Quantum Gravity

Maso*

March 23, 2024

Contents

Lecture 1

mer 06 mar 2024 14:30

Exam. Presentation on topic not seen during class, even a follow up.
Lecture notes by Tom Hartman.

Program. First week is general difficulties with quantum gravity, overview and approaches, some problems associated with quantum gravity. The following three weeks are black hole thermodynamics, black hole information paradox. The next three weeks are introduction to CFT. The following week is an instruction to AdS. The last three weeks are an introduction to AdS/CFT.

1 Introduction

Classical gravity. The action of general relativity is the Einstein–Hilbert action

$$S = \frac{1}{16\pi G_N} \int \mathrm{d}^D x \sqrt{-g}[R - 2\Lambda] + S_{\text{matter}}$$

The equations of motion are Einstein's field equations

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}^{\text{matter}}$$

Classical gravity is understood well. Solutions to these equations include black holes.

*https://github.com/M-a-s-o/notes

Quantum field theory. Quantum dynamics without gravity is quantum field theory. It describes the matter sector of the EH Lagrangian. It describes the three other fundamental forces: electromagnetic, weak and strong forces. They are unified in the Standard Model which has worked very well. Though, quantum chromodynamics is a strongly couple theory and perturbation theory is limited.

Quantum gravity. Quantum gravity is supposed to unify general relativity with the Standard Model by quantizing the gravitational field in a consistent manner.

However, there are some difficulties. The naive approach is taking GR action, take a metric, pick a background and expand it around such background

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{16\pi G_N} h_{\mu\nu}\,, \quad \Lambda = 0$$

As such one gets

$$S = \int \mathrm{d}^D x \sqrt{-g} \left[\frac{1}{2}[\partial_\mu h^{\mu\nu}\,\partial_\nu h - \partial_\mu h^{\rho\sigma}\,\partial_\rho h^\mu{}_\sigma + \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h^{\rho\sigma})(\partial_\nu h_{\rho\sigma}) - \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h)(\partial_\nu h) + \sum_{n\geq 3} G_N^{\frac{n}{2}} P_n(\partial_\mu, h_{\mu\nu})\right]$$

where $h = \operatorname{Tr} h_{\mu\nu}$. The first three terms is the kinetic term of the graviton, a spin-2 particle. There are also an infinite number of terms (given by the sum) which can be thought of as interactions. [r] Now one can think about this theory as a QFT for the field h. The theory has one coupling constant G_N with length dimensions $D-2$. Every time one adds the coupling, one has to add momentum to have a dimensionless quantity $G_N p^{D-2}$ so at higher order the theory diverges more severely. This theory is not renormalizable, more precisely it is not power-counting renormalizable.

This means that GR should be seen as an effective field theory (EFT), because this problems involves higher energies.

The mass scale relevant is the Planck's mass

$$S = \frac{1}{16\pi G_N}\int \mathrm{d}^4 x\sqrt{-g}R = \frac{1}{16\pi\hbar}\int \mathrm{d}^4 x\sqrt{-g}M_P^2 R\,, \quad M_P = \sqrt{\frac{\hbar}{G_N}} \approx 1.22\times 10^{19}\,\mathrm{GeV}$$

So the action looks like the following

$$S = \frac{1}{16\pi\hbar}\int \mathrm{d}^4 x\sqrt{-g}[M_P^2 R + R^2 + M_p^{-2}R^3 + \cdots)$$

The last terms give contributions in the UV energy spectrum, while in IR then the terms are just tiny corrections. The Planck scale is also the scale at which the Schwartzschild radius is the Compton wavelength.

Non-renormalizability. To solve the problem of renormalizability:

- the naive power-counting fails and there are infinite cancellations in the Feynman diagrams (for GR it has been computed at two-loops and the cancellations do not happen, same happens in maximal supergravity but higher orders);
- there is a fixed point in the RG flow but cannot be understood through power-counting: the perturbative QFT is naive and there is a non-perturbative fixed point;
- pure quantum GR is not a consistent theory, so one must add other degrees of freedom besides the graviton (string theory works in this way).

The above is also dependent on the background of the expansion, but even for the simplest, Minkowski, there are problems.

One would like to understand the path integral of quantum gravity. The partition function is

$$Z[\text{b.c.}] = \int_{\partial g=\text{b.c.}} [\mathcal{D}g]\,\mathrm{e}^{-\frac{i}{\hbar}S_{\mathrm{EH}}}$$

where b.c. stands for boundary condition. [r] one fixes the metric infinitely far away. For AdS the integral is understood non perturbatively.

Different approaches to quantum gravity. Different approaches to quantum gravity are

- string theory; originally string theory was developed for strong interaction;
- holography and AdS/CFT; it originated from string theory and will be developed from here on;
- loop quantum gravity; space-time is discrete and there are critical length scales, the degrees of freedom are loops like in gauge theory, one formulates a [r];
- asymptotic safety; there is a fixed point of GR in the UV, but the power-counting argument fails bc it is perturbative, but the fixed point is not perturbative;
- causal dynamical triangulation; it is a lattice approach to QG but it needs to rely on asymptotic safety.

1.1 Black hole entropy

The best hint towards quantum gravity is without doubt the Bekenstein–Hawking formula for black hole entropy

$$S_{\mathrm{BH}} = \frac{1}{4}\frac{k_{\mathrm{B}}c^3}{\hbar G_N} A_{\mathrm{H}}$$

Natural units are used in the following $c = \hbar = k_B = G_N = 1$. A quarter of the area of a black hole's event horizon A_{H} is the entropy. In this formula, there appears Boltzmann constant for statistical mechanics, the speed of light for special relativity, the universal gravitation constant for gravity and Planck's constant for quantum physics.

The formula first came from a thought experiment of Bekenstein. Entropy always increases thanks to the second law of thermodynamics. Black holes must have entropy because the information of falling objects must be kept. Hawking proved that the area of a black hole always increases

$$\Delta A \geq 0$$

So the entropy of the black hole should be proportional to the area. Hawking found the proportionality constant to be $\frac{1}{4}$.

In thermodynamics, entropy is extensive so it scales with volume, but in this case it scales with the area. The property of being extensive is related to degrees of freedoms of local systems so it is an important property. The concept of holography is that the degrees of freedom of a local quantum gravity are located on the boundary of the black hole, not its bulk.

Black holes have a lot of entropy. In statistical mechanics, entropy counts the fundamental different degrees of freedoms

$$S = \log \rho_{\mathrm{micro}}$$

For example, Sagittarius A* the entropy is

$$S_{\mathrm{BH}} = 2.69 \times 10^{67}\,\mathrm{J\,K^{-1}}$$

which is about 10^{20} the entropy of the sun.

A success of string theory is that for supersymmetric five-dimensional black holes, a count of individual degrees of freedom coincides with the formula above.

1.2 Open problems

The hierarchy problem. The mass of the Higgs boson $m_{\mathrm{H}} \approx 125\,\mathrm{GeV}$, but the Standard Model predicts infinity (for the bare value of the mass). The mass is computed from the propagator and there are loop diagrams like the self-interaction that give

$$\int \mathrm{d}^4k \, \frac{1}{k^2 - m^2} \sim k_{\mathrm{max}}^2$$

The mass correction to the mass is $\delta m_{\mathrm{H}}^2 \sim \# k_{\mathrm{max}}^2$. However $\delta m_{\mathrm{H}} \gg m_{\mathrm{H}}$. The cancellation of the bare mass and true mass happens to many digits. This problem is related to fine tuning and naturalness problem.

This is related to quantum gravity because there may be not new physics up to the quantum gravity energy scale and QG has to solve such hierarchy problem. If $k_{\max} \sim M_p$ then one has a fine-tuning of 10^{17} digits.

There are many resolutions within quantum field theory like supersymmetry.

The cosmological constant problem. In quantum field theory the energy of the vacuum is not very much important, but in general relativity energy bends space-time. The energy density of the vacuum is

$$\begin{aligned}\rho_{\text{vac}} &= \langle 0|\, \rho \,|0\rangle = \langle 0|\, T_{00} \,|0\rangle = \frac{1}{2}\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\sqrt{k^2+m^2} \\ &= \frac{1}{4\pi^2}\int^{k_{\max}} \mathrm{d}k\, k^2\sqrt{k^2+m^2} = \frac{k_{\max}^4}{16\pi^2}\left[1+\frac{m^2}{k_{\max}}+\cdots\right]\end{aligned}$$

The observed energy density is

$$\Lambda_{\text{eff}} = \Lambda_{\text{bare}} + 8\pi G_N \rho_{\text{vac}}\,, \quad \rho_{\text{obs}} = \frac{\Lambda_{\text{eff}}}{8\pi G_N} \approx 10^{-47}\,\text{GeV}^4$$

At the Planck's scale one has

$$\rho_{\text{vac}} \sim 10^{71}\,\text{GeV}^4$$

Lecture 2

gio 07 mar 2024 14:30

This argument is missing a part. The pressure density is

$$\langle 0|\, p_{\text{vac}} \,|0\rangle = \langle 0|\, T_{0i} \,|0\rangle \implies \frac{\rho_{\text{vac}}}{p_{\text{vac}}} = \frac{1}{3} \neq -1$$

The energy density is not -1 so it cannot be interpreted as the cosmological constant. The UV cutoff regularization breaks Lorentz invariance.

Proper argument. The energy density of the vacuum is proportional to the vacuum bubble diagram. To preserve Lorentz invariance one may use dimensional regularization:

$$\rho_{\text{vac}} = \frac{m^4}{64\pi^2}\ln\frac{m^2}{4\pi\mu^2}$$

The quartic divergence appears in the mass, while the divergence in the momentum μ is only logarithmic. In quantum field theory one does not care about the vacuum energy because all disconnected diagrams are cancelled. Computing in this way the pressure, the ratio is -1. So observed energy density is

$$\rho_{\text{obs}} = \rho_{\text{bare}} + \rho_{\text{vac}}$$

for the tau lepton mass one has $\rho_{\text{obs}} \sim 10^{-47}\,\text{GeV}^4$ and $\rho_{\text{vac}} \sim 10^8\,\text{GeV}^4$; so there are still 40 orders of magnitude.

Resolution in AdS/CFT. Starting from the Einstein–Hilbert action

$$S = \frac{1}{16\pi G_N}\int \mathrm{d}^D x\,\sqrt{-g}(R-2\Lambda) = \frac{1}{16\pi G_N}\int \mathrm{d}^D x\,\sqrt{-g}\left[R + \frac{(D-1)(D-2)}{l_{\text{AdS}}^2}\right]$$

where l_{AdS} is a length scale of the cosmological constant. There are two length scales

$$l_p \sim G_N^{\frac{1}{D-2}}\,, \quad l_{\text{AdS}}$$

The effective field theory suggests that the Planck scale is the same as the AdS. [r] However gravity is semi-classical if the curvature of AdS is very little and as such the length scale is very large: this happens for $l_{\mathrm{AdS}} \gg l_p$. This is the cosmological constant problem, the separation of the scale is not natural from the point of view of effective field theory.

AdS/CFT is a dictionary and one of its entries is

$$\frac{l_{\mathrm{AdS}}^{D-2}}{G_N} = C_T$$

where C_T is the central charge. In conformal field theory [r]

$$\langle T_{\mu\nu}(x_1) T_{\rho\sigma}(x_2) \rangle = C_T I_{\mu\nu\rho\sigma}(x_1, x_2)$$

The second factor is fixed by CFT symmetry up to one number. The central charge should be $C_T \gg 1$ to get semi-classical gravity. In CFT, the central charge counts the number of degrees of freedom. Therefore, to solve the cosmological problem one has to consider CFT with many degrees of freedom.

2 Black hole thermodynamics

Black holes carry an entropy proportional to the event horizon

$$S_{\mathrm{BH}} = \frac{A_{\mathrm{H}}}{4G_N}$$

Black holes also have temperature. The numerical factor was found by Hawking by calculating the temperature of a black hole.

There are two approaches to study the thermodynamics of black holes:

- euclidean methods, which involve thermal physics;
- QFT on a curved background.

2.1 Euclidean quantum field theory

The partition function is

$$Z(\beta) = \operatorname{Tr} \mathrm{e}^{-\beta H} \qquad \beta = \frac{1}{T}$$

In thermal physics everything can be obtained from it. The entropy is

$$S = -\partial_t (T \log Z)$$

61G4|o

One would like to find the connection with path integrals. In ordinary quantum mechanics, a Schrödinger state evolves through the Lorentzian time evolution operator

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht} |\psi(0)\rangle$$

This operator is unitary. The Euclidean time τ is given by $t = -\mathrm{i}\tau$. The state is then

$$|\psi(t)\rangle = \mathrm{e}^{-\tau H} |\psi(0)\rangle$$

The Euclidean time evolution operator is not unitary. A Euclidean path integral is a transition amplitude

$$\langle \phi_2 | \mathrm{e}^{-\tau H} | \phi_1 \rangle = \int_{\phi_1}^{\phi_2} [\mathcal{D}\phi] \, \mathrm{e}^{-S_{\mathrm{E}}} \,, \quad \phi_1 = \phi(\tau = 0) \,, \quad \phi_2 = \phi(\tau)$$

One integrates all field configurations with some boundary conditions weighed by some the Euclidean action.

One can also use path integrals to prepare states. Consider a quantum state

$$|\psi\rangle = \mathrm{e}^{-\beta H} |\phi_1\rangle$$

It has a wave-functional

$$\Psi(\phi) = \langle \phi | \mathrm{e}^{-\beta H} | \phi_1 \rangle$$

[r] The state ϕ is not fixed, but depends on the specific boundary conditions. In terms of the path integral one has

$$\int_{\phi_1}^{\phi(\beta)} [\mathcal{D}\phi] \, \mathrm{e}^{-S_{\mathrm{E}}}$$

Ground state. Consideri a generic state

$$|\psi\rangle \sum_n c_n |n\rangle \ , \quad H|n\rangle = E_n |n\rangle$$

The evolved state is

$$e^{-\beta H}|\psi\rangle = \sum_n c_n e^{-\beta E_n}|n\rangle$$

Taking $\beta \to \infty$, one can extract the vacuum (which means $T \to 0$ where T is the temperature). The vacuum is given by the Euclidean path integral with open boundary conditions at $\tau = 0$ and path integrating from $-\infty$; this is half of $\mathbb{R}^d$. A bra can be prepared from ∞ to zero. The vacuum normalization is just a path integral form $-\infty$ to ∞. The boundary conditions get integrated, it is similar to inserting a complete set of states

$$\langle 0|0\rangle = \sum_{\phi_1} \langle 0|\phi_1\rangle \langle \phi_1|0\rangle$$

This is the starting point to generate everything else.

Correlation functions. A two-point correlation function in the vacuum is

$$\langle 0|\phi(x_1)\phi(x_2)|0\rangle = \int_{\mathbb{R}^d} [\mathcal{D}\phi]\, e^{-S_E}\phi(x_1)\phi(x_2)$$

One prepares the vacuum and inserts two operators at two specific points. In Euclidean space there is no Hilbert space because one uses fields and computes functions in Euclidean coordinates.

Thermal physics. One would like to compute the partition function

$$Z = \operatorname{Tr} e^{-\beta H} = \sum_{\phi_1} \langle \phi_1| e^{-\beta H}|\phi_1\rangle$$

The sum over same boundary conditions glues them together, so one obtains a cylinder with circumference β: periodic boundary conditions. Thermal physics corresponds to periodicity in Euclidean time. The period is $\beta = \frac{1}{T}$ where T is the temperature.

[r] The Green function is

$$G(\tau, x) = \langle O(\tau, x)O(0,0)|_\beta = \operatorname{Tr}\big[e^{-\beta H}O(\tau,x)O(0,0)\big] = \operatorname{Tr}\big[O(0,0)e^{-\beta H}O(\tau,x)\big]$$

Knowing that

$$O(\beta, 0) = e^{-\beta H}O(0,0)e^{\beta H}$$

one has

$$G(\tau, x) = \operatorname{Tr}\big[e^{-\beta H}O(\beta,0)O(\tau,x)\big] = G(\tau-\beta, x)$$

This is periodicity.

The path integral with open boundary conditions on one side and closed on the other is a state. If both are open then it is a density matrix of the thermal state

$$\rho = e^{-\beta H}$$

The amplitude is given by definite boundary conditions

$$\langle \phi_2|\rho|\phi_1\rangle$$

Density matrix. An entangled state

$$|EPR\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) \neq |\psi\rangle_1 \otimes |\psi\rangle_2$$

cannot be written as two separate qubit states. [r] Tracing over one state one has

$$\langle\uparrow|_2 |EPR\rangle\langle EPR| |\uparrow\rangle_2 + \langle\downarrow|_2 |EPR\rangle\langle EPR| |\downarrow\rangle_2 = \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

It is a density matrix.

2.2 Rindler space and polar coordinates

The metric of two-dimensional Rindler space

$$\mathrm{d}s^2 = \mathrm{d}R^2 - R^2\,\mathrm{d}\eta^2\,, \quad R > 0\,, \quad \eta \in \mathbb{R}$$

The horizon of the space is $R = 0$ where the metric is singular. Setting $\eta = \mathrm{i}\theta$, the metric becomes

$$\mathrm{d}s^2 = \mathrm{d}R^2 + R^2\,\mathrm{d}\theta^2$$

that is polar coordinates. Rindler space is the Lorentzian equivalent of polar coordinates. Using the change of coordinates

$$x = R\cosh\eta\,, \quad t = R\sinh\eta$$

one obtains flat space

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2$$

One has

$$x^2 - t^2 = R^2 > 0$$

so on the Penrose diagram one occupies only the right Rindler patch. In higher dimension, the Rindler space is

$$\mathrm{d}s^2 = \mathrm{d}R^2 - R^2\,\mathrm{d}\eta^2 + \sum_{i=2}^{d} \left|\mathrm{d}x^i\right|^2$$

The trajectories of fixed R are not geodesics, but are trajectories of constant acceleration.

In Minkowski space, the Hamiltonian generates time translation

$$H = \int_{\mathbb{R}} \mathrm{d}x\, T_{tt}$$

In Rindler space, the η-time translation is a boost

$$H_R = \int_{\mathbb{R}} \mathrm{d}x\, x^2 T_{tt}$$

Euclidean path integral. The overlap of the vacuum with itself is the path integral from $-\infty$ to ∞ generated by $\mathrm{e}^{-\beta H}$ for $\beta \to \infty$. Instead of opening the path integral and inserting a configuration, one opens the path integral as a pizza slice and uses a time evolution operator that gives rotations not translations

$$\mathrm{e}^{-2\pi\,\partial_\phi}\,, \quad \eta = \mathrm{i}\phi\,, \quad H_{\mathrm{Rindler}} = \partial_\phi$$

A rotation is a Euclidean-Rindler time translation.

One particular matrix element of the Rindler Hamiltonian is

$$\langle\phi_2|\,\mathrm{e}^{-2\pi H_{\mathrm{Rindler}}}\,|\phi_1\rangle$$

To compute the path integral over all of space, one has to glue together the two states so sum over the states

$$\langle 0|0\rangle = \sum_{\phi_1} \langle\phi_1|\,\mathrm{e}^{-2\pi H_{\mathrm{Rindler}}}\,|\phi_1\rangle = \mathrm{Tr}\,\mathrm{e}^{-2\pi H_{\mathrm{Rindler}}} = Z_{\mathrm{Rindler}}$$

This is the partition function with temperature $\beta = 2\pi$. The vacuum state can be interpreted as a thermal state of Rindler space of temperature $T = \frac{1}{2\pi}$. The Rindler space is half of Minkowski space, the other half is not know and it is a temperature mass [r]. The Rindler horizon is not very different from the black hole's.

Lecture 3

mer 13 mar 2024 14:30

2.3 Hawking temperature

The entropy of a black hole is related to its area. The Schwarzschild metric is given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2MG}{r}} + r^2\,\mathrm{d}\Omega^2$$

Going to Euclidean time, one has

$$\mathrm{d}s_{\mathrm{E}}^2 = \left(1 - \frac{2GM}{r}\right)\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{1 - \frac{2MG}{r}} + r^2\,\mathrm{d}\Omega_2^2$$

One expands around the event horizon to get the near horizon expansion

$$r_{\mathrm{H}} = 2MG\,, \quad r = 2MG(1+\varepsilon^2)\,, \quad \varepsilon \ll 1$$

The line element is

$$\begin{aligned}\mathrm{d}s_{\mathrm{E}}^2 &= \left(1 - \frac{1}{1+\varepsilon^2}\right)\mathrm{d}\tau^2 + \frac{(2MG)^2 4\varepsilon^2\,\mathrm{d}\varepsilon^2}{1 - \frac{1}{1+\varepsilon^2}} + (2MG)^2(1+\varepsilon^2)^2\,\mathrm{d}\Omega_2^2 \\ &= \varepsilon^2\,\mathrm{d}\tau^2 + 16M^2G^2\,\varepsilon^2 + 4M^2G^2\,\mathrm{d}\Omega_2^2\end{aligned}$$

The first two addenda are the Rindler Euclidean metric, while the last is the two-sphere. The full solutions expands the Einstein's field equations, Rindler is flat space, but the sphere has positive metric. In the vacuum, Einstein's equation give $R = 0$, but here $R \neq 0$. There is no horizon in the Euclidean Rindler metric. [r]

Employing a change of coordinates

$$\varepsilon = \frac{R}{4MG}\,, \quad \tau = \frac{\theta}{4MG}$$

one gets

$$\mathrm{d}s^2 = R^2\,\mathrm{d}\theta^2 + \mathrm{d}R^2 + 4M^2G^2\,\mathrm{d}\Omega_2^2$$

This metric is called cigar metric. The coordinate θ is 2π-periodic, being smaller or greater implies a conical singularity. Going back to Euclidean time τ of an observer at infinity, one has

$$\tau \sim \tau + 8\pi MG$$

The Euclidean time has the period above, so the temperature is

$$\beta = 8\pi MG \implies T_{\mathrm{H}} = \frac{1}{8\pi MG}$$

For a black hole of the mass of the sun, the temperature is $T_{\mathrm{H}} \approx 6 \times 10^{-8}\,\mathrm{K}$. The cosmic microwave background as a temperature of $T_{\mathrm{CMB}} \approx 3\,\mathrm{K}$.

To compute the entropy, one uses the following thermodynamic identity

$$\mathrm{d}E = T\,\mathrm{d}S$$

Remembering that energy is mass, one has

$$\mathrm{d}M = T\,\mathrm{d}S = \frac{1}{8\pi MG}\,\mathrm{d}S$$

The entropy is proportional to the area of the black holed

$$S = \alpha A_{\mathrm{H}} = \alpha 4\pi r_{\mathrm{H}}^2 = \alpha 16\pi M^2G^2$$

Therefore

$$\mathrm{d}S = \alpha 32\pi G^2 M\,\mathrm{d}M \implies \mathrm{d}M = \frac{1}{8\pi MG}\alpha 32\pi MG^2\,\mathrm{d}M \implies \alpha = \frac{1}{4G}$$

The entropy of a black hole is

$$S_{\mathrm{BH}} = \frac{A_{\mathrm{H}}}{4G}$$

This answer was found with the anstaz of entropy been linear in the area: only this ansatz works because masses cancels.

This derivation is semi-classical: one has gone into Euclidean space and has applied classical thermodynamical equations to obtain a valid answer. Why is happens is yet unknown.

2.4 Quantum field theory in curved space-time

One picks a frozen background metric $g^{\mathrm{b}}_{\mu\nu}$ which solves Einstein's field equations and applies quantum field theory on such background without Einstein's equations. One consider a state of the QFT, it may have a non trivial stress-tensor, but this is not input into Einstein's equation to solved for a new metric: the metric is frozen.

This method is easier than quantum gravity, however it is not trivial.

Revision of quantum field theory. Consider a free scalar field ϕ of mass m. The equation of motion are Klein–Gordon's

$$(\Box + m^2)\phi = 0 \implies \phi(x^\mu) = \phi_0 \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\mathbf{k}\cdot\mathbf{x}}\,, \quad \omega^2 = |\mathbf{k}|^2 + m^2$$

The above is just one solution. The most general one is a superposition of plane waves. One needs to find an orthonormal set of motions onto which expand the solutions. The Klein–Gordon inner product

$$(\phi_1, \phi_2) = -\mathrm{i}\int_{\Sigma_t} (\phi_1\,\partial_t\phi_2^* - \phi_2^*\,\partial_t\phi_1)\,\mathrm{d}^{d-1}x\,, \quad \Sigma_t\,, \quad t = 0$$

where Σ is a slice [r] a solution is given by

$$f_{\mathbf{k}}(x^\mu) = \frac{\mathrm{e}^{\mathrm{i}kx}}{\sqrt{(2\pi)^{d-1}2\omega}}$$

The dispersion relation does not fix the sign of the energy.

The positive-frequency modes have

$$\partial_t f_{\mathbf{k}} = -\mathrm{i}\omega f_{\mathbf{k}}\,, \quad \omega > 0$$

The negative-frequency modes have

$$\partial_t f_{\mathbf{k}}^* = \mathrm{i}\omega f_{\mathbf{k}}^*\,, \quad \omega < 0$$

One has the inner products

$$(f_{\mathbf{k}_1}, f_{\mathbf{k}_2}^*) = 0\,, \quad (f_{\mathbf{k}_1}^*, f_{\mathbf{k}_2}^*) = -\delta^{(d-1)}(\mathbf{k}_1 - \mathbf{k}_2)$$

The most general solution is given by

$$\phi(t, \mathbf{x}) = \int \mathrm{d}^{d-1}k\,[a_{\mathbf{k}} f_{\mathbf{k}}(t, \mathbf{x}) + a_{\mathbf{k}}^* f_{\mathbf{k}}^*(t, x)]$$

One promotes the field to operator which implies the Fourier coefficient to be operators as well. The positive frequency modes are associated with the annihilation operators $\hat{a}_{\mathbf{k}}$. The vacuum is the state which is annihilated by all annihilation operators

$$\hat{a}_k\,|0\rangle = 0$$

A Fork space is built upon the vacuum

$$\hat{a}_{\mathbf{k}_1}^\dagger \cdots \hat{a}_{\mathbf{k}_n}^\dagger\,|0\rangle$$

The number operator counts the number of particle in a state

$$N_{\mathbf{k}} = \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}$$

Performing a boost, the momenta k^μ change, but the vacuum is Lorentz-invariant. The number of particles is conserved. In curved space-time this conservation no longer holds.

Curved space-time. The Lagrangian is made covariant

$$\mathcal{L} = -\frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \rightsquigarrow \mathcal{L} = \sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\,\nabla_\nu\phi - \frac{1}{2}m^2\phi^2 - \xi R\phi^2\right]$$

where R is the Ricci scalar; with $\xi = 0$ is the minimal coupling, while the conformal coupling is

$$\xi = \frac{d-2}{4(d-1)}$$

This last coupling means that the Lagrangian is invariant under conformal transformations (scaling and translating). The equations of motion are

$$(\nabla_\mu\nabla^\mu - m^2)\phi = 0$$

The covariant Klein–Gordon inner product is

$$(\phi_1, \phi_2) = -\mathrm{i}\int_\Sigma (\phi_1\,\nabla_\mu\phi_2^* - \phi_2^*\,\nabla_\mu\phi_1)\,n^\mu\sqrt{\gamma}\,\mathrm{d}^{d-1}x$$

where n^μ is a unit vector pointing outwards from the slice Σ and γ is the induced metric on the slice Σ. One needs to find an orthonormal set of solution with respect to the inner product above.

In general, the solution is not separable, as opposed to flat space-time $f_t(t)f_{\mathbf{k}}(\mathbf{x})$. One needs to pick some functions $f(t, \mathbf{x})$ such that

$$(f_i, f_j) = \delta_{ij}\,, \quad (f_i^*, f_j^*) = -\delta_{ij}$$

The scalar field is then the superposition

$$\phi = \sum_i a_i f_i + a_i^* f_i^*$$

Similar to flat space-time, one promotes the field to operators and defines the vacuum as the state annihilated by all destruction operators

$$a_i\,|0\rangle_f = 0\,, \quad \forall i$$

The number operator is

$$N_{fi} = a_i^\dagger a_i$$

One could have also done a different expansion

$$\phi = \sum_j (b_j g_j + b_j^\dagger g_j^*)$$

with different vacuum and different number operator

$$b\,|0\rangle_g = 0\,, \quad N_{gj} = b_j^\dagger b_j$$

Though no basis is the preferred one. The important property is that, in flat space-time the vacuum is unique, while in this case the vacuum is not unique

$$|0\rangle_f \neq |0\rangle_g$$

Even if there is no preferred basis, two observers cannot agree on the number of particles.

[r] To get from one basis to another, one employs Bogoliubov transformation

$$g_i = \sum_j (\alpha_{ij} f_j + \beta_{ij} f_j^*)\,, \quad f_i = \sum_j (\alpha_{ji}^* g_j - \beta_{ji} g_j^*)$$

[r] which is a change of basis. The coefficients are defined as the overlap of the wave functions

$$\alpha_{ij} = (g_i, f_j)\,, \quad \beta_{ij} = -(g_i, f_j^*)$$

They are normalized as

$$\sum_k (\alpha_{ik}\alpha^*_{jk} - \beta_{ik}\beta^*_{jk}) = \delta_{ij} \,, \quad \sum_k (\alpha^*_{ik}\beta_{jk} - \beta^*_{ik}\alpha_{jk}) = 0$$

The transformation can be also used to change the destruction and creation operators

$$a_i = \sum_j (\alpha_{ij} b_j + \beta^*_{ji} b^\dagger_j) \,, \quad b_i = \sum_j (\alpha^*_{ij} a_j - \beta^*_{ij} a^\dagger_j)$$

[r] The different choices do not agree. The expectation value of the number of the g_i mode is

$$\begin{aligned} {}_f\langle 0| n_{gi} |0\rangle_f &= {}_j\langle 0| \sum_{jk} (\alpha^*_{ij} a^\dagger_j - \beta_{ij} a_j)(\alpha_{ik} a_k - \beta^*_{ik} a^\dagger_k) |0\rangle_f \\ &= {}_f\langle 0| \sum_{jk} \beta_{ij} a_j \beta^*_{ik} a^\dagger_k |0\rangle_f = {}_f\langle 0| \sum_{jk} \beta_{ij}\beta^*_{ik} (\delta_{jk} + a^\dagger_k a_j) |0\rangle_f \\ &= {}_f\langle 0| \sum_j |\beta_{ij}|^2 |0\rangle = \sum_j |\beta_{ij}|^2 > 0 \end{aligned}$$

Different observers do not agree on the particle count. The vacuum for one gives a net positive number of particle for another observer. In the context of black hole, a geodesic measures no particles, but at constant radius one feels a flux of particle coming from Hawking radiation.

Lecture 4

gio 14 mar 2024 14:30

This is similar to Rindler space: the Minkowski vacuum is a thermal state. The particle number is not conserved because one does not perform Lorentz boosts. By the equivalence principle, Rindler space is similar to a curved background.

Rindler space. Consider a trajectory

$$t(\tau) = \frac{1}{\alpha} \sinh \alpha\tau \,, \quad x(\tau) = \frac{1}{\alpha} \cos \alpha\tau$$

The acceleration is

$$a^\mu = \mathrm{d}^2_\tau x^\mu \,, \quad a^t = \alpha \sinh \alpha\tau \,, \quad a^x = \alpha \cosh \alpha\tau \,, \quad \sqrt{a^\mu a_\mu} = \alpha$$

The above trajectory is a hyperbola in the right quadrant of Minkowski space xt

$$x^2(\tau) = t^2(\tau) + \frac{1}{\alpha^2}$$

The point at $t = 0$ has $x = \alpha^{-1}$. The radius is constant and given by $R = \alpha^{-1}$. Rindler time is the time perceived when constantly accelerating in Minkowski space.

Wave equation in Rindler space. The wave equation for a massless scalar field is given by

$$\Box\,\phi = 0$$

The line element is

$$\mathrm{d}s^2 = -R^2\,\mathrm{d}\eta^2 - \mathrm{d}R^2$$

The solutions are

$$g^{\mathrm{R}}_k = \frac{1}{\sqrt{4\pi\omega}} \mathrm{e}^{-\mathrm{i}\omega\eta} R^{\mathrm{i}k} \,, \quad \omega = |\mathbf{k}|$$

The positive frequency modes are

$$\partial_\eta g^R = -\mathrm{i}\omega g^R \,, \quad \omega > 0$$

[r] There is another change of coordinate

$$t = -R\sinh\eta\,, \quad x = -R\cosh\eta$$

It describes the left quadrant of Minkowski space. In this coordinate system, the solution to the wave equation is

$$g_k^{\mathrm{L}} = \frac{1}{\sqrt{4\pi\omega}\mathrm{e}^{\mathrm{i}\omega\eta}R^{\mathrm{i}k}}$$

The positive frequency modes are

$$\partial_{-\eta}g^{\mathrm{L}} = -\mathrm{i}\omega g^{\mathrm{L}}\,, \quad \omega > 0$$

The two solutions obtain are true for different coordinate systems [r]. The Rindler expansion of the field is

$$\varphi = \int \mathrm{d}k\,[g_k^{\mathrm{R}}b_k^{\mathrm{R}} + (g_k^R)^*(b_k^R)^\dagger + g_k^Lb_k^L + (g_k^L)^*(b_k^L)^\dagger]$$

where one has

$$[b_{k_1}^L, b_{k_2}^R] = [b_{k_1}^L, (b_{k_2}^R)^\dagger] = 0$$

The Minkowski expansion of the field is

$$\varphi = \int \mathrm{d}k\,[a_kf_k + a_k^\dagger f_k^*]$$

Since there are two sets of modes, there are two inequivalent vacua, one for Rindler and one for Minkowski. The expectation values of the hamiltonian are

$$\langle 0|\,H\,|0\rangle_{\mathrm{Rindler}} = \infty$$

This vacuum is not part of a Hilbert space because it is singular. The Bogoliubov coefficients are

$$\alpha_{ij} = (g_i, f_j)\,, \quad b_i = \sum_j \alpha_{ij}a_j + \cdots$$

from which one can compute

$$\langle 0|\,b_i^\dagger b_i\,|0\rangle_M$$

However this is lengthy and one can use a trick to see that the vacuum is thermal. One patches together the Rindler modes to obtain something related to Minkowski modes. In fact

$$\sqrt{4\pi\omega}g_k^R = (-t+x)^{\mathrm{i}\omega}\,, \quad x > |t|\,, \quad k > 0$$

for $x < 0$ one has $g_k^R = 0$. The right modes look like waves that disappear in the left patch of Minkowski space. The left modes are

$$\sqrt{4\pi\omega}g_k^L = (-t-x)^{-\mathrm{i}\omega}$$

However this does not work because the way propagates in the opposite direction to the right modes. Though, one may notice

$$\sqrt{4\pi\omega}(g_{-k}^L)^* = \mathrm{e}^{\pi\omega}(-t+x)^{\mathrm{i}\omega}$$

The factor in front can be absorbed

$$\sqrt{4\pi\omega}[g_k^L + \mathrm{e}^{-\pi\omega}(g_{-k}^R)^*] = (-t+x)^{\mathrm{i}\omega}$$

In this way the modes are not normalized. So to normalize them one defines

$$h_k^{(1)} = \frac{1}{[\mathrm{e}^{\frac{\pi\omega}{2}}}g_k^R + \mathrm{e}^{-\frac{\pi\omega}{2}}(g_{-k}^L)^*] \sim (-t+x)^{\mathrm{i}\omega}$$

$$h_k^{(2)} = \frac{1}{[\mathrm{e}^{\frac{\pi\omega}{2}}}g_k^L + \mathrm{e}^{-\frac{\pi\omega}{2}}(g_{-k}^R)^*] \sim (t+x)^{\mathrm{i}\omega}$$

These are not the Minkowski modes because there an exponential between. Therefore, one has

$$\varphi = \int \mathrm{d}k\, [c_k^{(1)} h_k^{(1)} + c_k^{(1)} (h_k^{(1)})^* + c_k^{(2)} h_k^{(2)} + (c_k^{(2)})^\dagger (h_k^{(2)})^*]$$

One may show that

$$c_k^{(1)} \left|0\right\rangle_M = 0 = c_k^{(2)} \left|0\right\rangle_M$$

The coefficients c_k are a superposition of only destruction operators a_k.

One would like to compute the expectation value of the number operator on the Minkowski vacuum

$$\left\langle 0\right| n_k^R \left|0\right\rangle_M = \left\langle 0\right| (b_k^R)^\dagger b_k^R \left|0\right\rangle_M$$

Inverting the expressions, one has

$$b_k^R = \frac{1}{\sqrt{2\sinh \pi\omega}} [\mathrm{e}^{\frac{\pi\omega}{2}} c_k^{(1)} + \mathrm{e}^{-\frac{\pi\omega}{2}} (c_k^{(2)})^\dagger]$$

Inserting this expression in the expectation value above, one obtains

$$\left\langle 0\right| n_k^R \left|0\right\rangle_M = \frac{1}{2\sinh \pi\omega} \mathrm{e}^{-\pi\omega} \left\langle 0\right| c_k^{(2)} (c_k^{(2)})^\dagger \left|0\right\rangle = \frac{1}{\mathrm{e}^{2\pi\omega} - 1} \delta(0)$$

The term $\delta(0)$ would not appear if one computes the expectation value for a wave packet instead of a single mode. In the second equality one has applied the commutation relation

$$[c_{k_1}^{(2)}, (c_{k_2}^{(2)})^\dagger] = \delta(k_1 - k_2)$$

The above is the Boltzmann–Maxwell distribution with temperature

$$T = \frac{1}{2\pi}$$

This temperature is valid for an observer at infinity. For a finite radius one would have $T = \frac{a}{2\pi}$. This is in agreement with the Euclidean Rindler time. The presence of such temperature is called Unruh effect.

The Euclidean method is faster and simpler, therefore more powerful.

2.5 Quantum entanglement

One has seen that the vacuum of Minkowski space is the same as a density matrix given by $\mathrm{e}^{-2\pi H_{\mathrm{Rindler}}} = \rho_{\mathrm{R}}$ of the right patch. [r] The state that has support in both left and right patches is called thermofield-double state

$$\left|\mathrm{TFD}(\beta)\right\rangle = \sum_n \mathrm{e}^{-\frac{1}{2}\beta E_n} \left|n\right\rangle_R \otimes \left|n\right\rangle_L^*$$

where $\left|n\right\rangle$ are Rindler energy eigenstates. The factors in the exponent are half Boltzmann factors. The star denotes CPT conjugation. This form already appeared

$$g_k^R + (g_{-k}^L)^*$$

The minus sign is about parity, the star is about charge conjugation, and the flow of time (up for right patch, down for left patch) is time reversal.

The Minkowski vacuum is the thermofield-double of the Rindler modes

$$\left|0\right\rangle_M = \int \mathrm{d}E_n\, \mathrm{e}^{-\pi E_n} \left|n\right\rangle_R \otimes \left|n\right\rangle_L^*$$

The TFD state is an entangled pure state. The boost operator kills the Minkowski vacuum and it is

$$K = H_R - H_L\,, \quad K\left|0\right\rangle_M = 0$$

The trace of the TFD state on the left wedge recovers a thermal state. The entropy of thermodynamics is the entropy between the left and right patch.

Entanglement. In quantum mechanics, a Hilbert space can be often decomposed as product of Hilbert spaces

$$\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$$

A state $|\psi\rangle$ of the Hilbert space $\mathcal{H}$ is entangled if

$$|\psi\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle$$

Entanglement implies correlation. The two point function of two operators is

$$\langle\psi| O_A O_B |\psi\rangle$$

If the state can be factorized, then the correlation function of operators with support only on one Hilbert space is factorized

$$\langle\psi| O_A O_B |\psi\rangle = \langle\psi_A| O_A |\psi_A\rangle \langle\psi_B| O_B |\psi_B\rangle$$

The entanglement can be quantified. The density matrix is

$$\rho = |\psi\rangle\langle\psi|$$

[r] the reduced density matrix is

$$\rho_A = \mathrm{Tr}_B\, \rho$$

One can count the logarithm of entangled q-bits using [r]

$$S_{\mathrm{EE}} = -\,\mathrm{Tr}\, \rho_A \log \rho_A$$

Example. Consider the following EPR pair

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle)$$

The density matrix is

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\uparrow\uparrow\rangle\langle\downarrow\downarrow| + |\downarrow\downarrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|) = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Its trace is $\mathrm{Tr}\,\rho = 1$. If ρ is pure, in the eigebasis it looks like

$$\rho = \mathrm{diag}(1, 0, 0, 0)$$

Using the trace trick, one has

$$\rho_A = \mathrm{Tr}_B\, \rho = \langle\uparrow|_B\, \rho\, |\uparrow\rangle_B + \langle\downarrow|_B\, \rho\, |\downarrow\rangle_B = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|) = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{\dim \mathcal{H}_A} I$$

This is the maximally mixed stated. The [r] entropy is

$$S_{\mathrm{EE}} = -\,\mathrm{Tr}\, \rho_A \log \rho_A = -2\left[\frac{1}{2} \log \frac{1}{2}\right] = \log 2$$

This corresponds to one EPR pair.

The state of the example is also highly correlated, exactly because it is entangled.

Proposition. A few properties. If a state $|\psi\rangle$ is pure, then $S_A = S_B$. If the state $|\psi\rangle$ is a density matrix ρ then $S_A \neq S_B$ and S_{EE} is not the number of EPR pairs.

Multi-partite quantum entanglement is very hard.

Quantum field theory. In quantum field theory, one splits space-time in two, but one finds $S_{\mathrm{EE}} = \infty$. This is because the oscillators are highly entangled between the barrier of the two Hilbert spaces. This is because one cannot split the Hilbert space. A regularization can be done on the lattice.

Rindler space. Going back to Rindler space, one has

$$|\mathrm{TFD}(\beta)\rangle = \sum_n \mathrm{e}^{-\frac{1}{2}\beta E_n} |n\rangle_L \otimes |n\rangle_R^*$$

From which

$$|\mathrm{TFD}\rangle\langle\mathrm{TFD}| = \sum_{nm} \mathrm{e}^{-\frac{1}{2}\beta(E_n+E_m)} |n\rangle_L |n\rangle_R \langle m|_L \langle m|_R$$

Taking the trace, one has

$$\mathrm{Tr}_{H_L} |\mathrm{TFD}\rangle\langle\mathrm{TFD}| = \sum_k \langle k|_L (\cdots) |k\rangle_L = \sum_k \mathrm{e}^{-\beta E_k} |k\rangle_R \langle k|_R (\cdots) |k\rangle_L = \mathrm{e}^{-\beta H_R}$$

[r] The thermal entropy is

$$S_{\mathrm{th}} = -\operatorname{Tr} \rho_{\mathrm{th}} \log \rho_{\mathrm{th}} = S_{\mathrm{EE}}(R)$$

The thermal entropy can be interpreted as entanglement entropy that purifies the state [r]. Applying this to Rindler space, one has

$$|0\rangle_M = |\mathrm{TFD}\rangle_R = \int \mathrm{d}n \, \mathrm{e}^{-\frac{1}{2}\beta E_n} |n\rangle_L \otimes |n\rangle_R^*$$

[r] So

$$S_{\mathrm{th}}(\text{Rindler}) = S_{\mathrm{EE}}(\text{half space-time}) = \infty$$

This infinite entanglement is important in QFT because one can compute correlation functions for space-like separation

$$\langle \phi_L(x_1)\phi_R(x_2)\rangle = \frac{1}{(x_1-x_2)^2}$$

This is only possible because the vacuum is a very entangled state. In an unentangled scenario, one would have

$$\langle\phi\phi\rangle = \langle\phi\rangle\langle\phi\rangle = 0$$

This state has infinite energy.

There is a relation between thermal physics and entanglement. In Minkowski space, the left region is a heat bath for the right patch. [r]

Lecture 5

mer 20 mar 2024 14:30

2.6 Schwarzschild black hole

The radial acceleration needed to remain at a fixed radius in the Schwarzschild metric is

$$a_r = \frac{GM}{r^2\sqrt{1-\frac{2GM}{r}}}$$

One assume that the state is like the vacuum. In free falling one sees the vacuum everywhere. The temperature at the horizon is

$$T_H = \lim_{r\to r_H} \frac{a_r}{2\pi}$$

but at the horizon the acceleration diverges to infinity. One does not measure the local temperature at the horizon, but one is at infinity and measures the temperature there. However, due to the gravitational potential, the temperature at infinity is shifted and related to the temperature at the horizon. The temperature at infinity is

$$T_\infty = \frac{V_H}{V_\infty} T_H = V_H T_H = \frac{V_H a_H}{2\pi}$$

The numerator is called surface gravity. A surface gravity can be defined generally: a Killing vector which is null on a hypersurface Σ [r]. Simply one can think of it as the acceleration needed to remain at a certain radius. The temperature at infinity is

$$T_\infty = \frac{1}{8\pi GM}$$

which is in accordance with the Euclidean trick. In this setup it was assumed that near the horizon one is locally in the vacuum. This is a particular choice of state, but there are other choices.

Hadamard states. Looking at a two-point function on a scale smaller than anything else, one should recover the Minkowski's UV structure from any curved space. The correlation function is a function of the geodesic distance between two points

$$\langle\psi|\,\phi(x)\phi(x_2)\,|\psi\rangle = f(d_{\text{geodesic}}(x_1,x_2)) = f(g^{\mu\nu}x_{12\mu}x_{12\nu})$$

One defines a universal prescription

$$\sigma_\varepsilon(x_1,x_2) = g^{\mu\nu}x_{12\mu}x_{12\nu} + 2\mathrm{i}\varepsilon(t_1-t_2) + \varepsilon^2$$

The Hadamard state is

$$\langle\psi|\,\phi(x_1)\phi(x_2)\,|\psi\rangle = \frac{U(x_1,x_2)}{4\pi^2\sigma_\varepsilon} + V(x_1,x_2)\ln\sigma_\varepsilon + W(x_1,x_2)$$

where the three functions U, V and W are all smooth with $U(x_1,x_2)=1$. [r] Consequence is that the stress-energy tensor is regular on Hadamard state.

The Hartle–Hawking state is a unique state that is smooth everywhere and invariant under ∂_t. [r] There is a flux of particle from infinity that equilibrate the particles emitted from the black hole.

Boulmare state. The flux is null at future and past infinity. This state is singular on the past and future horizons. This is similar to the Rindler vacuum. The stress-energy tensor diverges.

Unruh state. One demands no flux at F^- and is singular on H^-, but there is flux at F^+ and is regular on H^+. This geometry evaporates.

The evaporation cannot yet be seen from the previous calculation [r] because the mass is fixed. For a dynamical system, the emission of particles diminishes the mass of the black hole. One needs to do backreaction and update the metric to get a lower mass of the black hole.

Greybody factors. The Schwarzschild metric is Rindler times a two-sphere. One may use the tortoise coordinates

$$\mathrm{d}r^* = \left[1-\frac{2GM}{r}\right]^{-1}\mathrm{d}r$$

from which

$$\mathrm{d}s^2 = \cdots$$

[r] the potential is a complicated term [r] due to spherical symmetry one can decompose the solution in spherical harmonics. One has a wave equation with some potential

$$[\partial_t^2 - \partial_{r^*}^2 + V(r)]\phi_l(t,r) = 0$$

Near the horizon, a particle still needs to tunnel through the potential to propagate to infinity. The peak is the greybody factor. When one measures the distribution of a particle number operator at infinity, the Bose–Einstein distribution gets corrected by the scattering potential

$$n_\omega = \frac{\Gamma_l(\omega)}{\mathrm{e}^{-\frac{\omega}{T_H}}-1}$$

Closed forms of this distribution are not known, but the limits are

$$\Gamma_l(\omega) \le 1$$

high frequency

$$\Gamma_l(\omega) \approx 1\,, \quad \omega \gg GM$$

low frequency

$$\Gamma_l(\omega) \approx \frac{A_H}{4\pi}\omega^2\,, \quad \omega \ll GM$$

So black holes are grey bodies.

3 Black hole thermodynamics

Black holes have a temperature associated to them and an entropy

$$T = \frac{\kappa}{2\pi}\,, \quad S = \frac{A_H}{4G}$$

where κ is the surface gravity of the black hole. Black holes are thermodynamical objects.

Law	Thermodynamics	BH thermodynamics
0-th	Equilibrium means homogeneous T	κ is homogeneous on horizon for stationary BH
1st	Conservation of energy $\mathrm{d}E = T\,\mathrm{d}S$	$\mathrm{d}M = T\frac{\mathrm{d}A}{8\pi G}$
2nd	$\mathrm{d}_t S \geq 0$	$\mathrm{d}_t A_H \geq 0$, $\frac{\Delta A}{4G} + \Delta S_{\text{ext}} \geq 0$
3rd	S constant as $T \to 0$	$\kappa = 0$ cannot be attained ?

A stationary black hole has a time-like Killing vector, a static metric has a time-like Killing vector orthogonal to spatial direction [r].
The second law for BH is called generalized second law; the third law for BH one cannot collapse all matter into an extremal $J = M$ black hole in finite time to achieve zero temperature black holes (this may be wrong, see arXiv). The relations for BH were originally formulated classically for general relativity, but can be interpreted in thermodynamics. The duality in CFT is sharper.

3.1 Quantum gravity path integral

The partition function is

$$Z[\text{b.c.}] = \int_{\partial g = \text{b.c.}} [\mathcal{D}g]\mathrm{e}^{-S[g]}$$

One would like to understand the above in the semi-classical limit $\hbar \to 0$.

Saddle-point method. One may use the saddle-point method (also known as steepest ascent or Laplace method; the name depends on the presence of an imaginary unit i). Consider an integral of the type

$$I = \int \mathrm{d}x\,\mathrm{e}^{\frac{1}{a}f(x)}\,, \quad a \ll 1$$

An approximation of such integral by finding the saddle-point. The dominant contribution comes from the maximum of $f(x)$

$$x_0 \mid f'(x_0) = 0\,, \quad f''(x_0) < 0$$

The approximation is

$$I \approx \int \mathrm{d}x \exp\left[\frac{1}{a}f(x_0) + \frac{1}{2a}(x - x_0)^2 f''(x_0) + \cdots\right] \approx \sum_{x_0} \mathrm{e}^{\frac{1}{a}f(x_0)} \sum_{n=0}^{\infty} f_n$$

where the first sum is over all the maxima while the second is the perturbations around the maxima. The second derivative is just a Gaussian integral. In QFT, since the first derivative is zero, means that the variation of the action is null. Therefore, the first sum above is the classical equations of motion, while the second are the perturbative $\hbar$ expansion.

In the semi-classical limit, the path integral becomes

$$Z[\text{b.c.}] \approx \sum_{g_0} \mathrm{e}^{-S[g_0]} \sum_{i=0}^{\infty} (g_0)_n^i$$

where the sum is over the solutions of the equations of motion. In general, when one varies the boundary conditions, the EOM solutions can exchange dominance.

3.2 Semi-classical limit

The Einstein–Hilbert action is

$$S = -\frac{1}{16\pi G}\int_M \mathrm{d}^4x\, R$$

where M is a manifold. To have a well-defined variational principle, it is not sufficient that the fields go to zero, because there are derivatives of the metric, so one has to add another boundary term called Gibbons–Hawking–York to have

$$S = -\frac{1}{16\pi G}\int_M \mathrm{d}^4x\, R - \frac{1}{8\pi G}\int_{\partial M} \mathrm{d}^3x\, \sqrt{h}K$$

where h is the induced metric and K is the trace of extrinsic curvature, it is the second fundamental form

$$K_{\mu\nu} = \nabla_{(\mu}n_{\nu)}\,, \quad K = h^{\mu\nu}K_{\mu\nu}$$

where n is a unit normal to ∂M vector pointing inwards. This is needed because the EH action for Schwarzschild solution is null since $R = 0$.

The metric is

$$\mathrm{d}s^2 = \left[1 - \frac{2GM}{r}\right]\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{1-\frac{2GM}{r}} + r^2\,\mathrm{d}\Omega_2^2$$

The boundary ∂M is at $r = r_{\text{max}}$. Therefore

$$h_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \left[1 - \frac{2GM}{r_{\text{max}}}\right]\mathrm{d}\tau^2 + r_{\text{max}}^2\,\mathrm{d}\Omega_2^2$$

The normal vector is

$$n_\mu = -\left[1 - \frac{2GM}{r}\right]^{-\frac{1}{2}}\partial_r$$

The minus sign is needed to be inwards, the derivative is needed to make it orthogonal. The second fundamental form is

$$K = \frac{2r - 3GM}{\sqrt{1 - \frac{2GM}{r_{\text{max}}}r_{\text{max}}^2}}$$

The square root determinant is

$$\sqrt{h} = r_{\text{max}}^2\left[1 - \frac{2GM}{r_{\text{max}}}\right]^{\frac{1}{2}}\sin\theta$$

The action is

$$\begin{aligned} S &= \frac{1}{8\pi G}\int_{r_{\text{max}}}(3GM - 2r_{\text{max}}) = \frac{1}{8\pi G}\int \mathrm{d}\Omega_2^2 \int_0^\beta \mathrm{d}\tau\,(3GM - 2r_{\text{max}}) \\ &= \frac{1}{8\pi G}4\pi\beta(3GM - 2r_{\text{max}}) \end{aligned}$$

This is the solution at a fix r_{max}. It diverges when taking all of space. To deal with this one uses a counter term

$$S_{\text{tot}} = S_{\text{EH}} + S_{\text{GHY}} + S_{\text{ct}}$$

The counter term is not know how to be found from first principles, but one uses the action on Minkowski

$$S_{\text{ct}} = -S_{\text{EH}}^{\text{M}} - S_{\text{CHY}}^{\text{M}}$$

but this does not always work like in AdS, where one should use holographic normalization instead [r]. The EH action on Minkowski is zero [r] and the other can be calculated from the metric

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 + r^2\,\mathrm{d}\Omega_2^2$$

One remembers that Euclidean time is cyclic $\tau \sim \tau + \beta$, but in [r] one has

$$\tau \sim \tau + \beta\sqrt{1 - \frac{2GM}{r_{\max}}}$$

[r] One may finally get

$$S_{\text{tot}} = \frac{3}{2}\beta M - \frac{\beta r_{\max}}{G} + \frac{\beta r_{\max}}{G}\sqrt{1 - \frac{2GM}{r_{\max}}} = \frac{3}{2}\beta M - \beta M = \frac{1}{2}\beta M + o(r_{\max}^{-1})$$

The partition function is then

$$Z[\beta] = \mathrm{e}^{-\frac{1}{2}\beta M} = \exp\left[-\frac{\beta^2}{16\pi G}\right]$$

The free energy is then

$$F = -\frac{1}{\beta}\log Z = \frac{\beta}{16\pi G} = \frac{1}{16\pi G T}$$

The entropy is then

$$S = -\partial_T F = (1 - \beta\,\partial_\beta)\log Z = -\frac{\beta^2}{16\pi G} + \frac{\beta^2}{8\pi G} = \frac{\beta^2}{16\pi G} = \frac{A_H}{4G}$$

Once one obtains the partition function, one can obtain everything. This procedure can also be done in AdS [r].

Lecture 6

gio 21 mar 2024 14:30

For more complicated stationary black holes, the partition function also has chemical potential and [r].

One may also compute the specific heat

$$c = -T\,\partial_T^2 F = -\frac{1}{8\pi G T^2} < 0$$

The specific heat is negative but it is not desired. This is thermodynamical instability. If the mass M decreases, then temperature T increases; the mass decreases more rapidly and so the temperature increases. This is a first intuition of the evaporation of black holes in flat space. In AdS the case is different: the potential attractive and it is like a box.

3.3 Black hole information paradox

Gravity is not renormalizable. The paradox appear in infrared limit.

Consider a classical argument. Consider a black hole made from an initial state of matter. By the no-hair theorem, a black hole is uniquely specified by M, Q and J. [r] The initial information seems to be lost in the black hole. This is not really a paradox since if one knew the inside the black hole, one could time reverse the system to obtain the information of the initial state.

The quantum argument is properly a paradox. It seems to violate one of the most fundamental property of quantum mechanics: unitarity. General relativity and quantum mechanics are not compatible due to unitary: probability is not conserved. One starts from a star at low-density in a pure state $|\psi\rangle$. One lets the star collapse to form the black hole. The quantum fields of this geometry are in the Unruh state, the state that is smooth in the future. Slowly, Hawking radiation depletes the black hole, so it evaporates; the particles emitted are thermal particle. Eventually the black hole disappears [r] diagr. In the end there are thermal quanta, a mixed state

$$\rho_{\text{thermal}} = \mathrm{e}^{-\beta H}$$

This is incompatible with unitarity. Time evolution implies

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht}\,|\psi(0)\rangle\ , \quad \rho(t) = \mathrm{e}^{-\mathrm{i}Ht}\,|\psi(0)\rangle\langle\psi(0)|\,\mathrm{e}^{\mathrm{i}Ht}$$

This is the density matrix of a pure state

$$-\operatorname{Tr}\rho(t)\log\rho(t) = 0$$

However, the entropy collected from the black hole is

$$S(t) = S_{\text{th}} \neq 0 \iff \rho(t \to \infty) = \rho_{\text{thermal}} \neq \rho_{\text{pure}}$$

Resolution. A first option is stating that quantum mechanics is wrong. This is logically possible, but there is not other alternative.

The second option is remnants: the black hole evaporates and at some point it gets to the Planck's scale, here though one may not claim to control the calculation. This is again an ultraviolet problem. One stops at the Planck's scale: one is left with a very compact object of Planck size that has an unbounded number of quantum states (since the original black hole can be arbitrarily large). This option is a drastic departure from quantum field theory where the number of quantum states, at fixed volume and energy, is always bounded. A loop diagram [r] is the ratio of the number of degrees of freedom to the coupling constant. However, this option present a problem before the Planck scale.

The third option is a problem with Hawking's calculations. Namely, the final state is pure

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht}\,|\psi(0)\rangle$$

but the calculation is an approximation. There are corrections ($\mathrm{e}^{-S_{\mathrm{EH}}}$) that add up to restore unitarity. This may be the right solution.

The Page curve. The aforementioned problem of the second option is related to the Page curve. Consider a black hole formed from a collapse of a pure state $|\psi\rangle$. At some time after, some radiation has been emitted. At any time t_0, the Hilbert space can be decompose as

$$\mathcal{H} = \mathcal{H}_{\mathrm{R}} \otimes \mathcal{H}_{\mathrm{BH}}$$

The Hilbert space of the black hole can be thought of as the radiation at $t > t_0$. One may study how this decomposition evolves in time. The entanglement entropy for the radiation $S^{\mathrm{R}}_{\mathrm{EE}}(t)$ increases forever according to Hawking, while the entropy of the black hole decreases. The curve compatible with unitarity is the Page curve [r] diagr. If the state is pure, then entropy cannot be bigger than the dimension of the Hilbert space.

Page's argument goes as follows. If [r] a Hilbert space can be decomposed

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B\,, \quad |\mathcal{H}_A| \gg |\mathcal{H}_B|$$

The typical state is

$$S_A = S_B = \log\dim\mathcal{H}_B$$

A black hole state is typical.

At an early time $|\mathcal{H}_{\mathrm{R}}| \ll |\mathcal{H}_{\mathrm{BH}}|$ so

$$S_{\mathrm{R}} \sim tT$$

For the Schwarzschild black hole, one has

$$J = 0 \implies l = m = 0$$

[r] One may forget about the two-sphere. The entropies are

$$S_{\mathrm{BH}} = A \sim M^2\,, \quad T \sim \frac{1}{M}$$

For $t \ll M^3$, one has tT; at $t \sim M^3$ is the turn-over point which has

$$|\mathcal{H}_{\mathrm{BH}}| < |\mathcal{H}_K|$$

then one has

$$\frac{A_H(t)}{4G} = S_0\left[1 - \frac{t}{M^3}\right]^{\frac{2}{3}}$$

Before the critical time, the photons emitted are entangled with photons inside the black hole; after the critical time the photons emitted are entangled with all photons already outside. This is what must happen to have a unitarity theory; but one needs to calculate if this is actually the case.

3.4 Exponential corrections and thermalization

Thermalization does not mean to take a pure state, evolve it and it becomes a thermal state. Thermalization is taking a state not in equilibrium, evolve it and take an expectation value of a simple operator (coarse-grained quantity) since they are well approximated by their expectation value

$$\langle\psi(t)|\,\mathcal{E}\,|\psi(t)\rangle = \mathrm{Tr}\,\mathrm{e}^{-\beta H}\mathcal{E} + e^{-S}$$

The exponential correction is important for the Page curve.

In AdS/CFT, eternal black holes are thermal states: a collapsing star is a pure state, it thermalizes, it becomes a state on which the expectation value of an operator is close to the expectation value of a thermal state. Hawking is missing the exponential corretions.