Quantum Gravity

Maso*

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Contents

Lecture 1

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Exam. Presentation on topic not seen during class, even a follow up.
Lecture notes by Tom Hartman.

Program. First week is general difficulties with quantum gravity, overview and approaches, some problems associated with quantum gravity. The following three weeks are black hole thermodynamics, black hole information paradox. The next three weeks are introduction to CFT. The following week is an instruction to AdS. The last three weeks are an introduction to AdS/CFT.

1 Introduction

Classical gravity. The action of general relativity is the Einstein–Hilbert action

$$S = \frac{1}{16\pi G_N} \int \mathrm{d}^D x \sqrt{-g}[R - 2\Lambda] + S_{\text{matter}}$$

The equations of motion are Einstein's field equations

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}^{\text{matter}}$$

Classical gravity is understood well. Solutions to these equations include black holes.

Quantum field theory. Quantum dynamics without gravity is quantum field theory. It describes the matter sector of the EH Lagrangian. It describes the three other fundamental forces: electromagnetic, weak and strong forces. They are unified in the Standard Model which has worked very well. Though, quantum chromodynamics is a strongly couple theory and perturbation theory is limited.

*`https://github.com/M-a-s-o/notes`

Quantum gravity. Quantum gravity is supposed to unify general relativity with the Standard Model by quantizing the gravitational field in a consistent manner.

However, there are some difficulties. The naive approach is taking GR action, take a metric, pick a background and expand it around such background

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{16\pi G_N} h_{\mu\nu} \,, \quad \Lambda = 0$$

As such one gets

$$S = \int \mathrm{d}^D x \sqrt{-g} \left[\frac{1}{2} [\partial_\mu h^{\mu\nu} \, \partial_\nu h - \partial_\mu h^{\rho\sigma} \, \partial_\rho h^\mu{}_\sigma + \frac{1}{2} \eta^{\mu\nu} (\partial_\mu h^{\rho\sigma})(\partial_\nu h_{\rho\sigma}) - \frac{1}{2} \eta^{\mu\nu} (\partial_\mu h)(\partial_\nu h) + \sum_{n\geq 3} G_N^{\frac{n}{2}} P_n(\partial_\mu, h_{\mu\nu}) \right]$$

where $h = \operatorname{Tr} h_{\mu\nu}$. The first three terms is the kinetic term of the graviton, a spin-2 particle. There are also an infinite number of terms (given by the sum) which can be thought of as interactions. [r] Now one can think about this theory as a QFT for the field h. The theory has one coupling constant G_N with length dimensions $D-2$. Every time one adds the coupling, one has to add momentum to have a dimensionless quantity $G_N p^{D-2}$ so at higher order the theory diverges more severely. This theory is not renormalizable, more precisely it is not power-counting renormalizable.

This means that GR should be seen as an effective field theory (EFT), because this problems involves higher energies.

The mass scale relevant is the Planck's mass

$$S = \frac{1}{16\pi G_N} \int \mathrm{d}^4 x \sqrt{-g} R = \frac{1}{16\pi\hbar} \int \mathrm{d}^4 x \sqrt{-g} M_P^2 R \,, \quad M_P = \sqrt{\frac{\hbar}{G_N}} \approx 1.22 \times 10^{19} \,\mathrm{GeV}$$

So the action looks like the following

$$S = \frac{1}{16\pi\hbar} \int \mathrm{d}^4 x \sqrt{-g} [M_P^2 R + R^2 + M_p^{-2} R^3 + \cdots)$$

The last terms give contributions in the UV energy spectrum, while in IR then the terms are just tiny corrections. The Planck scale is also the scale at which the Schwartzschild radius is the Compton wavelength.

Non-renormalizability. To solve the problem of renormalizability:

- the naive power-counting fails and there are infinite cancellations in the Feynman diagrams (for GR it has been computed at two-loops and the cancellations do not happen, same happens in maximal supergravity but higher orders);
- there is a fixed point in the RG flow but cannot be understood through power-counting: the perturbative QFT is naive and there is a non-perturbative fixed point;
- pure quantum GR is not a consistent theory, so one must add other degrees of freedom besides the graviton (string theory works in this way).

The above is also dependent on the background of the expansion, but even for the simplest, Minkowski, there are problems.

One would like to understand the path integral of quantum gravity. The partition function is

$$Z[\text{b.c.}] = \int_{\partial g = \text{b.c.}} [\mathcal{D}g] \, \mathrm{e}^{-\frac{i}{\hbar} S_{\mathrm{EH}}}$$

where b.c. stands for boundary condition. [r] one fixes the metric infinitely far away. For AdS the integral is understood non perturbatively.

Different approaches to quantum gravity. Different approaches to quantum gravity are

- string theory; originally string theory was developed for strong interaction;
- holography and AdS/CFT; it originated from string theory and will be developed from here on;
- loop quantum gravity; space-time is discrete and there are critical length scales, the degrees of freedom are loops like in gauge theory, one formulates a [r];
- asymptotic safety; there is a fixed point of GR in the UV, but the power-counting argument fails bc it is perturbative, but the fixed point is not perturbative;
- causal dynamical triangulation; it is a lattice approach to QG but it needs to rely on asymptotic safety.

1.1 Black hole entropy

The best hint towards quantum gravity is without doubt the Bekenstein–Hawking formula for black hole entropy

$$S_{\mathrm{BH}} = \frac{1}{4}\frac{k_{\mathrm{B}}c^3}{\hbar G_N}A_{\mathrm{H}}$$

Natural units are used in the following $c = \hbar = k_B = G_N = 1$. A quarter of the area of a black hole's event horizon A_{H} is the entropy. In this formula, there appears Boltzmann constant for statistical mechanics, the speed of light for special relativity, the universal gravitation constant for gravity and Planck's constant for quantum physics.

The formula first came from a thought experiment of Bekenstein. Entropy always increases thanks to the second law of thermodynamics. Black holes must have entropy because the information of falling objects must be kept. Hawking proved that the area of a black hole always increases

$$\Delta A \geq 0$$

So the entropy of the black hole should be proportional to the area. Hawking found the proportionality constant to be $\frac{1}{4}$.

In thermodynamics, entropy is extensive so it scales with volume, but in this case it scales with the area. The property of being extensive is related to degrees of freedoms of local systems so it is an important property. The concept of holography is that the degrees of freedom of a local quantum gravity are located on the boundary of the black hole, not its bulk.

Black holes have a lot of entropy. In statistical mechanics, entropy counts the fundamental different degrees of freedoms

$$S = \log \rho_{\mathrm{micro}}$$

For example, Sagittarius A* the entropy is

$$S_{\mathrm{BH}} = 2.69 \times 10^{67}\,\mathrm{J\,K^{-1}}$$

which is about 10^{20} the entropy of the sun.

A success of string theory is that for supersymmetric five-dimensional black holes, a count of individual degrees of freedom coincides with the formula above.

1.2 Open problems

The hierarchy problem. The mass of the Higgs boson $m_{\mathrm{H}} \approx 125\,\mathrm{GeV}$, but the Standard Model predicts infinity (for the bare value of the mass). The mass is computed from the propagator and there are loop diagrams like the self-interaction that give

$$\int \mathrm{d}^4k\,\frac{1}{k^2 - m^2} \sim k_{\mathrm{max}}^2$$

The mass correction to the mass is $\delta m_{\mathrm{H}}^2 \sim \# k_{\mathrm{max}}^2$. However $\delta m_{\mathrm{H}} \gg m_{\mathrm{H}}$. The cancellation of the bare mass and true mass happens to many digits. This problem is related to fine tuning and naturalness problem.

This is related to quantum gravity because there may be not new physics up to the quantum gravity energy scale and QG has to solve such hierarchy problem. If $k_{\max} \sim M_p$ then one has a fine-tuning of 10^{17} digits.

There are many resolutions within quantum field theory like supersymmetry.

The cosmological constant problem. In quantum field theory the energy of the vacuum is not very much important, but in general relativity energy bends space-time. The energy density of the vacuum is

$$\begin{aligned}\rho_{\text{vac}} &= \langle 0| \rho |0\rangle = \langle 0| T_{00} |0\rangle = \frac{1}{2}\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\sqrt{k^2+m^2} \\ &= \frac{1}{4\pi^2}\int^{k_{\max}} \mathrm{d}k\, k^2\sqrt{k^2+m^2} = \frac{k_{\max}^4}{16\pi^2}\left[1+\frac{m^2}{k_{\max}}+\cdots\right]\end{aligned}$$

The observed energy density is

$$\Lambda_{\text{eff}} = \Lambda_{\text{bare}} + 8\pi G_N \rho_{\text{vac}}\,, \quad \rho_{\text{obs}} = \frac{\Lambda_{\text{eff}}}{8\pi G_N} \approx 10^{-47}\,\text{GeV}^4$$

At the Planck's scale one has

$$\rho_{\text{vac}} \sim 10^{71}\,\text{GeV}^4$$

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This argument is missing a part. The pressure density is

$$\langle 0| p_{\text{vac}} |0\rangle = \langle 0| T_{0i} |0\rangle \implies \frac{\rho_{\text{vac}}}{p_{\text{vac}}} = \frac{1}{3} \neq -1$$

The energy density is not -1 so it cannot be interpreted as the cosmological constant. The UV cutoff regularization breaks Lorentz invariance.

Proper argument. The energy density of the vacuum is proportional to the vacuum bubble diagram. To preserve Lorentz invariance one may use dimensional regularization:

$$\rho_{\text{vac}} = \frac{m^4}{64\pi^2}\ln\frac{m^2}{4\pi\mu^2}$$

The quartic divergence appears in the mass, while the divergence in the momentum μ is only logarithmic. In quantum field theory one does not care about the vacuum energy because all disconnected diagrams are cancelled. Computing in this way the pressure, the ratio is -1. So observed energy density is

$$\rho_{\text{obs}} = \rho_{\text{bare}} + \rho_{\text{vac}}$$

for the tau lepton mass one has $\rho_{\text{obs}} \sim 10^{-47}\,\text{GeV}^4$ and $\rho_{\text{vac}} \sim 10^8\,\text{GeV}^4$; so there are still 40 orders of magnitude.

Resolution in AdS/CFT. Starting from the Einstein–Hilbert action

$$S = \frac{1}{16\pi G_N}\int \mathrm{d}^D x\,\sqrt{-g}(R-2\Lambda) = \frac{1}{16\pi G_N}\int \mathrm{d}^D x\,\sqrt{-g}\left[R + \frac{(D-1)(D-2)}{l_{\text{AdS}}^2}\right]$$

where l_{AdS} is a length scale of the cosmological constant. There are two length scales

$$l_p \sim G_N^{\frac{1}{D-2}}\,, \quad l_{\text{AdS}}$$

The effective field theory suggests that the Planck scale is the same as the AdS. [r] However gravity is semi-classical if the curvature of AdS is very little and as such the length scale is very large: this happens for $l_{\mathrm{AdS}} \gg l_p$. This is the cosmological constant problem, the separation of the scale is not natural from the point of view of effective field theory.

AdS/CFT is a dictionary and one of its entries is

$$\frac{l_{\mathrm{AdS}}^{D-2}}{G_N} = C_T$$

where C_T is the central charge. In conformal field theory [r]

$$\langle T_{\mu\nu}(x_1) T_{\rho\sigma}(x_2) \rangle = C_T I_{\mu\nu\rho\sigma}(x_1, x_2)$$

The second factor is fixed by CFT symmetry up to one number. The central charge should be $C_T \gg 1$ to get semi-classical gravity. In CFT, the central charge counts the number of degrees of freedom. Therefore, to solve the cosmological problem one has to consider CFT with many degrees of freedom.

2 Black hole thermodynamics

Black holes carry an entropy proportional to the event horizon

$$S_{\mathrm{BH}} = \frac{A_{\mathrm{H}}}{4G_N}$$

Black holes also have temperature. The numerical factor was found by Hawking by calculating the temperature of a black hole.

There are two approaches to study the thermodynamics of black holes:

- euclidean methods, which involve thermal physics;
- QFT on a curved background.

2.1 Euclidean quantum field theory

The partition function is

$$Z(\beta) = \operatorname{Tr} \mathrm{e}^{-\beta H} \qquad \beta = \frac{1}{T}$$

In thermal physics everything can be obtained from it. The entropy is

$$S = -\partial_t (T \log Z)$$

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One would like to find the connection with path integrals. In ordinary quantum mechanics, a Schrödinger state evolves through the Lorentzian time evolution operator

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht} |\psi(0)\rangle$$

This operator is unitary. The Euclidean time τ is given by $t = -\mathrm{i}\tau$. The state is then

$$|\psi(t)\rangle = \mathrm{e}^{-\tau H} |\psi(0)\rangle$$

The Euclidean time evolution operator is not unitary. A Euclidean path integral is a transition amplitude

$$\langle \phi_2 | \mathrm{e}^{-\tau H} | \phi_1 \rangle = \int_{\phi_1}^{\phi_2} [\mathcal{D}\phi]\, \mathrm{e}^{-S_{\mathrm{E}}}\,, \quad \phi_1 = \phi(\tau = 0)\,, \quad \phi_2 = \phi(\tau)$$

One integrates all field configurations with some boundary conditions weighed by some the Euclidean action.

One can also use path integrals to prepare states. Consider a quantum state

$$|\psi\rangle = \mathrm{e}^{-\beta H} |\phi_1\rangle$$

It has a wave-functional

$$\Psi(\phi) = \langle \phi | \mathrm{e}^{-\beta H} | \phi_1 \rangle$$

[r] The state ϕ is not fixed, but depends on the specific boundary conditions. In terms of the path integral one has

$$\int_{\phi_1}^{\phi(\beta)} [\mathcal{D}\phi]\, \mathrm{e}^{-S_{\mathrm{E}}}$$

Ground state. Consideri a generic state

$$|\psi\rangle \sum_n c_n |n\rangle \ , \quad H|n\rangle = E_n |n\rangle$$

The evolved state is

$$\mathrm{e}^{-\beta H}|\psi\rangle = \sum_n c_n \mathrm{e}^{-\beta E_n}|n\rangle$$

Taking $\beta \to \infty$, one can extract the vacuum (which means $T \to 0$ where T is the temperature). The vacuum is given by the Euclidean path integral with open boundary conditions at $\tau = 0$ and path integrating from $-\infty$; this is half of $\mathbb{R}^d$. A bra can be prepared from ∞ to zero. The vacuum normalization is just a path integral form $-\infty$ to ∞. The boundary conditions get integrated, it is similar to inserting a complete set of states

$$\langle 0|0\rangle = \sum_{\phi_1} \langle 0|\phi_1\rangle \langle \phi_1|0\rangle$$

This is the starting point to generate everything else.

Correlation functions. A two-point correlation function in the vacuum is

$$\langle 0|\, \phi(x_1)\phi(x_2)\, |0\rangle = \int_{\mathbb{R}^d} [\mathcal{D}\phi]\, \mathrm{e}^{-S_\mathrm{E}} \phi(x_1)\phi(x_2)$$

One prepares the vacuum and inserts two operators at two specific points. In Euclidean space there is no Hilbert space because one uses fields and computes functions in Euclidean coordinates.

Thermal physics. One would like to compute the partition function

$$Z = \mathrm{Tr}\, \mathrm{e}^{-\beta H} = \sum_{\phi_1} \langle \phi_1|\, \mathrm{e}^{-\beta H}\, |\phi_1\rangle$$

The sum over same boundary conditions glues them together, so one obtains a cylinder with circumference: periodic boundary conditions. Thermal physics corresponds to periodicity in Euclidean time. The period is $\beta = \frac{1}{T}$ where T is the temperature.

[r] The Green function is

$$G(\tau, x) = \langle O(\tau, x)O(0,0)|_\beta = \mathrm{Tr}\big[\mathrm{e}^{-\beta H} O(\tau, x)O(0,0)\big] = \mathrm{Tr}\big[O(0,0)\mathrm{e}^{-\beta H} O(\tau, x)\big]$$

Knowing that

$$O(\beta, 0) = \mathrm{e}^{-\beta H} O(0,0) \mathrm{e}^{\beta H}$$

one has

$$G(\tau, x) = \mathrm{Tr}\big[\mathrm{e}^{-\beta H} O(\beta, 0) O(\tau, x)\big] = G(\tau - \beta, x)$$

This is periodicity.

The path integral with open boundary conditions on one side and closed on the other is a state. If both are open then it is a density matrix of the thermal state

$$\rho = \mathrm{e}^{-\beta H}$$

The amplitude is given by definite boundary conditions

$$\langle \phi_2|\, \rho\, |\phi_1\rangle$$

Density matrix. An entangled state

$$|EPR\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) \neq |\psi\rangle_1 \otimes |\psi\rangle_2$$

cannot be written as two separate qubit states. [r] Tracing over one state one has

$$\langle \uparrow|_2\, |EPR\rangle\langle EPR|\, |\uparrow\rangle_2 + \langle \downarrow|_2\, |EPR\rangle\langle EPR|\, |\downarrow\rangle_2 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

It is a density matrix.

2.2 Rindler space and polar coordinates

The metric of two-dimensional Rindler space

$$\mathrm{d}s^2 = \mathrm{d}R^2 - R^2\,\mathrm{d}\eta^2\,, \quad R > 0\,, \quad \eta \in \mathbb{R}$$

The horizon of the space is $R = 0$ where the metric is singular. Setting $\eta = \mathrm{i}\theta$, the metric becomes

$$\mathrm{d}s^2 = \mathrm{d}R^2 + R^2\,\mathrm{d}\theta^2$$

that is polar coordinates. Rindler space is the Lorentzian equivalent of polar coordinates. Using the change of coordinates

$$x = R\cosh\eta\,, \quad t = R\sinh\eta$$

one obtains flat space

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2$$

One has

$$x^2 - t^2 = R^2 > 0$$

so on the Penrose diagram one occupies only the right Rindler patch. In higher dimension, the Rindler space is

$$\mathrm{d}s^2 = \mathrm{d}R^2 - R^2\,\mathrm{d}\eta^2 + \sum_{i=2}^{d} \left|\mathrm{d}x^i\right|^2$$

The trajectories of fixed R are not geodesics, but are trajectories of constant acceleration.

In Minkowski space, the Hamiltonian generates time translation

$$H = \int_{\mathbb{R}} \mathrm{d}x\, T_{tt}$$

In Rindler space, the η-time translation is a boost

$$H_R = \int_{\mathbb{R}} \mathrm{d}x\, x^2 T_{tt}$$

Euclidean path integral. The overlap of the vacuum with itself is the path integral from $-\infty$ to ∞ generated by $\mathrm{e}^{-\beta H}$ for $\beta \to \infty$. Instead of opening the path integral and inserting a configuration, one opens the path integral as a pizza slice and uses a time evolution operator that gives rotations not translations

$$\mathrm{e}^{-2\pi\,\partial_\phi}\,, \quad \eta = \mathrm{i}\phi$$

A rotation is a Euclidean-Rindler time translation.

One particular matrix element of the Rindler Hamiltonian is

$$\langle\phi_2|\,\mathrm{e}^{-2\pi H_{\mathrm{Rindler}}}\,|\phi_1\rangle$$

To compute the path integral over all of space, one has to glue together the two states so sum over the states

$$\langle 0|0\rangle = \sum_{\phi_1} \langle\phi_1|\,\mathrm{e}^{-2\pi H_{\mathrm{Rindler}}}\,|\phi_1\rangle = \mathrm{Tr}\,\mathrm{e}^{-2\pi H_{\mathrm{Rindler}}} = Z_{\mathrm{Rindler}}$$

This is the partition function with temperature $\beta = 2\pi$. The vacuum state can be interpreted as a thermal state of Rindler space of temperature $T = \frac{1}{2\pi}$. The Rindler space is half of Minkowski space, the other half is not know and it is a temperature mass [r]. The Rindler horizon is not very different from the black hole's.