Theory and Phenomenology of Fundamental Interactions

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Contents

Lecture 1

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The course will make full sense after knowing QFT 1 & 2.

Exam. Computation of tree-level amplitude. [r] The oral part is comprised of 3 questions discussing theoretical aspects and fundamental aspects of the theory.

1 Introduction

The first part of the course is about the completion of the theoretical description of the electroweak sector of the Standard Model. Then Yukawa interaction linked to the [r] matrix: weak interactions are not diagonal with respect to flavor families. [r] Then discussion of phenomenology of the electroweak and Higgs sectors at the LHC.

The second part treats QCD, its gauge invariance; perturbative regime, the subtleties, universal divergences, properties of non-abelian gauge theories.

The last part deals with hadronic collision and its non-trivial description.

Further topics are neutrino masses, etc.

1.1 The bigger picture

The course deals with the Standard Model. One needs to understand how it must be viewed from a historical perspective and a modern perspective. To study fundamental interactions (excluding gravity) the popolar choice is quantum field theory: quantum electrodynamics, electroweak theory, quantum chromodynamics. The first is an abelian gauge theory, while the last two are non-abelian.

In these theories, the Lagrangian density is the fundamental object: it is Lorentz invariant, invariant under Poincaré transformation, and describes quantized fields. The particles are excitations of the fields. Every Lagrangian can be divided in the free term and the interaction term

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\mathrm{I}$$

*`https://github.com/M-a-s-o/notes`

The free Lagrangian contains kinetic terms, the propagators [r]. The interaction Lagrangian contains the interaction terms, which are represented as vertices in Feynman diagrams. The quantum electrodynamics Lagrangian is

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(\mathrm{i}\not{\partial} - m)\psi - eQ\bar{\psi}\gamma^\mu\psi A_\mu$$

where e is the elementary charge and Q is the quantum number of charge.

If one assumes that fields and interactions to be truly fundamental:

- then the theory has to be unitary and predictive at all energies (so predict final results and amplitudes that do not exceed 1).
- The non-trivial link to quantum field theory is that the theory must be renormalizable. The lagrangian must be constrained: the coupling constants cannot have negative mass dimension. Since fundamental interactions are well described by gauge theories, gauge invariance is a fundamental requirement of the renormalizability of the theory. [r] Once one picks a gauge group, it must produce observable phenomena and one must keep abiding its rules to keep gauge invariance.

Effective Standard Model. The Standard Model [r] but it is not a complete theory. There is no natural candidate for dark matter, gravity is not accounted for, neutrino masses are not explained. The Model has to fail at some point, at some energy scale where a phenomenon cannot be described with the field content of the Model. Therefore, the Standard Model must be an effective theory (as opposed to a fundamental theory). Therefore, it is allowed to add non-renormalizable operators: terms in the Lagrangian with negative mass dimension. If one wants to understand the physics beyond the Standard Model from a bottom-up approach, this is a middle ground: one modifies the theory enough to compute phenomena [r]. A similar story happened when going from Fermi's four-interaction theory to the intermedia vector boson theory. The interaction term for Fermi theory is $\bar{\psi}\psi\bar{\psi}\psi$ with dimension six, so the coupling constant must have mass dimension -2. [r] One may add non-renormalizable operators built from Standard Model objects that respect its symmetry group. In this paradigm one may not use renormalizability and unitarity, so the predictions are valid up to some energies.

1.2 Weyl spinors

A massless Dirac field is made of two Weyl field. A massive Dirac spinor is made of left- and right-chiral components. A term like

$$\bar{\psi}_{\mathrm{L}}\psi_{\mathrm{R}}$$

is not invariant under $\mathrm{SU}(2)_{\mathrm{L}}$.

Lorentz group. The proper Lorentz group [r] has six generators. [r] through the exponential map as

$$R(\hat{e},\theta) = \exp(-\mathrm{i}\theta\hat{e}\cdot\mathbf{J})\,, \quad B(\hat{u},\eta) = \exp(-\mathrm{i}\eta\hat{u}\cdot\mathbf{K})$$

where $\mathbf{J}$ are the generators of rotations and $\mathbf{K}$ are the generators of boosts. The explicit form of the generators can be obtained from infinitesimal transformations. For example

$$J_z = \mathrm{i}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\,, \quad K_x = \mathrm{i}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

These are a fundamental representation of the Lorentz algebra $\mathrm{SO}(1,3)$

$$[J_i, J_j] = \mathrm{i}\varepsilon_{ijk}J_k\,, \quad [K_i, K_j] = -\mathrm{i}\varepsilon_{ijk}J_k\,, \quad [J_i, K_j] = \mathrm{i}\varepsilon_{ijk}K_k$$

The above algebra can be rewritten as

$$[M_{\mu\nu}, M_{\rho\sigma}] = \mathrm{i}(\eta_{\nu\rho}M_{\mu\sigma} + \cdots)$$

where $M^{\mu\nu}$ an anti-symmetric tensor such that

$$M^{0i} = K_i \,, \quad M^{ij} = \varepsilon_{ijk} J_k$$

[r] In order to label the representations one has to use mass and spin.

The Lorentz algebra can be decomposed in two other algebras

$$\mathrm{so}(1,3) \simeq \mathrm{su}(2) \oplus \mathrm{su}(2)$$

[r] In fact, one combines

$$J_k^{\pm} = \frac{1}{2}(J_k \pm \mathrm{i} K_k) \implies [J_i^+, J_j^-] = 0 \,, \quad [J_i^{\pm}, J_j^{\pm}] = \mathrm{i}\varepsilon_{ijk} J_k^{\pm}$$

In order to label the possible elementary fields associated to Lorentz group (in general the Poincaré group) one needs two non-negative half-integers (s_1, s_2). For $(0,0)$ the transformation under each su(2) is trivial, so they do not transform, they are a singlet: it is a scalar field. The next representations are $(1/2, 0)$ and $(0, 1/2)$ for right-chiral Weyl spinor and left-chiral Weyl spinor. For a vector field one has $(1/2, 1/2)$. P 146 Schwartz

For right-chiral Weyl spinors u_R, the generators are

$$J_i^+ = \frac{1}{2}\sigma_i \,, \quad J_i^- = 0 \implies J_i = \frac{1}{2}\sigma_i \,, \quad \mathrm{i}K_i = \frac{1}{2}\sigma_i$$

Therefore, for a rotation, one has

$$R = \exp(-\mathrm{i}\boldsymbol{\theta} \cdot \mathbf{J}) = \exp\left(-\frac{1}{2}\mathrm{i}\boldsymbol{\theta} \cdot \boldsymbol{\sigma}\right)$$

and for a boost

$$B = \exp(-\mathrm{i}\boldsymbol{\eta} \cdot \mathbf{K}) = \exp\left(-\frac{1}{2}\boldsymbol{\eta} \cdot \sigma\right)$$

Similarly for a left-chiral Weyl spinor. For a four-vector one has

$$J_i = J_i^+ + J_i^-$$

One may realize that there are two states that transforms as [r] The singlet component under rotation is A^0, while the triplet is A^i.

Parity. Under parity, the generators of rotations do not transform $J \to J$, while boosts do $K \to -K$. Also parity maps

$$(s_1, s_2) \to (s_2, s_1)$$

Therefore, left-chiral spinor becomes a right-chiral spinor, while a vector is still a vector.

Weyl spinors. Weyl spinors can be combined into a vector. Considering

$$\sigma_{\pm}^{\mu} = (I, \pm\boldsymbol{\sigma})$$

which is equivalent to the notation

$$\sigma^\mu = (I, \sigma^i) \,, \quad \bar{\sigma}^\mu = (I, -\sigma^i)$$

Therefore a vector is given by

$$u_\mathrm{R}^\dagger \sigma^\mu u_\mathrm{R} \,, \quad u_\mathrm{L}^\dagger \bar{\sigma}^\mu u_\mathrm{L}$$

These are bilinear objects in the spinor fields. One may use them to construct Lagrangians.

Lagrangian. One can build a Lagrangian from these fields by requiring that

$$u_{\mathrm{R,L}} \to \mathrm{e}^{\mathrm{i}\theta} u_{\mathrm{R,L}}$$

In fact, one may have

$$\mathcal{L}_{\mathrm{Weyl}} = \mathrm{i} u^\dagger_{\mathrm{R,L}} \sigma^\mu_\pm \, \partial_\mu u_{\mathrm{R,L}}$$

[r] The equations of motion are

$$\sigma^\mu \, \partial_\mu \psi_{\mathrm{R}} = 0 \,, \quad \bar{\psi}^\mu \, \partial_\mu \psi_{\mathrm{L}} = 0 \implies (\partial_0 \pm \boldsymbol{\sigma} \cdot \nabla) \psi_{\mathrm{R,L}} = 0$$

Acting on the last equation with $(\partial_0 \mp \boldsymbol{\sigma} \cdot \nabla)$ on the left side, one gets a massless Klein–Gordon equation

$$\Box \, \psi_{\mathrm{R,L}} = 0$$

In momentum space one has [r]

$$\psi_{\mathrm{R,L}} = \hat{\psi}_{\mathrm{R,L}}(\mathbf{k}) \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \,, \quad k^0 = |\mathbf{k}|$$

where the hat indicates the Fourier transform. In momentum space, the Weyl equations are

$$[k^0 \mp (\mathbf{k} \cdot \sigma)] \hat{\psi}_{\mathrm{R,L}} = 0$$

So the spinor is an eigenvector of the helicity operator

$$\frac{1}{2} \frac{\boldsymbol{\sigma} \cdot \mathbf{k}}{|\mathbf{k}|}$$

[r] from this, right-chiral spinors have positive helicity and similar.

Lecture 2

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Weyl equations are important because electroweak theory violates parity: left- and right-chiralities transform differently.

If one relaxes the global U(1) symmetry, one can add a Lorentz scalar to the Lagrangian. Composing two left-chiral representations

$$\left(\frac{1}{2}, 0\right) \otimes \left(\frac{1}{2}, 0\right) = (0, 0) \oplus (1, 0)$$

One has a part that transforms as a vector under SU(2) and one as a scalar. The singlet combination can be extracted as

$$\varepsilon_{ab} u^a_\pm u^b_\pm \,, \quad \varepsilon_{ab} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

where each spinor has two complex entries. The above combination is Lorentz invariant. When checking that it is Lorentz invariant, the Levi–Civita tensor gives the determinant of some matrix, which is the exponential of the Pauli matrices, so it is 1. The combination is not zero because the spinor product is not symmetric: the spinor components are Grassmann odd fields

$$\{u^a_\pm, u^b_\pm\} = 0$$

Such a term in a Lagrangian is allowed

$$\mathcal{L}^\pm_{\mathrm{Weyl}} = \mathrm{i} u^\dagger_\pm \sigma^\mu_\pm \, \partial_\mu u_\pm - \frac{1}{2} m [\varepsilon_{ab} u^a_\pm u^b_\pm + \mathrm{h.c.}]$$

This is a Majorana mass term, it is bilinear in the fields. This term is not invariant under U(1) of the spinors. If one would like to implement a global or local transformation such the previous cannot have a mass term like the one above. [r] For charged (under some symmetry group) chiral fermions, one cannot have a Majorana mass term. In QED a charged fermion transforms non trivially under $\mathrm{U}(1)_{\mathrm{EM}}$.

For a non charged particle of any symmetry of the Standard Model, such a term is allowed, like right-handed neutrinos.

1.3 Dirac spinors

Under parity, the right- and left-chiral representations are mapped into one another. Parity invariance means the presence of both chiralities. A Dirac spinor is a combination of Weyl spinors (in the Weyl basis)

$$\psi = \begin{bmatrix} u_- \\ u_+ \end{bmatrix} = \begin{bmatrix} \psi_\mathrm{L} \\ \psi_\mathrm{R} \end{bmatrix} , \quad \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$$

This is a direct sum of representations because the two chiral representations do not mix. In the Dirac basis, the Dirac spinor is

$$\psi = \frac{1}{\sqrt{2}} \begin{bmatrix} u_+ + u_- \\ u_+ - u_- \end{bmatrix}$$

The course uses Weyl basis (also called chiral basis). In this representation, the Dirac matrices are

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix} , \quad \gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} -I_2 & 0 \\ 0 & I_2 \end{bmatrix}$$

The projection operators are then

$$P_{\mathrm{L,R}} = \frac{1 \mp \gamma^5}{2} , \quad P_\mathrm{L} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} , \quad P_\mathrm{R} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} , \quad P_{\mathrm{L,R}}^2 = P_{\mathrm{L,R}} , \quad P_\mathrm{L} P_\mathrm{R} = P_\mathrm{R} P_\mathrm{L} = 0$$

The Dirac conjugate spinors are

$$\bar{\psi} = \psi^\dagger \gamma^0 , \quad \bar{\psi}_{\mathrm{L,R}} = \gamma_{\mathrm{L,R}}^\dagger \gamma^0 = \bar{\psi} \frac{1 \pm \gamma^5}{2}$$

notice how they have the opposite chirality of their non conjugate part. Some properties of the fifth gamma matrix are

$$\gamma_5 = \gamma_5^\dagger , \quad \gamma_5^2 = I , \quad \{\gamma_\mu, \gamma_5\} = 0 , \quad \{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$$

The free Dirac lagrangian in terms of Weyl spinors is

$$\mathcal{L}_\mathrm{Dirac} = \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi = \bar{\psi}_\mathrm{L} \mathrm{i}\,\partial\!\!\!/ \psi_\mathrm{L} + \bar{\psi}_\mathrm{R} \mathrm{i}\,\partial\!\!\!/ \psi_\mathrm{R} - m(\bar{\psi}_\mathrm{R}\psi_\mathrm{L} + \bar{\psi}_\mathrm{L}\psi_\mathrm{R})$$

This mass term is different from Majorana's. This term couples the two chiral fields. It is invariant under $\mathrm{U}(1)_\mathrm{EM}$

$$\psi \to \mathrm{e}^{\mathrm{i}\alpha}\psi$$

but it is not invariant under different transformations of the chiral fields (like $\mathrm{SU}(2)_\mathrm{L}$). In electroweak theory, one needs to [r].

Vector theory. A vector theory does not distinguish the chiral parts of a field. As a consequence, parity is conserved. This is the reason why the fermionic current is a vector

$$\bar{\psi}\gamma^\mu\psi = \bar{\psi}_\mathrm{L}\gamma^\mu\psi_\mathrm{L} + \bar{\psi}_\mathrm{R}\gamma^\mu\psi_\mathrm{R}$$

which is invariant under

$$\psi_{\mathrm{L,R}} \to U(x)\psi_{\mathrm{L,R}}$$

Chiral theory. A chiral theory treats fields differently based on their chirality. It is parity violating. A Dirac mass term is not gauge invariant for chiral theories.

For QED and QCD a Dirac mass term is allowed; only the weak sector of the Standard Model creates problems. Under $\mathrm{SU}(2)_\mathrm{L}$ a left-chiral field transforms non trivially

$$\psi_\mathrm{L} \to U(x)\psi_\mathrm{L}$$

while a right-chiral field remains the same.

1.4 Conventions

A Dirac field may be written in a Fourier series as

$$\psi(t,\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\sum_s [u_s(\mathbf{k})a_s(\mathbf{k})\mathrm{e}^{-\mathrm{i}kx} + v_s(\mathbf{k})b^\dagger(\mathbf{k})\mathrm{e}^{ikx}]$$

where u and v are wave functions? [r]. The normalization of the free spinors is

$$\sum_s u_s(\mathbf{p})\bar{u}_s(\mathbf{p}) = \not{p} + m\,, \quad \sum_s v_s(\mathbf{p})\bar{v}_s(\mathbf{p}) = \not{p} - m$$

which solve

$$(\not{p} - m)u_s = 0\,, \quad \bar{u}_s(\not{p} - m) = 0\,, \quad (\not{p} + m)v_s = 0\,, \quad \bar{v}_s(\not{p} + m) = 0$$

[r] signs
A fermion propagator is

$$\frac{\mathrm{i}(\not{p} + m)}{p^2 - m^2 + \mathrm{i}\varepsilon}$$

The Feynman amplitude is

$$\mathrm{i}\mathcal{M} = \sum \text{Feynman diagrams}$$

The sum over quantum numbers of external particles is

$$\sum |\mathcal{M}|^2$$

The sum and average is instead

$$\overline{\sum} |\mathcal{M}|^2$$

[r] The tree-level cross-section is obtained

$$\mathrm{d}\sigma = \mathcal{F}\overline{\sum}|\mathcal{M}|^2 \mathrm{d}\phi_n$$

where $\mathcal{F}$ is the flux factor and $\mathrm{d}\phi_n$ is the phase space

$$\mathrm{d}\phi_n = (2\pi)^4\delta^{(4)}(\Delta p^\mu)\prod_{i=1}^n [\mathrm{d}k_i]$$

where the Lorentz invariant phase space measure is

$$[\mathrm{d}k_i] = \frac{\mathrm{d}^3k_i}{(2\pi)^3 2E_i}\,, \quad E_i^2 = m_i^2 + |\mathbf{k}_i|^2$$

In general, the phase space contains also symmetry factors $\frac{1}{n!}$ if the final state particles are identical bosons.

The decay width of a particle M decaying is

$$\mathrm{d}\Gamma = \frac{1}{2M}\overline{\sum}|\mathcal{M}|^2\,\mathrm{d}\phi_n$$

The total width is

$$\Gamma = \int \mathrm{d}\Gamma = \frac{1}{2M}\int \overline{\sum}|\mathcal{M}|^2\,\mathrm{d}\phi_n$$

The Dirac traces is not explicitly written but a bracket is present [r] for Bhabha scattering one has [r] wrong. For computing the cross section one needs $\mathcal{M}^*$ and therefore

$$\sum |\mathcal{M}|^2 = [\cdots]$$

Four products of momenta get shortened

$$p_1^\mu p_{2\mu} = (12) = (p_1 \cdot p_2)$$

1.5 Quantum electrodynamics

One applies the gauge principle to go from a global symmetry to a local symmetry and make the lagrangian invariant. The interactions appear in the covariant derivative.

The QED lagrangian is

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - q\bar{\psi}\gamma^\mu\psi A_\mu$$

The free lagrangian

$$\mathcal{L}_0 = \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi$$

is invariant under global U(1). Making it local and requiring invariance, the fields transform as

$$\psi'(x) = \mathrm{e}^{\mathrm{i}\alpha(x)}\psi(x)\,, \quad A'_\mu = A_\mu - \frac{1}{q}\,\partial_\mu\alpha(x)$$

The covariant derivative is then

$$D_\mu = \partial_\mu + \mathrm{i}qA_\mu$$

So that the invariant lagrangian is

$$\mathcal{L} = \bar{\psi}(\mathrm{i}D\!\!\!\!/ - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

The derivative of the field does not transform the same way as the field, however the covariant derivative does

$$(\partial_\mu\psi)' \neq \mathrm{e}^{\mathrm{i}\alpha(x)}\partial_\mu\psi\,, \quad (D_\mu\psi)' = \mathrm{e}^{\mathrm{i}\alpha(x)}D_\mu\psi$$

Therefore the term $\bar{\psi}D_\mu\psi$ is gauge invariant. The field strength tensor is also gauge invariant.

Remark 1.1. The term

$$(F_{\mu\nu}F^{\mu\nu})^2$$

is not included because its mass dimension is 8 and the theory is not renormalizable.

Remark 1.2. The term $A^\mu A_\mu$ is not gauge invariant and it corresponds to a mass term, but the photon is massless.

Lecture 3

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1.6 Non-abelian gauge groups

The Standard Model is based in part on SU(n) symmetry groups. Consider $U \in \mathrm{SU}(n)$, it can be expressed as

$$U = \exp[\mathrm{i}\theta^a t^a]\,, \quad a = 1, \cdots, n^2 - 1$$

where t^a are the generators of the group and θ^a are real parameters. Elements of such group have the properties

$$UU^\dagger = U^\dagger U = I\,, \quad \det U = 1$$

The generators, which belong to the algebra, obey

$$t^a = (t^a)^\dagger\,, \quad 1 = \det e^{t^a} = e^{\operatorname{Tr} t^a} \implies \operatorname{Tr} t^a = 0$$

So the generators are hermitian traceless matrices. The generators are normalized to give

$$\operatorname{Tr}\big(t^a t^b\big) = T_R\delta^{ab}\,, \quad T_R = \frac{1}{2}$$

The commutation relations of the Lie algebra su(n) are

$$[t^a, t^b] = \mathrm{i}f^{abc}t^c$$

The coefficient f^{abc} are the structure constants of the Lie algebra. For non-abelian gauge groups, the commutator is not identically zero.

Exercise. Defining the matrix
$$\tau^{ab} \equiv \mathrm{i}[t^a, t^b]$$
where a, b do not label the components. Show that
$$\operatorname{Tr} \tau^{ab} = 0 \,, \quad (\tau^{ab})^\dagger = \tau^{ab}$$
and that f is totally anti-symmetric and real.

In general, a d-dimensional representation of an algebra is a set of $d \times d$ matrices that satisfy the commutation relation
$$[T^a, T^b] = \mathrm{i} f^{abc} T^c$$
The number of T^a is the dimension of the Lie group.

The important representations are the fundamental, anti-fundamental and the adjoint representations. The fundamental representation is an N-dimensional representation. It acts on N-dimensional objects. The adjoint representation is given by the structure constants
$$(T^a)_{bc} = \mathrm{i} f^{bac}$$
where bc are the components of the matrix T^a. It is a $N^2 - 1$ dimensional representation.

Gauge symmetry. The matrix
$$U(x) = \mathrm{e}^{\mathrm{i}\theta^a(x) t^a}$$
depends on space-time coordinates. In the fundamental representation, the matrix acts on N dimensional objects. A spinor does not transform trivially, but as
$$\psi'(x) = U(x)\psi(x)$$
The derivative transforms as
$$(\partial_\mu \psi)' = \partial_\mu[U(x)\psi(x)] = (\partial_\mu U)\psi + U\, \partial_\mu \psi$$
but this is not favorable. One uses the covariant derivative
$$(D_\mu)_{ij} = \partial_\mu \delta_{ij} + \mathrm{i} g t^a_{ij} A^a_\mu$$
where the ij indices treats gauge group components. Therefore, there are $N^2 - 1$ gauge fields A^a_μ. The covariant transforms in the same way as the field
$$(D_\mu \psi)' = D'_\mu \psi' = U(x) D_\mu \psi$$
This implies that the gauge fields transform as
$$(t^a A^a_\mu)' = t^a A'^a_\mu = U(t^a A^a_\mu) U^{-1} + \frac{\mathrm{i}}{g}(\partial_\mu U) U^{-1}$$
To define the field strength tensor one goes by analogy with quantum electrodynamics
$$\mathrm{i} q F^{\mathrm{QED}}_\mu = [D^{\mathrm{QED}}_\mu, D^{\mathrm{QED}}_\nu] \,, \quad D^{\mathrm{QED}}_\mu = \partial_\mu + \mathrm{i} q A_\mu$$
In general, one defines
$$\mathrm{i} g t^a F^a_{\mu\nu} = [D_\mu, D_\nu]$$
This gives
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$
Under gauge transformation one has
$$(t^a F^a_{\mu\nu})' = t^a F'^a_{\mu\nu} = U(t^a F^a_{\mu\nu}) U^{-1}$$
The kinetic term for gauges boson is
$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = -\frac{1}{2} \operatorname{Tr}(\mathbf{F}^{\mu\nu} \mathbf{F}_{\mu\nu}) \,, \quad \mathbf{F}_{\mu\nu} = t^a F^a_{\mu\nu} \,, \quad F^a_{\mu\nu} F^{a\mu\nu} = 2 \operatorname{Tr}(\mathbf{F}^{\mu\nu} \mathbf{F}_{\mu\nu})$$

One sees that

$$\mathbf{F}'^{\mu\nu} = U\mathbf{F}^{\mu\nu}U^{-1}$$

For a non-abelian gauge group the Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\psi}_j(x)[\mathrm{i}\not{D}_{jk} - m\delta_{jk}]\psi_k(x)$$

The indices jk are the one carried by the generators t^a_{jk}, not the four components of Dirac spinors. In the fundamental representation one has

$$a = 1, \dots, N^2 - 1\,, \quad j, k = 1, \dots, N$$

The covariant derivative is

$$D^\mu_{jk} = \partial^\mu \delta_{jk} + \mathrm{i}g t^a_{jk} A^a_\mu$$

There are $N^2 - 1$ gauge boson fields A^a_μ and they are massless because massive terms $m^2 A^a_\mu A^{a\mu}$ are not guage invariant.

A term $(F^a_{\mu\nu}F^{a\mu\nu})^n$ is not renormalizable for $n > 1$.

The covariant derivative for the electroweak sector

$$D_\mu = \partial_\mu - \mathrm{i}q t^a A^a_\mu$$

while for the quantum chromodynamics sector is

$$D_\mu = \partial_\mu + \mathrm{i}g t^a A^a_\mu$$

[r] DC
The interaction term

$$\bar{\psi}\not{D}\psi \rightsquigarrow -g(\bar{\psi}t^a\gamma^\mu\psi)A^a_\mu$$

generates a vertex equal to

$$-\mathrm{i}g t^a_{ji}\gamma^\mu$$

In the kinetic part FF there are three-vertices

$$FF \rightsquigarrow g f^{abc}(\partial_\mu A^a_\nu)A^b_\mu A^c_\nu$$

which is called derivative vertex. In momentum space, the derivative is a momentum [r]

$$-g f^{abc}[(p_a - p_b)^\gamma \eta^{\alpha\beta} + (p_b - p_c)^\alpha \eta^{\beta\gamma} + (p_c - p_a)^\beta \eta^{\alpha\gamma}]$$

The charged boson have to interact with the carrier of the force, which is another boson.

In the kinetic part there are also four-vertices

$$FF \rightsquigarrow g^2 f f AAAA$$

1.7 History of the Standard Model

Fermi theory. Fermi theory is a theory of the electroweak sector. The Lagrangian is

$$\mathcal{L} = -\frac{G_\mathrm{F}}{\sqrt{2}}J^\mu J^\dagger_\mu\,, \quad J_\mu = \bar{\psi}_l\gamma_\mu(1-\gamma_5)\psi_{\nu_l} + \bar{\psi}_d\gamma_\mu(1-\gamma_5)\psi_u = L_\mu + H_\mu$$

The current contains both the leptonic part and the hadronic part. Examples of weak decays are the muon beta decay and the neutron beta decay.

One can compute tree-level total decay widths Γ and differential widths $\mathrm{d}_\Omega\Gamma$ from which one obtains the Fermi constant $G_\mathrm{F} = 1.16 \times 10^{-5}\,\mathrm{GeV}^{-2}$. The Lagrangian correctly describes these two processes [r].

Fermi theory is a V-A theory because bilinears of the types

$$V^\mu = \bar{\psi}_1\gamma^\mu\psi_2$$

transforms like a polar vector while bilinears like

$$A^\mu \bar{\psi}_1 \gamma^\mu \gamma^5 \psi_2$$

transforms like an axial vector, or pseudo-vector. A theory of this type is in accordance with experiments. A V-A theory is maximally parity violating. Check that $V'^\mu = \Lambda^\mu{}_\nu V^\mu$ and $A'^\mu = \bar{\Lambda}^\mu{}_\nu A^\nu$ [r]. To see how it is maximally violating, one can see that from the Lagrangian one has terms like

$$A^\mu V_\mu \to -A^\mu V_\mu$$

Processes like neutrino deep-inelastic scattering. For example $\bar{\nu}_e u \to d e^+$ one has a differential cross section

$$\mathrm{d}_\Omega \sigma = \frac{G_\mathrm{F}^2}{8\pi^2} \frac{s}{4} (1 + \cos\theta)^2 \,, \quad s = (p_1 + p_2)^2$$

but this cross section implies that the scattering matrix is not unitary.

Therefore Fermi theory is not renormalizable because the coupling constant has negative mass dimensions and it is not unitary. To fix the second problem one introduces a vector gauge boson.