Quantum Field Theory I

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Contents

Lecture 1

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Exam. The exam can be done whenever, just send an email at least ten days before. Half an hour to calculate something like in class, or generalize something, just technical stuff. It is possible to do QFT 1 and 2 together.

Course structure. In order to go towards increasing complexity, one starts from real scalar boson fields, then fermions, then gauge fields and gauge theories. QFT I is about scalar bosons, while QFT II starts from fermions. Between the two parts there is a bit of conformal field theory.

1 Introduction

The aim is to develop an alternative description of elementary particles and fundamental interactions using functional methods instead of canonical quantization. The path integral formulation of quantum mechanics is an example of functional method. Canonical quantization works by taking classical observable quantities and promoting them to operators obeying certain commutator relations. This was done in the relativistic formulation of quantum field theory in Theoretical Physics I.

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*https://github.com/M-a-s-o/notes

The alternative, albeit equivalent, formulation of quantum mechanics is based on Feynman path integrals: the propagators are written in term of path integrals. Quantum Field Theory I and II reformulates quantum field theory in terms of a generalization of the quantum mechanical path integral to relativistic field theories. The content of Theoretical Physics I is studied using a functional approach.

One needs to formulate quantum field theories for particles of different spins: scalar bosons, spinor fermions and vector bosons. Gauge theories are quantized using functional methods. One does not look for spin $\frac{3}{2}$ and 2 fields, because quantum field theory is not suitable for their description: the theory is inconsistent because it is not renormalizable at every energy scale, in particular in the ultraviolet. For higher spins, there are problems in the propagation of particles in ordinary quantum field theory and one needs a more general approach.

The functional approach lets one study phenomena for which canonical quantization is not suitable.

1.1 Prerequisites

The prerequisites are the following.

Real Gaussian integral. Gaussian integrals are useful for many computations. In one dimension, one has

$$\int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-\frac{a}{2}x^2+bx} = \sqrt{\frac{2\pi}{a}}\mathrm{e}^{\frac{b^2}{2a}}$$

In n dimensions (considering only diagonal matrices) one has

$$\int_{\mathbb{R}^n} \mathrm{d}x_1 \cdots \mathrm{d}x_n\, \mathrm{e}^{-\frac{a_1}{2}x_1^2-\cdots-\frac{a_n}{2}x_n^2+b_1x_1+\cdots+b_nx_n} = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{a_1\cdots a_n}}\mathrm{e}^{\frac{b_1^2}{2a_1}+\cdots+\frac{b_n^2}{2a_n}}$$

Introducing the matrix and vectors

$$A = \mathrm{diag}(a_1, \dots, a_n)\,, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ a_n \end{bmatrix}\,, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

one obtains

$$\int_{\mathbb{R}^n} \mathrm{d}x_1 \cdots \mathrm{d}x_n\, \mathrm{e}^{-\frac{1}{2}x^\top Ax+b^\top x} = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}}\mathrm{e}^{\frac{1}{2}b^\top A^{-1}b}$$

which more generally holds for positive-definite symmetric matrices[1] A. This result is analytically continued to the cases where the exponent in the integrand has arbitrary signs or imaginary numbers while assuming the right-hand side keeps its validity.

Complex Gaussian integral. The complex Gaussian integral is the following

$$\int \prod_{j=1}^n \mathrm{d}z_j\, \mathrm{d}\bar{z}_j\, \mathrm{e}^{-\bar{z}^\top Az+\bar{b}^\top z+b^\top\bar{z}} = \frac{(2\pi)^n}{\det A}\mathrm{e}^{\bar{b}^\top A^{-1}b}$$

where A is a positive-definite hermitian matrix, b is a complex column vector and the bar is complex conjugation. Since $\mathbb{C} \simeq \mathbb{R}^2$, and the dimension is still n, one expects double the integrals.

There is also the generalization of Gaussian integrals for Grassmann odd variables suitable for spinors.

[1] See `https://en.wikipedia.org/wiki/Gaussian_integral`.

Average of a polynomial with Gaussian distribution. Consider the mean value of Gaussian-distributed variables (see Weinberg, eq. 9.A.10)

$$
\begin{aligned}
\langle x_{k_1} \cdots x_{k_l} \rangle &\equiv \int \prod_{j=1}^{n} \mathrm{d}x_j \, x_{k_1} \cdots x_{k_l} \mathrm{e}^{-\frac{1}{2} x^\top A x + b^\top x} \Big|_{b=0} \\
&= \int \prod_{j=1}^{n} \mathrm{d}x_j \, \frac{\partial^l}{\partial b_{k_1} \cdots \partial b_{k_l}} \mathrm{e}^{-\frac{1}{2} x^\top A x + b^\top x} \Big|_{b=0} \\
&= \frac{\partial^l}{\partial b_{k_1} \cdots \partial b_{k_l}} \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}} \mathrm{e}^{\frac{1}{2} b^\top A^{-1} b} \Big|_{b=0} \\
&= \begin{cases} 0 \,, & l \text{ odd} \\ \dfrac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}} \displaystyle\sum_{\substack{\text{pairings} \\ \text{of } \{k_i\}}} \prod_{\{k_i k_j\}} A^{-1}_{k_i k_j} \,, & l \text{ even} \end{cases}
\end{aligned}
$$

The last sum is over the pairings of $(k_1 \cdots k_l)$ — two pairings are the same if they differ only by the order of the pairs, or by the order of indices within a pair — and the product is over all such pairs.

One can compute averages through the derivatives of Gaussian integrals. From the first line (or the third), one may notice that if l is odd then the integral is null.

Functional calculus. Informally, functional calculus is need for infinite (non-countable) dimensional vectors (for example functions). From a discrete set of quantities q_i for which one knows how to take the derivative, one moves to the case of continuous variables where one works with functions $q(x)$.

Inner product. The inner product of two finite dimensional vector fields is

$$
u \cdot v \equiv \sum_{i=1}^{n} u_i v_i
$$

while for infinite dimensional vector fields one has

$$
u \cdot v \equiv \int \mathrm{d}x \, u(x) v(x)
$$

The inner product with respect to a matrix is

$$
u^\top M v = \sum_{ij=1}^{n} u_i M_{ij} v_j \rightsquigarrow u^\top M v = \int \mathrm{d}x \, \mathrm{d}y \, u(x) M(x,y) v(y)
$$

Identity operator. The identity operator is

$$
q = Iq \iff q_i = \sum_j \delta_{ij} q_j \rightsquigarrow q(x) = \int \mathrm{d}y \, \delta(x-y) q(y)
$$

The identity operator is the Dirac delta function.

Functional derivatives. The notation for functional derivatives is $\frac{\delta}{\delta q(x)}$. The concept of functional derivative can be defined by imposing a set of ansätze (pl. of ansatz):

- Linearity
$$
\delta_q [F_1(q) + F_2(q)] = \delta_q F_1 + \delta_q F_2
$$

- Leibniz rule
$$
\delta_q [F_1(q) F_2(q)] = (\delta_q F_1) F_2 + F_1 \delta_q F_2
$$

- It must hold

$$\delta_{q(x)}q(y) = \delta(y-x)$$

With these ansätze one can prove that the functional derivative enjoys the same properties as the regular derivative. For example

$$\delta_{q(x)} \int \mathrm{d}y\, q^P(y) = \int \mathrm{d}y\, Pq^{P-1}(y)\, \delta_{q(x)}q(y) = Pq^{P-1}(x)$$

Likewise

$$\delta_{q(x)} \int \mathrm{d}y\, f(y)\, \partial_y q(y) = \int \mathrm{d}y\, f(y)\, \partial_y \delta(x-y) = -\partial_x f(x)$$

where the result is obtained remembering that the Dirac delta is a distribution.

Generalization of Gaussian integrals to functional integrals. The Gaussian integral becomes

$$\int \prod_{j=1}^{n} \mathrm{d}q_j\, \mathrm{e}^{-\frac{1}{2}q^\top Aq} = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}} \rightsquigarrow \int [\mathcal{D}q]\, \mathrm{e}^{-\frac{1}{2}\int \mathrm{d}x\, \mathrm{d}y\, q(x)A(x,y)q(y)} = \frac{\text{const.}}{\sqrt{\text{“}\det A\text{”}}}$$

where $\mathcal{D}q$ is the functional measure that, along with the determinant, needs to be properly defined.

Useful property. Starting from

$$\mathrm{e}^{\mathrm{i}q\cdot J} \equiv \mathrm{e}^{\mathrm{i}\int \mathrm{d}y\, q(y)J(y)}$$

one may find its derivative

$$\begin{aligned}\delta_{J(x)}\mathrm{e}^{\mathrm{i}q\cdot J}|_{J=0} &= \delta_{J(x)}\mathrm{e}^{\mathrm{i}\int \mathrm{d}y\, q(y)J(y)}|_{J=0} = \delta_{J(x)}\left[\mathrm{i}\int \mathrm{d}y\, q(y)J(y)\right]\mathrm{e}^{\mathrm{i}q\cdot J}|_{J=0} \\ &= \mathrm{i}q(x)\mathrm{e}^{\mathrm{i}q\cdot J}|_{J=0} = \mathrm{i}q(x)\end{aligned}$$

A function can be expressed as the derivative of an exponential with respect to a parameter. This parameter J is called the source of q. Since

$$q(x) = -\mathrm{i}\, \delta_{J(x)}\mathrm{e}^{\mathrm{i}q\cdot J}|_{J=0}$$

then one may write any function of $q(x)$ as

$$G(q(x)) = G(-\mathrm{i}\, \delta_{J(x)})\mathrm{e}^{\mathrm{i}q\cdot J}|_{J=0}$$

To make sense of the right-hand expression, one may expand the function $G(x)$ in a Taylor series.

2 Feynman path integral in quantum mechanics

See Feynman–Hibbs, Srednicki, Anselmi, Cheng–Li. Quantum mechanics is a non-relativistic theory developed through the canonical quantization of observables (also called first quantization) $[\hat{x}, \hat{p}] = i\hbar$, but it can also be equivalently formulated through Feynman's path integrals. In quantum field theory one can use the canonical quantization of fields (also called second quantization): expanding the free field in terms of plane waves, one promotes the Fourier coefficients to operators obeying (anti-)commutation relations

$$[a(\mathbf{k}), a^\dagger(\mathbf{k})] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')$$

Though one may also use functional quantization which is developed in the following.

A massive particle. See Anselmi, Cheng–Li and Srednicki (for philosophy, application etc see Feynman–Hibbs). A simple case is the one of one massive particle in one dimension subject to a potential V. The Hamiltonian of the system is

$$H = \frac{p^2}{2m} + V(q)$$

where q is the position of the particle. The observables are promoted to operators

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}) \,, \quad [\hat{q}, \hat{p}] = \mathrm{i}\hbar$$

The propagation of the particle is described by Schrödinger's equation

$$\hat{H} \left|\psi(t)\right\rangle = \mathrm{i}\hbar\, \partial_t \left|\psi(t)\right\rangle$$

where $\left|\psi(t)\right\rangle$ is the state of the particle. Starting from the Hamiltonian, the corresponding Lagrangian is obtained by the Legendre transform

$$L(q, \dot{q}) = [p\dot{q} - H(p, q)]_{p=p(q,\dot{q})}$$

The expression of the momentum p in terms of q and $\dot{q}$ is obtain by solving Hamilton's equations

$$\dot{q} = \partial_p H \,, \quad \dot{p} = -\partial_q H$$

To describe the propagation of a particle one needs to know the probability of measuring such particle after some time. One would like an alternative mathematical formulation to ordinary quantum mechanics. Consider a particle at some initial point q_i and at some initial time t_i. Letting it propagate up to a final time t_f, one would like to compute the probability that the particle is at a final position q_f. Between the two measurements $t_i < t < t_f$, the particle is not being observed and it propagates. One can at most compute a probability amplitude given by

$${}_{\mathrm{S}}\langle q_f | \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} H(t_f - t_i)} \left|q_i\right\rangle_{\mathrm{S}} \equiv {}_{\mathrm{H}}\langle q_f, t_f | q_i, t_i\rangle_{\mathrm{H}}$$

where $\left|q, t\right\rangle_{\mathrm{H}} = \mathrm{e}^{\frac{\mathrm{i}}{\hbar} Ht} \left|q\right\rangle_{\mathrm{S}}$ is an instantaneous (and time independent) eigenvector of the time dependent position operator $Q(t)$ (see Srednicki, p. 43). Feynman observes that one does not know the precise path of the particle during its propagation, so the particle could take any path between the start and the end points. The only condition is that when $\hbar \to 0$, the classical limit, among the infinite number of possible paths between the start and the end, the only path available is the classical one, which is taken from the variational principle (called action principle in quantum field theory)

$$\delta S = 0 \,, \quad \forall q \mid \delta q(t_i) = \delta q(t_f) = 0$$

Feynman proposes that the probability amplitude should be the sum over all possible paths, each weighed by the probability of the particle to travel along such path: when taking the classical limit, the most probable path is the classical one. They weight of each path is

$$\mathrm{e}^{\frac{\mathrm{i}}{\hbar} S(q_f, t_f, q_i, t_i)}$$

where the action is the classical one. The reason the weight is the above can be seen as follows. Interpreting the sum as an average, in the classical limit the weight oscillates rapidly and its average is zero. The classical path is the only one that can survive because the action is at a minimum, so minimal oscillation. Therefore, the path integral formulation is

$${}_{\mathrm{S}}\langle q_f | \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} H(t_f - t_i)} \left|q_i\right\rangle_{\mathrm{S}} = \int [\mathcal{D}q]\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S(q_f, t_f, q_i, t_i)}$$

The right-hand side is just a formal definition of the idea of path integral.

Lecture 2

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One would like to give an explicit prescription for the sum of all possible paths between two points

$$\int [\mathcal{D}q]$$

Since one does not know how to deal with a continuum of paths, one can discretize the theory and then take the continuum limit. Time and space are discretized in equal-length steps

$$\delta t = \frac{t_f - t_f}{N+1}$$

The position of the particle is no longer $q(t)$ but $q(t_j)$. The smaller is the step $\delta t \ll 1$, the better the discrete path approximates the Feynman path. Consecutive discrete points can be joined by straight segments giving the discrete path as

$$q(t) = \frac{q_j - q_{j-1}}{\delta t}(t - t_{j-1}) + q_{j-1}\,, \quad j = 1, \dots, N+1$$

where $q_0 \equiv q_i$ and $q_{N+1} \equiv q_f$. From the equation above, the time interval is

$$t_f - t_i = (N+1)\delta t$$

so the probability amplitude becomes

$$\begin{aligned} A &= {}_{\mathrm{S}}\langle q_f | \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H(t_f - t_i)} | q_i \rangle_{\mathrm{S}} = {}_{\mathrm{S}}\langle q_f | \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\delta t\,(N+1)} | q_i \rangle_{\mathrm{S}} = {}_{\mathrm{S}}\langle q_f | \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\delta t} \cdots \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\delta t} | q_i \rangle_{\mathrm{S}} \\ &= \int \prod_{j=1}^{N} \mathrm{d}q_j \; \langle q_f | \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\,\delta t} \, | q_N \rangle\langle q_N | \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\,\delta t} \, | q_{N-1} \rangle \cdots \langle q_1 | \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\,\delta t} \, | q_i \rangle \end{aligned}$$

at the second line one has inserted N identities as the completeness relation in the position eigenstates

$$\int \mathrm{d}q \; |q\rangle\langle q| = I$$

Considering a generic factor, one notices

$$\langle q_{j+1} | \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\,\delta t} \, | q_j \rangle = \langle q_{j+1} | \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}[\frac{p^2}{2m} + V(q)]\,\delta t} \, | q_j \rangle$$

Since q and p do not compute, one cannot split the exponential the same way one does with numbers, but one has to apply the Zassenhaus formula (related to the Baker–Campbell–Hausdorff formula):

$$\mathrm{e}^{t(x+y)} = \mathrm{e}^{tx}\mathrm{e}^{ty}\mathrm{e}^{-\frac{1}{2}t^2[x,y] + o(t^2)}$$

The first order in t is just $\mathrm{e}^{t(x+y)} \approx \mathrm{e}^{tx}\mathrm{e}^{ty}$. Therefore, one has

$$\langle q_{j+1} | \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\,\delta t} \, | q_j \rangle = \langle q_{j+1} | \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{p^2}{2m}\,\delta t}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}V(q)\,\delta t} \, | q_j \rangle$$

The position states are eigenstates of the potential, but not of the momentum. One can insert a completeness relation in the momentum eigenstates between the two exponentials

$$\begin{aligned} \langle q_{j+1} | \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\,\delta t} \, | q_j \rangle &= \int \mathrm{d}p_j \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{p_j^2}{2m}\,\delta t}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}V(q_j)\,\delta t} \, \langle q_{j+1} | p_j \rangle \, \langle p_j | q_j \rangle \\ &= \int \frac{\mathrm{d}p_j}{2\pi\hbar} \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{p_j^2}{2m}\,\delta t - \frac{\mathrm{i}}{\hbar}V(q_j)\,\delta t}\mathrm{e}^{\mathrm{i}p_j(q_{j+1} - q_j)} \end{aligned}$$

the operators can then act on their eigenstates producing the associated eigenvalue. At the second line one remembers

$$\langle q | p \rangle = \frac{\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}}{\sqrt{2\pi\hbar}}$$

Since δt is infinitesimal, then one may substitute the potential with its value at the midpoint between two coordinates

$$V(q_j) \to V(\bar{q}_j)\,, \quad \bar{q}_j = \frac{q_{j+1}+q_j}{2}$$

Replacing every factor inside the probability amplitude, one gets

$$A = \int \prod_{j=1}^{N} \mathrm{d}q_j \prod_{k=0}^{N} \frac{\mathrm{d}p_k}{2\pi\hbar}\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p_k (q_{k+1}-q_k) - \frac{\mathrm{i}}{\hbar} H(p_k,\bar{q}_k)\,\delta t}$$

In the limit $N \to \infty$, equivalent to $\delta t \to 0$, one has

$$\begin{aligned} A &= \lim_{N\to\infty} \int \prod_{j=1}^{N} \mathrm{d}q_j \prod_{k=0}^{N} \frac{\mathrm{d}p_k}{2\pi\hbar}\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p_k \frac{q_{k+1}-q_k}{\delta t}\,\delta t - \frac{\mathrm{i}}{\hbar} H(p_k,\bar{q}_k)\,\delta t} \\ &= \lim_{N\to\infty} \int \prod_{j=1}^{N} \mathrm{d}q_j \prod_{k=0}^{N} \frac{\mathrm{d}p_k}{2\pi\hbar}\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p_k \dot{q}_k\,\delta t - \frac{\mathrm{i}}{\hbar} H(p_k,\bar{q}_k)\,\delta t} \\ &\equiv \int [\mathcal{D}q\,\mathcal{D}p] \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_i}^{t_f} \mathrm{d}t\,\big[p(t)\dot{q}(t) - H(p(t),q(t))\big]\right] \end{aligned}$$

The last line is the definition of the path integral.

Since the explicit formula of the Hamiltonian is known

$$H = \frac{p^2}{2m} + V(q)$$

one can integrate over a momenta p_k to obtain

$$\begin{aligned} \int \mathrm{d}p_k\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p_k\dot{q}_k - \frac{p_k^2}{2m})\,\delta t} &= \frac{1}{2\pi\hbar}\sqrt{\frac{2\pi\hbar m}{\mathrm{i}\,\delta t}} \exp\left[-\frac{\dot{q}_k^2\,(\delta t)^2}{\hbar^2}\left(\frac{2\mathrm{i}}{\hbar}\frac{\delta t}{m}\right)^{-1}\right] \\ &= \left[\frac{m}{2\pi\hbar\mathrm{i}\,\delta t}\right]^{\frac{1}{2}} \mathrm{e}^{\frac{\mathrm{i}m}{2\hbar}(\frac{q_{k+1}-q_k}{\delta t})^2\,\delta t} \end{aligned}$$

where one applies the analytic continuation of the Gaussian integral

$$\int \mathrm{d}x\, \mathrm{e}^{-\frac{a}{2}x^2+bx} = \sqrt{\frac{2\pi}{a}}\mathrm{e}^{\frac{b^2}{2a}}\,, \quad a = \frac{\mathrm{i}}{\hbar}\frac{\delta t}{m}\,, \quad b = \frac{\mathrm{i}}{\hbar}\dot{q}_k\,\delta t$$

At the second line, one replaces

$$\dot{q}_k \equiv \frac{q_{k+1}-q_k}{\delta t}$$

Performing the integration for every $k = 0, \dots, N$, the probability amplitude is then

$$A = \lim_{N\to\infty}\left[\frac{m}{2\pi\hbar\mathrm{i}\,\delta t}\right]^{\frac{N+1}{2}} \int \prod_{j=1}^{N} \mathrm{d}q_j\, \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{j=0}^{N}\delta t\left(\frac{m}{2}\left(\frac{q_{j+1}-q_j}{\delta t}\right)^2 - V\right)\right]$$

in the limit, the exponent is the classical Lagrangian (i.e. there are no operators)

$$\frac{\mathrm{i}}{\hbar}\sum_{j=0}^{N}\delta t\left[\frac{m}{2}\left(\frac{q_{j+1}-q_j}{\delta t}\right)^2 - V\right] = \frac{\mathrm{i}}{\hbar}\int_{t_i}^{t_f}\mathrm{d}t\, L(q,\dot{q})$$

The coefficient in square bracket of the amplitude is divergent but it compensates the infinitesimal nature of the integration measure. The path integral is then defined as

$$\int [\mathcal{D}q] \equiv \lim_{N\to\infty}\left[\frac{m}{2\pi\hbar\mathrm{i}\,\delta t}\right]^{\frac{N}{2}} \int \prod_{j=1}^{N-1} \mathrm{d}q_j$$

mind the rescaling $N+1 \to N$. To give a meaningful definition, time has to be discretized: continuous functions become discrete (like functions on a lattice).

Remark. See Feynman–Hibbs. If the path integral formulation is a consistent alternative prescription, then it concerns the computation of probability amplitudes, called kernels or Feynman propagators

$$K(q_f, t_f; q_i, t_i) \equiv \langle q_f | \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} H(t_f - t_i)} \, | q_i \rangle$$

The definition is independent of ordinary quantum mechanics and their operators. Probability amplitudes are weighed with the exponential of the classical action, the only sign of quantum mechanics is the appearance of Planck's constant $\hbar$.

One may notice that

- The weight $\mathrm{e}^{\frac{\mathrm{i}}{\hbar} S}$ is given in terms of the classical action

  $$S = \int_{t_i}^{t_f} \mathrm{d}t \, L(q(t), \dot{q}(t))$$

  The path integral is then referred to as the quantum integral.

- One may find a first equivalence with ordinary quantum mechanics. Consider the quantum mechanical wave function

  $$\begin{aligned} \psi(q, t) &= {}_{\mathrm{H}}\langle q, t | \psi \rangle = {}_{\mathrm{S}}\langle q | \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} H t} \, | \psi \rangle = \int \mathrm{d}q' \; {}_{\mathrm{S}}\langle q | \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} H(t - t')} \, | q' \rangle \, \langle q', t' | \psi \rangle \\ &= \int \mathrm{d}q' \, K(q, t; q', t') \psi(q', t') \end{aligned}$$

  The kernel has the meaning of evolution operator. One may check that the wave function $\psi(q, t)$ satisfies Schrödinger's equation (see Anselmi, Feynman–Hibbs) thanks to the properties of the kernel.

Kernel of a free particle. See Anselmi. The potential of a free particle is identically zero $V(q) = 0$. The kernel is

$$K_0(q_f, t_f; q_i, t_i) = \lim_{N \to \infty} \left[\frac{m}{2\pi\hbar\mathrm{i}\,\delta t} \right]^{\frac{N}{2}} \int \prod_{j=1}^{N-1} \mathrm{d}q_j \, \exp\left[\frac{\mathrm{i}m}{2\hbar\delta t} \sum_{j=1}^{N} (q_j - q_{j-1})^2 \right]$$

Its form is that of many Gaussian integrals, but they are not factorized, so a change of variables is necessary

$$\widetilde{q}_j = q_j - q_f \, , \quad \widetilde{q}_0 = q_i - q_f \, , \quad \widetilde{q}_N = 0 \, , \quad \mathrm{d}q_j = \mathrm{d}\widetilde{q}_j$$

remembering that after rescaling? [r] one has

$$q_0 = q_i \, , \quad q_N = q_f$$

The sum in the exponent is then

$$\begin{aligned} \sum_{j=1}^{N} (q_j - q_{j-1})^2 &= (q_1 - q_i)^2 + (q_2 - q_1)^2 + \cdots + (q_N - q_{N-1})^2 \\ &= [\widetilde{q}_1 - (q_i - q_f)]^2 + (\widetilde{q}_2 - \widetilde{q}_1)^2 + \cdots + (\cancel{\widetilde{q}_N} - \widetilde{q}_{N-1})^2 \\ &= (q_i - q_f)^2 - 2\widetilde{q}_1(q_i - q_f) + 2\widetilde{q}_1^2 + 2\widetilde{q}_2^2 + \cdots + 2\widetilde{q}_{N-1}^2 \\ &\quad -2\widetilde{q}_1\widetilde{q}_2 - 2\widetilde{q}_3\widetilde{q}_2 - \cdots - 2\widetilde{q}_{N-2}\widetilde{q}_{N-1} \\ &= (q_i - q_f)^2 + 2\widetilde{q}_1(q_f - q_i) + \widetilde{q}^\top \widetilde{M} \widetilde{q} \end{aligned}$$

where one has a $N - 1$ dimensional vector and square matrix

$$q = \begin{bmatrix} q \\ \vdots \\ q_{N-1} \end{bmatrix} , \quad \widetilde{M} = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots \\ -1 & 2 & -1 & 0 & \cdots \\ 0 & -1 & 2 & -1 & \cdots \\ 0 & 0 & -1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The kernel is then

$$
\begin{aligned}
K_0(q_f,t_f;q_i,t_i) &= \lim_{N\to\infty}\left[\frac{m}{2\pi\hbar\mathrm{i}\,\delta t}\right]^{\frac{N}{2}} \mathrm{e}^{\frac{\mathrm{i}m}{2\hbar\,\delta t}(q_f-q_i)^2}\int\prod_{j=1}^{N-1}\mathrm{d}\widetilde{q}_j\,\exp\left[\frac{\mathrm{i}m}{2\hbar\,\delta t}\left[\widetilde{q}^\top\widetilde{M}\widetilde{q}+2\widetilde{q}_1(q_f-q_i)\right]\right]\\
&= \lim_{N\to\infty}\left[\frac{m}{2\pi\hbar\mathrm{i}\,\delta t\,N}\right]^{\frac{1}{2}} \mathrm{e}^{\frac{\mathrm{i}m}{2\hbar\,\delta t}(q_f-q_i)^2}\mathrm{e}^{-\frac{\mathrm{i}m}{2\hbar\,\delta t}(q_f-q_i)^2\frac{N-1}{N}}\\
&= \lim_{N\to\infty}\left[\frac{m}{2\pi\hbar\mathrm{i}\,\delta t\,N}\right]^{\frac{1}{2}} \mathrm{e}^{\frac{\mathrm{i}m}{2\hbar\,\delta t\,N}(q_f-q_i)^2} = \left[\frac{m}{2\pi\hbar\mathrm{i}\,\Delta t}\right]^{\frac{1}{2}} \mathrm{e}^{\frac{\mathrm{i}m}{2\hbar\,\Delta t}(q_f-q_i)^2}
\end{aligned}
$$

at the second line one remembers that

$$
\int\prod_{i=1}^{N-1}\mathrm{d}x_i\,\mathrm{e}^{-\frac{1}{2}x^\top Mx+b^\top x} = \frac{(2\pi)^{\frac{N-1}{2}}}{\sqrt{\det M}}\mathrm{e}^{\frac{1}{2}b^\top M^{-1}b}
$$

where

$$
M = -\frac{\mathrm{i}m}{\hbar\,\delta t}\widetilde{M}\,,\quad \det\widetilde{M}=N\,,\quad (\widetilde{M}^{-1})_{11}=\frac{N-1}{N}\,,\quad b_1=\frac{\mathrm{i}m}{\delta t}(q_f-q_i)
$$

At the last line, one notices that

$$
\delta t = \frac{t_f-t_i}{N} = \frac{\Delta t}{N}
$$

With this specific example, one may notice that the divergence of δt is compensated to give a finite result.

Properties of the kernel. A few properties:

- The equal-time limit is

  $$
  \lim_{t_f\to t_i}K_0(q_f,t_f;q_i,t_i)=\delta(q_f-q_i)
  $$

  equivalent to $q_f\to q_i$. This is consistent with an expression of the Dirac delta given by

  $$
  \lim_{\varepsilon\to 0}\frac{1}{\sqrt{2\pi\varepsilon}}\mathrm{e}^{-\frac{q^2}{2\varepsilon}}=\delta(q)
  $$

  [r]

- The kernel satisfies Schrödinger's equation

  $$
  \mathrm{i}\hbar\,\partial_t K_0 = -\frac{\hbar^2}{2m}\,\partial_q^2 K_0
  $$

  This can be checked for

  $$
  K_0(q,t;0,0)=\sqrt{\frac{m}{2\pi\hbar\mathrm{i}t}}\mathrm{e}^{\frac{\mathrm{i}m}{2\hbar t}q^2}
  $$

  It follows that the wave function

  $$
  \psi(q,t)=\int\mathrm{d}q'\,K_0(q,t;q',t')\psi(q',t')
  $$

  satisfies Schrödinger's equation because its dependence on time comes from the kernel.

- Composition law

  $$
  K_0(q,t;q_0,t_0)=\int\mathrm{d}q'\,K_0(q,t;q',t')K_0(q',t';q_0,t_0)
  $$

  This can be seen from

  $$
  K_0(q,t,;q_0,t_0)=\left[\frac{m}{2\pi\hbar\mathrm{i}(t-t_0)}\right]^{\frac{1}{2}}\mathrm{e}^{\frac{\mathrm{i}m}{2\hbar}\frac{(q-q_0)^2}{t-t_0}}
  $$

but working backwards: integrating to obtain such expression. The integral is not trivial since in the exponents one has

$$\begin{aligned}\frac{im}{2}\left[\frac{(q-q')^2}{t-t'}+\frac{(q'-q_0)^2}{t'-t_0}\right] &= \frac{im}{2}\left[\frac{[(t'-t_0)q+(t-t')q_0+(t_0-t)q']^2}{(t-t_0)(t-t')(t'-t_0)}+\frac{(q-q_0)^2}{t-t_0}\right] \\ &= \frac{im}{2}\frac{(q-q_0)^2}{t-t_0}+\frac{im}{2}\left[-\frac{q'}{(t-t')(t'-t_0)}+\delta q\right]\end{aligned}$$

[r] idk if true
where δq is some shift and one can substitute the bracket with another variable.

Exercise. Finish the calculation.

Lecture 3

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2.1 Quadratic potential

See Srednicki, Schulman cap 6. From now on one sets $\hbar = 1$. Consider a one dimensional massive particle in a quadratic potential

$$V(q) = \frac{1}{2}c(t)q^2$$

The Lagrangian is

$$L(q,\dot{q}) = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}c(t)q^2$$

One would like to compute the kernel, the Feynman propagator

$$K(q_b,t_b;q_a,t_a) = \int_{q_a,t_a}^{q_b,t_b}[\mathcal{D}q]\,e^{iS}$$

The following procedure works well for quadratic potentials. One can expand a generic trajectory $q(t)$ around a classical trajectory $\bar{q}(t)$ (that is, solution of the classical equations of motion obtained from the variational principle)

$$q(t) = \bar{q}(t) + \delta q(t)$$

This fluctuation does not apply to the end points $\delta q(t_a) = \delta q(t_b) = 0$. The variation of the action is

$$\delta S = \int \mathrm{d}t\,\frac{\delta S}{\delta q(t)}\,\delta q(t) = 0\,, \quad \forall \delta q(t) \implies \frac{\delta S}{\delta q(t)} = 0$$

For the quadratic potential, the equations of motion are

$$m\ddot{\bar{q}} + c(t)\bar{q} = 0$$

The path integral sums all the fluctuations δq from the classical path. To find the kernel, one needs to compute the action

$$S[\bar{q}+\delta q, \dot{\bar{q}}+\delta\dot{q}]$$

First method. There are two methods to compute the kernel. The first one is

$$S = \int \mathrm{d}t\,\left[\frac{1}{2}m\dot{q}^2 - \frac{1}{2}c(t)q^2\right] = \int \mathrm{d}t\,\left[\frac{1}{2}m(\dot{\bar{q}}+\delta\dot{q})^2 - \frac{1}{2}c(t)(\bar{q}+\delta q)^2\right] = \cdots = S[\bar{q}] + S[\delta q]$$

[r] This result is particular to quadratic potentials.

Second method. The second method utilizes functional derivatives and the equivalent of the Taylor series

$$S[\bar{q}+\delta q] = S[\bar{q}] + \delta_q S|_{\bar{q}}\,\delta q + \frac{1}{2}\,\delta_q^2 S|_{\bar{q}}\,(\delta q)^2$$

The expansion ends at second order because the potential is quadratic. [r] The linear term is zero because $\bar{q}$ is the solution to the classical equations of motion, thus extremizing the action. Only the last term has to be computed

$$\delta_q^2 S\,(\delta q)^2 = \delta_q(\delta_q S\,\delta q)\,\delta q = \int \mathrm{d}t''\,\frac{\delta}{\delta q(t'')}\left[\int \mathrm{d}t'\,\frac{\delta S}{\delta q(t')}\delta q(t')\right]\delta q(t'')$$

The inner integral is

$$\begin{aligned}\int \mathrm{d}t'\,\frac{\delta S}{\delta q(t')}\,\delta q(t') &= \int \mathrm{d}t' \int \mathrm{d}t\,\frac{\delta L(q(t))}{\delta q(t')}\,\delta q(t') \\ &= \int \mathrm{d}t'\,\mathrm{d}t\,[m\dot{q}(t)\,\partial_t\delta(t-t') - c(t)q(t)\delta(t-t')]\,\delta q(t') \\ &= \int \mathrm{d}t\,\left[-m\dot{q}(t)\delta\dot{q}(t) - c(t)q(t)\delta q(t)\right]\end{aligned}$$

where one remembers

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}c(t)q^2\,, \quad \frac{\delta q(t)}{\delta q(t')} = \delta(t-t')$$

in particular

$$\frac{\delta\dot{q}^2(t)}{\delta q(t')} = \frac{\delta\dot{q}^2(t)}{\delta\dot{q}(t)}\,\frac{\delta\dot{q}(t)}{\delta q(t')} = 2\dot{q}(t)\,\partial_t\frac{\delta q(t)}{\delta q(t')} = 2\dot{q}(t)\,\partial_t\delta(t-t')$$

The second derivative is then

$$\begin{aligned}\delta_q^2 S\,(\delta q)^2 &= \int \mathrm{d}t''\,\frac{\delta}{\delta q(t'')}\left[\int \mathrm{d}t\,\left[-m\dot{q}(t)\delta\dot{q}(t) - c(t)q(t)\delta q(t)\right]\right]\delta q(t'') \\ &= \int \mathrm{d}t''\,\mathrm{d}t\,[-m\,\partial_t\delta(t-t'')\,\delta\dot{q}(t)\,\delta q(t'') - c(t)\delta(t-t'')\,\delta q(t)\,\delta q(t'')] \\ &= \int \mathrm{d}t\,\left[m(\delta\dot{q})^2 - c(t)[\delta q(t)]^2\right]\end{aligned}$$

This is exactly the original action but with δq instead of q (and a missing factor of $1/2$). Therefore

$$S[\bar{q}+\delta q] = S[\bar{q}] + \frac{1}{2}\,\delta_q^2 S|_{\bar{q}}\,(\delta q)^2 = S[\bar{q}] + S[\delta q]$$

Remark. This method of the Taylor expansion is more general and can be applied to arbitrary potentials of the form $V(q,g)$ where $g \ll 1$ is a coupling constant. Starting from the cubic term of the Taylor series, one has contributions only from the potential since the kinetic term is quadratic.

Kernel. The kernel becomes

$$K(q_b,t_b;q_a,t_a) = \mathrm{e}^{\mathrm{i}S[\bar{q}]}\int_{0,t_a}^{0,t_b}[\mathcal{D}\delta q]\,\mathrm{e}^{\mathrm{i}S[\delta q]} \equiv \mathrm{e}^{\mathrm{i}S[\bar{q}]}K(0,t_b;0,t_a)$$

To compute the kernel, one has to discretize time

$$\frac{t_b-t_a}{N+1} = \delta t \implies K(0,t_b;0,t_a) = \lim_{N\to\infty}\left[\frac{m}{2\pi\mathrm{i}\,\delta t}\right]^{\frac{1}{2}}\int\prod_{j=1}^{N}\mathrm{d}(\delta q_j)\,\mathrm{e}^{\mathrm{i}S[\delta q]}$$

where the action is discretized also

$$\mathrm{i}S[\delta q] = \sum_{j=0}^{N}\mathrm{i}\left[\frac{m}{2\,\delta t}(\delta q_{j+1}-\delta q_j)^2 - \frac{1}{2}\delta t\,c_j(\delta q_j)^2\right]\,,\quad c_j = c(t_a+\delta t\,j)\,,\quad \delta q_j = \delta q\,(t_a+\delta t\,j)$$

keeping in mind that the extremes have to be constant

$$\delta q_0 = \delta q(t_a) = 0\,, \quad \delta q_{N+1} = \delta q(t_b) = 0$$

[r] Considering a vector

$$\eta = \begin{bmatrix} \delta q_1 \\ \vdots \\ \delta q_N \end{bmatrix}$$

by completing the square? [r], the action can be rewritten as

$$\mathrm{i}S[\delta q] = -\eta^\top A \eta\,, \quad A = \frac{m}{2\pi \mathrm{i}\,\delta t} \begin{bmatrix} 2 - \frac{(\delta t^2)}{m} c_1 & -1 & 0 & \cdots \\ -1 & 2 - \frac{(\delta t^2)}{m} c_2 & -1 & \cdots \\ 0 & -1 & 2 - \frac{(\delta t)^2}{m} c_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The kernel is then

$$\begin{aligned} K(0, t_b; 0, t_a) &= \lim_{N \to \infty} \left[\frac{m}{2\pi \mathrm{i}\,\delta t}\right]^{\frac{N+1}{2}} \int \mathrm{d}^N \eta\, \mathrm{e}^{-\eta^\top A \eta} = \lim_{N \to \infty} \left[\frac{m}{2\pi \mathrm{i}\,\delta t}\right]^{\frac{N+1}{2}} \frac{\pi^{\frac{N}{2}}}{\sqrt{\det A}} \\ &= \lim_{N \to \infty} \left[\frac{m}{2\pi \mathrm{i}\,\delta t} \frac{\pi^N}{(\frac{2\pi \mathrm{i}\,\delta t}{m})^N \det A}\right]^{\frac{1}{2}} = \lim_{N \to \infty} \left[\frac{m}{2\pi \mathrm{i}} \frac{1}{\delta t (\frac{2\pi \mathrm{i}\,\delta t}{m})^N \det A}\right]^{\frac{1}{2}} \end{aligned}$$

At the second equality of the first line one applies the Gaussian integral (notice there is no $1/2$). Let the denominator be

$$F_N(t_b, t_a) \equiv \delta t \left(\frac{2\pi \mathrm{i}\,\delta t}{m}\right)^N \det A = \delta t\, P_N$$

and

$$P_N \equiv \left(\frac{2\pi \mathrm{i}\,\delta t}{m}\right)^N \det A = \det \begin{bmatrix} 2 - \frac{(\delta t)^2}{m} c_1 & -1 & 0 & \cdots \\ -1 & 2 - \frac{(\delta t)^2}{m} c_2 & -1 & \cdots \\ 0 & -1 & 2 - \frac{(\delta t)^2}{m} c_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

One can find a recursion relation for a generic P_j given by

$$P_{j+1} = \left(2 - \frac{(\delta t)^2}{m} c_{j+1}\right) P_j - P_{j-1}\,, \quad j = 1, \dots, N\,, \quad P_0 = 1\,, \quad P_1 = 2 - \frac{(\delta t)^2}{m} c_1$$

This relation can be rewritten as

$$\delta t\, \frac{P_{j+1} - 2P_j + P_{j-1}}{\delta t^2} = -\frac{c_{j+1}}{m} P_j\, \delta t$$

Letting

$$F_j = \delta t\, P_j \equiv \varphi_j = \varphi(t_a + \varepsilon j)$$

the previous equation becomes

$$\frac{\varphi_{j+1} - 2\varphi_j + \varphi_{j-1}}{(\delta t)^2} = -\frac{c_{j+1}}{m} \varphi_j$$

For $\delta t \to 0$ the identity above becomes a second order differential equation

$$\mathrm{d}_t^2 \varphi(t) = -\frac{c(t)}{m} \varphi(t)$$

The initial value problem is

$$\varphi(t = t_a) = \delta t\, P_0 \to 0\,, \quad \mathrm{d}_t \varphi(t = t_a) = \delta t\, \frac{P_1 - P_0}{\delta t} = 2 - \frac{\delta t^2}{m} c_1 - 1 \to 1$$

The quantity needed to compute in the kernel $K(0,t_b;0,t_a)$ is $F(t_b,t_a)=\varphi(t=t_b)$. The full kernel is thus

$$K(q_b,t_b;q_a,t_a)=\mathrm{e}^{\mathrm{i}S[\bar{q}]}K(0,t_b;0,t_a)=\mathrm{e}^{\mathrm{i}S[\bar{q}]}\left[\frac{m}{2\pi\mathrm{i}F(t_b,t_a)}\right]^{\frac{1}{2}}$$

where one has

$$F(t,t_a)=\varphi(t)$$

which satisfies

$$m\,\mathrm{d}_t^2F(t,t_a)+c(t)F(t,t_a)=0\,,\quad F(t_a,t_a)=0\,,\quad \mathrm{d}_tF(t_a,t_a)=1$$

Simple harmonic oscillator. The parameter is

$$c(t)=m\omega^2$$

The differential equation above becomes

$$\ddot{F}+\omega^2F=0\,,\quad F(t_a)=0\,,\quad \dot{F}(t_a)=1$$

whose solution is

$$F(t,t_a)=\frac{1}{\omega}\sin[\omega(t-t_a)]$$

The desired quantity is then

$$F(t_b,t_a)=\frac{1}{\omega}\sin[\omega(t_b-t_a)]=\frac{1}{\omega}\sin\omega T\,,\quad T\equiv t_b-t_a$$

To compute the kernel one has to find the action and path for the classical equations of motion. The Lagrangian is

$$L=\frac{1}{2}m\dot{q}^2-\frac{1}{2}m\omega^2q^2$$

The equations of motion are

$$m\ddot{q}+m\omega^2q=0\,,\quad q(t_a)=q_a\,,\quad q(t_b)=q_b$$

The solution is therefore

$$q(t)=\frac{1}{\sin\omega T}[q_b\sin\omega(t-t_a)+q_a\sin\omega(t_b-t)]$$

Its derivative is

$$\dot{q}(t)=\frac{\omega}{\sin\omega T}[q_b\cos\omega(t-t_a)-q_a\cos\omega(t_b-t)]$$

The action is

$$\begin{aligned}S&=\int\mathrm{d}t\left[\frac{1}{2}m\dot{q}^2-\frac{1}{2}m\omega^2q^2\right]=\frac{1}{2}mq\dot{q}\bigg|_{t_a}^{t_b}-\int\mathrm{d}t\left[\frac{1}{2}mq\ddot{q}+\frac{1}{2}m\omega^2q^2\right]\\&=\frac{1}{2}m\bar{q}(t_b)\dot{\bar{q}}(t_b)-\frac{1}{2}m\bar{q}(t_a)\dot{\bar{q}}(t_a)=\cdots\\&=\frac{m\omega}{2\sin\omega T}[(q_a^2+q_b^2)\cos\omega T-2q_aq_b]\end{aligned}$$

At the first equality one integrates by parts. The last bracket is zero

$$\frac{1}{2}mq(\ddot{q}+\omega^2q)|_{\bar{q}}=0$$

on the classical path since the equations of motion are

$$\ddot{\bar{q}}+\omega^2\bar{q}=0$$

Finally, the kernel is

$$K(q_b,t_b;q_a,t_a)=\left[\frac{m\omega}{2\pi\mathrm{i}\sin\omega T}\right]^{\frac{1}{2}}\exp\left[\frac{im\omega}{2\sin\omega T}\big((q_a^2+q_b^2)\cos\omega T-2q_aq_b\big)\right]$$

For any quadratic potential, the kernel can be computed by solving the differential equation of $F(t,t_a)$. For other potentials, one has to use perturbative approaches.

2.2 Partition function

The kernel is defined as

$$K(q_f, t_f; q_i, 0) = {}_\mathrm{S}\langle q_f|\mathrm{e}^{-\mathrm{i}Ht_f}\,|q_i\rangle_\mathrm{S} = \int [\mathcal{D}q]\,\mathrm{e}^{\mathrm{i}S[q]}$$

The partition function is the integral of a periodic kernel (i.e. its initial and final points coincide)

$$Z(t) = \int \mathrm{d}q\, K(q,t;q,0)$$

Relation to statistical mechanics. This definition coincides with the statistical mechanical definition

$$Z(t) = \mathrm{Tr}\,\mathrm{e}^{-\mathrm{i}\hat{H}t}$$

Setting the system in a box, one has a discrete energy spectrum E_n with corresponding energy states $|n\rangle$. The identity can be decomposed as

$$I = \sum_n |n\rangle\langle n|$$

The kernel is then

$$\begin{aligned} K(q_f, t_f; q_i, 0) &= \langle q_f|\,\mathrm{e}^{-\mathrm{i}Ht_f}\,|q_i\rangle = \sum_n \langle q_f|\,\mathrm{e}^{-\mathrm{i}Ht_f}\,|n\rangle\,\langle n|q_i\rangle = \sum_n \mathrm{e}^{-\mathrm{i}E_n t_f}\,\langle q_f|n\rangle\,\langle n|q_i\rangle \\ &= \sum_n \mathrm{e}^{-\mathrm{i}E_n t_f}\psi_n^\dagger(q_f)\psi_n(q_i) \end{aligned}$$

The partition function is then

$$\begin{aligned} Z(t) &= \int \mathrm{d}q\, K(q,t;q,0) = \int \mathrm{d}q\, \sum_n \mathrm{e}^{-\mathrm{i}E_n t}\psi_n^\dagger(q)\psi_n(q) \\ &= \sum_n \mathrm{e}^{-\mathrm{i}E_n t}\int \mathrm{d}q\,|\psi_n(q)|^2 = \sum_n \mathrm{e}^{-\mathrm{i}E_n t} \end{aligned}$$

remembering that the wave function is normalized such that the above integral (i.e. the probability) is unity. Finally one notices that

$$Z(t) = \mathrm{Tr}\,\mathrm{e}^{-\mathrm{i}\hat{H}t} = \sum_n \mathrm{Tr}\Big[\mathrm{e}^{-\mathrm{i}\hat{H}t}\,|n\rangle\langle n|\Big] = \sum_n \langle n|\,\mathrm{e}^{-\mathrm{i}\hat{H}t}\,|n\rangle = \sum_n \mathrm{e}^{-\mathrm{i}E_n t}$$

at the second equality one inserts the identity inside the trace (remembering that $\mathrm{Tr}(A+B) = \mathrm{Tr}\,A + \mathrm{Tr}\,B$) and the cyclic property of the trace noting that the Hamiltonian is diagonal on the energy eigenstates $|n\rangle$.

Exercise. .[r] Compute the partition function of the harmonic oscillator. Use Taylor expansion. Prove that the harmonic oscillator energy spectrum is

$$E_n = \hbar\omega\left[n + \frac{1}{2}\right]$$

Solution. Starting from the kernel

$$K(q_b, t_b; q_a, t_a) = \left[\frac{m\omega}{2\pi\mathrm{i}\sin\omega T}\right]^{\frac{1}{2}} \exp\left[\frac{\mathrm{i}m\omega}{2\sin\omega T}\big((q_a^2+q_b^2)\cos\omega T - 2q_a q_b\big)\right]$$

The relevant terms are

$$K(q,t;q,0) = \sqrt{\frac{A}{2\pi\mathrm{i}}}\exp\big[\mathrm{i}Aq^2(\cos\omega t - 1)\big]\,, \quad A \equiv \frac{m\omega}{\sin\omega t}$$

Therefore

$$\begin{aligned}
Z(t) &= \int \mathrm{d}q\, K(q,t;q,0) = \sqrt{\frac{A}{2\pi\mathrm{i}}}\int \mathrm{d}q\, \exp\left[\mathrm{i}Aq^2(\cos\omega t - 1)\right], \quad B \equiv A(\cos\omega t - 1) \\
&= \frac{1}{\sqrt{2\pi\mathrm{i}}}\frac{1}{\sqrt{\cos\omega t - 1}}\int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}x^2}, \quad \sqrt{B}q \equiv x, \quad \mathrm{d}q = \frac{\mathrm{d}x}{\sqrt{B}} \\
&= \frac{1}{\sqrt{2\pi\mathrm{i}(\cos\omega t - 1)}}\sqrt{\frac{2\pi}{-2\mathrm{i}}} = \frac{1}{\sqrt{2(\cos\omega t - 1)}} = [\mathrm{e}^{\mathrm{i}\omega t} - 2 + \mathrm{e}^{-\mathrm{i}\omega t}]^{-\frac{1}{2}} \\
&= [\mathrm{e}^{\frac{\mathrm{i}}{2}\omega t} - \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t}]^{-1} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t}\frac{1}{1-\mathrm{e}^{-\mathrm{i}\omega t}} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t}\sum_{n\geq 0}\mathrm{e}^{-\mathrm{i}n\omega t} \\
&= \sum_{n\geq 0}\exp\left[-\mathrm{i}\omega\left(n+\frac{1}{2}\right)t\right] = \sum_{n\geq 0}\mathrm{e}^{-\mathrm{i}E_n t} \implies E_n = \omega\left(n+\frac{1}{2}\right)
\end{aligned}$$

At the second line, one applies the Gaussian integral

$$\int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-\frac{a}{2}x^2 + bx} = \sqrt{\frac{2\pi}{a}}\mathrm{e}^{\frac{b^2}{2a}}, \quad a = -2\mathrm{i}$$

At the third line, one expresses the cosine in terms of complex exponentials. At the fourth line, one uses the analytic continuation of the geometric series.

The same calculation can be done in Euclidean time (and considering hyperbolic functions).

2.3 Correlation function

Two-point correlation function. A two-point correlation function for an operator O_H in the Heisenberg picture

$$O_\mathrm{H}(t) = \mathrm{e}^{\mathrm{i}Ht}O_\mathrm{S}\mathrm{e}^{-\mathrm{i}Ht}$$

is given by

$$C(t_1,t_2,t_f,t_i) = {}_\mathrm{H}\langle q_f,t_f|\mathcal{T}\{O_\mathrm{H}(t_1)O_\mathrm{H}(t_2)\}\,|q_i,t_i\rangle_\mathrm{H}$$

Considering $t_i < t_2 < t_1 < t_f$ one has

$$C(t_1,t_2,t_f,t_i) = \langle q_f,t_f|\, O_\mathrm{H}(t_1)O_\mathrm{H}(t_2)\,|q_i,t_i\rangle = \langle q_f|\,\mathrm{e}^{-\mathrm{i}H(t_f-t_1)}O_\mathrm{S}\mathrm{e}^{-\mathrm{i}H(t_1-t_2)}O_\mathrm{S}\mathrm{e}^{-\mathrm{i}H(t_2-t_i)}\,|q_i\rangle$$

Supposing that O_S is a function of $\hat{q}$ only, then one has

$$\begin{aligned}
&C(t_1,t_2,t_f,t_i) = \\
&\quad = \int \mathrm{d}q_1\,\mathrm{d}q_2\,\langle q_f|\,\mathrm{e}^{-\mathrm{i}H(t_f-t_1)}O_\mathrm{S}(q_1)\,|q_1\rangle\langle q_1|\,\mathrm{e}^{-\mathrm{i}H(t_1-t_2)}O_\mathrm{S}(q_2)\,|q_2\rangle\langle q_2|\,\mathrm{e}^{-\mathrm{i}H(t_2-t_i)}\,|q_i\rangle
\end{aligned}$$

where one has substituted the operators O with their eigenvalues, so the ones above are functions (as hinted by the presence of an explicit function argument). One obtains

$$C(t_1,t_2,t_f,t_i) = \int \mathrm{d}q_1\,\mathrm{d}q_2\, K(q_f,t_f;q_1,t_1)O(q_1)K(q_1,t_1;q_2,t_2)O(q_2)K(q_2,t_2;q_i,t_i)$$

This corresponds to a path integral where one integrates all fluctuations of the classical path γ_1 between q_f and q_1, then measures with the operator, integrates the fluctuations of the classical path γ_2 between q_1 and q_2, measures with other operator and finally integrates the fluctuations to the beginning of the classical path γ_3 between q_2 and q_i.

Lecture 4

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Inserting the definition of the kernels, one obtains

$$C(t_1,t_2) = \int \mathrm{d}q_1\,\mathrm{d}q_2\, O(q_1)O(q_2)\int[\mathcal{D}q^{\gamma_1}]\,\mathrm{e}^{\mathrm{i}S_{\gamma_1}}\int[\mathcal{D}q^{\gamma_2}]\,\mathrm{e}^{\mathrm{i}S_{\gamma_2}}\int[\mathcal{D}q^{\gamma_3}]\,\mathrm{e}^{\mathrm{i}S_{\gamma_3}}$$

Discretizing the path integrals

$$\int [\mathcal{D}q]\, \mathrm{e}^{\mathrm{i}S[q]} \to \frac{1}{A}\int \prod_{j=1}^{N} \frac{\mathrm{d}q_j}{A}\, \mathrm{e}^{\mathrm{i}S}\,, \quad A = \left[\frac{2\pi \mathrm{i}\, \delta t}{m}\right]^{\frac{1}{2}}$$

gives

$$\begin{aligned} C &= \frac{1}{A^3}\int \left[\prod_{j=1}^{N_1} \frac{\mathrm{d}q_j}{A}\right] \mathrm{d}q_1 \left[\prod_{k=1}^{N_2} \frac{\mathrm{d}q_k}{A}\right] \mathrm{d}q_2 \left[\prod_{l=1}^{N_3} \frac{\mathrm{d}q_l}{A}\right] O(q_1)O(q_2)\mathrm{e}^{\mathrm{i}S_\Gamma} \\ &= \frac{1}{A}\int \prod_{j=1}^{N_1} \frac{\mathrm{d}q_j}{A}\frac{\mathrm{d}q_1}{A}\prod_{k=1}^{N_2} \frac{\mathrm{d}q_k}{A}\frac{\mathrm{d}q_2}{A}\prod_{l=1}^{N_3} \frac{\mathrm{d}q_l}{A} O(q_1)O(q_2)\mathrm{e}^{\mathrm{i}S_\Gamma} \end{aligned}$$

where $\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$. Keep in mind that the product (capital pi) refers only to the fraction immediately after.

The first integral is for the first path $\gamma_1 : q_1 < q_j < q_f$, the second is for the path $\gamma_2 : q_2 < q_k < q_1$ and the third is for $\gamma_3 : q_i < q_l < q_2$. In the limit $N_i \to \infty$, the measure is

$$[\mathcal{D}q]_\Gamma = \frac{1}{A}\int \prod_{j=1}^{N_1} \frac{\mathrm{d}q_j}{A}\frac{\mathrm{d}q_1}{A}\prod_{k=1}^{N_2} \frac{\mathrm{d}q_k}{A}\frac{\mathrm{d}q_2}{A}\prod_{l=1}^{N_3} \frac{\mathrm{d}q_l}{A}$$

Therefore, the two-point correlation function is

$$C(t_1, t_2) = \langle q_f, t_f |\, \mathcal{T}\{O_\mathrm{H}(t_1)O_\mathrm{H}(t_2)\}\, | q_i, t_i\rangle = \int [\mathcal{D}q]\, O(q(t_1))O(q(t_2))\mathrm{e}^{\mathrm{i}S}$$

The correlation function is an intermediate step. The actual interesting quantity is the Green's function.

2.4 Green's function

See Cheng–Li. The Green's function is a particular correlation function where the initial and final states are the ground state $|0\rangle$

$$G(t_1, \ldots, t_n) = \langle 0 |\, \mathcal{T}\{O_\mathrm{H}(t_1)\cdots O_\mathrm{H}(t_n)\}\, |0\rangle$$

The definition can be generalized to different operators within the same time-ordered product, but, for simplicity, the following uses the same operator.

Two-point Green's function. One would like to compute the Green's function with the path integral. The two-point Green's function is given by

$$G(t_1, t_2) = \langle 0 |\, \mathcal{T}\{O_\mathrm{H}(t_1)O_\mathrm{H}(t_2)\}\, |0\rangle = \int \mathrm{d}q\, \mathrm{d}q'\ \langle 0 | q't'\rangle \langle q't' |\, \mathcal{T}\{O_\mathrm{H}(t_1)O_\mathrm{H}(t_2)\}\, |qt\rangle \langle qt | 0\rangle$$

The expectation value in the middle can be computed through the correlation function above. The bra-kets are the wave function

$$\langle qt | 0\rangle = \langle q |\, \mathrm{e}^{-\mathrm{i}Ht}\, |0\rangle = \langle q |\, \mathrm{e}^{-\mathrm{i}E_0 t}\, |0\rangle = \mathrm{e}^{-\mathrm{i}E_0 t}\varphi_0(q)$$

In quantum field theory, the energy levels can be arbitrarily shifted so one sets $E_0 = 0$ (as opposed to general relativity). Therefore

$$\langle qt | 0\rangle = \varphi_0(q)\,, \quad \langle 0 | q't'\rangle = \varphi_0^*(q')$$

The Green's function is then

$$\begin{aligned} G(t_1, t_2) &= \int \mathrm{d}q\, \mathrm{d}q'\, \varphi_0^*(q')\varphi'(q) \int [\mathcal{D}q]\, O(q(t_1))O(q(t_2))\mathrm{e}^{\mathrm{i}S} \\ &= \int [\mathcal{D}q]\, \varphi_0^*(q')\varphi_0(q) O(q(t_1))O(q(t_2))\mathrm{e}^{\mathrm{i}S} \end{aligned}$$

at the second line one incorporates the measures $\mathrm{d}q\,\mathrm{d}q'$ into the path integral's.

One would like to elaborate the above prescription in order to eliminate the wave functions $\varphi_0^*(q')\varphi_0(q)$ because they make calculations cumbersome. [r] The correlation function is

$$\begin{aligned}C(t_1,t_2) &= \langle q't'|\,\mathcal{T}\{O_\mathrm{H}(t_1)O_\mathrm{H}(t_2)\}\,|qt\rangle \\ &= \int \mathrm{d}Q\,\mathrm{d}Q'\;\langle q't'|Q'T'\rangle\,\langle Q'T'|\,\mathcal{T}\{O_\mathrm{H}(t_1)O_\mathrm{H}(t_2)\}\,|QT\rangle\,\langle QT|qt\rangle\end{aligned}$$

where one has inserted two completeness relations. [r] Letting E_n be the energy eigenvalue with corresponding eigenstate $|n\rangle$, the wave function of the eigenvector is

$$\langle q|n\rangle = \varphi_n(q)$$

[r] Therefore

$$\begin{aligned}\langle q't'|Q'T'\rangle &= \langle q'|\,\mathrm{e}^{-\mathrm{i}H(t'-T')}\,|Q'\rangle = \sum_n \langle q'|\,\mathrm{e}^{-\mathrm{i}H(t'-T')}\,|n\rangle\,\langle n|Q'\rangle = \sum_n \mathrm{e}^{-\mathrm{i}E_n(t'-T')}\varphi_n(q')\varphi_n^*(Q') \\ &= \varphi_0(q')\varphi_0^*(Q') + \sum_{n>0}\mathrm{e}^{-\mathrm{i}E_n(t'-T')}\varphi_n(q')\varphi_n^*(Q')\end{aligned}$$

where one has separated the contribution of the ground state from the rest. The times t and t' can (almost) be freely moved around the number line since they are not integrated. One observes that in the limit $t'\to-\mathrm{i}\infty$ (similar to a Wick's rotation), the exponential tends to zero (remember that the energy is positive since $E_0=0$). Therefore

$$\lim_{t'\to-\mathrm{i}\infty}\langle q't'|Q'T'\rangle = \varphi_0(q')\varphi_0^*(Q')$$

while for the non-primed bra-ket

$$\lim_{t\to+\mathrm{i}\infty}\langle QT|qt\rangle = \varphi_0(Q)\varphi_0^*(q)$$

Performing a similar calculation one has

$$\lim_{\substack{t'\to-\mathrm{i}\infty\\ t\to+\mathrm{i}\infty}}\langle q't'|qt\rangle = \varphi_0(q')\varphi_0^*(q)$$

Taking the above limit for the correlator, one has

$$\begin{aligned}\lim_{\substack{t'\to-\mathrm{i}\infty\\ t\to+\mathrm{i}\infty}} C(t_1,t_2) &= \lim_{\substack{t'\to-\mathrm{i}\infty\\ t\to+\mathrm{i}\infty}}\langle q't'|\,\mathcal{T}\{O_\mathrm{H}(t_1)O_\mathrm{H}(t_2)\}\,|qt\rangle \\ &= \int \mathrm{d}Q\,\mathrm{d}Q'\;\varphi_0(q')\varphi_0^*(Q')\,\langle Q'T'|\,\mathcal{T}\{O_\mathrm{H}(t_1)O_\mathrm{H}(t_2)\}\,|QT\rangle\,\varphi_0(Q)\varphi_0^*(q) \\ &= \varphi_0(q')\varphi_0^*(q)\int \mathrm{d}Q\,\mathrm{d}Q'\;\varphi_0^*(Q')\varphi_0(Q)\,\langle Q'T'|\,\mathcal{T}\{O_\mathrm{H}(t_1)O_\mathrm{H}(t_2)\}\,|QT\rangle \\ &= \lim_{\substack{t'\to-\mathrm{i}\infty\\ t\to+\mathrm{i}\infty}}\langle q't'|qt\rangle\,G(t_1,t_2)\end{aligned}$$

in the last line, the integral is the definition of Green's function but with Q,Q'. [r] Therefore, reorganizing the above equation, the Green's function is

$$G(t_1,t_2) = \lim_{\substack{t'\to-\mathrm{i}\infty\\ t\to+\mathrm{i}\infty}}\frac{\langle q't'|\,\mathcal{T}\{O_\mathrm{H}(t_1)O_\mathrm{H}(t_2)\}\,|qt\rangle}{\langle q't'|qt\rangle}$$

This operation is similar to a normalization: the Green's function is a correlation function normalized by the product of the two states.

This definition of two-point Green's function can be generalized to n points

$$G(t_1,\dots,t_n) = \lim_{\substack{t'\to-\mathrm{i}\infty\\ t\to+\mathrm{i}\infty}}\frac{\langle q't'|\,\mathcal{T}\{O_\mathrm{H}(t_1)\cdots O_\mathrm{H}(t_n)\}\,|qt\rangle}{\langle q't'|qt\rangle}$$

In this way, the Green's function does not explicitly depend on the ground state.

Generating functional. The generating functional for the Green's function is

$$W[J] = \lim_{\substack{t'\to -\mathrm{i}\infty \\ t\to +\mathrm{i}\infty}} \frac{1}{\langle q't'|qt\rangle} \int [\mathcal{D}q] \exp\left[\mathrm{i}\int_t^{t'} \mathrm{d}\tau \, [L(\tau) + J(\tau) O_{\mathrm{H}}(q(\tau))]\right]$$

Applying an arbitrary number of derivatives with respect to J brings the operator O_{H} in front of the exponential. In this way one obtains the desired Green's function

$$G(t_1, \ldots, t_n) = \left.\frac{(-\mathrm{i})^n \delta^n W[J]}{\delta J(t_1)\cdots\delta J(t_n)}\right|_{J=0}$$

The function J is called the source of the operator O_{H}.

The generating functional is a modification of a path integral that one would get without any insertion of operators and with a source different from zero. The functional is the transition amplitude from the ground state at time t to the ground state at time t' in the presence of an external source $J(\tau)$

$$W[J] = \langle 0|0\rangle_J$$

The normalization is given by the absence of the source

$$W[0] = \langle 0|0\rangle = 1$$

This is in accordance with the prescription of the Green's function above from the state $|qt\rangle$ to the state $|q't'\rangle$ but without any insertion of operators: the ratio is unity.

Euclidean Green's function. Working in Euclidean space is useful to make sense of the limits of the times t, t'. By performing a Wick's rotation, the time coordinate becomes $\tau = -\mathrm{i}\tau_{\mathrm{E}}$ while the limits are

$$\begin{aligned} t' \to -\mathrm{i}\infty &\rightsquigarrow t'_{\mathrm{E}} \to +\infty \\ t \to +\mathrm{i}\infty &\rightsquigarrow t_{\mathrm{E}} \to -\infty \end{aligned}$$

The exponential of the Lagrangian becomes

$$\exp\left[\mathrm{i}\int \mathrm{d}\tau \, L(\tau)\right] \rightsquigarrow \exp\left[\int \mathrm{d}\tau_{\mathrm{E}} \, L(-\mathrm{i}\tau_{\mathrm{E}})\right]$$

Considering the Lagrangian of a one-dimensional particle in a potential

$$L = \frac{1}{2}m\dot{q}^2 - V(q) = \frac{1}{2}m(\mathrm{d}_\tau q)^2 - V(q) \rightsquigarrow L(-\mathrm{i}\tau_{\mathrm{E}}) = -\frac{1}{2}m\dot{q}_{\mathrm{E}}^2 - V(q) \equiv -L_{\mathrm{E}}$$

the exponential is then

$$\exp\left[\mathrm{i}\int \mathrm{d}\tau \, L(\tau)\right] \rightsquigarrow \exp\left[-\int \mathrm{d}\tau_{\mathrm{E}} \, L_{\mathrm{E}}\right]$$

The ground energy can always be set to zero, so the energies are positive and the exponential is no longer a phase but tends to zero. The integral becomes well-defined.

Euclidean generating functional. The generating functional becomes

$$W_{\mathrm{E}}[J] = \lim_{\substack{t'\to \infty \\ t\to -\infty}} \frac{1}{\langle q't'|qt\rangle} \int [\mathcal{D}q] \exp\left[-\int \mathrm{d}\tau_{\mathrm{E}}[L_{\mathrm{E}} + JO_{\mathrm{H}}(\tau_{\mathrm{E}})]\right]$$

2.5 Arbitrary dimensions

The formalism developed can be generalized to D dimensions. If a particle propagates in D dimensions, the configuration point is

$$\mathbf{q} = (q^1, \ldots, q^D)$$

The kernel is

$$K(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i) = \int [\mathcal{D}\mathbf{q}]\, \mathrm{e}^{\mathrm{i}S[\mathbf{q}]} = \lim_{N\to\infty} \frac{1}{A} \int \prod_{j=1}^{N} \frac{\mathrm{d}\mathbf{q}_j}{A^D}\, \mathrm{e}^{\mathrm{i}S[\mathbf{q}_j]}$$
$$= \lim_{N\to\infty} \left[\frac{m}{2\pi\mathrm{i}\,\delta t}\right]^{\frac{ND+1}{2}} \int \prod_{j=1}^{N} \mathrm{d}\mathbf{q}_j\, \mathrm{e}^{\mathrm{i}S[\mathbf{q}_j]}$$

Conclusion. Some phenomena are easier described through the path integral formulation of quantum mechanics, like tunnelling and scattering.

3 Scalar boson fields

In quantum field theory, canonical quantization is limited and cumbersome. One uses the functional approach which generalizes the quantum mechanical path integral.

Spin zero particles are bosons and are described by scalar fields $\varphi(x)$. Neutral particles are real scalar fields.

3.1 Canonical quantization

One may start from neutral particles. For free massive scalar fields, the Lagrangian density and action are given by

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\, \partial_\mu\varphi\, \partial_\nu\varphi - \frac{1}{2}m^2\varphi^2\,, \quad S = \int \mathrm{d}^4x\, \mathcal{L} = \int \mathrm{d}^4x\, \left[\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\nabla\varphi)^2 - \frac{1}{2}m^2\varphi^2\right]$$

where the Minkowski metric η convention is timelike, mostly minus. Applying the action principle $\delta S = 0$ one obtains the classical equations of motion with some boundary conditions: when integrating over all space-time, one imposes that the fields go to zero sufficiently fast. For a scalar field, the equations of motion are Klein–Gordon's

$$(\Box + m^2)\varphi(x) = 0\,, \quad \Box \equiv \partial_\mu\partial^\mu = \partial_0^2 - \nabla^2$$

The canonical momentum of the field is

$$\pi(x) = \frac{\delta\mathcal{L}}{\delta\dot{\varphi}(x)} = \dot{\varphi}(x)$$

The most general solution for the Klein–Gordon equation is

$$\varphi(x) = \varphi_+(x) + \varphi_-(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2k^0}} [a_+(\mathbf{k})\mathrm{e}^{\mathrm{i}k^\mu x_\mu} + a_-(\mathbf{k})\mathrm{e}^{-\mathrm{i}k^\mu x_\mu}]\,, \quad k^0 = \sqrt{|\mathbf{k}|^2 + m^2}$$

where $\pm$ is not related to positive and negative frequency components, it just labels the sign of the exponent. The reality condition given by the neutral nature of the boson field implies

$$[a_-(\mathbf{k})]^\dagger = a_+(\mathbf{k})$$

The Fourier coefficients can be expressed in terms of the field as

$$a_\pm(\mathbf{k}) = \mp \frac{\mathrm{i}}{(2\pi)^{\frac{3}{2}}\sqrt{2k^0}} \int \mathrm{d}^3x\, \mathrm{e}^{\mp\mathrm{i}kx}\, \overleftrightarrow{\partial_0}\varphi(x)\,, \quad f\,\overleftrightarrow{\partial_0}g \equiv f\,\partial_0 g - (\partial_0 f)g$$

One can prove that, since the field $\varphi(x)$ is a solution of the free Klein–Gordon equation, the annihilation and creation operators are time-independent. For the interacting theory, the Klein–Gordon equation of motion is not zero. In fact, the Lagrangian has a potential

$$\mathcal{L} = \frac{1}{2}\, \partial_\mu\varphi\, \partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - V(\varphi)$$

from which the equations of motion are

$$(\Box + m^2)\varphi(x) = -\partial_\varphi V \equiv J(x)$$

and in general the annihilation and destruction operators are functions of time.

Exercise. Check that the previous expressions give back the field φ. Check that the operators of the free theory are time-independent

$$\partial^0 a(\mathbf{k}) = \frac{\mathrm{i}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}} \int \mathrm{d}^3x\, \partial^0[(\partial_0 \mathrm{e}^{\mathrm{i}kx})\varphi(x) - \mathrm{e}^{\mathrm{i}kx}\, \partial_0\varphi] = \cdots = 0$$

To get zero one has to apply the Klein–Gordon equation.

Canonical quantization. The canonical quantization of the fields (called second quantization) is implemented by promoting the Fourier coefficients $a_\pm(\mathbf{k})$ to be operators obeying the canonical commutation relations (CCR)

$$[a_-(\mathbf{k}), a_+(\mathbf{k}')] = \delta^{(3)}(\mathbf{k}-\mathbf{k}') \implies [a(\mathbf{k}), a^\dagger(\mathbf{k})] = \delta^{(3)}(\mathbf{k}-\mathbf{k}')$$

and all others zero. The number operator and the total number are

$$N(\mathbf{k}) \equiv a^\dagger(\mathbf{k})a(\mathbf{k})\,, \quad N = \int \mathrm{d}^3k\, N(\mathbf{k})$$

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One defines a vacuum state as the state annihilated by all destruction operators

$$a(\mathbf{k})\,|0\rangle = 0\,, \quad \forall\mathbf{k}$$

The excited states are constructed by acting on the vacuum with a creation operator

$$a^\dagger(\mathbf{k})\,|0\rangle = |\psi(\mathbf{k})\rangle$$

The excited states are interpreted as particle with momentum $\mathbf{k}$ and on-shell dispersion relation $E = \sqrt{|\mathbf{k}|^2 + m^2}$. Though it has no physical sense that a particle has a definite momentum $\mathbf{k}$, but it is more physical to consider a wave packet with a distribution of momenta centered around a momentum $\mathbf{k}$

$$|\psi(\mathbf{k})\rangle = \int \mathrm{d}^3k'\, f(\mathbf{k},\mathbf{k}')a^\dagger(\mathbf{k})\,|0\rangle$$

where f is a function that gives the distribution of the plane waves inside the wave packet, typically a Gaussian (see Srednicki, §5)

$$f(\mathbf{k},\mathbf{k}') \propto \exp\left[-\frac{1}{4\sigma^2}(\mathbf{k}'-\mathbf{k})^2\right], \quad f(\mathbf{k},\mathbf{k}') \to \delta^{(3)}(\mathbf{k}'-\mathbf{k})\,, \quad \sigma \to 0$$

Commutators. The commutators between the fields can be obtained by applying the canonical commutation relations of the annihilation and creation operators. The relations for the separate parts of the field are

$$[\varphi_\mp(x), \varphi_\pm(y)] = \pm \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0} \mathrm{e}^{\mp\mathrm{i}k(x-y)}$$

From these, the relations for the field are

$$[\varphi(x), \varphi(y)] = [\varphi_+(x), \varphi_-(y)] + [\varphi_-(x), \varphi_+(y)] = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}[\mathrm{e}^{\mathrm{i}k(x-y)} - \mathrm{e}^{-\mathrm{i}k(x-y)}]$$

The canonical commutations relations are then

$$[\varphi(x), \dot\varphi(y)]_{x^0=y^0} = [\varphi(x^0,\mathbf{x}), \pi(x^0,\mathbf{y})] = \mathrm{i}\delta^{(3)}(x-y)$$

because $\pi = \dot\varphi$ is the canonical momentum of the field.

The commutators for space-like separations are null

$$[\varphi(x), \varphi(y)] = 0\,, \quad |x-y|^2 < 0$$

The above commutators are the Pauli–Jordan functions, which are related to the Klein–Gordon propagator. The meaning of the above commutator is that two particles separated by arbitrary space-like distances cannot influence each other: it is the condition of micro-causality.

The (Feynman) propagator for a free scalar field is the following two-point function

$$\begin{aligned}
\mathrm{i}\Delta_0(x-y) &\equiv \langle 0|\,\mathcal{T}\{\varphi(x)\varphi(y)\}\,|0\rangle = \theta(x^0-y^0)\,\langle 0|\,\varphi(x)\varphi(y)\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,\varphi(y)\varphi(x)\,|0\rangle \\
&= \theta(x^0-y^0)\,\langle 0|\,\varphi_-(x)\varphi_+(y)\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,\varphi_-(y)\varphi_+(x)\,|0\rangle \\
&= \theta(x^0-y^0)\,\langle 0|\,[\varphi_-(x),\varphi_+(y)]\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,[\varphi_-(y),\varphi_+(x)]\,|0\rangle \\
&= \theta(x^0-y^0)\,\langle 0|\,[\varphi_-(x),\varphi_+(y)]\,|0\rangle - \theta(y^0-x^0)\,\langle 0|\,[\varphi_+(x),\varphi_-(y)]\,|0\rangle \\
&= \theta(x^0-y^0)\int\frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}\mathrm{e}^{-\mathrm{i}k(x-y)} + \theta(y^0-x^0)\int\frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}\mathrm{e}^{\mathrm{i}k(x-y)} \\
&= \mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}k(x-y)}}{k^2-m^2+\mathrm{i}\varepsilon}
\end{aligned}$$

At the second line only the non-zero contributions have been explicitly written; at the last line one has made the analytic continuation of k^0 in the complex plane and has integrated using Feynman's prescription and Jordan's lemmas.

Once quantum field theory is reformulated through functional methods, one finds the same expression of the propagator above using functional integrals.

The propagator is the probability amplitude of creating a particle at y, letting it propagate to x and destroying it there. The propagator above is a two-point function and it is the Green's function of the Klein–Gordon operator

$$(\Box + m^2)\Delta_0(x) = -\delta^{(4)}(x-y)$$

[r]

The propagator (which is related to the Green's function) is needed to justify the fact that the evaluation of a scattering amplitude is related to the computation of the Green's function. The scattering amplitude can be expressed in terms of the Green's function thanks to the Lehmann–Symanzik–Zimmermann (LSZ) reduction formula.

3.2 Lehmann–Symanzik–Zimmermann reduction formula

See Srednicki §5, Weinberg, Itzykson–Zuber. One would like to compute the scattering amplitude between an initial and a final state $\langle f|i\rangle$ which is related to the probability that a system prepared in the state $|i\rangle$ at $t\to-\infty$ transitions to the state $|f\rangle$ at $t\to+\infty$.

Adiabatic hypothesis. The interaction occurs only in a finite volume of space. At infinity the interaction is negligible and there are only the free asymptotic states $|i\rangle$ and $|f\rangle$ given by

$$|i\rangle = \lim_{t\to-\infty} a_1^\dagger(t)a_2^\dagger(t)\cdots|0\rangle\ ,\quad |f\rangle = \lim_{t\to\infty} a_{1'}^\dagger(t)a_{2'}^\dagger(t)\cdots|0\rangle$$

where

$$a_1^\dagger(t)\,|0\rangle = \int \mathrm{d}^3k\, f_1(\mathbf{k})a^\dagger(\mathbf{k},t)\,|0\rangle$$

remembering that the operators depend on time because the theory has interaction and is not free. Therefore it is useful to compute $a_j^\dagger(+\infty) - a_j^\dagger(-\infty)$. Starting from the first one, it follows

$$\begin{aligned}
a_1^\dagger(+\infty) - a_1^\dagger(-\infty) &= \int_{\mathbb{R}} \mathrm{d}t\, \partial_0 a_1(t) = \int_{\mathbb{R}} \mathrm{d}t\, \partial_0 \int \mathrm{d}^3k\, f_1(\mathbf{k}) a^\dagger(\mathbf{k}, t) \\
&= -\mathrm{i} \int \mathrm{d}^3k\, f_1(\mathbf{k}) \int_{\mathbb{R}} \mathrm{d}t\, \partial_0 \int \frac{\mathrm{d}^3x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}} \mathrm{e}^{-\mathrm{i}kx}\, \overleftrightarrow{\partial_0} \varphi(x) \\
&= -\mathrm{i} \int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}}\, \partial_0 [\mathrm{e}^{-\mathrm{i}kx}\, \overleftrightarrow{\partial_0} \varphi(x)] \\
&= -\mathrm{i} \int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}}\, [\mathrm{e}^{-\mathrm{i}kx}\, \partial_0^2 \varphi + k_0^2 \mathrm{e}^{-\mathrm{i}kx} \varphi] \\
&= -\mathrm{i} \int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}}\, [\mathrm{e}^{-\mathrm{i}kx}\, \partial_0^2 \varphi + (|\mathbf{k}|^2 + m^2) \mathrm{e}^{-\mathrm{i}kx} \varphi] \\
&= -\mathrm{i} \int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}} \mathrm{e}^{-\mathrm{i}kx} (\partial_0^2 - \nabla^2 + m^2) \varphi(x) \\
&= -\mathrm{i} \int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}} \mathrm{e}^{-\mathrm{i}kx} (\Box + m^2) \varphi(x)
\end{aligned}$$

at the second line one expresses the creation operator with its expansions in terms of the fields; at the second to last line, one notices

$$|\mathbf{k}|^2 \mathrm{e}^{-\mathrm{i}kx} = -\nabla^2 \mathrm{e}^{-\mathrm{i}kx}$$

and integrates by parts supposing that at infinity the fields to go zero fast enough to produce no boundary terms. At the last line, in the free theory, the field is a solution of the Klein–Gordon equation and the desired difference is zero which is consistent with the fact that the two operators at infinity are the same; however, in the interacting theory this is not true because the Klein–Gordon operator does not necessarily give zero.

Similarly, for the destruction operators

$$a_1(+\infty) - a_1(-\infty) = \mathrm{i} \int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}} \mathrm{e}^{\mathrm{i}kx} (\Box + m^2) \varphi(x)$$

Scattering amplitude. The following considers a probability amplitude for a $2 \to 2$ scattering process. The initial and final states are

$$|i\rangle = a_1^\dagger(-\infty) a_2^\dagger(-\infty) |0\rangle\ , \quad |f\rangle = a_{1'}^\dagger(+\infty) a_{2'}^\dagger(+\infty) |0\rangle$$

The scattering amplitude is then

$$\begin{aligned}
\langle f|i\rangle &= \langle 0| a_{1'}(+\infty) a_{2'}(+\infty) a_1^\dagger(-\infty) a_2^\dagger(-\infty) |0\rangle \\
&= \langle 0| \mathcal{T}\{a_{1'}(+\infty) a_{2'}(+\infty) a_1^\dagger(-\infty) a_2^\dagger(-\infty)\} |0\rangle
\end{aligned}$$

At the second line, one notices that the product is already time-ordered. One may insert the expression of the operators in terms of the fields

$$\begin{aligned}
a_j^\dagger(-\infty) &= a_j^\dagger(+\infty) + \mathrm{i} \int \mathrm{d}^3k\, f_j(\mathbf{k}) \int \frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}} \mathrm{e}^{-\mathrm{i}kx} (\Box + m^2) \varphi(x) \\
a_{j'}(+\infty) &= a_{j'}(-\infty) + \mathrm{i} \int \mathrm{d}^3k\, f_{j'}(\mathbf{k}) \int \frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}} \mathrm{e}^{\mathrm{i}kx} (\Box + m^2) \varphi(x)
\end{aligned}$$

Looking at the second expression, the time-ordered product above pushes the primed destruction operators to the right and they annihilate the vacuum, so only the integral remains. Therefore

$$\begin{aligned}
\langle f|i\rangle &= \int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \mathrm{d}^3k'\, f_2(\mathbf{k}') \int \mathrm{d}^3k''\, f_{1'}(\mathbf{k}'') \int \mathrm{d}^3k'''\, f_{2'}(\mathbf{k}''') \\
&\quad \times \int \frac{\mathrm{d}^4x_1\, \mathrm{e}^{-\mathrm{i}kx_1}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_1^0}} \int \frac{\mathrm{d}^4x_2\, \mathrm{e}^{-\mathrm{i}k'x_2}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_2^0}} \int \frac{\mathrm{d}^4x_3\, \mathrm{e}^{\mathrm{i}k''x_3}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_{1'}^0}} \int \frac{\mathrm{d}^4x_4\, \mathrm{e}^{\mathrm{i}k'''x_4}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_{2'}^0}} \\
&\quad \times \langle 0|\, \mathcal{T}\{J(x_1)J(x_2)J(x_3)J(x_4)\}\, |0\rangle \\
&= \int \frac{\mathrm{d}^4x_1\, \mathrm{e}^{-\mathrm{i}k_1x_1}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_1^0}} \int \frac{\mathrm{d}^4x_2\, \mathrm{e}^{-\mathrm{i}k_2x_2}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_2^0}} \int \frac{\mathrm{d}^4x_{1'}\, \mathrm{e}^{\mathrm{i}k_{1'}x_{1'}}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_{1'}^0}} \int \frac{\mathrm{d}^4x_{2'}\, \mathrm{e}^{\mathrm{i}k_{2'}x_{2'}}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_{2'}^0}} \\
&\quad \times \langle 0|\, \mathcal{T}\{J(x_1)J(x_2)J(x_{1'})J(x_{2'})\}\, |0\rangle \\
&= \int \frac{\mathrm{d}^4x_1\, \mathrm{e}^{-\mathrm{i}k_1x_1}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_1^0}} \int \frac{\mathrm{d}^4x_2\, \mathrm{e}^{-\mathrm{i}k_2x_2}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_2^0}} \int \frac{\mathrm{d}^4x_{1'}\, \mathrm{e}^{\mathrm{i}k_{1'}x_{1'}}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_{1'}^0}} \int \frac{\mathrm{d}^4x_{2'}\, \mathrm{e}^{\mathrm{i}k_{2'}x_{2'}}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_{2'}^0}} \\
&\quad \times (\Box_1 + m_1^2)(\Box_2 + m_2^2)(\Box_{1'} + m_{1'}^2)(\Box_{2'} + m_{2'}^2)\, \langle 0|\, \mathcal{T}\{\varphi(x_1)\varphi(x_2)\varphi(x_{1'})\varphi(x_{2'})\}\, |0\rangle
\end{aligned}$$

At the second equality, one takes the limit where the distributions tend to the Dirac delta function

$$f_j(\mathbf{k}^{(l)}) \to \delta^{(3)}(\mathbf{k}^{(l)} - \mathbf{k}_j)$$

At the third equality, source is replaced by using the equations of motion and the d'Alembertian is brought outside the expectation value.

This last expectation value is the four-point Green's function. The generalization of the scattering amplitude to N particles is straightforward

$$\begin{aligned}
\langle f_{n'}|i_n\rangle &= \mathrm{i}^{n+n'} \prod_{j=1}^{n} \int \frac{\mathrm{d}^4x_j\, \mathrm{e}^{-\mathrm{i}k_jx_j}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_j^0}} \prod_{l=1}^{n'} \int \frac{\mathrm{d}^4x'_l\, \mathrm{e}^{-\mathrm{i}k'_lx'_l}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_{l'}^0}} \\
&\quad \times \prod_j (\Box_j + m_j^2) \prod_l (\Box_l + m_l'^2)\, \langle 0|\, \mathcal{T}\{\varphi(x_j)\varphi(x'_l)\}\, |0\rangle
\end{aligned}$$

The last expectation value is the $(n+n')$-point Green's function. This is the LSZ reduction formula: the scattering amplitudes are related to the computation of Green's functions.

Caveat. The calculation was based on the assumption that the particles are always created from time-dependent operators $a^\dagger(t)$ acting on the unique vacuum $|0\rangle$. This is true for the free theory, but in the interacting theory it may not be. The following discussion is not a proof, but a motivation of the validity of the assumption.

If the assumption is true for the interacting theory, then one should have

$$\langle 0|\, \varphi(x)\, |0\rangle = 0$$

because one creates a particle and projects it onto the vacuum. One may obtain the above by writing the field in terms of the field in the origin

$$\langle 0|\, \varphi(x)\, |0\rangle = \langle 0|\, \mathrm{e}^{\mathrm{i}Px}\varphi(x)\mathrm{e}^{-\mathrm{i}Px}\, |0\rangle = \langle 0|\, \mathrm{e}^{0}\varphi(0)\mathrm{e}^{-0}\, |0\rangle = \langle 0|\, \varphi(0)\, |0\rangle = v\,, \quad P^\mu\, |0\rangle = 0$$

where P is the translation generator (the momentum) and the vacuum is assumed to be unique; at the second equality, the momentum of the vacuum is $p = 0$ which then implies its translational invariance. It may be that $v \neq 0$, but one can shift the field φ to obtain a zero expectation value

$$\varphi'(x) = \varphi(x) - v \implies \langle 0|\, \varphi'(x)\, |0\rangle = \langle 0|\, \varphi'(0)\, |0\rangle = 0$$

This is justified because the physics of the Lagrangian does not change: it is just a renaming of the operator of interest.

Consider now a one particle state. One expects to find

$$\langle p|\, \varphi(x)\, |0\rangle = \mathrm{e}^{\mathrm{i}px}$$

like in the free case. In fact

$$\langle p|\,\varphi(x)\,|0\rangle = \langle p|\,\mathrm{e}^{\mathrm{i}Px}\varphi(0)\mathrm{e}^{-\mathrm{i}Px}\,|0\rangle = \mathrm{e}^{\mathrm{i}px}\,\langle p|\,\varphi(x)\,|0\rangle = A\mathrm{e}^{\mathrm{i}px}$$

By rescaling the field, one has

$$\langle p|\,\frac{1}{A}\varphi(x)\,|0\rangle = \langle p|\,\varphi'(x)\,|0\rangle = \mathrm{e}^{\mathrm{i}px}$$

Consider now a multi-particle state $|p,n\rangle$ where p is the total momentum and n are the other relevant quantum numbers. One expects to find

$$\langle p,n|\,\varphi(x)\,|0\rangle = 0$$

To this end, one may write

$$\langle p,n|\,\varphi(x)\,|0\rangle = \mathrm{e}^{\mathrm{i}px}\,\langle p,n|\,\varphi(0)\,|0\rangle = \mathrm{e}^{\mathrm{i}px}A_n(p)$$

where $A_n(p)$ is a function of products of Lorentz-invariants. For more than one particle, the total energy is

$$E_{\mathrm{tot}} = p_0 = \sqrt{p^2+M^2} = \sum_j \sqrt{p_j^2+m_j^2}$$

with momenta p_j and masses m_j. Considering two particles with equal mass, one has

$$p = p_1 + p_2\,, \quad E_{\mathrm{tot}} = \sqrt{p_1^2+m^2} + \sqrt{p_2^2+m^2}$$

Therefore, the minimum energy is $E_{\mathrm{tot}} = 2m$ and above there is a continuum of hyperbolae of energies (given by $\sqrt{E^2-p^2} = m = \text{const.}$, see Srednicki Fig. 5.1). On the other hand, the energy of a one-particle state is completely determined by its momentum and it lies on an isolated hyperbola. Instead of

$$\langle p,n|\,\varphi(0)\,|0\rangle$$

being zero, one should strictly consider

$$\langle p,n|\,a^\dagger(\pm\infty)\,|0\rangle$$

Considering only renormalizable states $|\psi\rangle$

$$|\psi\rangle = \sum_n \int \mathrm{d}^3p\,\psi_n(p)\,|p,n\rangle$$

[r] it follows

$$\langle\psi|\,a^\dagger(t)\,|0\rangle = \sum_n \int \mathrm{d}^3p\,\psi_n^*(f)\,\langle p,n|\,a^\dagger(t)\,|0\rangle$$

Inserting the expression of the operator in terms of the fields, one may finally get

$$\langle\psi|\,a^\dagger(t)\,|0\rangle = \sum_n \int \mathrm{d}^3p\,\psi_n^*(p)\frac{(2\pi)^3}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}}f_1(\mathbf{p})(p_0+k_0)A_n(\mathbf{p})\mathrm{e}^{-\mathrm{i}(k_0-p_0)t}$$

where

$$p_0 = \sqrt{p^2+M^2}\,, \quad k_0 = \sqrt{p^2+m^2}\,, \quad M \geq 2m\,, \quad p_0 > k_0$$

When taking the limit $t \to \pm\infty$, the exponential becomes a very fast oscillating function: integrating over momenta gives zero (due to the Riemann–Lebesgue lemma).

Conclusion. The LSZ reduction formula works under the assumptions that

$$\langle 0|\,\varphi(0)\,|0\rangle = 0\,, \quad \langle p|\,\varphi(x)\,|0\rangle = \mathrm{e}^{\mathrm{i}px}$$

which are related to translating and scaling the field.

Lecture 6

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3.3 Free field theory

Comparison with quantum mechanics. In quantum mechanics one finds similar observables to the ones present in the LSZ reduction formula? [r]

$$G(t_1, \dots, t_n) = \langle 0| \mathcal{T}\{O(t_1) \cdots O(t_n)\} |0\rangle = (-\mathrm{i})^n \frac{\delta^n W[J]}{\delta J(t_1) \cdots \delta J(t_n)}\bigg|_{J=0}$$

where $|0\rangle$ is the fundamental state (there is no notion of Fock vacuum in ordinary quantum mechanics) and the generating functional is

$$W[J] = \lim_{\substack{t' \to -\mathrm{i}\infty \\ t \to +\mathrm{i}\infty}} \frac{1}{\langle q't'|qt\rangle} \int [\mathcal{D}q] \exp\left[\mathrm{i}\int \mathrm{d}\tau\, [L(\tau) + J(\tau)O(\tau)]\right]$$

One would like to generalize the above prescription to quantum field theory: the operators must be functions of position too because time and space must be treated on the same level.

The discrete positions $q_i(t)$ become a field $q_x(t) = \varphi(\mathbf{x}, t)$ with continuous index $i \to x$. The ordinary operators $O(t) = O(q_i(t))$ becomes a field operators $O(\varphi(x))$. The Lagrangian $L(q_i, \dot{q}_i)$ becomes a Lagrangian density $\mathcal{L}(\varphi(x), \partial_\mu \varphi)$. The path integral in quantum mechanics has a well-defined prescription. The functional measure before and after integrating the momenta is

$$\int [\mathrm{d}q\, \mathrm{d}p] \equiv \lim_{N\to\infty} \int \prod_{j=1}^N \mathrm{d}q_j \prod_{k=1}^{N+1} \frac{\mathrm{d}p_k}{2\pi} \implies \int [\mathrm{d}q] = \lim_{N\to\infty} \frac{1}{A} \int \prod_{j=1}^N \frac{\mathrm{d}q_j}{A}\,, \quad A = \frac{2\pi\mathrm{i}\,\delta t}{m}$$

which becomes

$$[\mathcal{D}\varphi(x)\, \mathcal{D}\pi(x)] \implies [\mathcal{D}\varphi]$$

whose rigorous definition has not been presented. To define the measure in quantum field theory, one can discretize the theory on a lattice to obtain ordinary quantum mechanics with the identity

$$\varphi(x_i) = \varphi_i$$

This way can be efficient for some problems. Though the approach taken next is different: one leaves the above just as a formal definition of the measure since one almost never explicitly computes functional integrals.

Generalization. The definitions of the Green's function and the generating functional can be generalized. [r] The first one is

$$G^{(n)}(x_1, \dots, x_n) = \langle 0| \mathcal{T}\{\varphi(x_1) \cdots \varphi(x_n)\} |0\rangle = (-\mathrm{i})^n \frac{\delta^n W[J]}{\delta J(t_1) \cdots \delta J(t_n)}\bigg|_{J=0}$$

where the generating functional for Green's functions is

$$W[J] = N \int [\mathcal{D}\varphi] \exp\left[\mathrm{i}\int \mathrm{d}^4x\, [\mathcal{L}(\varphi(x), \partial_\mu\varphi(x)) + J(x)\varphi(x)]\right]$$

where N is a suitable normalization. If one is able to evaluate the generating functional W, then one may compute the Green's function of an arbitrary number of observables. The normalization N is typically chosen such that

$$W[J=0] \equiv \langle 0|0\rangle = 1$$

One would like to study how to compute the generating functional $W[J]$ for a real scalar field.

Generating functional. The simplest case is the free field theory. The Lagrangian density is

$$\mathcal{L}(\varphi) = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2$$

The generating functional is

$$W_0[J] = N\int[\mathcal{D}\varphi]\,\exp\left[\mathrm{i}\int \mathrm{d}^4x\,(\mathcal{L}+J\varphi)\right]$$

To study the above one may use the momentum space. The Fourier conventions are

$$\widetilde{\varphi}(k) = \int \mathrm{d}^4x\,\mathrm{e}^{-\mathrm{i}kx}\varphi(x)\,,\quad \varphi(x) = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{\mathrm{i}kx}\widetilde{\varphi}(k)$$

For the source, one has

$$\begin{aligned}\int \mathrm{d}^4x\,J(x)\varphi(x) &= \int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\,\mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}(k_1+k_2)x}\widetilde{J}(k_1)\widetilde{\varphi}(k_2)\\ &= \int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}(2\pi)^4\delta^{(4)}(k_1+k_2)\widetilde{J}(k_1)\widetilde{\varphi}(k_2)\\ &= \int\frac{\mathrm{d}^4k}{(2\pi)^4}\widetilde{J}(k)\widetilde{\varphi}(-k) = \frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}[\widetilde{J}(k)\widetilde{\varphi}(-k)+\widetilde{J}(-k)\widetilde{\varphi}(k)]\end{aligned}$$

The same can be done for the mass term, while for the kinetic term one should integrate by parts first to get

$$\int \mathrm{d}^4x\,\frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi = -\int\mathrm{d}^4x\,\frac{1}{2}\varphi\,\Box\,\varphi$$

To quickly transform, one substitutes the function with its Fourier transform and absorbs the Jacobian into the normalization constant [r].

The generating functional is

$$W_0[J] = N\int[\mathcal{D}\widetilde{\varphi}]\,\exp\left[\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}[\widetilde{\varphi}(k)(k^2-m^2)\widetilde{\varphi}(-k)+\widetilde{J}(k)\widetilde{\varphi}(-k)+\widetilde{J}(-k)\widetilde{\varphi}(k)]\right]$$

Without formal development, one uses the Gaussian integrals for the functional integral above. One completes the square in the exponent

$$\begin{aligned}E &= \widetilde{\varphi}(k)(k^2-m^2)\widetilde{\varphi}(-k)+\widetilde{J}(k)\widetilde{\varphi}(-k)+\widetilde{J}(-k)\widetilde{\varphi}(k)\\ &= \left[\widetilde{\varphi}(k)+\widetilde{J}(k)\frac{1}{k^2-m^2}\right](k^2-m^2)\left[\widetilde{\varphi}(-k)+\frac{1}{k^2-m^2}\widetilde{J}(-k)\right]-\widetilde{J}(k)\frac{1}{k^2-m^2}\widetilde{J}(-k)\end{aligned}$$

By making a change of variables

$$\widetilde{\varphi}'(-k)\equiv\widetilde{\varphi}(-k)+\frac{1}{k^2-m^2}\widetilde{J}(-k)$$

the generating functional becomes

$$\begin{aligned}W_0[J] &= N\int[\mathcal{D}\widetilde{\varphi}']\,\exp\left[\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\widetilde{\varphi}'(k^2-m^2)\widetilde{\varphi}'\right]\exp\left[-\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\widetilde{J}(k)\frac{1}{k^2-m^2}\widetilde{J}(-k)\right]\\ &= \exp\left[-\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\widetilde{J}(k)\frac{1}{k^2-m^2}\widetilde{J}(-k)\right]N\int[\mathcal{D}\widetilde{\varphi}']\,\exp\left[\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\widetilde{\varphi}'(k^2-m^2)\widetilde{\varphi}'\right]\\ &= \exp\left[-\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\widetilde{J}(k)\frac{1}{k^2-m^2}\widetilde{J}(-k)\right]W_0[J=0]\,,\quad W_0[J=0]=1\\ &= \exp\left[-\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\widetilde{J}(k)\frac{1}{k^2-m^2}\widetilde{J}(-k)\right]\end{aligned}$$

Going back to the configuration space, one notices that $(k^2-m^2)^{-1}$ is the propagator. Recalling that it is

$$\Delta_0(x-x') = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}k(x-x')}}{k^2-m^2+\mathrm{i}\varepsilon}$$

then the generating functional is

$$W_0[J] = \exp\left[-\frac{\mathrm{i}}{2}\int\mathrm{d}^4x\,\mathrm{d}^4x'\,J(x)\Delta_0(x-x')J(x')\right]$$

Two-point function. The two-point Green's function is

$$\begin{aligned}\langle 0|\,\mathcal{T}\{\varphi(x_1)\varphi(x_2)\}\,|0\rangle &= (-\mathrm{i})^2\frac{\delta^2 W_0[J]}{\delta J(x_1)\,\delta J(x_2)}\bigg|_{J=0}\\ &= (-\mathrm{i})^2\frac{\delta}{\delta J(x_1)}\bigg[-\mathrm{i}\int \mathrm{d}^4x\, J(x)\Delta(x-x_2)\,W_0[J]\bigg]_{J=0}\\ &= [(-\mathrm{i})^3\Delta_0(x_1-x_2) + (\text{terms with } J\text{s})]W_0[J]\Big|_{J=0}\\ &= \mathrm{i}\Delta_0(x_1-x_2)\end{aligned}$$

at the second line one notices

$$\frac{\delta J(x')}{\delta J(x_2)} = \delta^{(4)}(x'-x_2)$$

Also there is a factor of 2 that is simplified with the fraction because the derivative can act on both $J(x)$ and $J(x')$? [r].

Euclidean space. The Lagrangian is

$$\mathcal{L} = -\frac{1}{2}\varphi\,\Box\,\varphi - \frac{1}{2}m^2\varphi^2$$

Applying Wick's rotation $x^0 = -\mathrm{i}x^0_{\mathrm{E}}$, the d'Alembertian operator is [r]

$$\Box = \partial_\mu\partial^\mu = \partial_0^2 - \nabla^2 = -\partial_{0\mathrm{E}}^2 - \nabla^2 = \Box_{\mathrm{E}}$$

In this way, in the Fourier transform one has

$$\Box \rightsquigarrow -k^2\,,\quad \Box_{\mathrm{E}} \rightsquigarrow k_{\mathrm{E}}^2$$

The generating function is

$$\begin{aligned}W_0^{\mathrm{E}}[0] &= N\int [\mathcal{D}\varphi]\,\exp\bigg[-\frac{1}{2}\int \mathrm{d}^4x_{\mathrm{E}}\,\varphi(\Box_{\mathrm{E}}+m^2)\varphi\bigg]\\ &= N\int [\mathcal{D}\widetilde{\varphi}]\,\exp\bigg[-\frac{1}{2}\int \frac{\mathrm{d}^4k_{\mathrm{E}}}{(2\pi)^4}\widetilde{\varphi}(k_{\mathrm{E}})(k_{\mathrm{E}}^2+m^2)\widetilde{\varphi}(-k_{\mathrm{E}})\bigg] = 1\end{aligned}$$

[r] The formulation above is well-defined since the exponential is decaying. With the source, one has

$$\begin{aligned}W_0^{\mathrm{E}}[J] &= N\int [\mathcal{D}\widetilde{\varphi}]\,\exp\bigg[-\frac{1}{2}\int \frac{\mathrm{d}^4k_{\mathrm{E}}}{(2\pi)^4}[\widetilde{\varphi}(k_{\mathrm{E}})(k_{\mathrm{E}}^2+m^2)\widetilde{\varphi}(-k_{\mathrm{E}}) + 2\widetilde{J}(k_{\mathrm{E}})\widetilde{\varphi}(-k_{\mathrm{E}})]\bigg]\\ &= \exp\bigg[\frac{1}{2}\int \frac{\mathrm{d}^4k_{\mathrm{E}}}{(2\pi)^4}\widetilde{J}(k_{\mathrm{E}})\frac{1}{k_{\mathrm{E}}^2+m^2}\widetilde{J}(-k_{\mathrm{E}})\bigg]\\ &= \exp\bigg[\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4x'\,J(x)\Delta_0(x-x')J(x')\bigg]\end{aligned}$$

The Euclidean Green's function is

$$G_{\mathrm{E}}^{(n)}(x_1,\ldots,x_n) = \frac{\delta^n W_0^{\mathrm{E}}[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0}$$

The propagator is the two-point function

$$\Delta_0(x_1-x_2) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}k(x_1-x_2)}}{k^2+m^2} = \langle 0|\,\mathcal{T}\{\varphi(x_1)\varphi(x_2)\}\,|0\rangle$$

where all variables are in Euclidean space (see Cheng–Li, eq. 1.84).

Exercise. Check that the propagator is the two-point function.

Green's function. [r] The Green's function is

$$G_{\mathrm{E}}^{(n)}(x_1,\ldots,x_n) = \langle 0|\,\mathcal{T}\{\varphi(x_1)\cdots\varphi(x_n)\}\,|0\rangle = \left.\frac{\delta^n W_0^{\mathrm{E}}[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right|_{J=0}$$

The Lagrangian has a $\mathbb{Z}_2$ symmetry and the Green's function must have that symmetry too. So only even-point functions are non-zero

$$G^{(2k+1)} = 0\,,\quad k\in\mathbb{N}_0$$

The four-point function is

$$\begin{aligned} G_{\mathrm{E}}^{(4)}(x_1,\ldots,x_4) &= \langle 0|\,\mathcal{T}\{\varphi(x_1)\cdots\varphi(x_4)\}\,|0\rangle \\ &= \frac{\delta^4}{\delta J(x_1)\cdots\delta J(x_4)}\exp\left[\frac{1}{2}\int \mathrm{d}^4y\,\mathrm{d}^4y'\,J(y)\Delta_0(y-y')J(y')\right]_{J=0} \end{aligned}$$

One may use the fact that

$$\frac{\delta}{\delta J(x_j)}\int \mathrm{d}^4y\,\mathrm{d}^4y'\,J(y)\Delta_0(y-y')J(y') = 2\int \mathrm{d}^4y\,J(y)\Delta(y-x_j)$$

and the only terms that survive after plugging in $J=0$ are the ones without the factor J. The Green's function is

$$\begin{aligned} G_{\mathrm{E}}^{(4)}(x_1,\ldots,x_4) = \frac{\delta^2}{\delta J(x_1)J(x_2)}\Big[&\Delta(x_3-x_4)W_0^{\mathrm{E}}[J] \\ &+\int \mathrm{d}^4y\,J(y')\Delta(y-x_3)\int \mathrm{d}^4y'\,J(y')\Delta(y-x_4)W_0^{\mathrm{E}}[J]\Big] \\ = \Delta(x_3-x_4)&\Delta(x_1-x_2)+\Delta(x_2-x_3)\Delta(x_1-x_4)+\Delta(x_1-x_3)\Delta(x_2-x_4) \end{aligned}$$

At the last line [r]. The last line are all the combinations of products of propagators. This is Wick's theorem: all the possible two contractions of four terms.

3.4 Interacting field theory

Generating functional. A simple interacting theory is the $\lambda\varphi^4$ theory. Its Lagrangian is

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4 = \mathcal{L}_0+\mathcal{L}_1$$

where the first two terms are the free part and the last term is the interacting part. The Euclidean time Lagrangian is

$$\mathcal{L}_{\mathrm{E}} = \frac{1}{2}\varphi(\Box_{\mathrm{E}}+m^2)\varphi + \frac{\lambda}{4!}\varphi^4$$

The generating functional is

$$\begin{aligned}
W^{\mathrm{E}}[J] &= N \int [\mathcal{D}\varphi] \exp\left[-\int \mathrm{d}^4x \left(\frac{1}{2}\varphi(\Box_{\mathrm{E}} + m^2)\varphi + \frac{\lambda}{4!}\varphi^4 + J\varphi\right)\right] \\
&= N \int [\mathcal{D}\varphi] \exp\left[-\int \mathrm{d}^4x \left(\frac{1}{2}\varphi(\Box_{\mathrm{E}} + m^2)\varphi + J\varphi\right)\right] \exp\left[-\int \mathrm{d}^4x \,\frac{\lambda}{4!}\varphi^4\right] \\
&= N \int [\mathcal{D}\varphi] \sum_{n=0}^{\infty} \frac{1}{n!} \left[-\int \mathrm{d}^4x \,\frac{\lambda}{4!}\varphi^4\right]^n \exp\left[-\int \mathrm{d}^4x \left(\frac{1}{2}\varphi(\Box_{\mathrm{E}} + m^2)\varphi + J\varphi\right)\right] \\
&= N \int [\mathcal{D}\varphi] \sum_{n=0}^{\infty} \frac{1}{n!} \left[-\int \mathrm{d}^4x \,\frac{\lambda}{4!}\left(-\frac{\delta}{\delta J(x)}\right)^4\right]^n \\
&\quad \times \exp\left[-\int \mathrm{d}^4x \left(\frac{1}{2}\varphi(\Box_{\mathrm{E}} + m^2)\varphi + J\varphi\right)\right] \\
&= \exp\left[-\int \mathrm{d}^4x \,\frac{\lambda}{4!}(-\delta_{J(x)})^4\right] N \int [\mathcal{D}\varphi] \exp\left[-\int \mathrm{d}^4x \left(\frac{1}{2}\varphi(\Box_{\mathrm{E}} + m^2)\varphi + J\varphi\right)\right] \\
&= \exp\left[-\int \mathrm{d}^4x \,\mathcal{L}_1(-\delta_{J(x)})\right] N \int [\mathcal{D}\varphi] \exp\left[-\int \mathrm{d}^4x \left(\frac{1}{2}\varphi(\Box_{\mathrm{E}} + m^2)\varphi + J\varphi\right)\right] \\
&= \exp\left[-\int \mathrm{d}^4x \,\mathcal{L}_1(-\delta_{J(x)})\right] W_0[J]
\end{aligned}$$

where the variables are all in Euclidean time, the index is dropped from now on. [r]

3.5 Perturbation theory

See Cheng–Li, Sdrenicki. For small a parameter λ one can Taylor expand the exponential in the generating functional up to a desired order. The generating functional is

$$\begin{aligned}
W[J] &= \exp\left[-\frac{\lambda}{4!}\int \mathrm{d}^4x \,(-\delta_{J(x)})^4\right] W_0[J] \\
&= \left[1 - \frac{\lambda}{4!}\int \mathrm{d}^4x \,(\delta_{J(x)})^4 + \frac{1}{2!}\frac{\lambda^2}{(4!)^2}\int \mathrm{d}^4x \,(\delta_{J(x)})^4 \int \mathrm{d}^4x' \,(\delta_{J(x')})^4 + o(\lambda^2)\right] W_0[J] \\
&\equiv W_0[J](1 + \lambda\omega_1[J] + \lambda^2\omega_2[J] + \cdots)
\end{aligned}$$

where

$$\begin{aligned}
\omega_1[J] &= -\frac{1}{4!}W_0^{-1}[J]\int \mathrm{d}^4x \,(\delta_{J(x)})^4 W_0[J] \\
\omega_2[J] &= \frac{1}{2!}\frac{1}{(4!)^2}W_0^{-1}[J]\int \mathrm{d}^4x \,(\delta_{J(x)})^4 \int \mathrm{d}^4x' \,(\delta_{J(x')})^4 W_0[J] \\
W_0[J] &= \exp\left[\frac{1}{2}\int \mathrm{d}^4y \,\mathrm{d}^4y' \, J(y)\Delta(y - y')J(y')\right] \equiv \mathrm{e}^{k[J]}
\end{aligned}$$

and so on. The derivatives of the exponent of the generating functional are

$$\frac{\delta k[J]}{\delta J(x)} = \int \mathrm{d}^4y \, J(y)\Delta(y - x)\,, \quad \frac{\delta^2 k[J]}{\delta J(x)^2} = \Delta_0(x - x) = \Delta_0(0)\,, \quad \frac{\delta^n k[J]}{\delta J(x)^n} = 0\,, \quad n \geq 3$$

also

$$\frac{\delta W_0[J]}{\delta J(x)} = \frac{\delta k[J]}{\delta J(x)}\, W_0[J] \iff \frac{\delta \mathrm{e}^{k[J]}}{\delta J(x)} = \frac{\delta k[J]}{\delta J(x)}\, \mathrm{e}^{k[J]}$$

Lecture 7

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3.5.1 First-order expansion coefficient

One would like to compute the first-order expansion coefficient

$$\omega_1[J] = -\frac{1}{4!}W_0^{-1}[J]\int \mathrm{d}^4x \,(\delta_{J(x)})^4 W_0[J]$$

Omitting the argument x and compacting the derivative notation, the integrand is

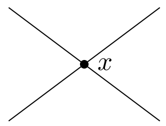
$$\begin{aligned}\delta_J^4 \mathrm{e}^{k[J]} &= \delta_J^3(\delta_J k\, \mathrm{e}^k) = \delta_J^2[\delta_J^2 k\, \mathrm{e}^k + (\delta_J k)^2 \mathrm{e}^k] = \delta_J[3\delta_J^2 k\, \delta_J k\, \mathrm{e}^k + (\delta_J k)^3 \mathrm{e}^k] \\ &= 3(\delta_J^2 k)^2 \mathrm{e}^k + 6\, \delta_J^2 k\, (\delta_J k)^2 \mathrm{e}^k + (\delta_J k)^4 \mathrm{e}^k \\ &= W_0[J][3(\delta_J^2 k)^2 + 6\, \delta_J^2 k\, (\delta_J k)^2 + (\delta_J k)^4]\end{aligned}$$

At the first two lines one applies one derivative at a time remembering that derivatives of order higher than two are zero. The coefficient is then

$$\begin{aligned}\omega_1[J] = -\frac{1}{4!}\int \mathrm{d}^4 x\, \Big[&3\Delta(x-x)\Delta(x-x) \\ &+ 6\Delta(x-x)\int \mathrm{d}^4 y_1\, \mathrm{d}^4 y_2\, J(y_1)\Delta(x-y_1)J(y_2)\Delta(x-y_2) \\ &+ \int \mathrm{d}^4 y_1\, \mathrm{d}^4 y_2\, \mathrm{d}^4 y_3\, \mathrm{d}^4 y_4\, J(y_1)\Delta(x-y_1)J(y_2)\Delta(x-y_2)J(y_3)\Delta(x-y_3)J(y_4)\Delta(x-y_4)\Big]\end{aligned}$$

Feynman diagrams may be used to quickly obtain the expression above. [r] One may interpret the sum of the terms above in a convenient diagrammatic way:

- The internal point x is associated to

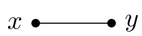


$$-\frac{1}{4!}\int \mathrm{d}^4 x$$

- The external points y_i are associated to

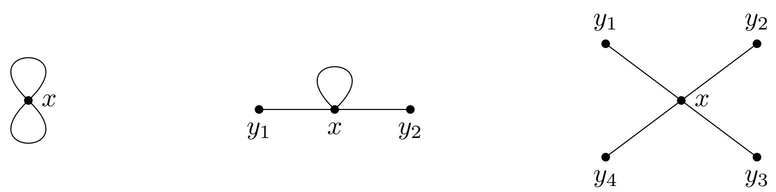
$$\int \mathrm{d}^4 y_i\, J(y_i)$$

- The propagators $\Delta(x-y)$ are internal lines associated to



$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}k(x-y)}}{k^2+m^2}$$

In the first addendum of the coefficient ω_1 there is a single internal point with two propagators. The second addendum has one internal point and two external points with three propagators. The third addendum has one internal point and four external points with four propagators.



Notice that there are no odd numbers of external points: this is again related to the $\mathbb{Z}_2$ symmetry of the Lagrangian.

At fixed order in the λ expansion, one has a corresponding number of internal vertices: first order, one internal vertex; second order, two vertices, etc.

One observes:

- The number of powers of λ is equal to the number of internal vertices.
- The internal vertex is connected to four lines which corresponds to the four derivatives of the source J, which is related to the study of the $\lambda\varphi^4$ theory. The number of lines coming out of an internal vertex is determined by the choice of the interaction $\mathcal{L}_1$.

- The way one represents the coefficient ω_1 in terms of diagrams — which is related to the algebraic structure of the theory [r] — is a linear combination of diagrams of increasing number of external points. At a given order in the parameter λ, the corresponding coefficient ω is a linear combination of diagrams with increasing number of external points.

- The diagrams without any external points survive when setting $J = 0$: these are vacuum diagrams. They are the only ones contributing to $W[J = 0]$. The normalization with $W[0] = 1$ is equivalent to dividing the integral within $W[J]$ by $W[0]$:

$$W[J] = \frac{1}{W[J=0]} \int [\mathcal{D}\varphi] \exp\left[-\frac{1}{2}\int \mathcal{L}_\mathrm{E} + J\varphi\right]$$

  In this way one cancels all vacuum diagrams, so one may forget about them during calculations:

$$\begin{aligned} W[J] &= \frac{1 + 3D_1 + 3!D_2 + D_3 + o(\lambda)}{1 + 3D_1 + o(\lambda)_\text{vacuum}} \\ &= [1 + 3D_1 + 3!D_2 + D_3 + o(\lambda)][1 - 3D_1 + o(\lambda)_\text{vacuum}] \\ &= 1 + 3D_1 - 3D_1 + \cdots \end{aligned}$$

  where D_j are the algebraic expressions related to the diagrams. [r]

- Each diagram has an associated combinatorial factor. To find the factor one must find all the ways one can connect an internal point with four lines to n external points through one internal line each. The number obtained must be divided by the symmetry factor of the diagram — it is the number of equivalent ways one can draw the diagram. This factor can be computed in a more efficient way.

One would like to use the diagrams to compute the expansion terms [r].

For the coefficient ω_1, the factor $6 = 3!$ of the second addendum is obtained by the $12 = 4 \cdot 3$ ways one can connect an internal point to two external points divided by a factor of 2 because the external points can be swapped while giving the same diagram. For the third addendum, the internal point can be connected in $4!$ ways to the four external points. The symmetry factor comes from the ways in which one can permute the indices of the external line which is $4!$. The factor of the addendum is then just 1. In both of these examples, the common factor of $(4!)^{-1}$ has been omitted, but it should always be included.

In general, for the order λ^p, the expansion coefficient is

$$\omega_p[J] = \frac{W_0^{-1}[J]}{p!}\left[-\frac{1}{4!}\right]^p \left[\int \mathrm{d}^4x\, (\delta_{J(x)})^4\right]^p W_0[J]$$

The diagrams have p internal points. The number of external points is even. The combinatorial factor is

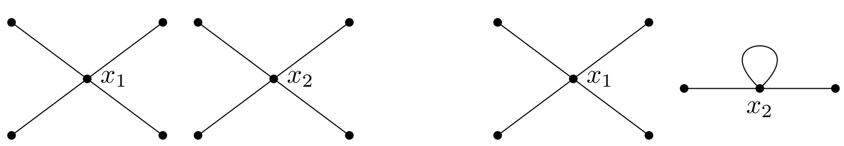
$$s = \frac{1}{p!}\left[-\frac{1}{4!}\right]^p \frac{N}{D}$$

where N is the number of equivalent ways to connect the points with propagators and D is the symmetry factor of the diagram (which is the number of equivalent topologies).

3.5.2 Second-order expansion coefficient

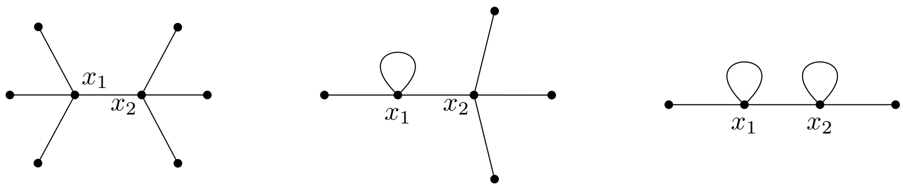
The coefficient contains two internal points x_1 and x_2. One may first draw the diagram and derive from it the algebraic expression.

Disconnected diagrams. The disconnected diagrams are



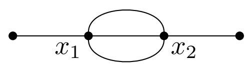
These sum up to $\frac{1}{2}\omega_1^2$.

Connected diagrams. The connected diagrams have lines connecting the internal points. With one line, one has



With two lines, one has

With three lines, one has



The vacuum diagrams have not been considered.

The diagrams can be translated into algebraic expressions. Consider the second connected diagram:

$$\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2 \int \mathrm{d}^4y_1\,\mathrm{d}^4y_2\,\mathrm{d}^4y_3\,\mathrm{d}^4y_4 J(y_1)J(y_2)J(y_3)J(y_4)\Delta(x_1-x_2)\Delta(x_2-x_2)$$
$$\times\,\Delta(x_1-y_1)\Delta(x_1-y_2)\Delta(x_1-y_3)\Delta(x_2-y_4)$$

One still needs the combinatorial factor. All diagrams have a common factor of

$$\frac{1}{2!}\frac{1}{(4!)^2}$$

One has to compute N and D for each diagram.

First diagram. The first line can be connected to the first internal point in 8 possible ways. The second line must connect to the same point as the first line, so there are 3 ways. Similarly, the third line has 2 ways. The first line on the second internal point can be put in 4 ways, the second in 3 and the third in 2. The last line of each internal point can be connected in one way only. Therefore

$$N = 8\cdot 3!\cdot 4!$$

The symmetries are:

- exchanging the internal points, so two configurations;
- permutations of the three line of each internal vertex, 3! configurations each.

Therefore

$$D = 2\cdot 3!\cdot 3!$$

The combinatorial factor is

$$S = \frac{1}{2!}\frac{1}{(4!)^2}\frac{N}{D} = \frac{1}{2(3!)^2}$$

The diagram translates to

$$\begin{aligned}&\frac{1}{2(3!)^2}\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\prod_{j=1}^{6}\mathrm{d}y_j\,\Delta(x_1-y_1)\Delta(x_1-y_2)\Delta(x_1-y_3)\\ &\times\Delta(x_2-y_4)\Delta(x_2-y_5)\Delta(x_2-y_6)J(y_1)J(y_2)J(y_3)J(y_4)J(y_5)J(y_6)\end{aligned}$$

Second diagram. One has

$$N=8\cdot(4!)\cdot 3\,,\quad D=2\cdot(3!)\,,\quad s=\frac{1}{4\cdot 3!}=\frac{1}{4!}$$

Third diagram. One has

$$N=8\cdot(4)\cdot(9)\,,\quad D=2\,,\quad s=\frac{1}{8}$$

Fourth diagram. The fourth diagram has

$$N=8\cdot(3)\cdot(4\cdot 3)\cdot 2$$

and the symmetry factor is

$$D=2\cdot(2!)\cdot(2!)$$

The combinatorial factor is [r]

$$s=\frac{1}{16}$$

Fifth diagram. One may compute the factor for two copies of the diagram with the internal points swapped and sum their contributions. The first one gives

$$N=4\cdot 3\cdot(4\cdot 3)\,,\quad D=2\,,\quad s=\frac{1}{16}$$

Summing up with the second, one obtains

$$s=\frac{1}{8}$$

Sixth diagram. One has

$$N=8\cdot 4\cdot(3!)\,,\quad D=2\,,\quad s=\frac{1}{12}$$

Exercise. Translate each diagram into an algebraic expression.

Lecture 8

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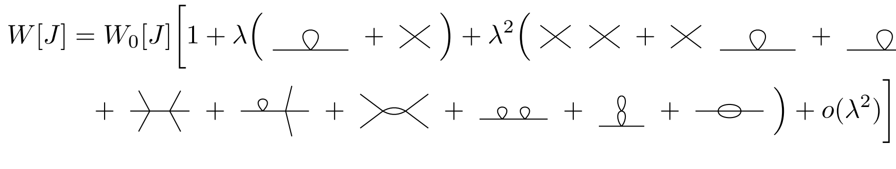
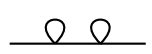
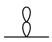
Remark. The generating functional is given by



The number of the external points in a diagram corresponds to the number of sources J appearing in the associated algebraic expression. The Green's function is

$$G^{(n)}(z_1,\dots,z_n)=\langle 0|\,\mathcal{T}\{\varphi(z_1)\cdots\varphi(z_n)\}\,|0\rangle=\left.\frac{\delta^n W[J]}{\delta J(z_1)\cdots\delta J(z_n)}\right|_{J=0}$$

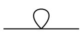
The two-point function is

$$G^{(2)}(z_1,z_2)=\left.\frac{\delta^2 W[J]}{\delta J(z_1)\,\delta J(z_2)}\right|_{J=0}=\lambda$$  $+\lambda^2\Big($  $+$  $+$  $\Big)+o(\lambda^2)$

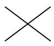
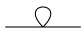
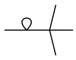
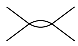
When evaluating at $J=0$, the non-vanishing terms are the ones that have exactly two external points because the derivatives remove precisely that many sources from the generating functional. When taking the derivative, one has

$$\delta_{J(z_1)}\int \mathrm{d}^4y_1\,J(y_1)\Delta(x-y_1)=\int \mathrm{d}^4y_1\,\delta^{(4)}(y_1-z_1)\Delta(x-y_1)=\Delta(x-z_1)$$

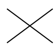
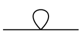
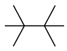
[r] The contributions to the Green's function are the propagators coming from the external points [r]

$G^{(2)}(z_1,z_2)\rightsquigarrow\lambda$  $\rightsquigarrow\int \mathrm{d}^4x\,\Delta(x-z_1)\Delta(x-z_2)\Delta(x-x)$

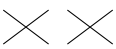
The four-point function is

$G^{(4)}(z_1,\dots,z_4)=\lambda$  $+\lambda^2\Big($   $+$  $+$  $\Big)$

The six-point function is

$G^{(6)}(z_1,\dots,z_6)=\lambda^2\Big($   $+$  $\Big)$

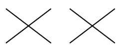
The eight-point function is

$G^{(8)}(z_1,\dots,z_8)=\lambda^2$ 

The ten-point function is, at least, of order λ^3. Higher point Green's functions get contributions only from higher order terms in the coupling constant λ.

3.5.3 Disconnected diagrams

The above Green's functions have taken into account disconnected diagrams. Consider the eight-point function

$G^{(8)}(z_1,\dots,z_8)=\langle 0|\,\mathcal{T}\{\varphi(z_1)\cdots\varphi(z_8)\}\,|0\rangle=\lambda^2$  $+$ permutations

Up to combinatorial contributions, the Green's function is

$$\begin{aligned}G^{(8)}(z_1,\dots,z_8)&=\lambda^2\int \mathrm{d}^4x_1\,\Delta(x_1-z_1)\Delta(x_1-z_2)\Delta(x_1-z_3)\Delta(x_1-z_4)\\&\quad\times\int \mathrm{d}^4x_2\,\Delta(x_2-z_5)\Delta(x_2-z_6)\Delta(x_2-z_7)\Delta(x_2-z_8)\\&\sim\lambda^2\,\langle 0|\,\mathcal{T}\{\varphi(z_1)\cdots\varphi(z_4)\}\,|0\rangle\,\langle 0|\,\mathcal{T}\{\varphi(z_5)\cdots\varphi(z_8)\}\,|0\rangle\end{aligned}$$

From this one notices that disconnected Feynman diagrams contribute to disconnected Green's functions. These can be built from lower-order connected Green's functions.

To eliminate the disconnected contributions both in the generating functional and the Green's function, one notices that the connected Green's functions are given by

$$G_{\mathrm{c}}^{(n)}(z_1,\dots,z_n)=\left.\frac{\delta^n\ln W[J]}{\delta J(z_1)\cdots\delta J(z_n)}\right|_{J=0}$$

One may check the above for the order λ^2. Rewriting the generating functional as

$$W[J]=W_0[J]\Big[1+W_0^{-1}[J](W[J]-W_0[J])\Big]$$

One sees that the second addendum is the perturbative expansion

$$W_0^{-1}[J](W[J]-W_0[J])=\lambda\omega_1[J]+\lambda^2\omega_2[J]+o(\lambda^2)$$

[r] Taking the logarithm, one has

$$\begin{aligned}\ln W[J] &= \ln W_0[J] + \ln[1 + W_0^{-1}[J](W[J] - W_0[J])] \\ &= \ln W_0 + W_0^{-1}(W - W_0) - \frac{1}{2}[W_0^{-1}(W - W_0)]^2 + o(\lambda^2) \\ &= \ln W_0 + \lambda\omega_1 + \lambda^2\omega_2 + o(\lambda^2) - \frac{1}{2}[\lambda\omega_1 + \lambda^2\omega_2 + o(\lambda^2)]^2 + o(\lambda^2) \\ &= \ln W_0 + \lambda\omega_1 + \lambda^2\left[\omega_2 - \frac{1}{2}\omega_1^2\right] + o(\lambda^2)\end{aligned}$$

At the first line, the perturbative expansion is of order λ, so one Taylor expands

$$\ln[1 + \varepsilon] = 1 + \varepsilon - \frac{1}{2}\varepsilon^2 + o(\varepsilon^2)$$

The second addendum in the bracket is the negative sum of the disconnected diagrams, therefore the whole bracket is the sum of only the connected diagrams at order λ^2.

One defines the generating functional of connected Green's functions $Z[J]$ as

$$W[J] \equiv e^{Z[J]} \implies Z[J] \equiv \ln W[J]$$

Therefore, the connected Green's function prescription is

$$G_c^{(n)}(z_1, \ldots, z_n) = \left.\frac{\delta^n Z[J]}{\delta J(z_1) \cdots \delta J(z_n)}\right|_{J=0}$$

The normalization follows from the one of the generating functional W

$$W[J = 0] = 1 \implies Z[J = 0] = 0$$

From the expansion above of $\ln W[J]$ one can see that

$$Z[J] = Z_0[J] + \lambda(\cdots) + \lambda^2(\cdots) + o(\lambda^2)$$

At the present moment, one has eliminated vacuum diagrams and disconnected diagrams.

3.6 Green's functions in momentum space

The integrals of Green's functions are easier to compute in momentum space rather than configuration space. One needs to reformulate the Feynman rules after applying the Fourier transform.

Definition. The n-point Green's function in momentum space is

$$\widetilde{G}_c^{(n)}(p_1, \ldots, p_n)(2\pi)^4\delta^{(4)}(p_1 + p_2 + \cdots + p_n) = \int \prod_{j=1}^{n} d^4x_j\, e^{-ip_jx_j}\, G_c^{(n)}(x_1, \ldots, x_n)$$

One may apply this prescription to the two-point Green's function. In configuration space, the free Green's function is

$$G_0^{(2)}(x_1, x_2) = \Delta(x_1 - x_2) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x_1 - x_2)}}{k^2 + m^2}$$

In momentum space one has

$$\begin{aligned}\widetilde{G}_0^{(2)}(p_1, p_2)(2\pi)^4\delta^{(4)}(p_1 + p_2) &= \int d^4x_1\, d^4x_2\, e^{-i(p_1x_1 + p_2x_2)} \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x_1 - x_2)}}{k^2 + m^2} \\ &= \int \frac{d^4k}{(2\pi)^4} (2\pi)^4\delta^{(4)}(k - p_1)(2\pi)^4\delta^{(4)}(k + p_2)\frac{1}{k^2 + m^2} \\ &= (2\pi)^4\delta^{(4)}(p_1 + p_2)\frac{1}{p_1^2 + m^2}\end{aligned}$$

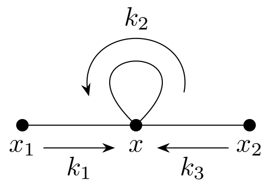
from which

$$\widetilde{G}_0^{(2)}(p_1, p_2) = \frac{1}{p_1^2 + m^2}\,, \quad p_1 = -p_2 \iff \boxed{\widetilde{G}_0^{(2)}(p, -p) = \frac{1}{p^2 + m^2}}$$

One would like to find the momentum space version of Feynman rules. Consider the correction of order λ to the two-point Green's function $G^{(2)}$. There is only one contributing diagram [r]. One has

$$\begin{aligned} G^{(2)}(x_1, x_2)\Big|_\lambda &= -\frac{\lambda}{2}\int \mathrm{d}^4x\, \Delta(x - x_1)\Delta(x - x_2)\Delta(x - x) \\ &= -\frac{\lambda}{2}\int \mathrm{d}^4x \int \frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}k_1(x-x_1)}}{k_1^2 + m^2}\int \frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{1}{k_2^2 + m^2}\int \frac{\mathrm{d}^4k_3}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}k_3(x-x_2)}}{k_3^2 + m^2} \end{aligned}$$

Since one can always send $k \to -k$ in the integral, one has to choose in which way to assign momentum and has to be consistent with the assignment. For example, in the above, the momentum k_1 goes from an external point x_1 towards an internal point x. The momentum in the loop can be put in either direction.



Performing the x-integral, one obtains

$$\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}x(k_1+k_3)} = (2\pi)^4\delta^{(4)}(k_1 + k_3)$$

Therefore, the Green's function is

$$G^{(2)}(x_1, x_2)\Big|_\lambda = -\frac{\lambda}{2}\int \frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}k_1(x_1-x_2)}}{(k_1^2 + m^2)^2(k_2^2 + m^2)}$$

The Fourier transform is then

$$\begin{aligned} &\widetilde{G}^{(2)}(p_1, p_2)\Big|_\lambda (2\pi)^4\delta^{(4)}(p_1 + p_2) = \\ &\quad = -\frac{\lambda}{2}\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\mathrm{e}^{-\mathrm{i}p_1x_1 - \mathrm{i}p_2x_2}\int \frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}k_1(x_1-x_2)}}{(k_1^2 + m^2)^2(k_2^2 + m^2)} \\ &\quad = -\frac{\lambda}{2}\int \frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{(2\pi)^4\delta^{(4)}(k_1 - p_2)(2\pi)^4\delta^{(4)}(p_1 + k_1)}{(k_1^2 + m^2)^2(k_2^2 + m^2)} \\ &\quad = -\frac{\lambda}{2}(2\pi)^4\delta^{(4)}(p_1 + p_2)\int \frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{1}{k_2^2 + m^2}\frac{1}{(p_2^2 + m^2)^2} \end{aligned}$$

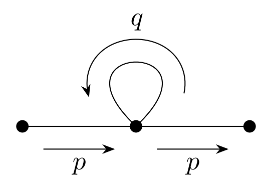
Keeping in mind the momentum conservation, one has

$$\widetilde{G}^{(2)}(p_1, p_2)\Big|_\lambda = -\frac{\lambda}{2}\frac{1}{(p_2^2 + m^2)^2}\int \frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{1}{k_2^2 + m^2}\,, \quad p_1 = -p_2$$

Tidying up the expression, one can write

$$\widetilde{G}^{(2)}(p, -p) = -\frac{\lambda}{2}\frac{1}{(p^2 + m^2)^2}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{q^2 + m^2}$$

The corresponding diagram becomes



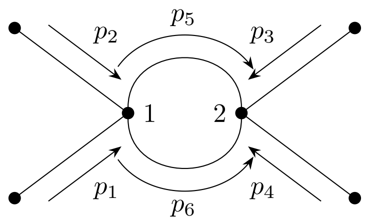
[r] In this way one guarantees momentum conservation at the internal vertex.

Feynman rules. One has to assign a set of momenta to guarantee momentum conservation at each vertex and total momentum conservation using the direction of the arrows. [r] Every time a loop appears, there is an integral on the loop momentum.

Therefore

- an internal vertex corresponds to
$$-\frac{\lambda}{4!}(2\pi)^4\delta^{(4)}(p_1+p_2+p_3+p_4)$$
- a propagator is an internal line corresponding to
$$\frac{1}{p^2+m^2}$$
- a loop corresponds to an integral
$$\int\frac{\mathrm{d}^4q}{(2\pi)^4}$$

Exercise. Write the momentum integral corresponding to the following diagram

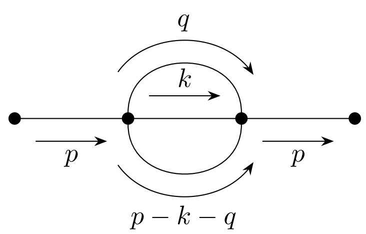


One should obtain a λ^2 contribution to the Green's function $\widetilde{G}^{(4)}(p_1,\dots,p_4)$.

One should assign a momentum to every external line and internal line. One imposes momentum conservation at each internal vertex. At the first vertex, one has
$$p_1+p_2=p_5+p_6$$
At the second vertex, one has
$$p_5+p_6=-p_3-p_4$$
From these two conditions, one has the total momentum conservation and the dependence of one momentum on the others
$$p_1+p_2+p_3+p_4=0\,,\quad p_6=p_1+p_2-p_5$$
There is only a single integral in p_5.

Briefly, one draws the diagram, assigns the momenta remembering total momentum conservation; there is only one loop which corresponds to an integral, the other line is given by momentum conservation.

Example. Consider the following two-loop diagram



The algebraic expression without the combinatorial factor is
$$\widetilde{G}^{(2)}(p,-p)\Big|_{\lambda^2}\rightsquigarrow\frac{1}{(p^2+m^2)^2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(q^2+m^2)(k^2+m^2)[(p-k-q)^2+m^2]}$$

The main point is to compute the integrals. One does not care about external propagators because they are spectators. One moves towards the evaluation of cut diagrams where one focuses only on the integrals of the diagram.

3.7 Loop expansion

Previously, the generating functional has been perturbatively expanded in powers of the coupling constant λ. One may reshuffle the terms and organize them by how may loops are present. In this way one can obtain an expansion in powers of the reduced Planck's constant $\hbar$ [r].

One should first fix the Green's function, so fix the number of external points. Writing explicitly the Planck's constant, the generating functional is

$$\begin{aligned}
W[J] &= N \int [\mathcal{D}\varphi] \exp\left[-\int \mathrm{d}^4x \left(\frac{1}{\hbar}(\mathcal{L}_0 + \mathcal{L}_1) + J\varphi\right)\right] \\
&= N \int [\mathcal{D}\varphi] \exp\left[-\frac{1}{\hbar}\int \mathrm{d}^4x \,(\mathcal{L}_0 + \mathcal{L}_1 + \hbar J\varphi)\right] \\
&= \exp\left[-\frac{1}{\hbar}\int \mathrm{d}^4x \,\mathcal{L}_1(-\delta_{J(x)})\right] N \int [\mathcal{D}\varphi] \exp\left[-\frac{1}{\hbar}\int \mathrm{d}^4x \,(\mathcal{L}_0 + \hbar J\varphi)\right] \\
&= \exp\left[-\frac{1}{\hbar}\int \mathrm{d}^4x \,\mathcal{L}_1(-\delta_{J(x)})\right] W_0[J]
\end{aligned}$$

The propagator is the inverse of the kinetic term, [r] therefore

$$W_0[J] = \exp\left[\frac{\hbar}{2}\int \mathrm{d}^4x \,\mathrm{d}^4x' \,J(x)\Delta(x-x')J(x')\right]$$

The Feynman rules have to be modified. The propagator is $\hbar\Delta(x-y)$ and so it carries one Planck's constant $\hbar$. Each internal point brings one inverse of the Planck's constant $\hbar^{-1}$ since the internal points come from the expansion of the exponential of the integral of the interaction Lagrangian $\mathcal{L}_1$.

Considering a diagram with V internal vertices and I internal lines. Its order is

$$\hbar^{I-V} = \frac{1}{\hbar}\hbar^{I-V+1} = \frac{1}{\hbar}\hbar^{L}$$

where $L = I - V + 1$ the number of loops. In this way one can reorganize the perturbative expansion in powers of the Planck's constant and as such in terms of the number of loops.

As long as there is only one coupling constant, the loop expansion is the same as the coupling constant λ expansion.