

# Statistical Mechanics

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## Contents

<b>I</b>	<b>Macroscopic systems</b>	<b>2</b>
<b>1</b>	<b>Gibbs ensembles</b>	<b>2</b>
1.1	Statistical ensemble in quantum mechanics . . . . .	4
1.2	Review of probability theory . . . . .	5
1.3	Micro-canonical ensemble . . . . .	8
1.3.1	Ideal gas . . . . .	12
1.4	Canonical ensemble . . . . .	14
1.4.1	Ideal gas . . . . .	17
1.4.2	Fluctuations . . . . .	18
1.5	Gran-canonical ensemble . . . . .	19
1.5.1	Fluctuations . . . . .	21
1.5.2	Ideal gas . . . . .	21
<b>2</b>	<b>Interacting particles</b>	<b>22</b>
2.1	Classical ideal polyatomic gas . . . . .	22
2.2	Weakly interacting classical gas . . . . .	25
2.2.1	Cluster expansion . . . . .	28
<b>3</b>	<b>Ideal quantum gas</b>	<b>31</b>
3.1	Equations of state . . . . .	33
3.2	High temperatures . . . . .	35
3.3	Low temperatures . . . . .	36
3.3.1	Degenerate Fermi gas . . . . .	36
3.3.2	Low temperature Fermi gas and Sommerfeld expansion . . . . .	37
3.3.3	Bose-Einstein condensate . . . . .	39
3.4	Photon gas . . . . .	43
3.5	Phonon gas . . . . .	44

## Lecture 1

**Exam.** Oral exam on every part of the course in detail with proofs. Extra topics are not evaluated. The exam can be done whenever with enough notice.

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**References.** Some references are

- Mehran Kardar, also Tong (which are based on it): statistical physics, statistical field theory, kinetic theory. Most of the material is covered in the first set of notes.
- Kerson Huang;
- Le Bellac, for the renormalization group.

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\*<https://github.com/M-a-s-o/notes>

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**Program.** The first part is statistical mechanics of equilibrium: one discusses the problem of equilibrium in statistical mechanics. One studies the statistical ensembles: micro-canonical, canonical and grand-canonical. One discusses ideal gasses, interacting gasses and quantum gasses (Fermi–Dirac, Bose–Einstein distributions and Bose–Einstein condensation).

In the second part of the course one studies critical phenomena: liquid-gas transitions and the study of spin-systems (like Ising). One studies critical exponents and universality. One studies Landau theory and Landau–Ginsburg theory. One way to deal with the quantum field theory formalism is to study statistical systems.

The third part concerns the renormalization group: block spin transformations, relevant and irrelevant couplings. Extra?: quantum field theory approach with beta functions and the Wilson–Fisher fixed point.

The fourth part is kinetic theory of gasses, study the problem of the approach to equilibrium, Boltzmann equation. Extra: hydrodynamics.

## Part I

# Macroscopic systems

Statistical mechanics deals with systems with a very large number of particles  $N \sim 10^{23}$ . From analytical mechanics, one would study the phase space of this enormous system. The state of the system is called microstate, it is a point in  $D = 6N$  dimensional phase space  $\Gamma$ . The coordinates of the point is

$$(\mathbf{p}, \mathbf{q}) \in \mathbb{R}^{6N}$$

The equation of motion for the system are

$$\dot{\mathbf{q}} = \partial_{\mathbf{p}} H, \quad \dot{\mathbf{p}} = -\partial_{\mathbf{q}} H$$

Solving the equations is virtually impossible. However, it is not even interesting. For a macroscopic system, only a few quantities are interesting: the temperature, the pressure, etc. One would like to characterize the system with a few quantities, one does not need to know the motion of every particle in phase space. Typically, a macroscopic state (or macrostate) is defined by fixing few macroscopic quantities: energy, pressure, volume, temperature, etc. Even if a system is macroscopically at equilibrium — the macrostate is fixed — the point (i.e., the microstate) in the phase space  $\Gamma$  can move. One moves away from the analytical mechanical approach to study a system and one focuses on the macroscopic behaviour. This is possible with the theory of statistical ensembles introduced by Gibbs.

## 1 Gibbs ensembles

There are many to one correspondences between microstates and macrostates. Each point in the phase space  $\Gamma$  corresponds to a microstate. There are many  $\mathcal{N}$  points which are compatible with a given macrostate. Since there is a many-to-one correspondence and as  $N \rightarrow \infty$  also  $\mathcal{N} \rightarrow \infty$ . The collection of compatible microstates is called “ensemble”. This collection is a distribution of points in the phase space  $\Gamma$ . One may define the probability distribution  $\rho(\mathbf{p}, \mathbf{q}, t)$  as function of the position in the phase space and time. The number of compatible microstates is then

$$\rho(\mathbf{p}, \mathbf{q}, t) d^{3N}p d^{3N}q = \rho(\mathbf{p}, \mathbf{q}, t) d\Gamma = \frac{dN}{N}$$

[r] curly  $\mathcal{N}$ ?

Integrating the distribution over the entire phase space, one must find unity

$$\int_{\Gamma} \rho(\mathbf{p}, \mathbf{q}, t) d\Gamma = 1$$

If one knows the solutions to the equations of motion, knowing the distribution at some initial time  $t_i$ , one may extract the distribution at some later final time  $t_f$ . [r] One assumes that the

Hamiltonian of the system is time-independent. This implies that the trajectories in the phase space are time-reversal invariant, they do not cross. Therefore, the total number of points in the phase space, hence the probability, is conserved: it satisfies a continuity equation. The probability distribution is essentially an incompressible fluid. Consider a spherical region  $\gamma$  in the phase space and a direction  $\mathbf{n}$ . Integrating the probability distribution in such region gives

$$I = \partial_t \int_{\gamma} \rho(\mathbf{p}, \mathbf{q}, t) d\Gamma$$

Since the Hamiltonian is time-independent, what leaves the volume is what passes through its boundary, so

$$I = - \int_{\partial\gamma} \rho \mathbf{v} \cdot \mathbf{n} dS, \quad \mathbf{v} = (\dot{\mathbf{p}}, \dot{\mathbf{q}})$$

This is the continuity equation. By applying Stokes theorem, one finds

$$\partial_t \int_{\gamma} \rho(\mathbf{p}, \mathbf{q}, t) d\Gamma = - \int_{\gamma} \nabla \cdot (\rho \mathbf{v}) d\Gamma$$

This argument must be true for every place in the phase space: it must hold locally. It must hold for every  $\gamma$ , therefore

$$\boxed{\partial_t \rho(\mathbf{p}, \mathbf{q}, t) + \nabla \cdot (\rho \mathbf{v}) = 0}$$

The divergence can be rewritten as

$$\begin{aligned} \nabla \cdot (\rho \mathbf{v}) &= \sum_{i=1}^N \partial_{\mathbf{p}_i} (\rho \dot{\mathbf{p}}_i) + \sum_{i=1}^N \partial_{\mathbf{q}_i} (\rho \dot{\mathbf{q}}_i) = \sum_{i=1}^N (\dot{\mathbf{p}}_i \partial_{\mathbf{p}_i} \rho + \dot{\mathbf{q}}_i \partial_{\mathbf{q}_i} \rho) + \sum_{i=1}^N \rho (\partial_{\mathbf{p}_i} \dot{\mathbf{p}}_i + \partial_{\mathbf{q}_i} \dot{\mathbf{q}}_i) \\ &= \sum_{i=1}^N (\dot{\mathbf{p}}_i \partial_{\mathbf{p}_i} \rho + \dot{\mathbf{q}}_i \partial_{\mathbf{q}_i} \rho) = \sum_{i=1}^N (-\partial_{\mathbf{q}_i} H \partial_{\mathbf{p}_i} \rho + \dots) = \{\rho, H\} \end{aligned}$$

At the first line, in the last parenthesis, one applies the equations of motion

$$\partial_{\mathbf{p}_i} \dot{\mathbf{p}}_i = -\partial_{\mathbf{q}_i}^2 H$$

One finds the Poisson bracket of the probability distribution with the Hamiltonian. [r] Therefore

$$\boxed{d_t \rho(\mathbf{p}, \mathbf{q}, t) = \partial_t \rho(\mathbf{p}, \mathbf{q}, t) + \{\rho, H\} = 0}$$

This is Liouville theorem. The probability distribution in a statistical ensemble behaves as an incompressible fluid.

From the probability distribution one may compute the averages of the ensemble, so that one may identify them with the macroscopic quantities line energy, pressure, etc. The expectation values

$$\langle O \rangle = \int_{\Gamma} \rho(\mathbf{p}, \mathbf{q}, t) O(\mathbf{p}, \mathbf{q}) d\Gamma$$

are identified with the thermodynamic quantities. Skipping the question of whether a system can be in equilibrium, one defines the equilibrium as the condition

$$d_t \langle O \rangle = 0$$

The total time derivative of an expectation value of  $O(\mathbf{p}, \mathbf{q})$  is

$$\begin{aligned} d_t \langle O \rangle &= \int [\partial_t \rho(\mathbf{p}, \mathbf{q}, t)] O(\mathbf{p}, \mathbf{q}) d\Gamma \\ &= \int d\Gamma O(\mathbf{p}, \mathbf{q}) \sum_{\alpha=1}^{3N} (\partial_{p_{\alpha}} \rho \partial_{q_{\alpha}} H - \partial_{q_{\alpha}} \rho \partial_{p_{\alpha}} H) \end{aligned}$$

At the first line, the time dependence comes only from the explicit dependence on  $t$  of  $\rho$ , since the dependence from  $\mathbf{p}$  and  $\mathbf{q}$  is eliminated due to the fact that the two variables are integrated.

At the second line, one has used the continuity equation. Consider the first addendum in the parenthesis

$$\int d\Gamma O(\mathbf{p}, \mathbf{q}) \partial_{p_\alpha} \rho \partial_{q_\alpha} H = - \int d\Gamma \rho(\mathbf{p}, \mathbf{q}, t) \partial_{p_\alpha} O \partial_{q_\alpha} H - \int d\Gamma \rho(\mathbf{p}, \mathbf{q}, t) O \partial_{q_\alpha p_\alpha}^2 H$$

where one supposes that  $\rho$  vanishes at the boundary. The last term cancels a similar contribution from the second addendum. Therefore

$$d_t \langle O \rangle = \sum_{\alpha=1}^{3N} \int d\Gamma \rho(\mathbf{p}, \mathbf{q}, t) (\partial_{p_\alpha} O \partial_{q_\alpha} H - \partial_{q_\alpha} O \partial_{p_\alpha} H) = - \langle \{O, H\} \rangle$$

The last parenthesis is again a Poisson bracket. At equilibrium, the total time derivative is zero there the partial derivative of the probability density  $\rho$  must be identically zero

$$d_t \langle O \rangle = 0 \iff \partial_t \rho(\mathbf{p}, \mathbf{q}, t) = 0$$

Therefore, at equilibrium, the probability density has no explicit time dependence, but is only a function of  $\mathbf{p}$  and  $\mathbf{q}$ . The existence of an equilibrium distribution is the basic assumption of statistical mechanics. This condition also is not enough to prove that a system eventually evolves to equilibrium.

## 1.1 Statistical ensemble in quantum mechanics

In quantum mechanics, the state of a system is ray in Hilbert space. If the system is large, the wave function has many variables and so is statistical: there are still the same type of problems. Again, there are many microstates (which are rays in the Hilbert space) which are compatible with a given macrostate. This macrostate is fixed by few macroscopic variables: energy, pressure, etc. In this case one must also develop a statistical approach. There are microstates  $|\psi_i\rangle$  each with a statistical (ensemble) probability  $p_i$  that is normalized

$$\sum_i p_i = 1$$

This is a discrete version of the classical case. One may compute ensemble averages as

$$\langle A \rangle = \sum_{q \in \text{CMS}} p_q \langle \psi_i | A | \psi_i \rangle$$

where CMS stands for compatible microstates. One is taking two averages: the quantum expectation value, a fundamental unknown, and the statistical expectation value, which is a practical unknown. The state  $|\psi_i\rangle$  is a microstate, also called pure state, even if it is a superposition of other states. Instead, a collection of pure states  $|\psi_i\rangle$  with (statistical) probability  $p_i$  is a mixed state.

**Density matrix.** The density matrix is an operator defined as

$$\hat{\rho} \equiv \sum_i p_i |\psi_i\rangle \langle \psi_i|$$

It is useful for the computation of a trace

$$\begin{aligned} \text{Tr}(\rho A) &= \sum_n \langle n | \hat{\rho} A | n \rangle = \sum_n \sum_i p_i \langle n | \psi_i \rangle \langle \psi_i | A | n \rangle = \sum_i \sum_n p_i \langle \psi_i | A | n \rangle \langle n | \psi_i \rangle \\ &= \sum_i p_i \langle \psi_i | A | \psi_i \rangle = \langle A \rangle \end{aligned}$$

where  $|n\rangle$  is a basis. In quantum statistical mechanics, traces can be replaced with averages.

Since the state  $|\psi_i\rangle$  evolves with the Hamiltonian  $e^{-iHt} |\psi_i\rangle$ , then the density matrix evolves due to the evolution of the pure states

$$\hat{\rho}(0) = \sum_i p_i |\psi_i\rangle \langle \psi_i| \rightarrow \sum_i p_i e^{-iHt} |\psi_i\rangle \langle \psi_i| e^{iHt} = e^{-iHt} \hat{\rho}(0) e^{iHt} = \hat{\rho}(t)$$

The partial derivative with respect to time is

$$\partial_t \hat{\rho}(t) = -iH\hat{\rho}(t) + \hat{\rho}(t)iH = i[\hat{\rho}, H]$$

Therefore, the density matrix satisfies

$$\partial_t \hat{\rho}(t) + \frac{1}{i\hbar} [\hat{\rho}, H] = 0$$

This is basically Liouville theorem with the naive replacement of the Poisson bracket with the  $(i\hbar)^{-1}$  commutator.

**Semi-classical limit.** Comparing classical and quantum statistical mechanics, there is a parallel between a quantum microstate  $|\psi_i\rangle$  and a classical microstate  $(\mathbf{p}, \mathbf{q} \in \Gamma)$ . One would like to understand how to go from quantum to classical: the semi-classical limit.

Consider  $N$  particles in a box of volume  $V = L^3$ . A particle in this box is described by a wave function

$$\frac{1}{\sqrt{V}} e^{i\mathbf{p} \cdot \mathbf{x}}$$

where the momentum is quantized

$$\mathbf{p} = \frac{2\pi}{L} \hbar(n_1, n_2, n_3)$$

The number of states in  $\mathbf{p} + [0, d\mathbf{p}]$  is

$$d^3n = \frac{L^3}{(2\pi\hbar)^3} d^3p = \frac{d^3q d^3p}{(2\pi\hbar)^3}$$

One should also add a  $(N!)^{-1}$  to account for identical particles. [r] One may observe that the sum  $\sum_i$  gives a very precise measure in the semi-classical limit

$$\int \frac{d^{3N}p d^{3N}q}{(2\pi\hbar)^{3N} N!}$$

The reduced Planck's constant is present only for dimensional reason, because in the semi-classical limit it cancels out. On the other hand, the factorial is important and does not cancel. This is related to the Gibbs's paradox. When doing statistical analysis, the factorial does not appear, but this brings problems and one has to manually add it. The only explanation for its necessity comes from the semi-classical limit.

## 1.2 Review of probability theory

Given a function  $f$  taking values  $f_i$  at the states  $i$  of the system, with probability  $p_i$ , one defines

- the mean value

$$\langle f \rangle = \sum_i f_i p_i$$

- the variance

$$(\Delta f)^2 = \langle f - \langle f \rangle \rangle^2 = \langle f^2 \rangle - \langle f \rangle^2 = \sum_i f_i^2 p_i - \left[ \sum_i f_i p_i \right]^2$$

- the probability of taking some value  $F$  is

$$P(f = F) = \sum_i p_i \delta(f_i - F) = \langle \delta(f_i - F) \rangle$$

- the entropy of a distribution is

$$S = - \sum_i p_i \ln p_i = - \langle \ln p_i \rangle$$

One may see the meaning of entropy. Consider a distribution with  $p_1 = 1$  and  $p_i = 0$  for  $i \neq 1$ . This distribution has zero entropy,  $S = 0$ : there is zero uncertainty. Viceversa, if all outcomes are equally probable,  $p_i = N^{-1}$ , then the entropy is

$$S = - \sum_i \frac{1}{N} \ln \frac{1}{N} = \ln N$$

**Proposition.** The entropy takes the maximum value for the random distribution.

*Proof.* One may utilize a constrained extremization using Lagrange multipliers

$$\begin{aligned} 0 \equiv \partial_{p_j} \left[ - \sum_i p_i \ln p_i + \lambda \left( \sum_i p_i - 1 \right) \right] &= - \sum_i \delta_{ij} \ln p_i - \sum_i p_i \frac{1}{p_i} \delta_{ij} + \lambda \sum_i \delta_{ij} \\ &= - \ln p_j - 1 + \lambda \implies p_j = e^{\lambda-1} \end{aligned}$$

Using the constraint

$$1 = \sum_{j=1}^N p_j = N e^{\lambda-1}$$

one finds

$$p_j = \frac{1}{N}$$

Therefore, the completely random distribution maximizes the entropy.  $\square$

**Recap.** There is a distribution  $\rho$  of compatible microstates with a given macrostate. The equilibrium condition is

$$d_t \langle O \rangle = 0 \iff \partial_t \rho_{\text{eq}} = 0 = \{ \rho_{\text{eq}}, H \}$$

This condition does not specify the distribution. One assumes that the equilibrium exists (as one may see at the end of the course). Given a macrostate, the equilibrium distribution is such that

- it maximizes the entropy
- does not introduce any other bias
- treats all the compatible microstates as equiprobable

The three propositions are equivalent.

One fixes the macrostate (by fixing the energy, temperature or chemical potential), the other request to determine the equilibrium distribution is to require the distribution to treat all microstates as equiprobable (in practice one maximizes the entropy).

## Lecture 2

**Previous lecture.** The idea of the statistical approach is to define the Gibbs ensemble as the collection of microstates (a microstate is a point in the phase space  $\Gamma = \mathbb{R}^{6N}$  compatible with a given macrostate which is a state fixed by defining a few microscopic quantities like energy, temperature and pressure). There are many particles in the system and so many compatible microstates. The distribution of microstates integrated gives the fraction of compatible microstates

$$\rho(\mathbf{p}, \mathbf{q}, t) d\Gamma = \frac{d\mathcal{N}}{\mathcal{N}}$$

The distribution is normalized. The distribution behaves as an incompressible fluid

$$d_t \rho = 0 = \partial_t \rho + \{ \rho, H \}$$

From the distribution one may compute the ensemble averages

$$\langle O(\mathbf{p}, \mathbf{q}) \rangle = \int \rho(\mathbf{p}, \mathbf{q}, t) O(\mathbf{p}, \mathbf{q}) d\Gamma$$

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Equilibrium is defined by requiring

$$d_t \langle O(\mathbf{p}, \mathbf{q}) \rangle = 0 \iff \partial_t \rho_{\text{eq}} = 0 = \{\rho_{\text{eq}}, H\}$$

The value of the macroscopic quantities do not change in time.

One assumes that the distribution at equilibrium  $\rho_{\text{eq}}$  exists and that it has the properties:

- it maximizes the entropy,
- does not introduce any other bias,
- treats all the compatible microstates as equiprobable.

This condition is enough to obtain the equilibrium distribution once the macrostate is fixed.

**Macrostates.** The macrostates can be fixed by fixing the energy, the temperature and the chemical potential  $\mu$ , to obtain respectively the microcanonical, the canonical and the gran-canonical ensembles. Each has a different distribution at equilibrium. These different distributions give different averages of the same observable. However, this is a problem: there must be only one average since it is associated to a macroscopic quantity. This problem is resolved by noting that, in the thermodynamic limit  $N, V \rightarrow \infty$ , the three averages tend to the same value. Therefore, macroscopic quantities like the energy have a non-zero possibility of taking a value different from the average but the variance of the samples must behave asymptotically as

$$\frac{\Delta E}{\langle E \rangle} \sim \frac{1}{\sqrt{N}} \rightarrow 0, \quad N \rightarrow \infty$$

This is not proven but can be seen through a parallel with the central limit theorem: in the thermodynamic limit, the three averages tend to the same value and the possibility that the system is not at the average is extremely suppressed. To compute, one may use whatever ensemble is easier because one is interested in the thermodynamic limit anyway.

**Central limit theorem.** The mechanism of the three averages tending to the same value is similar to the central limit theorem. Consider a system with  $N$  random variables  $\mathbf{x} = (x_1, \dots, x_N)$  with joint probability

$$p(\mathbf{x}) = \prod_{i=1}^N p(x_i)$$

Let

$$X \equiv \sum_{i=1}^N x_i$$

The probability that the system obtains the value  $X$  is

$$p_N(X) = \int dx_1 \cdots dx_N \prod_{i=1}^N p(x_i) \delta(x - \sum_i x_i)$$

Taking the Fourier transform gives

$$\begin{aligned} p_N(k) &= \int dX e^{-ikX} p_N(X) = \int dX \int dx_1 \cdots dx_N \prod_{i=1}^N p(x_i) \delta(x - \sum_i x_i) e^{-ikX} \\ &= \int dx_1 \cdots dx_N \prod_{i=1}^N p(x_i) e^{-ik(x_1 + \cdots + x_N)} = [p(k)]^N \end{aligned}$$

with

$$p(k) = \int dx_i e^{-ikx_i} p(x_i)$$

One may expand this expression to have

$$p(k) = \int dx p(x) \left[ 1 - ikx - \frac{1}{2}k^2x^2 + i\frac{k^3}{3!}x^3 + \dots \right] = 1 - ik\langle x \rangle - \frac{1}{2}k^2\langle x^2 \rangle + i\frac{k^3}{3!}\langle x^3 \rangle + \dots$$

Taking the logarithm, expanding it in a power series and resumming the terms, one may reorganize them to obtain the cumulants (see Lab of Computational Physics)

$$\ln p(k) = -ik\langle x \rangle_c - \frac{k^2}{2}\langle x^2 \rangle_c + \dots$$

where

$$\langle x \rangle_c = \langle x \rangle, \quad \langle x^2 \rangle_c = \langle x^2 \rangle - \langle x \rangle^2 = \sigma^2, \quad \dots$$

One may rewrite

$$\begin{aligned} p_N(X) &= \int \frac{dk}{2\pi} e^{ikX} [p_N(k)]^N = \int \frac{dk}{2\pi} e^{ikX} e^{N \ln p_N(k)} \\ &= \int \frac{dk}{2\pi} e^{ikX} e^{-ikN\langle x \rangle} e^{-\frac{1}{2}Nk^2\sigma^2} \exp \left[ \frac{i}{6}Nk^3\langle x^3 \rangle_c + \dots \right] \\ &= \int \frac{dk}{2\pi} e^{ikX} e^{-ikN\langle x \rangle} e^{-\frac{1}{2}Nk^2\sigma^2} \left[ 1 + \frac{i}{6}Nk^3\langle x^3 \rangle_c + \dots \right] \\ &= \int \frac{dy}{2\pi\sqrt{N}} \exp \left[ i\frac{y}{\sqrt{N}}(X - N\langle x \rangle) \right] e^{-\frac{1}{2}y^2\sigma^2} \left[ 1 + \frac{i}{6}N\frac{y^3}{N^{\frac{3}{2}}}\langle x^3 \rangle_c \right] \\ &\sim \int \frac{dy}{2\pi\sqrt{N}} \exp \left[ i\frac{y}{\sqrt{N}}(X - N\langle x \rangle) \right] e^{-\frac{1}{2}y^2\sigma^2} = \frac{1}{\sqrt{2\pi\sigma^2N}} \exp \left[ -\frac{(X - N\langle x \rangle)^2}{2\sigma^2N} \right] \end{aligned}$$

At the second line one has inserted the found expression for  $p(k)$ . At the second line, the exponential contains subleading terms. At the fourth line one may change  $k = \frac{y}{\sqrt{N}}$ . At the fifth line, one is in the limit  $N \rightarrow \infty$ .

The integral is a Gaussian integral. Even if the probability of measuring  $p(x_i)$  is generic with mean  $\langle x \rangle$  and standard deviation  $\sigma$ , the probability of measuring  $p_N(X)$  for  $N \rightarrow \infty$  becomes a Gaussian with mean  $N\langle x \rangle$  and standard deviation  $\sqrt{N}\sigma$ . Therefore, in the thermodynamic limit, the relative error scale as

$$\lim_{N \rightarrow \infty} \frac{\sigma\sqrt{N}}{N\langle x \rangle} \sim \frac{1}{\sqrt{N}} \rightarrow 0$$

Regardless of the ensemble used, the result in the thermodynamic limit is the same. The central limit theorem holds with very general assumptions.

### 1.3 Micro-canonical ensemble

The macrostate is characterized by requiring that the energy is fixed

$$H(p, q) = E = \text{const.}$$

One collects all the microstates compatible with the macrostate. Though, one should not introduce any bias. The equilibrium distribution is

$$\rho_{\text{eq}}(p, q) = \frac{1}{\Omega(E)} \delta(H(p, q) - E)$$

where the number of microstates with fixed energy  $E$  is

$$\Omega(E) = \int d\Gamma \delta(H(p, q) - E)$$

There are some technical issue which can be avoided. It is more convenient to find the state within an infinitesimal range.

**Remark 1.1.** The number of microstates with a given energy  $\Omega(E)$  is typically of the order  $e^N$  where  $N$  is the total number of the particles.



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This observation implies that the entropy is

$$\begin{aligned} S &= -k \langle \ln \rho \rangle = -k \int d\Gamma \frac{\delta(H-E)}{\Omega(E)} \ln \frac{1}{\Omega(E)} = -k \ln \frac{1}{\Omega(E)} \frac{1}{\Omega(E)} \int d\Gamma \delta(H-E) \\ &= -k \ln \frac{1}{\Omega(E)} = k \ln \Omega(E) \end{aligned}$$

where  $k \approx 1.3 \times 10^{-23} \text{ J K}^{-1}$  is the Boltzmann constant. The entropy is the logarithm of the total number of microstates. Since  $\Omega(E) \sim e^N$  then the entropy goes like

$$S \sim kN$$

From this one sees that the entropy is an extensive quantity and the normalization by the Boltzmann constant is natural

$$S \sim kN \sim O(N^0)$$

**Remark 1.2.** Recall  $\Omega(E) \sim e^N$ . Consider two isolated systems with energies  $E_1$  and  $E_2$ . After combining them, the total number of accessible microstates is

$$\Omega(E_T) = \Omega(E_1)\Omega(E_2)$$

Therefore, the entropy of the combined system is

$$S_T = k \ln \Omega(E_T) = k \ln [\Omega(E_1)\Omega(E_2)] = k \ln \Omega(E_1) + k \ln \Omega(E_2) = S_1 + S_2$$

This means that entropy is additive.

**Remark 1.3.** [r] The property of being extensive and additive only holds when including the  $N!$  from the semi-classical limit.

**Remark 1.4.** The microcanonical equilibrium distribution  $\rho_{\text{eq}}$  is compatible with the equilibrium condition of Liouville's

$$\partial_t \rho_{\text{eq}} = 0 = \{\rho_{\text{eq}}, H\}$$

because the distribution just depends on the energy.

**Second law of thermodynamics.** The microcanonical ensemble means working with combinatorics. Consider two separated systems with energies  $E_1$  and  $E_2$ . The total energy of the combined system is just the sum. Let the two system come into thermal contact. Let the interaction be weak, i.e. the interaction Hamiltonian is small with respect to the free Hamiltonian. The energy of the first system is  $E_i$  while the one of the second is  $E_T - E_i$ . The total system is still isolated. The entropy of the total system before the interaction is just the sum of the entropies. After the thermal contact, the total number of accessible microstates is

$$\Omega_{\text{final}}(E_T) = \sum_i \Omega_1(E_i)\Omega_2(E_T - E_i)$$

The number of accessible states is much larger due to the presence of the sum. The final entropy is

$$S_{\text{final}}(E_T) = k \ln \Omega_{\text{final}}(E_T) \geq S_{\text{initial}} = S_1(E_1) + S_2(E_2)$$

This is the second law of thermodynamics: entropy grows since there more accessible compatible microstates.

**Zeroth law of thermodynamics.** The zeroth law is the law of equilibrium. Recall that  $\Omega(E) \sim e^N$  and that the entropy is extensive  $S(E) \sim N$ . Therefore, the number of microstates is

$$\Omega(E) = \sum_i \Omega_1(E_i)\Omega_2(E_T - E_i) = \sum_i \exp \left[ \frac{1}{k} [S_1(E_i) + S_2(E_T - E_i)] \right]$$

Since the exponent is big, then the value of the integral is dominated by the saddle point (or stationary point) of the energy  $E_i$ . Any variation away from such value produces very suppressed contributions. The stationary point can be found as

$$0 = \partial_{E_i}[S_1(E_i) + S_2(E_T - E_i)]_{E_i=E^*} = [\partial_E S_1 - \partial_E S_2]_{E=E^*}$$

Since

$$\partial_E S = \frac{1}{T}$$

then one finds

$$\boxed{T_1 = T_2}$$

when the two systems settle at  $E = E^*$ . The thermal equilibrium coincides with the situation that the energy extremizes the number of total accessible microstates. The two system can exchange energy, heat and may reach the equilibrium configuration with energies  $E^*$  and  $E_T - E^*$  corresponding to the situation the equilibrium temperature  $T_1 = T_2 = T_{\text{eq}}$ .

**Remark 1.5.** The energy  $E^*$  is the dominant value. One would like to find this quantity in a statistical framework. One may study the probability that the system 1 has energy  $E_1$

$$\begin{aligned} p_1(E_1) &= \int \rho_T(E_T) \delta(H_1 - E_1) d\Gamma_1 d\Gamma_2 = \int \frac{\delta(H_T - E_T)}{\Omega(E_T)} \delta(H_1 - E_1) d\Gamma_1 d\Gamma_2 \\ &= \frac{1}{\Omega(E_T)} \int (\delta(H_2 - (E_T - E_1))) d\Gamma_2 \int \delta(H_1 - E_1) d\Gamma_1 \\ &= \frac{1}{\Omega(E_T)} \Omega_2(E_T - E_1) \Omega_1(E_1) = \frac{1}{\Omega(E_T)} \exp \left[ \frac{1}{k} [S_2(E_T - E_1) + S_1(E_1)] \right] \end{aligned}$$

at the second line, one has noticed that

$$\int \delta(H_T - E_T) \delta(H_1 - E_1) d\Gamma_1 d\Gamma_2 = \int \delta(H_2 - (E_T - E_1)) \delta(H_1 - E_1) d\Gamma_1 d\Gamma_2$$

As noted before, the exponential is dominated by the entropy at the energy  $E^*$ . One may expand

$$E_1 = E^* + \Delta E$$

Therefore

$$\begin{aligned} p_1(E_1) &= \frac{1}{\Omega(E_T)} \exp \left[ \frac{1}{k} [S_2(E_T - E^*) + S_1(E^*)] \right] e^{\frac{1}{k} \Delta E \cdot 0} \\ &\quad \times \exp \left[ \frac{1}{2k} [S_2''(E_T - E^*) + S_1''(E^*)] (\Delta E)^2 \right] \times \text{sub-leading} \end{aligned}$$

The linear term in  $\Delta E$  is zero since  $E^*$  is an extremum. Since

$$S''(E) = \partial_E^2 S(E) = \partial_E \frac{1}{T} = -\frac{1}{T^2} \partial_E T = -\frac{1}{T^2} \frac{1}{C}$$

with  $C$  being the heat capacity, then the probability that the system 1 has energy  $E_1$

$$p_1(E_1) \approx \exp \left[ -\frac{1}{2kT^2} \left( \frac{1}{c_1} + \frac{1}{c_2} \right) (\Delta E)^2 \right]$$

is a Gaussian with mean value  $\langle E_1 \rangle = E^*$  and standard deviation

$$\sigma = T \sqrt{\frac{c_1 c_2}{c_1 + c_2}} \sim \sqrt{N}$$

recalling that the heat capacity is positive and extensive,  $c \sim N$ . As expected the relative error is

$$\frac{\sigma}{\langle E_1 \rangle} = \frac{\sigma}{E^*} \sim \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}} \rightarrow 0, \quad N \rightarrow \infty$$

**Generalization.** Consider two systems separated. Each is characterized by a fixed energy  $E_i$  and volume  $V_i$ . When the two systems come into contact, they can exchange heat and work. Both energy and volume may redistribute. The equilibrium situation is given by the first system with values  $(E_i, V_i)$  and the second with  $(E_T - E_i, V_T - V_i)$ . The number of accessible microstates is

$$\Omega_{\text{final}}(E_T, V_T) = \sum_i \Omega_1(E_i, V_i) \Omega_2(E_T - E_i, V_T - V_i) = \sum_i \exp \left[ \frac{1}{k} [S_1(E_i, V_i) + S_2(E_T - E_i, V_T - V_i)] \right]$$

The procedure is the same as before, but with one interaction more. Since the entropy scales like  $S \sim N$ , one looks for the saddle point to obtain two equations

$$\left. \partial_E S_1(E, V) \right|_{E^*} = - \left. \partial_E S_2(E, V) \right|_{E^*} \implies T_1 = T_2$$

and

$$\left. \partial_V S_1(E, V) \right|_{V^*} = - \left. \partial_V S_2(E, V) \right|_{V^*} \implies P_1 = P_2$$

where one the pressure is

$$P = T \partial_V S$$

There is both thermal equilibrium and mechanical equilibrium.

This procedure can be generalized. For every extensive quantity  $X$  there is a conjugate intensive quantity

$$J = -T \partial_X S$$

The entropy is a function of the extensive quantities

$$S = S(E, V, N, M, \dots)$$

For example, for the volume  $V$  is the pressure  $-P$ , for the number of particles  $N$  is the chemical potential  $\mu$ , for the magnetization  $M$  is the magnetic field  $B$ .

For the case in which  $S(E, V, N)$ , one repeats the procedure above and one finds three equilibrium conditions

$$T_1 = T_2, \quad P_1 = P_2, \quad \mu_1 = \mu_2$$

Thermal, mechanical and chemical equilibrium are associated to the dominant values  $E^*$ ,  $V^*$  and  $N^*$ .

### Lecture 3

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2024 14:30

**Previous lecture.** The saddle point extremization in the limit of large  $N$  corresponds to the equilibrium conditions.

Since  $\Omega_{\text{final}} \geq \Omega_{\text{initial}}$  then one finds that entropy is always increasing in thermodynamical processes

$$S = k \ln \Omega \implies \Delta S \geq 0$$

**Example 1.6** (Two-state system). Consider  $N$  non-interacting particles taking spin  $|+\rangle$  or  $|-\rangle$  with energies

$$E_+ = \varepsilon \quad E_- = 0$$

Let  $N_+$  be the number of spin up particles and similarly for  $N_-$  and spin down particles. The total energy is

$$E_{\text{tot}} = N_+ \varepsilon + N_- \cdot 0 = N_+ \varepsilon$$

The system is isolated and one may use the micro-canonical ensemble. The number of states of the total system which have energy  $E_{\text{tot}}$  is

$$\Omega(E_{\text{tot}}) = \binom{N}{N_+} = \frac{N!}{(N_+)!(N - N_+)!}$$

The entropy is therefore

$$S(E_{\text{tot}}) = k \ln \frac{N!}{(N_+)!(N - N_+)!}$$

Notice that the entropy is zero  $S = 0$  for  $N = N_+$  or  $N = N_-$ : there is no ambiguity. Also, it is maximal when the configurations are maximally random

$$N_+ = N_- = \frac{1}{2}N$$

### 1.3.1 Ideal gas

See Kardar, §4.4. Consider a system of  $N$  non-interacting point-like particles. The Hamiltonian is

$$H(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^N \frac{\mathbf{p}_i^2}{2m}$$

Assume that the system is isolated

$$H(\mathbf{p}, \mathbf{q}) = E = \text{const.}$$

but allow  $E \leq H \leq E + \Delta E$ : this hypothesis allows a simpler derivation and it is also physically sound. The number of micro-states is given by

$$\Omega(E) = \int d\Gamma \delta(H - E), \quad d\Gamma = \frac{d^{3N}p d^{3N}q}{(2\pi\hbar)^{3N} N!}$$

Firstly, notice that the Hamiltonian depends only on the momenta, so

$$\int d^{3N}q = V^N$$

secondly, the Dirac delta implies that it holds

$$\sum_{i=1}^N \mathbf{p}_i^2 = 2mE = \text{const.}$$

however, since  $E \leq H \leq E + \Delta E$ , the integration over the momenta is an integration over a  $3N$ -dimensional spherical shell of radius

$$R = \sqrt{2mE}$$

with thickness

$$\Delta R = \sqrt{\frac{2m}{E}} \Delta E$$

The surface area of a  $d$ -dimensional ball is

$$A = S_d R^{d-1}$$

where  $S_d$  is the coefficient obtained from integrating over the angles. For example, the surface of a 3-ball, i.e. a three dimensional figure, is a 2-sphere and the area is  $A = S_3 R^2 = 4\pi R^2$ . The coefficient can be obtained in the following way

$$\begin{aligned} \pi^{\frac{d}{2}} &= \left[ \int_{\mathbb{R}} e^{-x^2} dx \right]^d = \int d^d x e^{-(x_1^2 + \dots + x_d^2)} = S_d \int_0^\infty r^{d-1} e^{-r^2} dr, \quad r^2 = y \\ &= \frac{1}{2} S_d \int_0^\infty y^{\frac{d}{2}-1} e^{-y} dy = \frac{1}{2} S_d \Gamma(d/2) \end{aligned}$$

where  $\Gamma(z)$  is the Euler's Gamma function. Therefore

$$S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(d/2)}$$

The number of micro-states is

$$\Omega(E, V, N) = \frac{V^N}{(2\pi\hbar)^{3N} N!} S_d R^{d-1} = \frac{V^N}{(2\pi\hbar)^{3N} N!} \frac{2\pi^{\frac{3N}{2}}}{\Gamma(3N/2)} (2mE)^{\frac{3N-1}{2}} \Delta R$$

One is interested in studying the entropy

$$S = k \ln \Omega(E, V, N)$$

in the limit  $N \rightarrow \infty$ . One may assume that  $3N/2$  is an even integer and may drop the  $-1$  factors in the exponent and in the Gamma function expressed as factorial. One may use the approximations

$$\ln \Delta R \sim \ln \sqrt{E} \sim \ln \sqrt{N}$$

the second relation is motivated by the fact that the energy is extensive and so it scales with the number of particles. This contribution is sub-leading. One may also utilize Stirling's formula

$$\ln N! \sim N \ln N - N$$

Thus, one obtains the Sackur-Tetrode equation

$$\begin{aligned} S = k \ln \Omega &\sim k \left[ N \ln V + \frac{3N}{2} \ln \frac{2mE\pi}{4\pi^2\hbar^2} - \frac{3N}{2} \ln \frac{3N}{2} + \frac{3N}{2} - N \ln N + N \right] \\ &= kN \ln \left[ \frac{V}{N} e \left( \frac{2meE}{2\pi\hbar^2 3N} \right)^{\frac{3}{2}} \right] \end{aligned}$$

Now from the entropy, one may derive the thermodynamics. The energy is

$$\frac{1}{T} = \partial_E S = \partial_E [kN \ln E^{\frac{3}{2}} + \dots] = kN \frac{3}{2} \frac{1}{E} \implies \boxed{E = \frac{3}{2} NkT}$$

This result is expected from the equipartition theorem. Notice also that Planck's constant does not appear. The ideal gas law is

$$p = T \partial_V S = T \partial_V [kN \ln V + \dots] = T \frac{kN}{V} \implies \boxed{pV = NkT}$$

**Gibbs's paradox.** The factor of  $N!$  is related to Gibbs's paradox. The entropy can be written as

$$S = kN \ln \left[ \frac{V}{N} \sigma(T) \right]$$

where

$$\sigma(T) \equiv e \left( \frac{meE}{\pi\hbar^2 3N} \right)^{\frac{3}{2}} = e \left( \frac{e}{\pi\hbar^2} \frac{mkT}{2} \right)^{\frac{3}{2}}$$

at the second equality one insert the expression from the energy and one sees that  $\sigma$  is just a function of the temperature. The entropy is a homogeneous function of degree 1, it is well-behaved under a rescaling of extensive quantities

$$S(\lambda E, \lambda V, \lambda N) = \lambda S(E, V, N)$$

However, if the factor  $N!$  is not present in the phase space measure  $d\Gamma$ , then one has

$$\tilde{S} = kN \ln[V\sigma(T)]$$

The expression of the energy and the ideal gas law can still be both obtained, but the entropy is no longer a homogeneous function

$$\tilde{S}(\lambda E, \lambda V, \lambda N) = \lambda \tilde{S}(E, V, N) + kN \ln \lambda$$

Consider two ideal gasses  $(N_1, V_1)$  and  $(N_2, V_2)$ , adjacent to each other, but separated. If their temperature is equal  $T_1 = T_2 = T$ , then by letting the two gasses mix, one finds a variation of the entropy  $\Delta \tilde{S}$ . Before the mixing, the system has entropy

$$\tilde{S}_i = kN_1 \ln[V_1 \sigma(T)] + kN_2 \ln[V_2 \sigma(T)] = kN_1 \ln V_1 + kN_2 \ln V_2 + k(N_1 + N_2) \ln \sigma(T)$$

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After the mixing, one finds

$$\tilde{S}_f = kN_1 \ln[(V_1 + V_2)\sigma(T)] + kN_2 \ln[(V_1 + V_2)\sigma(T)]$$

The variation in the entropy is

$$\Delta\tilde{S} = \tilde{S}_f - \tilde{S}_i = kN_1 \ln \frac{V_1 + V_2}{V_1} + kN_2 \ln \frac{V_1 + V_2}{V_2} > 0$$

The variation is positive since there are more accessible states. When mixing two identical gasses

$$\frac{N_1}{V_1} = \frac{N_2}{V_2}, \quad T_1 = T_2$$

the variation should be zero  $\Delta\tilde{S} = 0$ , because it is just the same gas in two different compartments. However, the formula above states that the variation must be strictly positive in this case. This is Gibbs's paradox.

The correct result is obtained when considering the factor  $N!$  in the entropy

$$\begin{aligned} \Delta S = S_f - S_i &= k(N_1 + N_2) \ln \frac{V_1 + V_2}{N_1 + N_2} \sigma(T) - kN_1 \ln \frac{V_1}{N_1} \sigma(T) - kN_2 \ln \frac{V_2}{N_2} \sigma(T) \\ &= k(N_1 + N_2) \ln \frac{V_1 + V_2}{N_1 + N_2} + k(N_1 + N_2) \ln \sigma(T) \\ &\quad - kN_1 \ln \frac{V_1}{N_1} - kN_2 \ln \frac{V_2}{N_2} - k(N_1 + N_2) \ln \sigma(T) \\ &= k(N_1 + N_2) \ln \frac{V_1 + V_2}{N_1 + N_2} - k(N_1 + N_2) \ln \frac{V_1}{N_1} \\ &= k(N_1 + N_2) \left[ \ln \frac{V_1 + V_2}{N_1 + N_2} - \ln \frac{V_1}{N_1} \right] = 0 \end{aligned}$$

In fact, let

$$x \equiv \frac{N_1}{V_1} = \frac{N_2}{V_2}$$

then

$$N_1 = xV_1, \quad N_2 = xV_2 \implies N_1 + N_2 = x(V_1 + V_2)$$

from which

$$\frac{N_1 + N_2}{V_1 + V_2} = x = \frac{N_1}{V_1}$$

From this one observes that quantum mechanics carries non-trivial information even in the semi-classical limit.

## 1.4 Canonical ensemble

See Tong, Statistical Physics, §1.3. In the canonical ensemble, the macro-state has a fixed temperature, while the energy (i.e. not the average energy) can vary. The situation of the canonical ensemble is that of a system  $S$  immersed in a big reservoir  $R$ . The total Hamiltonian is

$$H = H_R + H_S + H_{\text{int}}, \quad H_{\text{int}} \ll H_S \ll H_R$$

The system and the reservoir are isolated from the environment, therefore the total energy is constant and the combined system can be described with the micro-canonical approach

$$E_T = E_R + E_S = \text{const.}$$

The total number of micro-state must consider all the possible ways in which the energy can be distributed between the system and the reservoir

$$\Omega(E_T) = \sum_i \Omega_R(E_T - E_i) = \sum_i \exp \left[ \frac{S_R(E_T - E_i)}{k} \right]$$

where  $S_R$  is the entropy of the reservoir. The sum is over the micro-states of the system  $S$  with energy  $E_i$ , not over the possible energies: many micro-states may have the same energy. The probability that the system  $S$  is in the state  $i$  is

$$P_S(i) = \frac{\Omega_R(E_T - E_i)}{\Omega(E_T)}$$

The numerator may be rewritten as

$$\begin{aligned}\Omega_R(E_T - E_i) &= \exp\left[\frac{S_R(E_T - E_i)}{k}\right] \approx \exp\left[\frac{S_R(E_T)}{k} - \frac{E_i}{k} \partial_{E_i} S_R(E_T)\right] \\ &= \exp\left[\frac{S_R(E_T)}{k}\right] \exp\left[-\frac{E_i}{k} \frac{1}{T}\right]\end{aligned}$$

where one Taylor expands knowing that the reservoir implies

$$E_i \ll E_T, \quad T = T_R = T_S$$

The denominator is just the sum of the numeration over the states, therefore one finds

$$\Omega(E_T) = \sum_i \exp\left[\frac{S_R(E_T)}{k}\right] \exp\left[-\frac{E_i}{kT}\right]$$

One finds the Boltzmann distribution

$$P_S(i) \approx \frac{e^{-\frac{E_i}{kT}}}{\sum_i e^{-\frac{E_i}{kT}}}$$

Its denominator is the Boltzmann partition function

$$Z = \sum_i e^{-\beta E_i}, \quad \beta = \frac{1}{kT}$$

where the sum is over the compatible micro-states.

**Helmholtz free energy.** See Tong, SP, §1.3.4. The free energy can be defined from the partition function

$$F = -kT \ln Z$$

From the free energy one may find all the thermodynamic quantities of interest. The average energy is

$$\langle E \rangle = \sum_i P(i) E_i = \frac{1}{Z} \sum_i E_i e^{-\beta E_i} = -\partial_\beta \ln Z$$

The entropy is

$$S = -k \sum_i P(i) \ln P_i = -\frac{k}{Z} \sum_i e^{-\beta E_i} \ln \frac{e^{-\beta E_i}}{Z} = \frac{1}{T} \langle E \rangle + k \ln Z = \frac{E}{T} - \frac{F}{T}$$

The symbol of average is typically dropped, but always understood since the average is always identified with the macroscopic quantity. The free energy can be rewritten as

$$F = E - TS$$

The free energy  $F(T, V, N)$  is a thermodynamic potential used to work with  $T$ ,  $V$  and  $N$ . Computing its variation gives

$$dF = dE - dT S - T dS = (T dS - p dV + \mu dN) - S dT - T dS = -p dV + \mu dN - S dT$$

where one recalls that

$$dE = T dS - p dV + \mu dN$$

obtained by rearranging the variation of the entropy. Thus, the pressure, the entropy and the chemical potential are

$$p = -\partial_V F, \quad S = -\partial_T F, \quad \mu = \partial_N F$$

In the canonical ensemble, the partition function gives the macroscopic quantities of interest.

There are also other thermodynamic potentials

- The enthalpy to work with  $S$ ,  $p$  and  $V$

$$H = E + pV$$

- The Gibbs free energy to work with  $T$ ,  $p$  and  $N$

$$G = E - TS + pV = H - TS$$

All the thermodynamic potentials are related by Legendre transforms. Notice that the fourth potential is the internal energy  $E$ . To easily remember the derivatives of the thermodynamic potentials, one may utilize the Born square<sup>1</sup>.

## Lecture 4

**Previous lecture.** In the canonical ensemble, the macro-state as fixed  $T$ ,  $V$  and  $N$ . The partition function is a sum over the states and not over the energy levels

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2024 16:30

$$Z(T, V, N) = \sum_i e^{-\beta E_i}$$

The density distribution is

$$\rho(i) = \frac{1}{Z} e^{-\beta E_i}$$

From the partition function one may define the free energy and from it obtain the macroscopic physical observables.

**Comments on the Boltzmann distribution.** The Boltzmann distribution is compatible with the Liouville theorem

$$\partial_t \rho_S(i) = 0 \iff \{\rho_S(i), H\} = 0$$

since the distribution is function of the energies  $E_i$  only. One has to check that it maximizes the entropy with the constraints

$$\sum_i P(i) = 1, \quad \sum_i P(i) E_i = \langle E \rangle$$

In fact

$$\begin{aligned} 0 &= \partial_{P(j)} \left[ \sum_i P(i) \ln P(i) + \lambda_1 \left( \sum_i P(i) - 1 \right) + \lambda_2 \left( \sum_i P(i) E_i - \langle E \rangle \right) \right] \\ &= -\ln P(j) - 1 + \lambda_1 + \lambda_2 E_j \end{aligned}$$

from which

$$P_j = c e^{d E_j}$$

From the constraints, one may find the constants

$$c = \frac{1}{Z}, \quad d = -\beta$$

to obtain the Boltzmann distribution.

<sup>1</sup>See [https://en.wikipedia.org/wiki/Thermodynamic\\_square](https://en.wikipedia.org/wiki/Thermodynamic_square).



### 1.4.1 Ideal gas

See Kardar, §4.7, Tong, SP, §2.2. Instead of repeating the procedure of the micro-canonical ensemble, one may compute the partition function. Consider  $N$  non-interacting particles. The Hamiltonian of the system is given by

$$H = \sum_{i=1}^N \frac{\mathbf{p}_i^2}{2m}$$

The state  $i$  is a part in the phase space  $\Gamma = \mathbb{R}^{6N}$ . The probability distribution is

$$P_S(\mathbf{p}, \mathbf{q}) = \frac{1}{Z} \exp \left[ -\beta \sum_{i=1}^N \frac{\mathbf{p}_i^2}{2m} \right]$$

One may compute the partition function

$$\begin{aligned} Z &= \sum_i P_S(i) = \int d\Gamma \exp \left[ -\beta \sum_{i=1}^N \frac{\mathbf{p}_i^2}{2m} \right] = \int \left[ \prod_{i=1}^N \frac{d\mathbf{q}_i d\mathbf{p}_i}{(2\pi\hbar)^{3N} N!} \right] \exp \left[ -\beta \sum_{i=1}^N \frac{\mathbf{p}_i^2}{2m} \right] \\ &= \frac{V^N}{(2\pi\hbar)^{3N} N!} \left[ \frac{2\pi m}{\beta} \right]^{\frac{3N}{2}} \equiv \frac{1}{N!} \left[ \frac{V}{\lambda^3} \right]^N = \frac{1}{N!} (Z_1)^N \end{aligned}$$

Recall that one is summing the compatible micro-states (which are points in the phase space such that  $H = E_i$ ). At the first line the sum is over the states  $i$ . At the second line,  $Z_1$  is the single-particle partition function. One also defines the de Broglie thermal wave length

$$\lambda \equiv \frac{2\pi\hbar}{\sqrt{2\pi m k T}}$$

which schematically embodies the Heisenberg uncertainty principle

$$E \sim kT, \quad p \sim \sqrt{2mE} \implies \lambda \sim \frac{\hbar}{p}$$

From the partition function one may derive the thermodynamic quantities.

**Helmholtz free energy.** The free energy is

$$\begin{aligned} F &= -kT \ln Z = -kTN \ln \frac{V}{\lambda^3} + kT \ln N! \sim -kTN [\ln V - \ln N + 1 - 3 \ln \lambda], \quad N \rightarrow \infty \\ &= -kTN \left[ \ln V - \ln N + 1 + \frac{3}{2} \ln \frac{2\pi m k T}{\hbar^2} \right] \end{aligned}$$

**Entropy and energy.** The entropy is

$$S = -\partial_T F = -\frac{F}{T} + \frac{3}{2} Nk$$

from which the energy is

$$E = F + TS = \frac{3}{2} NkT$$

This agrees again with the equipartition theorem.

**Pressure.** The pressure is

$$p = -\partial_V F = \frac{NkT}{V} \implies pV = NkT$$

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**Chemical potential.** The chemical potential is

$$\mu = \partial_N F = \frac{F}{N} + kT = kT \ln \frac{N\lambda^3}{V}$$

From the chemical potential, one may define the fugacity

$$z \equiv e^{\frac{\mu}{kT}} = \frac{N\lambda^3}{V}$$

Notice that  $V/N$  is the classical volume per particle and  $\lambda^3$  is the quantum volume per particle. If

$$\frac{V}{N} \gg \lambda^3$$

then the system can be treated classically and the temperature is high, so there is no wave function overlap or thermal boson effects. If

$$\frac{V}{N} \sim \lambda^3$$

then the quantum statistics effects are important and the temperature is low. The classical (or ideal) case is the high temperature case. In such situation

$$\frac{N}{V}\lambda^3 \ll 1 \implies \mu < 0$$

#### 1.4.2 Fluctuations

See Tong, SP, §1.3.2. The average energy is

$$\langle E \rangle = \frac{1}{Z} \sum_i E_i e^{-\beta E_i} = -\partial_\beta \ln Z$$

One may compute the variance. Consider the second derivative

$$\begin{aligned} \partial_\beta^2 \ln Z &= \partial_\beta \left[ -\frac{1}{Z} \sum_i E_i e^{-\beta E_i} \right] = -\frac{1}{Z^2} \left[ \sum_i E_i e^{-\beta E_i} \right]^2 + \frac{1}{Z} \sum_i E_i^2 e^{-\beta E_i} \\ &= -\langle E \rangle^2 + \langle E^2 \rangle = (\Delta E)^2 \end{aligned}$$

from which

$$(\Delta E)^2 = -\partial_\beta \langle E \rangle$$

Recalling that the heat capacity is

$$C = d_t \langle E \rangle$$

then one finds

$$(\Delta E)^2 = -\partial_\beta \langle E \rangle = -\partial_\beta T d_T \langle E \rangle = kT^2 C$$

therefore

$$(\Delta E)^2 \sim C, \quad \langle E \rangle \sim N \implies C \sim N$$

The relative fluctuations scale as

$$\frac{\Delta E}{\langle E \rangle} \sim \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}} \rightarrow 0, \quad N \rightarrow \infty$$

They are suppressed in the thermodynamic limit and  $\langle E \rangle$  is the most probable value of the energy  $E$ . To see this, one may consider the probability that the system has energy  $E_i$

$$P(E_i) = \frac{1}{Z} \Omega(E_i) e^{-\beta E_i}$$

where the partition function is

$$Z = \sum_i e^{-\beta E_i} = \sum_{E_i} \Omega(E_i) e^{-\beta E_i} \approx \Omega(E^*) e^{-\beta E^*}, \quad N \rightarrow \infty$$

where the sum is over the states. The approximation holds because, in the thermodynamic limit, the sum is dominated by the maximum value  $E^*$  (see Tong, SP, §1.3.3). Therefore, the most probable value of the energy is

$$\langle E \rangle = -\partial_\beta \ln Z \approx -\partial_\beta \ln[\Omega(E^*)e^{-\beta E^*}] = E^* - \frac{1}{Z} \partial_\beta E^* \partial_E [\Omega(E)e^{-\beta E}]_{E=E^*} = E^* + 0$$

The derivative of the bracket is zero because  $E^*$  extremizes the argument.

**Example 1.7** (Two-state system). See Kardar, §4.7, Tong, SP, §1.3.2. Consider  $N$  non-interacting particles taking spin  $|\pm\rangle$  with energies

$$E_+ = \varepsilon, \quad E_- = 0$$

The partition function is  $Z = (Z_1)^N$  because each particle is independent from the rest. The single-particle canonical partition function is

$$Z_1 = \sum_{i=\pm} e^{-\beta E_i} = e^{-\beta \varepsilon} + e^{-\beta \cdot 0} = 1 + e^{-\beta \varepsilon} = 2e^{-\frac{\beta}{2}\varepsilon} \frac{e^{\frac{\beta}{2}\varepsilon} + e^{-\frac{\beta}{2}\varepsilon}}{2} = 2e^{-\frac{\beta}{2}\varepsilon} \cosh \frac{\beta \varepsilon}{2}$$

Thus the partition function of the system is

$$Z = 2^N e^{-\frac{\beta}{2}N\varepsilon} \left[ \cosh \frac{\beta \varepsilon}{2} \right]^N$$

from which the energy is

$$\langle E \rangle = -\partial_\beta \ln Z = \dots = \frac{N\varepsilon}{1 + e^{\beta \varepsilon}} = \frac{N\varepsilon}{2} \left[ 1 - \tanh \frac{\beta \varepsilon}{2} \right]$$

## 1.5 Gran-canonical ensemble

See Kardar, §4.9, Tong, SP, §1.4.2. In the gran-canonical ensemble, the macro-state has fixed volume, temperature and chemical potential, while the energy and number of particles can vary. The situation is similar to the canonical ensemble: the system is immersed in a reservoir with which it exchanges energy and particles. The total Hamiltonian is

$$H_T = H_R + H_S + H_{\text{int}}$$

The reservoir implies

$$E_R \gg E_S, \quad N_R \gg N_S$$

The whole system is isolated from the environment so the total quantities are conserved

$$E_T = E_R + E_S = \text{const.}, \quad N_T = N_R + N_S = \text{const.}$$

The probability distribution of a state is

$$P_S(i) = \frac{\Omega_R(E_T - E_i, N_T - N_i)}{\Omega(E_T, N_T)}$$

The total number of micro-states of the reservoir is

$$\begin{aligned} \Omega_R(E_T - E_i, N_T - N_i) &= \exp \left[ \frac{1}{k} S_R(E_T - E_i, N_T - N_i) \right] \\ &\approx \exp \left[ \frac{1}{k} S_R(E_T, N_T) \right] \exp \left[ -\frac{1}{k} (E_i \partial_{E_i} S_R + N_i \partial_{N_i} S_R) \right] \end{aligned}$$

Since

$$T = T_S = T_R = [\partial_E S_R]^{-1}, \quad \mu = \mu_S = \mu_R = -T \partial_N S_R$$

one finds the gran-canonical distribution

$$P_S(i) = \frac{e^{-\beta E_i + \beta \mu N_i}}{\sum_i e^{-\beta E_i + \beta \mu N_i}}$$

and the gran-canonical partition function

$$\mathcal{Z}(T, V, \mu) = \sum_i e^{-\beta E_i + \beta \mu N_i}$$

where the sum is over the micro-states compatible with fixed  $T$ ,  $V$  and  $\mu$ . One may relate this partition function to the one of the canonical ensemble

$$\mathcal{Z}(T, V, \mu) = \sum_{N_i=0}^{\infty} e^{\beta \mu N_i} \sum_{\substack{i \text{ w/} \\ \text{fixed } N_i}} e^{-\beta E_i} = \sum_{N_i=0}^{\infty} e^{\beta \mu N_i} Z(T, V, N_i) = \sum_{N=0}^{\infty} z^N Z(T, V, N)$$

where one finds the fugacity  $z \equiv e^{\beta \mu}$  and  $Z(T, V, N)$  is the canonical partition function.

One may define the gran-canonical potential

$$\Phi = -kT \ln \mathcal{Z}$$

One may extract the information about the system from the partition function and the gran-canonical potential. The derivative with respect to  $\beta$  gives

$$-\partial_\beta \ln \mathcal{Z}(T, V, \mu) = \langle E \rangle - \mu \langle N \rangle$$

**Number of particles.** The number of particles is

$$\langle N \rangle = \frac{1}{\beta} \partial_\mu \ln \mathcal{Z}$$

**Entropy.** The entropy is

$$S = -k \sum_i P(i) \ln P(i) = -\frac{k}{\mathcal{Z}} \sum_i e^{-\beta(E_i - \mu N_i)} \ln \frac{1}{\mathcal{Z}} e^{-\beta(E_i - \mu N_i)} = k \ln \mathcal{Z} + \frac{1}{T} (\langle E \rangle - \mu \langle N \rangle)$$

**Gran-canonical potential.** Inserting the above  $k \ln \mathcal{Z}$  inside the potential, one finds (where the average is understood)

$$\Phi(T, V, \mu) = -kT \ln \mathcal{Z} = -TS + E - \mu N = F(T, V, N) - \mu N$$

This is a Legendre transform between  $\mu$  and  $N$ .

Recalling that

$$dF = -S dT - p dV + \mu dN$$

one obtains

$$d\Phi = dF - \mu dN - N d\mu = -S dT - p dV - N d\mu$$

from which one reads

$$\boxed{p = -\partial_V \Phi, \quad S = -\partial_T \Phi, \quad N = -\partial_\mu \Phi}$$

Notice that the potential depends on one extensive variable, the volume. The potential is extensive too

$$\Phi(T, \lambda V, \mu) = \lambda \Phi(T, V, \mu)$$

Therefore the potential is linear in the volume

$$\Phi(T, V, \mu) = V f(T, \mu) = -pV$$

since

$$p = -\partial_V \Phi$$

This argument is not valid for the Helmholtz free energy  $F(T, V, N)$  because it depends on two extensive variables, the volume and the number of particles. For the Gibbs free energy one has

$$G(T, p, N) = \mu N$$

since only the number of particles is extensive.

### 1.5.1 Fluctuations

Consider the average number of particles

$$\langle N \rangle = \frac{1}{\beta} \partial_\mu \ln \mathcal{Z}$$

Its variance is

$$(\Delta N)^2 = \langle N^2 \rangle - \langle N \rangle^2 = \frac{1}{\beta} \partial_\mu \langle N \rangle = \frac{1}{\beta^2} \partial_\mu^2 \ln \mathcal{Z} \sim N$$

Therefore, the relative fluctuations behave as

$$\frac{\Delta N}{\langle N \rangle} \sim \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}} \rightarrow 0, \quad N \rightarrow \infty$$

**Comments on the distribution.** One notices that the distribution of states satisfies

$$\partial_i \rho(i) = 0 \iff \{\rho(i), H\} = 0$$

One may check that the gran-canonical distribution maximizes the entropy with the constraints

$$\langle E \rangle = \sum_i E_i P(i), \quad \langle N \rangle = \sum_i N_i P(i), \quad \sum_i P(i) = 1$$

In fact

$$\begin{aligned} 0 &= \partial_{P(i)} \left[ -k \sum_j P(j) \ln P(j) + \lambda_1 \left( \sum_j P(j) - 1 \right) \right. \\ &\quad \left. + \lambda_2 \left( \sum_j E_j P(j) - E \right) + \lambda_3 \left( \sum_j N_j P(j) - N \right) \right] \\ &= -k + k \ln P(i) + \lambda_1 + \lambda_2 E_i + \lambda_3 N_i \end{aligned}$$

from this one obtains

$$P(i) = c \exp \left[ -\frac{\lambda_2 E_i + \lambda_3 N_i}{k} \right]$$

### 1.5.2 Ideal gas

The partition function is

$$\mathcal{Z}(T, V, \mu) = \sum_{N=0}^{\infty} z^N Z(T, V, N) = \sum_{N=0}^{\infty} \frac{1}{N!} \left[ \frac{zV}{\lambda^3} \right]^N = \exp \left[ \frac{zV}{\lambda^3} \right]$$

while the gran-canonical potential is

$$\Phi = -kT \ln \mathcal{Z} = -kT \frac{zV}{\lambda^3}$$

Thus, the average pressure is

$$p = -\frac{\Phi}{V} = \frac{kT}{\lambda^3} z$$

The average number of particles is

$$N = \frac{1}{\beta} \partial_\mu \ln \mathcal{Z} = \frac{1}{\beta} \frac{V}{\lambda^3} \partial_\mu z = \frac{V}{\lambda^3} z, \quad z = e^{\beta\mu}$$

The value of the fugacity can be extracted from the equation of the average number of particles (i.e. the equation above). Inserting its expression into the equation for the average pressure, one finds the ideal gas law

$$p = \frac{kT}{\lambda^3} z = \frac{kT}{V} N \implies pV = NkT$$

For the ideal gas, the system is classical when the temperature is high

$$z = e^{\beta\mu} = \frac{N}{V} \lambda^3 \ll 1 \implies \mu < 0$$

## 2 Interacting particles

See Kardar, §4.4?, sources? [r]. One would like to describe  $N$  classical particles that interact with one another with potential  $U(\mathbf{q}_1, \dots, \mathbf{q}_N)$  at fixed temperature. The focus is just on weak interactions, but the problem can be equally formulated for stronger interactions. One may use the canonical ensemble. The partition function is

$$Z = \int \frac{d^{3N}p d^{3N}q}{(2\pi\hbar)^{3N} N!} \exp \left[ -\beta \left( \sum_{i=1}^N \frac{\mathbf{p}_i^2}{2m} + U(\mathbf{q}_1, \dots, \mathbf{q}_N) \right) \right] = \int \frac{d^{3N}p d^{3N}q}{(2\pi\hbar)^{3N} N!} e^{-\beta H}$$

### Lecture 5

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2024 14:30

The distribution of micro-states is

$$\rho(\mathbf{p}, \mathbf{q}) = \frac{1}{Z} e^{-\beta H}$$

One may calculate the probability of having a particle with momentum  $\mathbf{p}$

$$\rho(\mathbf{p}) = \int d\Gamma \rho(\mathbf{p}, \mathbf{q}) \delta(\mathbf{p}_i - \mathbf{p}) = \mathcal{N} e^{-\beta \frac{\mathbf{p}^2}{2m}}$$

The integration in the position is complicated due to the presence of the potential  $U$ . To compute the constant  $\mathcal{N}$ , one may utilize the normalization condition

$$\int d^3p \rho(\mathbf{p}) = 1$$

from which one finds

$$1 \equiv \mathcal{N} \int d^3p e^{-\frac{\mathbf{p}^2}{2mkT}} = \mathcal{N} (2\pi kTm)^{\frac{3}{2}} \implies \mathcal{N} = \frac{1}{(2\pi kTm)^{\frac{3}{2}}}$$

Therefore, one obtains the Maxwell–Boltzmann distribution of the momentum

$$\boxed{\rho(\mathbf{p}) = \frac{1}{(2\pi kTm)^{\frac{3}{2}}} e^{-\frac{\mathbf{p}^2}{2mkT}}}$$

which is valid for any interaction  $U(\mathbf{q}_1, \dots, \mathbf{q}_N)$ . From this one computes the average kinetic energy of a particle

$$\langle E_{\text{kin}} \rangle = \frac{1}{2m} \langle \mathbf{p}^2 \rangle = \int d^3p \frac{\mathbf{p}^2}{2m} \rho(\mathbf{p}) = \frac{1}{2m} \frac{4\pi}{(2\pi kTm)^{\frac{3}{2}}} \int dp p^4 e^{-\frac{p^2}{2mkT}} = \frac{3}{2} kT$$

This is expected from the equipartition theorem: there are three degrees of freedom with quadratic terms in the Hamiltonian.

### 2.1 Classical ideal polyatomic gas

See Tong, SP, §§2.4, 3.4, sources? [r]. Consider a gas of  $N$  non-interacting molecules with an internal structure. Assume that there is no long-range interaction between such molecules and the system is classical, so no wave function overlap effects

$$\frac{V}{N} \gg \lambda^3$$

This means that the density of the gas is low enough to have no quantum effects.

The Hamiltonian of the system is the sum of the single-molecule Hamiltonians

$$H = \sum_i H_i$$

and as such the canonical partition function is factorized

$$Z = \frac{1}{N!} (Z_1)^N$$

where  $Z_1$  is the single-molecule partition function. Let the gas be diatomic and contain  $L$  particles, one finds

$$\begin{aligned} Z_1 &= \int \frac{d^{3L}p d^{3L}q}{(2\pi\hbar)^{3L}} \exp \left[ -\beta \left( \sum_{j=1}^L \frac{\mathbf{p}_j^2}{2m} + U(\mathbf{q}_1, \dots, \mathbf{q}_L) \right) \right] \\ &= Z_{\text{int}}(T) \int \frac{d^3p_{\text{CM}} d^3q_{\text{CM}}}{(2\pi\hbar)^3} \exp \left[ -\beta \frac{\mathbf{p}_{\text{CM}}^2}{2M} \right] = \frac{V}{\lambda^3} Z_{\text{int}}(T) \end{aligned}$$

where  $M$  is the mass of the molecule and as such  $\lambda$  depends on it. The partition function is factorized into a contribution from the center of mass and a contribution from the interaction. Therefore, the partition function is

$$Z(T, V, N) = \frac{1}{N!} (Z_1)^N = \frac{1}{N!} \left[ \frac{V}{\lambda^3} \right]^N Z_{\text{int}}^N(T)$$

while the gran-canonical partition function is

$$\mathcal{Z}(T, V, \mu) = \sum_{N=0}^{\infty} z^N Z(T, V, N) = \sum_{N=0}^{\infty} \frac{1}{N!} \left[ \frac{zV}{\lambda^3} Z_{\text{int}}(T) \right]^N = \exp \left[ \frac{zV}{\lambda^3} Z_{\text{int}}(T) \right]$$

The gran-canonical potential is

$$-pV = \Phi = -kT \ln \mathcal{Z} = -kT \frac{zV}{\lambda^3} Z_{\text{int}}(T)$$

The average number of particles is

$$N = \frac{1}{\beta} \partial_{\mu} \ln \mathcal{Z} = \frac{zV}{\lambda^3} Z_{\text{int}}(T)$$

Recognizing this expression inside the potential, one finds the ideal gas law

$$\Phi = -pV = -kT \frac{zV}{\lambda^3} Z_{\text{int}}(T) = -kTN \implies pV = kNT$$

The law is not sensitive to the internal structure of the molecules. The partition function of the interaction does not appear, but it is clear that the average energy depends on such partition function.

Consider a simple molecule: two atoms interacting through a quadratic potential (e.g. a spring). The Hamiltonian contains three terms

$$H = H_{\text{CM}} + H_{\text{rot}} + H_{\text{vib}} \implies Z_1 = Z_{\text{CM}} Z_{\text{rot}} Z_{\text{vib}}$$

**Center-of-mass contribution.** The contribution from the center of mass is

$$H_{\text{CM}} = \frac{p_{\text{CM}}^2}{2M}$$

which, by the equipartition theorem, gives the contribution

$$E_{\text{CM}} = \frac{3}{2} kT$$

---

**Rotational contribution.** The rotational contributions comes from two degrees of freedom. The rotational Lagrangian is

$$\mathcal{L}_{\text{rot}} = \frac{1}{2}I(\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2)$$

where  $I$  is the moment of inertia. The conjugate momenta are

$$p_{\theta} = \partial_{\dot{\theta}} \mathcal{L}_{\text{rot}} = I\dot{\theta}, \quad p_{\varphi} = \partial_{\dot{\varphi}} \mathcal{L}_{\text{rot}} = I \sin^2 \theta \dot{\varphi}$$

The Hamiltonian is then

$$H_{\text{rot}} = \dot{\theta}p_{\theta} + \dot{\varphi}p_{\varphi} - \mathcal{L}_{\text{rot}} = \frac{p_{\theta}^2}{2I} + \frac{p_{\varphi}^2}{2I \sin^2 \theta} = \frac{1}{2I} \mathbf{L}^2$$

where  $\mathbf{L}$  is the angular momentum. The rotational partition function is then

$$\begin{aligned} Z_{\text{rot}} &= \frac{1}{(2\pi\hbar)^2} \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \int dp_{\theta} \int dp_{\varphi} e^{-\beta H_{\text{rot}}} \\ &= \frac{1}{(2\pi\hbar)^2} \sqrt{\frac{2\pi I}{\beta}} \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \sqrt{\frac{2\pi I \sin^2 \theta}{\beta}} \\ &= \frac{2I}{\hbar^2} kT = \frac{2I}{\hbar^2} \frac{1}{\beta} \end{aligned}$$

Therefore, the rotational energy is

$$E_{\text{rot}} = -\partial_{\beta} \ln Z_{\text{rot}} = -\partial_{\beta} \ln \left[ \frac{2I}{\hbar^2} \frac{1}{\beta} \right] = kT = 2 \cdot \frac{1}{2} kT$$

The first 2 correctly represents the number of degrees of freedom.

**Vibrational contribution.** The vibrational Hamiltonian is

$$H_{\text{vib}} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 s^2$$

From the equipartition theorem one finds

$$E_{\text{vib}} = 2 \cdot \frac{1}{2} kT$$

**Total contribution.** The total contribution to the energy is

$$E_{\text{tot}} = N[E_{\text{CM}} + E_{\text{rot}} + E_{\text{vib}}] = N \left[ \frac{3}{2}kT + kT + kT \right] = \frac{7}{2}NkT$$

Therefore, the heat capacity is

$$C = \partial_T E = \frac{7}{2}kN$$

**Quantum effects.** In reality, the heat capacity changes with the temperature: it is an increasing step function. In order of increasing temperature, the contributions come from translation, rotation and vibration and so the energy and heat capacity change accordingly. See Tong, SP, fig. 11.

The rotational quantum Hamiltonian is

$$\hat{H}_{\text{rot}} = \frac{\hbar^2}{2I} \hat{\mathbf{L}}^2$$

The single-particle partition function is

$$Z_1^{\text{rot}} = \sum_{l=0}^{\infty} (2l+1) \exp \left[ -\beta \frac{\hbar^2}{2I} l(l+1) \right]$$



where the parenthesis are the degeneracy of the level  $l$ . At high temperatures  $\beta \ll 1$ , the sum can be approximated by an integral

$$Z_1^{\text{rot}} \approx \int_0^\infty dx (2x+1) \exp\left[-\beta \frac{\hbar^2}{2I} x(x+1)\right] \approx \frac{2I}{\beta \hbar^2}$$

Therefore the rotational energy is

$$E_{\text{rot}} = -\partial_\beta \ln Z_1^{\text{rot}} \approx \frac{1}{\beta} = kT$$

This agrees with what one has found above: the classical result found with the equipartition theorem. For low temperatures  $\beta \gg 1$ , all states effectively decouple and only  $l=0$  contributes

$$Z_{\text{rot}} \approx 1 \implies E_{\text{rot}} \approx 0$$

The rotational modes are frozen and only the translational modes contribute to the heat capacity.

Similarly, the Hamiltonian of the vibrations is the simple quantum harmonic oscillator and the single-particle partition function is

$$Z_1^{\text{vib}} = \sum_{n=0}^{\infty} \exp\left[-\beta \hbar \omega \left(n + \frac{1}{2}\right)\right] = e^{-\frac{1}{2}\beta \hbar \omega} \frac{1}{1 - e^{-\beta \hbar \omega}}$$

At high temperatures, one may approximate

$$e^{-\frac{1}{2}\beta \hbar \omega} \approx 1 - \beta \hbar \omega$$

so that the partition function and the energy are

$$Z_1^{\text{vib}} \approx \frac{1}{\beta \hbar \omega} \implies E_{\text{vib}} \approx kT$$

At low temperatures, one finds

$$e^{-\beta \hbar \omega} \approx 0 \implies Z_1^{\text{vib}} \approx e^{-\frac{1}{2}\beta \hbar \omega} \implies E_{\text{vib}} \approx \frac{1}{2} \hbar \omega$$

Therefore, at low temperatures, the non-translational contributions are

$$E_{\text{rot}} + E_{\text{vib}} \approx \frac{1}{2} \hbar \omega$$

and give no contributions to the heat capacity

$$C = d_T E$$

## 2.2 Weakly interacting classical gas

See Kardar, §§5.2, 5.3, Tong, SP, §2.5. Consider  $N$  point particles with a long range interaction. Suppose no quantum overlap effects

$$\frac{V}{N} \gg \lambda^3$$

and let  $l$  be the range of the interaction

$$\frac{V}{N} \gtrsim l^3$$

This assumption implies low density and one may only consider a two-body interaction, while three-body and higher interactions are unlikely and suppressed.

The Hamiltonian of the system is

$$H = \sum_{i=1}^N \frac{\mathbf{p}_i^2}{2m} + \sum_{i<j} U(|\mathbf{q}_i - \mathbf{q}_j|)$$

The gas consists of neutral particles, but they can get an instantaneous dipole  $\mathbf{p}_1$  that induces an electric field

$$E \sim \frac{p_1}{r^3}$$

and as such a dipole, so one may have dipole-dipole interactions

$$\frac{p_1 p_2}{r^3} \sim \frac{1}{r^6}$$

This is called van der Waals interaction.

At short distances, one assumes a repulsive force to account for the Pauli exclusion principle. One very common potential that is often used to model the force between atoms is the Lennard-Jones potential

$$U(r) \sim \left[ \frac{r_0}{r} \right]^{12} - \left[ \frac{r_0}{r} \right]^6$$

which can be simplified to

$$U(r) = \begin{cases} \infty, & r \leq r_0 \\ -U_0 \left[ \frac{r_0}{r} \right]^6, & r > r_0 \end{cases}$$

Due to the form of the interaction, one expects that the equation of state of the gas is a modification of the ideal gas equation of state

$$\frac{p}{kT} = \frac{N}{V} + B_2(T) \left[ \frac{N}{V} \right]^2 + B_3(T) \left[ \frac{N}{V} \right]^3 + \dots$$

The correction to the ideal gas law is expressed in terms of a density expansion  $N/V$ , called virial expansion. One would like to find the coefficients from the ensemble.

The canonical partition function is

$$\begin{aligned} Z &= \frac{1}{N!} \int \prod_{i=1}^N \frac{d\mathbf{p}_i d\mathbf{q}_i}{(2\pi\hbar)^{3N}} \exp \left[ -\beta \left( \sum_{i=1}^N \frac{\mathbf{p}_i^2}{2m} + \sum_{i<j} U(q_{ij}) \right) \right] \\ &= \frac{1}{N!} \frac{1}{\lambda^{3N}} \int \prod_{i=1}^N d\mathbf{q}_i \exp \left[ -\beta \sum_{i<j} U(q_{ij}) \right] \end{aligned}$$

where  $q_{ij} \equiv |\mathbf{q}_i - \mathbf{q}_j|$ . Since the density is low, meaning the system is classical and the temperature is high, one may first attempt to expand

$$\exp \left[ -\beta \sum_{i<j} U(q_{ij}) \right] \sim 1 - \beta \sum_{i<j} U(q_{ij}) + \beta^2 \dots$$

This does not work at short distances

$$U(r) \rightarrow \infty, \quad r < r_0$$

A second attempt defines the Mayer function

$$f(q) \equiv e^{-\beta U(q)} - 1$$

Its limits are

$$\lim_{q \rightarrow r_0} f(q) = -1, \quad \lim_{q \rightarrow \infty} f(q) = 0, \quad |f(q)| \in [0, 1]$$

This function can be used as an expansion parameter. One may rewrite the interaction as

$$\exp \left[ \sum_{i<j} U(q_{ij}) \right] = \prod_{i<j} e^{-\beta U(q_{ij})} = \prod_{i<j} [1 + f(q_{ij})] = 1 + \sum_{i<j} f(q_{ij}) + \sum_{i<j} \sum_{l<m} f(q_{ij}) f(q_{lm}) + \dots$$

Therefore, the partition function is

$$Z = \frac{1}{N!} \frac{1}{\lambda^{3N}} \int \prod_{i=1}^N d\mathbf{q}_i \left[ 1 + \sum_{i<j} f_{ij} + \sum_{i<j} \sum_{l<m} f_{ij} f_{lm} + \dots \right], \quad f_{ij} \equiv f(q_{ij})$$

The first addendum is just an integral in the position

$$\int \prod_{i=1}^N d\mathbf{q}_i = V^N$$

Let  $i = 1$  and  $j = 2$ , then the second addendum gives

$$\begin{aligned} \int \left[ \prod_{i=1}^N d\mathbf{q}_i \right] f_{ij} &= \left[ \int d\mathbf{q}_1 d\mathbf{q}_2 f_{12} \right] \left[ \int d\mathbf{q}_3 \cdots d\mathbf{q}_N \right] = \left[ \int d\mathbf{k} f_{12}(k) \right] \left[ \int d\mathbf{r} \right] V^{N-2} \\ &= \left[ \int d\mathbf{k} f_{12}(k) \right] V^{N-1}, \quad \mathbf{k} = \mathbf{q}_1 - \mathbf{q}_2, \quad \mathbf{r} = \mathbf{q}_1 + \mathbf{q}_2 \end{aligned}$$

Every term in the sum gives the same contribution, therefore

$$\sum_{i < j} \int \left[ \prod_{i=1}^N d\mathbf{q}_i \right] f(q_{ij}) = \frac{1}{2} N(N-1) V^{N-1} \int d\mathbf{k} f(k) \approx \frac{N^2}{2V} V^N \int d\mathbf{k} f(k)$$

where the approximation hold in the thermodynamic limit. Thus, the partition function is

$$Z \approx \frac{V^N}{\lambda^{3N} N!} \left[ 1 + \frac{N^2}{2V} \int d\mathbf{k} f(k) + \cdots \right]$$

Recalling that for  $x \ll 1$  holds

$$(1+x)^N \approx (1+Nx)$$

one may expand the partition function in  $N/V$

$$Z \approx \frac{V^N}{\lambda^{3N} N!} \left[ 1 + \frac{N}{2V} \int d\mathbf{k} f(k) \right]^N$$

The free energy is then

$$\begin{aligned} F &= -kT \ln Z \approx -kT \ln \frac{V^N}{\lambda^{3N} N!} - kTN \ln \left[ 1 + \frac{N}{2V} \int d\mathbf{k} f(k) \right] \\ &\approx -kT \ln \frac{V^N}{\lambda^{3N} N!} - kTN \frac{N}{2V} \int d\mathbf{k} f(k) = \text{ideal gas} + \text{corrections} \end{aligned}$$

At the second line one has applied

$$\ln(1+x) \approx x$$

The pressure is

$$p = -\partial_V F = kT \frac{N}{V} \left[ 1 - \frac{N}{2V} \int d\mathbf{k} f(k) \right] + \cdots$$

from this one finds the first coefficient

$$B_2(T) = -\frac{1}{2} \int d\mathbf{k} f(k)$$

The first coefficient depends on the first term in the expansion. There still is no information about the particular interaction.

One is interested in the next coefficients. The naive idea is to expand in more terms and collect them accordingly. A more clever way is the cluster expansion: it is a systematic analysis of the perturbative expansion without assuming any detail of the interaction. The form of the interaction is needed only at the end when one is interested in the numerical values.

## Lecture 6

### 2.2.1 Cluster expansion

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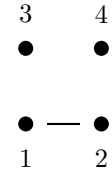
Recall the partition function

$$Z = \frac{1}{N!} \frac{1}{\lambda^{3N}} \int \prod_{i=1}^N d\mathbf{q}_i \left[ 1 + \sum_{i<j} f_{ij} + \sum_{i<j} \sum_{l<m} f_{ij} f_{lm} + \cdots \right]$$

One may check by writing explicitly the sums that it holds:

- all indices are distinct,  $f_{ij} f_{lm} f_{kn}$ ;
- repeated indices are allowed,  $f_{ij} f_{lm} f_{mk}$ ;
- the same pair cannot appear more than once,  $f_{ij} f_{lm} f_{ij}$ ;
- on the same  $f$  the indices cannot have the same value,  $f_{ii} f_{lm} f_{kj}$ .

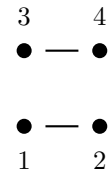
One can graphically represent the integrals in the partition function. The number  $N$  of integration variables  $d\mathbf{q}_i$  corresponds to the number  $N$  of points and each  $f_{ij}$  is a link connecting the two points  $i$  and  $j$ . An example for  $N = 4$  is the following

$$\int d\mathbf{q}_1 d\mathbf{q}_2 d\mathbf{q}_3 d\mathbf{q}_4 f(q_{12}) = \int d\mathbf{q}_3 d\mathbf{q}_4 \int d\mathbf{q}_1 d\mathbf{q}_2 f(q_{12}) =$$


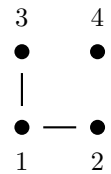
The bottom dots are a 2-cluster, while the upper dots each are a 1-cluster. Let  $m_l$  be the number of  $l$ -clusters, then

$$m_1 = 2, \quad m_2 = 1, \quad m_3 = m_4 = 0 \implies \sum_l l m_l = N = 4$$

Another example is

$$\int d\mathbf{q}_1 d\mathbf{q}_2 d\mathbf{q}_3 d\mathbf{q}_4 f_{12} f_{34} = \left[ \int d\mathbf{q}_1 d\mathbf{q}_2 f_{12} \right] \left[ \int d\mathbf{q}_3 d\mathbf{q}_4 f_{34} \right] =$$


where one has  $m_2 = 2$ . Finally

$$\int d\mathbf{q}_1 d\mathbf{q}_2 d\mathbf{q}_3 d\mathbf{q}_4 f_{12} f_{13} = \left[ \int d\mathbf{q}_1 d\mathbf{q}_2 d\mathbf{q}_3 f_{12} f_{13} \right] \left[ \int d\mathbf{q}_4 \right] =$$


with  $m_1 = 1$  and  $m_3 = 1$ .

**Definition 2.1.** An  $l$ -cluster is a connected part of the graph with  $l$  connected points.

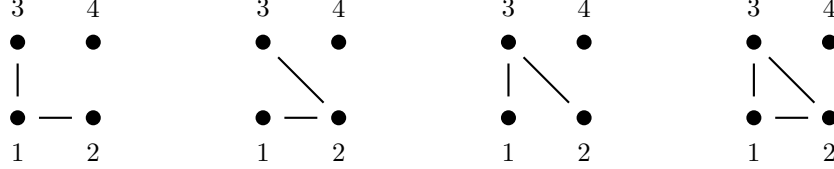
A graph with  $N$  points can be decomposed into  $l$ -clusters with the condition

$$N = \sum_{l=1}^N m_l l$$

with  $m_l$  the number of  $l$ -clusters. The graphs above are some partitions of four points. For example, for  $N = 4$  one finds

- $m_1 = 4, m_2 = m_3 = m_4 = 0$  so  $4 \cdot 1 = 4$ ;
- $m_1 = 2, m_2 = 1, m_3 = m_4 = 0$ , so  $2 \cdot 1 + 1 \cdot 2 = 4$ ;
- $m_2 = 2, m_1 = m_3 = m_4 = 0$ , so  $2 \cdot 2 = 4$ ;
- $m_1 = m_3 = 1, m_2 = m_4 = 0$ , so  $1 \cdot 1 + 1 \cdot 3 = 4$ ;
- $m_4 = 1, m_1 = m_2 = m_3 = 0$ , so  $1 \cdot 4 = 4$ .

Notice that there are four 3-clusters connecting 1, 2 and 3 with different numbers of links (which correspond to powers of  $f$ )



The last graph has three links,  $f^3$ , while the others have two,  $f^2$ . The coefficients of the virial expansion are not organized in powers of the Mayer function  $f$ . The importance is on how many points are connected and not on how many links there are. This is reflected in the cluster expansion. The  $l$ -clusters can have different numbers of links.

Defining the contribution of the  $l$ -clusters to be

$$U_l \equiv \int \left[ \prod_{j=1}^N d\mathbf{q}_j \right] (\text{all } l\text{-clusters})$$

the multiplicity of the contribution of an  $l$ -cluster  $U_l$  is

$$N! \prod_l \frac{1}{(l!)^{m_l} m_l!}$$

The factor  $l!$  counts the permutations of points in the  $l$ -cluster, while the  $m_l$  factor counts the permutation of  $l$ -clusters. An example contribution is  $U_3$  which is the sum of the four diagrams above.

The partition function is then

$$Z(T, V, N) = \frac{1}{\lambda^{3N} N!} \sum_{\{m_l\}} N! \prod_l \frac{(U_l)^{m_l}}{(l!)^{m_l} m_l!} = \frac{1}{\lambda^{3N}} \sum_{\{m_l\}} \prod_l \frac{(U_l)^{m_l}}{(l!)^{m_l} m_l!}$$

where the sum is over all distinct divisions of  $N$  points into a set of clusters  $\{m_1, m_2, m_3, \dots\}$  such that

$$\sum_{l=1}^N m_l l = N$$

This constraint complicates computations, but can be removed by using the gran-canonical partition function<sup>2</sup>

$$\begin{aligned} \mathcal{Z}(T, V, \mu) &= \sum_{N=0}^{\infty} z^N Z(T, V, N) = \sum_{N=0}^{\infty} \frac{z^N}{\lambda^{3N}} \sum_{\{m_l\}} \prod_l \frac{(U_l)^{m_l}}{(l!)^{m_l} m_l!} \\ &= \sum_{N=0}^{\infty} \sum_{\substack{\{m_l\} \\ \text{constraint}}} \left[ \frac{z}{\lambda^3} \right]^{\sum_l m_l l} \prod_l \frac{(U_l)^{m_l}}{(l!)^{m_l} m_l!} = \sum_{N=0}^{\infty} \sum_{\substack{\{m_l\} \\ \text{constraint}}} \prod_l \left[ \frac{z}{\lambda^3} \right]^{m_l l} \frac{(U_l)^{m_l}}{(l!)^{m_l} m_l!} \\ &= \sum_{\{m_l\}} \prod_l \left[ \frac{z}{\lambda^3} \right]^{m_l l} \frac{1}{m_l!} \left[ \frac{U_l}{l!} \right]^{m_l} = \prod_l \sum_{m_l=0}^{\infty} \frac{1}{m_l!} \left[ \left( \frac{z}{\lambda^3} \right)^l \frac{U_l}{l!} \right]^{m_l} \\ &= \prod_{l=1}^{\infty} \exp \left[ \left( \frac{z}{\lambda^3} \right)^l \frac{U_l}{l!} \right] = \exp \left[ \frac{V}{\lambda^3} \sum_{l=1}^{\infty} b_l z^l \right] \end{aligned}$$

<sup>2</sup>This is done because it is an equivalent description to the canonical ensemble in the thermodynamic limit,  $N, V \rightarrow \infty$ .

At the third line one removes the constraint. At the last line, one defines

$$b_l \equiv \frac{\lambda^3}{V} \frac{U_l}{l!} \frac{1}{\lambda^{3l}}$$

The sum over all diagrams has been rewritten as the exponential of the sum over all connected diagrams, meaning all clusters.

**Observables and coefficients.** The average pressure can be found from

$$pV = -\Phi = kT \ln \mathcal{Z}(T, V, \mu) = kT \frac{V}{\lambda^3} \sum_{l=1}^{\infty} b_l z^l$$

while the average number of particles is

$$N = \frac{1}{\beta} \partial_{\mu} \ln \mathcal{Z}(T, V, \mu) = \frac{V}{\lambda^3} \sum_{l=1}^{\infty} l b_l z^l$$

One may find the fugacity  $z$  in terms of the average number of particles  $N$  at some order from the equation above. Inserting this result into the equation for average pressure, one may find the pressure in terms of the density  $N/V$  at a given order. At the second order in  $z$  one finds

$$\lambda^3 \frac{N}{V} = z + 2b_2 z^2 + o(z^2) \implies z \approx \lambda^3 \frac{N}{V} - 2b_2 \left[ \lambda^3 \frac{N}{V} \right]^2 + \dots$$

where one isolates  $z$  and substitutes the result into  $z^2$  (this is equivalent to solving the quadratic equation, expanding the square root and truncating at the desired order). Substituting this expression into the pressure gives

$$\frac{p}{kT} \approx \frac{1}{\lambda^3} (z + b_2 z^2) \approx \frac{N}{V} \left[ 1 - b_2 \lambda^3 \frac{N}{V} \right]$$

where  $b_1 = 1$ . By comparison with the virial expansion

$$\frac{p}{kT} = \frac{N}{V} + B_2(T) \left[ \frac{N}{V} \right]^2 + B_3(T) \left[ \frac{N}{V} \right]^3 + \dots$$

one sees

$$B_2(T) = -b_2 \lambda^3$$

Expanding to higher orders, one finds

$$B_3 = \lambda^6 (4b_2^2 - 2b_3), \quad \dots$$

**Computation of coefficients.** Expanding the coefficients gives

$$\begin{aligned} B_2(T) &= -b_2 \lambda^3 = -\frac{U_2}{2V} = -\frac{1}{2V} \int d\mathbf{q}_1 d\mathbf{q}_2 f(q_{12}) = -\frac{1}{2V} \int d\mathbf{q} \int d\mathbf{k} f_{12}(k) \\ &= -\frac{1}{2} \int d\mathbf{k} f_{12}(k) = -\frac{1}{2} \int d\mathbf{k} [e^{-\beta U(k)} - 1] \end{aligned}$$

where one sets

$$q = q_1 + q_2, \quad k = q_1 - q_2$$

Using the potential

$$U(r) = \begin{cases} \infty, & r \leq r_0 \\ -U_0 \left[ \frac{r_0}{r} \right]^6, & r > r_0 \end{cases}$$

one sees

$$\begin{aligned}
\int \mathrm{d}\mathbf{r} [e^{-\beta U(r)} - 1] &= 4\pi \int_0^\infty r^2 \mathrm{d}r [e^{-\beta U(r)} - 1] \\
&= 4\pi \int_0^{r_0} r^2 \mathrm{d}r (-1) + 4\pi \int_{r_0}^\infty r^2 \mathrm{d}r [e^{-\beta U(r)} - 1] \\
&= -\frac{4}{3}\pi r_0^3 - 4\pi \int_{r_0}^\infty r^2 \mathrm{d}r \beta U(r) = -\frac{4}{3}\pi r_0^3 + 4\pi \int_{r_0}^\infty r^2 \mathrm{d}r \beta U_0 \left[\frac{r_0}{r}\right]^6 \\
&= -\frac{4}{3}\pi r_0^3 + \frac{4}{3}\pi \beta U_0 r_0^3 = \frac{4}{3}\pi r_0^3 \left[\frac{U_0}{kT} - 1\right]
\end{aligned}$$

At the third line, one recalls the high temperature condition  $\beta \ll 1$  inside the second integral and expands the exponential noting that the potential is not problematic in the region  $r > r_0$ . Therefore, the coefficient is

$$B_2(T) = -\left[\frac{a}{kT} - b\right], \quad a = \frac{2\pi}{3}r_0^3 U_0, \quad b = \frac{2\pi}{3}r_0^3$$

the two parameters contain informations about the potential, in particular about the hard repulsion. Thus, the virial expansion at the first non-trivial order reads

$$\begin{aligned}
\frac{p}{kT} &= \frac{N}{V} - \left[\frac{N}{V}\right]^2 \left[\frac{a}{kT} - b\right] = \frac{N}{V} \left[1 + b\frac{N}{V}\right] - \frac{a}{kT} \left[\frac{N}{V}\right]^2 \\
&= \frac{N}{V} \left[\frac{1}{1 - b\frac{N}{V}}\right] - \frac{a}{kT} \left[\frac{N}{V}\right]^2 + o(N^2/V^2) \approx \frac{N}{V - bN} - \frac{a}{kT} \left[\frac{N}{V}\right]^2
\end{aligned}$$

where at the second line one has used the geometric series

$$1 + x \approx \frac{1}{1 - x}, \quad x \ll 1$$

So, at order  $(N/V)^2$ , the average pressure is

$$p = \frac{kTN}{V - bN} - a \left[\frac{N}{V}\right]^2$$

The parameter  $b$  reduces the volume  $V$  due to the volume of each particle (i.e. the repulsion). The parameter  $a$  reduces the pressure due to the long range interaction (i.e.  $U_0$ ) weighted by  $(N/V)^2$  which is approximately the number of interaction pairs. The equation can also be rewritten as the van der Waals equation

$$kT = \left[p + a \left(\frac{N}{V}\right)^2\right] \left[\frac{V}{N} - b\right]$$

This is the first correction to the ideal gas law which is a qualitative description of the phase transition from gas to liquid [r].

**Remark 2.2.** The cluster expansion is very general. It applies to any interaction with asymptotic behavior

$$U(r) \sim \frac{1}{r^n}, \quad n \geq 4$$

The number 4 is specific to three spatial dimensions and can be read from the convergence of the integrals.

## Lecture 7

### 3 Ideal quantum gas

See Tong, SP, §§3.1, 3.5, 3.6. Consider the ideal gas without long range interactions, but in the case where quantum effects are relevant

$$\frac{V}{N} \approx \lambda^3$$

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To use the canonical ensemble, one needs to compute the partition function  $Z$  by summing over all compatible micro-states of the system.

Assume the system has energy levels  $|k\rangle$  with eigenvalue  $E_k$ . To specify a micro-state, one has to give the set of occupation numbers  $\{n_k\}$  of each energy level, i.e. the number of particles in each level. For fermions, the possible values of the occupation number are just two  $n_k = 0, 1$  due to the Pauli exclusion principle while bosons can appear in any quantity  $n_k \in \mathbb{N}_0$ . The total number of particles and the total energy is then

$$N = \sum_k n_k, \quad E = \sum_k n_k E_k$$

So the partition function is

$$Z(T, V, N) = \sum_{\substack{\{n_k\} \\ \text{constraint}}} \exp \left[ -\beta \sum_k n_k E_k \right]$$

where the sum is over all possible sets  $\{n_k\}$  that obey the constraint that their sum is  $N$ . To remove the constraint, one utilizes the gran-canonical partition function

$$\begin{aligned} \mathcal{Z}(T, V, \mu) &= \sum_{N=0}^{\infty} z^N Z(T, V, N) = \sum_{N=0}^{\infty} z^{\sum_k n_k} \sum_{\substack{\{n_k\} \\ \text{constraint}}} e^{-\beta \sum_k n_k E_k} \\ &= \sum_{N=0}^{\infty} \sum_{\substack{\{n_k\} \\ \text{constraint}}} \exp \left[ -\beta \sum_k n_k (E_k - \mu) \right], \quad z = e^{\beta \mu} \\ &= \sum_{\{n_k\}} \prod_k e^{-\beta n_k (E_k - \mu)} = \prod_k \sum_{n_k} e^{-\beta (E_k - \mu) n_k} = \prod_k \mathcal{Z}_k \end{aligned}$$

where  $\mathcal{Z}_k$  is the single-level gran-canonical partition function. This is similar to what happens in the classical ideal gas where the canonical partition function factorizes into single-particle partition functions

$$Z(T, V, N) = (Z_1)^N$$

The ideal quantum gas partition function does not factorize into single-particle partition functions because the gas inherently has an interaction, but which different from the classical long range interaction. The gran-canonical potential is

$$\Phi = -kT \ln \mathcal{Z}(T, V, \mu) = -kT \sum_k \ln \left[ \sum_{n_k} e^{\beta(\mu - E_k) n_k} \right]$$

For bosons, the inner summation is

$$\sum_{n_k=0}^{\infty} e^{\beta(\mu - E_k) n_k} = \frac{1}{1 - e^{\beta(\mu - E_k)}}$$

Assuming that the ground state has energy  $E_0 = 0$ , the radius of convergence of the geometric series requires

$$|e^{\beta(\mu - E_k)}| < 1 \implies \boxed{\mu < 0}$$

This property of the chemical potential is also true classically. For fermions, the sum has two terms

$$\sum_{n_k=0}^1 e^{\beta(\mu - E_k) n_k} = 1 + e^{\beta(\mu - E_k)}$$

Therefore, the gran-canonical potential is

$$\Phi = -kT \left[ \pm \sum_k \ln(1 \pm e^{\beta(\mu - E_k)}) \right]$$



where the  $+$  sign is for fermions, while the  $-$  sign is for bosons.

The average number of particles is then

$$\langle N \rangle = \frac{1}{\beta} \partial_\mu \ln \mathcal{Z}(T, V, \mu) = -\partial_\mu \Phi = \sum_k \frac{e^{\beta(\mu - E_k)}}{1 \pm e^{\beta(\mu - E_k)}} = \sum_k \frac{1}{e^{-\beta(\mu - E_k)} \pm 1} = \sum_k \langle n_k \rangle$$

From this one reads the fermions' Fermi–Dirac distribution

$$\langle n_k \rangle = \frac{1}{e^{-\beta(\mu - E_k)} + 1}$$

and the bosons' Bose–Einstein distribution

$$\langle n_k \rangle = \frac{1}{e^{-\beta(\mu - E_k)} - 1}$$

At high temperatures  $\beta \ll 1$ , one expects that the two distributions tend to the Boltzmann distribution due to a lower density. However, one cannot simply expand the exponential because the chemical potential has a non-trivial dependence on the temperature  $T$ . Since one is interested in the high temperature limit, one may use the result of the ideal gas and suppose that its leading contribution is the same as in a regime not quite in such limit. Recall that in the ideal gas ( $T \rightarrow \infty$ ) in the canonical ensemble, from the fugacity one has found

$$\frac{\mu}{kT} = \ln \frac{\lambda^3 N}{V} \sim -\frac{3}{2} \ln T$$

or equivalently

$$z = e^{\beta\mu} = \lambda^3 \frac{N}{V} \ll 1, \quad T \rightarrow \infty$$

this limit can be equivalently expressed as  $z \rightarrow 0$ . Therefore, the two distributions behave as

$$[e^{-\beta(\mu - E_k)} \pm 1]^{-1} = \left[ \frac{e^{\beta E_k}}{z} \pm 1 \right]^{-1} \sim z e^{-\beta E_k}, \quad z \rightarrow 0$$

where one recognizes the Boltzmann distribution.

In the high temperature limit, the information about the statistic is lost. One would like to study both high temperature and corrections to infinite temperature. One looks at a virial expansion of the ideal gas. One would like also to study the low temperature regime. Finally, one obtains a master set of equation from which one can study the behaviour of the quantum gas in either low or high temperatures.

### 3.1 Equations of state

In the following, one approximates sums with integrals since they are easier to evaluate. Consider a gas in a very large cubic box of volume  $V = L^3$ . The energy eigenstate of a non-interacting particle is

$$\psi = \frac{1}{\sqrt{V}} e^{i \mathbf{p} \cdot \mathbf{x}}$$

where the momentum is quantized

$$p_i = \frac{2\pi\hbar}{L} n_i, \quad n_i \in \mathbb{Z}$$

The sum over the energy levels can be approximated as

$$\sum_{\mathbf{n}} \approx \int d^3 n = \frac{V}{(2\pi\hbar)^3} \int d^3 p = \frac{V}{(2\pi\hbar)^3} 4\pi \int_0^\infty p^2 dp \equiv \int dE g(E)$$

where the density of states  $g(E)$  is the number of states with energy in the interval  $[E, E + dE]$  which takes the form of

$$g(E) = \frac{V}{2\pi^2\hbar^3} p^2(E) d_E p$$

The momentum  $p(E)$  is given by the dispersion relation. In the non-relativistic case, one has

$$E = \frac{p^2}{2m} \implies p = \sqrt{2mE}, \quad dp = \sqrt{\frac{m}{2E}} dE$$

so the density of states is

$$g(E) = \frac{gV}{2\pi^2\hbar^3} m\sqrt{2mE} \sim \sqrt{E}$$

where  $g$  is the  $g$ -factor related to the spin magnetic moment and intrinsic angular momentum. In the relativistic case, the dispersion relation is Einstein's

$$E^2 = m^2c^4 + p^2c^2 \implies p^2c^2 = E^2 - m^2c^4 \implies 2pc^2 dp = 2E dE$$

from which the density is

$$g(E) = \frac{gV}{2\pi^2\hbar^3c^3} \sqrt{E^2 - m^2c^4} E$$

In the massless and ultra-relativistic cases, the behaviour of the density is  $g(E) \sim E^2$ .

**Pressure.** The average pressure can be computed from the gran-canonical potential

$$\begin{aligned} pV = -\Phi &= \pm kT \sum_k \ln(1 \pm e^{\beta(\mu - E_k)}) \approx \frac{\pm kT gV m \sqrt{2m}}{2\pi^2\hbar^3} \int_0^\infty dE \sqrt{E} \ln(1 \pm ze^{-\beta E}) \\ &= \frac{\pm kT gV m \sqrt{2m}}{2\pi^2\hbar^3} (kT)^{\frac{3}{2}} \int_0^\infty dx \sqrt{x} \ln(1 \pm ze^{-x}), \quad x \equiv \beta E \\ &= \frac{(kT)^{\frac{5}{2}}}{2\pi^2\hbar^3} gV m \sqrt{2m} \frac{2}{3} \int_0^\infty \frac{x^{\frac{3}{2}} ze^{-x}}{1 \pm ze^{-x}} = \frac{4kTV}{3\lambda^3} \frac{g}{\sqrt{\pi}} \int_0^\infty dx \frac{x^{\frac{3}{2}}}{(e^x/z) \pm 1} \\ &= \mp \frac{gkTV}{\lambda^3} \text{Li}_{\frac{5}{2}}(\mp z) \end{aligned}$$

Only at the last line, the  $-$  sign is for fermions while the  $+$  sign is for bosons. At the first line one approximates the sum with an integral

$$\sum_k \approx \int dE g(E)$$

At the third line one has integrated by parts deriving the logarithm and integrating the square root: the boundary term vanishes. At the last line, one uses an integral representation of the polylogarithm function.

**Polylogarithm and Gamma functions.** The polylogarithm is defined by the series

$$\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

convergent in  $|z| < 1$ , which can be analytically continued through the integrals

$$\text{Li}_s(z) = \frac{1}{\Gamma(s)} \int_0^\infty dt \frac{t^{s-1}}{(e^t/z) - 1}, \quad \text{Li}_s(-z) = -\frac{1}{\Gamma(s)} \int_0^\infty dt \frac{t^{s-1}}{(e^t/z) + 1}$$

with domains of convergence, in which it is also holomorphic, given respectively by

$$z \in \mathbb{C} \setminus [1, \infty), \quad z \in \mathbb{C} \setminus (-\infty, -1]$$

with both having  $\text{Re } s > 0$ . For  $s = 1$  it reduces to the logarithm

$$\text{Li}_1(z) = \sum_{k=1}^{\infty} \frac{z^k}{k} = -\ln(1 - z)$$

The Euler Gamma function is defined by

$$\Gamma(z) = \int_0^\infty dt t^{z-1} e^{-t}, \quad \text{Re } z > 0$$

On the natural numbers it reduces to the factorial

$$k \in \mathbb{N} \implies \Gamma(k+1) = k!$$

A few important values are

$$\Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(3/2) = \frac{1}{2}\sqrt{\pi}, \quad \Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$$

**Number of particles.** The average number of particles is

$$\begin{aligned} N = \sum_k n_k &\approx \int_0^\infty dE g(E) n(E) = \frac{gV m \sqrt{2m}}{2\pi^2 \hbar^3} \int_0^\infty dE \frac{\sqrt{E}}{(e^{\beta E}/z) \pm 1} \\ &= \frac{gV m \sqrt{2m}}{2\pi^2 \hbar^3} (kT)^{\frac{3}{2}} \int_0^\infty dx \frac{x^{\frac{1}{2}}}{(e^x/z) \pm 1}, \quad x \equiv \beta E \\ &= \mp \frac{gV}{\lambda^3} \text{Li}_{\frac{3}{2}}(\mp z) \end{aligned}$$

**Energy.** The average energy is

$$E = \sum_k n_k E_k \approx \int_0^\infty dE g(E) n(E) E = \frac{gV m \sqrt{2m}}{2\pi^2 \hbar^3} \int_0^\infty dE \frac{E^{\frac{3}{2}}}{(e^{\beta E}/z) \pm 1} = \mp \frac{3}{2} \frac{gV}{\lambda^3} kT \text{Li}_{\frac{5}{2}}(\mp z)$$

**Summary.** For a non-relativistic quantum gas, the observables are

$$\begin{aligned} \frac{p}{kT} &= \mp \frac{g}{\lambda^3} \text{Li}_{\frac{5}{2}}(\mp z) \\ \frac{E}{kT} &= \mp \frac{3}{2} \frac{gV}{\lambda^3} \text{Li}_{\frac{5}{2}}(\mp z) \\ N &= \mp \frac{gV}{\lambda^3} \text{Li}_{\frac{3}{2}}(\mp z) \end{aligned}$$

where the  $-$  sign is for fermions while the  $+$  sign is for bosons.

From the first two equations one finds

$$\boxed{pV = \frac{2}{3}E}$$

which is also true for the classical ideal gas. In fact, in the classical case, one has

$$E = \frac{3}{2} N kT, \quad pV = N kT \implies pV = \frac{2}{3} E$$

For an ultra-relativistic quantum gas, one finds instead

$$g(E) \sim E^2 \implies pV = \frac{1}{3} E$$

### 3.2 High temperatures

For a classical ideal gas,  $T \sim \infty$ , one has

$$z = \lambda^3 \frac{N}{V} \ll 1$$

So one would like to study the equations of state for small values of the fugacity  $|z| < 1$ . One may utilize the series definition of the polylogarithm. For example, at second order one sees

$$\begin{aligned}\frac{p}{kT} &= \mp \frac{g}{\lambda^3} \left[ \mp z + \frac{z^2}{2^{\frac{5}{2}}} + \dots \right] \\ \frac{N}{V} &= \mp \frac{g}{\lambda^3} \left[ \mp z + \frac{z^2}{2^{\frac{3}{2}}} + \dots \right]\end{aligned}$$

From the second equation, one finds

$$z = \frac{\lambda^3 N}{gV} \pm \frac{1}{2^{\frac{3}{2}}} z^2 \approx \frac{\lambda^3 N}{gV} \pm \frac{1}{2^{\frac{3}{2}}} \left[ \frac{\lambda^3 N}{gV} \right]^2$$

This is the result of the ideal gas plus a correction. This expression can be inserted into the first equation to have

$$\frac{p}{kT} = \frac{g}{\lambda^3} \left[ \frac{\lambda^3 N}{gV} \pm \frac{1}{2^{\frac{3}{2}}} \left( \frac{\lambda^3 N}{gV} \right)^2 \right] \mp \frac{g}{\lambda^3 2^{\frac{5}{2}}} \left( \frac{\lambda^3 N}{gV} \right)^2 = \frac{N}{V} \pm \left[ \frac{N}{V} \right]^2 \frac{\lambda^3}{\sqrt{2}g} \left[ \frac{1}{2} - \frac{1}{4} \right] + o(N^2/V^2)$$

again this is the ideal gas plus a correction. From this one reads the first virial coefficient

$$B_2(T) = \frac{\lambda^3}{\sqrt{2}g} \left[ \frac{1}{2} - \frac{1}{4} \right]$$

The correction is proportional to

$$\frac{N}{V} \lambda^3 \ll 1$$

which is small. The fermions have a  $+$  sign due to the Pauli exclusion principle: by confining the particles, the pressure rises quicker. On the other hand, bosons have a  $-$  sign since they do not exhibit such behaviour. Both types of particles are similar in the classical limit. However, at low temperature the behaviour is very different.

## Lecture 8

### 3.3 Low temperatures

gio 24 ott  
2024 16:30

For bosons, one has  $\mu < 0$  and the low temperature limit corresponds to

$$z = e^{\beta\mu} \rightarrow 1$$

For fermions, one has  $z \in \mathbb{R}$  and the limit corresponds to  $z \rightarrow \infty$ .

#### 3.3.1 Degenerate Fermi gas

See §3.6.2. The average occupation number for the Fermi–Dirac distribution is

$$\langle n_E \rangle = \frac{1}{e^{\beta(E-\mu)} + 1} \rightarrow \begin{cases} 1, & E < \mu \\ 0, & E > \mu \end{cases}$$

In the low temperature limit, all the particles are located below a minimum energy called Fermi energy

$$E_F = \mu(0)$$

This energy can be computed from the average number of particles at zero temperature

$$\begin{aligned}N &= \int_0^\infty dE g(E) n(E) = \int_0^\infty dE \frac{g(E)}{e^{\beta(E-\mu)} + 1} \rightarrow \int_0^{E_F} dE g(E) \\ &= \frac{gV m \sqrt{2m}}{2\pi^2 \hbar^3} \int_0^{E_F} dE \sqrt{E} = \frac{gV m \sqrt{2m}}{3\pi^2 \hbar^3} (E_F)^{\frac{3}{2}}\end{aligned}$$

from which

$$E_F = \frac{\hbar^2}{2m} \left[ \frac{6\pi^2 N}{g V} \right]^{\frac{2}{3}} = \frac{\lambda^2}{4\pi} kT \left[ \frac{6\pi^2 N}{g V} \right]^{\frac{2}{3}}$$

One may define the Fermi temperature

$$T_F = \frac{E_F}{k}$$

which is the temperature where the effects due to the fermions' statistics become relevant. For the electrons in a metal it is  $T_F \sim 10^4$  K, while in a white dwarf it is  $T_F \sim 10^7$  K.

The average energy is

$$\langle E \rangle = \int_0^\infty dE \frac{g(E)E}{e^{\beta(E-\mu)} + 1} \rightarrow K \int_0^{E_F} dE \sqrt{E}E = K \frac{2}{5} E_F^{\frac{5}{2}}$$

where  $K$  is a constant, while the average number of particles is

$$\langle N \rangle = K \int_0^{E_F} dE \sqrt{E} = K \frac{2}{3} E_F^{\frac{3}{2}}$$

from which

$$\langle E \rangle = \frac{3}{5} E_F \langle N \rangle$$

For a non-relativistic quantum gas it holds

$$pV = \frac{2}{3} E = \frac{2}{5} E_F N$$

The pressure is non-zero even at zero temperature due to the fermionic statistic.

### 3.3.2 Low temperature Fermi gas and Sommerfeld expansion

See §§3.6.3, 3.6.4. One may study what happens when the temperature  $T$  is low, but not zero. One expects a correction to the degenerate Fermi gas behaviour. One expects a quantity of active electrons of order  $g(E_F)kT$ . The distribution of occupation numbers differs from the one at  $T = 0$  only for a short range of energies  $kT$  around  $E_F$ . See Tong, SP, fig. 22.

One expects a quadratic correction to the energy of the degenerate case

$$\langle E \rangle = \frac{3}{5} E_F N + K g(E_F)(kT)^2 \sim T^2$$

where  $K$  is a constant. In the zero temperature limit, the fugacity is

$$z = e^{\beta\mu} \sim e^{\beta E_F} \rightarrow \infty$$

One needs to expand the polylogarithm  $\text{Li}_s(z)$  for  $z \rightarrow \infty$ .

**Sommerfeld expansion.** To this end one utilizes the Sommerfeld expansion. Let

$$f_n(z) \equiv -\text{Li}_n(-z)$$

then

$$\begin{aligned}
\Gamma(n)f_n(z) &= \int_0^\infty dx \frac{x^{n-1}}{(e^x/z) + 1} = \int_0^{\ln z} dx \frac{x^{n-1}}{(e^x/z) + 1} + \int_{\ln z}^\infty dx \frac{x^{n-1}}{(e^x/z) + 1} \\
&= \int_0^{\ln z} dx x^{n-1} - \int_0^{\ln z} dx \frac{x^{n-1}}{1 + ze^{-x}} + \int_{\ln z}^\infty dx \frac{x^{n-1}}{(e^x/z) + 1} \\
&= \frac{1}{n} \ln^n z - \int_0^{\ln z} du \frac{(\ln z - u)^{n-1}}{1 + e^u} + \int_0^\infty dv \frac{(\ln z + v)^{n-1}}{1 + e^v} \\
&\approx \frac{1}{n} \ln^n z - \int_0^\infty du \frac{(\ln z - u)^{n-1}}{1 + e^u} + \int_0^\infty dv \frac{(\ln z + v)^{n-1}}{1 + e^v} \\
&= \frac{1}{n} \ln^n z + \int_0^\infty dy \frac{(\ln z + y)^{n-1} - (\ln z - y)^{n-1}}{1 + e^y} \\
&= \frac{1}{n} \ln^n z + \int_0^\infty dy \frac{(\ln z)^{n-1}}{1 + e^y} \left[ \left(1 + \frac{y}{\ln z}\right)^{n-1} - \left(1 - \frac{y}{\ln z}\right)^{n-1} \right] \\
&= \frac{1}{n} \ln^n z + \int_0^\infty dy \frac{(\ln z)^{n-1}}{1 + e^y} \left[ 1 + (n-1) \frac{y}{\ln z} - 1 + (n-1) \frac{y}{\ln z} + o(y/\ln z) \right] \\
&\sim \frac{1}{n} \ln^n z + \int_0^\infty dy 2(n-1) \frac{y}{1 + e^y} (\ln z)^{n-2} \\
&= \frac{1}{n} \ln^n z + 2(n-1)(\ln z)^{n-2} \int_0^\infty dy \frac{y}{1 + e^y} \\
&= \frac{1}{n} \ln^n z + \frac{\pi^2}{6} (n-1)(\ln z)^{n-2}
\end{aligned}$$

At the second line, one has rewritten the first integrand as

$$\frac{x^{n-1}}{(e^x/z) + 1} = x^{n-1} \frac{ze^{-x}}{1 + ze^{-x}} = x^{n-1} \left[ 1 - \frac{1}{1 + ze^{-x}} \right]$$

At the third line, one has substituted in the second integral

$$e^u = ze^{-x}, \quad x = \ln z - u$$

and in the third integral

$$e^v = e^x z^{-1}, \quad x = \ln z + v$$

At the fourth line, the upper integration limit in the second integral is approximated with infinity since  $\ln z \gg 1$ . At the seventh line, one uses the binomial series to expand the argument of the bracket since

$$\frac{y}{\ln z} \ll 1$$

Therefore

$$\boxed{f_n(z) \sim \frac{(\ln z)^n}{n\Gamma(n)} \left[ 1 + \frac{\pi^2}{6} \frac{n(n-1)}{(\ln z)^2} \right]} = \frac{(\ln z)^n}{\Gamma(n+1)} \left[ 1 + \frac{\pi^2}{6} \frac{n(n-1)}{(\ln z)^2} \right], \quad z \rightarrow \infty$$

**Equations of state.** One may expand the equations of state for fermions

$$\frac{p}{kT} = \frac{g}{\lambda^3} f_{\frac{5}{2}}(z), \quad \frac{N}{V} = \frac{g}{\lambda^3} f_{\frac{3}{2}}(z), \quad pV = \frac{2}{3} E$$

The density is

$$\frac{N}{V} = \frac{g}{\lambda^3} f_{\frac{3}{2}}(z) \sim \frac{g}{\lambda^3} \frac{(\ln z)^{\frac{3}{2}}}{\frac{3}{4}\sqrt{\pi}} \left[ 1 + \frac{\pi^2}{8} \frac{1}{(\ln z)^2} \right]$$

from which one may extract

$$(\ln z)^{\frac{3}{2}} \sim \frac{3}{4} \frac{\sqrt{\pi}\lambda^3}{g} \frac{N}{V} \left[ 1 - \frac{\pi^2}{8} \frac{1}{(\ln z)^2} \right] = \left[ \frac{E_F}{kT} \right]^{\frac{3}{2}} \left[ 1 + \frac{\pi^2}{8} \frac{1}{(\ln z)^2} \right]$$

and find

$$\ln z \sim \frac{E_F}{kT} \left[ 1 - \frac{\pi^2}{8} \frac{1}{(\ln z)^2} \right]^{\frac{2}{3}} \sim \frac{E_F}{kT} \left[ 1 - \frac{\pi^2}{12} \frac{1}{(\ln z)^2} \right] \sim \frac{E_F}{kT} \left[ 1 - \frac{\pi^2}{12} \left( \frac{kT}{E_F} \right)^2 \right]$$

From this, the chemical potential is

$$\mu(T) = \frac{1}{\beta} \ln z = kT \ln z \sim E_F \left[ 1 - \frac{\pi^2}{12} \frac{1}{(\ln z)^2} \right] \sim E_F \left[ 1 - \frac{\pi^2}{12} \left( \frac{kT}{E_F} \right)^2 \right]$$

one notices that the deviation from the degenerate Fermi gas is quadratic in the temperature. Recalling

$$pV = \frac{2}{3}E$$

the energy is

$$\begin{aligned} \frac{E}{N} &= \frac{3}{2} kT \frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} \sim \frac{3}{2} kT \frac{(\ln z)^{\frac{5}{2}} \Gamma(5/2)}{(\ln z)^{\frac{3}{2}} \Gamma(7/2)} \frac{1 + \frac{\pi^2}{6} \frac{1}{(\ln z)^2} \frac{15}{4}}{1 + \frac{\pi^2}{6} \frac{1}{(\ln z)^2} \frac{3}{4}} \\ &\sim \frac{3}{2} kT (\ln z) \frac{2}{5} \left[ 1 + \frac{\pi^2}{6} \frac{1}{(\ln z)^2} \left( \frac{15}{4} - \frac{3}{4} \right) \right] = \frac{3}{5} kT \ln z \left[ 1 + \frac{\pi^2}{2} \frac{1}{(\ln z)^2} \right] \\ &\sim \frac{3}{5} E_F \left[ 1 + \frac{\pi^2}{2} \left( \frac{kT}{E_F} \right)^2 \right] \left[ 1 - \frac{\pi^2}{12} \left( \frac{kT}{E_F} \right)^2 \right] \sim \frac{3}{5} E_F \left[ 1 + \frac{5\pi^2}{12} \left( \frac{kT}{E_F} \right)^2 \right] \end{aligned}$$

At the second line, one uses the binomial series to expand

$$(1+x)^{-1} \sim 1-x, \quad x \ll 1$$

At the third line, one has inserted the expression for  $\ln z$ .

From this, the heat capacity is

$$C = d_T E \sim Nk \frac{T}{T_F} \sim T$$

In a material at low temperature, the heat capacity

$$C = \alpha T + \beta T^3$$

gets contributions from the electron gas and the phonons, i.e. the vibrations of the material's lattice.

### 3.3.3 Bose-Einstein condensate

See §3.5.3. Remember that for bosons it holds

$$\sum_{n_k} e^{\beta(\mu - E_k)n_k} = \frac{1}{1 - e^{\beta(\mu - E_k)}}$$

from which

$$\mu < 0 \implies z \in (0, 1)$$

At low temperatures  $T$  one has  $z \rightarrow 1$ . The equations of state are

$$\frac{p}{kT} = \frac{g}{\lambda^3} \text{Li}_{\frac{5}{2}}(z), \quad \frac{N}{V} = \frac{g}{\lambda^3} \text{Li}_{\frac{3}{2}}(z), \quad pV = \frac{2}{3}E$$

Notice that the polylogarithm is a monotonous function of  $z$ . Moreover, at  $z = 1$  it is equal to the Riemann zeta function

$$\text{Li}_n(1) = \zeta(n)$$

For this section, it is useful to know that  $\zeta(3/2) \approx 2.612$ . Noting that

$$\lambda \sim \frac{1}{\sqrt{T}}$$

from the equation of state of the number of particles, one finds

$$\frac{N}{V} \sim T^{\frac{3}{2}} \text{Li}_{\frac{3}{2}}(z)$$

By keeping the density fixed, as the temperature  $T$  lowers, the fugacity  $z$  grows and so does the polylogarithm. When the fugacity reaches  $z = 1$ , the equation no longer makes sense. There is a problem: this is the hint of a new behaviour. Let the critical temperature  $T_c$  be the temperature at  $z = 1$ , then

$$\frac{N}{gV} = \frac{1}{\lambda^3(T_c)} \text{Li}_{\frac{3}{2}}(1) = \left[ \frac{mkT_c}{2\pi} \right]^{\frac{3}{2}} \frac{1}{\hbar^{\frac{3}{2}}} \text{Li}_{\frac{3}{2}}(1)$$

where one finds the temperature

$$T_c = \frac{2\pi\hbar^2}{mkT} \left[ \frac{N}{gV \text{Li}_{\frac{3}{2}}(1)} \right]^{\frac{2}{3}}$$

For temperatures lower than  $T_c$ , from the equation of state of the density  $N/V$ , one finds that the number of particles should decrease, but this cannot happen. The problem lies in the approximation of the sum over the energy states with an integral

$$\sum_k \rightarrow \int dE g(E)$$

The energy density goes like  $g(E) \sim \sqrt{E}$ : the contribution from the ground state is missing<sup>3</sup>. To solve the problem, one adds by hand such contribution

$$N = \frac{gV}{\lambda^3} \text{Li}_{\frac{3}{2}}(z) + \langle n_0 \rangle, \quad \langle n_0 \rangle = \frac{z}{1-z}$$

For  $T \rightarrow 0$ , one has  $z \rightarrow 1$  and  $\langle n_0 \rangle \rightarrow \infty$ . More precisely, the behaviour of the fugacity is

$$z \sim 1 - \frac{1}{N}$$

and so it approaches 1 in the thermodynamic limit  $N \rightarrow \infty$ . From the equation above, the average occupation number of the ground state is

$$\frac{\lambda^3}{gV} \langle n_0 \rangle = \frac{\lambda^3}{gV} N - \text{Li}_{\frac{3}{2}}(z)$$

It is positive when

$$\frac{\lambda^3}{gV} N > \text{Li}_{\frac{3}{2}}(1) = \frac{\lambda^3(T_c)}{gV} N \implies \lambda^3 > \lambda^3(T_c) \implies T < T_c$$

For  $T < T_c$ , the relative occupation number of the ground state is

$$\frac{\langle n_0 \rangle}{N} = 1 - \frac{gV}{\lambda^3 N} \text{Li}_{\frac{3}{2}}(z) \sim 1 - \frac{gV}{\lambda^3 N} \text{Li}_{\frac{3}{2}}(1) = 1 - \frac{\lambda^3(T_c)}{\lambda^3(T)} = 1 - \left[ \frac{T}{T_c} \right]^{\frac{3}{2}}, \quad z \rightarrow 1$$

Below the critical temperature  $T_c$ , a macroscopic number of particles join in the ground state to form the Bose–Einstein condensate.

The graph of the relative occupation number of the ground state as a function of the temperature has a discontinuity in the derivative at  $T = T_c$  which is indicative of a phase transition. The order of a phase transition is related to the order of the discontinuous derivative.

The first observation of a Bose–Einstein condensate was made at a critical temperature of  $T_c \sim 10^{-7}$  K.



## Lecture 9

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2024 14:30

**Dependence of the fugacity on the temperature.** See §3.5.4. From the correct equation of the average number of particles

$$\frac{N}{gV}\lambda^3 = \text{Li}_{\frac{3}{2}}(z) + \frac{\lambda^3}{gV} \frac{z}{1-z}$$

one may numerically find the fugacity  $z(T)$  as a function of the temperature  $T$ . It is a smooth solution. One may also find the fugacity  $z(T)$  in the thermodynamic limit  $N, V \rightarrow \infty$ .

One may start with the first objective. From the equation above one may subtract  $\text{Li}_{\frac{3}{2}}(1)$  on both sides to have

$$\frac{N}{gV}\lambda^3 - \frac{N}{gV}\lambda_c^3 = \frac{\lambda^3}{gV} \frac{z}{1-z} + \text{Li}_{\frac{3}{2}}(z) - \text{Li}_{\frac{3}{2}}(1)$$

recalling that the definition of  $T_c$  is

$$\frac{N}{gV}\lambda_c^3 = \text{Li}_{\frac{3}{2}}(1)$$

Consider the last two addenda in the limit  $z \rightarrow 1$ . From the series definition, one notices that

$$z \, d_z \text{Li}_n(z) = \text{Li}_{n-1}(z)$$

which can be inserted in the Taylor expansion of the polylogarithm

$$\text{Li}_{\frac{3}{2}}(z) = \text{Li}_{\frac{3}{2}}(1) + (z-1)[d_z \text{Li}_{\frac{3}{2}}(z)]_{z=1} = \text{Li}_{\frac{3}{2}}(1) + \left[ \frac{1}{z} \text{Li}_{\frac{1}{2}}(z) \right]_{z=1}$$

However, the polylogarithm diverges at  $z = 1$ , so one needs to study this divergence. From the integral representation one has

$$\begin{aligned} \text{Li}_{\frac{1}{2}}(z) &= \frac{1}{\Gamma(1/2)} \int_0^\infty dx \frac{x^{-\frac{1}{2}}}{(e^x/z) - 1} \sim \frac{1}{\Gamma(1/2)} \int_0^\epsilon dx \frac{x^{-\frac{1}{2}}}{(1+x)z^{-1} - 1}, \quad z \rightarrow 1 \\ &= \frac{z}{\Gamma(1/2)} \int_0^\epsilon dx \frac{x^{-\frac{1}{2}}}{1+x-z} \sim \frac{2z}{\sqrt{1-z}} \frac{1}{\Gamma(1/2)} \int_0^\epsilon \frac{du}{1+u^2}, \quad u \equiv \sqrt{\frac{x}{1-z}} \\ &\sim \frac{z}{\sqrt{1-z}} \end{aligned}$$

Therefore

$$\text{Li}_{\frac{3}{2}}(z) = \text{Li}_{\frac{3}{2}}(1) + c \frac{z-1}{z} \frac{z}{\sqrt{1-z}} = \text{Li}_{\frac{3}{2}}(1) - c\sqrt{1-z}$$

Thus the equation of interest is

$$\frac{N}{gV}(\lambda^3 - \lambda_c^3) = \frac{\lambda^3}{gV} \frac{z}{1-z} + \text{Li}_{\frac{3}{2}}(z) - \text{Li}_{\frac{3}{2}}(1) = \frac{\lambda^3}{gV} \frac{z}{1-z} - c\sqrt{1-z}$$

which can be written in an equivalent form

$$\boxed{A(T^{-\frac{3}{2}} - T_c^{-\frac{3}{2}}) = \frac{B}{V} \frac{z}{1-z} - c\sqrt{1-z}}$$

with  $A, B, C > 0$  constants. Solving numerically this equation, one finds the fugacity as function of the temperature  $z = z(T)$  and it is a smooth function, of class  $C^\infty$ .

The second objective,  $N, V \rightarrow \infty$ , has two cases

- Let  $T > T_c$ . The left-hand side of the above equation is negative, so one can drop the first addendum on the right-hand side for  $V \rightarrow \infty$ . This gives

$$A(T^{-\frac{3}{2}} - T_c^{-\frac{3}{2}}) = -c\sqrt{1-z} \implies 3A \frac{T - T_c}{2T_c^{\frac{5}{2}}} = -c\sqrt{1-z}$$

<sup>3</sup>Notice that the energy of the ground state is conventionally set to  $E_0 = 0$ .

where one has expanded the parenthesis in a Taylor series and considered the linear order. From this one obtains

$$1 - z \sim \alpha(T - T_c)^2 \implies z \sim 1 - \alpha(T - T_c)^2$$

Its first two derivatives are

$$d_T z = -2\alpha(T - T_c), \quad d_T^2 z = -2\alpha \neq 0$$

- Let  $T < T_c$ . The first addendum cannot be ignored since it diverges for  $z \rightarrow 1$  and also the left-hand side is positive. Both derivatives are zero since  $z \sim 1$ .

Therefore, in the thermodynamic limit, the second derivative  $d_T^2 z$  is discontinuous at  $T = T_c$ .

**Pressure and energy.** The pressure behaves as

$$p = \frac{kT}{\lambda^3} g \text{Li}_{\frac{5}{2}}(z) \sim \frac{kT}{\lambda^3} g \text{Li}_{\frac{5}{2}}(1) \sim T^{\frac{5}{2}}$$

The pressure does not depend on the density. The energy is

$$E = \frac{3}{2} p V = \frac{3}{2} \frac{kT}{\lambda^3} g V \text{Li}_{\frac{5}{2}}(z)$$

while the heat capacity is

$$C = d_T E = \frac{3}{2} \frac{kgV}{\lambda^3} \text{Li}_{\frac{5}{2}}(z) + \frac{3}{2} \frac{kTgV}{\lambda^3} d_z \text{Li}_{\frac{5}{2}}(z) d_T z$$

For  $T < T_c$ , the derivative is  $d_T z = 0$  and the heat capacity is

$$C = \frac{15}{4} \frac{kgV}{\lambda^3} \text{Li}_{\frac{5}{2}}(1) \sim c_1 T^{\frac{3}{2}}$$

while for  $T > T_c$ , the derivative is  $d_T z = -\alpha(T - T_c)$  and the heat capacity is

$$C = \frac{15}{4} \frac{kgV}{\lambda^3} \text{Li}_{\frac{5}{2}}(1) - c_2(T - T_c) \sim c_1 T^{\frac{3}{2}} - c_2(T - T_c)$$

The heat capacity as a function of the temperature  $T$  increases to a maximum at  $T_c$  where its derivative is discontinuous and then decreases to

$$C = \frac{3}{2} N k$$

See Tong, SP, fig. 20. The discontinuity in the derivative of the heat capacity is a sign of a phase transition. Notice that the discontinuity appears only in the thermodynamic limit: phase transitions with their associated discontinuities can only arise in strictly infinite systems. There are no phase transitions in finite systems.

**Pressure and ground state correction.** The ground state does not contribute to the pressure. Recall that the pressure is

$$p = -\frac{kT}{V} \sum_k \ln(1 - ze^{-\beta E_k}) = -\frac{kT}{V} \int dE g(E) \ln(1 - ze^{-\beta E_k})$$

One may add the contribution from the ground state

$$-\frac{kT}{V} \ln(1 - z)$$

but in the thermodynamic limit,  $N, V \rightarrow \infty$ , it is suppressed. In fact, knowing

$$z \sim 1 - \frac{1}{N} \iff \frac{1}{1 - z} \sim N \sim V$$

one finds

$$-\frac{kT}{V} \ln(1 - z) \sim kT \frac{\ln V}{V} \rightarrow 0, \quad V \rightarrow \infty$$

### 3.4 Photon gas

Photons are massless spin-one particles. Their  $g$ -factor is  $g = 2$  since there are two physical polarizations. There are no self-interactions, so photons constitute an ideal gas. A few initial observations are

- The equilibrium distribution is due to the interaction of the photons with the boundary of a cavity.
- The total number of photons  $N$  is not conserved.
- Photons are massless, so the gas has an ultra-relativistic dispersion relation

$$\frac{gV}{(2\pi\hbar)^3} d^3p = \frac{gV}{2\pi^2\hbar^3} p^2 dp = \frac{gV}{2\pi^2\hbar^3} \frac{E^2}{c^3} dE = g(E) dE = \frac{V}{\pi^2 c^3} \omega^2 d\omega = g(\omega) d\omega$$

where at the second equality one has inserted  $E = pc$  and at the third  $E = \hbar\omega$ .

Since the total number of photons is not conserved, one may calculate again the canonical partition function

$$Z = \sum_{\{n_k\}} e^{-\beta E_k n_k}$$

from which one may compute the pressure and the energy (or use the results of the boson gas)

$$pV = kT \sum_n \ln(1 - e^{\beta(\mu - E_k)}) , \quad E = \sum_k \frac{E_k}{e^{\beta(\mu - E_k)} - 1}$$

with  $\mu = 0$ .

The gran-canonical partition function is

$$\begin{aligned} \mathcal{Z} &= \sum_{N=0}^{\infty} z^{\sum_k n_k} \sum_{\{n_k\}} e^{-\beta \sum_k n_k E_k} = \sum_{N=0}^{\infty} \sum_{\{n_k\}} \prod_k z^{n_k} e^{-\beta n_k E_k} \\ &= \sum_{n_k} \prod_k z^{n_k} e^{-\beta n_k E_k} = \sum_{n_k} \prod_k e^{-\beta n_k E_k} \end{aligned}$$

where one notices  $z = e^{\beta\mu} = 1$ . Therefore, the equations of state of a photon gas are

$$pV = kT \sum_k \ln(1 - e^{-\beta E_k}) , \quad E = \sum_k \frac{E_k}{e^{\beta E_k} - 1}$$

**Average energy.** Approximating the sum with an integral

$$\sum_k \approx \int d\omega g(\omega)$$

The average energy is

$$\begin{aligned} \langle E \rangle &= \int_0^{\infty} d\omega g(\omega) \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} = \frac{V\hbar}{\pi^2 c^3} \int_0^{\infty} d\omega \frac{\omega^3}{e^{\beta\hbar\omega} - 1} \\ &= \frac{V\hbar}{\pi^2 c^3} \frac{1}{(\beta\hbar)^4} \int_0^{\infty} dx \frac{x^3}{e^x - 1} = (kT)^4 \frac{V\pi^2}{15c^3\hbar^3} , \quad x = \beta\hbar\omega \end{aligned}$$

The integrand in the first line is the Planck distribution, that is the energy in the interval  $[\omega, \omega + d\omega]$

$$P_{\text{planck}}(\omega) = E(\omega) d\omega = \frac{V\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1}$$

Its graph is the typical black-body radiation graph. The peak is given by Wien's displacement law

$$\omega_{\text{peak}} = 2.8 \frac{kT}{\hbar}$$

the numerical constant is the non-trivial solution to

$$3 - x = 3e^{-x} \implies x = W(-3/e^3) + 3 \approx 2.82144$$

where  $W(x)$  is the Lambert  $W$  function. From the above law, one sees that the peak wavelength is longer for a cooler gas.

**Average pressure.** The average pressure

$$\begin{aligned} p &= \frac{kT}{V} \int_0^\infty d\omega g(\omega) \ln(1 - e^{-\beta\hbar\omega}) = \frac{kT}{V} \int_0^\infty d\omega \frac{1}{3} \omega g(\omega) \frac{(-\beta\hbar)e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} \\ &= \frac{1}{3V} \int_0^\infty d\omega g(\omega) \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} = \frac{1}{3V} \langle E \rangle \end{aligned}$$

At the first line, one integrates by parts integrating  $g(\omega)$  and differentiating the logarithm. The equation of state is therefore

$$\boxed{pV = \frac{1}{3} E}$$

**High temperatures.** In the high temperature limit  $\beta \rightarrow 0$ , the Planck distribution reduces to the Rayleigh–Jeans law for the distribution of classical radiation. The average energy diverges in the ultraviolet

$$\langle E \rangle = \int_0^\infty d\omega g(\omega) \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} \sim \int_0^\infty d\omega g(\omega) \frac{\hbar\omega}{\beta\hbar\omega} = \int_0^\infty d\omega g(\omega) kT$$

This is the ultraviolet catastrophe.

### 3.5 Phonon gas

See Tong, SP, §3.3. Consider a solid one dimensional periodic lattice with spacing  $a$ . The Hamiltonian is

$$H = \sum_{i=1}^N \frac{p_i^2}{2m} + \frac{1}{2} m\omega_0^2 (q_{i+1} - q_i)^2$$

The equations of motion are

$$m\ddot{q}_i = m\omega_0^2 (2q_i - q_{i+1} - q_{i-1})$$

which can be solved with the discrete Fourier transform. Consider the ansatz

$$q_l = \frac{1}{\sqrt{N}} \sum_k \hat{q}_k e^{i(kla - \omega_k t)}$$

the equation becomes

$$-\omega_k^2 = -\omega_0^2 (2 - e^{ika} - e^{-ika}) \implies \omega_k^2 = 4\omega_0^2 \sin^2 \frac{ka}{2}$$

from which the dispersion relation is

$$\boxed{\omega_k = 2\omega_0 \left| \sin \frac{ka}{2} \right|}$$

At low frequencies, one has

$$\omega_k \approx \omega_0 ka = c_s k$$

where  $c_s = \omega_0 a$  is the speed of sound in the lattice.

One may draw a parallel between photons and phonons. The former have speed  $c$ ,  $g$ -factor of  $g = 2$  and there is no ultraviolet cutoff for the energy. The latter have speed  $c_s$ ,  $g$ -factor of  $g = 3$  in three dimensions and there is a natural cutoff given by  $a^{-1}$  which corresponds to the Debye wavelength  $\lambda_D \approx a$  (or equivalently the frequency). One may use the results of the photon gas, but operating the above replacements.

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**Maximum frequency.** The way to determine the maximum frequency  $\omega_D$  was found by Debye. The number of single phonon states is

$$\int_0^{\omega_D} d\omega g(\omega) = \frac{gV}{2\pi^2 c_s^3} \frac{\omega_D^3}{3}$$

and it is equal to the number of degrees of freedom  $3N$  of the lattice of  $N$  points (see Tong for intuition). From this equality one finds

$$3N = \frac{gV}{2\pi^2 c_s^3} \frac{\omega_D^3}{3} \implies \omega_D = \left[ \frac{6\pi^2 N}{V} \right]^{\frac{1}{3}} c_s, \quad g = 3$$

From this one defines the Debye temperature

$$T_D = \frac{\hbar \omega_D}{k}$$

For lead it is about 100 K while for diamond it is about 2000 K. The Debye temperature divides the high and low temperature regimes.

**Average energy and heat capacity.** The average energy is

$$\langle E \rangle = \int_0^{\omega_D} g(\omega) \frac{\hbar \omega}{e^{\beta \hbar \omega} - 1} = \frac{3V}{2\pi^2 c_s^3} \frac{(kT)^4}{\hbar^3} \int_0^{T_D/T} dx \frac{x^3}{e^x - 1}, \quad x = \beta \hbar \omega$$

In the low temperature regime,  $T \ll T_D$ , the integral is

$$\int_0^{T_D/T} dx \frac{x^3}{e^x - 1} \approx \int_0^\infty dx \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}$$

from which the average energy and the heat capacity are

$$E \approx \frac{3V}{2\pi^2} \frac{(kT)^4}{(c_s \hbar)^3} \frac{\pi^4}{15} \sim T^4, \quad C = d_T E \sim T^3$$

In the high temperature regime,  $T \gg T_D$ , the integral is

$$\int_0^{T_D/T} dx \frac{x^3}{e^x - 1} \approx \int_0^{T_D/T} dx \frac{x^3}{x} = \frac{1}{3} \left[ \frac{T_D}{T} \right]^3$$

so that the average energy and the heat capacity are

$$E \approx \frac{3V}{2\pi^2} \frac{(kT)^4}{(c_s \hbar)^3} \frac{1}{3} \left[ \frac{T_D}{T} \right]^3 = \frac{V k^3 T_D^3}{2\pi^2 \hbar^3 c_s^3} kT = 3NkT, \quad C = d_T E = 3Nk$$

At low temperatures, the heat capacity of metals has a contribution from the fermionic nature of electrons and from phonons

$$C \sim \alpha T + \beta T^3$$