# Theory and Phenomenology of Fundamental Interactions

Maso*

March 16, 2024

## Contents



### Lecture 1

lun 04 mar 2024 12:30

The course will make more sense after taking QFT 1 and 2.

**Exam.** Computation of tree-level amplitude. [r] The oral part is comprised of 3 questions discussing theoretical aspects and fundamental aspects of the theory.

## 1 Introduction

The first part of the course is about the completion of the theoretical description of the electroweak sector of the Standard Model. Then Yukawa interaction linked to the [r] matrix: weak interactions are not diagonal with respect to flavor families. [r] Then discussion of phenomenology of the electroweak and Higgs sectors at the LHC.

The second part treats QCD, its gauge invariance; perturbative regime, the subtleties, universal divergences, properties of non-abelian gauge theories.

The last part deals with hadronic collision and its non-trivial description.

Further topics are neutrino masses, etc.

### 1.1 The bigger picture

The course deals with the Standard Model. One needs to understand how it must be viewed from a historical perspective and a modern perspective. To study fundamental interactions (excluding gravity) the popolar choice is quantum field theory: quantum electrodynamics, electroweak theory, quantum chromodynamics. The first is an abelian gauge theory, while the last two are non-abelian.

In these theories, the Lagrangian density is the fundamental object: it is Lorentz invariant, invariant under Poincaré transformation, and describes quantized fields. The particles are excitations of the fields. Every Lagrangian can be divided in the free term and the interaction term

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\mathrm{I}$$

*https://github.com/M-a-s-o/notes

The free Lagrangian contains kinetic terms, the propagators [r]. The interaction Lagrangian contains the interaction terms, which are represented as vertices in Feynman diagrams. The quantum electrodynamics Lagrangian is

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(\mathrm{i}\not{\partial} - m)\psi - eQ\bar{\psi}\gamma^\mu\psi A_\mu$$

where $e$ is the elementary charge and $Q$ is the quantum number of charge.

If one assumes that fields and interactions to be truly fundamental:

- then the theory has to be unitary and predictive at all energies (so predict final results and amplitudes that do not exceed 1).
- The non-trivial link to quantum field theory is that the theory must be renormalizable. The lagrangian must be constrained: the coupling constants cannot have negative mass dimension. Since fundamental interactions are well described by gauge theories, gauge invariance is a fundamental requirement of the renormalizability of the theory. [r] Once one picks a gauge group, it must produce observable phenomena and one must keep abiding its rules to keep gauge invariance.

**Effective Standard Model.** The Standard Model [r] but it is not a complete theory. There is no natural candidate for dark matter, gravity is not accounted for, neutrino masses are not explained. The Model has to fail at some point, at some energy scale where a phenomenon cannot be described with the field content of the Model. Therefore, the Standard Model must be an effective theory (as opposed to a fundamental theory). Therefore, it is allowed to add non-renormalizable operators: terms in the Lagrangian with negative mass dimension. If one wants to understand the physics beyond the Standard Model from a bottom-up approach, this is a middle ground: one modifies the theory enough to compute phenomena [r]. A similar story happened when going from Fermi's four-interaction theory to the intermedia vector boson theory. The interaction term for Fermi theory is $\bar{\psi}\psi\bar{\psi}\psi$ with dimension six, so the coupling constant must have mass dimension $-2$. [r] One may add non-renormalizable operators built from Standard Model objects that respect its symmetry group. In this paradigm one may not use renormalizability and unitarity, so the predictions are valid up to some energies.

## 1.2 Weyl spinors

A massless Dirac field is made of two Weyl field. A massive Dirac spinor is made of left- and right-chiral components. A term like

$$\bar{\psi}_{\mathrm{L}}\psi_{\mathrm{R}}$$

is not invariant under $\mathrm{SU}(2)_{\mathrm{L}}$.

**Lorentz group.** The proper Lorentz group [r] has six generators. [r] through the exponential map as

$$R(\hat{e},\theta) = \exp(-\mathrm{i}\theta\hat{e}\cdot\mathbf{J})\,, \quad B(\hat{u},\eta) = \exp(-\mathrm{i}\eta\hat{u}\cdot\mathbf{K})$$

where $\mathbf{J}$ are the generators of rotations and $\mathbf{K}$ are the generators of boosts. The explicit form of the generators can be obtained from infinitesimal transformations. For example

$$J_z = \mathrm{i}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\,, \quad K_x = \mathrm{i}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

These are a fundamental representation of the Lorentz algebra $\mathrm{SO}(1,3)$

$$[J_i, J_j] = \mathrm{i}\varepsilon_{ijk}J_k\,, \quad [K_i, K_j] = -\mathrm{i}\varepsilon_{ijk}J_k\,, \quad [J_i, K_j] = \mathrm{i}\varepsilon_{ijk}K_k$$

The above algebra can be rewritten as

$$[M_{\mu\nu}, M_{\rho\sigma}] = \mathrm{i}(\eta_{\nu\rho}M_{\mu\sigma} + \cdots)$$

where $M^{\mu\nu}$ an anti-symmetric tensor such that

$$M^{0i} = K_i \,, \quad M^{ij} = \varepsilon_{ijk} J_k$$

[r] In order to label the representations one has to use mass and spin.

The Lorentz algebra can be decomposed in two other algebras

$$\mathrm{so}(1,3) \simeq \mathrm{su}(2) \oplus \mathrm{su}(2)$$

[r] In fact, one combines

$$J_k^\pm = \frac{1}{2}(J_k \pm \mathrm{i} K_k) \implies [J_i^+, J_j^-] = 0 \,, \quad [J_i^\pm, J_j^\pm] = \mathrm{i}\varepsilon_{ijk} J_k^\pm$$

In order to label the possible elementary fields associated to Lorentz group (in general the Poincaré group) one needs two non-negative half-integers $(s_1, s_2)$. For $(0,0)$ the transformation under each su(2) is trivial, so they do not transform, they are a singlet: it is a scalar field. The next representations are $(1/2, 0)$ and $(0, 1/2)$ for right-chiral Weyl spinor and left-chiral Weyl spinor. For a vector field one has $(1/2, 1/2)$. P 146 Schwartz

For right-chiral Weyl spinors $u_\mathrm{R}$, the generators are

$$J_i^+ = \frac{1}{2}\sigma_i \,, \quad J_i^- = 0 \implies J_i = \frac{1}{2}\sigma_i \,, \quad \mathrm{i}K_i = \frac{1}{2}\sigma_i$$

Therefore, for a rotation, one has

$$R = \exp(-\mathrm{i}\boldsymbol{\theta} \cdot \mathbf{J}) = \exp\left(-\frac{1}{2}\mathrm{i}\boldsymbol{\theta} \cdot \boldsymbol{\sigma}\right)$$

and for a boost

$$B = \exp(-\mathrm{i}\boldsymbol{\eta} \cdot \mathbf{K}) = \exp\left(-\frac{1}{2}\boldsymbol{\eta} \cdot \sigma\right)$$

Similarly for a left-chiral Weyl spinor. For a four-vector one has

$$J_i = J_i^+ + J_i^-$$

One may realize that there are two states that transforms as [r] The singlet component under rotation is $A^0$, while the triplet is $A^i$.

**Parity.** Under parity, the generators of rotations do not transform $J \to J$, while boosts do $K \to -K$. Also parity maps

$$(s_1, s_2) \to (s_2, s_1)$$

Therefore, left-chiral spinor becomes a right-chiral spinor, while a vector is still a vector.

**Weyl spinors.** Weyl spinors can be combined into a vector. Considering

$$\sigma^\mu_\pm = (I, \pm\boldsymbol{\sigma})$$

which is equivalent to the notation

$$\sigma^\mu = (I, \sigma^i) \,, \quad \bar{\sigma}^\mu = (I, -\sigma^i)$$

Therefore a vector is given by

$$u_\mathrm{R}^\dagger \sigma^\mu u_\mathrm{R} \,, \quad u_\mathrm{L}^\dagger \bar{\sigma}^\mu u_\mathrm{L}$$

These are bilinear objects in the spinor fields. One may use them to construct Lagrangians.

**Lagrangian.** One can build a Lagrangian from these fields by requiring that

$$u_{\mathrm{R,L}} \to \mathrm{e}^{\mathrm{i}\theta} u_{\mathrm{R,L}}$$

In fact, one may have

$$\mathcal{L}_{\mathrm{Weyl}} = \mathrm{i} u^{\dagger}_{\mathrm{R,L}} \sigma^{\mu}_{\pm} \partial_{\mu} u_{\mathrm{R,L}}$$

[r] The equations of motion are

$$\sigma^{\mu} \partial_{\mu} \psi_{\mathrm{R}} = 0 \,, \quad \bar{\psi}^{\mu} \partial_{\mu} \psi_{\mathrm{L}} = 0 \implies (\partial_0 \pm \boldsymbol{\sigma} \cdot \nabla) \psi_{\mathrm{R,L}} = 0$$

Acting on the last equation with $(\partial_0 \mp \boldsymbol{\sigma} \cdot \nabla)$ on the left side, one gets a massless Klein–Gordon equation

$$\Box \, \psi_{\mathrm{R,L}} = 0$$

In momentum space one has [r]

$$\psi_{\mathrm{R,L}} = \hat{\psi}_{\mathrm{R,L}}(\mathbf{k}) \mathrm{e}^{-\mathrm{i}\mathbf{k} \cdot \mathbf{x}} \,, \quad k^0 = |\mathbf{k}|$$

where the hat indicates the Fourier transform. In momentum space, the Weyl equations are

$$[k^0 \mp (\mathbf{k} \cdot \sigma)] \hat{\psi}_{\mathrm{R,L}} = 0$$

So the spinor is an eigenvector of the helicity operator

$$\frac{1}{2} \frac{\boldsymbol{\sigma} \cdot \mathbf{k}}{|\mathbf{k}|}$$

[r] from this, right-chiral spinors have positive helicity and similar.

## Lecture 2

Weyl equations are important because electroweak theory violates parity: left- and right-chiralities transform differently. mar 05 mar 2024 12:30

If one relaxes the global U(1) symmetry, one can add a Lorentz scalar to the Lagrangian. Composing two left-chiral representations

$$\left(\frac{1}{2}, 0\right) \otimes \left(\frac{1}{2}, 0\right) = (0, 0) \oplus (1, 0)$$

One has a part that transforms as a vector under SU(2) and one as a scalar. The singlet combination can be extracted as

$$\varepsilon_{ab} u^a_{\pm} u^b_{\pm} \,, \quad \varepsilon_{ab} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

where each spinor has two complex entries. The above combination is Lorentz invariant. When checking that it is Lorentz invariant, the Levi–Civita tensor gives the determinant of some matrix, which is the exponential of the Pauli matrices, so it is 1. The combination is not zero because the spinor product is not symmetric: the spinor components are Grassmann odd fields

$$\{u^a_{\pm}, u^b_{\pm}\} = 0$$

Such a term in a Lagrangian is allowed

$$\mathcal{L}^{\pm}_{\mathrm{Weyl}} = \mathrm{i} u^{\dagger}_{\pm} \sigma^{\mu}_{\pm} \partial_{\mu} u_{\pm} - \frac{1}{2} m [\varepsilon_{ab} u^a_{\pm} u^b_{\pm} + \mathrm{h.c.}]$$

This is a Majorana mass term, it is bilinear in the fields. This term is not invariant under U(1) of the spinors. If one would like to implement a global or local transformation such the previous cannot have a mass term like the one above. [r] For charged (under some symmetry group) chiral fermions, one cannot have a Majorana mass term. In QED a charged fermion transforms non trivially under $\mathrm{U}(1)_{\mathrm{EM}}$.

For a non charged particle of any symmetry of the Standard Model, such a term is allowed, like right-handed neutrinos.

## 1.3 Dirac spinors

Under parity, the right- and left-chiral representations are mapped into one another. Parity invariance means the presence of both chiralities. A Dirac spinor is a combination of Weyl spinors (in the Weyl basis)

$$\psi = \begin{bmatrix} u_- \\ u_+ \end{bmatrix} = \begin{bmatrix} \psi_\mathrm{L} \\ \psi_\mathrm{R} \end{bmatrix} , \quad \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$$

This is a direct sum of representations because the two chiral representations do not mix. In the Dirac basis, the Dirac spinor is

$$\psi = \frac{1}{\sqrt{2}} \begin{bmatrix} u_+ + u_- \\ u_+ - u_- \end{bmatrix}$$

The course uses Weyl basis (also called chiral basis). In this representation, the Dirac matrices are

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix} , \quad \gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} -I_2 & 0 \\ 0 & I_2 \end{bmatrix}$$

The projection operators are then

$$P_{\mathrm{L,R}} = \frac{1 \mp \gamma^5}{2} , \quad P_\mathrm{L} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} , \quad P_\mathrm{R} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} , \quad P_{\mathrm{L,R}}^2 = P_{\mathrm{L,R}} , \quad P_\mathrm{L} P_\mathrm{R} = P_\mathrm{R} P_\mathrm{L} = 0$$

The Dirac conjugate spinors are

$$\bar{\psi} = \psi^\dagger \gamma^0 , \quad \bar{\psi}_{\mathrm{L,R}} = \gamma_{\mathrm{L,R}}^\dagger \gamma^0 = \bar{\psi} \frac{1 \pm \gamma^5}{2}$$

notice how they have the opposite chirality of their non conjugate part. Some properties of the fifth gamma matrix are

$$\gamma_5 = \gamma_5^\dagger , \quad \gamma_5^2 = I , \quad \{\gamma_\mu, \gamma_5\} = 0 , \quad \{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$$

The free Dirac lagrangian in terms of Weyl spinors is

$$\mathcal{L}_\mathrm{Dirac} = \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi = \bar{\psi}_\mathrm{L} \mathrm{i}\,\partial\!\!\!/ \psi_\mathrm{L} + \bar{\psi}_\mathrm{R} \mathrm{i}\,\partial\!\!\!/ \psi_\mathrm{R} - m(\bar{\psi}_\mathrm{R}\psi_\mathrm{L} + \bar{\psi}_\mathrm{L}\psi_\mathrm{R})$$

This mass term is different from Majorana's. This term couples the two chiral fields. It is invariant under $\mathrm{U}(1)_\mathrm{EM}$

$$\psi \to \mathrm{e}^{\mathrm{i}\alpha}\psi$$

but it is not invariant under different transformations of the chiral fields (like $\mathrm{SU}(2)_\mathrm{L}$). In electroweak theory, one needs to [r].

**Vector theory.** A vector theory does not distinguish the chiral parts of a field. As a consequence, parity is conserved. This is the reason why the fermionic current is a vector

$$\bar{\psi}\gamma^\mu\psi = \bar{\psi}_\mathrm{L}\gamma^\mu\psi_\mathrm{L} + \bar{\psi}_\mathrm{R}\gamma^\mu\psi_\mathrm{R}$$

which is invariant under

$$\psi_{\mathrm{L,R}} \to U(x)\psi_{\mathrm{L,R}}$$

**Chiral theory.** A chiral theory treats fields differently based on their chirality. It is parity violating. A Dirac mass term is not gauge invariant for chiral theories.

For QED and QCD a Dirac mass term is allowed; only the weak sector of the Standard Model creates problems. Under $\mathrm{SU}(2)_\mathrm{L}$ a left-chiral field transforms non trivially

$$\psi_\mathrm{L} \to U(x)\psi_\mathrm{L}$$

while a right-chiral field remains the same.

## 1.4 Conventions

A Dirac field may be written in a Fourier series as

$$\psi(t,\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\sum_s [u_s(\mathbf{k})a_s(\mathbf{k})\mathrm{e}^{-\mathrm{i}kx} + v_s(\mathbf{k})b^\dagger(\mathbf{k})\mathrm{e}^{ikx}]$$

where $u$ and $v$ are wave functions? [r]. The normalization of the free spinors is

$$\sum_s u_s(\mathbf{p})\bar{u}_s(\mathbf{p}) = \not{p} + m\,, \quad \sum_s v_s(\mathbf{p})\bar{v}_s(\mathbf{p}) = \not{p} - m$$

which solve

$$(\not{p} - m)u_s = 0\,, \quad \bar{u}_s(\not{p} - m) = 0\,, \quad (\not{p} + m)v_s = 0\,, \quad \bar{v}_s(\not{p} + m) = 0$$

[r] signs
A fermion propagator is

$$\frac{\mathrm{i}(\not{p} + m)}{p^2 - m^2 + \mathrm{i}\varepsilon}$$

The Feynman amplitude is

$$\mathrm{i}\mathcal{M} = \sum \text{Feynman diagrams}$$

The sum over quantum numbers of external particles is

$$\sum |\mathcal{M}|^2$$

The sum and average is instead

$$\overline{\sum} |\mathcal{M}|^2$$

[r] The tree-level cross-section is obtained

$$\mathrm{d}\sigma = \mathcal{F}\overline{\sum} |\mathcal{M}|^2 \mathrm{d}\phi_n$$

where $\mathcal{F}$ is the flux factor and $\mathrm{d}\phi_n$ is the phase space

$$\mathrm{d}\phi_n = (2\pi)^4\delta^{(4)}(\Delta p^\mu)\prod_{i=1}^n [\mathrm{d}k_i]$$

where the Lorentz invariant phase space measure is

$$[\mathrm{d}k_i] = \frac{\mathrm{d}^3k_i}{(2\pi)^3 2E_i}\,, \quad E_i^2 = m_i^2 + |\mathbf{k}_i|^2$$

In general, the phase space contains also symmetry factors $\frac{1}{n!}$ if the final state particles are identical bosons.

The decay width of a particle $M$ decaying is

$$\mathrm{d}\Gamma = \frac{1}{2M}\overline{\sum} |\mathcal{M}|^2 \,\mathrm{d}\phi_n$$

The total width is

$$\Gamma = \int \mathrm{d}\Gamma = \frac{1}{2M}\int \overline{\sum} |\mathcal{M}|^2 \,\mathrm{d}\phi_n$$

The Dirac traces is not explicitly written but a bracket is present [r] for Bhabha scattering one has [r] wrong. For computing the cross section one needs $\mathcal{M}^*$ and therefore

$$\sum |\mathcal{M}|^2 = [\cdots]$$

Four products of momenta get shortened

$$p_1^\mu p_{2\mu} = (12) = (p_1 \cdot p_2)$$

## 1.5 Quantum electrodynamics

One applies the gauge principle to go from a global symmetry to a local symmetry and make the lagrangian invariant. The interactions appear in the covariant derivative.

The QED lagrangian is

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - q\bar{\psi}\gamma^\mu\psi A_\mu$$

The free lagrangian

$$\mathcal{L}_0 = \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi$$

is invariant under global U(1). Making it local and requiring invariance, the fields transform as

$$\psi'(x) = \mathrm{e}^{\mathrm{i}\alpha(x)}\psi(x)\,, \quad A'_\mu = A_\mu - \frac{1}{q}\,\partial_\mu\alpha(x)$$

The covariant derivative is then

$$D_\mu = \partial_\mu + \mathrm{i}qA_\mu$$

So that the invariant lagrangian is

$$\mathcal{L} = \bar{\psi}(\mathrm{i}D\!\!\!\!/ - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

The derivative of the field does not transform the same way as the field, however the covariant derivative does

$$(\partial_\mu\psi)' \neq \mathrm{e}^{\mathrm{i}\alpha(x)}\partial_\mu\psi\,, \quad (D_\mu\psi)' = \mathrm{e}^{\mathrm{i}\alpha(x)}D_\mu\psi$$

Therefore the term $\bar{\psi}D_\mu\psi$ is gauge invariant. The field strength tensor is also gauge invariant.

**Remark.** The term

$$(F_{\mu\nu}F^{\mu\nu})^2$$

is not included because its mass dimension is 8 and the theory is not renormalizable.

**Remark.** The term $A^\mu A_\mu$ is not gauge invariant and it corresponds to a mass term, but the photon is massless.

## Lecture 3

mer 06 mar 2024 12:30

## 1.6 Non-abelian gauge groups

The Standard Model is based in part on $\mathrm{SU}(n)$ symmetry groups. Consider $U \in \mathrm{SU}(n)$, it can be expressed as

$$U = \exp[\mathrm{i}\theta^a t^a]\,, \quad a = 1, \cdots, n^2 - 1$$

where $t^a$ are the generators of the group and $\theta^a$ are real parameters. Elements of such group have the properties

$$UU^\dagger = U^\dagger U = I\,, \quad \det U = 1$$

The generators, which belong to the algebra, obey

$$t^a = (t^a)^\dagger\,, \quad 1 = \det e^{t^a} = e^{\operatorname{Tr} t^a} \implies \operatorname{Tr} t^a = 0$$

So the generators are hermitian traceless matrices. The generators are normalized to give

$$\operatorname{Tr}\!\left(t^a t^b\right) = T_R\delta^{ab}\,, \quad T_R = \frac{1}{2}$$

The commutation relations of the Lie algebra $\mathrm{su}(n)$ are

$$[t^a, t^b] = \mathrm{i}f^{abc}t^c$$

The coefficient $f^{abc}$ are the structure constants of the Lie algebra. For non-abelian gauge groups, the commutator is not identically zero.

**Exercise.** Defining the matrix

$$\tau^{ab} \equiv \mathrm{i}[t^a, t^b]$$

where $a, b$ do not label the components. Show that

$$\mathrm{Tr}\,\tau^{ab} = 0\,, \quad (\tau^{ab})^\dagger = \tau^{ab}$$

and that $f$ is totally anti-symmetric and real.

In general, a $d$-dimensional representation of an algebra is a set of $d \times d$ matrices that satisfy the commutation relation

$$[T^a, T^b] = \mathrm{i} f^{abc} T^c$$

The number of $T^a$ is the dimension of the Lie group.

The important representations are the fundamental, anti-fundamental and the adjoint representations. The fundamental representation is an $N$-dimensional representation. It acts on $N$-dimensional objects. The adjoint representation is given by the structure constants

$$(T^a)_{bc} = \mathrm{i} f^{bac}$$

where $bc$ are the components of the matrix $T^a$. It is a $N^2 - 1$ dimensional representation.

**Gauge symmetry.** The matrix

$$U(x) = \mathrm{e}^{\mathrm{i}\theta^a(x) t^a}$$

depends on space-time coordinates. In the fundamental representation, the matrix acts on $N$ dimensional objects. A spinor does not transform trivially, but as

$$\psi'(x) = U(x)\psi(x)$$

The derivative transforms as

$$(\partial_\mu \psi)' = \partial_\mu[U(x)\psi(x)] = (\partial_\mu U)\psi + U\,\partial_\mu \psi$$

but this is not favorable. One uses the covariant derivative

$$(D_\mu)_{ij} = \partial_\mu \delta_{ij} + \mathrm{i} g t^a_{ij} A^a_\mu$$

where the $ij$ indices treats gauge group components. Therefore, there are $N^2 - 1$ gauge fields $A^a_\mu$. The covariant transforms in the same way as the field

$$(D_\mu \psi)' = D'_\mu \psi' = U(x) D_\mu \psi$$

This implies that the gauge fields transform as

$$(t^a A^a_\mu)' = t^a A'^a_\mu = U(t^a A^a_\mu)U^{-1} + \frac{\mathrm{i}}{g}(\partial_\mu U)U^{-1}$$

To define the field strength tensor one goes by analogy with quantum electrodynamics

$$\mathrm{i} q F^{\mathrm{QED}}_\mu = [D^{\mathrm{QED}}_\mu, D^{\mathrm{QED}}_\nu]\,, \quad D^{\mathrm{QED}}_\mu = \partial_\mu + \mathrm{i} q A_\mu$$

In general, one defines

$$\mathrm{i} g t^a F^a_{\mu\nu} = [D_\mu, D_\nu]$$

This gives

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$

Under gauge transformation one has

$$(t^a F^a_{\mu\nu})' = t^a F'^a_{\mu\nu} = U(t^a F^a_{\mu\nu})U^{-1}$$

The kinetic term for gauges boson is

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = -\frac{1}{2}\,\mathrm{Tr}(\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu})\,, \quad \mathbf{F}_{\mu\nu} = t^a F^a_{\mu\nu}\,, \quad F^a_{\mu\nu} F^{a\mu\nu} = 2\,\mathrm{Tr}(\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu})$$

One sees that
$$\mathbf{F}'^{\mu\nu} = U\mathbf{F}^{\mu\nu}U^{-1}$$
For a non-abelian gauge group the Lagrangian is
$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\psi}_j(x)[\mathrm{i}\not{D}_{jk} - m\delta_{jk}]\psi_k(x)$$
The indices $jk$ are the one carried by the generators $t^a_{jk}$, not the four components of Dirac spinors. In the fundamental representation one has
$$a = 1, \dots, N^2 - 1\,, \quad j, k = 1, \dots, N$$
The covariant derivative is
$$D^\mu_{jk} = \partial^\mu\delta_{jk} + \mathrm{i}gt^a_{jk}A^a_\mu$$
There are $N^2 - 1$ gauge boson fields $A^a_\mu$ and they are massless because massive terms $m^2A^a_\mu A^{a\mu}$ are not guage invariant.

A term $(F^a_{\mu\nu}F^{a\mu\nu})^n$ is not renormalizable for $n > 1$.

The covariant derivative for the electroweak sector
$$D_\mu = \partial_\mu - \mathrm{i}qt^aA^a_\mu$$
while for the quantum chromodynamics sector is
$$D_\mu = \partial_\mu + \mathrm{i}gt^aA^a_\mu$$
[r] DC
The interaction term
$$\bar{\psi}\not{D}\psi \rightsquigarrow -g(\bar{\psi}t^a\gamma^\mu\psi)A^a_\mu$$
generates a vertex equal to
$$-\mathrm{i}gt^a_{ji}\gamma^\mu$$
In the kinetic part $FF$ there are three-vertices
$$FF \rightsquigarrow gf^{abc}(\partial_\mu A^a_\nu)A^b_\mu A^c_\nu$$
which is called derivative vertex. In momentum space, the derivative is a momentum [r]
$$-gf^{abc}[(p_a - p_b)^\gamma\eta^{\alpha\beta} + (p_b - p_c)^\alpha\eta^{\beta\gamma} + (p_c - p_a)^\beta\eta^{\alpha\gamma}]$$
The charged boson have to interact with the carrier of the force, which is another boson.

In the kinetic part there are also four-vertices
$$FF \rightsquigarrow g^2ffAAAA$$

## 1.7 History of the Standard Model

**Fermi theory.** Fermi theory is a theory of the electroweak sector. The Lagrangian is
$$\mathcal{L} = -\frac{G_\mathrm{F}}{\sqrt{2}}J^\mu J^\dagger_\mu\,, \quad J_\mu = \bar{\psi}_l\gamma_\mu(1-\gamma_5)\psi_{\nu_l} + \bar{\psi}_d\gamma_\mu(1-\gamma_5)\psi_u = L_\mu + H_\mu$$
The current contains both the leptonic part and the hadronic part. Examples of weak decays are the muon beta decay and the neutron beta decay.

One can compute tree-level total decay widths $\Gamma$ and differential widths $\mathrm{d}_\Omega\Gamma$ from which one obtains the Fermi constant $G_\mathrm{F} = 1.16 \times 10^{-5}\,\mathrm{GeV}^{-2}$. The Lagrangian correctly describes these two processes [r].

Fermi theory is a V-A theory because bilinears of the types
$$V^\mu = \bar{\psi}_1\gamma^\mu\psi_2$$

transforms like a polar vector while bilinears like

$$A^\mu \bar{\psi}_1 \gamma^\mu \gamma^5 \psi_2$$

transforms like an axial vector, or pseudo-vector. A theory of this type is in accordance with experiments. A V-A theory is maximally parity violating. Check that $V'^\mu = \Lambda^\mu{}_\nu V^\mu$ and $A'^\mu = \bar{\Lambda}^\mu{}_\nu A^\nu$ [r]. To see how it is maximally violating, one can see that from the Lagrangian one has terms like

$$A^\mu V_\mu \to -A^\mu V_\mu$$

Processes like neutrino deep-inelastic scattering. For example $\bar{\nu}_e u \to d e^+$ one has a differential cross section

$$\mathrm{d}_\Omega \sigma = \frac{G_\mathrm{F}^2}{8\pi^2} \frac{s}{4} (1 + \cos\theta)^2 \,, \quad s = (p_1 + p_2)^2$$

but this cross section implies that the scattering matrix is not unitary.

Therefore Fermi theory is not renormalizable because the coupling constant has negative mass dimensions and it is not unitary. To fix the second problem one introduces a vector gauge boson.

## Lecture 4

lun 11 mar 2024 12:30

**Intermediate vector boson theory.** A neutrino scattering is an exchange in the $t$-channel of a vector boson [r] diagr. The diagram gives an amplitude

$$\mathcal{M} \sim g_W^2 J^\mu_{e\nu} \left[ -\eta_{\mu\nu} + \frac{Q_\mu Q_\nu}{M^2} \right] \frac{1}{Q^2 - M_W^2} J^2_{ud}$$

In the limit of the momentum going to zero

$$Q^\mu = P_1^\mu - P_3^\mu \to 0$$

the amplitude is asymptotic to

$$M \sim g_W^2 J^{e\nu}_\mu J^{ud}_\nu \frac{\eta^{\mu\nu}}{Q^2 - M_W^2}$$

Confronting this result with Fermi theory, one obtains a coupling constant of

$$-\frac{G_\mathrm{F}}{\sqrt{2}} = \frac{1}{8} \frac{g_W^2}{Q^2 - M_W^2}$$

The intermediate vector boson is an advanced theory respect to Fermi's. Also, in the low momentum limit $Q \to 0$ (equivalent to $Q^2 \ll M_W^2$) one obtains the Fermi theory

$$\frac{G_\mathrm{F}}{\sqrt{2}} = \frac{g_W^2}{8 M_W^2}$$

In the high energy limit, $Q^2 \gg M^2$, the differential cross section does not diverge

$$\mathrm{d}_\Omega \sigma = \frac{g_W^2}{4\pi^2} \frac{s}{(s - M_W^2)^2} \,, \quad s = (p_1 + p_2)^2$$

An explicit mass term in the Lagrangian breaks the gauge invariance

$$\mathcal{L} \sim M_W^2 W_\mu W^\mu$$

The theory presents a problem: the need of a theory with massive vector boson spoils the gauge invariance. Also a problem is that the theory is not renormalizable. The propagator of a massive vector boson is

$$G_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M^2} \left[ -\eta_{\mu\nu} + \frac{k_\mu k_\nu}{M^2} \right]$$

If one wants to understand the renormalizabilty through power counting arguments, one sees that the propagator is constant

$$G_{\mu\nu}(\mathbf{k}) \sim \frac{k_\mu k_\nu}{k^2}\,, \quad k \to \infty$$

The renormalizable nature of a theory is given by the convergence of the propagators in a Feynman diagram.

Another problem is the loss of unitary. When scattering two vector bosons in two vector bosons, one has to build the amplitude by summing all diagrams. One can check how the amplitude scales with the energy. The behaviour of $E^4$ cancels and remains $E^2$: the amplitude grows indefinitely.

**Electroweak sector.** One has to associate currents to the left-chiral part of the Dirac field. A leptonic current is

$$J_\mu^{\mathrm{lept}} = \frac{1}{2}\bar{\nu}\gamma_\nu(1-\gamma_5)e$$

The current must come from a covariant derivative. The current must be Noether current. Therefore

$$\partial_\mu - \mathrm{i}gT^a A_\mu^a$$

This term is always between two fermionic fields, so the current is

$$\bar{\psi}_i\gamma_\mu T^a_{ij}\psi_j$$

where the indices $ij$ are associated to the transformation of the gauge group.

One defines a left-chiral leptonic doublet

$$L(x) = P_{\mathrm{L}}\begin{bmatrix}\nu_e \\ e\end{bmatrix}$$

for which the current is

$$J_\mu^{\mathrm{lept}} = \bar{L}\gamma_\mu\tau^+ L$$

[r] Confronting with the current above one has

$$\tau^+ = \begin{bmatrix}0 & 1 \\ 0 & 0\end{bmatrix} = \frac{1}{2}(\tau_1 + \mathrm{i}\tau_2)$$

where $\tau_i$ are the Pauli matrices. So the gauge group is SU(2). Moreover, there is

$$(J_\mu^{\mathrm{lept}})^\dagger = \bar{L}\gamma_\mu\tau^- L\,, \quad \tau^- = \frac{1}{2}(\tau_1 - \mathrm{i}t_2) = \begin{bmatrix}0 & 0 \\ 1 & 0\end{bmatrix}$$

There is bijective correspondence between currents and generators. Since the algebra is close, there is a third current given by

$$(J_\mu^{\mathrm{lept}})^3 = \bar{L}\gamma_\mu[\tau^+, \tau^-]L = \bar{L}\gamma_\mu\tau^3 L$$

This current implies a bilinear combination of spinors of the type

$$e\gamma_\mu e\,, \quad \nu\gamma_\mu\nu$$

If one searches for more generators, one may compute the commutator. However, in this case the algebra is closed

$$[\tau^3, \tau^\pm] = 2\tau^\pm$$

So there are no other currents.

The leptonic electroweak sector has $\mathrm{SU}(2)_\mathrm{L} \times U(1)_Y$ gauge group. The Lagrangian respecting this symmetry is given by

$$\mathcal{L}_\mathrm{f} = \mathrm{i}\bar{L}\not{D}L + \mathrm{i}\bar{e}_\mathrm{R}\not{D} + \mathrm{i}\bar{\nu}_{e\mathrm{R}}\not{D}\nu_{e\mathrm{R}}$$

where

$$D_\mu = \partial_\mu - \mathrm{i}gW^j_\mu T^j - \frac{1}{2}\mathrm{i}g'Y(\psi)B_\mu$$

where $j = 1, 2, 3$ and $W_\mu$ is the field associated to the local SU(2) symmetry, while $B_\mu$ is associated to U(1). This last gauge group is not the one of electromagnetism.

The generators $T^i$ are the generators of the Lie algebra in the representation in which the fields transform

$$T^i = \begin{cases} \frac{1}{2}\tau^i\,, & \text{left-chiral fields} \\ 0\,, & \text{right-chiral fields} \end{cases}$$

For the hypercharge symmetry, the hypercharge is an abelian group and the associated charge can depend on the field itself.

The Lagrangian $\mathcal{L}_f$ is invariant under

$$U(x) = \exp\left[\frac{1}{2}\mathrm{i}g\theta(x)\tau^i\right], \quad U(x) = \exp\left[\frac{1}{2}\mathrm{i}g'\alpha(x)Y\right]$$

The kinetic part of the Lagrangian is

$$\mathcal{L}_f^{\mathrm{kin}} = \mathrm{i}\bar{L}\,\partial\!\!\!/ L + \mathrm{i}\bar{\nu}_{e\mathrm{R}}\,\partial\!\!\!/ \nu_{e\mathrm{R}} + \mathrm{i}\bar{e}_\mathrm{R}\,\partial\!\!\!/ e_\mathrm{R}$$

One may notice that the electrons and neutrinos are both massless, but when measured they are not massless.

The charge current interaction is

$$\begin{aligned}\mathcal{L}_{\mathrm{cc}} &= \frac{1}{2}gW^1_\mu\bar{L}\gamma^\mu\tau^1L + \frac{1}{2}gW^2_\mu\bar{L}\gamma^\mu\tau^2L = \frac{g}{\sqrt{2}}[W^+_\mu\bar{L}\gamma^\mu\tau^+L + W^-_\mu\gamma^\mu\tau^-L] \\ &= \frac{g}{\sqrt{2}}[W^+_\mu\bar{\nu}_\mathrm{L}\gamma^\mu e_\mathrm{L} + W^-_\mu\bar{e}_\mathrm{L}\gamma^\mu\nu_\mathrm{L}]\end{aligned}$$

where one defines

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(W^1_\mu \mp \mathrm{i}W^2_\mu)$$

The Feynman rule is [r] diagr

$$\mathrm{i}\frac{g}{\sqrt{2}}\gamma^\mu\frac{1-\gamma_5}{2}$$

These are the charged currents because the carrier boson is charged.

The neutral current part is

$$\mathcal{L}_{\mathrm{nc}} = \frac{1}{2}gW^\mu_3[\bar{\nu}_{e\mathrm{L}}\gamma_\mu\nu_{e\mathrm{L}} - \bar{e}_\mathrm{L}\gamma_\mu e_\mathrm{L}] + \frac{1}{2}g'B^\mu[Y(L)(\bar{\nu}_{e\mathrm{L}}\gamma_\mu\nu_{e\mathrm{L}} + \bar{e}_\mathrm{L}\gamma_\mu e_\mathrm{L}) + Y(e_\mathrm{R})(\bar{e}_\mathrm{R}\gamma_\mu e_\mathrm{R}) + Y(\nu_{e\mathrm{R}})(\bar{\nu}_{e\mathrm{R}}\gamma_\mu v_{e\mathrm{R}})]$$

Introducing

$$\Psi = \begin{bmatrix}\nu_{e\mathrm{L}} & e_\mathrm{L} & \nu_{e\mathrm{R}} & e_\mathrm{R}\end{bmatrix}^\top$$

the third value of the isospin is

$$T_3 = \frac{1}{2}\begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}$$

while the hypercharge is

$$Y = \mathrm{diag}[Y(L), Y(L), Y(\nu_{e\mathrm{R}}), Y(e_\mathrm{R})]$$

The Lagrangian becomes

$$\mathcal{L}_{\mathrm{nc}} = g(\bar{\Psi}\gamma^\mu T_3\Psi)W^3_\mu + \frac{1}{2}g'(\bar{\Psi}\gamma^\mu Y\Psi)B_\mu$$

From these fields one would like to recognize quantum electrodynamics. One considers linear combinations of the two fields through an orthogonal rotation: the kinetic term [r]. Therefore

$$\begin{bmatrix}B_\mu \\ W^3_\mu\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}A_\mu \\ Z_{mu}\end{bmatrix}$$

The angle is called Weinberg's angle. One needs to look at the kinetic terms and how the interactions change. The interaction Lagrangian is

$$L_{\mathrm{YM,int}} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}W^3_{\mu\nu}W_3^{\mu\nu} \to -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu}$$

where $F^{\mu\nu}$ is the field strength tensor of the electromagnetic field $A^\mu$, and $Z^{\mu\nu}$ for the $Z^\mu$ field. There are no mixed kinetic terms $F_{\mu\nu}Z^{\mu\nu}$.

The neutral current part of the Lagrangian is

$$\mathcal{L}_{\mathrm{nc}} = \bar{\Psi}\gamma_\mu[g\sin\theta\, T_3 + \frac{1}{2}g'\cos\theta\, Y]\Psi A^\mu + \bar{\Psi}\gamma_\mu[g\cos\theta\, T_3 - \frac{1}{2}g'\sin\theta\, Y]\Psi Z^\mu$$

Noting that the interaction Lagrangian of quantum electrodynamics is

$$\mathcal{L}^{\mathrm{int}}_{\mathrm{QED}} = \bar{\Psi}\gamma^\mu eQ\Psi A_\mu$$

where $e > 0$ is the elementary charge and

$$Q = \mathrm{diag}(Q_\nu, Q_e, Q_\nu, Q_e) = \mathrm{diag}(0, -1, 0, -1)$$

Therefore, one has

$$g\sin\theta\, T_3 + \frac{1}{2}g'\cos\theta\, Y \equiv eQ$$

Since $Y$ is always with $g'$, then one can fix a value for $Y$ and the other follow

$$Y(L) \equiv -1$$

therefore, for the left-chiral neutrino field $\nu_{e\mathrm{L}}$ one has $T_3 = \frac{1}{2}$ and the left-chiral electron field one has $T_3 = -\frac{1}{2}$. For the neutrino and electron one has

$$\frac{1}{2}g\sin\theta - \frac{1}{2}g\cos\theta = 0\,, \quad -\frac{1}{2}g\sin\theta - \frac{1}{2}g'\cos\theta$$

This implies

$$g\sin\theta + g'\cos\theta = e$$

Therefore one has the Gell-Mann–Nijishima equation

$$Q = T_3 + \frac{1}{2}Y$$

For the right-chiral fields, one has $T_3 = 0$ and

$$Y(\nu_{\mathrm{R}}) = 0\,, \quad Y(e_{\mathrm{R}}) = -2$$

The right-chiral neutrino behaves like it does not exist (apart from gravity).

The Feynman rules for quantum electrodynamics $e\bar{\Psi}\gamma_\mu Q\Psi A^\mu$ and neutral currents $e\bar{\Psi}\gamma_\mu Q_Z\Psi Z^\mu$ are [r] diagr

$$\mathrm{i}eQ_f\gamma^\mu\,, \quad \mathrm{i}e\gamma_\mu(c_{\mathrm{L}}P_{\mathrm{L}} + c_{\mathrm{R}}P_{\mathrm{R}}) = \mathrm{i}e\gamma^\mu(v_f - a_f\gamma_5)$$

where

$$Q_Z = \frac{1}{\sin\theta\cos\theta}[T_3 - Q\sin^2\theta]\,, \quad \Psi = \Psi_{\mathrm{L}} + \Psi_{\mathrm{R}}$$

and

$$c_{\mathrm{L}} = \frac{1}{\sin\theta\cos\theta}(T_f^3 - Q_f\sin^2\theta)\,, \quad c_{\mathrm{R}} = -\tan\theta\, Q_f$$

likewise

$$v_f = \frac{1}{2}(c_{\mathrm{L}} - c_{\mathrm{R}}) = \frac{T_f^3 - 2Q_f^2\sin^2\theta}{2\sin\theta\cos\theta}\,, \quad a_f = \frac{1}{2}(c_{\mathrm{L}} + c_{\mathrm{R}}) = \frac{T_f^3}{2\sin\theta\cos\theta}$$

## Lecture 5

mar 12 mar 2024 12:30

The pure Yang–Mills part is

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}W_{\mu\nu}W_i^{\mu\nu}\,, \quad i = 1,2,3$$

where one has

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f^{abc} W^b_\mu W^c_\nu$$

Inserting the physical fields through the relations

$$W^1_\mu = \frac{1}{\sqrt{2}}(W^+_\mu + W^-_\mu)\,, \quad W^2_\mu = \frac{\mathrm{i}}{\sqrt{2}}(W^+_\mu - W^-_\mu)\,, \quad W^3_\mu = A_\mu \sin\theta + Z_\mu \cos\theta\,, \quad B_\mu = A_\mu\cos\theta - Z_\mu \sin\theta$$

one obtains

$$\begin{aligned}\mathcal{L}_{\mathrm{YM}} = \mathcal{L}_{\mathrm{kin}} + \mathcal{L}_{\mathrm{int}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{2}W^+_{\mu\nu}W^{\mu\nu}_- \\ &+ \text{3-point interactions} + \text{4-point interactions}\end{aligned}$$

in the first line there are no mixed terms: the fields are mass eigenstates. The three-point vertex contains derivative so they scale with the energy of the momentum, while the four-point have no dependence on momentum.

**Isospin and hypercharge Hadronic sector.** See Ridolfi. This sector has a symmetry

$$\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_Y$$

Hadrons are not elementary particles, but are made of quarks. A neutron decays into

$$n \to p e^- \bar{\nu}_e\,, \quad |udd\rangle \to |uud\rangle + e^- + \bar{\nu}_e$$

In the low-energy limit, the decay can be explained using Fermi's four-point interaction. The interaction has to be generated from a term

$$J^{\mathrm{lept}}_\mu J^\mu_{\mathrm{had}}$$

where one has the charged currents

$$J^{\mathrm{had}}_\mu = \frac{1}{2}\bar{u}\gamma_\mu(1-\gamma_5)d$$

The up and down quarks are not the only ones. Experiments showed the presence of strange hadrons $K^\pm$, $K^0$, $\Lambda^0$, etc. These particles decay slowly, they have a short decay width so the interaction has a weak coupling constant: they decay weakly. One assumes that a kaon is made of another quark, the strange quark

$$\left|K^+\right\rangle = |u\bar{s}\rangle$$

One of its decay chains is

$$\left|K^+\right\rangle \to \left|\pi^0\right\rangle e^+\nu$$

It starts from a kaon and has leptons in the final state: the interaction is the weak. The strange quark is postulated to have electromagnetic charge of

$$Q_s = -\frac{1}{3}$$

The strangeness quantum number of the strange quark is $Q_s = -1$.

The natural phenomenological hypothesis is considering that the hadronic current has two parts

$$J^{\mathrm{had}}_\mu = \cos\theta\frac{1}{2}\bar{u}\gamma_\mu(1-\gamma_5)d + \sin\theta\frac{1}{2}\bar{u}\gamma_\mu(1-\gamma_5)s$$

where $\theta \approx 12°$ is the Cabibbo angle.

One may extend the model supposing the current is Noether's and one may find new interactions. The hadronic current is

$$J_\mu^{\mathrm{had}} = \begin{bmatrix} \bar{u}_\mathrm{L} & \bar{d}_\mathrm{L} & \bar{s}_\mathrm{L} \end{bmatrix} \gamma_\mu T^+ \begin{bmatrix} u_\mathrm{L} \\ d_\mathrm{L} \\ s_\mathrm{L} \end{bmatrix} , \quad T^+ = \begin{bmatrix} 0 & \cos\theta & \sin\theta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The hermitian conjugate current is

$$(J_\mu^{\mathrm{had}})^\dagger = \cdots , \quad T^- = \begin{bmatrix} 0 & 0 & 0 \\ \cos\theta & 0 & 0 \\ \sin\theta & 0 & 0 \end{bmatrix}$$

One imagines that the matrices $T^\pm$ are elements of a Lie group ([r] still Cabibbo angle? or which one is it?)

$$[T^+, T^-] = T^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\cos^2\theta & -\sin\theta\cos\theta \\ 0 & -\sin\theta\cos\theta & -\sin^2\theta \end{bmatrix}$$

This matrix is not diagonal. The current associated with this element is

$$J_\mu^{\mathrm{had},3} = \begin{bmatrix} \bar{u}_\mathrm{L} & \bar{d}_\mathrm{L} & \bar{s}_\mathrm{L} \end{bmatrix} \gamma_\mu T^3 \begin{bmatrix} u_\mathrm{L} \\ d_\mathrm{L} \\ s_\mathrm{L} \end{bmatrix} = \bar{u}_\mathrm{L}\gamma_\mu u_\mathrm{L} - \cos^2\theta \bar{d}_\mathrm{L}\gamma_\mu d_\mathrm{L} - \sin^2\theta \bar{s}_\mathrm{L}\gamma_\mu s_\mathrm{L} - \sin\theta\cos\theta[\bar{d}_\mathrm{L}\gamma_\mu s_\mathrm{L} + \bar{s}_\mathrm{L}\gamma_\mu d_\mathrm{L}]$$

The last parenthesis is the flavor-changing neutral current (FCNC). These currents exist in nature but are extremely suppressed. However, this theory does not predict a suppressed FCNC.

The flavor-changing charge current can be seen from

$$\left|K^+\right\rangle \to \left|\pi^0\right\rangle e^+\nu , \quad |u\bar{s}\rangle \to |u\bar{u}\rangle$$

mediated by

$$\sin\theta_c \, \bar{s}_\mathrm{L}\gamma^\mu u_\mathrm{L}$$

[r] Instead, for the mediating FCNC is

$$\sin\theta_c \cos\theta_c \bar{s}_\mathrm{L}\gamma^\mu d_\mathrm{L}$$

The ratio of the experimental widths is

$$\frac{\Gamma(K^+ \to \pi^+ e^+ e^-)}{\Gamma(K^+ \to \pi^0 e^+ \nu)} \approx 10^{-5}$$

while in theory one has

$$\sim \frac{(\sin\theta_c \cos\theta_c)^2}{\sin^2\theta_c} \sim 0.97$$

To resolve the issue, one may postulate the existence of a fourth quark, called charm quark. Its electric charge is

$$Q = \frac{2}{3}$$

It has to be heavy and coupled to the down and strange quarks through charged currents. The current is

$$\begin{aligned} J_\mu^{\mathrm{had}} &= \cos\theta_c \bar{u}_\mathrm{L}\gamma_\mu d_\mathrm{L} + \sin\theta_c \bar{u}_\mathrm{L}\gamma_\mu s_\mathrm{L} - \sin\theta_c \bar{c}_\mathrm{L}\gamma_\mu d_\mathrm{L} + \cos\theta_c \bar{c}_\mathrm{L}\gamma_\mu s_\mathrm{L} \\ &= \bar{u}_\mathrm{L}\gamma_\mu d'_\mathrm{L} + \bar{c}_\mathrm{L}\gamma_\mu s'_\mathrm{L} \end{aligned}$$

where one has

$$\begin{bmatrix} d' \\ s' \end{bmatrix} = \begin{bmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{bmatrix} \begin{bmatrix} d \\ s \end{bmatrix}$$

The current is then

$$J_\mu^{\mathrm{had}} = \begin{bmatrix} \bar{u}_\mathrm{L} & \bar{d}'_\mathrm{L} \end{bmatrix} \gamma_\mu \tau^+ \begin{bmatrix} u_\mathrm{L} \\ d'_\mathrm{L} \end{bmatrix} + \begin{bmatrix} \bar{c}_\mathrm{L} & \bar{s}'_\mathrm{L} \end{bmatrix} \gamma_\mu \tau^- \begin{bmatrix} c_\mathrm{L} \\ s'_\mathrm{L} \end{bmatrix}$$

The third current is

$$\begin{aligned} J_\mu^{\mathrm{had},3} &= \begin{bmatrix}\bar u_\mathrm{L} & \bar d'_\mathrm{L}\end{bmatrix}\gamma_\mu\tau^3\begin{bmatrix}u_\mathrm{L}\\ d'_\mathrm{L}\end{bmatrix} + \begin{bmatrix}\bar c_\mathrm{L} & \bar s'_\mathrm{L}\end{bmatrix}\gamma_\mu\tau^3\begin{bmatrix}c_\mathrm{L}\\ s'_\mathrm{L}\end{bmatrix} \\ &= \bar u_\mathrm{L}\gamma_\mu u_\mathrm{L} + \bar c_\mathrm{L}\gamma_\mu c_\mathrm{L} - \bar d_\mathrm{L}\gamma_\mu d_\mathrm{L} - \bar s_\mathrm{L}\gamma_\mu s_\mathrm{L} \end{aligned}$$

There are no flavour-changing neutral currents.

The existence of the charm quark was discovered four years after postulating its existence. It was discovered the bound state $|c\bar c\rangle$ which is the $J/\psi$ particle. The mass of the quark is $m_c \approx 1.5\,\mathrm{GeV}$.

Those who postulated its existence worked also on the GIM mechanics: there are no FCNC at tree-level, the FCNC are suppressed at 1 loop if $\Delta s \neq 0$, explains the mass difference of $K_L$ and $K_S$.

At present time, the mechanism involves the mixing between three quark families: there are three angles and one complex phase; this phase implies CP violation.

**Summary.** The Standard Model has the following fields. The quarks are

$$\begin{bmatrix}u_\mathrm{L}\\ d'_\mathrm{L}\end{bmatrix}, \quad \begin{bmatrix}c_\mathrm{L}\\ s'_\mathrm{L}\end{bmatrix}, \quad \begin{bmatrix}t_\mathrm{L}\\ b'_\mathrm{L}\end{bmatrix}, \quad u_\mathrm{R} d'_\mathrm{R} c_\mathrm{R} s'_\mathrm{R} t_\mathrm{R} b'_\mathrm{R}$$

The leptons are

$$\begin{bmatrix}\nu_{el}\\ e_L\end{bmatrix}, \quad \begin{bmatrix}\nu_{\mu L}\\ \mu_L\end{bmatrix}, \quad \begin{bmatrix}\nu_{\tau L}\\ \tau_L\end{bmatrix}, \quad e_R\mu_R\tau_R\nu_{eR}\nu_{\mu R}\nu_{\tau R}$$

The left-chiral leptons are organized in doublets of $\mathrm{SU}(2)_L$ with $T_3 = \pm\frac{1}{2}$, while the right-chiral are singlets with $T_j = 0$. One also has

$$Q = T_3 + \frac{1}{2}Y$$

where one has

$$Y(u_R) = \frac{4}{3}, \quad Y(d_R) = -\frac{2}{3}, \quad Y(Q_L) = \frac{1}{3}, \quad Q_u = \frac{2}{3}, \quad Q_d = -\frac{1}{3}, \quad Q_L = \begin{bmatrix}u_L\\ d'_L\end{bmatrix}, \cdots$$

The charged-current Lagrangian

$$\mathcal{L}_{cc} = \frac{g}{\sqrt{2}}\sum_f [\bar L_f\gamma_\mu\tau_+ L_f + \bar Q_f\gamma_\mu\tau_+ Q_f]W_\mu^+ + \mathrm{h.c.}$$

where $f = 1, 2, 3$ is the index associated with the three quark and leptonic families and

$$L_f \in \{\begin{bmatrix}v_{eL}\\ e_L\end{bmatrix}, \begin{bmatrix}v_{\mu L}\\ \mu_L\end{bmatrix}, \cdots\}, \quad Q_f = \{\begin{bmatrix}u_L\\ d'_L\end{bmatrix}, \cdots$$

the Lagrangian is

$$\mathcal{L}_{cc} = \mathcal{L}_\mathrm{lept} + \frac{g}{\sqrt{2}}\left[\sum_{f,g}(\bar u_L^f\gamma_\mu V^{fg} d_L^g)W_\mu^+ + \mathrm{h.c.}\right]$$

where $V_{fg}$ is a $3\times 3$ non-diagonal matrix called CKM matrix. [r] one has the Feynman

$$\sim V^{12}[\gamma_\mu\frac{1}{2}(1-\gamma_5)\frac{g}{\sqrt{2}}]$$

The neutral current Lagrangian has no FCNC but is diagonal

$$\mathcal{L}_\mathrm{nc} = e(\bar\Psi\gamma_\mu Q\Psi)A_\mu + (\bar\Psi\gamma^\mu Q_Z\Psi)Z_\mu$$

**Higgs mechanism.** None of the fields have any masses because gauge symmetry must be respected. Masses are given through the spontaneous symmetry breaking of gauge theories: the Higgs mechanism. This gives masses to the gauge boson.

A Yukawa interaction is allowed and a Higgs field with vacuum expectation value gives masses to fermions.

One may study the number of degrees of freedom needed. In the electroweak sector, the gauge fields are massless each with two transverse polarization; though in nature three are massive, so one more degree for each is needed.

One needs a scalar field charge under the gauge group $\mathrm{SU}(2)_L \times U(1)_Y$ with non zero vacuum expectation value. The gauge group must be broken, but there must remain a gauge subgroup of $U(1)_{\mathrm{EM}}$. The simplest and minimal choice is using a doublet of scalar complex fields

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

It has four real degrees of freedom. Three of them make up the masses of three gauge boson, while the last is the Higgs field. The doublet transforms as

$$T_3 \phi = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \phi \,, \quad Y(\phi) = \cdots$$

the hypercharge is not yet fixed.

To realize the spontaneous symmetry breaking one needs a potential [r], to respect gauge invariance, and the theory to be normalizable. The potential can only be

$$V(\phi) = m^2 |\phi|^2 + \lambda |\phi|^4 \,, \quad |\phi|^2 = |\phi_1|^2 + |\phi_2|^2$$

The mass dimension of $\lambda$ is zero, so the theory is renormalizable. A cubic term is not gauge invariant.

A spontaneous symmetry breaking implies a non zero vacuum expectation value. The minimum of the potential is

$$|\phi_{\mathrm{min}}|^2 = -\frac{m^2}{2\lambda} = \frac{1}{2} v^2 \implies m^2 < 0$$

The minimum configuration must be invariant under $U(1)$ so that this gauge group is unbroken and the photon does not gain mass. One parametrizes the minimum field as

$$\phi_{\mathrm{min}} = \frac{1}{\sqrt{2}} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \,, \quad |v_1|^2 + |v_2|^2 = v^2$$

The residual invariance implies

$$e^{i\alpha(x) Q} \phi_{\mathrm{min}} = \phi_{\mathrm{min}} \implies Q \phi_{\mathrm{min}} = 0$$

## Lecture 6

mer 13 mar 2024 12:30

One needs to have

$$\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \frac{1}{2} \begin{bmatrix} 1 + Y & 0 \\ 0 & -1 + Y \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

There are two choices $v_1 = 0$ and $v_2 = |v|$ or $v_1 = |v|$ and $v_2 = 0$. For the first one has $Y(\phi) = 1$ while for the second $Y(\phi) = -1$. One picks the first choice

$$Q_{\mathrm{EM}}(\phi_1) = 1 \,, \quad Q_{\mathrm{EM}}(\phi_2) = 0$$

The doublet can rewritten as

$$\phi = \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix}$$

To obtain the content of the theory, one parametrizes the field around the vacuum expectation value. This can be done through Cartesian coordinates

$$\phi = \begin{bmatrix} \xi_1 + \mathrm{i}\xi_2 \\ \frac{1}{\sqrt{2}}(v + H(x) + \mathrm{i}\chi) \end{bmatrix}$$

but it is more useful to use complex coordiantes

$$\phi = \frac{1}{2} \exp\left[\frac{1}{2}\mathrm{i}\tau^a\theta^a(x)\right] \begin{bmatrix} 0 \\ v + H(x) \end{bmatrix}$$

where $H(x)$ is the Higgs field. The four fields above have zero expectation value. One imposes the unitary gauge

$$\phi(x)' = U(x)\phi(x)\,, \quad U(x) = \exp\left[-\frac{1}{2}\mathrm{i}\tau^a\theta^a(x)\right]$$

This gauge is allowed because the Lagrangian is gauge invariant. The $\theta^a$ fields are absorbed into the $W^\pm$ and $Z$ fields as longitudinal polarization, so the associated bosons are massive. The field $H(x)$ is associated with a physical particle. The spontaneous symmetry patterns is such that three of the five degrees of freedom [r]

$$\tau_1\varphi_{\mathrm{min}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{bmatrix} \neq 0\,, \quad \tau_2\phi_{\mathrm{min}} \neq 0$$

whereas

$$\frac{1}{2}(\tau_3 + Y)\phi_{\mathrm{min}} = 0$$

**Physical consequences.** The Lagrangian of the field is

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi)\,, \quad \phi = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ v + H(x) \end{bmatrix}$$

The potential is

$$\begin{aligned} V = m^2|\phi|^2 + \lambda|\phi|^4 &= \frac{1}{2}m^2[H^2 + 2vH] + \frac{1}{4}\lambda[H^4 + 4H^3v + 6H^2v^2 + 4Hv^3] + \text{const.} \\ &= \frac{1}{2}(2\lambda v^2)H^2 + \lambda vH^3 + \frac{1}{4}\lambda H^4 \end{aligned}$$

at the second line one uses the fact that

$$-m^2 = \lambda v^2$$

The first term is a mass term $m_{\mathrm{H}}^2 = 2\lambda v^2$. The other two terms are a three-point and four-point self-couplings of the Higgs field. The two vertices are proportional to the free parameter $\lambda$.

Within the covariant derivative there is the Higgs kinetic term and the interactions with the vector boson HVV, HHVV. The covariant derivative is

$$\begin{aligned} D_\mu\phi &= \left[\partial_\mu - \frac{1}{2}\mathrm{i}gW_\mu^i\tau^i - \frac{1}{2}\mathrm{i}g'YB_\mu\right] \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ v + H(x) \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ \partial_\mu H(x) \end{bmatrix} - \frac{\mathrm{i}}{2}\left[1 + \frac{M}{v}\right] \begin{bmatrix} gvW_\mu^+ \\ -v\sqrt{\frac{g^2+g'^2}{2}}Z_\mu \end{bmatrix} \end{aligned}$$

where $Y = 1$ for the Higgs field. Therefore, one gets

$$(D_\mu\phi)^\dagger(D^\mu\phi) = \cdots = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \left(1 + \frac{M}{v}\right)^2 \left[\left(\frac{gv}{2}\right)^2 W_\mu^+W_\mu^- + \frac{v^2}{2}\frac{g^2 + g'^2}{4}Z_\mu Z^\mu\right]$$

The masses of the gauge bosons are then

$$m_W^2 = \frac{1}{2}gv\,, \quad m_Z^2 = v^2\frac{g^2 + g'^2}{4} = \frac{m_W^2}{\cos^2\theta}$$

while the mass of the electromagnetic field is zero because a massa term does not appear. [r]

The vacuum expectation value must be

$$\frac{G_\mathrm{F}}{\sqrt{2}} = \frac{g^2}{8M_W^2} \implies v = (\sqrt{2}G_\mathrm{F})^{-\frac{1}{2}} \approx 246\,\mathrm{GeV}$$

This energy is the energy scale associated to a symmetry breaking.

**Interactions.** The three-point interaction is

$$\begin{aligned} \mathrm{HVV} &= \frac{2m_W^2}{v} W_\mu^+ W_-^\mu H + \frac{m_Z^2}{v} Z_\mu Z^\mu H \\ &= g m_W \mathrm{WWH} + \frac{1}{2} g \frac{m_Z}{\cos\theta} \mathrm{ZZH} \end{aligned}$$

at the second equality one uses the relations above for the masses. The Higgs boson couples in a way proportional to the mass of the vector bosons. The vertex is [r]

$$i\eta_{\mu\nu} \begin{cases} m_W \,, & \mathrm{HWW} \\ \frac{m_W}{\cos\theta} \,, & \mathrm{HZZ} \end{cases}$$

Another interaction is the HHVV.

**Remark.** SSB produces a Lagrangian with massive vector boson in a gauge invariant way. This is done through a scalar field with non zero vev.

However, there are heavy fermions which have

$$\langle \bar{\psi}\psi \rangle \neq 0$$

If they were charged properly for SU(2) left [r] If such field exists, its energy is higher than the current probed energy. A model like this is called composite Higgs.

The Higgs mass is

$$m_H \approx 125\,\mathrm{GeV}$$

The mass is known from experiment and one has

$$m_H^2 = 2\lambda v^2$$

To measure directly the parameter $\lambda$ one has to obtain a three-point Higgs self-coupling. The Higgs can be obtain from gluon fusion. [r] however there is a background. The dependence on $\lambda$ comes from the destructive interference between the two diagrams.

**Yukawa interaction.** [r] One has to recover a Dirac mass. [r] The kinetic term for fermions, one can compactly write

$$\mathcal{L}_\mathrm{fermions} = \sum_{f=1}^{n} \sum_{k=1}^{5} \bar{\psi}_\mathrm{R}^{(f)} \mathrm{i}\not{D} \psi_\mathrm{R}^{(f)}$$

where

$$k \in \{ \begin{bmatrix} u_\mathrm{L} \\ d_\mathrm{L} \end{bmatrix}, u_\mathrm{R}, d_\mathrm{R}, \begin{bmatrix} \nu_{eL} \\ e_\mathrm{L} \end{bmatrix}, e_\mathrm{R} \}$$

and $f = 1, 2, 3$ so $n$ is the number of fermionic families.

The Lagrangian is invariant under the transformation U(n)

$$\psi_k^{(f)} \to U^{fg} \psi_k^{(g)}$$

This is an accidental global symmetry.

One has to mix different families, so the notation used is the following: the primed denotes the interaction eigenstates, the interactions with gauge bosons are diagonal; the unprimed denotes the mass eigenstates. A diagonal matrix is useful because one can read out the propagators. One would like to express everything in terms of the mass eigenstates. Therefore

$$Q'_{\mathrm{L}} = \begin{bmatrix} u'_{\mathrm{L}} \\ d'_{\mathrm{L}} \end{bmatrix}, u'_{\mathrm{R}}, d_{\mathrm{R}}, L'_{\mathrm{L}} = \begin{bmatrix} \nu'_{eL} \\ e'_{\mathrm{L}} \end{bmatrix}, e'_{\mathrm{R}}$$

With the SM field content, the only term that can be added that respects Lorentz invariance, gauge invariance and renormalizability as the Yukawa interaction

$$\bar{\psi}\psi\phi$$

The Yukawa Lagrangian is

$$\mathcal{L}_{\mathrm{Y}} = -\bar{Q}'_{\mathrm{L}} h'_D d'_{\mathrm{R}} \phi - \bar{Q}'_{\mathrm{L}} h'_U u'_{\mathrm{R}} \phi^c - \bar{L}'_{\mathrm{L}} h'_E e_{\mathrm{R}} \phi + \text{h.c.}$$

The $h'$ terms are $n \times n$ complex matrices in flavor space. Their entries are numbers so their mass dimension is zero. In the SM $n = 3$ so a total of 27 parameters; some are fixed by the masses and [r].

The terms are invariant under SU(2) left and hypercharge.

Also one has

$$\phi^c = \varepsilon\phi\,, \quad \phi = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \mathrm{i}\tau^2$$

The anti-fundamental representation of SU(2) is the same as the fundamental of SU(2). One has

$$\phi^c = \frac{1}{\sqrt{2}} \begin{bmatrix} v + H(x) \\ 0 \end{bmatrix}$$

In the unitary gauge, the Lagrangian is

$$\mathcal{L}_{\mathrm{Y}} = -\frac{v + H}{\sqrt{2}} [\bar{d}'_{\mathrm{L}} h'_D d'_{\mathrm{R}} + \bar{u}'_{\mathrm{L}} h'_U u'_{\mathrm{R}} + \bar{e}'_{\mathrm{L}} h'_E e'_{\mathrm{R}} + \text{h.c.}]$$

Imagining that the matrices are diagonal, one obtains the Dirac mass for each field. Though the matrices are not diagonal because there is flavour mixing. By the singular-value decomposition, for each $h'$ with complex entries, not necessarily square, it holds

$$h' = UhV^\dagger$$

where $U$ and $V$ are unitary matrices, and $h$ is a diagonal matrix with positive and real entries (i.e. eigenvalue). In this way one can write a diagonal matrix, with other two acting on the spinors, but this does not matter because spinor have to be rotated. In fact

$$\mathcal{L}_{\mathrm{Y}} = -\frac{v + H}{\sqrt{2}} [\bar{d}'_{\mathrm{L}} U_D h_D V_D^\dagger d'_{\mathrm{R}} + \bar{u}'_{\mathrm{L}} U_U h_U V_U^\dagger u'_{\mathrm{R}} + \bar{e}'_{\mathrm{L}} U_E h_E V_E^\dagger e'_{\mathrm{R}} + \text{h.c.}]$$

The matrices $h$ are all diagonal. One defines

$$d_{\mathrm{R,R}} = U_D^\dagger d'_{\mathrm{R,L}}$$

and same for $u$ and $e$. One has

$$\mathcal{L}_{\mathrm{Y}} = -\frac{v + H}{\sqrt{2}} [\bar{d}_{\mathrm{L}} h_D d_{\mathrm{R}} + \bar{u}_{\mathrm{L}} h_U u_{\mathrm{R}} + \bar{e}_{\mathrm{L}} h_E e_{\mathrm{R}} + \text{h.c.}]$$

These are the mass eigenstates. So

$$m_D = \frac{v}{\sqrt{2}} h_D = \mathrm{diag}(m_d, m_s, m_b)$$

same for $m_U$ and $m_E$.

After this rotation, the kinetic terms remains diagonal [r]