Quantum Field Theory II

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Lecture 1

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Topics. Generalize the formalism for complex scalar fields. Introduce fermions and their functional quantization. Discrete symmetries and PCT theorem. Gauge theories and their functional quantization. Anomalies. Possibly instantons, applications of RG flows, etc.

1 Complex scalar boson fields

[r] sources?

1.1 Generalization

Consider a complex scalar field φ. It has two degrees of freedom: either the fields φ and φ^* or the real and imaginary parts of the field φ. Both area useful for computation

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + \mathrm{i}\varphi_2)\,, \quad \varphi^* = \frac{1}{\sqrt{2}}(\varphi_1 - \mathrm{i}\varphi_2)$$

The action of the $\lambda\varphi^4$ theory is

$$\begin{aligned} S &= \int \mathrm{d}^4x\, [\partial_\mu\varphi\, \partial^\mu\varphi^* - m^2\varphi\varphi^* - \lambda(\varphi\varphi^*)^2] \\ &= \int \mathrm{d}^4x\, \left[\frac{1}{2}\,\partial_\mu\varphi_1\,\partial^\mu\varphi_1 - \frac{1}{2}m^2\varphi_1^2 + \frac{1}{2}\,\partial_\mu\varphi_2\,\partial^\mu\varphi_2 - \frac{1}{2}m^2\varphi_2^2 - \frac{\lambda}{4}(\varphi_1^2+\varphi_2^2)^2\right] \end{aligned}$$

Each degree of freedom has an equation of motion. Varying the action with respect to φ gives the equations of motion for φ^* and vice versa. For $\lambda = 0$, the fields components satisfy the Klein–Gordon equation.

Since the interaction term is real, it corresponds to a quartic vertex $\varphi\varphi^*\varphi\varphi^*$. For $\lambda = 0$, one can rewrite the action in terms of real φ_1 and imaginary φ_2 parts and see that the two degrees of freedom decouple since there is no mixed term $\varphi_1^2\varphi_2^2$.

*https://github.com/M-a-s-o/notes

With two degrees of freedom, the Euclidean generating functional depends on two source terms

$$W[J, J^*] = \int [\mathcal{D}\varphi \, \mathcal{D}\varphi^*] \, \exp\left[- \int \mathrm{d}^4 x \, [\mathcal{L}(\varphi, \varphi^*) - J\varphi - J^*\varphi^*] \right]$$
$$W[J_1, J_2] = \int [\mathcal{D}\varphi_1 \, \mathcal{D}\varphi_2] \, \exp\left[- \int \mathrm{d}^4 x \, [\mathcal{L}(\varphi_1, \varphi_2) - J_1\varphi_1 - J_2\varphi_2] \right]$$

The Euclidean Green's functions are

$$\langle 0 | \, \mathcal{T}\{\varphi(x_1) \cdots \varphi(x_l) \varphi^*(x_{l+1}) \cdots \varphi^*(x_n)\} \, |0\rangle = \\ = \left. \frac{\delta^n W[J, J^*]}{\delta J(x_1) \cdots \delta J(x_l) \, \delta J^*(x_{l+1}) \cdots \delta J^*(x_n)} \right|_{J = J^* = 0}$$

Similarly

$$\langle 0 | \, \mathcal{T}\{\varphi_1(x_1) \cdots \varphi_1(x_l) \varphi_2(x_{l+1}) \cdots \varphi_2(x_n)\} \, |0\rangle = \\ = \left. \frac{\delta^n W[J_1, J_2]}{\delta J_1(x_1) \cdots \delta J_1(x_l) \, \delta J_2(x_{l+1}) \cdots \delta J_2(x_n)} \right|_{J_1 = J_2 = 0}$$

Free theory. For a free real scalar theory, i.e. setting $\lambda = 0$, one can compute the free propagator exactly. In this case the situation is slightly different? [r]. In the complex fields formulation, one has to compute a complex Gaussian integral of the form

$$\int \left[\prod_{j=1}^n \mathrm{d}z_j \, \mathrm{d}z_j^* \right] \mathrm{e}^{-z^* A z} = \frac{\pi^n}{\det A}$$

where $z \equiv (z_1, \ldots, z_n)$ and A is a square matrix of dimension n (with hermitian part positive-definite). The Euclidean generating functional is then

$$\begin{aligned} W[J_1, J_2] &= \int [\mathcal{D}\varphi_1 \, \mathcal{D}\varphi_2] \, \mathrm{e}^{-S_0[\varphi_1] - S_0[\varphi_2]} \exp\left[\int \mathrm{d}^4 x \, (J_1\varphi_1 + J_2\varphi_2) \right] \\ &= \int [\mathcal{D}\varphi_1] \, \mathrm{e}^{-S[\varphi_1] + \int \mathrm{d}^4 x \, J_1\varphi_1} \int [\mathcal{D}\varphi_2] \, \mathrm{e}^{-S[\varphi_2] + \int \mathrm{d}^4 x \, J_2\varphi_2} \\ &= W_1[J_1] W_2[J_2] \end{aligned}$$

The theory factorizes.

The Lagrangian above has a U(1) symmetry of the fields

$$\varphi' = \mathrm{e}^{\mathrm{i}\alpha}\varphi \,, \quad \varphi'^* = \mathrm{e}^{-\mathrm{i}\alpha}\varphi^* \,, \quad \alpha \in \mathbb{R}$$

Equivalently, the Lagrangian is invariant under SO(2) of the components φ_1 and φ_2.

Exercise. Write the most general SO(2) transformation.

Exercise. See Peskin, Problem 12.3, p. 428. The interaction in the action $S[\varphi_1, \varphi_2]$ can be split into self-interaction and coupling between the fields. This form can be generalized

$$\mathcal{L} = \frac{1}{2} \, \partial_\mu\varphi_1 \, \partial^\mu\varphi_1 + \frac{1}{2} \, \partial_\mu\varphi_2 \, \partial^\mu\varphi_2 - \frac{\lambda}{4!}(\varphi_1^4 + \varphi_2^4) - \frac{2\rho}{4!}\varphi_1^2\varphi_2^2$$

Do the following:

- Write the Euclidean Feynman rules.
- Compute all one-loop corrections to the λ-vertex and the ρ-vertex in dimensional regularization $D = 4 - 2\varepsilon$, use massless integrals (discussed in the following).

- Impose the normalization conditions at $s = t = u = \Lambda^2$ such that

$$\Gamma^{(4)}(s = t = u = \Lambda^2) = -\lambda\,, \quad \Gamma^{(4)}_{\text{mixed}}(s = t = u = \Lambda^2) = -2\rho$$

  [r] minus?
  Notice that this implies that there are no one-loop finite corrections to the couplings.

- Compute the beta functions β_λ, β_ρ and

$$\beta_{\frac{\lambda}{\rho}} = \mathrm{d}_t \frac{\lambda}{\rho} = \frac{1}{\rho}\,\mathrm{d}_t\lambda - \frac{1}{\rho^2}\lambda\,\mathrm{d}_t\rho = \frac{1}{\rho}\beta_\lambda - \frac{\lambda}{\rho^2}\beta_\rho$$

- Find the fixed points and describe the RG flow.

- Explain what happens for

$$\frac{1}{3} < \frac{\lambda}{\rho} < 1$$

  and for $\lambda = \rho$.

1.2 Massless integrals in dimensional regularization

Proposition. In dimensional regularization, it holds

$$I_\alpha = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{(p^2)^\alpha} = 0\,, \quad \alpha \in \mathbb{C}$$

Notice that it is null also for $\alpha = 0$. This integral describes tadpoles for $\alpha = 1$: the two-point Green's function receives no correction at one-loop if the fields are massless.

The intuitive argument of why this is true can be seen for $D \neq 2\alpha$ where the integral is dimensionful: the result has to be dimensionful as well and must be written in terms of dimensionful parameters, however the integral does not depend on any parameter and so the result must be zero.

Proof. Consider the theorem below and the Euclidean formalism. One may rewrite the integrand

$$\frac{1}{(p^2)^\alpha} = \frac{1}{(p^2)^\alpha}\frac{p^2 + m^2}{p^2 + m^2} = \frac{1}{(p^2)^\alpha}\frac{m^2}{p^2 + m^2} + \frac{1}{(p^2)^{\alpha - 1}}\frac{1}{p^2 + m^2}$$

The region of convergence for the integral

$$\int \mathrm{d}^D p \, \frac{1}{(p^2)^\alpha}\frac{m^2}{p^2 + m^2}$$

is given by the following limits. At infinity, one has

$$|p| \to \infty\,, \quad I \sim \frac{1}{p^{2\alpha + 2 - D}}$$

There is no singularity for $2\alpha + 2 - \operatorname{Re} D > 0$. At the origin, one has

$$|p| \to 0\,, \quad I \sim p^{D - 2\alpha}$$

There is no singularity for $\operatorname{Re} D - 2\alpha > 0$. The integral of the first addendum is well-defined for

$$2\alpha < \operatorname{Re} D < 2\alpha + 2$$

Similarly, the region of convergence of the integral of the second addendum is

$$2\alpha - 2 < \operatorname{Re} D < 2\alpha$$

The two regions do not overlap and, by the theorem below, one has

$$I_\alpha = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{m^2}{(p^2)^\alpha (p^2+m^2)} + \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{(p^2)^{\alpha-1}(p^2+m^2)} \equiv I_\alpha^{(1)} + I_\alpha^{(2)}$$

Looking at tables of integrals [r], one has a general formula

$$\int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{(p^2)^a (p^2+m^2)^b} = (m^2)^{\frac{D}{2}-a-b} \frac{\Gamma({}^D\!/\!{}_2 - a)\Gamma(a+b-{}^D\!/\!{}_2)}{(4\pi)^{\frac{D}{2}}\Gamma(b)\Gamma({}^D\!/\!{}_2)}$$

For the first integral, one has $a = \alpha$, $b = 1$ and for the second $a = \alpha - 1$ and $b = 1$. Therefore

$$\begin{aligned} I_\alpha &= m^{D-2\alpha} \frac{\Gamma({}^D\!/\!{}_2 - \alpha)\Gamma(1+\alpha-{}^D\!/\!{}_2)}{(4\pi)^{\frac{D}{2}}\Gamma({}^D\!/\!{}_2)} + m^{D-2\alpha} \frac{\Gamma({}^D\!/\!{}_2 + 1 - \alpha)\Gamma(\alpha-{}^D\!/\!{}_2)}{(4\pi)^{\frac{D}{2}}\Gamma({}^D\!/\!{}_2)} \\ &= \frac{m^{D-2\alpha}}{(4\pi)^{\frac{D}{2}}\Gamma({}^D\!/\!{}_2)} \Gamma(\alpha - {}^D\!/\!{}_2)\Gamma({}^D\!/\!{}_2 - \alpha)\left[\alpha - \frac{D}{2} + \frac{D}{2} - \alpha\right] = 0 \end{aligned}$$

where one applies $\Gamma(z+1) = z\Gamma(z)$ to the Gamma function with three addenda in its argument. □

Theorem. If an integrand can be written as a sum of terms with non-overlapping regions of convergence (for their integrals), then the integral is a sum of their integrals.

One a-posteriori argument to see why the integral is null follows. One may rescale $p = sp'$ to have

$$I_\alpha = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{(p^2)^\alpha} = s^{D-2\alpha} \int \frac{\mathrm{d}^D p'}{(2\pi)^D} \frac{1}{(p'^2)^\alpha} \implies I_\alpha = s^{D-2\alpha} I_\alpha$$

For $D \neq 2\alpha$ then $I_\alpha = 0$. For the case $D = 2\alpha$, one can argue that there is a fine-tuning between the ultraviolet and the infrared divergences. In this case, there divergences are both present and give zero when summed. Consider the particular example of $D = 2 - \varepsilon$ and $\alpha = 1$. Then, the integral becomes

$$\begin{aligned} I_1 &= \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{p^2} = -\int \frac{\mathrm{d}^D p}{(2\pi)^D} \int_0^\infty \mathrm{d}\lambda \, \frac{1}{(p^2-\lambda)^2} = -\int_0^\infty \mathrm{d}\lambda \int \frac{\mathrm{d}^{2-\varepsilon} p}{(2\pi)^{2-\varepsilon}} \frac{1}{(p^2-\lambda)^2} \\ &= -\int_0^\infty \mathrm{d}\lambda \, \frac{\Gamma(1+{}^\varepsilon\!/\!{}_2)}{(4\pi)^{1-\frac{\varepsilon}{2}}\Gamma(2)} (-\lambda)^{-\frac{\varepsilon}{2}-1} = (-1)^{\frac{\varepsilon}{2}} \frac{\Gamma(1+{}^\varepsilon\!/\!{}_2)}{(4\pi)^{1-\frac{\varepsilon}{2}}} \int_0^\infty \frac{\mathrm{d}\lambda}{\lambda^{1+\frac{\varepsilon}{2}}} \end{aligned}$$

Ignoring the coefficients, the integral is

$$\begin{aligned} I_1 &\propto \int_{a_{\mathrm{IR}}}^1 \frac{\mathrm{d}\lambda}{\lambda^{1+\frac{\varepsilon}{2}}} + \int_1^{a_{\mathrm{UV}}} \frac{\mathrm{d}\lambda}{\lambda^{1+\frac{\varepsilon}{2}}} = -\frac{2}{\varepsilon}\lambda^{-\frac{\varepsilon}{2}}\bigg|_{a_{\mathrm{IR}}}^1 - \frac{2}{\varepsilon}\lambda^{-\frac{\varepsilon}{2}}\bigg|_1^{a_{\mathrm{UV}}} = -\frac{2}{\varepsilon}[-a_{\mathrm{IR}}^{-\frac{\varepsilon}{2}} + a_{\mathrm{UV}}^{-\frac{\varepsilon}{2}}] \\ &= -\frac{2}{\varepsilon}\left[-\frac{\varepsilon}{2}(-\ln a_{\mathrm{IR}} + \ln a_{\mathrm{UV}}) + o(\varepsilon)\right] = \ln a_{\mathrm{UV}} - \ln a_{\mathrm{IR}} + o(\varepsilon^0) = 0 \end{aligned}$$

At the second line, one has integrated up to a cutoff [r]. From this one sees that $a_{\mathrm{IR}} = a_{\mathrm{UV}}$ and $I_\alpha = 0$ is the product of a cancellation between the ultraviolet and infrared divergences.

When dealing with massless theories in two dimensions $D = 2$, the cancellation of the tadpole is due to a balance between infrared and ultraviolet divergences. When interested in either divergence, one has to remove the other divergence in order to renormalize the one of interest. For example, one replaces

$$I_1 = \int \frac{\mathrm{d}^{2-\varepsilon} p}{(2\pi)^{2-\varepsilon}} \frac{1}{p^2} \to \int \frac{\mathrm{d}^{2-\varepsilon} p}{(2\pi)^{2-\varepsilon}} \frac{1}{p^2+\mu^2} = \frac{\Gamma({}^\varepsilon\!/\!{}_2)}{(4\pi)^{1-\frac{\varepsilon}{2}}}\mu^{-\varepsilon} \sim \frac{2}{\varepsilon}\mathrm{e}^{-\varepsilon\ln\mu+\cdots} \sim \frac{2}{\varepsilon}, \quad \varepsilon \to 0$$

where μ^2 is an infrared regulator. This is the ultraviolet divergence of the tadpole in two dimensions.

Therefore, in dimensional regularization, massless tadpoles can be ignored when $D \neq 2$. In $D = 2$ the tadpole is dimensionless and gives a contribution when removing one divergence.

Part I
Spin-half fermion fields

Review – classical fields. See Srednicki. A fermion field is a field describing Dirac spinors. A Dirac spinor in the Weyl basis is comprised of two fixed-chirality Weyl spinors

$$\psi = \begin{bmatrix} \chi_\mathrm{L} \\ \chi_\mathrm{R} \end{bmatrix}$$

where the left-chiral spinor belongs to the $(\frac{1}{2}, 0)$ representation of the Lorentz group $\mathrm{SO}(1,3)$, while the right-chiral spinor belongs to $(0, \frac{1}{2})$.

The equation of motion of the Dirac field is the Dirac equation

$$(\mathrm{i}\,\partial\!\!\!/ - m)\psi(x) = 0$$

where the Dirac matrices are four square matrices of dimension four defining the Dirac algebra $\mathrm{Cl}_{1,3}(\mathbb{C})$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

The equation of motion can be obtained from the action principle with the Lagrangian

$$\mathcal{L} = \mathrm{i}\bar{\psi}\,\partial\!\!\!/\psi - m\bar{\psi}\psi = \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi\,, \quad \bar{\psi} = \psi^\dagger\gamma^0$$

If the field ψ satisfies the Dirac equation of motion, then each of its four components (each two of the Weyl spinors) satisfies the Klein–Gordon equation

$$(\Box + m^2)\psi_j(x) = 0$$

In momentum space, the above is an algebraic equation that gives the dispersion relation

$$(-p^2 + m^2)\psi(p) = 0 \implies p^2 = m^2 \iff p^0 = \sqrt{|\mathbf{p}|^2 + m^2} \equiv \omega$$

The most general solution of the Klein–Gordon equation has the form[1]

$$\psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[u(\mathbf{p})\mathrm{e}^{-\mathrm{i}px} + v(\mathbf{p})\mathrm{e}^{\mathrm{i}px}]_{p^0=\omega}$$

Imposing the Dirac equation one finds

$$(p\!\!\!/ - m)u(\mathbf{p}) = 0\,, \quad (p\!\!\!/ + m)v(\mathbf{p}) = 0$$

One solves these equation by going to the rest frame $\mathbf{p} = 0$ for which

$$p\!\!\!/ = \gamma^0 p_0 = \gamma^0 m \implies (\gamma^0 - 1)u(0) = 0\,, \quad (\gamma^0 + 1)v(0) = 0$$

Each equation has two independent solutions $u_\pm$ and $v_\pm$. To get the solution for an arbitrary momentum one has to perform a boost. Therefore, the general solution to the Dirac equation is

$$\psi(x) = \sum_{s=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[b_s(\mathbf{p})u_s(\mathbf{p})\mathrm{e}^{-\mathrm{i}px} + d_s^\dagger v_s(\mathbf{p})\mathrm{e}^{\mathrm{i}px}]_{p^0=\omega}$$

where b and d are numbers.

The two-spinors u_s and v_s satisfy several identities (see Peskin, p.48). The normalization is chosen to be

$$\bar{u}_r(\mathbf{p})u_s(\mathbf{p}) = 2m\delta_{rs}\,, \quad \bar{v}_r(\mathbf{p})v_s(\mathbf{p}) = -2m\delta_{rs}$$

Equivalent to

$$u_r^\dagger(\mathbf{p})u_s(\mathbf{p}) = 2p^0\delta_{rs}\,, \quad v_r^\dagger(\mathbf{p})v_s(\mathbf{p}) = 2p^0\delta_{rs}$$

[1] Understood as a group of four components that solve the equation.

The spinors are orthogonal

$$\bar{u}_r(\mathbf{p})v_s(\mathbf{p}) = \bar{v}_r(\mathbf{p})u_s(\mathbf{p}) = 0\,, \quad u_r^\dagger(\mathbf{p})v_s(-\mathbf{p}) = v_r^\dagger(-\mathbf{p})u_s(\mathbf{p}) = 0$$

The last two relations are not zero for both momenta being $+\mathbf{p}$. With these, one can obtain

$$b_s(\mathbf{p}) = \int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}px} u_s^\dagger(\mathbf{p})\psi(x)\,, \quad d_s(\mathbf{p}) = \int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}px} \psi^\dagger(x)v_s(\mathbf{p})$$

It is sufficient to Fourier transform $u_s^\dagger(\mathbf{p})\psi(x)$, substitute $\psi(x)$ and apply the above rules. Notice that

$$\int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}x(p-q)} = \mathrm{e}^{\mathrm{i}x^0(p^0-q^0)}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})$$