General Relativity

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Contents

Lecture 1

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Exam: three easy questions seen during the course, three questions somewhat new, two harder questions that require critical thinking.

1 Introduction

General relativity replaces the Newtonian description of gravity. The force of Newtonian gravity is proportional to the reciprocal of the distance squared between two masses. Changing the distance changes the force immediately, but this is not consistent with Special Relativity.

The problems with Newtonian gravity can be solved using field theoretical methods to construct theories in agreement with Special Relativity. Though, Einstein took a different route. He identified something often overlooked in what was already known: the equivalence principle. Einstein wanted to make sure that Maxwell's equations were in agreement with the principle of relativity. Newtonian mechanics was made possible by the equivalence principle: in its basic form, it states that gravitational mass — that is the mass that appears in Newton's law of gravitation — is equal to inertial mass — that is the mass that appears in Newton's second law of motion. A priori, these two masses can be different, but the equivalence principle postulates their equality. One can study the implications of the principle. Two thought experiments (gedankenexperimente) can help understand such implications.

First thought experiment. Consider a mass m inside a box with two propellers at the bottom. The box is accelerated upwards at a constant acceleration g equal to the sea-level Earth's acceleration. In the reference frame of the box, there is a downward force pushing the mass towards and then against the floor of the box. This behaviour is the exact same experienced on the surface of the Earth: holding a mass and letting it go, it experiences a downward force towards the floor. In short: it looks like gravity is present.

One can locally mimic gravity with an apparent force. The principle works locally: because of tidal forces — the force of gravity varies with distance — one can distinguish rocket-powered acceleration from a mass' gravitational field.

Second thought experiment. Consider a free falling box towards Earth. A mass inside the box experiences only gravity. In the frame of the box, the mass is floating and one may not distinguish the situation from the one where gravity is absent in the first place. In short: it looks like gravity is absent.

An example of a free falling experiment is the International Space Station (ISS): its altitude from the Earth's surface is 400 km, so the gravity is about 90% the one on the surface, but the feeling is that of weightlessness: the ISS is constantly falling, though it has enough lateral velocity that the Earth below moves away faster than the ISS can fall.

Einstein's equivalence principle. The equivalence principle is true for electromagnetism and all of physics. For a small enough box, one may not tell whether gravity is acting or not. As adding the constancy of the speed of light to Galilean relativity brings Special Relativity, then adding the equivalence principle to classical physics gives General Relativity.

Consequences. Since the principle applies to electromagnetism, then it also applies to light. Consider a laser shining a beam of light from left to right across a box. If the box is propelled upwards, the laser hits the right wall lower (relative to the floor) than it was shot from the left side. From the reference frame of the box, the laser is bending downwards. By the equivalence principle, the same should apply to a box immersed in a gravitational field. This prediction has been experimentally verified by gravitational lensing.

Curvature. Space has always been thought of as Euclidean space. However, if light curves, then the definition of straight line requires more caution. On a sphere, the sum of internal angles of a triangle is no longer π, but greater. After observing light curving, the geometry of space can no longer be Euclidean: since light travels in a curved trajectory, space itself is curved. On a sphere, the minimum distance between two points is given by the arc of the great circle passing between the two points. A straight line is then defined as the line that minimizes distance.

Since light is described differently by the equivalence principle, then massive objects need new equations also. These need to also explain the motion of planets: the precession of the perihelion of Mercury was not explained by Newtonian gravity.

A free particle in space-time maximizes its proper time which is proportional to the relativistic action. Objects that are only subject to gravity, either massive or massless, — that is, objects in free fall — follow geodesics, trajectories that maximize proper time. The trajectory of objects in free fall is described purely by geometrical ideas: there is no force of gravity.

Curvature is described in a way that matter bends space-time and space-time tells matter how to move.

Lecture 2

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2 Special relativity

Special relativity is a theory obtain from uniting Galilean principle of relativity and the idea that the speed of light is the same in every frame of reference. From the postulates one can derive time dilation, length contraction, relativity of simultaneity, etc. Lorentz transformations are used to go from one frame of reference to another. The Lorentz transformation Λ for a boost in the x direction is given by

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}, \quad \beta = \frac{v}{c}, \quad \gamma = (1-\beta^2)^{-\frac{1}{2}}$$

In this course, the natural unit $c = 1$ will be used. One can express Lorentz transformations also in terms of rapidity λ by setting

$$\gamma = \cosh\lambda, \quad \beta\gamma = \sinh\lambda$$

The Lorentz transformation above becomes

$$\Lambda = \begin{bmatrix} \cosh\lambda & -\sinh\lambda & 0 & 0 \\ -\sinh\lambda & \cosh\lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is a hyperbolic rotation: points in space-time are moved along hyperbolae. Ordinary three-dimensional rotation matrices are orthogonal matrices $R^\top R = RR^\top = I$ and leave the norm of $\mathbb{R}^3$ unchanged. Points are moved along circles. Similarly, Lorentz transformations preserve the Minkowski metric $\tau^2 = t^2 - |\vec{x}|^2$. This invariant is the proper time. For timelike vectors, it measures the time between two events happening in the same place. For spacelike vectors, it measures the spatial distance between two simultaneous events.

Points in Minkowski space are called events to highlight the fact that time is also a coordinate. The proper time of a trajectory is the time measured by an observer moving along that trajectory. Proper time is a physically meaningful quantity independent of frame of reference. The proper time between two timelike events is the shortest time one can hope to measure between those two events.

The reference frame of a moving object can be superimposed on one's own reference frame — using a Minkowski diagram — by tilting the object frame's axes by the same angle towards a bisector of the quadrants: the points on the axes are following hyperbolae. In four dimensional space-time, light rays define a light cone and Lorentz transformations define hyperboloids.

Length of a curve. Proper time is the length of a straight path in Minkowski space, but one can generalized the idea of length to more complicated paths. In Euclidean space, the length of a curve γ is given by the integral

$$\int_\gamma \mathrm{d}l = \int_\gamma \sqrt{\mathrm{d}x^i\,\mathrm{d}x^i} = \int_\gamma \mathrm{d}\lambda\,\sqrt{\partial_\lambda x^i\,\partial_\lambda x^i}$$

In Minkowski space, the time of a trajectory measured by an observer moving along such trajectory, the proper time, is

$$\tau(\gamma) = \int_\gamma \mathrm{d}\tau = \int_\gamma \sqrt{(\mathrm{d}t)^2 - \mathrm{d}x^i\,\mathrm{d}x^i} = \int_\gamma \mathrm{d}t\,\sqrt{1 - \partial_t x^i\,\partial_t x^i} = \int_\gamma \mathrm{d}t\,\sqrt{1-v^2} = \int_\gamma \frac{\mathrm{d}t}{\gamma} = \int_\gamma \mathrm{d}\tau$$

The trajectory and observer need not be inertial. In Euclidean space, a path between two points can be arbitrarily long, so the their distance is the shortest path. By analogy, because of the minus sign, the length of a path between two points in space-time, that is proper time, can be arbitrarily short — by going close to the speed of light —, so the distance between the two points is the longest path. Therefore, proper time is maximized by straight paths in space-time which correspond to inertial observers.

Proper time is proportional to the action of a relativistic free particle

$$S = -m\int \mathrm{d}\tau = -m\int \mathrm{d}t\,\sqrt{1-v^2} \sim -m\int \mathrm{d}t\,\left[1 - \frac{1}{2}v^2 + o(v^2)\right] \approx \int \mathrm{d}t\,\left[-m + \frac{1}{2}mv^2\right]$$

Taking the non-relativistic limit, one obtains the kinetic energy of a free particle. By varying the action, one can obtain the equations of motion $\ddot{x}^j = 0$ which is a straight line.

Four-momentum. Energy and momentum can be combined into a four-vector: $P^\mu = (E, \vec{p})$. The invariant associated with the norm of the vector is mass $m^2 = P_\mu P^\mu$. For an object at rest, $\vec{p} = 0$, one obtains $E = mc^2$. To get the explicit expression for the four-momentum in another reference frame, one can perform a boost from the rest frame:

$$\begin{bmatrix} E' \\ p'_x \\ p'_y \\ p'_z \end{bmatrix} = \begin{bmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma m \\ -\gamma m v \\ 0 \\ 0 \end{bmatrix}$$

By boosting in a direction, the object goes the other way. More generally, mixing the boost with a rotation, it follows $E = \gamma m$ and $\vec{p} = \gamma m \vec{v}$. In the limit of low speeds, one recovers the classical relations

$$E = \frac{m}{\sqrt{1-v^2}} = m\left[1 + \frac{1}{2}v^2 + o(v^2)\right] \approx m + \frac{1}{2}mv^2$$

The rest energy in this expression has a different sign from the one found in the action. The rest energy is a costane potential, so going from the energy to the Lagrangian one has to add a minus sign to the potential.

2.1 Covariant notation

Covariant notation lets one treat time in the same manner as space. The norm of a four-vector is a quadratic form. For the proper time, one has

$$\tau^2 = -\begin{bmatrix} t & x & y & z \end{bmatrix} \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = -x^\top \eta x$$

The metric follows a spacelike notation typical of General Relativity, opposite to the one in QFT. In matrix notation, Lorentz transformations are given by

$$x' = \Lambda x$$

Proper time is an invariant and, calculated in two reference frames, one gets

$$\tau^2 = -x'^\top \eta x' = -(\Lambda x)^\top \eta (\Lambda x) = -x^\top \Lambda^\top \eta \Lambda x \equiv -x^\top \eta x \implies \eta = \Lambda^\top \eta \Lambda$$

Since the equality must hold for all vectors, then the implication must be follow. This means that the quadratic form is invariant under Lorentz transformations — a change of basis — and so is the inner product associated with the quadratic form $x_1^\top \eta x_2$. In Euclidean space, the equation above is $R^\top R = I$, so changes of basis are given by orthogonal matrices.

Lorentz transformations are elements of the Lorentz group $\mathrm{O}(3,1)$. The most general transformation depends on six parameters: three for boosts and three for rotations.

The standard inner product between two four-vectors $x_1^\top x_2$ is not invariant. Though, by defining a four-vector that transforms as

$$y' = \eta \Lambda \eta y$$

one can obtain an invariant inner product

$$x'^\top y' = x^\top \Lambda^\top \eta \Lambda \eta y = x^\top \eta^2 y = x^\top y$$

Vectors transforming as the position vector x are called contravariant and are represented as column vectors, while the vectors defined above are their dual, are called covariant and are represented as row vectors. From a contravariant vector x, one can derive a covariant one by applying the metric ηx and vice versa. From this, the inner product between contravariant and covariant vectors is invariant. In fact, the inner product $x_1^\top \eta x_2^\top$ can be viewed as the inner product between a contravariant vector x_1 and a covariant vector x_2.

The nature of a vector under Lorentz transformations is an important property and is made apparent with notation. Upper indices denote contravariant components x^μ, while lower indices denote covariant components y_μ. Greek indices include all four coordinates, while Latin indices only spatial coordinates. The inner product in Minkowski space becomes $x^\mu y_\mu$ using Einstein's summation convention: upper and lower indices appearing once are summed. Free indices should appear on both sides of an equation in the same position. If these are not the cases, the equation written has not the same form in every reference frame.

Lecture 3

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One can derivate the tensorial structure of the metric tensor. Knowing that the position vector is contravariant, then following the previous rules, one has

$$\tau^2 = -x^\top \eta x = -x^\mu \eta_{\mu\nu} x^\nu$$

In the matrix representation of a two-index tensor, the left index gives the μ-th row and the right index gives the ν-th column. The formula is also in agreement with the formula of a covariant vector

$$y = \eta x \implies y_\mu = \eta_{\mu\nu} x^\nu$$

The indices of the identity matrices are

$$x = Ix \implies x^\mu = \delta^\mu{}_\nu x^\nu$$

where $\delta^\mu{}_\nu$ is the Krockener delta. The inverse of the metric tensor is defined as

$$\eta^{-1}\eta = I \implies (\eta^{-1})^{\mu\rho}\eta_{\rho\nu} = \delta^\mu{}_\nu$$

It also happens that $\eta^{-1} = \eta$, but this is only true for Special Relativity. The Lorentz transformation has the following indices

$$x' = \Lambda x \implies x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

A Lorentz transformation is not symmetric in general. The defining equation of the Lorentz transformation is

$$\eta = \Lambda^\top \eta \Lambda \implies \eta_{\mu\nu} = (\Lambda^\top)_\mu{}^\rho \eta_{\rho\sigma} \Lambda^\sigma{}_\nu = \Lambda^\rho{}_\mu \eta_{\rho\sigma} \Lambda^\sigma{}_\nu$$

3 Non-linear coordinate changes

One wants to generalize coordinate transformations beyond Lorentz's to be able to hop to non-inertial frames of reference.

The infinitesimal arc length, the line element, is given by

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

It has the opposite sign of proper time and as such it is used to measure spatial lengths. A non-linear coordinate is given by a general function of the coordinates of the starting frame

$$x'^\mu = x'^\mu(x^0, x^1, x^2, x^3) = x'^\mu(x)$$

The infinitesimal transformation is given by the Jacobian

$$\mathrm{d}x'^\mu = \partial_\nu x'^\mu\,\mathrm{d}x^\nu = J^\mu{}_\nu\,\mathrm{d}x^\nu \iff \mathrm{d}x^\mu = \partial_{\nu'} x^\mu\,\mathrm{d}x'^\nu = (J^{-1})^\mu{}_\nu\,\mathrm{d}x'^\nu$$

Inserting the above in the line element gives

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \eta_{\mu\nu}\,\partial_{\rho'} x^\mu\,\mathrm{d}x'^\rho\,\partial_{\sigma'} x^\nu\,\mathrm{d}x'^\sigma = (\eta_{\mu\nu}\,\partial_{\rho'} x^\mu\,\partial_{\sigma'} x^\nu)\,\mathrm{d}x'^\rho\,\mathrm{d}x'^\sigma$$

The factors in the expression above can be freely interchanged because sums are performed, it is only when using index-free notation that one needs to be careful about position. One can define the metric

$$g'_{\mu\nu} \equiv \eta_{\mu\nu}\,\partial_{\rho'} x^\mu\,\partial_{\sigma'} x^\nu$$

For Lorentz transformations, the Jacobian matrix is the Lorentz matrix $\Lambda^\mu{}_\nu$ and the metric reduces to the Minkowski metric $g'_{\mu\nu} = \eta_{\mu\nu}$. The metric g is not constant in space-time and across reference frames. Therefore, the line element depends on space-time points. A point-dependent quadratic form is called a metric. In general, one cannot assume the existence of a coordinate system in which the metric corresponds to the Minkowski metric. One has to deal with line elements

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

In this course, the metric $g_{\mu\nu}$ is always non-degenerate because so is $\eta_{\mu\nu}$ and the Jacobians (for a non-degenerate coordinate change).

The metric has signature $(1,3)$ and is called Lorentzian metric: it has only one negative eigenvalue in every point of space-time. Sometimes it is useful to only consider some coordinates and the metric does not have the negative eigenvalue: such metrics are called Riemannian.

With the understanding that line elements are the same in all reference frames, under a general change of coordinates, one gets

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = g_{\mu\nu}\,\partial_{\rho'}x^\mu\,\mathrm{d}x'^\rho\,\partial_{\sigma'}x^\nu\,\mathrm{d}x'^\sigma = (g_{\mu\nu}\,\partial_{\rho'}x^\mu\,\partial_{\sigma'}x^\nu)\,\mathrm{d}x'^\rho\,\mathrm{d}x'^\sigma = g'_{\rho\sigma}\,\mathrm{d}x'^\rho\,\mathrm{d}x'^\sigma$$

so that the metric transforms as

$$g'_{\mu\nu} = \partial_{\mu'}x^\rho\,\partial_{\nu'}x^\sigma\,g_{\rho\sigma} = (J^{-1})^\rho{}_\mu(J^{-1})^\sigma{}_\nu g_{\rho\sigma} = [(J^{-1})^\top]_\mu{}^\rho g_{\rho\sigma}(J^{-1})^\sigma{}_\nu = [(J^{-1})^\top g J^{-1}]_{\mu\nu}$$

This property can be added to the definition of a metric, but in this section it follows from previous definitions. In matrix notation, it is easy to see a resemblance with the defining equation for Lorentz matrices.

Example of an accelerated observer. Given the position vector, one can define the four-velocity

$$v^\mu \equiv \mathrm{d}_\tau x^\mu$$

In Special Relativity, the invariant associated with its norm is the speed of light

$$v^\mu v_\mu = \eta_{\mu\nu}v^\mu v^\nu = \eta_{\mu\nu}\,\mathrm{d}_\tau x^\mu\,\mathrm{d}_\tau x^\nu = -(\mathrm{d}_\tau\tau)^2 = -1$$

In General Relativity, the norm is

$$v^\mu v_\mu = g_{\mu\nu}v^\mu v^\nu = -1$$

Since the speed of light is the upper limit on velocity, it is clear that velocity does not increase linearly with time. With a constant acceleration, one can expect velocity to asymptotically approach the speed of light. Consider two velocity four-vector at infinitesimally close successive moments — $v^\mu(\tau)$ and $v^\mu(\tau+\mathrm{d}\tau)$ — of an accelerating particle in the x direction. And consider the reference frame instantaneously at rest with such particle. At the moment τ, the four-velocity is

$$v^\mu = \begin{pmatrix}1 & \vec{0}\end{pmatrix}^\top$$

The following velocity is slightly different. In the frame at time τ, it is

$$v^\mu(\tau+\mathrm{d}\tau) = \begin{pmatrix}1 & a\,\mathrm{d}\tau & 0 & 0\end{pmatrix}^\top$$

The acceleration is defined in this way. One way of getting the velocity at all times is to boost back at a frame not at rest with the particle where $v^\mu(\tau)$ and $v^\mu(\tau+\mathrm{d}\tau)$ are known and find a differential equations for the components. Though, there is another method.

A boost in the x direction has the form

$$\Lambda^\mu{}_\nu = \begin{bmatrix}\gamma & -v\gamma & 0 & 0\\ -v\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}\cosh\lambda & -\sinh\lambda & 0 & 0\\ -\sinh\lambda & \cosh\lambda & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

In Relativity, the velocity does not sum linearly, but the rapidity does:

$$\Lambda(\lambda_1)\Lambda(\lambda_2) = \Lambda(\lambda_1+\lambda_2)$$

To get the formula for the velocity, it is enough to note that $v = \tanh\lambda$ and then use the formula for the addition of the hyperbolic tangent. Let S be the rest frame at time τ and S' the rest frame at time $\tau+\mathrm{d}\tau$. In each rest frame, the velocity is $v^\mu = \begin{pmatrix}1 & \vec{0}\end{pmatrix}^\top$. The velocity of the particle at time $\tau+\mathrm{d}\tau$ in the frame S is calculated from the one in the frame S' by a Lorentz boost:

$$v^\mu(\tau+\mathrm{d}\tau) = \Lambda_x(a'\,\mathrm{d}\tau)v'^\mu(\tau+\mathrm{d}\tau) = \Lambda_x(-a\,\mathrm{d}\tau)v^\mu(\tau)$$

Using also the fact that $\cosh\lambda = 1 + o(\lambda^2)$ and $\sinh\lambda = \lambda + o(\lambda^3)$ for small enough λ. Iterating the boosts, one gets

$$v^\mu(\tau) = \Lambda_x(-a\,\mathrm{d}\tau)v^\mu(\tau - \mathrm{d}\tau) = \Lambda_x(-a\,\mathrm{d}\tau)\Lambda_x(-a\,\mathrm{d}\tau)v^\mu(\tau - 2\,\mathrm{d}\tau) = \cdots$$

The velocity at an arbitrary time is the composition of many infinitesimal boosts. Let $\tau = N\,\mathrm{d}\tau$ for $N \gg 1$. Then

$$v^\tau = v^{N\,\mathrm{d}\tau} = \Lambda^N(-a\,\mathrm{d}\tau)v^\mu(0) = \Lambda(-Na\,\mathrm{d}\tau)v^\mu(0) = \Lambda(-a\tau)v^\mu(0)$$

The power N becomes a coefficient because rapidities sum. For an object initially at rest, one gets

$$v^\mu(\tau) = \begin{bmatrix} \cosh a\tau & \sinh a\tau & 0 & 0 \\ \sinh a\tau & \cosh a\tau & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cosh(a\tau) \\ \sinh(a\tau) \\ 0 \\ 0 \end{bmatrix}$$

Integrating both sides in proper time, one finds

$$x^\mu = \frac{1}{a}\begin{pmatrix} \sinh(a\tau) & \cosh(a\tau) & 0 & 0 \end{pmatrix}^\top$$

Its trajectory, its world line is a spacelike hyperbola asymptotic to a light ray (with integrations constants chosen so that the ray goes through the origin) in the limit $\tau \to \infty$. It is approaching the speed of light.

One may want to find the coordinates of the accelerated observer. Its space coordinate ξ should be constant for trajectories of other accelerated objects, thus stationary in the observer's frame: the endpoints of an object at rest with the observer would produce lines of constant ξ. This is achieved by boosting the object by a parameter $\alpha = a\tau$. These lines are hyperbolae. Similarly, the time coordinate η should be constant for simultaneous events in the observer's frame. These lines are perpendicular to v^μ (with respect to the metric $\eta_{\mu\nu}$). Since the position x^μ is perpendicular to the velocity v^μ, then the lines at constant η pass through the origin [r].

These requirements are satisfied by

$$x^\mu = \frac{\mathrm{e}^{a\xi}}{a}\begin{pmatrix} \sinh\eta & \cosh\eta & 0 & 0 \end{pmatrix}^\top$$

Lecture 4

Consider an observer on one hyperbola at $t = 0$ and another one at the origin. The second observer sent a message at light speed that is received by the first observer that can then respond. If the second observer sends another message, then it can never receive the first one: it is possibile to outrun a photon given enough head start. Any light message sent at $t \geq 0$ can never reach the accelerated observer. gio 12 ott 2023 14:30

The horizon is the locus of points beyond which an observer cannot communicate. An accelerated observer should experience phenomena similar to gravity.

[r]

$$\mathrm{d}s^2 = g'_{\mu\nu}\,\mathrm{d}x'^\mu\,\mathrm{d}x'^\nu = \partial_\mu x'^\rho\,\partial_\nu x'^\sigma g_{\rho\sigma}\,\mathrm{d}x^\rho\,\mathrm{d}x^\sigma = \mathrm{e}^{2a\xi}(-\mathrm{d}\eta^2 + \mathrm{d}\xi^2)$$

[r] and image

$$(J^{-1})^\top g J^{-1} = \mathrm{e}^{2a\xi}\begin{pmatrix} \cosh a\tau & \sinh a\tau \\ \sinh a\tau & \cosh a\tau \end{pmatrix}\eta\begin{pmatrix} \cosh a\tau & \sinh a\tau \\ \sinh a\tau & \cosh a\tau \end{pmatrix} = \mathrm{e}^{2a\xi}\eta$$

It is possibile to arrive at the answer in another way. Consider

$$\begin{aligned} \mathrm{d}x^0 &= \partial_{\mu'}x^0\,\mathrm{d}x'^\mu = \cosh(a\eta)\mathrm{e}^{a\xi}\,\mathrm{d}\eta + \sinh(a\eta)\mathrm{e}^{a\xi}\,\mathrm{d}\xi \\ \mathrm{d}x^1 &= \sinh(a\eta)\mathrm{e}^{a\xi}\,\mathrm{d}\eta + \cosh(a\eta)\mathrm{e}^{a\xi}\,\mathrm{d}\xi \end{aligned}$$

then

$$\begin{aligned}\mathrm{d}s^2 &= \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^i\,\mathrm{d}x^i)\\ &= \mathrm{e}^{2a\xi}\left[-\mathrm{d}\eta^2 + \mathrm{d}\xi^2\right] + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 = g'_{\mu\nu}\,\mathrm{d}x'^\mu\,\mathrm{d}x'^\nu\end{aligned}$$

The metric is then

$$g'_{\mu\nu} = \begin{pmatrix} -\mathrm{e}^{2a\xi} & & & \\ & \mathrm{e}^{2a\xi} & & \\ & & 1 & \\ & & & 1\end{pmatrix}$$

This is also called Rindler metric. Rindler's space is the portion of space-time between the horizons [r].

One can study if given a metric one can always find a coordinate transformation such that the metric is Minkowski's [r].

3.1 Vector fields

The Minkowski metric was generalized to a point-dependent quadratic form. It no longer makes sense to talk about vectors, which have to be generalized to include point-dependence: vector fields. At every point in space-time there's a vector associated with it. One wants to study how a vector field transforms by looking at some invariant.

A partial derivative ∂_μ keeps constant all coordinates, but one in which a limit is taken

$$\partial_1 f = \lim_{\varepsilon\to 0}\frac{f(x^0, x^1+\varepsilon, x^2, x^3) - f(x^0, x^1, x^2, x^3)}{\varepsilon}$$

To take a derivative in an arbitrary direction one take the directional derivative, a linear combination of the partial derivatives along the axes

$$v^\mu\,\partial_\mu f = \lim_{\varepsilon\to 0}\frac{f(x^\mu + \varepsilon v^\mu) - f(x^\mu)}{\varepsilon}$$

The result of the limit does not depend on the coordinate system, but depends on a point and the direction of the vector:

$$v'^\mu\,\partial'_\mu f = v^\nu\,\partial_\nu f$$

The new derivative can be written using the chain rule

$$\partial'_\mu = \partial'_\mu x^\nu\,\partial_\nu \implies v'^\mu\,\partial'_\mu x^\nu\,\partial_\nu f = v^\nu\,\partial_\nu f \implies v^\nu = v'^\nu\,\partial'_\mu x^\nu \implies v'^\mu = \partial_\nu x'^\mu\,v^\nu = J^\mu{}_\nu\,x^\nu$$

In General Relativity, a vector field is a quadruple that transforms in the way above. Sometimes one writes

$$v \equiv v^\mu\,\partial_\mu$$

In quantum mechanics, $v^\mu\,\partial_\mu$ is also the generator of translations. While, the generator of rotations is the angular momentum. The generators can be promoted to finite transformations through an exponential map. An integral course is a line tangent to the vector field at any point.

It is natural to look at the commutator of a vector field

$$[v, w] = [v^\mu\,\partial_\mu, w^\nu\,\partial_\nu]$$

Acting on a test function one gets

$$\begin{aligned}[v^\mu\,\partial_\mu, w^\nu\,\partial_\nu]f &= v^\mu\,\partial_\mu(w^\nu\,\partial_\nu f) - w^\nu\,\partial_\nu(v^\mu\,\partial_\mu f) = v^\mu\,\partial_\mu w^\nu\,\partial_\nu f - w^\nu\,\partial_\nu\,\partial_\mu f\\ &= (v^\nu\,\partial_\nu w^\mu - w^\nu\,\partial_\nu v^\mu)\,\partial_\mu f\end{aligned}$$

from which

$$[v^\mu\,\partial_\mu, w^\nu\,\partial_\nu]^\rho = (v^\nu\,\partial_\nu w^\rho - w^\nu\,\partial_\nu v^\rho)$$

This is a Lie bracket and gives a new vector field. One should check that this expression transforms as a vector field.

In Special Relativity, an invariant is

$$\eta_{\mu\nu} v^\mu v^\nu = \|v\|^2 \equiv v^2$$

In General Relativity, one defines the norm

$$g_{\mu\nu} v^\mu v^\nu \equiv \|v\|^2$$

that is no longer invariant, but is a function: it depends on space-time points and it transforms without being multiplied by anything. [r] In fact

$$g'_{\mu\nu} v'^\mu v'^\nu = \partial'_\mu x^\rho \, \partial'_\nu x^\sigma g_{\rho\sigma} \, \partial_\lambda x'^\mu \, v^\lambda \, \partial_\alpha x'^\nu \, v^\alpha = \delta^\rho{}_\lambda \delta^\sigma{}_\alpha g_{\rho\sigma} v^\lambda v^\alpha = g_{\rho\sigma} v^\rho v^\sigma = v^2$$

3.2 Tensors

One can introduce the dual of a vector field. A (one-)form is an object ω_μ that trasforms as

$$\omega'_\mu = \partial'_\mu x^\nu \, \omega_\nu$$

The product between a form and a vector field is a function

$$\omega'_\mu v'^\mu = \partial'_\mu x^\nu \, \omega_\nu \, \partial_\rho x'^\mu \, v^\rho = \delta^\nu{}_\rho \omega_\nu v^\rho = \omega_\rho v^\rho$$

Combining a form with an infinitesimal displacement one defines

$$\omega_\mu \, \mathrm{d}x^\mu \equiv \omega$$

Lecture 5

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Example. An example of form is given by the derivative of a function $\partial_\mu f$. Given a vector and a form, there's a natural pairing between the two. The directional derivative is a function $v^\mu \, \partial_\mu f$.

Example. A vector field is the generalization of contravariant vectors from Special Relativity, while a form is the generalization of covariant vectors. Similarly to Special Relativity, one can construct a form from a vector field

$$\omega_\mu = g_{\mu\nu} v^\nu$$

In fact, its transformation law is

$$\omega'_\mu = g'_{\mu\nu} v'^\nu = \partial'_\mu x^\rho \, \partial'_\nu x^\sigma \, g_{\rho\sigma} \, \partial_\lambda x'^\nu v^\lambda = \partial'_\mu x^\rho \, \delta^\sigma{}_\lambda g_{\rho\sigma} v^\lambda = \partial'_\mu x^\rho g_{\rho\sigma} v^\sigma = \partial'_\mu x^\rho \, \omega_\rho$$

It is typical to denote the form constructed from a vector with the same symbol: $v_\mu = g_{\mu\nu} v^\nu$.

Tensors. In general, one can define a tensor field $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}$ of type (k, l) as an object that transforms as

$$(T')^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} = \partial_{\rho_1} x'^{\mu_1} \cdots \partial_{\rho_k} x'^{\mu_k} T^{\rho_1 \cdots \rho_k}{}_{\sigma_1 \cdots \sigma_l} \, \partial'_{\nu_1} x^{\sigma_1} \cdots \partial'_{\nu_l} x^{\sigma_l}$$

It has k contravariant components and l covariant components. The word field is often omitted, but it is understood in almost any case.

A function is type $(0, 0)$ tensor. A metric is a type $(0, 2)$ tensor. A vector field is a type $(1, 0)$ tensor. A form is a type $(0, 1)$ tensor.

The Kronecker delta $\delta^\mu{}_\nu$ is a type $(1, 1)$ tensor with the same expression in all coordinate systems. In fact, its transformation is

$$(\delta')^\mu{}_\nu = \partial_\rho x'^\mu \, \partial'_\nu x^\sigma \delta^\rho{}_\sigma = \partial_\rho x'^\mu \, \partial'_\nu x^\rho = \partial'_\nu x'^\mu = \delta^\mu{}_\nu$$

Metric. Given a metric $g_{\mu\nu}$, its inverse $(g^{-1})^{\mu\nu}$ satisfies

$$(g^{-1})^{\mu\rho} g_{\rho\nu} = \delta^\mu{}_\nu$$

The inverse is typically denoted as $g^{\mu\nu}$ and it is clear that the metric is different from $g_{\mu\nu}$. The inverse metric is a type $(2,0)$ tensor.

Derivative. Given a vector field v^ν, its derivative $\partial_\mu v^\nu$ is not a tensor. In fact, its transformation is

$$\partial'_\mu v'^\nu = \partial'_\mu x^\rho \, \partial_\rho (\partial_\sigma x'^\nu \, v^\sigma) = \partial'_\mu x^\rho \, \partial_\sigma x'^\nu \, \partial_\rho v^\sigma + \partial'_\mu x^\rho \, \partial_\rho \partial_\sigma x'^\nu \, v^\sigma$$

For it to be a tensor, one expects only the first addendum to appear. There is no natural notion of derivative.

Anti-symmetrization. Given a form ω_ν, its derivative $\partial_\mu \omega_\nu$ is not a tensor. However, the anti-symmetrization is a type $(0,2)$ tensor

$$\partial_\mu \omega_\nu - \partial_\nu \omega_\mu \equiv 2\partial_{[\mu}\omega_{\nu]}$$

The transformation of the derivative is

$$\begin{aligned}
\partial'_\mu \omega'_\nu &= \partial'_\mu x^\rho \, \partial_\rho (\partial'_\nu x^\sigma \, \omega_\sigma) = \partial'_\mu x^\rho \, \partial'_\nu x^\sigma \, \partial_\rho \omega_\sigma + \partial'_\mu x^\rho \, \partial_\rho \partial'_\nu x^\sigma \, \omega_\sigma \\
&= \partial'_\mu x^\rho \, \partial'_\nu x^\sigma \, \partial_\rho \omega_\sigma + \partial'^2_{\mu\nu} x^\sigma \, \omega_\sigma
\end{aligned}$$

The transformation of the anti-symmetrization is

$$\begin{aligned}
2\partial'_{[\mu}\omega'_{\nu]} &= \partial'_{[\mu} x^\rho \, \partial'_{\nu]} x^\sigma \, \partial_\rho \omega_\sigma + \partial'^2_{[\mu\nu]} x^\sigma \, \omega_\sigma = \partial'_{[\mu} x^\rho \, \partial'_{\nu]} x^\sigma \, \partial_\rho \omega_\sigma + \partial'^2_{\mu\nu} x^\sigma \, \omega_\sigma - \partial'^2_{\nu\mu} x^\sigma \, \omega_\sigma \\
&= \partial'_{[\mu} x^\rho \, \partial'_{\nu]} x^\sigma \, \partial_\rho \omega_\sigma = \partial'_\mu x^\rho \, \partial'_\nu x^\sigma \, \partial_\rho \omega_\sigma - \partial'_\nu x^\rho \, \partial'_\mu x^\sigma \, \partial_\rho \omega_\sigma \\
&= \partial'_\mu x^\rho \, \partial'_\nu x^\sigma \, \partial_\rho \omega_\sigma - \partial'_\nu x^\sigma \, \partial'_\mu x^\rho \, \partial_\sigma \omega_\rho = \partial'_\mu x^\rho \, \partial'_\nu x^\sigma (\partial_\rho \omega_\sigma - \partial_\sigma \omega_\rho) \\
&= 2\, \partial'_\mu x^\rho \, \partial'_\nu x^\sigma \, \partial_{[\rho}\omega_{\sigma]}
\end{aligned}$$

At the second line, as long as the transformation between coordinates is a smooth function, then the mixed derivates are equal. At the third line, the indices σ and ρ are interchanged since they are summed over. An example of such a tensor is the field-strength tensor

$$F_{\mu\nu} = 2\, \partial_{[\mu} A_{\nu]}$$

where A_μ is the four-potential and also a form.

If $\omega_{\mu\nu}$ is a type $(0,2)$ tensor and it is anti-symmetric, then the following is a tensor

$$\begin{aligned}
\partial_{[\mu}\omega_{\nu\rho]} &= \frac{1}{3!}(\partial_\mu \omega_{\nu\rho} - \partial_\mu \omega_{\rho\nu} + \partial_\nu \omega_{\rho\mu} - \partial_\nu \omega_{\mu\rho} + \partial_\rho \omega_{\mu\nu} - \partial_\rho \omega_{\nu\mu}) \\
&= \frac{1}{3}(\partial_\mu \omega_{\nu\rho} + \partial_\nu \omega_{\rho\mu} + \partial_\rho \omega_{\mu\nu})
\end{aligned}$$

The homogeneous Maxwell's equations, written as Bianchi's identity, are an example of such a tensor

$$\partial_{[\mu} F_{\nu\rho]} = 0$$

This tensor is zero in every coordinate system. The aim is to formulate equations that have the same form in every reference frame. The inhomogeneous Maxwell's equations in the vacuum are

$$\partial_\mu F^{\mu\nu} = J^\nu = 0$$

but this is not a tensor in General Relativity: it needs to be modified.

More generally, given an anti-symmetric type $(0,k)$ tensor $A_{\mu_1 \cdots \mu_k}$, its anti-symmetrized derivative $\partial_{[\mu} A_{\mu_1 \cdots \mu_k]}$ is also a tensor. Anti-symmetric type $(0,k)$ tensors are called k-forms.

3.3 Lie derivatives

The directional derivative is given by $vf \equiv v^\mu \, \partial_\mu f$ and it acts on a function. One can define a similar derivative acting on a vector field. Given a vector field w^ν, the expression $v^\mu \, \partial_\mu w^\nu$ is not a vector field. The directional derivative is defined as

$$v^\mu \, \partial_\mu f = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon v) - f(x)}{\varepsilon}$$

But for the derivative of a vector field it doesn't make sense to compare the vector field at the transformed point $w^\mu(x+\varepsilon v)$ with the vector field at the given point $w^\mu(x)$. Instead, one wants to transform the vector field at the transformed point, then compare it

$$(L_v w)^\mu = \lim_{\varepsilon \to 0} \frac{J^\mu{}_\nu \, w^\nu(x+\varepsilon v) - w^\mu(x)}{\varepsilon}$$

where J is the Jacobian of the transformation generated by $-\vec{v}$. Let $x'^\mu = x^\mu - \varepsilon v^\mu$, the Jacobian is

$$J^\mu{}_\nu = \partial_\nu x'^\mu = \delta^\mu{}_\nu - \varepsilon \, \partial_\nu v^\mu$$

By expanding in a Taylor series, one gets

$$w^\nu(x+\varepsilon v) \approx w^\nu(x) + \varepsilon v^\mu \, \partial_\mu w^\nu$$

Then one has

$$\begin{aligned}
(L_v w)^\mu &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[(\delta^\mu{}_\nu - \varepsilon \, \partial_\nu v^\mu)(w^\nu + \varepsilon v^\rho \, \partial_\rho w^\nu) - w^\mu](x) \\
&= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[w^\mu - \varepsilon \, (\partial_\nu v^\mu) w^\nu + \varepsilon v^\rho \, \partial_\rho w^\mu - w^\mu](x) \\
&= \lim_{\varepsilon \to 0}[-\varepsilon \, (\partial_\nu v^\mu) w^\nu + \varepsilon v^\rho \, \partial_\rho w^\mu](x) = v^\nu \, \partial_\nu w^\mu - w^\nu \, \partial_\nu v^\mu = [v, w]^\mu
\end{aligned}$$

which is a vector field. The negative term came from the Jacobian matrix and has a geometrical meaning: when one applies the map generated by v to bring the point $x + \varepsilon v$ back to x, the vector w^ν might rotate a little.

This idea and method can be applied to any tensor: the derivative of a tensor along a vector field is again a tensor. This type of derivative is called Lie derivative.

A second method to determine the form of the more general Lie derivative imposes the Leibniz identity. Consider a one-form ω_μ and a vector field z^μ. It is already known that $\omega_\mu z^\mu$ is a function. The Lie derivative is

$$\begin{aligned}
L_v(\omega_\mu z^\mu) &= (L_v \omega)_\mu z^\mu + \omega_\mu (L_v z)^\mu \\
v^\nu \, \partial_\nu(\omega_\mu z^\mu) &= (L_v \omega)_\mu z^\mu + \omega_\mu (v^\nu \, \partial_\nu z^\mu - z^\nu \, \partial_\nu v^\mu) \\
v^\nu (\partial_\nu \omega_\mu) z^\mu + v^\nu \omega_\mu \, \partial_\nu z^\mu &= (L_v \omega)_\mu z^\mu + \omega_\mu (v^\nu \, \partial_\nu z^\mu - z^\nu \, \partial_\nu v^\mu) \\
v^\nu (\partial_\nu \omega_\mu) z^\mu &= (L_v \omega)_\mu z^\mu - \omega_\mu z^\nu \, \partial_\nu v^\mu \\
(L_v \omega)_\mu z^\mu &= v^\nu (\partial_\nu \omega_\mu) z^\mu + \omega_\mu z^\nu \, \partial_\nu v^\mu \\
(L_v \omega)_\mu z^\mu &= v^\nu (\partial_\nu \omega_\mu) z^\mu + \omega_\nu z^\mu \, \partial_\mu v^\nu \\
(L_v \omega)_\mu z^\mu &= [v^\nu \partial_\nu \omega_\mu + \omega_\nu \, \partial_\mu v^\nu] z^\mu
\end{aligned}$$

Since z^μ is an arbitrary vector, one has the following one-form

$$(L_v \omega)_\mu = v^\nu \partial_\nu \omega_\mu + \omega_\nu \, \partial_\mu v^\nu$$

The second addendum of this formula has the opposite sign as the one in the Lie derivative of a vector: this is a consequence of the Jacobian. If one evaluates the limit definition, one finds the inverse Jacobian and as such it has $\delta^\mu{}_\nu + \varepsilon \, \partial_\nu v^\mu$. The Lie derivative of a scalar is just the directional derivative

$$L_v f = v^\nu \partial_\nu f$$

One may notice that in these three examples of Lie derivates, the directional derivative is always present.

Using either method, one can obtain the Lie derivative of any tensor. In general, for a tensor field one has

$$
\begin{aligned}
(L_v T)^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} = {} & v^\rho \partial_\rho T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} \\
& - (\partial_\rho v^{\mu_1}) T^{\rho\mu_2\cdots\mu_k}{}_{\nu_1\cdots\nu_l} - (\partial_\rho v^{\mu_2}) T^{\mu_1\rho\cdots\mu_k}{}_{\nu_1\cdots\nu_l} - \cdots - (\partial_\rho v^{\mu_k}) T^{\mu_1\cdots\rho}{}_{\nu_1\cdots\nu_l} \\
& + (\partial_{\nu_1} v^\rho) T^{\mu_1\cdots\mu_k}{}_{\rho\nu_2\cdots\nu_l} + (\partial_{\nu_2} v^\rho) T^{\mu_1\cdots\mu_k}{}_{\nu_1\rho\cdots\nu_l} + \cdots + (\partial_{\nu_l} v^\rho) T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\rho}
\end{aligned}
$$

In general, the derivatives of tensors are not tensors. Lie derivatives let one differentiate a tensor along a vector field through a linear combination of partial derivatives. There is a way, the covariant derivative, to improve the notion of partial derivative without involving the linear combinations above.

The Lie derivative has also a geometrical meaning: it is the comparison between the tensor at a point and the tensor at another point that has been dragged back to the first point.

Metric tensor. The Lie derivative of the metric is

$$(L_v g)_{\mu\nu} = v^\rho \partial_\rho g_{\mu\nu} + (\partial_\mu v^\rho) g_{\rho\nu} + (\partial_\nu v^\rho) g_{\mu\rho}$$

A vector field v^μ such that the Lie derivative above is zero, is an infinitesimal transformation that leaves invariant the metric and is a symmetry of the metric.

Lecture 6

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Not every coordinate system can cover all space-time. An example of this is Rindler space.

3.4 Manifolds

Topological spaces are considered to be equivalent if there is a homeomorphism connecting them. A homeomorphism is a continuous bijective map whose inverse is also continuous. A more advanced concept of being equivalent is the homotopic concept.

A manifold of dimension N is a topological space such that every point has a neighbourhood that is homeomorphic to an open set of $\mathbb{R}^n$. [r]

A smooth manifold M is a manifold with a choice of open sets U_i (called charts) and maps $\phi \to \mathbb{R}^n$ such that

$$\bigcup_i U_i = M$$

and in regions of overlap

$$U_i \cap U_j \neq \emptyset \implies \phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

the transition functions are smooth. This means that two coordinate systems transition well when going when one to another.

The collection of all charts and maps is called atlas.

A smooth map f from a smooth manifold M of dimension N to another smooth manifold M' of dimension N' is a map such that at any point $p \in M$, taking one set U in which p belongs to and the associated map ϕ, and taking one V which $f(p)$ belongs to and the associated map ψ, then $\psi \circ f \circ \phi^{-1}$ is smooth.

A diffeomorphism from a smooth manifold M to another smooth manifold M' is a smooth bijective map whose inverse is also smooth.

Two manifolds can be considered the same once one can find a diffeomorphism between them. An interesting case is considering maps from a space to itself $M = M'$.

Exercise. Consider the two-sphere

$$S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3$$

It can be covered by two charts:

$$U_N = S^2 \setminus \{(0,0,-1)\}\,, \quad U_S = S^2 \setminus \{(0,0,1)\}$$

Every point p of the sphere can be mapped to a point $\phi(p)$ in the tangent plane at one pole using a ray starting from the other pole

$$\phi_S : U_S \to \mathbb{R}^2 \,, \quad \phi_N : U_N \to \mathbb{R}^2$$

It is similar to the Riemann sphere. Find the transition from one set to the other.

Another way of working on the sphere employs spherical coordinates (θ, φ) but it only works on $S^2 \setminus \{(0,0,\pm 1)\}$.

Metric. Since multiple coordinates systems are needed, one needs to study the metric. A metric on a manifold M is a choice of metrics g_i on each chart U_i such that on each intersection $U_i \cap U_j$ the line elements coincide $\mathrm{d}s_i^2 = \mathrm{d}s_i^2$:

$$(g_i)_{\mu\nu} = \frac{\partial x_j^\rho}{\partial x_i^\mu} \frac{\partial x_j^\sigma}{\partial x_i^\nu} (g_j)_{\rho\sigma}$$

Example. Consider again the two-sphere S^2. There is a natural metric induced by the euclidean metric in $\mathbb{R}^3$. The euclidean metric is

$$\mathrm{d}s^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2$$

Given the transformation from polar coordinates to Cartesian, one gets

$$x_1 = r \sin\theta \cos\varphi \,, \quad x_2 = r \sin\theta \sin\varphi \,, \quad x_3 = r\cos\theta$$

Calculating each infinitesimal, the line element becomes

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2)$$

When interested only in the surface of the sphere, the radius stays constant:

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2 \implies g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}$$

There are three symmetries that leave the sphere invariant: rotations. This is the round metric. It is two-dimensional without time. In this case, one is interested in just a part of space-time, not all of it. This is a Riemannian metric because it is positive definite. This metric is degenerate for $\theta = 0, \pi$, at the poles. This degeneracy is an artifact of the coordinate system.

One can change to a coordinate system that does not present these artifacts. For the two-sphere, one can utilize the stereographic projection and the coordinates of the projective plane

$$x = \tan\frac{\theta}{2} \cos\varphi \,, \quad y = \tan\frac{\theta}{2} \sin\varphi \implies \mathrm{d}s^2 = 4\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{(1 + x^2 + y^2)^2}$$

This is the Fubini–Study metric. This metric applies to projective spaces. It is well-defined except at the south pole. The metric is

$$g_{\mu\nu} = \frac{4}{(1 + x^2 + y^2)^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By changing coordinate systems, the degeneracies vanish. This phenomenon also happens with more complicated metrics.

The formalism built patches multiple $\mathbb{R}^n$ together to describe a general space-time. One can also use arbitrary tensor fields.