenstates. Therefore

$$Q'_{\text{L}} = \begin{bmatrix} u'_{\text{L}} \\ d'_{\text{L}} \end{bmatrix}, u'_{\text{R}}, d_{\text{R}}, L'_{\text{L}} = \begin{bmatrix} \nu'_{eL} \\ e'_{\text{L}} \end{bmatrix}, e'_{\text{R}}$$

With the SM field content, the only term that can be added that respects Lorentz invariance, gauge invariance and renormalizability as the Yukawa interaction

$$\bar{\psi}\psi\phi$$

The Yukawa Lagrangian is

$$\mathcal{L}_{\text{Y}} = -\bar{Q}'_{\text{L}} h'_D d'_{\text{R}}\phi - \bar{Q}'_{\text{L}} h'_U u'_{\text{R}}\phi^c - \bar{L}'_{\text{L}} h'_E e_{\text{R}}\phi + \text{h.c.}$$

The h' terms are $n \times n$ complex matrices in flavor space. Their entries are numbers so their mass dimension is zero. In the SM $n = 3$ so a total of 27 parameters; some are fixed by the masses and [r].

The terms are invariant under SU(2) left and hypercharge.

Also one has

$$\phi^c = \varepsilon\phi\,, \quad \phi = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \mathrm{i}\tau^2$$

The anti-fundamental representation of SU(2) is the same as the fundamental of SU(2). One has

$$\phi^c = \frac{1}{\sqrt{2}} \begin{bmatrix} v + H(x) \\ 0 \end{bmatrix}$$

In the unitary gauge, the Lagrangian is

$$\mathcal{L}_{\text{Y}} = -\frac{v+H}{\sqrt{2}}[\bar{d}'_{\text{L}} h'_D d'_{\text{R}} + \bar{u}'_{\text{L}} h'_U u'_{\text{R}} + \bar{e}'_{\text{L}} h'_E e'_{\text{R}} + \text{h.c.}]$$

Imagining that the matrices are diagonal, one obtains the Dirac mass for each field. Though the matrices are not diagonal because there is flavour mixing. By the singular-value decomposition, for each h' with complex entries, not necessarily square, it holds

$$h' = UhV^\dagger$$

where U and V are unitary matrices, and h is a diagonal matrix with positive and real entries (i.e. eigenvalue). In this way one can write a diagonal matrix, with other two acting on the spinors, but this does not matter because spinor have to be rotated. In fact

$$\mathcal{L}_\mathrm{Y} = -\frac{v+H}{\sqrt{2}}[\bar{d}'_\mathrm{L} U_D h_D V_D^\dagger d'_\mathrm{R} + \bar{u}'_\mathrm{L} U_U h_U V_U^\dagger u'_\mathrm{R} + \bar{e}'_\mathrm{L} U_E h_E V_E^\dagger e'_\mathrm{R} + \text{h.c.}]$$

The matrices h are all diagonal. One defines

$$d_{\mathrm{R,R}} = U_D^\dagger d'_{\mathrm{R,L}}$$

and same for u and e. One has

$$\mathcal{L}_\mathrm{Y} = -\frac{v+H}{\sqrt{2}}[\bar{d}_\mathrm{L} h_D d_\mathrm{R} + \bar{u}_\mathrm{L} h_U u_\mathrm{R} + \bar{e}_\mathrm{L} h_E e_\mathrm{R} + \text{h.c.}]$$

These are the mass eigenstates. So

$$m_D = \frac{v}{\sqrt{2}} h_D = \mathrm{diag}(m_d, m_s, m_b)$$

same for m_U and m_E.

After this rotation, the kinetic terms remains diagonal [r]

Lecture 7

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In the quark sector there is a mixing in the charged current. There is also a source of CP violation.

One has

$$m_U = \frac{v}{\sqrt{2}} h_U = \mathrm{diag}(m_u, m_c, m_t)\,, \quad m_E = \frac{v}{\sqrt{2}} h_E = \mathrm{diag}(m_e, m_\mu, m_\tau)$$

The Yukawa Lagrangian is

$$\mathcal{L}_\mathrm{Y} = -\sum_f m_f (\bar{\psi}_L^{(f)} \psi_R^{(f)} + \bar{\psi}_R^{(f)} \psi_L^{(f)}) \left[1 + \frac{H(x)}{v}\right]$$

In this way one has got the mass terms. There is also a three-point interaction called Yukawa coupling [r] diagr

$$-\mathrm{i}\frac{m_f}{v}$$

The coupling of the Higgs field is proportional to the mass of the particle. Mass is a consequence of the strength of the coupling.

Consequences. The kinetic term of u-type quarks is

$$\mathcal{L}_\mathrm{kin} \sim \bar{u}'_L \mathrm{i}\not{\partial} u'_L = \bar{u}_L \mathrm{i}\not{\partial} u_L$$

Since one has

$$d_R = V_D^\dagger d'_R\,, \quad d_L = U_D^\dagger d'_L\,, \quad u_R = V_U^\dagger u'_R\,, \quad u_L = U_U^\dagger u'_L\,, \quad e_R = V_E^\dagger e'_R\,, \quad e_R = U_E^\dagger e'_L$$

from which one has

$$d'_L = U_D d_L \implies \bar{d}'_L = \bar{d}_L U_D^\dagger$$

This also happens for d' and e', and for right-chiral fields.

The neutral current Lagrangian is diagonal in flavor space, so the same as above happens: FCNC are suppressed.

However there are flavor changing charged currents. The quark part of such currents is

$$\mathcal{L}_{\text{cc}}^{\text{hadron}} = \frac{g}{\sqrt{2}}[W_\mu^\dagger \bar{u}'_L \gamma^\mu d'_L + \text{h.c.}]$$

In the interaction basis (the primed fields) the above is diagonal

$$(\bar{u}'_L)^{(f)} \gamma^\mu \delta^{(f)(g)} (d'_L)^{(g)}$$

[r] The physical fields are mass eigenstates?

$$\mathcal{L}_{\text{cc}}^{\text{hadron}} = \frac{g}{\sqrt{2}}[W_\mu^\dagger \bar{u}_L \gamma^\mu U_U^\dagger U_D d_l + \text{h.c.}]\,, \quad U_U^\dagger U_D \neq I$$

The product above of unitary matrices are the Cabibbo–Kobayashi–Mashawa (CKM) matrix

$$V_{\text{CKM}} = U_U^\dagger U_D$$

It is the generalization of the Cabibbo 2×2 matrix. This matrix is unitary

$$V^\dagger V = U_D^\dagger U_U U_U^\dagger U_D = I$$

The rotation of the elementary fields is needed to accomodate empirical evidences. The entry of the matrix must come from experiment. This matrix encodes flavour mixing.

The leptonic charged currents are given by

$$\mathcal{L}_{\text{cc}}^{\text{lepton}} = \frac{g}{\sqrt{2}}[W_\mu^\dagger \bar{\nu}'_L \gamma^\mu e'_L + \text{h.c.}]$$

If neutrino ν are massless, one has the freedom to mix [r] with same [r] of charged leptonis

$$\nu_L = U_E \nu'_L$$

This works because there is no explicit ϕ^c term in the leptonic sector (because there is no ν_R). [r] Therefore

$$\mathcal{L}_{\text{cc}}^{\text{lepton}} = \frac{g}{\sqrt{2}}[W_\mu^\dagger \bar{\nu}_L \gamma^\mu e_L + \text{h.c.}]$$

where there is an identity matrix in leptonic families. There is no generator of mixing in the leptonic sector.

Therefore, there can be vertices of the type

$$\bar{d}u\,, \quad e^+\bar{\nu}_e$$

but not

$$\bar{\nu}_e \mu^+$$

Neutrino masses. One can include the masses of neutrinos. To construct the Lagrangian one should know if neutrinos are Dirac or Majorana particles. To the Lagrangian of the Standard Model, one may add

$$\bar{N}_R \mathrm{i}\not{\partial} N_R - \bar{L}'_L h'_N N_R \phi^c - \frac{1}{2} N_R M N_R$$

The third addendum is a Majorana mass term. Therefore, one can parametrize the mass terms in a compact way

$$\mathcal{L}_{\nu\text{mass}} \sim \begin{bmatrix} \nu^\top & N^\top \end{bmatrix} \begin{bmatrix} 0 & h_N v \\ h_N^\top v & M \end{bmatrix} \begin{bmatrix} \nu \\ N \end{bmatrix}\,, \quad \begin{bmatrix} \nu \\ N \end{bmatrix} = \begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \\ N_e \\ N_\mu \\ N_\tau \end{bmatrix}$$

where one consider directly the mass eigenstates. One should diagonalize the matrix and write the neutrino masses. One notices that neutrino mixing is present which has been observed.

CKM parameters. The CKM matrix is 3×3 because there are three families. Counting the parameters, one needs to add a complex number. The matrix is intrinsically complex and as such there is CP violation. The matrix is unitary

$$VV^\dagger = I_n$$

for n families. The independent parameters of the matrix are n^2. Not all these real parameters are physical. One may think the parameters as angles and phases

$$n^2 = N_{\text{angle}} + N_{\text{phase}}$$

The number of angles N_{angle} is the number of coordinate $2D$-planes of n-dimensional space

$$N_{\text{angle}} = \binom{n}{2} = \frac{1}{2}n(n-1)$$

The phases are then

$$N_{\text{phase}} = n^2 - N_{\text{angle}} = \frac{1}{2}n(n+1)$$

Not all the phases are measurable. For example, the transformation

$$u_L^{(f)} \to e^{i\alpha_f} u_L^{(f)}\,, \quad d_L^{(f)} \to e^{i\beta_f} d_L^{(f)}$$

leaves the Lagrangian invariant except in the charged currents $\mathcal{L}_{\text{cc}}^{\text{hadron}}$. In this way

$$V_{fg} \to V_{fg} e^{i(\beta_g - \alpha_f)}$$

One can re-absorb $n + (n-1)$ phases into the phases of the quark fields

$$\begin{bmatrix} u & c & t \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} d \\ s \\ b \end{bmatrix}$$

So the number of physical phases are

$$N_{\text{phys phas}} = \frac{1}{2}n(n+1) - (2n-1) = \frac{1}{2}(n-1)(n-2)$$

For $N = 1, 2$ there is no true physical phase. For $N = 3$ there is one physical phase in the CKM matrix. Therefore

$$V_{\text{CKM}} \neq V_{\text{CKM}}^*$$

This implies CP violation which is made possible only for $n \geq 3$. This CP violation is not enough to explain the violation on cosmological scale, but it is enough to explain low energy phenomenology.

The matrix is

$$V_{\text{CKM}} = \begin{bmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{bmatrix}$$

The Feynman rules is [r]

$$i\frac{g}{\sqrt{2}}\gamma^\mu \frac{1-\gamma_5}{2} V_{fg}$$

The matrix is unitary therefore

$$\sum_{f=1}^{3} |V_{fg}|^2 = \sum_{g=1}^{3} |V_{fg}|^2 = 1\,, \quad \sum_f V_{fg}^* V_{fh} = 0\,, \quad g \neq h$$

Geometrically, the fact that the matrix is unitary implies that the sum of the components must be a triangle in the complex plane. The entries of the matrix are measurable. The measurement give a circumference on which a vertex lies. This is the test of unitarity of the CKM matrix.

Lecture 8

2.2 Charge-parity violation

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See Schwartz, §29.5. QED is a vectorial theory, so it conserves parity and charge conjugation; CP symmetry is conserved. QCD is also a vectorial theory, however CP is violated: it is the strong CP violation. According to the theory is room for CP violation which is not particularly suppress, but in experiments it is.

Also the weak sector exhibits CP violation. One may study spinor bilinear fields because they appear in interaction terms. First, one is concerned with how the spinor bilinear operators transform under CP

$$\begin{gathered}(\bar{\psi}_f\psi_g)(t,\mathbf{x}) \to \bar{\psi}_g\psi_f(t,-\mathbf{x}) \\ \bar{\psi}_f\gamma_5\psi_g \to -\bar{\psi}_g\gamma_5\psi_f \\ \bar{\psi}_f \not{A}\psi_g \to \bar{\psi}_g\not{A}\psi_f \\ \bar{\psi}_f\not{A}\gamma_5\psi_g \to \bar{\psi}_g\not{A}\gamma_5\psi_f\end{gathered}$$

[r] The charged current Lagrangian in the mass eigenbasis is

$$\mathcal{L}_{\mathrm{cc}} = \frac{g}{\sqrt{2}}[W_\mu^+\bar{u}^f V_{fg}\gamma^\mu\frac{1}{2}(1-\gamma_5)d^g + W_\mu^-\bar{d}^f(V_{gf})^*\gamma^\mu\frac{1}{2}(1-\gamma_5)u^g] = \frac{g}{\sqrt{2}}[\mathcal{W}^+ + \mathcal{W}^-]$$

Under CP transformation, one obtains

$$\begin{aligned}\mathcal{W}^- &\to W_\mu^+\bar{u}^g(V_{gf})^*\gamma^\mu\frac{1}{2}(1-\gamma_5)d^f \\ \mathcal{W}^+ &\to W_\mu^-\bar{d}^g(V_{fg})^*\gamma^\mu\frac{1}{2}(1-\gamma_5)u^f\end{aligned}$$

CP symmetry is preserved if $\mathcal{W}^\mp \to \mathcal{W}^\pm$. One may check if this is the case

$$\mathrm{CP}(\mathcal{W}^+) = \bar{d}^g V_{fg}\Gamma u_f\,, \quad \Gamma = \frac{1}{2}\gamma^\mu(1-\gamma_5)$$

while

$$\mathcal{W}^- = \bar{d}^g V_{fg}^*\Gamma u^f$$

For the two to be equal, one has to have

$$V_{fg} = V_{fg}^*$$

This does not happen for three families of matter because the V matrix has a complex phase: it implies CP violation.

This violation happens in nature. In particular, in the oscillation and decay of $\bar{K}^0$-K^0 systems. The quark content of these kaons is

$$\left|K^0\right\rangle \sim |d\bar{s}\rangle\,, \quad \left|\bar{K}^0\right\rangle \sim \left|\bar{d}s\right\rangle$$

One may define the following CP eigenstates

$$\left|K^\pm\right\rangle = \frac{1}{\sqrt{2}}(\left|K^0\right\rangle \pm \left|\bar{K}^0\right\rangle)$$

with eigenvalue ± 1. The state long $|K_L\rangle$ is dominated by the eigenstate $|K^-\rangle$. If CP is conserved one would assumes that the eigenvalue is conserved. However, rarely it decays to

$$K_L \to \pi^0\pi^0$$

which has CP eigenvalue $+1$. In neutrally charged, strange mesons there is CP violation.

The parameter ε_k in the superposition of states of kaon long can be measured

$$|K_L\rangle = \left|K^-\right\rangle + \varepsilon_k\left|K^+\right\rangle$$

2.3 CKM matrix and Beyond the Standard Model tests

One of the possible ways of parametrizing the CKM matrix is

$$\begin{aligned}
V_{\mathrm{CKM}} &= \begin{bmatrix} c_{12}c_{13} & s_{12}s_{13} & s_{12}\mathrm{e}^{\mathrm{i}\delta} \\ \cdots & \cdots & s_{23}c_{12} \\ \cdots & \cdots & c_{23}c_{13} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix} \begin{bmatrix} c_{13} & 0 & s_{13}\mathrm{e}^{\mathrm{i}\delta} \\ 0 & 1 & 0 \\ -s_{13}\delta^{-\mathrm{i}\delta} & 0 & c_{13} \end{bmatrix} \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

where ij are flavour indices and s and c are sine and cosine of θ_{ij}. The last matrix on the right is the Cabibbo matrix. The measurements are

$$s_{12} \approx 0.225\,, \quad s_{23} \approx 0.04\,, \quad s_{13} \approx 0.003\,, \quad \delta \approx 65^\circ$$

One notices that the θ_{ij} angles are not big

$$\theta_{13} \ll \theta_{23} \ll \theta_{12}$$

and the δ angle is not small. One may check that

$$|V| \approx \begin{bmatrix} 0.97 & 0.226 & 0.003 \\ 0.226 & 0.97 & 0.04 \\ 0.008 & 0.04 & 0.99 \end{bmatrix} \approx \begin{bmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{bmatrix} + o(\lambda^3)\,, \quad \lambda \approx 0.22$$

This matrix is kind of the identity. One may write the matrix in an approximate way to understand how the physics enter in the matrix.

Wolfenstein parametrization. [r] One introduces four parameters A, λ, ρ, η to get

$$s_{12} = \lambda = \frac{|V_{us}|}{\sqrt{|V_{ud}|^2 + |V_{us}|^2}}\,, \quad s_{13} = A\lambda^2 = \lambda \left| \frac{V_{cb}}{V_{us}} \right|, \quad s_{13}\mathrm{e}^{\mathrm{i}\delta} = A\lambda^3(\rho + \mathrm{i}\eta) \equiv V_{ub}^*$$

The exact CKM matrix becomes

$$V_{\mathrm{CKM}} = \begin{bmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - \mathrm{i}\eta) \\ -\lambda & 1 - \frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1 - \rho - \mathrm{i}\eta) & -A\lambda^2 & 1 \end{bmatrix} + o(\lambda^3)$$

The pattern becomes clear and the quark mixing is

$$(12) \gg (23) \gg (13)$$

There is also a CP violating phase.

Jarlskog invariant. One would like to quantify CP violation without using an explicit parametrization

$$\mathrm{Im}(V_{ij}V_{kl}V_{il}^*V_{kj}^*) = J \sum_{mn} \varepsilon_{ikm}\varepsilon_{jln}$$

where J is the invariant. If $J \neq 0$ then there is CP violation. This parameter can be written as

$$J = c_{12}c_{23}c_{13}^2 s_{12}s_{13}s_{23}(\sin\delta) \approx \lambda^6 A^2 \eta$$

Remark. The Jarlskog invariant is not zero because all mixing angles are not zero.

Remark. The invariant is small (so CP violation is suppressed) because the mixing (angles) are small (and not because δ is small). In fact $J \sim \lambda^6$.

Unitary triangle. Since the matrix is unitary, there are six combinations of entries that must be zero. One in particular is the (db) combination

$$0 = \sum_{i=1}^{3} V_{id} V_{ib}^* = V_{ud} V_{ub}^* + V_{cd} V_{cb}^* + V_{td} V_{tb}^*$$

For this particular combination, the three addenda are of the same order λ^3. The triangle is almost equilateral. Since J does not depend on the parametrization, then the area of the triangles is the same since it is proportional to J.

Dividing by $V_{cd} V_{cb}^*$, one obtains

$$1 + \frac{V_{ub}^* V_{ud}}{V_{cd} V_{cb}^*} + \frac{V_{tb}^* V_{td}}{V_{cd} V_{cb}^*} = 0$$

[r] diagr
This is the unitary triangle. The edges have lengths of order 1. The lengths of the sides are

$$|AB| = \sqrt{(1-\rho)^2 + \eta^2}\,, \quad |OA| = \sqrt{\rho^2 + \eta^2}$$

The angles are related to quantities describing how quark flavours mix. One may check if one can over-constrain the triangle (by taking more measurements than parameters) and see if everything is consistent.

3 Unitarity and optical theorem

One would like to understand why a decaying particle has a decay width and why the Higgs boson gives longitudinal degrees of freedom to vector gauge bosons and why scattering these degrees of freedom give a unitary phenomenon.

Quantum mechanics. In ordinary quantum mechanics, a Hilbert space is complete

$$I = \sum_x \int \mathrm{d}\Pi_x \, |x\rangle\langle x| \,, \quad \mathrm{d}\Pi_x = \prod_{j=x} \frac{\mathrm{d}^3 p_j}{(2\pi)^3 2E_j} = \prod_{j\in x} [\mathrm{d}p_j]$$

[r] where one sums over all possible states x of the theory. The phase-space $\mathrm{d}\Pi_x$ is Lorentz-invariant. The scattering matrix S may be obtained through

$$|\psi, t\rangle = \mathrm{e}^{-\mathrm{i}Ht} |\psi, 0\rangle \equiv S |\psi, 0\rangle$$

This relates the unitary of the theory to the properties of the scattering matrix. The transition density probability is

$$|\langle f| S |i\rangle|^2$$

Summing over all final states, one has

$$1 = \sum_f |\langle f| S |i\rangle|^2 = \sum_f \langle f| S |i\rangle (\langle f| S |i\rangle)^* = \sum_f \langle i| S^\dagger |f\rangle \langle f| S |i\rangle = \langle i| S^\dagger S |i\rangle \implies S^\dagger S = I$$

if one assumes that $\langle i|i\rangle = 1$.

When computing transition amplitudes, one is interested in the non-identity part of the S-matrix given by the transition matrix T

$$S = I + \mathrm{i}T\,, \quad \langle f| T |i\rangle = (2\pi)^4 \delta^{(4)}(p_i - p_f)\mathcal{M}$$

In the Feynman amplitude there are only connected diagrams contributions. In momentum space, one has

$$\mathrm{i}\mathcal{M} = \sum \text{Feynman diagrams}$$

From the unitarity of the S-matrix, one obtains

$$(I + \mathrm{i}T)(I - \mathrm{i}T^\dagger) = I \implies \mathrm{i}(T^\dagger - T) = T^\dagger T$$

Calculating the expectation value between the states $\langle f|$ and $|i\rangle$, one obtains

$$\mathcal{M}(i \to f) - \mathcal{M}^*(f \to i) = \mathrm{i} \sum_x \int \mathrm{d}\Pi_x \, (2\pi)^4 \delta^{(4)}(p_i - p_x) \mathcal{M}(i \to x) \mathcal{M}^*(f \to x)$$

This is the generalized optical theorem. There are non trivial links between amplitudes with initial and final states.

The expansion of the amplitudes in the left-hand side has powers higher of the coupling constant than the amplitudes in the right-hand side.

Lecture 9

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The above identity must hold order by order in perturbation theory. There is a link between tree-level and one-loop amplitudes (to compute complicated loop amplitudes, one may use concepts from tree-level amplitudes).

One may see a few applications of the theorem. Consider $|i\rangle = |f\rangle = |A\rangle$. From the optical theorem one has

$$2\mathrm{i} \operatorname{Im} \mathcal{M}(A \to A) = \mathrm{i} \sum_x \int \mathrm{d}\Pi_x \, (2\pi)^4 \delta^{(4)}(p_A - p_x) |\mathcal{M}(A \to x)|^2$$

One may assume that the state $|A\rangle$ is a one-particle state. The transition width is

$$\Gamma(A \to x) = \frac{1}{2m_A} \int \mathrm{d}\Pi_x \, (2\pi)^4 \delta^{(4)}(p_A - p_x) |\mathcal{M}(A \to x)|^2$$

From the above relation from the optical theorem, one obtains

$$\operatorname{Im} \mathcal{M}(A \to A) = m_A \Gamma_{A,\mathrm{tot}}$$

One may instead assume that the state $|A\rangle$ is a two-particle state. [r] The cross-section is

$$\begin{aligned} \sigma(A \to x) &= \frac{1}{4E_{\mathrm{CM}}|\mathbf{p}_{\mathrm{in}}|} \int \mathrm{d}\Pi_x \, (2\pi)^4 \delta^{(4)}(p_A - p_x) |\mathcal{M}(A \to x)|^2 \\ &= \frac{1}{2s} \int \mathrm{d}\Pi_x \, (2\pi)^4 \delta^{(4)}(p_A - p_x) |\mathcal{M}(A \to x)|^2 \end{aligned}$$

At the second equality one one assumes that the two particles are massless, so the notation simplifies (see Schwartz).

From the consequence of the optical theorem, one obtains

$$\operatorname{Im} \mathcal{M}(A \to A) = s \sum_x \sigma(A \to x) = s \sigma_{\mathrm{tot}}(A)$$

If A is a state of two colliding particles, then the final state is a scattering without any deflection: this is called forward scattering amplitude.

Example. Consider two scalar fields

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{1}{2} M^2 \Phi^2 + \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2} M^2 \varphi^2 + \frac{\lambda}{2} \Phi \varphi \varphi$$

If $M > 2m$, then the decay $\Phi \to \varphi\varphi$ is kinematically possible. One would like to explicitly verify that

$$\operatorname{Im} \mathcal{M}(\Phi \to \Phi) = M \Gamma(\Phi \to \varphi\varphi) + o(\lambda)$$

One needs to compute the amplitude of the left-hand side [r] diagr

$$\mathrm{i}\mathcal{M} = 2 \left[\frac{\mathrm{i}\lambda}{2}\right]^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\varepsilon} \frac{\mathrm{i}}{(k-p)^2 - m^2 + \mathrm{i}\varepsilon} = 2 \left[\frac{\mathrm{i}\lambda}{2}\right]^2 I$$

Using Feynman parameters

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \, \frac{1}{[A + (B - A)x]^2}$$

then the above becomes

$$I \sim \int \frac{\mathrm{d}^4k}{(2\pi)^4}\left[\frac{1}{(k^2 - A^2 + \mathrm{i}\varepsilon)^2}\right]$$

One may apply the Pauli–Villars regularization to get

$$I \sim \int \frac{\mathrm{d}^4k}{(2\pi)^4}\left[\frac{1}{(k^2 - A^2 + \mathrm{i}\varepsilon)^2} - \frac{1}{(k^2 - \Lambda^2 + \mathrm{i}\varepsilon)^2}\right] = -\frac{\mathrm{i}}{16\pi^2}\log\frac{A^2}{\Lambda^2}$$

Finally

$$\mathrm{i}\mathcal{M} = -\mathrm{i}\frac{\lambda^2}{32\pi^2}\int_0^1 \mathrm{d}x \log\frac{m^2 - \mathrm{i}\varepsilon - p^2x(1-x)}{\Lambda^2}$$

Noting that $p^2 = M^2$ since the field Φ is on-shell. So

$$\mathcal{M}(\Phi \to \Phi) = -\frac{\lambda^2}{32\pi^2}\int_0^1 \mathrm{d}x \log\frac{m^2 - M^2x(1-x) - \mathrm{i}\varepsilon}{\Lambda^2}$$

One may notice that

$$x \in [0,1] \implies x(1-x) \leq \frac{1}{4}$$

If the numerator is negative, one has to be careful about the analytic continuation signaled by $\mathrm{i}\varepsilon$. If $M < 2m$ then the numerator is positive and the logarithm is real

$$\operatorname{Im}\mathcal{M} = 0 \implies \Gamma = 0$$

This is the case where no decay happens. If $M > 2m$ then the logarithm is of the type

$$\log(-A - \mathrm{i}\varepsilon) = \log A - \mathrm{i}\pi\,, \quad A > 0$$

In fact, one may see that

$$-A - \mathrm{i}\varepsilon = A(-1 - \mathrm{i}\varepsilon) = A\mathrm{e}^{-\mathrm{i}\pi}$$

where the branch cut is on $(-\infty, 0]$. A point infinitesimally under the branch cut is rewritten in polar coordinates. One then obtains the final result

$$\operatorname{Im}\mathcal{M}_{1\mathrm{Loop}} = \frac{\lambda^2}{32\pi}\int_0^1 \mathrm{d}x\,\theta[M^2x(1-x) - m^2] = \frac{\lambda^2}{32\pi}\sqrt{1 - \frac{4m^2}{M^2}}$$

Since, noting Heaviside's theta function, the support of the integral is dictated by

$$x(1-x) > \frac{m^2}{M^2} \implies x_{1,2} = \frac{1}{2}\left[1 \pm \sqrt{1 - \frac{4m^2}{M^2}}\right]$$

The right-hand side is

$$\Gamma(\Phi \to \varphi\varphi) = \frac{1}{2}\frac{1}{2M}\int |\mathcal{M}|^2\,\mathrm{d}\phi_2\,, \quad \mathrm{d}\phi_2 = \frac{|\mathbf{p}_f|}{M}\frac{\mathrm{d}\Omega}{16\pi^2}\,, \quad M = E_1 + E_2$$

where $\frac{1}{2}$ comes about because there are two identical bosons in the final state; also

$$E_1 = E_2\,, \quad |\mathbf{p}_1| = |\mathbf{p}_2|\,, \quad |\mathbf{p}_f|^2 = \frac{1}{4}M^2 - m^2$$

The partial width is then

$$\Gamma = \frac{1}{2}\frac{1}{2M}\left[\frac{1}{4} - \frac{m^2}{M^2}\right]^{\frac{1}{2}}\frac{\mathrm{d}\Omega}{16\pi^2}\lambda^2 = \frac{\lambda^2}{32\pi}\frac{1}{M}\sqrt{1 - \frac{4m^2}{M^2}}$$

This is exactly the same result as before, apart from the extra factor M. Therefore

$$\operatorname{Im} M_{1\mathrm{L}} = M\Gamma$$

Width of unstable particles. One may see a first application. From this one may see why a decay particle has a width. Consider a one-particle state and the one-particle irreducible amplitude [r] diagr

$$\cdots \equiv \mathrm{i}\Pi(p^2)$$

The full propagator is obtained through Dyson resummation [r] diagr

$$\cdots = \frac{\mathrm{i}}{p^2 - m^2} + \frac{\mathrm{i}}{p^2 - m^2}(\mathrm{i}\Pi)\frac{\mathrm{i}}{p^2 - m^2} + \cdots = \frac{\mathrm{i}}{p^2 - m^2}\left[1 - \mathrm{i}\Pi\frac{\mathrm{i}}{p^2 - m^2}\right]^{-1} = \frac{\mathrm{i}}{p^2 - m^2 + \Pi(p^2)}$$

One may assume that Π is an imaginary number

$$\cdots = \frac{\mathrm{i}}{p^2 - m^2 + \operatorname{Re}\Pi + \mathrm{i}\operatorname{Im}\Pi}$$

The quantity m is a parameter in the Lagrangian. The physical mass is not this parameter, but the pole of the real part of the propagator. The physical mass is then

$$m^2_{\mathrm{Phys}} = m^2 - \operatorname{Re}\Pi(m^2_{\mathrm{Ph}})$$

Therefore, the full propagator is

$$\cdots = \frac{\mathrm{i}}{p^2 - m^2_{\mathrm{Ph}} + \mathrm{i}\operatorname{Im}\Pi(p^2)}$$

A particle is unstable if

$$\operatorname{Im}\Pi(p^2) \neq 0$$

From the result of the optical theorem for one-particle states, one has

$$\operatorname{Im}\mathcal{M}(A \to A) = m_A\Gamma_{\mathrm{tot}}$$

The one-particle irreducible amplitude is then

$$\mathrm{i}\Pi \equiv \cdots = \mathrm{i}\mathcal{M}(A \to A) \implies \operatorname{Im}\Pi = m_A\Gamma_{\mathrm{tot}}$$

The full propagator for an unstable particle is

$$\cdots = \frac{\mathrm{i}}{p^2 - m^2_{\mathrm{Ph}} + \mathrm{i}m_{\mathrm{Ph}}\Gamma_{\mathrm{tot}}}$$

The modulus squared has a Breit–Wigner distribution. The width Γ is the same width of the distribution.

Example. One may see a second application. The numerator of the propagators is the sum over the physical spin states. For a scalar particle, one has

$$\frac{\mathrm{i}}{p^2 - m^2} \to 1$$

In fact, scalar particles have only one polarization and it is not dependent on momentum. For spin $\frac{1}{2}$ fermions, one has

$$\frac{\mathrm{i}}{p^2 - m^2}(\not{p} + m) = \frac{\mathrm{i}}{p^2 - m^2}\sum_{\mathrm{spin}} u\bar{u}$$

For massive vector boson, one has

$$\frac{\mathrm{i}}{p^2 - m^2}\left[-\eta^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}\right]$$

The bracket can be seen as a matrix with three 1 eigenvalue and one 0 eigenvalue: of the first three, two are transverse and one is longitudinal.

The fact that the numerator is the sum of the spins can be proven from the unitarity of the theory (See Schwartz).

Proposition (Unitarity bounds)**.** One can find a third application. The optical theorem can be seen as a constrained between the amplitude and its modulus squared. Given that the modulus is

$$|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^*$$

then the amplitude cannot be arbitrarily large but has to be bounded.

One may study elastic scattering in the center-of-mass frame

$$A(p_1) + B(p_2) \to A(p_3) + B(p_4)$$

Assuming particles to be massless, one has

$$\sigma_{\text{tot}}(A + B \to A + B) = \frac{1}{2s}\frac{1}{16\pi}\int \mathrm{d}(\cos\theta) \sum |\mathcal{M}|^2$$

One assumes that $\mathcal{M}$ is a function of the polar angle θ only due to cylindrical symmetry. In this scattering, one has

$$|\mathbf{p}_3| = |\mathbf{p}_4| = |\mathbf{p}_1| = |\mathbf{p}_2|$$

The amplitude can be decomposed through Legendre polynomials

$$\mathcal{M}(\theta) = 16\pi \sum_{j=0}^{\infty} a_j (2j + 1) P_j(\cos\theta)$$

where they are normalized as

$$P_j(1) = 1 \iff \theta = 0\,, \qquad \int_{-1}^{1} P_j(\cos\theta) P_k(\cos\theta)\, \mathrm{d}(\cos\theta) = \frac{2}{2j + 1}\delta_{jk}$$

Through the above decomposition, one may write

$$\sigma_{\text{tot}} = \frac{16\pi}{s} \sum_{j=0}^{\infty} (2j + 1)|a_j|^2$$

Comparing this with the optical theorem, one gets constraints on a_j and the amplitude.

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The optical theorem applied to the cross-section gives

$$\operatorname{Im} \mathcal{M}(AB \to AB, \theta = 0) = s \sum_x \sigma_{\text{tot}}(AB \to x) \geq s\sigma_{\text{tot}}(AB \to AB)$$

Substituting the above expressions, one obtains

$$16\pi \sum_{j=0}^{\infty} (2j + 1) \operatorname{Im} a_j \geq 16\pi \sum_{j=0}^{\infty} (2j + 1)|a_j|^2$$

This is the partial wave unitary bound. Since, in general it holds

$$|a_j| \geq \operatorname{Im} a_j$$

Then $|a_j|$ cannot be arbitrarily large since its square must be smaller than the imaginary part.

Special case. Consider

$$\sigma_{\text{tot}}(AB \to x) \approx \sigma_{\text{tot}}(AB \to AB)$$

It follows

$$\operatorname{Im} a_j = |a_j|^2$$

Letting $a_j = x + \mathrm{i}y$, one sees that

$$y = x^2 + y^2 \implies x^2 + \left[y - \frac{1}{2}\right]^2 = \frac{1}{4}$$

which is a circumference of radius $\frac{1}{2}$ centered at $(0, \frac{1}{2})$.

In general, one can prove that

$$\forall j\,, \quad |a_j| \leq 1\,, \quad 0 \leq \operatorname{Im} a_j \leq 1\,, \quad |\operatorname{Re} a| \leq \frac{1}{2}$$

These constraints can be proved to show that certain cross-sections with Fermi interaction are not unitary.

Propagators for massive vector bosons and polarization states. Consider the Proca Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu\,, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

The equations of motion are

$$(\Box + m^2) A_\mu = 0\,, \quad \partial_\mu A^\mu = 0$$

The second equation is not the gauge-fixing Lorenz condition, but comes from the equations of motion. In fact

$$\partial_{A_\sigma} \mathcal{L} = m^2 A^\sigma\,, \quad \partial_{\partial_\rho A_\sigma} \mathcal{L} = -F^{\rho\sigma}\,, \quad \partial_\rho \partial_{\partial_\rho A_\sigma} \mathcal{L} = \partial_{A_\sigma} \mathcal{L}$$

The Euler–Lagrange equations are

$$\partial_\rho F^{\rho\sigma} + m^2 A^\sigma = 0$$

Expanding $F^{\rho\sigma}$ one obtains

$$(\Box + m^2) A^\sigma - \partial^\sigma (\partial_\rho A^\rho) = 0$$

Acting with a derivative, one has

$$(\Box + m^2) \partial_\sigma A^\sigma - \Box(\partial_\rho A^\rho) = 0 \implies \partial_\sigma A^\sigma = 0$$

The solution of the equations of motion in configuration space is

$$A^\mu(x) = \sum_j \frac{\mathrm{d}^3 k}{(2\pi)^3} \widetilde{a}(\mathbf{k}) \varepsilon_j^\mu(\mathbf{k}) \mathrm{e}^{\mathrm{i}kx}\,, \quad k^0 = \sqrt{\mathbf{k}^2 + m^2}$$

where j is the polarization index [r] where sum over k?. The base upon which the solution is decomposed is

$$\varepsilon_0^\mu = (1,0,0,0)\,, \quad \varepsilon_1^\mu = (0,1,0,0)\,, \quad \varepsilon_2^\mu = (0,0,1,0)\,, \quad \varepsilon_3^\mu = (0,0,0,1)$$

The constraint found from the equations of motion implies

$$\partial_\mu A^\mu = 0 \implies k_\mu \varepsilon_j^\mu(\mathbf{k}) = 0\,, \quad k^2 = m^2$$

Therefore, there are three possible solutions with the following normalization convention

$$\varepsilon_i^\mu (\varepsilon_\mu^i)^* = -1$$

Letting the wave number be

$$k^\mu = (E, 0, 0, k_z)$$

One has

$$\varepsilon_1^\mu = (0,1,0,0)\,, \quad \varepsilon_1^\mu = (0,0,1,0)\,, \quad \varepsilon_3^\mu = \frac{1}{m}(k_z, 0, 0, E)\,, \quad \varepsilon_i^\mu k_\mu = 0$$

In the high-energy limit, the longitudinal polarization vector is

$$\varepsilon_3 \sim \frac{E}{m}(1,0,0,1) \sim k^\mu$$

The vector scales with the momentum itself. If there is no Higgs [r] the amplitude breaks unitarity. The Higgs boson makes unitary the high-energy limit.

From the above, one can verify that the sum of the physical polarization is equal to the numerator of the propagator

$$\sum_i \varepsilon_i^\mu \varepsilon_i^\nu = -\eta^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}$$

This does not mean that the propagator is on-shell. One may compute the above sum. One notices that

$$\sum_{j=1}^2 \varepsilon_i^\mu \varepsilon_i^\nu = \operatorname{diag}(0,1,1,0)$$

Also

$$\varepsilon_3^\mu \varepsilon_3^\nu = \frac{1}{m^2}\operatorname{diag}(k_z^2,0,0,E^2) + \frac{1}{m^2}\operatorname{antidiag}(k_zE,0,0,k_zE)$$

Therefore

$$\sum_{i=1}^3 \varepsilon_i^\mu \varepsilon_i^\nu = \operatorname{diag}(-1,1,1,1)+\frac{1}{m^2}\operatorname{diag}(k_z^2+m^2,0,0,E^2-m^2) = -\eta^{\mu\nu}+\frac{k^\mu k^\nu}{m^2}+\frac{1}{m^2}\operatorname{antidiag}(k_zE,0,0,k_zE)$$

[r] For massless vector bosons (like gluons) the situation requires more care.

4 Electroweak sector – phenomenology application

4.1 Kinematics

The differential cross-section of a two-scattering into n products is

$$\mathrm{d}\sigma = \mathcal{F}\overline{\sum}|\mathcal{M}|^2\,\mathrm{d}\phi_n$$

where $\mathcal{F}$ is the flux factor and the Lorentz-invariant phase-space is

$$\mathrm{d}\phi_n = (2\pi)^4\delta^{(4)}(p_i - k_f)\prod_{i=1}^n [\mathrm{d}k_i]\,, \quad [\mathrm{d}k_i] = \frac{\mathrm{d}^3k_i}{(2\pi)^3 2E_i}$$

If the particles with momenta p_1 and p_2, then

$$\mathcal{F} = \frac{1}{2s}\,, \quad s = (p_1+p_2)^2$$

Two-body final state. If the final state has two particles $Q \to k_1 + k_2$, then the phase-space is

$$\mathrm{d}\phi_2 = \frac{1}{16\pi^2}\frac{\widetilde{k}}{Q_0}\,\mathrm{d}\cos\theta\,\mathrm{d}\varphi\,, \quad Q = p_1 + p_2$$

where

$$\widetilde{k}^2 = \frac{1}{4Q_0^2}[Q_0^2 - (m_1+m_2)^2][Q_0^2-(m_1-m_2)^2]\,, \quad k_j^2 = m_j^2$$

For equal masses, one has

$$\widetilde{k}^2 = \frac{1}{4Q_0^2}Q_0^2(Q_0^2-4m^2) \implies \mathrm{d}\phi_2 = \frac{1}{32\pi^2}\sqrt{1-\frac{4m^2}{Q_0^2}}\,\mathrm{d}\cos\theta\,\mathrm{d}\varphi$$

In the center-of-mass frame one has

$$Q^\mu = (Q_0,0,0,0)\,, \quad Q_0^2 = Q^2\,, \quad \mathrm{d}\phi_2 = \frac{1}{32\pi^2}\sqrt{1-\frac{4m^2}{Q^2}}\,\mathrm{d}\cos\theta\,\mathrm{d}\varphi$$

Instead, for massless particles, one has

$$\mathrm{d}\phi_2 = \frac{1}{32\pi^2}\,\mathrm{d}\cos\theta\,\mathrm{d}\varphi = \frac{\mathrm{d}\Omega}{32\pi^2}$$

Notice how there are two variables: there are six degrees of freedom, but four conservation of four-momentum components.

4.2 Z boson decay width

One would like to compute the decay width of a Z boson into a fermion–anti-fermion pair [r] diagr

$$\mathrm{Z} \to \mathrm{f}\bar{\mathrm{f}} \,, \quad f = e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau, u, d, c, s, b$$

The decay into top quarks is not kinematically possible. The above vertex contributes with

$$\frac{\mathrm{i}e}{2\sin\theta\cos\theta}\gamma^\mu(v_f - a_f\gamma_5) \,, \quad v_f = T_3^f - 2Q^f\sin^2\theta \,, \quad a_f = T_3^f \,, \quad e = g\sin\theta$$

where θ is Weinberg's angle.

Particular final state. One computes the width for a given final state and the sums over all states

$$\Gamma_{ff} = \frac{1}{2M_Z}\int \mathrm{d}\phi_2 \overline{\sum}|\mathcal{M}|^2$$

[r] diagr From momentum conservation, one has

$$q = p + p' \,, \quad p^2 = p'^2 = m^2 \,, \quad q^2 = M_Z^2 = (p+p')^2 = 2pp' + 2m^2$$

The amplitude is

$$\mathrm{i}\mathcal{M} = \bar{u}(p)\frac{\mathrm{i}g}{2\cos\theta}\gamma^\mu(v - a\gamma_5)v(p')\varepsilon_\mu(q)$$

Its complex conjugate is

$$\mathcal{M}^* = \frac{g}{2\cos\theta}\bar{v}(p')\gamma^\nu(v - a\gamma_5)u(p)\varepsilon^*_\nu(p)$$

Therefore, the sum over the spin is

$$\begin{aligned}\sum|\mathcal{M}|^2 &= \frac{g^2}{4\cos^2\theta}\operatorname{Tr}[(\not{p} - m)\gamma^\mu(v - a\gamma_5)(\not{p}' + m)\gamma^\nu(v - a\gamma_5)]\sum_{\text{pol}}\varepsilon_\mu\varepsilon^*_\nu \\ &= \frac{g^2}{4\cos^2\theta}4\left[(M_Z^2 + 2m^2)v^2 + (M_Z^2 - 4m^2)a^2\right]\end{aligned}$$

where one has

$$\sum_{\text{pol}}\varepsilon_\mu\varepsilon^*_\nu = -\eta_{\mu\nu} + \frac{q_\mu q_\nu}{M_Z^2}$$

Exercise: compute the sum over spins above.

One may apply the approximation $m \approx 0$ since the heaviest particle is

$$m_\mathrm{b} \approx 4.5\,\mathrm{GeV} \ll M_Z$$

The second term in the polarization sum does not contribute. In fact, the amplitude is

$$\mathcal{M} \sim \bar{u}(p)\gamma^\mu(v - a\gamma_5)v(p')\varepsilon_\mu$$

when squaring it, one obtains

$$|\mathcal{M}|^2 \rightsquigarrow \bar{u}(p)\gamma^\mu(v - a\gamma_5)v(p')q_\mu = \bar{u}(p)(\not{p} + \not{p}')(v - a\gamma_5)v(p') = 0 \impliedby \bar{u}(p)\not{p} = 0$$

Therefore, one has

$$\sum|\mathcal{M}|^2 = G\operatorname{Tr}[\not{p}\gamma^\mu(v - a\gamma_5)\not{p}'\gamma^\nu(v - a\gamma_5)](-\eta_{\mu\nu}) \,, \quad G = \frac{g^2}{4\cos^2\theta}$$

The trace is

$$\begin{aligned}-\eta_{\mu\nu}\operatorname{Tr} &= -\operatorname{Tr}[\not{p}\gamma^\mu(v - a\gamma_5)\not{p}'\gamma_\mu(v - a\gamma_5)] = -\operatorname{Tr}\left[\not{p}\gamma^\mu\not{p}'\gamma_\mu(v - a\gamma_5)^2\right] \\ &= -\operatorname{Tr}\left[\not{p}\gamma^\mu\not{p}'\gamma_\mu(v^2 + a^2)\right] + 2va\operatorname{Tr}[\not{p}\gamma^\mu\not{p}'\gamma_\mu\gamma_5] \\ &= 8(v^2 + a^2)pp' + 0\end{aligned}$$

At the third line one applies

$$\operatorname{Tr}[\not{p}\gamma^\mu\not{p}'\gamma_\mu] = -2\operatorname{Tr}[\not{p}\not{p}'] = -8pp' \,, \quad \gamma^\mu\gamma_\alpha\gamma_\mu = -2\gamma_\alpha$$

and also

$$\operatorname{Tr}\left[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma_\mu\gamma_5\right] \propto \varepsilon^{\alpha\mu\beta}{}_\mu = 0$$

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The decay width is then

$$\mathrm{d}\Gamma = \frac{1}{2M_Z}\frac{1}{3}\sum|\mathcal{M}|^2\left[\frac{1}{32\pi^2}\,\mathrm{d}\cos\theta\,\mathrm{d}\varphi\right]$$

The factor $\frac{1}{3}$ comes from the averaging of the initial polarization states. One notices that

$$2pp' = (p+p')^2 = M_Z^2$$

There is no dependence of the matrix element on the polar angle. So one may integrate the phase space

$$\Gamma = \frac{1}{2M_Z}\frac{1}{3}(v_f^2 + a_f^2)8pp'\frac{g^2}{4\cos^2\theta}\frac{4\pi}{32\pi^2} = (v_f^2 + a^2_{)}M_Z\frac{g^2}{48\pi\cos^2\theta}$$

The total decay width is obtained by summing over all decay channels

$$\Gamma_{\mathrm{tot}} = \Gamma_{ee} + \Gamma_{\mu\mu} + \Gamma_{\tau\tau} + \Gamma_{\mathrm{hadrons}} + N\Gamma_{\nu\nu}$$

In the limit of massless fermions, the first three widths are the same. The width of the adrons is the sum over the available quark flavours. The Z boson decays in quarks but only hadrons are detected: this does not change the width, only the distribution. So

$$\Gamma_{\mathrm{hadrons}} = \Gamma_{uu} + \Gamma_{dd} + \Gamma_{cc} + \Gamma_{ss} + \Gamma_{bb} = 2\Gamma_{uu} + 3\Gamma_{dd}$$

Quarks carry color and they live in the fundamental representation of SU(3). When one sums over the quantum numbers, the final state has a multiplicity of 3 due to the colors.

Using the parameters v_f and a_f for u, d, ν and e, and using $N_\nu = 3$, one obtains a theoretical value of

$$\Gamma_{\mathrm{th}} \approx 2.44\,\mathrm{GeV}$$

while the experimental value is

$$\Gamma_{\mathrm{exp}} \approx 2.49\,\mathrm{GeV}$$

The result has been obtain with some approximations. One has neglected the effects of the masses that give the following contribution to the phase-space $\mathrm{d}\phi_2$

$$\sqrt{1 - \frac{4m^2}{M_Z^2}}$$

One should also include the loop effects from the electroweak sector and the QCD sector.

One may measure the cross-section as a function of the center-of-mass energy $\sqrt{s}$. Since the Z boson decays, one see a Breit–Wigner centered around M_Z. From the distribution, one can measure the width.

The neutrino decays are not detected because they interact weakly, so only sees effects [r]. If one leaves the number of neutrinos as an unknown, one may fit the distribution of the Z boson to extrapolate a number. The number obtained is 3.

If this is not the case, then there may be new physics: the Z boson may couple to another field whose particles are not seen.

4.3 Forward-backward asymmetry in electron-positron annihilation

Consider the scattering

$$e^- e^+ \to \mu^+\mu^-$$

in QED [r] diagr. The amplitude is

$$\mathrm{i}\mathcal{M}_\gamma = [\bar{u}(q)\mathrm{i}e\gamma^\mu v(\bar{q})]\frac{-\mathrm{i}\eta^{\mu\nu}}{Q^2}[\bar{v}(\bar{p})\mathrm{i}e\gamma^\nu u(p)] \implies \mathcal{M}_\gamma = \frac{e^2}{Q^2}[\bar{u}(q)\gamma^\mu v(\bar{q})][\bar{v}(\bar{p})\gamma_\mu u(p)]$$

The sum over the spins is

$$\sum\mathcal{M}^2 = \frac{e^4}{{}^{2}{}_{8}}(u^2+t^2)$$

where one has

$$t = (q-p)^2 = 2 - E_e E_\mu (1-\cos\theta)\,, \quad u = (\bar{q}-p)^2 = -2E_e E_\mu (1+\cos\theta)$$

where θ is the angle with respect to the initial collision line. The cross-section is symmetric for $\theta \to \pi - \theta$: the distribution of muons should be the same.

If one includes full electroweak effect, there is an asymmetry [r] diagr. The amplitude is

$$\begin{aligned} \mathrm{i}\mathcal{M} &= (\mathrm{i}e)^2[\bar{u}(q)\gamma_\mu v(\bar{q})]\frac{-\mathrm{i}\eta^{\mu\nu}}{Q^2}[\bar{v}(\bar{p})\gamma_\nu u(p)] \\ &+ \left[\frac{\mathrm{i}e}{2\sin\theta\cos\theta}\right]^2[\bar{u}(q)\gamma_\mu(v-a\gamma_5)v(\bar{q})]\frac{-\mathrm{i}\eta^{\mu\nu}+m^{-2}Q^\mu Q^\nu}{Q^2-m^2+\mathrm{i}m\Gamma}[\bar{v}(\bar{p})\gamma_\nu(v-a\gamma_5)u(p)] \end{aligned}$$

One may notice that $Q^\mu Q^\nu$ does not contribute because it gives

$$\bar{u}\not{Q}(v-a\gamma_5)v = \bar{u}(\not{p}+\not{\bar{p}})(v-a\gamma_5)v = 0$$

it is null due to Dirac equation. Letting

$$\bar{g} = \frac{e}{2\sin\theta\cos\theta}$$

and 1 be the electron line, 2 be the muon line; then the electron currents are

$$J_1^\mu = \bar{u}(\bar{p})\gamma^\mu u(p)\,, \quad J_1^{\mu 5} = \bar{v}(\bar{p})\gamma^\mu\gamma_5 u(p)$$

Similarly for the muon currents. The amplitude is then

$$\mathcal{M} = \frac{e^2}{s}J_1^\mu J_{2\mu} + \frac{\bar{g}^2}{s-M_Z^2+\mathrm{i}M_Z\Gamma_Z}(v_2J_2 - a_2J_2^5)^\mu(v_1J_1-a_1J_1^5)_\mu$$

Letting

$$X = \left(\frac{\bar{g}}{e}\right)^2\frac{s}{s-M_Z^2+\mathrm{i}M_Z\Gamma_Z}$$

the amplitude is

$$\begin{aligned} \mathcal{M} &= \frac{e^2}{s}[J_1J_2 + X(v_2J_2-a_2J_2^5)^\mu(v_1J_1-a_1J_1^5)_\mu] \\ &= \frac{e^2}{s}[J_1J_2(1+Xv_1v_2) + X(-v_1a_2J_1J_2^5 - v_2a_1J_2J_1^5) + Xa_1a_2J_1^5J_2^5] \end{aligned}$$

The complex conjugate is

$$\mathcal{M}^* = \frac{e^2}{s}[J_1^*J_2^*(1+X^*v_1v_2) + X^*(-v_1a_2J_1^*J_2^{5*} - v_2a_1J_2^*J_1^{5*}) + X^*a_1a_2J_1^{5*}J_2^{5*}]$$

One may study the electronic line. The terms appearing are of the type

$$\sum_{\mathrm{pol}} J_\mu J_\nu^* = \sum_{\mathrm{tot}}[\bar{v}(\bar{p})\gamma_\mu u(p)\bar{u}(p)\gamma_\nu v(\bar{p})] = \mathrm{Tr}[\not{\bar{p}}\gamma_\mu\not{p}\gamma_\nu] = 4[\bar{p}_\mu p_\nu + p_\mu\bar{p}_\nu - p\bar{p}\eta_{\mu\nu}] = V_{\mu\nu}(p,\bar{p})$$

Also

$$\sum_{\mathrm{pol}} J_\mu^5 J_\nu^{5*} = \mathrm{Tr}[\not{\bar{p}}\gamma_\mu\gamma_5\not{p}\gamma_\nu\gamma_5] = \mathrm{Tr}[\not{\bar{p}}\gamma_\mu\not{p}\gamma_\nu] = V_{\mu\nu}(p,\bar{p})$$

The axial terms are

$$\sum_{\mathrm{pol}} J_\mu^5 J_\nu^* = \mathrm{Tr}[\not{\bar{p}}\gamma_\mu\gamma_5\not{p}\gamma_\nu] = \mathrm{Tr}[\not{\bar{p}}\gamma_\mu\not{p}\gamma_\nu\gamma_5] = -4\mathrm{i}\varepsilon_{\alpha\mu\beta\nu}\bar{p}^\alpha p^\beta = 4\mathrm{i}\varepsilon_{\mu\nu\alpha\beta}\bar{p}^\alpha p^\beta \equiv A_{\mu\nu}(p,\bar{p})$$

Contractions. The amplitude squared $|\mathcal{M}|^2$ gives various contractions

$$V_{\mu\nu}(p,\bar{p})V^{\mu\nu}(q,\bar{q}) = 32[(pq)(\bar{p}\bar{q}) + (p\bar{q})(\bar{p}q)] = 32[(pq)^2 + (p\bar{q})^2$$

where one has used

$$p\bar{q} = \frac{(p-\bar{q})^2}{-2} = \frac{(\bar{p}-q)^2}{-2} = \bar{p}q$$

Likewise

$$V_{\mu\nu}A^{\mu\nu} = 0$$

because the first is symmetric while the second is anti-symmetric. Finally

$$A_{\mu\nu}(p,\bar{p})(A^{\mu\nu}(q,\bar{q}) = 16(\varepsilon_{\mu\nu\alpha\beta}\bar{p}^\alpha p^\beta)(\varepsilon^{\mu\nu\rho\sigma}\bar{q}_\rho q_\sigma) = -32[(\bar{p}\bar{q})(pq) - (p\bar{q})(\bar{p}q)]$$

where one has used

$$\varepsilon_{\mu\nu\alpha\beta} = -\varepsilon^{\mu\nu\alpha\beta}\,, \quad \varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\mu\nu\rho\sigma} = -2(\delta^\rho_\alpha\delta^\sigma_\beta - \delta^\sigma_\alpha\delta^\rho_\beta)$$

Kinematics. Letting θ be the angle of the product with the respect to the collision line. Then

$$p^\mu = (E,0,0,E)\,, \quad \bar{p}^\mu = (E,0,0,-E)$$

also

$$q^\mu = (E,0,E\sin\theta,E\cos\theta)\,, \quad \bar{q}^\mu = (E,0,-E\sin\theta,-E\cos\theta)$$

Some products are

$$pq = E^2(1-\cos\theta)\,, \quad p\bar{q} = E^2(1+\cos\theta) \implies s = (p+\bar{p})^2 = 4E^2$$

One obtains

$$V_{\mu\nu}V^{\mu\nu} = 4s^2(1+\cos^2\theta)\,, \quad A_{\mu\nu}A^{\mu\nu} = 8s^2\cos\theta$$

Squared amplitude. The averaged squared amplitude is

$$\begin{aligned}
\overline{\sum}|\mathcal{M}|^2 &= \frac{1}{4}\sum_{\text{in}}\sum_{\text{out}}|\mathcal{M}|^2 \\
&= \frac{1}{4}\left(\frac{e^2}{s}\right)^2\Big[VV(|1+Xv_1v_2|^2 + |Xv_1a_2|^2 + |Xv_2a_1|^2 + |Xa_1a_2|^2) \\
&\quad + AA[(1+Xv_1v_2)(X^*a_1a_2) + 2|X|^2v_1v_2a_1a_2(1+X^*v_1v_2)(Xa_1a_2)]\Big] \\
&= \frac{1}{4}\left(\frac{e^2}{s}\right)^2 4s^2\Big[(1+\cos^2\theta)[1 + 2(\operatorname{Re}X)v_1v_2 + |X|^2(v_1^2+a_1^2)(v_2^2+a_2^2)] \\
&\quad + 4\cos\theta[(\operatorname{Re}X)a_1a_2 + 2|X|^2(v_1v_2a_1a_2)]\Big]
\end{aligned}$$

Inside the first bracket, the term 1 comes purely from QED. The terms with $|X|^2$ come from the interference between Z and Z, while the terms with $\operatorname{Re}X$ come from interferences of γ and Z [r] diagr.

This matrix element is not symmetric for $\theta \to \pi - \theta$. The second line is not trivial: depending on the energy (which is contained in X), one may have more muons going forward than backwards (or viceversa).

Lecture 12

Knowing that the flux and the phase space are mer 03 apr 2024 12:30

$$\frac{1}{2s}\,, \quad \mathrm{d}\phi_2 = \frac{\mathrm{d}\cos\theta\,\mathrm{d}\varphi}{32\pi^2}$$

the differential cross-section in the center-of-mass frame is

$$
\begin{aligned}
(\mathrm{d}_\Omega\sigma)_{\mathrm{CM}} &= \frac{1}{64\pi^2}\frac{1}{s}e^4\left[(1+\cos^2\theta)C_{VV}+4\cos\theta C_{AA}\right] \\
&= \frac{\alpha^2}{4s}[(1+\cos^2\theta)C_{VV}+4\cos\theta\, C_{AA}]\,, \quad \alpha^2 = \frac{e^2}{4\pi}
\end{aligned}
$$

where one has

$$
\begin{aligned}
C_{VV} &= 1+2\operatorname{Re}X\, v_1v_2 + |X|^2(v_1^2+a_1^2)(v_2^2+a_2^2) \\
C_{AA} &= \operatorname{Re}X\, a_1a_2 + 2|X|^2 v_1v_2a_1a_2
\end{aligned}
$$

The total cross-section is

$$
\sigma_{\mathrm{tot}} = \int \mathrm{d}_\Omega\sigma\,\mathrm{d}\Omega = \frac{\alpha^2}{4s}\int_0^{2\pi}\mathrm{d}\varphi\int_{-1}^{1}\mathrm{d}\cos\theta[(1+\cos^2\theta)C_{VV}+4\cos\theta\, C_{AA}] = \frac{4\pi}{3}\frac{\alpha^2}{s}C_{VV}
$$

The structure of C_{VV} implies that the cross-section, as a function of $\sqrt{s}$, has a peak corresponding to the Z boson mass m_Z which comes from the third addendum

$$
X = \left(\frac{\bar{g}}{e}\right)^2\frac{s}{s-M_Z^2+\mathrm{i}M_Z\Gamma_Z}
$$

However, for s small, then $C_{VV}\sim 1$ and the cross-section grows since it gets contributions from QED. There is a middle region that is a mix of QED and weak interactions.

The width of the Breit–Wigner has been already computed.

Asymmetry. The forward-backward asymmetry can be quantified by

$$
\begin{aligned}
A_s = A_{\mathrm{FB}} = \frac{\sigma_+-\sigma_-}{\sigma_++\sigma_-} &= \left[\int_{\theta=\frac{\pi}{2}}^{\theta=0}\mathrm{d}_\Omega\sigma - \int_{\theta=\pi}^{\theta=\frac{\pi}{2}}\mathrm{d}_\Omega\sigma\right]\left[\int_{\theta=\frac{\pi}{2}}^{\theta=0}\mathrm{d}_\Omega\sigma + \int_{\theta=\pi}^{\theta=\frac{\pi}{2}}\mathrm{d}_\Omega\sigma\right]^{-1} \\
&= \frac{4C_{AA}[\int_0^1\mathrm{d}\cos\theta\,\cos\theta - \int_{-1}^0\mathrm{d}\cos\theta\,\cos\theta]}{C_{VV}\int_{-1}^1(1+\cos^2\theta)\,\mathrm{d}\cos\theta} = \frac{3}{2}\frac{C_{AA}}{C_{VV}}
\end{aligned}
$$

Remark. Far from QED $s\gg 0$ and below the Z boson peak $s\ll m_Z^2$, one has

$$
X = \left(\frac{\bar{g}}{e}\right)^2\frac{s}{s-m_Z^2+\mathrm{i}m_Z\Gamma_Z}\sim\left(\frac{\bar{g}}{e}\right)^2\frac{-s}{m_Z^2} = \cdots = -s\frac{G_{\mathrm{F}}}{2\sqrt{2}\pi\alpha}
$$

where one remembers

$$
\alpha = \frac{e^2}{4\pi}\,,\quad \frac{G_{\mathrm{F}}}{\sqrt{2}} = \frac{g^2}{8m_W^2}\,,\quad a_1=a_2=T_3^f=-\frac{1}{2}\,,\quad v_1=v_2=T_3^f-2Q_f\sin^2\theta_W = -\frac{1}{2}+2\sin^2\theta_W
$$

One may assume

$$
X\lesssim 1 \implies \frac{sG_{\mathrm{F}}}{\alpha}\lesssim 1\implies\sqrt{s}\lesssim\sqrt{\frac{\alpha}{G_{\mathrm{F}}}}\approx 34\,\mathrm{GeV}
$$

which is exactly in the middle between pure QED and the Z boson peak. Noting that

$$
C_{VV} = 1+2\operatorname{Re}X\,v_1v_2+o(|X|)\,,\quad C_{AA} = \operatorname{Re}X\,a_1a_2+2|X|^2a_1a_2v_1v_2
$$

one obtains

$$
C_{VV}\approx 1\,,\quad C_{AA}\approx\operatorname{Re}X\,a_1a_2 = \frac{1}{4}\operatorname{Re}X
$$

Therefore, the asymmetry becomes

$$
A_s = \frac{3}{2}\frac{C_{AA}}{C_{VV}}\approx\frac{3}{8}\operatorname{Re}X = \frac{3}{8}\frac{-s}{2\sqrt{2}}\frac{G_{\mathrm{F}}}{\pi\alpha}
$$

It is small (because of X) and it is negative. There must be more backward scattering than forward.

Remark. In the region near the peak, $s \approx m_Z^2$ then X is large. Therefore

$$C_{VV} \approx |X|^2 (v_1^2 + a_1^2)(v_2^2 + a_2^2)\,, \quad C_{AA} \approx 2|X|^2 a_1 a_2 v_1 v_2$$

The asymmetry is then

$$A_s = \frac{3}{8}\frac{16v^2}{(1+4v^2)^2} \approx 12v^2 = 3(1 - 4\sin^2\theta_W)\,, \quad v = v_1 = v_2 \ll 1$$

The asymmetry is positive: there is more forward scattering.

Remark. In pure QED $\sqrt{s} \approx 30\,\mathrm{GeV}$ one has a symmetric parabola $1 + \cos^2\theta$ in $\cos\theta$. Since there is more backward scattering, the data is to the right of the parabola.

Computing the same observable close to the Z boson peak, the data is to the left of the parabola.

Through measurements, one can constrain the free parameters of the Standard model.

5 Higgs boson phenomenology

Higgs decay. Consider the decay of the Higgs boson into a particle, anti-particle final state $H \to A\bar{A}$ with $m_A = m_{\bar{A}} = m$

$$H(q) \to A(p_1) + \bar{A}(p_2)$$

Since the Higgs boson is a scalar particle, there is no preferred direction in its reference frame.

The decay width is

$$\mathrm{d}\Gamma = \frac{1}{2m_H}\overline{\sum}|\mathcal{M}|^2\,\mathrm{d}\phi_2$$

The Feynman amplitude must be a function of the only scalars that can be built. Only $p_1 p_2$ is independent:

$$(p, q) = (p_1 + p_2)p_1 = m^2 + p_1 p_2$$

Therefore

$$|\mathcal{M}|^2 = F(p_1 p_2, m_H^2, m_A^2)$$

One finds that

$$p_1 p_2 = \frac{(p_1 + p_2)^2 - 2m_A^2}{2} = \frac{m_H^2 - 2m_A^2}{2}$$

The phase space is

$$\mathrm{d}\phi_2 = \frac{1}{8\pi}\sqrt{1 - \frac{4m^2}{m_H^2}}$$

where one has already integrated in the angles.

The decay width is then

$$\Gamma = \frac{1}{16\pi m_H}\overline{\sum}|\mathcal{M}|^2\sqrt{1 - \frac{4m^2}{m_H^2}}$$

Higgs to fermions. There is only one Feynman diagram [r]. The amplitude is

$$\mathrm{i}\mathcal{M} = -\mathrm{i}\frac{m_f}{v}\bar{u}(p_1)v(p_2)$$

The squared amplitude is

$$\begin{aligned}\overline{\sum}|\mathcal{M}|^2 &= N_C^{(f)}\left(\frac{m_f}{v}\right)^2 \sum_{\mathrm{pol}} \bar{u}(p_1)v(p_2)\bar{v}(p_2)u(p_1) = N_C^{(f)}\left(\frac{m_f}{v}\right)^2 \mathrm{Tr}[(\not{p}_1 + m_f)(\not{p}_2 - m_f)] \\ &= N_C^{(f)}\left(\frac{m_f}{v}\right)^2 \mathrm{Tr}[\not{p}_1\not{p}_2 - m_f^2] = N_C^{(f)}\left(\frac{m_f}{v}\right)^2 [4p_1 p_2 - 4m_f^2] \\ &= N_C^{(f)}\left(\frac{m_f}{v}\right)^2 2m_H^2\left[1 - \frac{4m_f^2}{m_H^2}\right]\end{aligned}$$

where the color summation is

$$N_C^{(\text{leptons})} = 1\,, \quad N_C^{(\text{quarks})} = 3$$

The decay width is then

$$\Gamma = \frac{1}{16\pi m_H}\left[1 - \frac{4m_f^2}{m_H^2}\right]^{\frac{3}{2}} N_C^{(f)} 2m_H^2 \left(\frac{m_f}{v}\right)^2 = N_C^{(f)} \frac{G_\mathrm{F}}{4\pi\sqrt{2}} m_H m_f^2 \left[1 - \frac{4m_f^2}{m_H^2}\right]^{\frac{3}{2}}$$

The coupling of the Higgs boson is proportional to the mass of the objects, in fact there is the term m_f^2. One expects that the partial width of the decays to heavier particles is greater than the one of lighter particles. Also, due to the color charge, quarks have greater width than similar mass fermions.

Exercise. Example of exam exercises are decays to $H \to ZZ$ and $H \to W^-W^+$ assuming $m_H > 2m_V$.

Remark. The decay width of the Higgs boson is very small with respect to the mass, contrary to the Z boson. When the Higgs boson has energy for the production of WW then the decay width becomes much greater and these decays dominate.

The reason why it grows is that the vector boson have three polarizations and the longitudinal polarization is of the type

$$\varepsilon_3^\mu = (p_z, 0, 0, E)\frac{1}{m_W}$$

The more the vector bosons are energetic, the more the longitudinal polarization grows because it depends on the energy: this is not the cause for transverse polarizations.

Remark. One also notes that the decay width to the charm quarks is smaller than the tau decay, but the former is heavier than the latter. So this is surprising.

The decay width to fermions is proportional to the number of QCD colors

$$\Gamma \sim N_C^{(f)} m_f^2$$

Since the charm is heavier than tau, then

$$\Gamma_c \sim N_c m_c^2\,, \quad N_\tau \sim m_\tau^2 \implies \left.\frac{\Gamma_c}{\Gamma_\tau}\right|_{\text{naive}} \approx 1.7$$

However, in QFT the mass of a field is a running parameter and has to be renormalized like the coupling. The mass of the charm quark has different values depending on the energy it is probed. The charm and the bottom receive large quantum correction. The running masses in the $\overline{\text{MS}}$ scheme are

$$m_b^2(\mu = m_H) \approx 0.45 m_b^2|_{\text{pole}}\,, \quad m_c^2(\mu = m_H) \approx 0.25 m_c^2|_{\text{pole}}$$

where $m^2|_{\text{pole}}$ is the kinematic mass, the one that is measured. For this reason

$$\left.\frac{\Gamma_c}{\Gamma_\tau}\right|_{\text{true}} = \frac{1}{4}\left.\frac{\Gamma_c}{\Gamma_\tau}\right|_{\text{naive}} = 0.42$$

Remark. The Higgs boson may decay to photons, but since these are massless, the decay cannot be tree-level and, in fact, it is given by a loop effect [r] diagr. The decay is the product of a destructive interference of two loop diagrams. Their sum scale like $|\mathcal{M}|^2 \propto \alpha^2$ and this is way it is suppressed. This decay channel has a distinct signature since the decay product is just photons which are easy to detect.

Remark. The Higgs may decay to four leptons through two virtual vector boson [r] diagr. This amplitude is suppressed because one boson is suppressed. The amplitude scales like

$$\frac{1}{(Q_{12}^2 - m_Z^2) + m_Z^2\Gamma_Z^2}\frac{1}{(Q_{34}^2 - m_Z^2) + m_Z^2\Gamma_Z^2}$$

If $Q_{12}^2 \approx m_Z^2$ then there is not enough energy for the other propagator. One pair of leptons is close to the Z boson peak, while the other pair is in a generic configuration. It holds always

$$Q_{1234}^2 = m_H^2$$

The Higgs boson was discovered through gluon fusion which produces the Higgs which then decays in two photons. At the LHC there is a dominant background of photons given by $q\bar{q}$. On top of the background there is a very narrow Breit–Wigner of the Higgs.

Another channel, clearer to detect, is the four-fermion decay. There are various backgrounds which produce two peaks and the Higgs produces a peak in the middle.