Quantum Field Theory I

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Lecture 1

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Exam. The exam can be done whenever, just send an email at least ten days before. Half an hour to calculate something like in class, or generalize something, just technical stuff. It is possible to do QFT 1 and 2 together.

Course structure. In order to go towards increasing complexity, one starts from real scalar boson fields, then fermions, then gauge fields and gauge theories. QFT I is about scalar bosons, while QFT II starts from fermions. Between the two parts there is a bit of conformal field theory.

1 Introduction

The aim is to develop an alternative description of elementary particles and fundamental interactions using functional methods instead of canonical quantization. The path integral formulation of quantum mechanics is an example of functional method. Canonical quantization works by taking classical observable quantities and promoting them to operators obeying certain commutator relations. This was done in the relativistic formulation of quantum field theory in Theoretical Physics I.

The alternative, albeit equivalent, formulation of quantum mechanics is based on Feynman path integrals: the propagators are written in term of path integrals. Quantum Field Theory I and II reformulates quantum field theory in terms of a generalization of the quantum mechanical path integral to relativistic field theories. The content of Theoretical Physics I is studied using a functional approach.

One needs to formulate quantum field theories for particles of different spins: scalar bosons, spinor fermions and vector bosons. Gauge theories are quantized using functional methods. One does not look for spin $\frac{3}{2}$ and 2 fields, because quantum field theory is not suitable for their description: the theory is inconsistent because it is not renormalizable at every energy scale, in particular in the ultraviolet. For higher spins, there are problems in the propagation of particles in ordinary quantum field theory and one needs a more general approach.

The functional approach lets one study phenomena for which canonical quantization is not suitable.

1.1 Prerequisites

The prerequisites are the following.

Real Gaussian integral. Gaussian integrals are useful for many computations. In one dimension, one has

$$\int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-\frac{a}{2}x^2+bx} = \sqrt{\frac{2\pi}{a}}\mathrm{e}^{\frac{b^2}{2a}}$$

In n dimensions (considering only diagonal matrices) one has

$$\int_{\mathbb{R}^n} \mathrm{d}x_1 \cdots \mathrm{d}x_n\, \mathrm{e}^{-\frac{a_1}{2}x_1^2-\cdots-\frac{a_n}{2}x_n^2+b_1x_1+\cdots+b_nx_n} = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{a_1\cdots a_n}}\mathrm{e}^{\frac{b_1^2}{2a_1}+\cdots+\frac{b_n^2}{2a_n}}$$

Introducing the matrix and vectors

$$A = \mathrm{diag}(a_1,\ldots,a_n)\,, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ a_n \end{bmatrix}\,, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

one obtains

$$\int_{\mathbb{R}^n} \mathrm{d}x_1 \cdots \mathrm{d}x_n\, \mathrm{e}^{-\frac{1}{2}x^\top Ax+b^\top x} = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}}\mathrm{e}^{\frac{1}{2}b^\top A^{-1}b}$$

which more generally holds for positive-definite symmetric matrices[1] A. This result is analytically continued to the cases where the exponent in the integrand has arbitrary signs or imaginary numbers while assuming the right-hand side keeps its validity.

Complex Gaussian integral. The complex Gaussian integral is the following[2]

$$\int \prod_{j=1}^{n} \mathrm{d}z_j\, \mathrm{d}\bar{z}_j\, \mathrm{e}^{-\bar{z}^\top Az+\bar{b}^\top z+b^\top \bar{z}} = \frac{\pi^n}{\det A}\mathrm{e}^{\bar{b}^\top A^{-1}b}\,, \quad \int \mathrm{d}z\, \mathrm{d}\bar{z} \equiv \int \mathrm{d}x\, \mathrm{d}y$$

where A is a positive-definite hermitian matrix, b is a complex column vector and the bar is complex conjugation. Since $\mathbb{C} \simeq \mathbb{R}^2$, and the dimension is still n, one expects double the integrals.

There is also the generalization of Gaussian integrals for Grassmann odd variables suitable for spinors.

Average of a polynomial with Gaussian distribution. Consider the mean value of Gaussian-distributed variables (see Weinberg, eq. 9.A.10)

$$\begin{aligned}
\langle x_{k_1}\cdots x_{k_l}\rangle &\equiv \int \prod_{j=1}^{n} \mathrm{d}x_j\, x_{k_1}\cdots x_{k_l}\mathrm{e}^{-\frac{1}{2}x^\top Ax+b^\top x}\Big|_{b=0} \\
&= \int \prod_{j=1}^{n} \mathrm{d}x_j\, \frac{\partial^l}{\partial b_{k_1}\cdots\partial b_{k_l}}\mathrm{e}^{-\frac{1}{2}x^\top Ax+b^\top x}\Big|_{b=0} \\
&= \frac{\partial^l}{\partial b_{k_1}\cdots\partial b_{k_l}}\frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}}\mathrm{e}^{\frac{1}{2}b^\top A^{-1}b}\Big|_{b=0} \\
&= \begin{cases} 0\,, & l \text{ odd} \\ \dfrac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}} \displaystyle\sum_{\substack{\text{pairings} \\ \text{of } \{k_i\}}} \prod_{\{k_ik_j\}} A^{-1}_{k_ik_j}\,, & l \text{ even} \end{cases}
\end{aligned}$$

The last sum is over the pairings of $(k_1\cdots k_l)$ — two pairings are the same if they differ only by the order of the pairs, or by the order of indices within a pair — and the product is over all such pairs.

One can compute averages through the derivatives of Gaussian integrals. From the first line (or the third), one may notice that if l is odd then the integral is null.

[1] See `https://en.wikipedia.org/wiki/Gaussian_integral`.

[2] A reference I could find is "A. Altland and B. Simons, Condensed Matter Field Theory, 3rd ed. Cambridge: Cambridge University Press, 2023, `https://doi.org/10.1017/9781108781244`". In particular, eq. 3.18.

Functional calculus. Informally, functional calculus is need for infinite (non-countable) dimensional vectors (for example functions). From a discrete set of quantities q_i for which one knows how to take the derivative, one moves to the case of continuous variables where one works with functions $q(x)$.

Inner product. The inner product of two finite dimensional vector fields is

$$u \cdot v \equiv \sum_{i=1}^{n} u_i v_i$$

while for infinite dimensional vector fields one has

$$u \cdot v \equiv \int \mathrm{d}x\, u(x) v(x)$$

The inner product with respect to a matrix is

$$u^\top M v = \sum_{ij=1}^{n} u_i M_{ij} v_j \rightsquigarrow u^\top M v = \int \mathrm{d}x\, \mathrm{d}y\, u(x) M(x,y) v(y)$$

Identity operator. The identity operator is

$$q = Iq \iff q_i = \sum_j \delta_{ij} q_j \rightsquigarrow q(x) = \int \mathrm{d}y\, \delta(x-y) q(y)$$

The identity operator is the Dirac delta function.

Functional derivatives. The notation for functional derivatives is $\frac{\delta}{\delta q(x)}$. The concept of functional derivative can be defined by imposing a set of ansätze (pl. of ansatz):

- Linearity
$$\delta_q[F_1(q) + F_2(q)] = \delta_q F_1 + \delta_q F_2$$
- Leibniz rule
$$\delta_q[F_1(q) F_2(q)] = (\delta_q F_1) F_2 + F_1 \delta_q F_2$$
- It must hold
$$\delta_{q(x)} q(y) = \delta(y - x)$$

With these ansätze one can prove that the functional derivative enjoys the same properties as the regular derivative. For example

$$\delta_{q(x)} \int \mathrm{d}y\, q^P(y) = \int \mathrm{d}y\, P q^{P-1}(y)\, \delta_{q(x)} q(y) = P q^{P-1}(x)$$

Likewise

$$\delta_{q(x)} \int \mathrm{d}y\, f(y)\, \partial_y q(y) = \int \mathrm{d}y\, f(y)\, \partial_y \delta(x-y) = -\partial_x f(x)$$

where the result is obtained remembering that the Dirac delta is a distribution.

Generalization of Gaussian integrals to functional integrals. The Gaussian integral becomes

$$\int \prod_{j=1}^{n} \mathrm{d}q_j\, \mathrm{e}^{-\frac{1}{2} q^\top A q} = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}} \rightsquigarrow \int [\mathcal{D}q]\, \mathrm{e}^{-\frac{1}{2}\int \mathrm{d}x\, \mathrm{d}y\, q(x) A(x,y) q(y)} = \frac{\text{const.}}{\sqrt{\text{“}\det A\text{”}}}$$

where $\mathcal{D}q$ is the functional measure that, along with the determinant, needs to be properly defined.

Useful property. Starting from

$$\mathrm{e}^{\mathrm{i}q\cdot J} \equiv \mathrm{e}^{\mathrm{i}\int \mathrm{d}y\, q(y)J(y)}$$

one may find its derivative

$$\begin{aligned}\delta_{J(x)}\mathrm{e}^{\mathrm{i}q\cdot J}|_{J=0} &= \delta_{J(x)}\mathrm{e}^{\mathrm{i}\int \mathrm{d}y\, q(y)J(y)}|_{J=0} = \delta_{J(x)}\left[\mathrm{i}\int \mathrm{d}y\, q(y)J(y)\right]\mathrm{e}^{\mathrm{i}q\cdot J}|_{J=0} \\ &= \mathrm{i}q(x)\mathrm{e}^{\mathrm{i}q\cdot J}|_{J=0} = \mathrm{i}q(x)\end{aligned}$$

A function can be expressed as the derivative of an exponential with respect to a parameter. This parameter J is called the source of q. Since

$$q(x) = -\mathrm{i}\,\delta_{J(x)}\mathrm{e}^{\mathrm{i}q\cdot J}|_{J=0}$$

then one may write any function of $q(x)$ as

$$G(q(x)) = G(-\mathrm{i}\,\delta_{J(x)})\mathrm{e}^{\mathrm{i}q\cdot J}|_{J=0}$$

To make sense of the right-hand expression, one may expand the function $G(x)$ in a Taylor series.

2 Feynman path integral in quantum mechanics

See Feynman–Hibbs, Srednicki (beware of different conventions), Anselmi, Cheng–Li. Quantum mechanics is a non-relativistic theory developed through the canonical quantization of observables (also called first quantization) $[\hat{x},\hat{p}] = i\hbar$, but it can also be equivalently formulated through Feynman's path integrals. In quantum field theory one can use the canonical quantization of fields (also called second quantization): expanding the free field in terms of plane waves, one promotes the Fourier coefficients to operators obeying (anti-)commutation relations

$$[a(\mathbf{k}), a^\dagger(\mathbf{k})] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')$$

Though one may also use functional quantization which is developed in the following.

A massive particle. See Anselmi, Cheng–Li and Srednicki (for philosophy, application etc see Feynman–Hibbs). A simple case is the one of one massive particle in one dimension subject to a potential V. The Hamiltonian of the system is

$$H = \frac{p^2}{2m} + V(q)$$

where q is the position of the particle. The observables are promoted to operators

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q})\,, \quad [\hat{q},\hat{p}] = \mathrm{i}\hbar$$

The propagation of the particle is described by Schrödinger's equation

$$\hat{H}\,|\psi(t)\rangle = \mathrm{i}\hbar\,\partial_t\,|\psi(t)\rangle$$

where $|\psi(t)\rangle$ is the state of the particle. Starting from the Hamiltonian, the corresponding Lagrangian is obtained by the Legendre transform

$$L(q,\dot{q}) = [p\dot{q} - H(p,q)]_{p=p(q,\dot{q})}$$

The expression of the momentum p in terms of q and $\dot{q}$ is obtain by solving Hamilton's equations

$$\dot{q} = \partial_p H\,, \quad \dot{p} = -\partial_q H$$

To describe the propagation of a particle one needs to know the probability of measuring such particle after some time. One would like an alternative mathematical formulation to ordinary

quantum mechanics. Consider a particle at some initial point q_i and at some initial time t_i. Letting it propagate up to a final time t_f, one would like to compute the probability that the particle is at a final position q_f. Between the two measurements $t_i < t < t_f$, the particle is not being observed and it propagates. One can at most compute a probability amplitude given by

$$ {}_{\mathrm{S}}\langle q_f|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H(t_f-t_i)}\,|q_i\rangle_{\mathrm{S}} \equiv {}_{\mathrm{H}}\langle q_f,t_f|q_i,t_i\rangle_{\mathrm{H}} $$

where $|q,t\rangle_{\mathrm{H}} = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}Ht}\,|q\rangle_{\mathrm{S}}$ is an instantaneous (and time independent) eigenvector of the time dependent position operator $Q(t)$ (see Srednicki, p. 43). Feynman observes that one does not know the precise path of the particle during its propagation, so the particle could take any path between the start and the end points. The only condition is that when $\hbar \to 0$, the classical limit, among the infinite number of possible paths between the start and the end, the only path available is the classical one, which is taken from the variational principle (called action principle in quantum field theory)

$$ \delta S = 0\,, \quad \forall q \mid \delta q(t_i) = \delta q(t_f) = 0 $$

Feynman proposes that the probability amplitude should be the sum over all possible paths, each weighed by the probability of the particle to travel along such path: when taking the classical limit, the most probable path is the classical one. They weight of each path is

$$ \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(q_f,t_f,q_i,t_i)} $$

where the action is the classical one. The reason the weight is the above can be seen as follows. Interpreting the sum as an average, in the classical limit the weight oscillates rapidly and its average is zero. The classical path is the only one that can survive because the action is at a minimum, so minimal oscillation. Therefore, the path integral formulation is

$$ {}_{\mathrm{S}}\langle q_f|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H(t_f-t_i)}\,|q_i\rangle_{\mathrm{S}} = \int [\mathcal{D}q]\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(q_f,t_f,q_i,t_i)} $$

The right-hand side is just a formal definition of the idea of path integral.

Lecture 2

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One would like to give an explicit prescription for the sum of all possible paths between two points

$$ \int [\mathcal{D}q] $$

Since one does not know how to deal with a continuum of paths, one can discretize the theory and then take the continuum limit. Time and space are discretized in equal-length steps

$$ \delta t = \frac{t_f - t_f}{N+1} $$

The position of the particle is no longer $q(t)$ but $q(t_j)$. The smaller is the step $\delta t \ll 1$, the better the discrete path approximates the Feynman path. Consecutive discrete points can be joined by straight segments giving the discrete path as

$$ q(t) = \frac{q_j - q_{j-1}}{\delta t}(t - t_{j-1}) + q_{j-1}\,, \quad j = 1,\dots,N+1 $$

where $q_0 \equiv q_i$ and $q_{N+1} \equiv q_f$. From the equation above, the time interval is

$$ t_f - t_i = (N+1)\delta t $$

so the probability amplitude becomes

$$ \begin{aligned} A &= {}_{\mathrm{S}}\langle q_f|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H(t_f-t_i)}\,|q_i\rangle_{\mathrm{S}} = {}_{\mathrm{S}}\langle q_f|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\delta t\,(N+1)}\,|q_i\rangle_{\mathrm{S}} = {}_{\mathrm{S}}\langle q_f|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\delta t}\cdots\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\delta t}\,|q_i\rangle_{\mathrm{S}} \\ &= \int \prod_{j=1}^{N} \mathrm{d}q_j\ \langle q_f|\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\,\delta t}\,|q_N\rangle\langle q_N|\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\,\delta t}\,|q_{N-1}\rangle\cdots\langle q_1|\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\,\delta t}\,|q_i\rangle \end{aligned} $$

at the second line one has inserted N identities as the completeness relation in the position eigenstates

$$\int \mathrm{d}q\ |q\rangle\langle q| = I$$

Considering a generic factor, one notices

$$\langle q_{j+1}|\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\,\delta t}\,|q_j\rangle = \langle q_{j+1}|\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}[\frac{p^2}{2m}+V(q)]\,\delta t}\,|q_j\rangle$$

Since q and p do not compute, one cannot split the exponential the same way one does with numbers, but one has to apply the Zassenhaus formula (related to the Baker–Campbell–Hausdorff formula):

$$\mathrm{e}^{t(x+y)} = \mathrm{e}^{tx}\mathrm{e}^{ty}\mathrm{e}^{-\frac{1}{2}t^2[x,y]+o(t^2)}$$

The first order in t is just $\mathrm{e}^{t(x+y)} \approx \mathrm{e}^{tx}\mathrm{e}^{ty}$. Therefore, one has

$$\langle q_{j+1}|\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\,\delta t}\,|q_j\rangle = \langle q_{j+1}|\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{p^2}{2m}\,\delta t}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}V(q)\,\delta t}\,|q_j\rangle$$

The position states are eigenstates of the potential, but not of the momentum. One can insert a completeness relation in the momentum eigenstates between the two exponentials

$$\begin{aligned}\langle q_{j+1}|\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\,\delta t}\,|q_j\rangle &= \int \mathrm{d}p_j\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{p_j^2}{2m}\,\delta t}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}V(q_j)\,\delta t}\,\langle q_{j+1}|p_j\rangle\,\langle p_j|q_j\rangle \\ &= \int \frac{\mathrm{d}p_j}{2\pi\hbar}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{p_j^2}{2m}\,\delta t-\frac{\mathrm{i}}{\hbar}V(q_j)\,\delta t}\mathrm{e}^{\mathrm{i}p_j(q_{j+1}-q_j)}\end{aligned}$$

the operators can then act on their eigenstates producing the associated eigenvalue. At the second line one remembers

$$\langle q|p\rangle = \frac{\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}}{\sqrt{2\pi\hbar}}$$

Since δt is infinitesimal, then one may substitute the potential with its value at the midpoint between two coordinates

$$V(q_j) \to V(\bar{q}_j)\,, \quad \bar{q}_j = \frac{q_{j+1}+q_j}{2}$$

Replacing every factor inside the probability amplitude, one gets

$$A = \int \prod_{j=1}^{N}\mathrm{d}q_j \prod_{k=0}^{N}\frac{\mathrm{d}p_k}{2\pi\hbar}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p_k(q_{k+1}-q_k)-\frac{\mathrm{i}}{\hbar}H(p_k,\bar{q}_k)\,\delta t}$$

In the limit $N \to \infty$, equivalent to $\delta t \to 0$, one has

$$\begin{aligned}A &= \lim_{N\to\infty}\int \prod_{j=1}^{N}\mathrm{d}q_j \prod_{k=0}^{N}\frac{\mathrm{d}p_k}{2\pi\hbar}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p_k\frac{q_{k+1}-q_k}{\delta t}\,\delta t-\frac{\mathrm{i}}{\hbar}H(p_k,\bar{q}_k)\,\delta t} \\ &= \lim_{N\to\infty}\int \prod_{j=1}^{N}\mathrm{d}q_j \prod_{k=0}^{N}\frac{\mathrm{d}p_k}{2\pi\hbar}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p_k\dot{q}_k\,\delta t-\frac{\mathrm{i}}{\hbar}H(p_k,\bar{q}_k)\,\delta t} \\ &\equiv \int [\mathcal{D}q\,\mathcal{D}p]\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_i}^{t_f}\mathrm{d}t\,\big[p(t)\dot{q}(t)-H(p(t),q(t))\big]\right]\end{aligned}$$

The last line is the definition of the path integral.

Since the explicit formula of the Hamiltonian is known

$$H = \frac{p^2}{2m} + V(q)$$

one can integrate over a momenta p_k to obtain

$$\begin{aligned}\int \mathrm{d}p_k\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p_k\dot{q}_k-\frac{p_k^2}{2m})\,\delta t} &= \frac{1}{2\pi\hbar}\sqrt{\frac{2\pi\hbar m}{\mathrm{i}\,\delta t}}\exp\left[-\frac{\dot{q}_k^2\,(\delta t)^2}{\hbar^2}\left(\frac{2\mathrm{i}}{\hbar}\frac{\delta t}{m}\right)^{-1}\right] \\ &= \left[\frac{m}{2\pi\hbar\mathrm{i}\,\delta t}\right]^{\frac{1}{2}}\mathrm{e}^{\frac{\mathrm{i}m}{2\hbar}(\frac{q_{k+1}-q_k}{\delta t})^2\,\delta t}\end{aligned}$$

where one applies the analytic continuation of the Gaussian integral

$$\int \mathrm{d}x\, \mathrm{e}^{-\frac{a}{2}x^2+bx} = \sqrt{\frac{2\pi}{a}}\mathrm{e}^{\frac{b^2}{2a}}\,, \quad a = \frac{\mathrm{i}}{\hbar}\frac{\delta t}{m}\,, \quad b = \frac{\mathrm{i}}{\hbar}\dot{q}_k\,\delta t$$

At the second line, one replaces

$$\dot{q}_k \equiv \frac{q_{k+1}-q_k}{\delta t}$$

Performing the integration for every $k = 0, \dots, N$, the probability amplitude is then

$$A = \lim_{N\to\infty}\left[\frac{m}{2\pi\hbar\mathrm{i}\,\delta t}\right]^{\frac{N+1}{2}} \int \prod_{j=1}^{N} \mathrm{d}q_j\, \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{j=0}^{N}\delta t\left(\frac{m}{2}\left(\frac{q_{j+1}-q_j}{\delta t}\right)^2 - V\right)\right]$$

in the limit, the exponent is the classical Lagrangian (i.e. there are no operators)

$$\frac{\mathrm{i}}{\hbar}\sum_{j=0}^{N}\delta t\left[\frac{m}{2}\left(\frac{q_{j+1}-q_j}{\delta t}\right)^2 - V\right] = \frac{\mathrm{i}}{\hbar}\int_{t_i}^{t_f}\mathrm{d}t\, L(q,\dot{q})$$

The coefficient in square bracket of the amplitude is divergent but it compensates the infinitesimal nature of the integration measure. The path integral is then defined as

$$\int [\mathcal{D}q] \equiv \lim_{N\to\infty}\left[\frac{m}{2\pi\hbar\mathrm{i}\,\delta t}\right]^{\frac{N}{2}} \int \prod_{j=1}^{N-1}\mathrm{d}q_j$$

mind the rescaling $N+1 \to N$. To give a meaningful definition, time has to be discretized: continuous functions become discrete (like functions on a lattice).

Remark. See Feynman–Hibbs. If the path integral formulation is a consistent alternative prescription, then it concerns the computation of probability amplitudes, called kernels or Feynman propagators

$$K(q_f,t_f;q_i,t_i) \equiv \langle q_f|\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H(t_f-t_i)}\,|q_i\rangle$$

The definition is independent of ordinary quantum mechanics and their operators. Probability amplitudes are weighed with the exponential of the classical action, the only sign of quantum mechanics is the appearance of Planck's constant $\hbar$.

One may notice that

- The weight $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}$ is given in terms of the classical action

  $$S = \int_{t_i}^{t_f}\mathrm{d}t\, L(q(t),\dot{q}(t))$$

  The path integral is then referred to as the quantum integral.

- One may find a first equivalence with ordinary quantum mechanics. Consider the quantum mechanical wave function

  $$\begin{aligned}\psi(q,t) = {}_{\mathrm{H}}\langle q,t|\psi\rangle = {}_{\mathrm{S}}\langle q|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}Ht}\,|\psi\rangle &= \int \mathrm{d}q'\ {}_{\mathrm{S}}\langle q|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H(t-t')}\,|q'\rangle\,\langle q',t'|\psi\rangle \\ &= \int \mathrm{d}q'\, K(q,t;q',t')\psi(q',t')\end{aligned}$$

  The kernel has the meaning of evolution operator. One may check that the wave function $\psi(q,t)$ satisfies Schrödinger's equation (see Anselmi, Feynman–Hibbs) thanks to the properties of the kernel.

Kernel of a free particle. See Anselmi. The potential of a free particle is identically zero $V(q) = 0$. The kernel is

$$K_0(q_f, t_f; q_i, t_i) = \lim_{N\to\infty} \Big[\frac{m}{2\pi\hbar \mathrm{i}\, \delta t}\Big]^{\frac{N}{2}} \int \prod_{j=1}^{N-1} \mathrm{d}q_j \, \exp\Big[\frac{\mathrm{i}m}{2\hbar\delta t}\sum_{j=1}^{N}(q_j - q_{j-1})^2\Big]$$

Its form is that of many Gaussian integrals, but they are not factorized, so a change of variables is necessary

$$\widetilde{q}_j = q_j - q_f\,, \quad \widetilde{q}_0 = q_i - q_f\,, \quad \widetilde{q}_N = 0\,, \quad \mathrm{d}q_j = \mathrm{d}\widetilde{q}_j$$

remembering that after rescaling? [r] one has

$$q_0 = q_i\,, \quad q_N = q_f$$

The sum in the exponent is then

$$\begin{aligned}\sum_{j=1}^{N}(q_j - q_{j-1})^2 &= (q_1 - q_i)^2 + (q_2 - q_1)^2 + \cdots + (q_N - q_{N-1})^2 \\ &= [\widetilde{q}_1 - (q_i - q_f)]^2 + (\widetilde{q}_2 - \widetilde{q}_1)^2 + \cdots + (\cancel{\widetilde{q}_N} - \widetilde{q}_{N-1})^2 \\ &= (q_i - q_f)^2 - 2\widetilde{q}_1(q_i - q_f) + 2\widetilde{q}_1^2 + 2\widetilde{q}_2^2 + \cdots + 2\widetilde{q}_{N-1}^2 \\ &\quad -2\widetilde{q}_1\widetilde{q}_2 - 2\widetilde{q}_3\widetilde{q}_2 - \cdots - 2\widetilde{q}_{N-2}\widetilde{q}_{N-1} \\ &= (q_i - q_f)^2 + 2\widetilde{q}_1(q_f - q_i) + \widetilde{q}^{\top}\widetilde{M}\widetilde{q}\end{aligned}$$

where one has a $N-1$ dimensional vector and square matrix

$$q = \begin{bmatrix} q \\ \vdots \\ q_{N-1}\end{bmatrix}\,, \quad \widetilde{M} = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots \\ -1 & 2 & -1 & 0 & \cdots \\ 0 & -1 & 2 & -1 & \cdots \\ 0 & 0 & -1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The kernel is then

$$\begin{aligned}K_0(q_f, t_f; q_i, t_i) &= \lim_{N\to\infty} \Big[\frac{m}{2\pi\hbar \mathrm{i}\, \delta t}\Big]^{\frac{N}{2}} \mathrm{e}^{\frac{\mathrm{i}m}{2\hbar\,\delta t}(q_f - q_i)^2} \int \prod_{j=1}^{N-1} \mathrm{d}\widetilde{q}_j \exp\Big[\frac{\mathrm{i}m}{2\hbar\,\delta t}\big[\widetilde{q}^{\top}\widetilde{M}\widetilde{q} + 2\widetilde{q}_1(q_f - q_i)\big]\Big] \\ &= \lim_{N\to\infty} \Big[\frac{m}{2\pi\hbar \mathrm{i}\, \delta t\, N}\Big]^{\frac{1}{2}} \mathrm{e}^{\frac{\mathrm{i}m}{2\hbar\,\delta t}(q_f - q_i)^2} \mathrm{e}^{-\frac{\mathrm{i}m}{2\hbar\,\delta t}(q_f - q_i)^2 \frac{N-1}{N}} \\ &= \lim_{N\to\infty} \Big[\frac{m}{2\pi\hbar \mathrm{i}\, \delta t\, N}\Big]^{\frac{1}{2}} \mathrm{e}^{\frac{\mathrm{i}m}{2\hbar\,\delta t\, N}(q_f - q_i)^2} = \Big[\frac{m}{2\pi\hbar \mathrm{i}\, \Delta t}\Big]^{\frac{1}{2}} \mathrm{e}^{\frac{\mathrm{i}m}{2\hbar\,\Delta t}(q_f - q_i)^2}\end{aligned}$$

at the second line one remembers that

$$\int \prod_{i=1}^{N-1} \mathrm{d}x_i \, \mathrm{e}^{-\frac{1}{2}x^{\top}Mx + b^{\top}x} = \frac{(2\pi)^{\frac{N-1}{2}}}{\sqrt{\det M}} \mathrm{e}^{\frac{1}{2}b^{\top}M^{-1}b}$$

where

$$M = -\frac{\mathrm{i}m}{\hbar\,\delta t}\widetilde{M}\,, \quad \det\widetilde{M} = N\,, \quad (\widetilde{M}^{-1})_{11} = \frac{N-1}{N}\,, \quad b_1 = \frac{\mathrm{i}m}{\delta t}(q_f - q_i)$$

At the last line, one notices that

$$\delta t = \frac{t_f - t_i}{N} = \frac{\Delta t}{N}$$

With this specific example, one may notice that the divergence of δt is compensated to give a finite result.

Properties of the kernel. A few properties:

- The equal-time limit is
$$\lim_{t_f \to t_i} K_0(q_f, t_f; q_i, t_i) = \delta(q_f - q_i)$$
equivalent to $q_f \to q_i$. This is consistent with an expression of the Dirac delta given by
$$\lim_{\varepsilon \to 0} \frac{1}{\sqrt{2\pi\varepsilon}} \mathrm{e}^{-\frac{q^2}{2\varepsilon}} = \delta(q)$$
[r]

- The kernel satisfies Schrödinger's equation
$$\mathrm{i}\hbar\, \partial_t K_0 = -\frac{\hbar^2}{2m}\, \partial_q^2 K_0$$
This can be checked for
$$K_0(q, t; 0, 0) = \sqrt{\frac{m}{2\pi\hbar \mathrm{i} t}} \mathrm{e}^{\frac{\mathrm{i}m}{2\hbar t} q^2}$$
It follows that the wave function
$$\psi(q, t) = \int \mathrm{d}q'\, K_0(q, t; q', t')\psi(q', t')$$
satisfies Schrödinger's equation because its dependence on time comes from the kernel.

- Composition law
$$K_0(q, t; q_0, t_0) = \int \mathrm{d}q'\, K_0(q, t; q', t') K_0(q', t'; q_0, t_0)$$
This can be seen from
$$K_0(q, t, ; q_0, t_0) = \left[\frac{m}{2\pi\hbar \mathrm{i}(t - t_0)}\right]^{\frac{1}{2}} \mathrm{e}^{\frac{\mathrm{i}m}{2\hbar} \frac{(q-q_0)^2}{t-t_0}}$$
but working backwards: integrating to obtain such expression. The integral is not trivial since in the exponents one has
$$\begin{aligned} \frac{\mathrm{i}m}{2}\left[\frac{(q-q')^2}{t-t'} + \frac{(q'-q_0)^2}{t'-t_0}\right] &= \frac{\mathrm{i}m}{2}\left[\frac{[(t'-t_0)q + (t-t')q_0 + (t_0-t)q']^2}{(t-t_0)(t-t')(t'-t_0)} + \frac{(q-q_0)^2}{t-t_0}\right] \\ &= \frac{\mathrm{i}m}{2}\frac{(q-q_0)^2}{t-t_0} + \frac{\mathrm{i}m}{2}\left[-\frac{q'}{(t-t')(t'-t_0)} + \delta q\right] \end{aligned}$$
[r] idk if true
where δq is some shift and one can substitute the bracket with another variable.

Exercise. Finish the calculation.

Lecture 3

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2.1 Quadratic potential

See Srednicki, Schulman cap 6. From now on one sets $\hbar = 1$. Consider a one dimensional massive particle in a quadratic potential
$$V(q) = \frac{1}{2}c(t)q^2$$
The Lagrangian is
$$L(q, \dot{q}) = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}c(t)q^2$$

One would like to compute the kernel, the Feynman propagator

$$K(q_b, t_b; q_a, t_a) = \int_{q_a, t_a}^{q_b, t_b} [\mathcal{D}q]\, \mathrm{e}^{\mathrm{i}S}$$

The following procedure works well for quadratic potentials. One can expand a generic trajectory $q(t)$ around a classical trajectory $\bar{q}(t)$ (that is, solution of the classical equations of motion obtained from the variational principle)

$$q(t) = \bar{q}(t) + \delta q(t)$$

This fluctuation does not apply to the end points $\delta q(t_a) = \delta q(t_b) = 0$. The variation of the action is

$$\delta S = \int \mathrm{d}t\, \frac{\delta S}{\delta q(t)}\, \delta q(t) = 0\,, \quad \forall \delta q(t) \implies \frac{\delta S}{\delta q(t)} = 0$$

For the quadratic potential, the equations of motion are

$$m\ddot{\bar{q}} + c(t)\bar{q} = 0$$

The path integral sums all the fluctuations δq from the classical path. To find the kernel, one needs to compute the action

$$S[\bar{q} + \delta q, \dot{\bar{q}} + \delta \dot{q}]$$

First method. There are two methods to compute the kernel. The first one is

$$S = \int \mathrm{d}t\, \left[\frac{1}{2}m\dot{q}^2 - \frac{1}{2}c(t)q^2\right] = \int \mathrm{d}t\, \left[\frac{1}{2}m(\dot{\bar{q}} + \delta\dot{q})^2 - \frac{1}{2}c(t)(\bar{q} + \delta q)^2\right] = \cdots = S[\bar{q}] + S[\delta q]$$

[r] This result is particular to quadratic potentials.

Second method. The second method utilizes functional derivatives and the equivalent of the Taylor series

$$S[\bar{q} + \delta q] = S[\bar{q}] + \delta_q S|_{\bar{q}}\, \delta q + \frac{1}{2}\, \delta_q^2 S|_{\bar{q}}\, (\delta q)^2$$

The expansion ends at second order because the potential is quadratic. [r] The linear term is zero because $\bar{q}$ is the solution to the classical equations of motion, thus extremizing the action. Only the last term has to be computed

$$\delta_q^2 S\, (\delta q)^2 = \delta_q(\delta_q S\, \delta q)\, \delta q = \int \mathrm{d}t''\, \frac{\delta}{\delta q(t'')} \left[\int \mathrm{d}t'\, \frac{\delta S}{\delta q(t')} \delta q(t')\right] \delta q(t'')$$

The inner integral is

$$\begin{aligned}
\int \mathrm{d}t'\, \frac{\delta S}{\delta q(t')}\, \delta q(t') &= \int \mathrm{d}t' \int \mathrm{d}t\, \frac{\delta L(q(t))}{\delta q(t')}\, \delta q(t') \\
&= \int \mathrm{d}t'\, \mathrm{d}t\, [m\dot{q}(t)\, \partial_t \delta(t - t') - c(t)q(t)\delta(t - t')]\, \delta q(t') \\
&= \int \mathrm{d}t\, [-m\dot{q}(t)\delta\dot{q}(t) - c(t)q(t)\delta q(t)]
\end{aligned}$$

where one remembers

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}c(t)q^2\,, \quad \frac{\delta q(t)}{\delta q(t')} = \delta(t - t')$$

in particular

$$\frac{\delta \dot{q}^2(t)}{\delta q(t')} = \frac{\delta \dot{q}^2(t)}{\delta \dot{q}(t)} \frac{\delta \dot{q}(t)}{\delta q(t')} = 2\dot{q}(t)\, \partial_t \frac{\delta q(t)}{\delta q(t')} = 2\dot{q}(t)\, \partial_t \delta(t - t')$$

The second derivative is then

$$\begin{aligned}\delta_q^2 S\,(\delta q)^2 &= \int \mathrm{d}t''\,\frac{\delta}{\delta q(t'')}\left[\int \mathrm{d}t\;[-m\dot{q}(t)\delta\dot{q}(t) - c(t)q(t)\delta q(t)]\right]\delta q(t'')\\ &= \int \mathrm{d}t''\,\mathrm{d}t\,[-m\,\partial_t\delta(t-t'')\,\delta\dot{q}(t)\,\delta q(t'') - c(t)\delta(t-t'')\,\delta q(t)\,\delta q(t'')]\\ &= \int \mathrm{d}t\;\left[m(\delta\dot{q})^2 - c(t)[\delta q(t)]^2\right]\end{aligned}$$

This is exactly the original action but with δq instead of q (and a missing factor of $1/2$). Therefore

$$S[\bar{q}+\delta q] = S[\bar{q}] + \frac{1}{2}\,\delta_q^2 S|_{\bar{q}}\,(\delta q)^2 = S[\bar{q}] + S[\delta q]$$

Remark. This method of the Taylor expansion is more general and can be applied to arbitrary potentials of the form $V(q,g)$ where $g \ll 1$ is a coupling constant. Starting from the cubic term of the Taylor series, one has contributions only from the potential since the kinetic term is quadratic.

Kernel. The kernel becomes

$$K(q_b,t_b;q_a,t_a) = \mathrm{e}^{\mathrm{i}S[\bar{q}]}\int_{0,t_a}^{0,t_b}[\mathcal{D}\delta q]\,\mathrm{e}^{\mathrm{i}S[\delta q]} \equiv \mathrm{e}^{\mathrm{i}S[\bar{q}]}K(0,t_b;0,t_a)$$

To compute the kernel, one has to discretize time

$$\frac{t_b-t_a}{N+1} = \delta t \implies K(0,t_b;0,t_a) = \lim_{N\to\infty}\left[\frac{m}{2\pi\mathrm{i}\,\delta t}\right]^{\frac{1}{2}}\int\prod_{j=1}^{N}\mathrm{d}(\delta q_j)\,\mathrm{e}^{\mathrm{i}S[\delta q]}$$

where the action is discretized also

$$\mathrm{i}S[\delta q] = \sum_{j=0}^{N}\mathrm{i}\left[\frac{m}{2\,\delta t}(\delta q_{j+1}-\delta q_j)^2 - \frac{1}{2}\delta t\,c_j(\delta q_j)^2\right],\quad c_j = c(t_a+\delta t\,j),\quad \delta q_j = \delta q\,(t_a+\delta t\,j)$$

keeping in mind that the extremes have to be constant

$$\delta q_0 = \delta q(t_a) = 0\,,\quad \delta q_{N+1} = \delta q(t_b) = 0$$

[r] Considering a vector

$$\eta = \begin{bmatrix}\delta q_1\\ \vdots\\ \delta q_N\end{bmatrix}$$

by completing the square? [r], the action can be rewritten as

$$\mathrm{i}S[\delta q] = -\eta^\top A\eta\,,\quad A = \frac{m}{2\pi\mathrm{i}\,\delta t}\begin{bmatrix}2-\frac{(\delta t^2)}{m}c_1 & -1 & 0 & \cdots\\ -1 & 2-\frac{(\delta t^2)}{m}c_2 & -1 & \cdots\\ 0 & -1 & 2-\frac{(\delta t)^2}{m}c_3 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{bmatrix}$$

The kernel is then

$$\begin{aligned}K(0,t_b;0,t_a) &= \lim_{N\to\infty}\left[\frac{m}{2\pi\mathrm{i}\,\delta t}\right]^{\frac{N+1}{2}}\int\mathrm{d}^N\eta\,\mathrm{e}^{-\eta^\top A\eta} = \lim_{N\to\infty}\left[\frac{m}{2\pi\mathrm{i}\,\delta t}\right]^{\frac{N+1}{2}}\frac{\pi^{\frac{N}{2}}}{\sqrt{\det A}}\\ &= \lim_{N\to\infty}\left[\frac{m}{2\pi\mathrm{i}\,\delta t}\frac{\pi^N}{(\frac{2\pi\mathrm{i}\,\delta t}{m})^N\det A}\right]^{\frac{1}{2}} = \lim_{N\to\infty}\left[\frac{m}{2\pi\mathrm{i}}\frac{1}{\delta t(\frac{2\pi\mathrm{i}\,\delta t}{m})^N\det A}\right]^{\frac{1}{2}}\end{aligned}$$

At the second equality of the first line one applies the Gaussian integral (notice there is no $1/2$). Let the denominator be

$$F_N(t_b,t_a) \equiv \delta t \left(\frac{2\pi \mathrm{i}\,\delta t}{m}\right)^N \det A = \delta t\, P_N$$

and

$$P_N \equiv \left(\frac{2\pi \mathrm{i}\,\delta t}{m}\right)^N \det A = \det \begin{bmatrix} 2-\frac{(\delta t)^2}{m}c_1 & -1 & 0 & \cdots \\ -1 & 2-\frac{(\delta t)^2}{m}c_2 & -1 & \cdots \\ 0 & -1 & 2-\frac{(\delta t)^2}{m}c_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

One can find a recursion relation for a generic P_j given by

$$P_{j+1} = \left(2-\frac{(\delta t)^2}{m}c_{j+1}\right)P_j - P_{j-1}\,, \quad j=1,\ldots,N\,, \quad P_0 = 1\,, \quad P_1 = 2-\frac{(\delta t)^2}{m}c_1$$

This relation can be rewritten as

$$\delta t\,\frac{P_{j+1}-2P_j+P_{j-1}}{\delta t^2} = -\frac{c_{j+1}}{m}P_j\,\delta t$$

Letting

$$F_j = \delta t\, P_j \equiv \varphi_j = \varphi(t_a+\varepsilon j)$$

the previous equation becomes

$$\frac{\varphi_{j+1}-2\varphi_j+\varphi_{j-1}}{(\delta t)^2} = -\frac{c_{j+1}}{m}\varphi_j$$

For $\delta t \to 0$ the identity above becomes a second order differential equation

$$\mathrm{d}_t^2\varphi(t) = -\frac{c(t)}{m}\varphi(t)$$

The initial value problem is

$$\varphi(t=t_a) = \delta t\, P_0 \to 0\,, \quad \mathrm{d}_t\varphi(t=t_a) = \delta t\,\frac{P_1-P_0}{\delta t} = 2-\frac{\delta t^2}{m}c_1 - 1 \to 1$$

The quantity needed to compute in the kernel $K(0,t_b;0,t_a)$ is $F(t_b,t_a)=\varphi(t=t_b)$. The full kernel is thus

$$K(q_b,t_b;q_a,t_a) = \mathrm{e}^{\mathrm{i}S[\bar{q}]}K(0,t_b;0,t_a) = \mathrm{e}^{\mathrm{i}S[\bar{q}]}\left[\frac{m}{2\pi\mathrm{i}F(t_b,t_a)}\right]^{\frac{1}{2}}$$

where one has

$$F(t,t_a) = \varphi(t)$$

which satisfies

$$m\,\mathrm{d}_t^2F(t,t_a) + c(t)F(t,t_a) = 0\,, \quad F(t_a,t_a)=0\,, \quad \mathrm{d}_tF(t_a,t_a)=1$$

Simple harmonic oscillator. The parameter is

$$c(t) = m\omega^2$$

The differential equation above becomes

$$\ddot{F} + \omega^2 F = 0\,, \quad F(t_a)=0\,, \quad \dot{F}(t_a)=1$$

whose solution is

$$F(t,t_a) = \frac{1}{\omega}\sin[\omega(t-t_a)]$$

The desired quantity is then

$$F(t_b, t_a) = \frac{1}{\omega} \sin[\omega(t_b - t_a)] = \frac{1}{\omega} \sin \omega T \,, \quad T \equiv t_b - t_a$$

To compute the kernel one has to find the action and path for the classical equations of motion. The Lagrangian is

$$L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2$$

The equations of motion are

$$m\ddot{q} + m\omega^2 q = 0 \,, \quad q(t_a) = q_a \,, \quad q(t_b) = q_b$$

The solution is therefore

$$q(t) = \frac{1}{\sin \omega T} [q_b \sin \omega(t - t_a) + q_a \sin \omega(t_b - t)]$$

Its derivative is

$$\dot{q}(t) = \frac{\omega}{\sin \omega T} [q_b \cos \omega(t - t_a) - q_a \cos \omega(t_b - t)]$$

The action is

$$\begin{aligned} S &= \int \mathrm{d}t \left[\frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2 \right] = \frac{1}{2} m q \dot{q} \Big|_{t_a}^{t_b} - \int \mathrm{d}t \left[\frac{1}{2} m q \ddot{q} + \frac{1}{2} m \omega^2 q^2 \right] \\ &= \frac{1}{2} m \bar{q}(t_b) \dot{\bar{q}}(t_b) - \frac{1}{2} m \bar{q}(t_a) \dot{\bar{q}}(t_a) = \cdots \\ &= \frac{m\omega}{2 \sin \omega T} [(q_a^2 + q_b^2) \cos \omega T - 2 q_a q_b] \end{aligned}$$

At the first equality one integrates by parts. The last bracket is zero

$$\frac{1}{2} m q (\ddot{q} + \omega^2 q)|_{\bar{q}} = 0$$

on the classical path since the equations of motion are

$$\ddot{\bar{q}} + \omega^2 \bar{q} = 0$$

Finally, the kernel is

$$K(q_b, t_b; q_a, t_a) = \left[\frac{m\omega}{2\pi \mathrm{i} \sin \omega T} \right]^{\frac{1}{2}} \exp\left[\frac{\mathrm{i} m\omega}{2 \sin \omega T} \big((q_a^2 + q_b^2) \cos \omega T - 2 q_a q_b \big) \right]$$

For any quadratic potential, the kernel can be computed by solving the differential equation of $F(t, t_a)$. For other potentials, one has to use perturbative approaches.

2.2 Partition function

The kernel is defined as

$$K(q_f, t_f; q_i, 0) = {}_{\mathrm{S}}\langle q_f | \mathrm{e}^{-\mathrm{i} H t_f} \, | q_i \rangle_{\mathrm{S}} = \int [\mathcal{D}q] \, \mathrm{e}^{\mathrm{i} S[q]}$$

The partition function is the integral of a periodic kernel (i.e. its initial and final points coincide)

$$Z(t) = \int \mathrm{d}q \, K(q, t; q, 0)$$

Relation to statistical mechanics. This definition coincides with the statistical mechanical definition

$$Z(t) = \operatorname{Tr} \mathrm{e}^{-\mathrm{i}\hat{H}t}$$

Setting the system in a box, one has a discrete energy spectrum E_n with corresponding energy states $|n\rangle$. The identity can be decomposed as

$$I = \sum_n |n\rangle\langle n|$$

The kernel is then

$$\begin{aligned} K(q_f, t_f; q_i, 0) &= \langle q_f | \, \mathrm{e}^{-\mathrm{i}Ht_f} \, |q_i\rangle = \sum_n \langle q_f | \, \mathrm{e}^{-\mathrm{i}Ht_f} \, |n\rangle \, \langle n|q_i\rangle = \sum_n \mathrm{e}^{-\mathrm{i}E_n t_f} \, \langle q_f|n\rangle \, \langle n|q_i\rangle \\ &= \sum_n \mathrm{e}^{-\mathrm{i}E_n t_f} \psi_n^\dagger(q_f)\psi_n(q_i) \end{aligned}$$

The partition function is then

$$\begin{aligned} Z(t) &= \int \mathrm{d}q \, K(q,t;q,0) = \int \mathrm{d}q \, \sum_n \mathrm{e}^{-\mathrm{i}E_n t} \psi_n^\dagger(q)\psi_n(q) \\ &= \sum_n \mathrm{e}^{-\mathrm{i}E_n t} \int \mathrm{d}q \, |\psi_n(q)|^2 = \sum_n \mathrm{e}^{-\mathrm{i}E_n t} \end{aligned}$$

remembering that the wave function is normalized such that the above integral (i.e. the probability) is unity. Finally one notices that

$$Z(t) = \operatorname{Tr} \mathrm{e}^{-\mathrm{i}\hat{H}t} = \sum_n \operatorname{Tr}\left[\mathrm{e}^{-\mathrm{i}\hat{H}t} \, |n\rangle\langle n|\right] = \sum_n \langle n| \, \mathrm{e}^{-\mathrm{i}\hat{H}t} \, |n\rangle = \sum_n \mathrm{e}^{-\mathrm{i}E_n t}$$

at the second equality one inserts the identity inside the trace (remembering that $\operatorname{Tr}(A+B) = \operatorname{Tr} A + \operatorname{Tr} B$) and the cyclic property of the trace noting that the Hamiltonian is diagonal on the energy eigenstates $|n\rangle$.

Exercise. .[r] Compute the partition function of the harmonic oscillator. Use Taylor expansion. Prove that the harmonic oscillator energy spectrum is

$$E_n = \hbar\omega \left[n + \frac{1}{2}\right]$$

Solution. Starting from the kernel

$$K(q_b, t_b; q_a, t_a) = \left[\frac{m\omega}{2\pi\mathrm{i}\sin\omega T}\right]^{\frac{1}{2}} \exp\left[\frac{\mathrm{i}m\omega}{2\sin\omega T}\left((q_a^2 + q_b^2)\cos\omega T - 2q_a q_b\right)\right]$$

The relevant terms are

$$K(q,t;q,0) = \sqrt{\frac{A}{2\pi\mathrm{i}}} \exp\left[\mathrm{i}Aq^2(\cos\omega t - 1)\right], \quad A \equiv \frac{m\omega}{\sin\omega t}$$

Therefore

$$\begin{aligned} Z(t) &= \int \mathrm{d}q \, K(q,t;q,0) = \sqrt{\frac{A}{2\pi\mathrm{i}}} \int \mathrm{d}q \, \exp\left[\mathrm{i}Aq^2(\cos\omega t - 1)\right], \quad B \equiv A(\cos\omega t - 1) \\ &= \frac{1}{\sqrt{2\pi\mathrm{i}}} \frac{1}{\sqrt{\cos\omega t - 1}} \int \mathrm{d}x \, \mathrm{e}^{\mathrm{i}x^2}, \quad \sqrt{B}q \equiv x, \quad \mathrm{d}q = \frac{\mathrm{d}x}{\sqrt{B}} \\ &= \frac{1}{\sqrt{2\pi\mathrm{i}(\cos\omega t - 1)}} \sqrt{\frac{2\pi}{-2\mathrm{i}}} = \frac{1}{\sqrt{2(\cos\omega t - 1)}} = [\mathrm{e}^{\mathrm{i}\omega t} - 2 + \mathrm{e}^{-\mathrm{i}\omega t}]^{-\frac{1}{2}} \\ &= [\mathrm{e}^{\frac{\mathrm{i}}{2}\omega t} - \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t}]^{-1} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t} \frac{1}{1 - \mathrm{e}^{-\mathrm{i}\omega t}} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t} \sum_{n\geq 0} \mathrm{e}^{-\mathrm{i}n\omega t} \\ &= \sum_{n\geq 0} \exp\left[-\mathrm{i}\omega\left(n + \frac{1}{2}\right)t\right] = \sum_{n\geq 0} \mathrm{e}^{-\mathrm{i}E_n t} \implies E_n = \omega\left(n + \frac{1}{2}\right) \end{aligned}$$

At the second line, one applies the Gaussian integral

$$\int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-\frac{a}{2}x^2+bx} = \sqrt{\frac{2\pi}{a}}\mathrm{e}^{\frac{b^2}{2a}}\,, \quad a=-2\mathrm{i}$$

At the third line, one expresses the cosine in terms of complex exponentials. At the fourth line, one uses the analytic continuation of the geometric series.

The same calculation can be done in Euclidean time (and considering hyperbolic functions).

2.3 Correlation function

Two-point correlation function. A two-point correlation function for an operator O_{H} in the Heisenberg picture

$$O_{\mathrm{H}}(t) = \mathrm{e}^{\mathrm{i}Ht}O_{\mathrm{S}}\mathrm{e}^{-\mathrm{i}Ht}$$

is given by

$$C(t_1,t_2,t_f,t_i) = {}_{\mathrm{H}}\langle q_f,t_f|\mathcal{T}\{O_{\mathrm{H}}(t_1)O_{\mathrm{H}}(t_2)\}\,|q_i,t_i\rangle_{\mathrm{H}}$$

Considering $t_i<t_2<t_1<t_f$ one has

$$C(t_1,t_2,t_f,t_i) = \langle q_f,t_f|\,O_{\mathrm{H}}(t_1)O_{\mathrm{H}}(t_2)\,|q_i,t_i\rangle = \langle q_f|\,\mathrm{e}^{-\mathrm{i}H(t_f-t_1)}O_{\mathrm{S}}\mathrm{e}^{-\mathrm{i}H(t_1-t_2)}O_{\mathrm{S}}\mathrm{e}^{-\mathrm{i}H(t_2-t_i)}\,|q_i\rangle$$

Supposing that O_{S} is a function of $\hat{q}$ only, then one has

$$\begin{aligned}C(t_1,t_2,t_f,t_i) &= \\ &= \int \mathrm{d}q_1\,\mathrm{d}q_2\ \langle q_f|\,\mathrm{e}^{-\mathrm{i}H(t_f-t_1)}O_{\mathrm{S}}(q_1)\,|q_1\rangle\langle q_1|\,\mathrm{e}^{-\mathrm{i}H(t_1-t_2)}O_{\mathrm{S}}(q_2)\,|q_2\rangle\langle q_2|\,\mathrm{e}^{-\mathrm{i}H(t_2-t_i)}\,|q_i\rangle\end{aligned}$$

where one has substituted the operators O with their eigenvalues, so the ones above are functions (as hinted by the presence of an explicit function argument). One obtains

$$C(t_1,t_2,t_f,t_i) = \int \mathrm{d}q_1\,\mathrm{d}q_2\,K(q_f,t_f;q_1,t_1)O(q_1)K(q_1,t_1;q_2,t_2)O(q_2)K(q_2,t_2;q_i,t_i)$$

This corresponds to a path integral where one integrates all fluctuations of the classical path γ_1 between q_f and q_1, then measures with the operator, integrates the fluctuations of the classical path γ_2 between q_1 and q_2, measures with other operator and finally integrates the fluctuations to the beginning of the classical path γ_3 between q_2 and q_i.

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Inserting the definition of the kernels, one obtains

$$C(t_1,t_2) = \int \mathrm{d}q_1\,\mathrm{d}q_2\,O(q_1)O(q_2)\int[\mathcal{D}q^{\gamma_1}]\,\mathrm{e}^{\mathrm{i}S_{\gamma_1}}\int[\mathcal{D}q^{\gamma_2}]\,\mathrm{e}^{\mathrm{i}S_{\gamma_2}}\int[\mathcal{D}q^{\gamma_3}]\,\mathrm{e}^{\mathrm{i}S_{\gamma_3}}$$

Discretizing the path integrals

$$\int[\mathcal{D}q]\,\mathrm{e}^{\mathrm{i}S[q]} \to \frac{1}{A}\int\prod_{j=1}^{N}\frac{\mathrm{d}q_j}{A}\,\mathrm{e}^{\mathrm{i}S}\,, \quad A=\left[\frac{2\pi\mathrm{i}\,\delta t}{m}\right]^{\frac{1}{2}}$$

gives

$$\begin{aligned}C &= \frac{1}{A^3}\int\left[\prod_{j=1}^{N_1}\frac{\mathrm{d}q_j}{A}\right]\mathrm{d}q_1\left[\prod_{k=1}^{N_2}\frac{\mathrm{d}q_k}{A}\right]\mathrm{d}q_2\left[\prod_{l=1}^{N_3}\frac{\mathrm{d}q_l}{A}\right]O(q_1)O(q_2)\mathrm{e}^{\mathrm{i}S_\Gamma} \\ &= \frac{1}{A}\int\prod_{j=1}^{N_1}\frac{\mathrm{d}q_j}{A}\,\frac{\mathrm{d}q_1}{A}\prod_{k=1}^{N_2}\frac{\mathrm{d}q_k}{A}\,\frac{\mathrm{d}q_2}{A}\prod_{l=1}^{N_3}\frac{\mathrm{d}q_l}{A}O(q_1)O(q_2)\mathrm{e}^{\mathrm{i}S_\Gamma}\end{aligned}$$

where $\Gamma=\gamma_1\cup\gamma_2\cup\gamma_3$. Keep in mind that the product (capital pi) refers only to the fraction immediately after.

The first integral is for the first path $\gamma_1 : q_1 < q_j < q_f$, the second is for the path $\gamma_2 : q_2 < q_k < q_1$ and the third is for $\gamma_3 : q_i < q_l < q_2$. In the limit $N_i \to \infty$, the measure is

$$[\mathcal{D}q]_\Gamma = \frac{1}{A} \int \prod_{j=1}^{N_1} \frac{\mathrm{d}q_j}{A} \frac{\mathrm{d}q_1}{A} \prod_{k=1}^{N_2} \frac{\mathrm{d}q_k}{A} \frac{\mathrm{d}q_2}{A} \prod_{l=1}^{N_3} \frac{\mathrm{d}q_l}{A}$$

Therefore, the two-point correlation function is

$$C(t_1, t_2) = \langle q_f, t_f | \, \mathcal{T}\{O_\mathrm{H}(t_1) O_\mathrm{H}(t_2)\} \, | q_i, t_i \rangle = \int [\mathcal{D}q] \, O(q(t_1)) O(q(t_2)) \mathrm{e}^{\mathrm{i}S}$$

The correlation function is an intermediate step. The actual interesting quantity is the Green's function.

2.4 Green's function

See Cheng–Li. The Green's function is a particular correlation function where the initial and final states are the ground state $|0\rangle$

$$G(t_1, \ldots, t_n) = \langle 0 | \, \mathcal{T}\{O_\mathrm{H}(t_1) \cdots O_\mathrm{H}(t_n)\} \, | 0 \rangle$$

The definition can be generalized to different operators within the same time-ordered product, but, for simplicity, the following uses the same operator.

Two-point Green's function. One would like to compute the Green's function with the path integral. The two-point Green's function is given by

$$G(t_1, t_2) = \langle 0 | \, \mathcal{T}\{O_\mathrm{H}(t_1) O_\mathrm{H}(t_2)\} \, | 0 \rangle = \int \mathrm{d}q \, \mathrm{d}q' \; \langle 0 | q' t' \rangle \, \langle q' t' | \, \mathcal{T}\{O_\mathrm{H}(t_1) O_\mathrm{H}(t_2)\} \, | q t \rangle \, \langle q t | 0 \rangle$$

The expectation value in the middle can be computed through the correlation function above. The bra-kets are the wave function

$$\langle q t | 0 \rangle = \langle q | \, \mathrm{e}^{-\mathrm{i}Ht} \, | 0 \rangle = \langle q | \, \mathrm{e}^{-\mathrm{i}E_0 t} \, | 0 \rangle = \mathrm{e}^{-\mathrm{i}E_0 t} \varphi_0(q)$$

In quantum field theory, the energy levels can be arbitrarily shifted so one sets $E_0 = 0$ (as opposed to general relativity). Therefore

$$\langle q t | 0 \rangle = \varphi_0(q) \,, \quad \langle 0 | q' t' \rangle = \varphi_0^*(q')$$

The Green's function is then

$$\begin{aligned} G(t_1, t_2) &= \int \mathrm{d}q \, \mathrm{d}q' \, \varphi_0^*(q') \varphi'(q) \int [\mathcal{D}q] \, O(q(t_1)) O(q(t_2)) \mathrm{e}^{\mathrm{i}S} \\ &= \int [\mathcal{D}q] \, \varphi_0^*(q') \varphi_0(q) O(q(t_1)) O(q(t_2)) \mathrm{e}^{\mathrm{i}S} \end{aligned}$$

at the second line one incorporates the measures $\mathrm{d}q \, \mathrm{d}q'$ into the path integral's.

One would like to elaborate the above prescription in order to eliminate the wave functions $\varphi_0^*(q') \varphi_0(q)$ because they make calculations cumbersome. [r] The correlation function is

$$\begin{aligned} C(t_1, t_2) &= \langle q' t' | \, \mathcal{T}\{O_\mathrm{H}(t_1) O_\mathrm{H}(t_2)\} \, | q t \rangle \\ &= \int \mathrm{d}Q \, \mathrm{d}Q' \; \langle q' t' | Q' T' \rangle \, \langle Q' T' | \, \mathcal{T}\{O_\mathrm{H}(t_1) O_\mathrm{H}(t_2)\} \, | Q T \rangle \, \langle Q T | q t \rangle \end{aligned}$$

where one has inserted two completeness relations. [r] Letting E_n be the energy eigenvalue with corresponding eigenstate $|n\rangle$, the wave function of the eigenvector is

$$\langle q | n \rangle = \varphi_n(q)$$

[r] Therefore

$$\begin{aligned}\langle q't'|Q'T'\rangle &= \langle q'|\,\mathrm{e}^{-\mathrm{i}H(t'-T')}\,|Q'\rangle = \sum_n \langle q'|\,\mathrm{e}^{-\mathrm{i}H(t'-T')}\,|n\rangle\,\langle n|Q'\rangle = \sum_n \mathrm{e}^{-\mathrm{i}E_n(t'-T')}\varphi_n(q')\varphi_n^*(Q') \\ &= \varphi_0(q')\varphi_0^*(Q') + \sum_{n>0}\mathrm{e}^{-\mathrm{i}E_n(t'-T')}\varphi_n(q')\varphi_n^*(Q')\end{aligned}$$

where one has separated the contribution of the ground state from the rest. The times t and t' can (almost) be freely moved around the number line since they are not integrated. One observes that in the limit $t' \to -\mathrm{i}\infty$ (similar to a Wick's rotation), the exponential tends to zero (remember that the energy is positive since $E_0 = 0$). Therefore

$$\lim_{t'\to-\mathrm{i}\infty}\langle q't'|Q'T'\rangle = \varphi_0(q')\varphi_0^*(Q')$$

while for the non-primed bra-ket

$$\lim_{t\to+\mathrm{i}\infty}\langle QT|qt\rangle = \varphi_0(Q)\varphi_0^*(q)$$

Performing a similar calculation one has

$$\lim_{\substack{t'\to-\mathrm{i}\infty\\ t\to+\mathrm{i}\infty}}\langle q't'|qt\rangle = \varphi_0(q')\varphi_0^*(q)$$

Taking the above limit for the correlator, one has

$$\begin{aligned}\lim_{\substack{t'\to-\mathrm{i}\infty\\ t\to+\mathrm{i}\infty}} C(t_1,t_2) &= \lim_{\substack{t'\to-\mathrm{i}\infty\\ t\to+\mathrm{i}\infty}}\langle q't'|\,\mathcal{T}\{O_{\mathrm{H}}(t_1)O_{\mathrm{H}}(t_2)\}\,|qt\rangle \\ &= \int \mathrm{d}Q\,\mathrm{d}Q'\,\varphi_0(q')\varphi_0^*(Q')\,\langle Q'T'|\,\mathcal{T}\{O_{\mathrm{H}}(t_1)O_{\mathrm{H}}(t_2)\}\,|QT\rangle\,\varphi_0(Q)\varphi_0^*(q) \\ &= \varphi_0(q')\varphi_0^*(q)\int \mathrm{d}Q\,\mathrm{d}Q'\,\varphi_0^*(Q')\varphi_0(Q)\,\langle Q'T'|\,\mathcal{T}\{O_{\mathrm{H}}(t_1)O_{\mathrm{H}}(t_2)\}\,|QT\rangle \\ &= \lim_{\substack{t'\to-\mathrm{i}\infty\\ t\to+\mathrm{i}\infty}}\langle q't'|qt\rangle\,G(t_1,t_2)\end{aligned}$$

in the last line, the integral is the definition of Green's function but with Q, Q'. [r] Therefore, reorganizing the above equation, the Green's function is

$$G(t_1,t_2) = \lim_{\substack{t'\to-\mathrm{i}\infty\\ t\to+\mathrm{i}\infty}}\frac{\langle q't'|\,\mathcal{T}\{O_{\mathrm{H}}(t_1)O_{\mathrm{H}}(t_2)\}\,|qt\rangle}{\langle q't'|qt\rangle}$$

This operation is similar to a normalization: the Green's function is a correlation function normalized by the product of the two states.

This definition of two-point Green's function can be generalized to n points

$$G(t_1,\dots,t_n) = \lim_{\substack{t'\to-\mathrm{i}\infty\\ t\to+\mathrm{i}\infty}}\frac{\langle q't'|\,\mathcal{T}\{O_{\mathrm{H}}(t_1)\cdots O_{\mathrm{H}}(t_n)\}\,|qt\rangle}{\langle q't'|qt\rangle}$$

In this way, the Green's function does not explicitly depend on the ground state.

Generating functional. The generating functional for the Green's function is

$$W[J] = \lim_{\substack{t'\to-\mathrm{i}\infty\\ t\to+\mathrm{i}\infty}}\frac{1}{\langle q't'|qt\rangle}\int[\mathcal{D}q]\,\exp\left[\mathrm{i}\int_t^{t'}\mathrm{d}\tau\,[L(\tau)+J(\tau)O_{\mathrm{H}}(q(\tau))]\right]$$

Applying an arbitrary number of derivatives with respect to J brings the operator O_{H} in front of the exponential. In this way one obtains the desired Green's function

$$G(t_1,\dots,t_n) = \left.\frac{(-\mathrm{i})^n\delta^n W[J]}{\delta J(t_1)\cdots\delta J(t_n)}\right|_{J=0}$$

The function J is called the source of the operator O_{H}.

The generating functional is a modification of a path integral that one would get without any insertion of operators and with a source different from zero. The functional is the transition amplitude from the ground state at time t to the ground state at time t' in the presence of an external source $J(\tau)$

$$W[J] = \langle 0|0\rangle_J$$

The normalization is given by the absence of the source

$$W[0] = \langle 0|0\rangle = 1$$

This is in accordance with the prescription of the Green's function above from the state $|qt\rangle$ to the state $|q't'\rangle$ but without any insertion of operators: the ratio is unity.

Euclidean Green's function. Working in Euclidean space is useful to make sense of the limits of the times t, t'. By performing a Wick's rotation, the time coordinate becomes $\tau = -\mathrm{i}\tau_{\mathrm{E}}$ while the limits are

$$\begin{aligned} t' \to -\mathrm{i}\infty &\rightsquigarrow t'_{\mathrm{E}} \to +\infty \\ t \to +\mathrm{i}\infty &\rightsquigarrow t_{\mathrm{E}} \to -\infty \end{aligned}$$

The exponential of the Lagrangian becomes

$$\exp\left[\mathrm{i}\int \mathrm{d}\tau\, L(\tau)\right] \rightsquigarrow \exp\left[\int \mathrm{d}\tau_{\mathrm{E}}\, L(-\mathrm{i}\tau_{\mathrm{E}})\right]$$

Considering the Lagrangian of a one-dimensional particle in a potential

$$L = \frac{1}{2}m\dot{q}^2 - V(q) = \frac{1}{2}m(\mathrm{d}_\tau q)^2 - V(q) \rightsquigarrow L(-\mathrm{i}\tau_{\mathrm{E}}) = -\frac{1}{2}m\dot{q}_{\mathrm{E}}^2 - V(q) \equiv -L_{\mathrm{E}}$$

the exponential is then

$$\exp\left[\mathrm{i}\int \mathrm{d}\tau\, L(\tau)\right] \rightsquigarrow \exp\left[-\int \mathrm{d}\tau_{\mathrm{E}}\, L_{\mathrm{E}}\right]$$

The ground energy can always be set to zero, so the energies are positive and the exponential is no longer a phase but tends to zero. The integral becomes well-defined.

Euclidean generating functional. The generating functional becomes

$$W_{\mathrm{E}}[J] = \lim_{\substack{t'\to\infty \\ t\to-\infty}} \frac{1}{\langle q't'|qt\rangle} \int [\mathcal{D}q]\, \exp\left[-\int \mathrm{d}\tau_{\mathrm{E}}[L_{\mathrm{E}} + JO_{\mathrm{H}}(\tau_{\mathrm{E}})]\right]$$

2.5 Arbitrary dimensions

The formalism developed can be generalized to D dimensions. If a particle propagates in D dimensions, the configuration point is

$$\mathbf{q} = (q^1, \ldots, q^D)$$

The kernel is

$$\begin{aligned} K(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i) &= \int [\mathcal{D}\mathbf{q}]\, \mathrm{e}^{\mathrm{i}S[\mathbf{q}]} = \lim_{N\to\infty} \frac{1}{A} \int \prod_{j=1}^{N} \frac{\mathrm{d}\mathbf{q}_j}{A^D}\, \mathrm{e}^{\mathrm{i}S[\mathbf{q}_j]} \\ &= \lim_{N\to\infty} \left[\frac{m}{2\pi\mathrm{i}\,\delta t}\right]^{\frac{ND+1}{2}} \int \prod_{j=1}^{N} \mathrm{d}\mathbf{q}_j\, \mathrm{e}^{\mathrm{i}S[\mathbf{q}_j]} \end{aligned}$$

Conclusion. Some phenomena are easier described through the path integral formulation of quantum mechanics, like tunnelling and scattering.

Part I
Scalar boson fields

In quantum field theory, canonical quantization is limited and cumbersome. One uses the functional approach which generalizes the quantum mechanical path integral.

Spin zero particles are bosons and are described by scalar fields $\varphi(x)$. Neutral particles are real scalar fields.

3 Canonical quantization

One may start from neutral particles. For free massive scalar fields, the Lagrangian density and action are given by

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\,\partial_\mu\varphi\,\partial_\nu\varphi - \frac{1}{2}m^2\varphi^2\,, \quad S = \int \mathrm{d}^4x\,\mathcal{L} = \int \mathrm{d}^4x\,\left[\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\nabla\varphi)^2 - \frac{1}{2}m^2\varphi^2\right]$$

where the Minkowski metric η convention is timelike, mostly minus. Applying the action principle $\delta S = 0$ one obtains the classical equations of motion with some boundary conditions: when integrating over all space-time, one imposes that the fields go to zero sufficiently fast. For a scalar field, the equations of motion are Klein–Gordon's

$$(\Box + m^2)\varphi(x) = 0\,, \quad \Box \equiv \partial_\mu\partial^\mu = \partial_0^2 - \nabla^2$$

The canonical momentum of the field is

$$\pi(x) = \frac{\delta\mathcal{L}}{\delta\dot{\varphi}(x)} = \dot{\varphi}(x)$$

The most general solution for the Klein–Gordon equation is

$$\varphi(x) = \varphi_+(x) + \varphi_-(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2k^0}}[a_+(\mathbf{k})\mathrm{e}^{\mathrm{i}k^\mu x_\mu} + a_-(\mathbf{k})\mathrm{e}^{-\mathrm{i}k^\mu x_\mu}]\,, \quad k^0 = \sqrt{|\mathbf{k}|^2 + m^2}$$

where $\pm$ is not related to positive and negative frequency components, it just labels the sign of the exponent. The reality condition given by the neutral nature of the boson field implies

$$[a_-(\mathbf{k})]^\dagger = a_+(\mathbf{k})$$

The Fourier coefficients can be expressed in terms of the field as

$$a_\pm(\mathbf{k}) = \mp\frac{\mathrm{i}}{(2\pi)^{\frac{3}{2}}\sqrt{2k^0}}\int \mathrm{d}^3x\,\mathrm{e}^{\mp\mathrm{i}kx}\overset{\leftrightarrow}{\partial_0}\varphi(x)\,, \quad f\overset{\leftrightarrow}{\partial_0}g \equiv f\,\partial_0 g - (\partial_0 f)g$$

One can prove that, since the field $\varphi(x)$ is a solution of the free Klein–Gordon equation, the annihilation and creation operators are time-independent. For the interacting theory, the Klein–Gordon equation of motion is not zero. In fact, the Lagrangian has a potential

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - V(\varphi)$$

from which the equations of motion are

$$(\Box + m^2)\varphi(x) = -\partial_\varphi V \equiv J(x)$$

and in general the annihilation and destruction operators are functions of time.

Exercise. Check that the previous expressions give back the field φ. Check that the operators of the free theory are time-independent

$$\partial^0 a(\mathbf{k}) = \frac{\mathrm{i}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}}\int \mathrm{d}^3x\,\partial^0[(\partial_0\mathrm{e}^{\mathrm{i}kx})\varphi(x) - \mathrm{e}^{\mathrm{i}kx}\,\partial_0\varphi] = \cdots = 0$$

To get zero one has to apply the Klein–Gordon equation.

Canonical quantization. The canonical quantization of the fields (called second quantization) is implemented by promoting the Fourier coefficients $a_\pm(\mathbf{k})$ to be operators obeying the canonical commutation relations (CCR)

$$[a_-(\mathbf{k}), a_+(\mathbf{k}')] = \delta^{(3)}(\mathbf{k}-\mathbf{k}') \implies [a(\mathbf{k}), a^\dagger(\mathbf{k})] = \delta^{(3)}(\mathbf{k}-\mathbf{k}')$$

and all others zero. The number operator and the total number are

$$N(\mathbf{k}) \equiv a^\dagger(\mathbf{k})a(\mathbf{k})\,, \quad N = \int \mathrm{d}^3k\, N(\mathbf{k})$$

Lecture 5

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One defines a vacuum state as the state annihilated by all destruction operators

$$a(\mathbf{k})\,|0\rangle = 0\,, \quad \forall \mathbf{k}$$

The excited states are constructed by acting on the vacuum with a creation operator

$$a^\dagger(\mathbf{k})\,|0\rangle = |\psi(\mathbf{k})\rangle$$

The excited states are interpreted as particle with momentum $\mathbf{k}$ and on-shell dispersion relation $E = \sqrt{|\mathbf{k}|^2 + m^2}$. Though it has no physical sense that a particle has a definite momentum $\mathbf{k}$, but it is more physical to consider a wave packet with a distribution of momenta centered around a momentum $\mathbf{k}$

$$|\psi(\mathbf{k})\rangle = \int \mathrm{d}^3k'\, f(\mathbf{k},\mathbf{k}')a^\dagger(\mathbf{k})\,|0\rangle$$

where f is a function that gives the distribution of the plane waves inside the wave packet, typically a Gaussian (see Srednicki, §5)

$$f(\mathbf{k},\mathbf{k}') \propto \exp\left[-\frac{1}{4\sigma^2}(\mathbf{k}'-\mathbf{k})^2\right], \quad f(\mathbf{k},\mathbf{k}') \to \delta^{(3)}(\mathbf{k}'-\mathbf{k})\,, \quad \sigma \to 0$$

Commutators. The commutators between the fields can be obtained by applying the canonical commutation relations of the annihilation and creation operators. The relations for the separate parts of the field are

$$[\varphi_\mp(x), \varphi_\pm(y)] = \pm \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0} \mathrm{e}^{\mp \mathrm{i}k(x-y)}$$

From these, the relations for the field are

$$[\varphi(x), \varphi(y)] = [\varphi_+(x), \varphi_-(y)] + [\varphi_-(x), \varphi_+(y)] = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0} [\mathrm{e}^{\mathrm{i}k(x-y)} - \mathrm{e}^{-\mathrm{i}k(x-y)}]$$

The canonical commutations relations are then

$$[\varphi(x), \dot\varphi(y)]_{x^0=y^0} = [\varphi(x^0,\mathbf{x}), \pi(x^0,\mathbf{y})] = \mathrm{i}\delta^{(3)}(x-y)$$

because $\pi = \dot\varphi$ is the canonical momentum of the field.

The commutators for space-like separations are null

$$[\varphi(x), \varphi(y)] = 0\,, \quad |x-y|^2 < 0$$

The above commutators are the Pauli–Jordan functions, which are related to the Klein–Gordon propagator. The meaning of the above commutator is that two particles separated by arbitrary space-like distances cannot influence each other: it is the condition of micro-causality.

The (Feynman) propagator for a free scalar field is the following two-point function

$$\begin{aligned}
\mathrm{i}\Delta_0(x-y) &\equiv \langle 0|\,\mathcal{T}\{\varphi(x)\varphi(y)\}\,|0\rangle = \theta(x^0-y^0)\,\langle 0|\,\varphi(x)\varphi(y)\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,\varphi(y)\varphi(x)\,|0\rangle \\
&= \theta(x^0-y^0)\,\langle 0|\,\varphi_-(x)\varphi_+(y)\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,\varphi_-(y)\varphi_+(x)\,|0\rangle \\
&= \theta(x^0-y^0)\,\langle 0|\,[\varphi_-(x),\varphi_+(y)]\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,[\varphi_-(y),\varphi_+(x)]\,|0\rangle \\
&= \theta(x^0-y^0)\,\langle 0|\,[\varphi_-(x),\varphi_+(y)]\,|0\rangle - \theta(y^0-x^0)\,\langle 0|\,[\varphi_+(x),\varphi_-(y)]\,|0\rangle \\
&= \theta(x^0-y^0)\int\frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}\mathrm{e}^{-\mathrm{i}k(x-y)} + \theta(y^0-x^0)\int\frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}\mathrm{e}^{\mathrm{i}k(x-y)} \\
&= \mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}k(x-y)}}{k^2-m^2+\mathrm{i}\varepsilon}
\end{aligned}$$

At the second line only the non-zero contributions have been explicitly written; at the last line one has made the analytic continuation of k^0 in the complex plane and has integrated using Feynman's prescription and Jordan's lemmas.

Once quantum field theory is reformulated through functional methods, one finds the same expression of the propagator above using functional integrals.

The propagator is the probability amplitude of creating a particle at y, letting it propagate to x and destroying it there. The propagator above is a two-point function and it is the Green's function of the Klein–Gordon operator

$$(\Box + m^2)\Delta_0(x) = -\delta^{(4)}(x-y)$$

[r]

The propagator (which is related to the Green's function) is needed to justify the fact that the evaluation of a scattering amplitude is related to the computation of the Green's function. The scattering amplitude can be expressed in terms of the Green's function thanks to the Lehmann–Symanzik–Zimmermann (LSZ) reduction formula.

4 Functional quantization

4.1 Lehmann–Symanzik–Zimmermann reduction formula

See Srednicki §5, Weinberg, Itzykson–Zuber. One would like to compute the scattering amplitude between an initial and a final state $\langle f|i\rangle$ which is related to the probability that a system prepared in the state $|i\rangle$ at $t\to-\infty$ transitions to the state $|f\rangle$ at $t\to+\infty$.

Adiabatic hypothesis. The interaction occurs only in a finite volume of space. At infinity the interaction is negligible and there are only the free asymptotic states $|i\rangle$ and $|f\rangle$ given by

$$|i\rangle = \lim_{t\to-\infty} a_1^\dagger(t)a_2^\dagger(t)\cdots|0\rangle\ , \quad |f\rangle = \lim_{t\to\infty} a_{1'}^\dagger(t)a_{2'}^\dagger(t)\cdots|0\rangle$$

where

$$a_1^\dagger(t)\,|0\rangle = \int \mathrm{d}^3k\, f_1(\mathbf{k})a^\dagger(\mathbf{k},t)\,|0\rangle$$

remembering that the operators depend on time because the theory has interaction and is not free. Therefore it is useful to compute $a_j^\dagger(+\infty)-a_j^\dagger(-\infty)$. Starting from the first one, it follows

$$\begin{aligned}
a_1^\dagger(+\infty)-a_1^\dagger(-\infty) &= \int_{\mathbb{R}}\mathrm{d}t\,\partial_0 a_1(t) = \int_{\mathbb{R}}\mathrm{d}t\,\partial_0\int\mathrm{d}^3k\,f_1(\mathbf{k})a^\dagger(\mathbf{k},t) \\
&= -\mathrm{i}\int\mathrm{d}^3k\,f_1(\mathbf{k})\int_{\mathbb{R}}\mathrm{d}t\,\partial_0\int\frac{\mathrm{d}^3x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}}\mathrm{e}^{-\mathrm{i}kx}\overset{\leftrightarrow}{\partial}_0\varphi(x) \\
&= -\mathrm{i}\int\mathrm{d}^3k\,f_1(\mathbf{k})\int\frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}}\,\partial_0[\mathrm{e}^{-\mathrm{i}kx}\overset{\leftrightarrow}{\partial}_0\varphi(x)] \\
&= -\mathrm{i}\int\mathrm{d}^3k\,f_1(\mathbf{k})\int\frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}}\,[\mathrm{e}^{-\mathrm{i}kx}\,\partial_0^2\varphi + k_0^2\mathrm{e}^{-\mathrm{i}kx}\varphi]
\end{aligned}$$

$$
\begin{aligned}
&= -\mathrm{i}\int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}} \left[\mathrm{e}^{-\mathrm{i}kx}\,\partial_0^2\varphi + (|\mathbf{k}|^2 + m^2)\mathrm{e}^{-\mathrm{i}kx}\varphi\right] \\
&= -\mathrm{i}\int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}} \mathrm{e}^{-\mathrm{i}kx}(\partial_0^2 - \nabla^2 + m^2)\varphi(x) \\
&= -\mathrm{i}\int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}} \mathrm{e}^{-\mathrm{i}kx}(\Box + m^2)\varphi(x)
\end{aligned}
$$

at the second line one expresses the creation operator with its expansions in terms of the fields; at the second to last line, one notices

$$
|\mathbf{k}|^2 \mathrm{e}^{-\mathrm{i}kx} = -\nabla^2 \mathrm{e}^{-\mathrm{i}kx}
$$

and integrates by parts supposing that at infinity the fields to go zero fast enough to produce no boundary terms. At the last line, in the free theory, the field is a solution of the Klein–Gordon equation and the desired difference is zero which is consistent with the fact that the two operators at infinity are the same; however, in the interacting theory this is not true because the Klein–Gordon operator does not necessarily give zero.

Similarly, for the destruction operators

$$
a_1(+\infty) - a_1(-\infty) = \mathrm{i}\int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}} \mathrm{e}^{\mathrm{i}kx}(\Box + m^2)\varphi(x)
$$

Scattering amplitude. The following considers a probability amplitude for a $2 \to 2$ scattering process. The initial and final states are

$$
|i\rangle = a_1^\dagger(-\infty)a_2^\dagger(-\infty)\,|0\rangle \ , \quad |f\rangle = a_{1'}^\dagger(+\infty)a_{2'}^\dagger(+\infty)\,|0\rangle
$$

The scattering amplitude is then

$$
\begin{aligned}
\langle f|i\rangle &= \langle 0|\, a_{1'}(+\infty)a_{2'}(+\infty)a_1^\dagger(-\infty)a_2^\dagger(-\infty)\,|0\rangle \\
&= \langle 0|\, \mathcal{T}\{a_{1'}(+\infty)a_{2'}(+\infty)a_1^\dagger(-\infty)a_2^\dagger(-\infty)\}\,|0\rangle
\end{aligned}
$$

At the second line, one notices that the product is already time-ordered. One may insert the expression of the operators in terms of the fields

$$
\begin{aligned}
a_j^\dagger(-\infty) &= a_j^\dagger(+\infty) + \mathrm{i}\int \mathrm{d}^3k\, f_j(\mathbf{k}) \int \frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}} \mathrm{e}^{-\mathrm{i}kx}(\Box + m^2)\varphi(x) \\
a_{j'}(+\infty) &= a_{j'}(-\infty) + \mathrm{i}\int \mathrm{d}^3k\, f_{j'}(\mathbf{k}) \int \frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}} \mathrm{e}^{\mathrm{i}kx}(\Box + m^2)\varphi(x)
\end{aligned}
$$

Looking at the second expression, the time-ordered product above pushes the primed destruction operators to the right and they annihilate the vacuum, so only the integral remains. Therefore

$$
\begin{aligned}
\langle f|i\rangle = &\int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \mathrm{d}^3k'\, f_2(\mathbf{k}') \int \mathrm{d}^3k''\, f_{1'}(\mathbf{k}'') \int \mathrm{d}^3k'''\, f_{2'}(\mathbf{k}''') \\
&\times \int \frac{\mathrm{d}^4x_1\, \mathrm{e}^{-\mathrm{i}kx_1}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_1^0}} \int \frac{\mathrm{d}^4x_2\, \mathrm{e}^{-\mathrm{i}k'x_2}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_2^0}} \int \frac{\mathrm{d}^4x_3\, \mathrm{e}^{\mathrm{i}k''x_3}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_{1'}^0}} \int \frac{\mathrm{d}^4x_4\, \mathrm{e}^{\mathrm{i}k'''x_4}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_{2'}^0}} \\
&\times \langle 0|\, \mathcal{T}\{J(x_1)J(x_2)J(x_3)J(x_4)\}\,|0\rangle \\
= &\int \frac{\mathrm{d}^4x_1\, \mathrm{e}^{-\mathrm{i}k_1x_1}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_1^0}} \int \frac{\mathrm{d}^4x_2\, \mathrm{e}^{-\mathrm{i}k_2x_2}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_2^0}} \int \frac{\mathrm{d}^4x_{1'}\, \mathrm{e}^{\mathrm{i}k_{1'}x_{1'}}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_{1'}^0}} \int \frac{\mathrm{d}^4x_{2'}\, \mathrm{e}^{\mathrm{i}k_{2'}x_{2'}}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_{2'}^0}} \\
&\times \langle 0|\, \mathcal{T}\{J(x_1)J(x_2)J(x_{1'})J(x_{2'})\}\,|0\rangle \\
= &\int \frac{\mathrm{d}^4x_1\, \mathrm{e}^{-\mathrm{i}k_1x_1}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_1^0}} \int \frac{\mathrm{d}^4x_2\, \mathrm{e}^{-\mathrm{i}k_2x_2}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_2^0}} \int \frac{\mathrm{d}^4x_{1'}\, \mathrm{e}^{\mathrm{i}k_{1'}x_{1'}}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_{1'}^0}} \int \frac{\mathrm{d}^4x_{2'}\, \mathrm{e}^{\mathrm{i}k_{2'}x_{2'}}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_{2'}^0}} \\
&\times (\Box_1 + m_1^2)(\Box_2 + m_2^2)(\Box_{1'} + m_{1'}^2)(\Box_{2'} + m_{2'}^2)\,\langle 0|\, \mathcal{T}\{\varphi(x_1)\varphi(x_2)\varphi(x_{1'})\varphi(x_{2'})\}\,|0\rangle
\end{aligned}
$$

At the second equality, one takes the limit where the distributions tend to the Dirac delta function

$$f_j(\mathbf{k}^{(l)}) \to \delta^{(3)}(\mathbf{k}^{(l)} - \mathbf{k}_j)$$

At the third equality, source is replaced by using the equations of motion and the d'Alembertian is brought outside the expectation value.

This last expectation value is the four-point Green's function. The generalization of the scattering amplitude to N particles is straightforward

$$\langle f_{n'}|i_n\rangle = \mathrm{i}^{n+n'} \prod_{j=1}^{n} \int \frac{\mathrm{d}^4 x_j \, \mathrm{e}^{-\mathrm{i}k_j x_j}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_j^0}} \prod_{l=1}^{n'} \int \frac{\mathrm{d}^4 x'_l \, \mathrm{e}^{-\mathrm{i}k'_l x'_l}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_{l'}^0}}$$
$$\times \prod_j (\Box_j + m_j^2) \prod_l (\Box_l + m_l'^2) \langle 0| \mathcal{T}\{\varphi(x_j)\varphi(x'_l)\} |0\rangle$$

The last expectation value is the $(n + n')$-point Green's function. This is the LSZ reduction formula: the scattering amplitudes are related to the computation of Green's functions.

Caveat. The calculation was based on the assumption that the particles are always created from time-dependent operators $a^\dagger(t)$ acting on the unique vacuum $|0\rangle$. This is true for the free theory, but in the interacting theory it may not be. The following discussion is not a proof, but a motivation of the validity of the assumption.

If the assumption is true for the interacting theory, then one should have

$$\langle 0| \varphi(x) |0\rangle = 0$$

because one creates a particle and projects it onto the vacuum. One may obtain the above by writing the field in terms of the field in the origin

$$\langle 0| \varphi(x) |0\rangle = \langle 0| \mathrm{e}^{\mathrm{i}Px} \varphi(x) \mathrm{e}^{-\mathrm{i}Px} |0\rangle = \langle 0| \mathrm{e}^{0} \varphi(0) \mathrm{e}^{-0} |0\rangle = \langle 0| \varphi(0) |0\rangle = v \,, \quad P^\mu |0\rangle = 0$$

where P is the translation generator (the momentum) and the vacuum is assumed to be unique; at the second equality, the momentum of the vacuum is $p = 0$ which then implies its translational invariance. It may be that $v \neq 0$, but one can shift the field φ to obtain a zero expectation value

$$\varphi'(x) = \varphi(x) - v \implies \langle 0| \varphi'(x) |0\rangle = \langle 0| \varphi'(0) |0\rangle = 0$$

This is justified because the physics of the Lagrangian does not change: it is just a renaming of the operator of interest.

Consider now a one particle state. One expects to find

$$\langle p| \varphi(x) |0\rangle = \mathrm{e}^{\mathrm{i}px}$$

like in the free case. In fact

$$\langle p| \varphi(x) |0\rangle = \langle p| \mathrm{e}^{\mathrm{i}Px} \varphi(0) \mathrm{e}^{-\mathrm{i}Px} |0\rangle = \mathrm{e}^{\mathrm{i}px} \langle p| \varphi(x) |0\rangle = A \mathrm{e}^{\mathrm{i}px}$$

By rescaling the field, one has

$$\langle p| \frac{1}{A} \varphi(x) |0\rangle = \langle p| \varphi'(x) |0\rangle = \mathrm{e}^{\mathrm{i}px}$$

Consider now a multi-particle state $|p, n\rangle$ where p is the total momentum and n are the other relevant quantum numbers. One expects to find

$$\langle p, n| \varphi(x) |0\rangle = 0$$

To this end, one may write

$$\langle p, n| \varphi(x) |0\rangle = \mathrm{e}^{\mathrm{i}px} \langle p, n| \varphi(0) |0\rangle = \mathrm{e}^{\mathrm{i}px} A_n(p)$$

where $A_n(p)$ is a function of products of Lorentz-invariants. For more than one particle, the total energy is

$$E_{\text{tot}} = p_0 = \sqrt{p^2 + M^2} = \sum_j \sqrt{p_j^2 + m_j^2}$$

with momenta p_j and masses m_j. Considering two particles with equal mass, one has

$$p = p_1 + p_2\,, \quad E_{\text{tot}} = \sqrt{p_1^2 + m^2} + \sqrt{p_2^2 + m^2}$$

Therefore, the minimum energy is $E_{\text{tot}} = 2m$ and above there is a continuum of hyperbolae of energies (given by $\sqrt{E^2 - p^2} = m = \text{const.}$, see Srednicki Fig. 5.1). On the other hand, the energy of a one-particle state is completely determined by its momentum and it lies on an isolated hyperbola. Instead of

$$\langle p, n|\, \varphi(0)\, |0\rangle$$

being zero, one should strictly consider

$$\langle p, n|\, a^\dagger(\pm\infty)\, |0\rangle$$

Considering only renormalizable states $|\psi\rangle$

$$|\psi\rangle = \sum_n \int \mathrm{d}^3p\, \psi_n(p)\, |p, n\rangle$$

[r] it follows

$$\langle\psi|\, a^\dagger(t)\, |0\rangle = \sum_n \int \mathrm{d}^3p\, \psi_n^*(f)\, \langle p, n|\, a^\dagger(t)\, |0\rangle$$

Inserting the expression of the operator in terms of the fields, one may finally get

$$\langle\psi|\, a^\dagger(t)\, |0\rangle = \sum_n \int \mathrm{d}^3p\, \psi_n^*(p) \frac{(2\pi)^3}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}} f_1(\mathbf{p})(p_0 + k_0) A_n(\mathbf{p}) \mathrm{e}^{-\mathrm{i}(k_0 - p_0)t}$$

where

$$p_0 = \sqrt{p^2 + M^2}\,, \quad k_0 = \sqrt{p^2 + m^2}\,, \quad M \geq 2m\,, \quad p_0 > k_0$$

When taking the limit $t \to \pm\infty$, the exponential becomes a very fast oscillating function: integrating over momenta gives zero (due to the Riemann–Lebesgue lemma).

Conclusion. The LSZ reduction formula works under the assumptions that

$$\langle 0|\, \varphi(0)\, |0\rangle = 0\,, \quad \langle p|\, \varphi(x)\, |0\rangle = \mathrm{e}^{\mathrm{i}px}$$

which are related to translating and scaling the field.

Lecture 6

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4.2 Free field theory

Comparison with quantum mechanics. In quantum mechanics one finds similar observables to the ones present in the LSZ reduction formula? [r]

$$G(t_1, \ldots, t_n) = \langle 0|\, \mathcal{T}\{O(t_1) \cdots O(t_n)\}\, |0\rangle = (-\mathrm{i})^n \frac{\delta^n W[J]}{\delta J(t_1) \cdots \delta J(t_n)}\bigg|_{J=0}$$

where $|0\rangle$ is the fundamental state (there is no notion of Fock vacuum in ordinary quantum mechanics) and the generating functional is

$$W[J] = \lim_{\substack{t' \to -\mathrm{i}\infty \\ t \to +\mathrm{i}\infty}} \frac{1}{\langle q't'|qt\rangle} \int [\mathcal{D}q]\, \exp\left[\mathrm{i} \int \mathrm{d}\tau\, [L(\tau) + J(\tau)O(\tau)]\right]$$

One would like to generalize the above prescription to quantum field theory: the operators must be functions of position too because time and space must be treated on the same level.

The discrete positions $q_i(t)$ become a field $q_x(t) = \varphi(\mathbf{x}, t)$ with continuous index $i \to x$. The ordinary operators $O(t) = O(q_i(t))$ becomes a field operators $O(\varphi(x))$. The Lagrangian $L(q_i, \dot{q}_i)$ becomes a Lagrangian density $\mathcal{L}(\varphi(x), \partial_\mu \varphi)$. The path integral in quantum mechanics has a well-defined prescription. The functional measure before and after integrating the momenta is

$$\int [\mathrm{d}q\, \mathrm{d}p] \equiv \lim_{N\to\infty} \int \prod_{j=1}^{N} \mathrm{d}q_j \prod_{k=1}^{N+1} \frac{\mathrm{d}p_k}{2\pi} \implies \int [\mathrm{d}q] = \lim_{N\to\infty} \frac{1}{A} \int \prod_{j=1}^{N} \frac{\mathrm{d}q_j}{A}\,, \quad A = \frac{2\pi \mathrm{i}\, \delta t}{m}$$

which becomes

$$[\mathcal{D}\varphi(x)\, \mathcal{D}\pi(x)] \implies [\mathcal{D}\varphi]$$

whose rigorous definition has not been presented. To define the measure in quantum field theory, one can discretize the theory on a lattice to obtain ordinary quantum mechanics with the identity

$$\varphi(x_i) = \varphi_i$$

This way can be efficient for some problems. Though the approach taken next is different: one leaves the above just as a formal definition of the measure since one almost never explicitly computes functional integrals.

Generalization. The definitions of the Green's function and the generating functional can be generalized. [r] The first one is

$$G^{(n)}(x_1, \ldots, x_n) = \langle 0 | \mathcal{T}\{\varphi(x_1) \cdots \varphi(x_n)\} | 0 \rangle = (-\mathrm{i})^n \frac{\delta^n W[J]}{\delta J(t_1) \cdots \delta J(t_n)}\bigg|_{J=0}$$

where the generating functional for Green's functions is

$$W[J] = N \int [\mathcal{D}\varphi] \exp\left[\mathrm{i} \int \mathrm{d}^4x \, [\mathcal{L}(\varphi(x), \partial_\mu \varphi(x)) + J(x)\varphi(x)]\right]$$

where N is a suitable normalization. If one is able to evaluate the generating functional W, then one may compute the Green's function of an arbitrary number of observables. The normalization N is typically chosen such that

$$W[J = 0] \equiv \langle 0 | 0 \rangle = 1$$

One would like to study how to compute the generating functional $W[J]$ for a real scalar field.

Generating functional. The simplest case is the free field theory. The Lagrangian density is

$$\mathcal{L}(\varphi) = \frac{1}{2}\, \partial_\mu \varphi\, \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2$$

The generating functional is

$$W_0[J] = N \int [\mathcal{D}\varphi] \, \exp\left[\mathrm{i} \int \mathrm{d}^4x \, (\mathcal{L} + J\varphi)\right]$$

To study the above one may use the momentum space. The Fourier conventions are

$$\widetilde{\varphi}(k) = \int \mathrm{d}^4x \, \mathrm{e}^{-\mathrm{i}kx} \varphi(x)\,, \quad \varphi(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \mathrm{e}^{\mathrm{i}kx} \widetilde{\varphi}(k)$$

For the source, one has

$$\begin{aligned}
\int \mathrm{d}^4x \, J(x)\varphi(x) &= \int \frac{\mathrm{d}^4k_1}{(2\pi)^4} \frac{\mathrm{d}^4k_2}{(2\pi)^4} \, \mathrm{d}^4x \, \mathrm{e}^{\mathrm{i}(k_1+k_2)x} \widetilde{J}(k_1)\widetilde{\varphi}(k_2) \\
&= \int \frac{\mathrm{d}^4k_1}{(2\pi)^4} \frac{\mathrm{d}^4k_2}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(k_1 + k_2) \widetilde{J}(k_1)\widetilde{\varphi}(k_2) \\
&= \int \frac{\mathrm{d}^4k}{(2\pi)^4} \widetilde{J}(k)\widetilde{\varphi}(-k) = \frac{1}{2} \int \frac{\mathrm{d}^4k}{(2\pi)^4} [\widetilde{J}(k)\widetilde{\varphi}(-k) + \widetilde{J}(-k)\widetilde{\varphi}(k)]
\end{aligned}$$

The same can be done for the mass term, while for the kinetic term one should integrate by parts first to get

$$\int \mathrm{d}^4x\,\frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi = -\int \mathrm{d}^4x\,\frac{1}{2}\varphi\,\Box\,\varphi$$

To quickly transform, one substitutes the function with its Fourier transform and absorbs the Jacobian into the normalization constant [r].

The generating functional is

$$W_0[J] = N\int[\mathcal{D}\widetilde{\varphi}]\,\exp\left[\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}[\widetilde{\varphi}(k)(k^2-m^2)\widetilde{\varphi}(-k)+\widetilde{J}(k)\widetilde{\varphi}(-k)+\widetilde{J}(-k)\widetilde{\varphi}(k)]\right]$$

Without formal development, one uses the Gaussian integrals for the functional integral above. One completes the square in the exponent

$$\begin{aligned}E &= \widetilde{\varphi}(k)(k^2-m^2)\widetilde{\varphi}(-k)+\widetilde{J}(k)\widetilde{\varphi}(-k)+\widetilde{J}(-k)\widetilde{\varphi}(k)\\ &= \left[\widetilde{\varphi}(k)+\widetilde{J}(k)\frac{1}{k^2-m^2}\right](k^2-m^2)\left[\widetilde{\varphi}(-k)+\frac{1}{k^2-m^2}\widetilde{J}(-k)\right]-\widetilde{J}(k)\frac{1}{k^2-m^2}\widetilde{J}(-k)\end{aligned}$$

By making a change of variables

$$\widetilde{\varphi}'(-k)\equiv\widetilde{\varphi}(-k)+\frac{1}{k^2-m^2}\widetilde{J}(-k)$$

the generating functional becomes

$$\begin{aligned}W_0[J] &= N\int[\mathcal{D}\widetilde{\varphi}']\,\exp\left[\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\widetilde{\varphi}'(k^2-m^2)\widetilde{\varphi}'\right]\exp\left[-\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\widetilde{J}(k)\frac{1}{k^2-m^2}\widetilde{J}(-k)\right]\\ &= \exp\left[-\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\widetilde{J}(k)\frac{1}{k^2-m^2}\widetilde{J}(-k)\right]N\int[\mathcal{D}\widetilde{\varphi}']\,\exp\left[\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\widetilde{\varphi}'(k^2-m^2)\widetilde{\varphi}'\right]\\ &= \exp\left[-\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\widetilde{J}(k)\frac{1}{k^2-m^2}\widetilde{J}(-k)\right]W_0[J=0]\,,\quad W_0[J=0]=1\\ &= \exp\left[-\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\widetilde{J}(k)\frac{1}{k^2-m^2}\widetilde{J}(-k)\right]\end{aligned}$$

Going back to the configuration space, one notices that $(k^2-m^2)^{-1}$ is the propagator. Recalling that it is

$$\Delta_0(x-x') = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}k(x-x')}}{k^2-m^2+\mathrm{i}\varepsilon}$$

then the generating functional is

$$W_0[J] = \exp\left[-\frac{\mathrm{i}}{2}\int\mathrm{d}^4x\,\mathrm{d}^4x'\,J(x)\Delta_0(x-x')J(x')\right]$$

Two-point function. The two-point Green's function is

$$\begin{aligned}\langle 0|\,\mathcal{T}\{\varphi(x_1)\varphi(x_2)\}\,|0\rangle &= (-\mathrm{i})^2\frac{\delta^2W_0[J]}{\delta J(x_1)\,\delta J(x_2)}\bigg|_{J=0}\\ &= (-\mathrm{i})^2\frac{\delta}{\delta J(x_1)}\left[-\mathrm{i}\int\mathrm{d}^4x\,J(x)\Delta(x-x_2)\,W_0[J]\right]_{J=0}\\ &= [(-\mathrm{i})^3\Delta_0(x_1-x_2)+(\text{terms with } J\text{s})]W_0[J]\Big|_{J=0}\\ &= \mathrm{i}\Delta_0(x_1-x_2)\end{aligned}$$

at the second line one notices

$$\frac{\delta J(x')}{\delta J(x_2)} = \delta^{(4)}(x'-x_2)$$

Also there is a factor of 2 that is simplified with the fraction because the derivative can act on both $J(x)$ and $J(x')$? [r].

Euclidean space. The Lagrangian is

$$\mathcal{L} = -\frac{1}{2}\varphi\,\Box\,\varphi - \frac{1}{2}m^2\varphi^2$$

Applying Wick's rotation $x^0 = -\mathrm{i}x^0_{\mathrm{E}}$, the d'Alembertian operator is [r]

$$\Box = \partial_\mu\partial^\mu = \partial_0^2 - \nabla^2 = -\partial_{0\mathrm{E}}^2 - \nabla^2 = \Box_{\mathrm{E}}$$

In this way, in the Fourier transform one has

$$\Box \rightsquigarrow -k^2\,, \quad \Box_{\mathrm{E}} \rightsquigarrow k_{\mathrm{E}}^2$$

The generating function is

$$\begin{aligned} W_0^{\mathrm{E}}[0] &= N\int[\mathcal{D}\varphi]\,\exp\left[-\frac{1}{2}\int \mathrm{d}^4x_{\mathrm{E}}\,\varphi(\Box_{\mathrm{E}}+m^2)\varphi\right] \\ &= N\int[\mathcal{D}\widetilde{\varphi}]\,\exp\left[-\frac{1}{2}\int\frac{\mathrm{d}^4k_{\mathrm{E}}}{(2\pi)^4}\widetilde{\varphi}(k_{\mathrm{E}})(k_{\mathrm{E}}^2+m^2)\widetilde{\varphi}(-k_{\mathrm{E}})\right] = 1 \end{aligned}$$

[r] The formulation above is well-defined since the exponential is decaying. With the source, one has

$$\begin{aligned} W_0^{\mathrm{E}}[J] &= N\int[\mathcal{D}\widetilde{\varphi}]\,\exp\left[-\frac{1}{2}\int\frac{\mathrm{d}^4k_{\mathrm{E}}}{(2\pi)^4}[\widetilde{\varphi}(k_{\mathrm{E}})(k_{\mathrm{E}}^2+m^2)\widetilde{\varphi}(-k_{\mathrm{E}}) - 2\widetilde{J}(k_{\mathrm{E}})\widetilde{\varphi}(-k_{\mathrm{E}})]\right] \\ &= \exp\left[\frac{1}{2}\int\frac{\mathrm{d}^4k_{\mathrm{E}}}{(2\pi)^4}\widetilde{J}(k_{\mathrm{E}})\frac{1}{k_{\mathrm{E}}^2+m^2}\widetilde{J}(-k_{\mathrm{E}})\right] \\ &= \exp\left[\frac{1}{2}\int\mathrm{d}^4x\,\mathrm{d}^4x'\,J(x)\Delta_0(x-x')J(x')\right] \end{aligned}$$

The Euclidean Green's function is

$$G_{\mathrm{E}}^{(n)}(x_1,\dots,x_n) = \left.\frac{\delta^n W_0^{\mathrm{E}}[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right|_{J=0}$$

The propagator is the two-point function

$$\Delta_0(x_1-x_2) = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}k(x_1-x_2)}}{k^2+m^2} = \langle 0|\,\mathcal{T}\{\varphi(x_1)\varphi(x_2)\}\,|0\rangle$$

where all variables are in Euclidean space (see Cheng–Li, eq. 1.84).

Exercise. Check that the propagator is the two-point function.

Green's function. [r] The Green's function is

$$G_{\mathrm{E}}^{(n)}(x_1,\dots,x_n) = \langle 0|\,\mathcal{T}\{\varphi(x_1)\cdots\varphi(x_n)\}\,|0\rangle = \left.\frac{\delta^n W_0^{\mathrm{E}}[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right|_{J=0}$$

The Lagrangian has a $\mathbb{Z}_2$ symmetry and the Green's function must have that symmetry too. So only even-point functions are non-zero

$$G^{(2k+1)} = 0\,, \quad k\in\mathbb{N}_0$$

The four-point function is

$$\begin{aligned} G_{\mathrm{E}}^{(4)}(x_1,\dots,x_4) &= \langle 0|\,\mathcal{T}\{\varphi(x_1)\cdots\varphi(x_4)\}\,|0\rangle \\ &= \frac{\delta^4}{\delta J(x_1)\cdots\delta J(x_4)}\exp\left[\frac{1}{2}\int\mathrm{d}^4y\,\mathrm{d}^4y'\,J(y)\Delta_0(y-y')J(y')\right]_{J=0} \end{aligned}$$

One may use the fact that

$$\frac{\delta}{\delta J(x_j)} \int \mathrm{d}^4y\, \mathrm{d}^4y'\, J(y)\Delta_0(y-y')J(y') = 2\int \mathrm{d}^4y\, J(y)\Delta(y-x_j)$$

and the only terms that survive after plugging in $J=0$ are the ones without the factor J. The Green's function is

$$\begin{aligned} G^{(4)}_{\mathrm{E}}(x_1,\ldots,x_4) &= \frac{\delta^2}{\delta J(x_1)J(x_2)}\bigg[\Delta(x_3-x_4)W_0^{\mathrm{E}}[J] \\ &\quad + \int \mathrm{d}^4y\, J(y')\Delta(y-x_3)\int \mathrm{d}^4y'\, J(y')\Delta(y-x_4)W_0^{\mathrm{E}}[J]\bigg] \\ &= \Delta(x_3-x_4)\Delta(x_1-x_2) + \Delta(x_2-x_3)\Delta(x_1-x_4) + \Delta(x_1-x_3)\Delta(x_2-x_4) \end{aligned}$$

At the last line [r]. The last line are all the combinations of products of propagators. This is Wick's theorem: all the possible two contractions of four terms.

4.3 Interacting field theory

Generating functional. A simple interacting theory is the $\lambda\varphi^4$ theory. Its Lagrangian is

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4 = \mathcal{L}_0 + \mathcal{L}_1$$

where the first two terms are the free part and the last term is the interacting part. The Euclidean time Lagrangian is

$$\mathcal{L}_{\mathrm{E}} = \frac{1}{2}\varphi(\Box_{\mathrm{E}} + m^2)\varphi + \frac{\lambda}{4!}\varphi^4$$

The generating functional is

$$\begin{aligned} W^{\mathrm{E}}[J] &= N\int[\mathcal{D}\varphi]\,\exp\bigg[-\int \mathrm{d}^4x\,\bigg(\frac{1}{2}\varphi(\Box_{\mathrm{E}}+m^2)\varphi + \frac{\lambda}{4!}\varphi^4 + J\varphi\bigg)\bigg] \\ &= N\int[\mathcal{D}\varphi]\,\exp\bigg[-\int \mathrm{d}^4x\,\bigg(\frac{1}{2}\varphi(\Box_{\mathrm{E}}+m^2)\varphi + J\varphi\bigg)\bigg]\exp\bigg[-\int \mathrm{d}^4x\,\frac{\lambda}{4!}\varphi^4\bigg] \\ &= N\int[\mathcal{D}\varphi]\sum_{n=0}^{\infty}\frac{1}{n!}\bigg[-\int \mathrm{d}^4x\,\frac{\lambda}{4!}\varphi^4\bigg]^n \exp\bigg[-\int \mathrm{d}^4x\,\bigg(\frac{1}{2}\varphi(\Box_{\mathrm{E}}+m^2)\varphi + J\varphi\bigg)\bigg] \\ &= N\int[\mathcal{D}\varphi]\sum_{n=0}^{\infty}\frac{1}{n!}\bigg[-\int \mathrm{d}^4x\,\frac{\lambda}{4!}\bigg(-\frac{\delta}{\delta J(x)}\bigg)^4\bigg]^n \\ &\quad\times\exp\bigg[-\int \mathrm{d}^4x\,\bigg(\frac{1}{2}\varphi(\Box_{\mathrm{E}}+m^2)\varphi + J\varphi\bigg)\bigg] \\ &= \exp\bigg[-\int \mathrm{d}^4x\,\frac{\lambda}{4!}(-\delta_{J(x)})^4\bigg]N\int[\mathcal{D}\varphi]\,\exp\bigg[-\int \mathrm{d}^4x\,\bigg(\frac{1}{2}\varphi(\Box_{\mathrm{E}}+m^2)\varphi + J\varphi\bigg)\bigg] \\ &= \exp\bigg[-\int \mathrm{d}^4x\,\mathcal{L}_1(-\delta_{J(x)})\bigg]N\int[\mathcal{D}\varphi]\,\exp\bigg[-\int \mathrm{d}^4x\,\bigg(\frac{1}{2}\varphi(\Box_{\mathrm{E}}+m^2)\varphi + J\varphi\bigg)\bigg] \\ &= \exp\bigg[-\int \mathrm{d}^4x\,\mathcal{L}_1(-\delta_{J(x)})\bigg]W_0[J] \end{aligned}$$

where the variables are all in Euclidean time, the index is dropped from now on. [r]

4.4 Perturbation theory

See Cheng–Li, Sdrenicki. For small a parameter λ one can Taylor expand the exponential in the generating functional up to a desired order. The generating functional is

$$\begin{aligned} W[J] &= \exp\bigg[-\frac{\lambda}{4!}\int \mathrm{d}^4x\,(-\delta_{J(x)})^4\bigg]\,W_0[J] \\ &= \bigg[1 - \frac{\lambda}{4!}\int \mathrm{d}^4x\,(\delta_{J(x)})^4 + \frac{1}{2!}\frac{\lambda^2}{(4!)^2}\int \mathrm{d}^4x\,(\delta_{J(x)})^4\int \mathrm{d}^4x'\,(\delta_{J(x')})^4 + o(\lambda^2)\bigg]\,W_0[J] \\ &\equiv W_0[J](1 + \lambda\omega_1[J] + \lambda^2\omega_2[J] + \cdots) \end{aligned}$$

where

$$\omega_1[J] = -\frac{1}{4!}W_0^{-1}[J]\int \mathrm{d}^4x\,(\delta_{J(x)})^4 W_0[J]$$
$$\omega_2[J] = \frac{1}{2!}\frac{1}{(4!)^2}W_0^{-1}[J]\int \mathrm{d}^4x\,(\delta_{J(x)})^4\int \mathrm{d}^4x'\,(\delta_{J(x')})^4 W_0[J]$$
$$W_0[J] = \exp\left[\frac{1}{2}\int \mathrm{d}^4y\,\mathrm{d}^4y'\,J(y)\Delta(y-y')J(y')\right] \equiv \mathrm{e}^{k[J]}$$

and so on. The derivatives of the exponent of the generating functional are

$$\frac{\delta k[J]}{\delta J(x)} = \int \mathrm{d}^4y\,J(y)\Delta(y-x)\,,\quad \frac{\delta^2 k[J]}{\delta J(x)^2} = \Delta_0(x-x) = \Delta_0(0)\,,\quad \frac{\delta^n k[J]}{\delta J(x)^n} = 0\,,\quad n\geq 3$$

also

$$\frac{\delta W_0[J]}{\delta J(x)} = \frac{\delta k[J]}{\delta J(x)}\,W_0[J] \iff \frac{\delta \mathrm{e}^{k[J]}}{\delta J(x)} = \frac{\delta k[J]}{\delta J(x)}\,\mathrm{e}^{k[J]}$$

Lecture 7

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4.4.1 First-order expansion coefficient

One would like to compute the first-order expansion coefficient

$$\omega_1[J] = -\frac{1}{4!}W_0^{-1}[J]\int \mathrm{d}^4x\,(\delta_{J(x)})^4 W_0[J]$$

Omitting the argument x and compacting the derivative notation, the integrand is

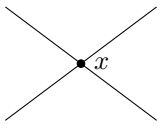
$$\begin{aligned}\delta_J^4\mathrm{e}^{k[J]} &= \delta_J^3(\delta_J k\,\mathrm{e}^k) = \delta_J^2[\delta_J^2 k\,\mathrm{e}^k + (\delta_J k)^2\mathrm{e}^k] = \delta_J[3\,\delta_J^2 k\,\delta_J k\,\mathrm{e}^k + (\delta_J k)^3\mathrm{e}^k]\\ &= 3(\delta_J^2 k)^2\mathrm{e}^k + 6\,\delta_J^2 k\,(\delta_J k)^2\mathrm{e}^k + (\delta_J k)^4\mathrm{e}^k\\ &= W_0[J][3(\delta_J^2 k)^2 + 6\,\delta_J^2 k\,(\delta_J k)^2 + (\delta_J k)^4]\end{aligned}$$

At the first two lines one applies one derivative at a time remembering that derivatives of order higher than two are zero. The coefficient is then

$$\begin{aligned}\omega_1[J] = -\frac{1}{4!}\int \mathrm{d}^4x\,\Big[&3\Delta(x-x)\Delta(x-x)\\ &+ 6\Delta(x-x)\int \mathrm{d}^4y_1\,\mathrm{d}^4y_2\,J(y_1)\Delta(x-y_1)J(y_2)\Delta(x-y_2)\\ &+ \int \mathrm{d}^4y_1\,\mathrm{d}^4y_2\,\mathrm{d}^4y_3\,\mathrm{d}^4y_4\,J(y_1)\Delta(x-y_1)J(y_2)\Delta(x-y_2)J(y_3)\Delta(x-y_3)J(y_4)\Delta(x-y_4)\Big]\end{aligned}$$

Feynman diagrams may be used to quickly obtain the expression above. [r] One may interpret the sum of the terms above in a convenient diagrammatic way:

- The internal point x is associated to



$$-\frac{1}{4!}\int \mathrm{d}^4x$$

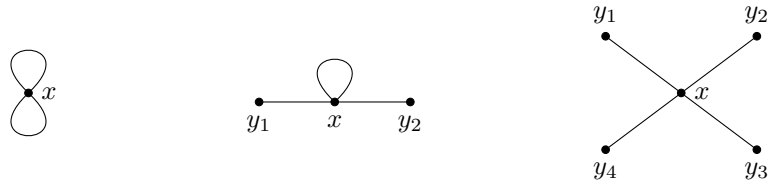
- The external points y_i are associated to

$$\int \mathrm{d}^4y_i\,J(y_i)$$

- The propagators $\Delta(x-y)$ are internal lines associated to

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{\mathrm{e}^{\mathrm{i}k(x-y)}}{k^2+m^2}$$

In the first addendum of the coefficient ω_1 there is a single internal point with two propagators. The second addendum has one internal point and two external points with three propagators. The third addendum has one internal point and four external points with four propagators.



Notice that there are no odd numbers of external points: this is again related to the $\mathbb{Z}_2$ symmetry of the Lagrangian.

At fixed order in the λ expansion, one has a corresponding number of internal vertices: first order, one internal vertex; second order, two vertices, etc.

One observes:

- The number of powers of λ is equal to the number of internal vertices.
- The internal vertex is connected to four lines which corresponds to the four derivatives of the source J, which is related to the study of the $\lambda\varphi^4$ theory. The number of lines coming out of an internal vertex is determined by the choice of the interaction $\mathcal{L}_1$.
- The way one represents the coefficient ω_1 in terms of diagrams — which is related to the algebraic structure of the theory [r] — is a linear combination of diagrams of increasing number of external points. At a given order in the parameter λ, the corresponding coefficient ω is a linear combination of diagrams with increasing number of external points.
- The diagrams without any external points survive when setting $J = 0$: these are vacuum diagrams. They are the only ones contributing to $W[J = 0]$. The normalization with $W[0] = 1$ is equivalent to dividing the integral within $W[J]$ by $W[0]$:

$$W[J] = \frac{1}{W[J=0]} \int [\mathcal{D}\varphi] \exp\left[-\frac{1}{2}\int \mathcal{L}_{\mathrm{E}} + J\varphi\right]$$

  In this way one cancels all vacuum diagrams, so one may forget about them during calculations:

$$\begin{aligned} W[J] &= \frac{1 + 3D_1 + 3!D_2 + D_3 + o(\lambda)}{1 + 3D_1 + o(\lambda)_{\text{vacuum}}} \\ &= [1 + 3D_1 + 3!D_2 + D_3 + o(\lambda)][1 - 3D_1 + o(\lambda)_{\text{vacuum}}] \\ &= 1 + 3D_1 - 3D_1 + \cdots \end{aligned}$$

  where D_j are the algebraic expressions related to the diagrams. [r]

- Each diagram has an associated combinatorial factor. To find the factor one must find all the ways one can connect an internal point with four lines to n external points through one internal line each. The number obtained must be divided by the symmetry factor of the diagram — it is the number of equivalent ways one can draw the diagram. This factor can be computed in a more efficient way.

One would like to use the diagrams to compute the expansion terms [r].

For the coefficient ω_1, the factor $6 = 3!$ of the second addendum is obtained by the $12 = 4 \cdot 3$ ways one can connect an internal point to two external points divided by a factor of 2 because the external points can be swapped while giving the same diagram. For the third addendum, the internal point can be connected in 4! ways to the four external points. The symmetry factor comes from the ways in which one can permute the indices of the external line which is 4!. The factor of the addendum is then just 1. In both of these examples, the common factor of $(4!)^{-1}$ has been omitted, but it should always be included.

In general, for the order λ^p, the expansion coefficient is

$$\omega_p[J] = \frac{W_0^{-1}[J]}{p!}\left[-\frac{1}{4!}\right]^p\left[\int \mathrm{d}^4x\,(\delta_{J(x)})^4\right]^p W_0[J]$$

The diagrams have p internal points. The number of external points is even. The combinatorial factor is

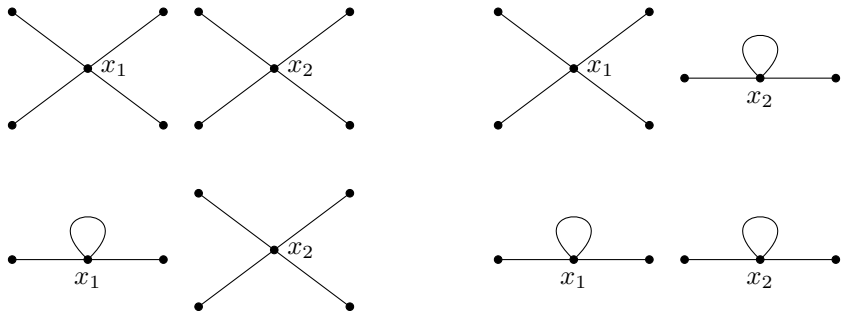
$$s = \frac{1}{p!}\left[-\frac{1}{4!}\right]^p \frac{N}{D}$$

where N is the number of equivalent ways to connect the points with propagators and D is the symmetry factor of the diagram (which is the number of equivalent topologies).

4.4.2 Second-order expansion coefficient

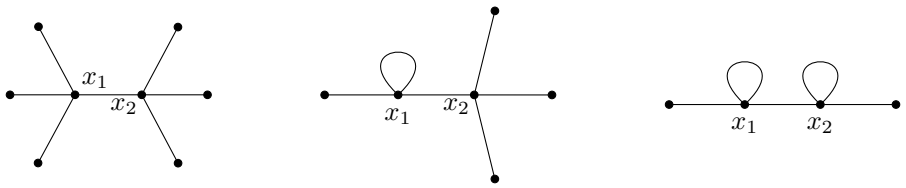
The coefficient contains two internal points x_1 and x_2. One may first draw the diagram and derive from it the algebraic expression.

Disconnected diagrams. The disconnected diagrams are



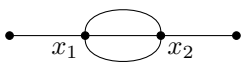
These sum up to $\frac{1}{2}\omega_1^2$.

Connected diagrams. The connected diagrams have lines connecting the internal points. With one line, one has



With two lines, one has

With three lines, one has



The vacuum diagrams have not been considered.

The diagrams can be translated into algebraic expressions. Consider the second connected diagram:

$$\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2 \int \mathrm{d}^4y_1\,\mathrm{d}^4y_2\,\mathrm{d}^4y_3\,\mathrm{d}^4y_4 J(y_1)J(y_2)J(y_3)J(y_4)\Delta(x_1-x_2)\Delta(x_2-x_2)$$
$$\times\,\Delta(x_1-y_1)\Delta(x_1-y_2)\Delta(x_1-y_3)\Delta(x_2-y_4)$$

One still needs the combinatorial factor. All diagrams have a common factor of

$$\frac{1}{2!}\frac{1}{(4!)^2}$$

One has to compute N and D for each diagram.

First diagram. The first line can be connected to the first internal point in 8 possible ways. The second line must connect to the same point as the first line, so there are 3 ways. Similarly, the third line has 2 ways. The first line on the second internal point can be put in 4 ways, the second in 3 and the third in 2. The last line of each internal point can be connected in one way only. Therefore

$$N = 8 \cdot 3! \cdot 4!$$

The symmetries are:

- exchanging the internal points, so two configurations;
- permutations of the three line of each internal vertex, 3! configurations each.

Therefore

$$D = 2 \cdot 3! \cdot 3!$$

The combinatorial factor is

$$S = \frac{1}{2!}\frac{1}{(4!)^2}\frac{N}{D} = \frac{1}{2(3!)^2}$$

The diagram translates to

$$\frac{1}{2(3!)^2}\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\prod_{j=1}^{6}\mathrm{d}y_j\,\Delta(x_1-y_1)\Delta(x_1-y_2)\Delta(x_1-y_3)$$
$$\times\,\Delta(x_2-y_4)\Delta(x_2-y_5)\Delta(x_2-y_6)J(y_1)J(y_2)J(y_3)J(y_4)J(y_5)J(y_6)$$

Second diagram. One has

$$N = 8\cdot(4!)\cdot 3\,,\quad D = 2\cdot(3!)\,,\quad s = \frac{1}{4\cdot 3!} = \frac{1}{4!}$$

Third diagram. One has

$$N = 8\cdot(4)\cdot(9)\,,\quad D = 2\,,\quad s = \frac{1}{8}$$

Fourth diagram. The fourth diagram has

$$N = 8\cdot(3)\cdot(4\cdot 3)\cdot 2$$

and the symmetry factor is

$$D = 2\cdot(2!)\cdot(2!)$$

The combinatorial factor is [r]

$$s = \frac{1}{16}$$

Fifth diagram. One may compute the factor for two copies of the diagram with the internal points swapped and sum their contributions. The first one gives

$$N = 4\cdot 3\cdot(4\cdot 3)\,,\quad D = 2\,,\quad s = \frac{1}{16}$$

Summing up with the second, one obtains

$$s = \frac{1}{8}$$

Sixth diagram. One has

$$N = 8 \cdot 4 \cdot (3!)\,, \quad D = 2\,, \quad s = \frac{1}{12}$$

Exercise. Translate each diagram into an algebraic expression.

Lecture 8

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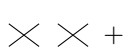
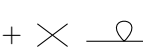
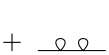
Remark. The generating functional is given by

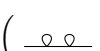
$W[J] = W_0[J]\Big[1 + \lambda\Big($  $+$  $\Big) + \lambda^2\Big($  $+$  $+$  $+$ 

$+$ $+$ $+$ $+$  $+$ $+$  $\Big) + o(\lambda^2)\Big]$

The number of the external points in a diagram corresponds to the number of sources J appearing in the associated algebraic expression. The Green's function is

$$G^{(n)}(z_1, \dots, z_n) = \langle 0 |\, \mathcal{T}\{\varphi(z_1) \cdots \varphi(z_n)\}\, | 0 \rangle = \frac{\delta^n W[J]}{\delta J(z_1) \cdots \delta J(z_n)}\bigg|_{J=0}$$

The two-point function is

$G^{(2)}(z_1, z_2) = \dfrac{\delta^2 W[J]}{\delta J(z_1)\, \delta J(z_2)}\Big|_{J=0} = \lambda$  $+ \lambda^2\Big($  $+$  $+$  $\Big) + o(\lambda^2)$

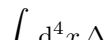
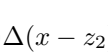
When evaluating at $J = 0$, the non-vanishing terms are the ones that have exactly two external points because the derivatives remove precisely that many sources from the generating functional. When taking the derivative, one has

$$\delta_{J(z_1)} \int \mathrm{d}^4 y_1\, J(y_1) \Delta(x - y_1) = \int \mathrm{d}^4 y_1\, \delta^{(4)}(y_1 - z_1) \Delta(x - y_1) = \Delta(x - z_1)$$

[r] The contributions to the Green's function are the propagators coming from the external points [r]

$$G^{(2)}(z_1, z_2) \rightsquigarrow \lambda$$  $$\rightsquigarrow \int \mathrm{d}^4 x\, \Delta(x - z_1) \Delta(x - z_2) \Delta(x - x)$$

The four-point function is

$G^{(4)}(z_1, \dots, z_4) = \lambda$  $+ \lambda^2\Big($  $+$  $+$  $\Big)$

The six-point function is

$G^{(6)}(z_1, \dots, z_6) = \lambda^2\Big($ $+$ $\Big)$

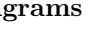
The eight-point function is

$G^{(8)}(z_1, \dots, z_8) = \lambda^2$ 

The ten-point function is, at least, of order λ^3. Higher point Green's functions get contributions only from higher order terms in the coupling constant λ.

4.4.3 Disconnected diagrams

The above Green's functions have taken into account disconnected diagrams. Consider the eight-point function

$G^{(8)}(z_1, \dots, z_8) = \langle 0 |\, \mathcal{T}\{\varphi(z_1) \cdots \varphi(z_8)\}\, | 0 \rangle = \lambda^2$  $+ \text{permutations}$

Up to combinatorial contributions, the Green's function is

$$\begin{aligned} G^{(8)}(z_1,\ldots,z_8) &= \lambda^2 \int \mathrm{d}^4x_1\, \Delta(x_1-z_1)\Delta(x_1-z_2)\Delta(x_1-z_3)\Delta(x_1-z_4) \\ &\quad \times \int \mathrm{d}^4x_2\, \Delta(x_2-z_5)\Delta(x_2-z_6)\Delta(x_2-z_7)\Delta(x_2-z_8) \\ &\sim \lambda^2 \langle 0|\, \mathcal{T}\{\varphi(z_1)\cdots\varphi(z_4)\}\, |0\rangle \langle 0|\, \mathcal{T}\{\varphi(z_5)\cdots\varphi(z_8)\}\, |0\rangle \end{aligned}$$

From this one notices that disconnected Feynman diagrams contribute to disconnected Green's functions. These can be built from lower-order connected Green's functions.

To eliminate the disconnected contributions both in the generating functional and the Green's function, one notices that the connected Green's functions are given by

$$G_{\mathrm{c}}^{(n)}(z_1,\ldots,z_n) = \left.\frac{\delta^n \ln W[J]}{\delta J(z_1)\cdots\delta J(z_n)}\right|_{J=0}$$

One may check the above for the order λ^2. Rewriting the generating functional as

$$W[J] = W_0[J]\Big[1 + W_0^{-1}[J](W[J]-W_0[J])\Big]$$

One sees that the second addendum is the perturbative expansion

$$W_0^{-1}[J](W[J]-W_0[J]) = \lambda\omega_1[J] + \lambda^2\omega_2[J] + o(\lambda^2)$$

[r] Taking the logarithm, one has

$$\begin{aligned} \ln W[J] &= \ln W_0[J] + \ln[1 + W_0^{-1}[J](W[J]-W_0[J])] \\ &= \ln W_0 + W_0^{-1}(W-W_0) - \frac{1}{2}[W_0^{-1}(W-W_0)]^2 + o(\lambda^2) \\ &= \ln W_0 + \lambda\omega_1 + \lambda^2\omega_2 + o(\lambda^2) - \frac{1}{2}[\lambda\omega_1 + \lambda^2\omega_2 + o(\lambda^2)]^2 + o(\lambda^2) \\ &= \ln W_0 + \lambda\omega_1 + \lambda^2\left[\omega_2 - \frac{1}{2}\omega_1^2\right] + o(\lambda^2) \end{aligned}$$

At the first line, the perturbative expansion is of order λ, so one Taylor expands

$$\ln[1+\varepsilon] = 1 + \varepsilon - \frac{1}{2}\varepsilon^2 + o(\varepsilon^2)$$

The second addendum in the bracket is the negative sum of the disconnected diagrams, therefore the whole bracket is the sum of only the connected diagrams at order λ^2.

One defines the generating functional of connected Green's functions $Z[J]$ as

$$W[J] \equiv \mathrm{e}^{Z[J]} \implies Z[J] \equiv \ln W[J]$$

Therefore, the connected Green's function prescription is

$$G_{\mathrm{c}}^{(n)}(z_1,\ldots,z_n) = \left.\frac{\delta^n Z[J]}{\delta J(z_1)\cdots\delta J(z_n)}\right|_{J=0}$$

The normalization follows from the one of the generating functional W

$$W[J=0] = 1 \implies Z[J=0] = 0$$

From the expansion above of $\ln W[J]$ one can see that

$$Z[J] = Z_0[J] + \lambda(\cdots) + \lambda^2(\cdots) + o(\lambda^2)$$

At the present moment, one has eliminated vacuum diagrams and disconnected diagrams.

4.5 Green's functions in momentum space

The integrals of Green's functions are easier to compute in momentum space rather than configuration space. One needs to reformulate the Feynman rules after applying the Fourier transform.

Definition. The n-point Green's function in momentum space is

$$\widetilde{G}_{\rm c}^{(n)}(p_1,\ldots,p_n)(2\pi)^4\delta^{(4)}(p_1+p_2+\cdots+p_n)=\int\prod_{j=1}^{n}\mathrm{d}^4x_j\,\mathrm{e}^{-\mathrm{i}p_jx_j}\,G_{\rm c}^{(n)}(x_1,\ldots,x_n)$$

One may apply this prescription to the two-point Green's function. In configuration space, the free Green's function is

$$G_0^{(2)}(x_1,x_2)=\Delta(x_1-x_2)=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}k(x_1-x_2)}}{k^2+m^2}$$

In momentum space one has

$$\begin{aligned}\widetilde{G}_0^{(2)}(p_1,p_2)(2\pi)^4\delta^{(4)}(p_1+p_2)&=\int\mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\mathrm{e}^{-\mathrm{i}(p_1x_1+p_2x_2)}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}k(x_1-x_2)}}{k^2+m^2}\\&=\int\frac{\mathrm{d}^4k}{(2\pi)^4}(2\pi)^4\delta^{(4)}(k-p_1)(2\pi)^4\delta^{(4)}(k+p_2)\frac{1}{k^2+m^2}\\&=(2\pi)^4\delta^{(4)}(p_1+p_2)\frac{1}{p_1^2+m^2}\end{aligned}$$

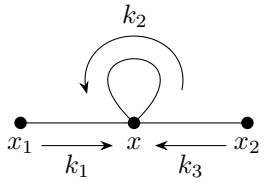
from which

$$\widetilde{G}_0^{(2)}(p_1,p_2)=\frac{1}{p_1^2+m^2}\,,\quad p_1=-p_2\iff\boxed{\widetilde{G}_0^{(2)}(p,-p)=\frac{1}{p^2+m^2}}$$

One would like to find the momentum space version of Feynman rules. Consider the correction of order λ to the two-point Green's function $G^{(2)}$. There is only one contributing diagram [r]. One has

$$\begin{aligned}G^{(2)}(x_1,x_2)\Big|_\lambda&=-\frac{\lambda}{2}\int\mathrm{d}^4x\,\Delta(x-x_1)\Delta(x-x_2)\Delta(x-x)\\&=-\frac{\lambda}{2}\int\mathrm{d}^4x\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}k_1(x-x_1)}}{k_1^2+m^2}\int\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{1}{k_2^2+m^2}\int\frac{\mathrm{d}^4k_3}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}k_3(x-x_2)}}{k_3^2+m^2}\end{aligned}$$

Since one can always send $k\to-k$ in the integral, one has to choose in which way to assign momentum and has to be consistent with the assignment. For example, in the above, the momentum k_1 goes from an external point x_1 towards an internal point x. The momentum in the loop can be put in either direction.



Performing the x-integral, one obtains

$$\int\mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}x(k_1+k_3)}=(2\pi)^4\delta^{(4)}(k_1+k_3)$$

Therefore, the Green's function is

$$G^{(2)}(x_1,x_2)\Big|_\lambda=-\frac{\lambda}{2}\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}k_1(x_1-x_2)}}{(k_1^2+m^2)^2(k_2^2+m^2)}$$

The Fourier transform is then

$$\begin{aligned}\widetilde{G}^{(2)}(p_1,p_2)\Big|_\lambda (2\pi)^4\delta^{(4)}(p_1+p_2) = \\ &= -\frac{\lambda}{2}\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\mathrm{e}^{-\mathrm{i}p_1x_1-\mathrm{i}p_2x_2}\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}k_1(x_1-x_2)}}{(k_1^2+m^2)^2(k_2^2+m^2)}\\ &= -\frac{\lambda}{2}\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{(2\pi)^4\delta^{(4)}(k_1-p_2)(2\pi)^4\delta^{(4)}(p_1+k_1)}{(k_1^2+m^2)^2(k_2^2+m^2)}\\ &= -\frac{\lambda}{2}(2\pi)^4\delta^{(4)}(p_1+p_2)\int\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{1}{k_2^2+m^2}\frac{1}{(p_2^2+m^2)^2}\end{aligned}$$

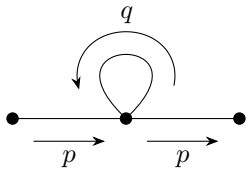
Keeping in mind the momentum conservation, one has

$$\widetilde{G}^{(2)}(p_1,p_2)\Big|_\lambda = -\frac{\lambda}{2}\frac{1}{(p_2^2+m^2)^2}\int\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{1}{k_2^2+m^2}\,,\quad p_1=-p_2$$

Tidying up the expression, one can write

$$\widetilde{G}^{(2)}(p,-p) = -\frac{\lambda}{2}\frac{1}{(p^2+m^2)^2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{q^2+m^2}$$

The corresponding diagram becomes



[r] In this way one guarantees momentum conservation at the internal vertex.

Feynman rules. One has to assign a set of momenta to guarantee momentum conservation at each vertex and total momentum conservation using the direction of the arrows. [r] Every time a loop appears, there is an integral on the loop momentum.

Therefore

- an internal vertex corresponds to

$$-\frac{\lambda}{4!}(2\pi)^4\delta^{(4)}(p_1+p_2+p_3+p_4)$$

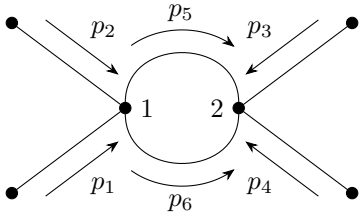
- a propagator is an internal line corresponding to

$$\frac{1}{p^2+m^2}$$

- a loop corresponds to an integral

$$\int\frac{\mathrm{d}^4q}{(2\pi)^4}$$

Exercise. Write the momentum integral corresponding to the following diagram



One should obtain a λ^2 contribution to the Green's function $\widetilde{G}^{(4)}(p_1,\dots,p_4)$.

One should assign a momentum to every external line and internal line. One imposes momentum conservation at each internal vertex. At the first vertex, one has

$$p_1 + p_2 = p_5 + p_6$$

At the second vertex, one has

$$p_5 + p_6 = -p_3 - p_4$$

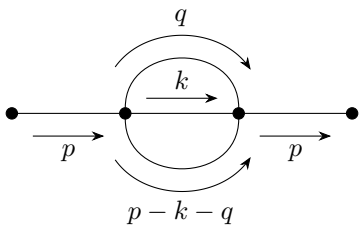
From these two conditions, one has the total momentum conservation and the dependence of one momentum on the others

$$p_1 + p_2 + p_3 + p_4 = 0\,, \quad p_6 = p_1 + p_2 - p_5$$

There is only a single integral in p_5.

Briefly, one draws the diagram, assigns the momenta remembering total momentum conservation; there is only one loop which corresponds to an integral, the other line is given by momentum conservation.

Example. Consider the following two-loop diagram



The algebraic expression without the combinatorial factor is

$$\widetilde{G}^{(2)}(p,-p)\Big|_{\lambda^2} \rightsquigarrow \frac{1}{(p^2+m^2)^2}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(q^2+m^2)(k^2+m^2)[(p-k-q)^2+m^2]}$$

The main point is to compute the integrals. One does not care about external propagators because they are spectators. One moves towards the evaluation of cut diagrams where one focuses only on the integrals of the diagram.

4.6 Loop expansion

Previously, the generating functional has been perturbatively expanded in powers of the coupling constant λ. One may reshuffle the terms and organize them by how may loops are present. In this way one can obtain an expansion in powers of the reduced Planck's constant $\hbar$ [r].

One should first fix the Green's function, so fix the number of external points. Writing explicitly the Planck's constant, the generating functional is

$$\begin{aligned}
W[J] &= N\int [\mathcal{D}\varphi]\,\exp\left[-\int \mathrm{d}^4x\,\left(\frac{1}{\hbar}(\mathcal{L}_0+\mathcal{L}_1)+J\varphi\right)\right]\\
&= N\int [\mathcal{D}\varphi]\,\exp\left[-\frac{1}{\hbar}\int \mathrm{d}^4x\,(\mathcal{L}_0+\mathcal{L}_1+\hbar J\varphi)\right]\\
&= \exp\left[-\frac{1}{\hbar}\int \mathrm{d}^4x\,\mathcal{L}_1(-\delta_{J(x)})\right] N\int [\mathcal{D}\varphi]\,\exp\left[-\frac{1}{\hbar}\int \mathrm{d}^4x\,(\mathcal{L}_0+\hbar J\varphi)\right]\\
&= \exp\left[-\frac{1}{\hbar}\int \mathrm{d}^4x\,\mathcal{L}_1(-\delta_{J(x)})\right] W_0[J]
\end{aligned}$$

The propagator is the inverse of the kinetic term, [r] therefore

$$W_0[J] = \exp\left[\frac{\hbar}{2}\int \mathrm{d}^4x\,\mathrm{d}^4x'\,J(x)\Delta(x-x')J(x')\right]$$

The Feynman rules have to be modified. The propagator is $\hbar\Delta(x-y)$ and so it carries one Planck's constant $\hbar$. Each internal point brings one inverse of the Planck's constant $\hbar^{-1}$ since the internal points come from the expansion of the exponential of the integral of the interaction Lagrangian $\mathcal{L}_1$.

Considering a diagram with V internal vertices and I internal lines. Its order is

$$\hbar^{I-V} = \frac{1}{\hbar}\hbar^{I-V+1} = \frac{1}{\hbar}\hbar^{L}$$

where $L = I - V + 1$ is the number of loops. In this way one can reorganize the perturbative expansion in powers of the Planck's constant and as such in terms of the number of loops.

Lecture 9

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One would like to show that for a theory with only one coupling constant, the loop expansion is equivalent to the expansion in powers of the coupling constant λ. In the following, one analyzes the $\lambda\varphi^4$ theory. One considers a contribution to the Green's function of order λ^V where V is the number of internal vertices.

A relation particular of this theory is the total number of lines exiting from the vertices

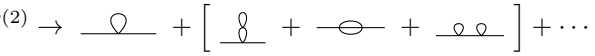
$$4V = E + 2I$$

where E is the number of external lines and I is the number of internal lines. Combining this equation with the formula for the number of loops, one finds

$$4V = E + 2(L + V - 1) \implies V = \frac{1}{2}E + L - 1$$

One may remember that the number of external lines E characterizes the Green's functions, so one may keep it fixed. Therefore, from λ^V one can obtain the order in powers of the Planck's constant $\lambda^{L-1} \sim \hbar^{L-1}$.

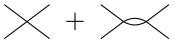
Example. Considering the two-point Green's function, it has contribution from diagrams of order λ and λ^2

$G^{(2)} \to$  $+\cdots$

Since $E = 2$ then one finds $\hbar$ and $\hbar^2$ respectively.

Remark. The Taylor expansion of a Green's function $G^{(n)}$ organized in powers of Planck's constant $\hbar^L$ contains terms of order $\hbar^0$ which are the classical contributions. These correspond to diagrams without any loop. These diagrams are called tree diagrams (tree in the sense of graph theory).

Example. The tree diagram for the two-point Green's function is the classical propagator, also called free propagator. For the four-point Green's function, the tree diagram is the cross diagram. In fact, one has $E = 4$ for which

$G^{(4)} \to$  + permutations $\sim \lambda + \lambda^2 \sim \hbar^0 + \hbar$

The Taylor expansion is done in powers of Planck's constant keeping in mind the correspondence with the loop expansion.

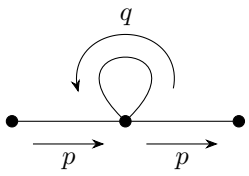
4.7 Ultraviolet divergences

[r] write discussion lec 08.

The functional measure is not well-defined for high energies. [r] For example, one finds ill-defined integrals associated to the two-point Green's function. Letting

$$\widetilde{G}^{(2)}(p,-p) \equiv \widetilde{G}^{(2)}(p)$$

one finds 1-loop contributions given by



$$\rightarrow \quad \frac{1}{(p^2+m^2)^2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{1}{q^2+m^2}$$

This integral is divergent for large momenta q. The superficial degree of divergence D is the difference between the number of momenta in the numerator and the number of momenta in the denominator. For the above integral one has

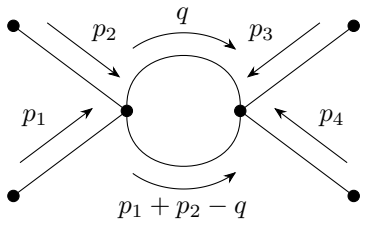
$$D = 4 - 2 = 2$$

When $D \geq 0$, ultraviolet divergences occur. The integral diverges quadratically and its dimension is D.

The 1-loop correction to the four-point Green's function

$$\widetilde{G}^{(4)}(p_1, p_2, p_3, -p_1 - p_2 - p_3)$$

is given by the diagram



+ permutations

whose algebraic expression is proportional to

$$\prod_{i=1}^{4}\frac{1}{p_i^2+m^2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{1}{(q^2+m^2)[(p_1+p_2-q)^2+m^2]}$$

The superficial degree of divergence is

$$D = 4 - 4 = 0$$

The integral diverges logarithmically.

One would like to study ultraviolet divergences through a general method instead of computing the superficial degree of freedom for each diagram. Given a certain theory, one would like to know how many kinds of diagrams are divergent and how many diagrams are divergent. Given the infinite Taylor expansion, there are three scenarios: the theory contains a finite number of ultraviolet divergent diagrams, there is a finite number of topologies of divergent diagrams, there is an infinite number of different topologies of divergent diagrams. The nature of the theory depends on these three possibilities. One needs a way to compute how many diagrams and topologies are divergent given a theory, an interaction and a dimension.

Number of divergent diagrams. Consider a theory in d space-time dimensions with a vertex connected to N lines (for example $\lambda\varphi^N$). Given a diagram with V_N internal vertices, E external lines, I internal lines and L loops, the superficial degree of divergence is

$$D = dL - 2I$$

The factor 2 depends on the spin of the field: scalar fields have 2 while fermion fields have 1. Noting that

$$L = I - V_N + 1\,, \quad NV_N = E + 2I$$

one obtains the superficial degree of divergence of a scalar $\lambda\varphi^N$ theory

$$\boxed{D = d - \frac{1}{2}(d-2)E + V_N\left[\frac{N-2}{2}d - N\right]}$$

Divergent diagrams correspond to $D \geq 0$.

Fixing $d = 4$, one finds that

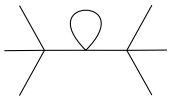
$$D = 4 - E + (N-4)V_N$$

The last term is negative only for $N = 3$ because for $N = 2$ one has a mass term, not a vertex. In this case, higher order terms are more convergent because the number of internal points increases. For $N = 4$, the superficial degree of divergence

$$D = 4 - E$$

does not depend on the number of internal vertices: at every order, the degree of divergence is the same. Also, the degree is $D \geq 0$ only for $E = 0, 2, 4$. For $E = 0$ one has vacuum diagrams, for $E = 2$ one has the two-point Green's function $\widetilde{G}^{(2)}$ and for $E = 3$ one has the four-point $\widetilde{G}^{(4)}$. Therefore, for the $\lambda\varphi^4$ theory, there are only two divergent topologies each with an infinite number of diagrams. There are ultraviolet divergences at any loop and they are of the type

However the analysis is not conclusive. Consider a six-point Green's function. A 1-loop correction is



$$\rightarrow \quad \text{external propagators} \times \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{q^2 + m^2}$$

Its superficial degree of divergence is $D = -2$, but the diagram is divergent as it has a divergent sub-diagram. If the superficial degree of divergence is negative $D < 0$, it is not enough to conclude that the diagram is convergent.

Theorem (Weinberg)**.** A Feynman diagram is ultraviolet finite if

- the superficial degree of divergence is negative $D < 0$,
- the degree of every sub-diagram is negative $D_i < 0$.

[r] One may split a diagram into sub-diagrams by cutting free propagators.

One-particle irreducible diagrams. Considering again the six-point Green's function, one may see that the loop sub-diagram can isolated by cutting two free propagators: the whole diagram is 2-particle reducible.

Generally, consider a loop diagram with a given number of external lines connected to another loop diagram by n free propagators. The diagram is then n-particle reducible: it can be split into two sub-diagrams by cutting n free propagators. The superficial degree of divergence of the diagram is

$$D = D_1 + D_2 + 4(n-1) - 2n$$

[r]

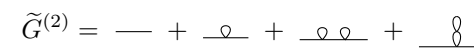
For example, if $D_1 = D_2 = 0$, then the sub-diagrams are logarithmically divergent, but the total degree is

$$D = 2n - 4$$

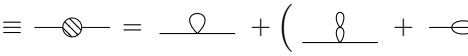
which is negative for $n = 1$. So the total diagram seems to be convergent, but the sub-diagrams diverge.

One would like to focus only on diagrams that cannot be reduced by cutting free propagators: these are called one-particle irreducible (1PI) diagrams. To compute divergences, it suffices to compute one-particle irreducible diagrams because the reducible diagrams can be constructed from the irreducible ones by using free propagators.

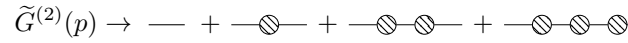
Consider the corrections to the two-point Green's function

$\widetilde{G}^{(2)} =$  $+ o(\hbar^2)$

The third diagram is the only one reducible diagram given by two one-loop contributions connected by a free propagator. One defines the sum of all irreducible diagrams also called self-energy

$1\mathrm{PI} = -\Sigma(p^2) \equiv$ 

from which one can see that [r]

$\widetilde{G}^{(2)}(p) \to$ 

The associated algebraic expression is

$$\begin{aligned}\widetilde{G}^{(2)}(p) &= \frac{1}{p^2+m^2} + \frac{1}{p^2+m^2}[-\Sigma(p^2)]\frac{1}{p^2+m^2} \\ &\quad + \frac{1}{p^2+m^2}[-\Sigma(p^2)]\frac{1}{p^2+m^2}[-\Sigma(p^2)]\frac{1}{p^2+m^2} + \cdots \\ &= \frac{1}{p^2+m^2}\left[1+\frac{\Sigma(p^2)}{p^2+m^2}\right]^{-1} = \frac{1}{p^2+m^2+\Sigma(p^2)}\end{aligned}$$

At the second line, the addenda contain a resummation of higher order loops given by the resummation $\Sigma(p^2)$. At the second equality one applies the geometric series. The resummation is a function of p^2 due to Lorentz invariance. The Green's function contains all the quantum corrections, so it becomes the quantum propagator.

From this discussion, one sees that one can study only the one-particle irreducible diagrams. From all diagrams, one has restricted oneself to the study of non-vacuum diagrams, connected diagrams and one-particle irreducible diagrams. One also notices that the external propagators can be omitted since they are recovered in the final resummation: one computes amputated one-particle irreducible diagrams. The resummation of these amputated diagrams is

$$\Gamma^{(n)}_{\mathrm{c}}(p_1,\dots,p_n) = G^{(n)}_{\mathrm{c}}(p_1,\dots,p_n)_{1\mathrm{PI}}\left[\prod_{k=1}^{n} G^{(2)}(p_k)\right]^{-1}$$

One may now drop the tilde to denote the Fourier transform of a Green's function and use the argument as a marker. Up to one-loop one notices that

$$\Gamma^{(2)}(p^2) = p^2 + m^2 - \Sigma(p^2)$$

[r]

In the $\lambda\varphi^4$ theory, the only ultraviolet divergent diagrams are the ones in $\Gamma^{(2)}$ and $\Gamma^{(4)}$.

4.7.1 Excursus: superficial degree of divergence

Two dimensions. The general case for the superficial degree of divergence is

$$D = d - \frac{1}{2}(d-2)E + V_N\left[\frac{N-2}{2}d - N\right]$$

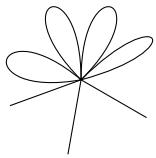
Consider two dimensions $d = 2$. The degree becomes

$$D = 2(1 - V_N)$$

It does not depend on the number of external lines. Therefore the divergences appear in the same way in any Green's function. Also, the degree does not depend on the number N of lines of the interaction vertex. The divergences appear for

$$2(1 - V_N) \geq 0 \implies V_N = 0, 1$$

Only the n-point Green's function with only one vertex is divergent

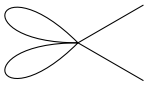


For a vertex $\lambda\varphi^N$, one connects n of lines to other points and contracts the remaining $N-n$ lines into loops. These diagrams are divergent.

These diagrams can be avoided by regularizing the theory with normal ordering (see Theoretical Physics I)

$$\lambda\varphi^N \rightsquigarrow :\lambda\varphi^N:$$

Example. For $\lambda\varphi^6$, an example of two-point function diagram is the following



Four dimensions. For four dimensions $d=4$, the degree is

$$D = 4 - E + (N-4)V_N$$

For $\lambda\varphi^3$ theory, $N=3$, one has

$$D = 4 - E - V_3$$

Fixing the number of external lines E, that is fixing the Green's function, then

$$D \geq 0 \implies 4 - E \geq V_3 \implies E \leq 4 - V_3 < 4$$

The divergences appear only for $E = 1, 2, 3$ and the degree is positive only for a restricted number of vertices

$$\begin{aligned} &E = 1\,, \quad D = 3 - V_3 \geq 0\,, \quad V_3 \leq 3 \\ &E = 2\,, \quad D = 2 - V_3 \geq 0\,, \quad V_3 \leq 2 \\ &E = 3\,, \quad D = 1 - V_3 \geq 0\,, \quad V_3 \leq 1 \end{aligned}$$

The number of topologies is finite and, in each, the number of divergent diagrams is finite.

Six dimensions. In six dimensions $d=6$, one has

$$D = 6 - 2E + 2V_N(N-3)$$

Setting $N=3$, one finds

$$D = 6 - 2E$$

This is a similar situation situation as $\lambda\varphi^4$ in $d=4$: no dependence on internal vertices. In fact

$$D \geq 0 \implies E = 1, 2, 3$$

So only the Green's functions $G^{(1)}$, $G^{(2)}$ and $G^{(3)}$ are divergent with an infinite number of divergent diagrams each.

Exercise. Study $d=3$ and $d=5$.

Lecture 10

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5 Renormalization

5.1 Conventional renormalization

See Cheng–Li, §2.1. Before evaluating the integrals, one would like to isolate the divergent part.

Historically, conventional renormalization was the first method of renormalization, however it is not suitable for higher orders: beyond two loops, the counter terms renormalization is more efficient.

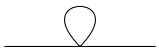
As a prototype, one considers the $\lambda\varphi^4$ theory in $d = 4$ dimensions. The divergences appear only for the Green's functions $G^{(2)}$ and $G^{(4)}$ since the superficial degree of divergence is

$$D = 4 - E$$

5.1.1 Regularization

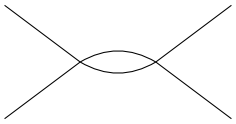
In order to compute the integrals one employs regularization by using a cutoff or through a dimensional parameter: one introduces a parameter that makes the integral finite. The divergent integrals depend on some parameters in such a way that when one removes these regulators, the integrals diverge again.

Example. The following one-loop diagram is divergent since its degree is $D = 2$



One integrates up to a finite cutoff Λ. This regularization parameter has mass dimension 1. The integral associated to the diagram is quadratically divergent and asymptotic to $c_1\Lambda^2 \to \infty$.

Example. The following two-loop diagram [r] has degree $D = 0$



It is asymptotic to

$$c_2 \log \frac{\Lambda^2}{m^2} \to \infty$$

One may use effective field theory [r] like in phenomenology. However, this method does not permit to work at any energy scale.

5.1.2 Renormalization

Before removing the regulator, one has to find a consistent way to remove the divergent part of ultraviolet divergent integrals. For example, the previous integrals are a sum of a divergent part and a finite part. One would like to remove the divergent part and keep the finite one.

Isolating the divergence — four-point connected amputated Green's function. The first step is understand how to isolate the divergent part in the result of a given integral. Consider the diagram in the second example above. It is given by

$$\Gamma^{(4)}(s) \propto \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(q^2 + m^2)[(p-q)^2 + m^2]}, \quad s \equiv p^2 = (p_1 + p_2)^2$$

where s is the first Mandelstam variable. The integral depends on the external momenta p_1 and p_2. Applying any number of derivatives with respect to the external momenta improves the

convergence of the integral

$$\begin{aligned}\partial_{p^2}\Gamma^{(4)}(p^2) &= \partial_{p^2}p^\mu\,\partial_{p^\mu}\Gamma^{(4)}(p^2) = \frac{p^\mu}{2p^2}\,\partial_{p^\mu}\Gamma^{(4)}(p^2)\\ &\propto \frac{p^\mu}{2p^2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\partial_{p^\mu}\frac{1}{(q^2+m^2)[(p-q)^2+m^2]}\\ &= -\frac{p^\mu}{2p^2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{2(p_\mu-q_\mu)}{(q^2+m^2)[(p-q)^2+m^2]^2}\\ &= -\frac{1}{p^2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{p(p-q)}{(q^2+m^2)[(p-q)^2+m^2]^2}\end{aligned}$$

at the first line, one as

$$\partial_{p^\mu}p^2 = \partial_{p^\mu}(p^\nu p_\nu) = 2p_\mu \implies \partial_{p^2}p^\mu = \frac{1}{2p^2}p^\mu$$

The degree of divergence in the integration variable q is

$$D = 4+1-6 = -1$$

However, one would like to separate the various contribution to the integral [r]. One may note that

$$\begin{aligned}p(p-q) &= -\frac{1}{2}[(p-(p-q))^2 - p^2 - (p-q)^2] = -\frac{1}{2}[q^2 - p^2 - (p-q)^2]\\ &= -\frac{1}{2}[-p^2 + (q^2+m^2) - ((p-q)^2+m^2)]\end{aligned}$$

The idea is to either get a term depending only on the external momenta or get something resembling the denominator. Therefore

$$\begin{aligned}\partial_{p^2}\Gamma^{(4)}(p^2) \propto -\frac{1}{2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{(q^2+m^2)[(p-q)^2+m^2]^2} + \frac{1}{2p^2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{[(p-q)^2+m^2]^2}\\ - \frac{1}{2p^2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{(q^2+m^2)[(p-q)^2+m^2]}\end{aligned}$$

The respective degrees of divergence of these three integrals are

$$D = -2\,,\quad D = 0\,,\quad D = 0$$

The first integral is finite. The last two are separately ultraviolet divergent, however their difference is finite. Taking the derivative with respect to the external momentum p^2 lowers the degree of divergence by 2 and gives a convergent integral.

To separate the divergent part from the finite part, one may Taylor expand the amputated function $\Gamma^{(4)}$ around a point. Choosing the origin, $p^2 = 0$, one obtains

$$\Gamma^{(4)}(p^2) = \Gamma^{(4)}(0) + \mathrm{d}_{p^2}\Gamma^{(4)}\Big|_{p^2=0}p^2 + o(p^2) \equiv \Gamma^{(4)}(0) + \widetilde{\Gamma}^{(4)}(p^2)$$

The first addendum is logarithmically divergent while the other terms are finite (denoted by the tilde). One may notice that

$$\widetilde{\Gamma}^{(4)}(p^2=0) = 0$$

To isolate the finite part of the function $\Gamma^{(4)}$ around $p^2 = 0$ one needs to evaluate

$$\widetilde{\Gamma}^{(4)}(p^2) = \Gamma^{(4)}(p^2) - \Gamma^{(4)}(0)$$

The choice of the point $p^2 = 0$ is arbitrary. One may take any other point $p^2 = \mu^2$ and obtain

$$\Gamma^{(4)}(p^2) = \Gamma^{(4)}(\mu^2) + \mathrm{d}_{p^2}\Gamma^{(4)}\Big|_{p^2=\mu^2}(p^2-\mu^2) + o(p^2-\mu^2) \equiv \Gamma^{(4)}(\mu) + \widetilde{\Gamma}^{(4)}(p^2,\mu^2)$$

where, once again, the first addendum is logarithmically divergent while the other terms are finite. One similarly has

$$\widetilde{\Gamma}^{(4)}(p^2=\mu^2,\mu^2)=0$$

Though, the finite part is different based on the chosen point. One may want to understand how to relate these various finite parts (without explicitly setting $\mu^2=0$).

The chosen expansion point μ^2 is called subtraction point: a point in momentum space around which one performs the Taylor expansion. When one chooses a subtraction point, one subsequently fixes an energy scale.

Isolating the divergence — two-point connected amputated Green's function. Since the function $\Gamma^{(2)}$ has a degree $D=2$ one needs to differentiate at least twice to obtain a finite result.

Consider the first example above. One needs two derivatives to reach a finite integral

$$\Gamma^{(2)}(p^2)=\Gamma^{(2)}(\mu^2)+\partial_{p^2}\Gamma^{(2)}\Big|_{p^2=\mu^2}(p^2-\mu^2)+\frac{1}{2}\partial^2_{p^2}\Gamma^{(2)}\Big|_{p^2=\mu^2}(p^2-\mu^2)^2+o(\cdots)$$

The first two addenda are divergent, while the rest are finite. Recalling that $\Gamma^{(2)}=p^2+m^2-\Sigma$ up to one-loop [r] one then has

$$\Sigma(p^2)=\Sigma(\mu^2)+\Sigma'(\mu^2)(p^2-m^2)+\widetilde{\Sigma}(p^2,\mu^2)$$

with

$$\widetilde{\Sigma}(p^2=\mu^2,\mu^2)=0\,,\quad \widetilde{\Sigma}'(p^2=\mu^2,\mu^2)=0$$

One may notice that, considering only one-loop correction to Σ, one has

$$\Sigma^{(1)}\propto\int\frac{\mathrm{d}^4q}{q^2+m^2}$$

So it does not depend on p^2 and it follows

$$[\Sigma^{(1)}]'(p^2=\mu^2)=0$$

Removal of divergences. The second step is remove the divergent parts of the integrals by redefining the parameters of the theory: the field φ, the mass m and the coupling constant λ. This is the renormalization step.

In the conventional normalization, one expresses the Lagrangian in terms of non-renormalized quantities (called bare) φ_0, m_0 and λ_0; the Feynman rules are given in terms of these bare parameters. On the other hand, in the counter terms renormalization, one uses renormalized quantities to begin with.

Renormalization — two-point Green's function. Following Cheng–Li, one may rotate back to Minkowski. The sum of one-particle irreducible corrections is $-\mathrm{i}\Sigma(p^2)$ and the two-point Green's function is

$$\mathrm{i}G^{(2)}(p)=\frac{\mathrm{i}}{p^2-m_0^2-\Sigma(p^2)+\mathrm{i}\varepsilon}$$

One may choose a subtraction point $p^2=\mu^2$. The corrections are divergent

$$\Sigma(p^2)=\Sigma(\mu^2)+\Sigma'(\mu^2)(p^2-\mu^2)+\widetilde{\Sigma}(p^2,\mu^2)$$

The first two addenda are ultraviolet divergent with degrees $D=2$ and $D=0$ (which correspond to their mass dimensions). Expanding the corrections, the Green's function is

$$\mathrm{i}G^{(2)}(p)=\frac{\mathrm{i}}{p^2-m_0^2-\Sigma(\mu^2)-(p^2-\mu^2)\Sigma'(\mu^2)-\widetilde{\Sigma}(p^2)+\mathrm{i}\varepsilon}$$

Looking at the dimensions, one redefines the mass parameter

$$\boxed{m_0^2+\Sigma(\mu^2)=\mu^2}$$

On the right-hand side, one could choose an arbitrary parameter since one can always add finite terms that are reabsorbed into the finite term $\widetilde{\Sigma}$.

In the expression above, there is a sum between a parameter and a divergent quantity that gives a finite term: one assumes that the bare parameter m_0 is itself divergent in a way exactly opposite to the corrections Σ. The arbitrariness above can also be seen as a consequence of the subtraction of two infinite quantities.

The employment of the subtraction point μ^2 as the mass parameter is purely utilitarian: one could choose any other parameter. In this way one obtains

$$\begin{aligned} \mathrm{i}G^{(2)}(p) &= \frac{\mathrm{i}}{(p^2-\mu^2)[1-\Sigma'(\mu^2)]-\widetilde{\Sigma}(p^2)+\mathrm{i}\varepsilon} = \frac{\mathrm{i}}{[1-\Sigma'(\mu^2)][p^2-\mu^2-\widetilde{\Sigma}(p^2)]+\mathrm{i}\varepsilon} \\ &\equiv \frac{\mathrm{i}Z_\varphi}{p^2-\mu^2-\widetilde{\Sigma}(p^2)+\mathrm{i}\varepsilon} \end{aligned}$$

One would like to isolate the divergent bracket in the first equality. The second equality holds at a given order in the coupling constant λ_0, because higher order terms appear and they are neglected. One defines the field renormalization function

$$Z_\varphi = [1-\Sigma'(\mu^2)]^{-1} \sim 1+\Sigma'(\mu^2)$$

This ultraviolet divergent term can be removed by renormalizing the bare field

$$\boxed{\varphi = Z_\varphi^{-\frac{1}{2}}\varphi_0}$$

One may study how this renormalization works. Consider the renormalized Green's function

$$\begin{aligned} \mathrm{i}G_{\mathrm{R}}^{(2)}(p) &\equiv \int \mathrm{d}^4x\, \mathrm{e}^{-\mathrm{i}px} \langle 0|\, \mathcal{T}\{\varphi(x)\varphi(0)\}\, |0\rangle = Z_\varphi^{-1} \int \mathrm{d}^4x\, \mathrm{e}^{-\mathrm{i}px} \langle 0|\, \mathcal{T}\{\varphi_0(x)\varphi_0(0)\}\, |0\rangle \\ &= Z_\varphi^{-1}\mathrm{i}G^{(2)}(p) = \frac{\mathrm{i}}{p^2-\mu^2-\widetilde{\Sigma}(p^2)+\mathrm{i}\varepsilon} \end{aligned}$$

where one omits the Dirac delta function of conservation.

For the two-point Green's function, one has removed the divergences by performing mass and field renormalization.

Remark. Remembering that

$$\widetilde{\Sigma}(p^2=\mu^2) = 0$$

the denominator of the two-point Green's function is

$$[p^2-\mu^2-\widetilde{\Sigma}(p^2)]_{p^2=\mu^2} = 0$$

Therefore $p^2=\mu^2$ a pole of the propagator and the parameter μ is the physical mass. This may not be true depending on the choice of the subtraction point.

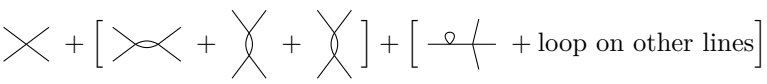
This procedure also applies to higher-point Green's functions. The renormalized Green's function is

$$G_{\mathrm{R}}^{(n)}(p_1,\dots,p_n) = Z_\varphi^{-\frac{n}{2}} G^{(n)}(p_1,\dots,p_n)$$

The amputated Green's function is

$$\begin{aligned} \Gamma_{\mathrm{R}}^{(n)}(p_1,\dots,p_n) &= G_{\mathrm{R}}^{(n)}(p_1,\dots,p_n)\left[\prod_{j=1}^n G_{\mathrm{R}}^{(2)}(p_j)\right]^{-1} \\ &= Z_\varphi^{-\frac{n}{2}} G^{(n)}(p_1,\dots,p_n)\left[\prod_{j=1}^n Z_\varphi^{-1}G^{(2)}(p_j)\right]^{-1} \\ &= Z_\varphi^{\frac{n}{2}} G^{(n)}(p_1,\dots,p_n)\left[\prod_{j=1}^n G^{(2)}(p_j)\right]^{-1} = Z_\varphi^{\frac{n}{2}}\Gamma^{(n)}(p_1,\dots,p_n) \end{aligned}$$

Renormalization — four-point Green's function. The following discussion deals mainly with one-loop contributions. The four-point Green's function up to one-loop contributions is

$$G^{(4)}(p_1,\dots,p_n)\Big|_{\text{1-loop}} =$$  $$+ \text{loop on other lines}$$

The diagrams in the first bracket are one-particle irreducible, while the diagrams in the second bracket are one-particle reducible. The amputated Green's function $\Gamma^{(4)}$ is obtained by removing the external lines — including their corrections (by multiplying by $[G^{(2)}]^{-1}$) — of the one-particle irreducible diagrams.

In Minkowski, the amputated four-point Green's function up to one-loop contributions is

$$\Gamma^{(4)}(s,t,u)\Big|_{\text{1-loop}} = -\mathrm{i}\lambda_0 + \Gamma(s) + \Gamma(t) + \Gamma(u)$$

where the first addendum corresponds to the four-point vertex and the last three addenda are the vertex functions Γ corresponding to the three Mandelstam variables s, t and u. The corresponding diagrams diverge logarithmically and their derivatives are convergent. One has to choose a subtraction point for each diagram: s_0, t_0, u_0. To make it easier, on-shell one knows that

$$s + t + u = 4\mu^2$$

where μ is a mass; so one may choose

$$s_0 = t_0 = u_0 = \frac{4}{3}\mu^2$$

Therefore, the Taylor expansion of each term gives

$$\begin{aligned}\Gamma^{(4)}(s,t,u)\Big|_{\text{1L}} &= -\mathrm{i}\lambda_0 + [\Gamma(s_0) + \widetilde{\Gamma}(s)] + [\Gamma(t_0) + \widetilde{\Gamma}(t)] + [\Gamma(u_0) + \widetilde{\Gamma}(u)] \\ &= -\mathrm{i}\lambda_0 + 3\Gamma(s_0) + \widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u)\end{aligned}$$

The first addendum in each bracket is divergent since the four-point function is logarithmically divergent $D = 0$. Also it holds

$$\widetilde{\Gamma}(s = s_0) = 0\,, \quad \widetilde{\Gamma}(t = t_0) = 0\,, \quad \widetilde{\Gamma}(u = u_0) = 0$$

The divergent terms can be absorbed into a redefinition of the coupling constant since their mass dimensions coincide.

Lecture 11

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Since the amputated Green's function is dimensionless, its divergent terms may be reabsorbed inside the coupling constant

$$-\mathrm{i}\lambda_0 + 3\Gamma(s_0) = -\mathrm{i}\lambda_0 Z_\lambda^{-1}$$

from which one has

$$-\mathrm{i}\lambda_0\left[1 + \frac{3\mathrm{i}}{\lambda_0}\Gamma(s_0)\right] = -\mathrm{i}\lambda_0 Z_\lambda^{-1} \implies Z_\lambda = \left[1 + \frac{3\mathrm{i}}{\lambda_0}\Gamma(s_0)\right]^{-1} \sim 1 - \frac{3\mathrm{i}}{\lambda_0}\Gamma(s_0)$$

The renormalized amputated Green's function is

$$\begin{aligned}\Gamma^{(4)}_{\text{R}}(s,t,u) &= Z_\varphi^2\Gamma^{(4)}(s,t,u) = Z_\varphi^2[-\mathrm{i}\lambda_0 Z_\lambda^{-1} + \widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u)] \\ &= -\mathrm{i}\lambda_0 Z_\lambda^{-1} Z_\varphi^2 + Z_\varphi^2[\widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u)] \\ &= -\mathrm{i}\lambda_0 Z_\lambda^{-1} Z_\varphi^2 + [\widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u)] + o(\lambda_0^2)\end{aligned}$$

At the second line, the bracket is of order λ_0^2 but the field renormalization function is $Z_\varphi = 1 + O(\lambda_0)$, so, at one loop, one may consider only the unity.

The first addendum contains two divergent terms (the renormalization functions Zs). The renormalized coupling constant is defined as

$$\boxed{\lambda \equiv \lambda_0 Z_\lambda^{-1} Z_\varphi^2}$$

Like for the mass, the bare coupling constant is ultraviolet divergent in a way that exactly cancels the other terms' divergence giving a finite renormalized coupling. Therefore, the renormalized amputated Green's function up to one-loop contributions is

$$\Gamma_{\mathrm{R}}^{(4)}(s,t,u) = -\mathrm{i}\lambda + \widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u) + o(\lambda^2)$$

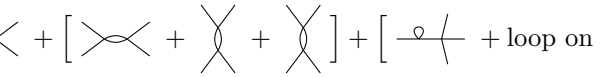
which is finite.

[r] At this point one has a list of renormalization prescriptions

$$\varphi_0 = Z_\varphi^{\frac{1}{2}} \varphi\,, \quad m_0^2 = \mu^2 - \Sigma(\mu^2)\,, \quad \lambda_0 = Z_\lambda Z_\varphi^{-2} \lambda$$

which make the one-particle irreducible contributions to the Green's function $G^{(2)}$ and $G^{(4)}$ finite. The Z and $\Sigma(\mu^2)$ terms are called renormalization functions [r].

Correctness check. One may check that the above prescription is sufficient to make one-particle reducible diagrams finite. The discussion is done for one-loop. The four-point Green's function contains diagrams of the type

$\mathrm{i}G^{(4)}(p_1,\dots,p_n) \to$  + loop on other lines

$$= \left[\prod_{j=1}^{4} \frac{\mathrm{i}}{p_j^2 - m_0^2 + \mathrm{i}\varepsilon}\right]\Big[-\mathrm{i}\lambda_0 + [3\Gamma(s_0) + \widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u)]$$

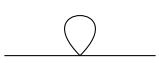
$$+ (-\mathrm{i}\lambda_0) \sum_{k=1}^{4} (-\mathrm{i}\Sigma(p_k^2)) \frac{\mathrm{i}}{p_k^2 - m_0^2 + \mathrm{i}\varepsilon}\Big]$$

Performing a partial resummation of the tree-level diagram and the one-particle reducible diagrams, one obtains

$$\mathrm{T} + 1\mathrm{PR} = -\mathrm{i}\lambda_0 \left[\prod_{j=1}^{4} \frac{\mathrm{i}}{p_j^2 - m_0^2 + \mathrm{i}\varepsilon}\right]\left[1 + \sum_{k=1}^{4} (-\mathrm{i}\Sigma(p_k^2)) \frac{\mathrm{i}}{p_k^2 - m_0^2 + \mathrm{i}\varepsilon}\right]$$

$$= -\mathrm{i}\lambda \prod_{j=1}^{4} \frac{\mathrm{i}}{p_j^2 - m_0^2 - \Sigma(p_j^2) + \mathrm{i}\varepsilon} + o(\lambda_0^2)$$

The second line is obtained by noting that the self-energy function $\Sigma(p^2)$ at one loop is given by the diagram



and then applying the geometric series.

The contribution from the one-particle irreducible diagrams is

$$1\mathrm{PI} = \left[\prod_{j=1}^{4} \frac{\mathrm{i}}{p_j^2 - m_0^2 - \Sigma(p_j^2) + \mathrm{i}\varepsilon}\right][3\Gamma(s_0) + \widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u)] + o(\lambda^2)$$

since $\Gamma \sim \lambda_0^2$ and $\widetilde{\Gamma} \sim \lambda_0^2$: one adds Σ and applies the geometric series.

Therefore, the four-point Green's function is

$$\mathrm{i}G^{(4)} = \left[\prod_{j=1}^{4} \frac{\mathrm{i}}{p_j^2 - m_0^2 - \Sigma(p_j^2) + \mathrm{i}\varepsilon}\right]\Big[-\mathrm{i}\lambda_0 + [3\Gamma(s_0) + \widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u)]\Big]$$

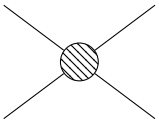
$$= \left[\prod_{j=1}^{4} \mathrm{i}G^{(2)}(p_j)\right]\Gamma^{(4)}(s,t,u)$$

The one-particle reducible diagrams contain a (quantum) two-point propagator at one-loop order λ_0, while one-particle irreducible diagrams do not, since otherwise their order would be higher.

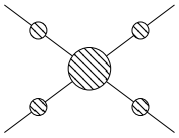
Now one may renormalize the above quantities

$$G_{\rm R}^{(4)} = Z_\varphi^{-2} G^{(4)} = Z_\varphi^{-2}\left[\prod_{j=1}^{4} G^{(2)}(p_j)\right]\Gamma^{(4)}(s,t,u) = Z_\varphi^{-2}\left[\prod_{j=1}^{4} Z_\varphi G_{\rm R}^{(2)}(p_j)\right] Z_\varphi^{-2}\Gamma_{\rm R}^{(4)}(s,t,u)$$
$$= Z_\varphi^{-4} Z_\varphi^4\left[\prod_{j=1}^{4} G_{\rm R}^{(2)}(p_j)\right]\Gamma_{\rm R}^{(4)}(s,t,u) = \left[\prod_{j=1}^{4} G_{\rm R}^{(2)}(p_j)\right]\Gamma_{\rm R}^{(4)}(s,t,u)$$

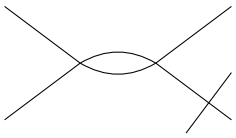
The diagram corresponding to the above expression is equivalent to computing the renormalized Green's function using renormalized Feynman rules: the propagator $G_{\rm R}^{(2)}$ and the vertex $\Gamma_{\rm R}^{(4)}$ are quantum because they contain all the corrections

$G_{\rm R}^{(2)}(p) =$ $\Gamma_{\rm R}^{(4)} =$ 

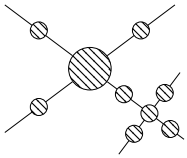
The four-point Green's function is then

$G_{\rm R}^{(4)} =$ 

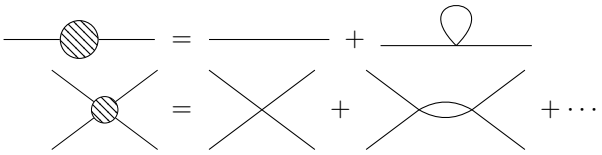
Higher-point Green's functions. The six-point Green's function at one loop gives only one-particle reducible diagrams, for example



The total degree of divergence is negative $D < 0$, but the loop sub-diagram is divergent. The Green's function has to be renormalized by adding the loop corrections (the blobs) on every line and vertex.

$G_{\rm R}^{(6)} =$ 

The renormalized vertex contains the loop. This Green's function is finite. This diagram is the sum of many diagrams each with one loop correction on one line (or vertex) only (otherwise the order increases). At one-loop, the corrected propagator and vertex are



The same is true for higher-point Green's functions. The procedure involves starting from the tree-level diagram and correct every line and vertex.

Exercise. Expand the above corrected $G_{\rm R}^{(6)}$ diagram in terms of all possible one-loop contributions. Do the same for $G^{(8)}$.

Higher loops. At one loop, for the $\lambda\varphi^4$ theory there is no Σ' term in the self-energy function Σ and as such no field renormalization. At one loop, the first term $\Sigma(\mu^2)$ of the renormalization function is proportional to the coupling constant λ_0. The renormalization of the mass $\delta\mu^2$ contains both λ_0 and m_0. One has also found that the amputated four-point Green's function is $\Gamma(s) \sim \lambda_0^2$ from which $Z_\lambda \sim \lambda_0, m_0$.

At one loop, the renormalization functions depend on the bare parameters. However, at this order, one can replace the bare parameters with the physical ones and ignore the higher orders.

At two loops one has to consider

- genuine two-loop diagrams contributing to the Green's functions $G^{(2)}$ and $G^{(4)}$;
- revisit the lower-loop calculations and replace the bare parameters up to the desired loop order.

An L-loop calculation requires revisiting all lower-loop results.

5.2 Bogoliubov–Parasiuk–Hepp–Zimmermann renormalization

See Cheng–Li, Ramond, Anselmi. The Bogoliubov–Parasiuk–Hepp–Zimmermann (BPHZ) renormalization is a more efficient procedure from a more technical point of view, but it is equivalent to the conventional renormalization. It is also called counter terms renormalization.

The procedure is the same as before: regularize the ultraviolet divergences, choose a subtraction point, perform the Taylor expansion and remove the divergences by renormalizing the parameters

$$\varphi_0 = Z_\varphi^{\frac{1}{2}}\varphi\,, \quad m_0^2 = m^2 - \delta m^2\,, \quad \lambda_0 = Z_\lambda Z_\varphi^{-2}\lambda$$

The counter terms renormalization is based on a simple manipulation of the original Lagrangian. The bare Lagrangian is

$$\mathcal{L}_0 = \frac{1}{2}\,\partial_\mu\varphi_0\,\partial^\mu\varphi_0 - \frac{m_0^2}{2}\varphi_0^2 - \frac{\lambda_0}{4!}\varphi_0^4$$

One expresses the bare parameters in terms of the physical ones

$$\begin{aligned}
\mathcal{L}_0 &= \frac{1}{2}Z_\varphi\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}(m^2-\delta m^2)Z_\varphi\varphi^2 - \frac{\lambda}{4!}Z_\lambda\varphi^4 \\
&= \frac{1}{2}Z_\varphi\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}(m^2-\delta m^2)Z_\varphi\varphi^2 - \frac{\lambda}{4!}Z_\lambda\varphi^4 + \mathcal{L} - \mathcal{L} \\
&= \mathcal{L} + \frac{1}{2}(Z_\varphi - 1)[\,\partial_\mu\varphi\,\partial^\mu\varphi - m^2\varphi^2] + \frac{1}{2}\,\delta m^2\,Z_\varphi\varphi^2 - \frac{\lambda}{4!}(Z_\lambda - 1)\varphi^4 \\
&= \mathcal{L} + \frac{1}{2}A\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}Bm^2\varphi^2 - \frac{\lambda}{4!}C\varphi^4
\end{aligned}$$

where

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4\,, \quad A = Z_\varphi - 1\,, \quad B = A - \frac{\delta m^2}{m^2}Z_\varphi\,, \quad C = Z_\lambda - 1$$

[r] check

The renormalized parameters are the finite parameters in terms of which one can write the physical parameters (since they do not necessarily coincide). The Lagrangian $\mathcal{L}$ is welcome, but not the remaining terms. These are still local terms called counter terms

$$\mathcal{L}_0 = \mathcal{L} + \mathcal{L}_{\text{ct}}$$

The Feynman rules are read from the renormalized Lagrangian $\mathcal{L}$ and the integrals of the loops are still (ultraviolet) divergent but written in terms of renormalized parameters. One adds new Feynman rules for the counter terms so that they cancel the previous divergences. Therefore one may

- write a set of Feynman rules from the renormalized Lagrangian; the propagator and the vertex are

$$\frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon}\,, \quad -\mathrm{i}\lambda$$

- perform the one-loop calculations of the Green's functions $\Gamma^{(2)}$ and $\Gamma^{(4)}$; for the former one has

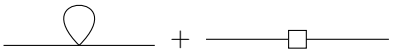
$$\Gamma^{(2)} \to \lambda I_{\text{div}} = \lambda \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathrm{i}}{q^2 - m^2 + \mathrm{i}\varepsilon}$$

[r] The divergent terms are $-\mathrm{i}\Sigma(\mu^2)$ with dimension 2. One may add a counter term to the renormalized Lagrangian $\mathcal{L}$ of the form

$$\mathcal{L}^{(1)} = \frac{1}{2} B m^2 \varphi^2$$

This modifies the Feynman rules by adding a two-point vertex indicated as follows

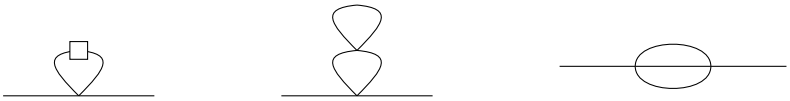
and one fixes B at one-loop such that the following sum is finite



This new (box) vertex is of order one-loop.

- At two loops one must construct all two-point diagrams also using the new vertex

For example



Lecture 12

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One may notice that the two-point amputated Green's function is the complete quantum kinetic term

$$\Gamma^{(2)} = G^{(2)}[G^{(2)}G^{(2)}]^{-1} = [G^{(2)}]^{-1}$$

While the four-point amputated Green's function is the complete quantum vertex. [r] Therefore, to balance the divergences, one has to add terms that counter the two previous Green's function. In fact, the counter term Lagrangian contains a classical kinetic part (the first two terms) and a classical vertex.

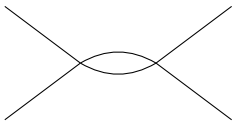
Perturbative procedure. The perturbative procedure is the following.

- Starting from the renormalized Lagrangian $\mathcal{L}$, one computes the one-loop corrections. The two-point Green's function $\Gamma^{(2)}$ has contributions from the diagram of order λ

because it has one vertex, and is therefore balanced by the kinetic counter term

$$(\mathcal{L}^{(1)}_{\text{ct}})_{\text{I}} = \frac{1}{2} A (\partial_\mu \varphi)^2 - \frac{1}{2} m^2 B \varphi^2$$

where the index (1) indicates the loop order, equivalent to the order of the parameters A, B and C. The terms A and B are both order λ. The Green's function $\Gamma^{(4)}$ is of order λ^2



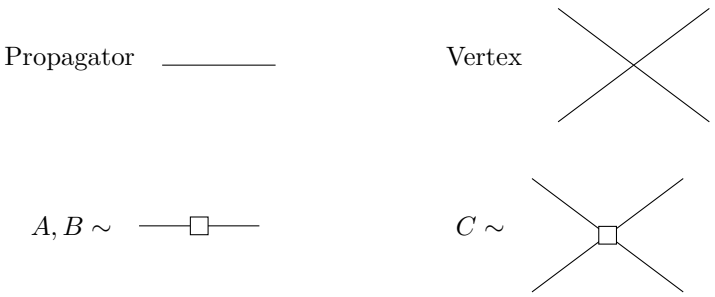
and one adds a vertex counter term

$$(\mathcal{L}^{(1)}_{\rm ct})_{\rm II} = -\frac{\lambda}{4!}C\varphi^4$$

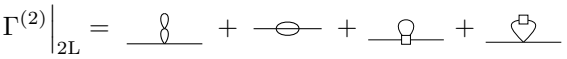
where the term C is order λ. The appropriate Lagrangian for one-loop calculation is then

$$\mathcal{L} + \mathcal{L}^{(1)}_{\rm ct}$$

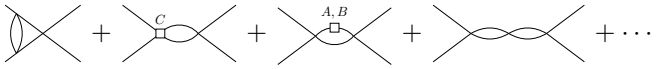
- At two loops, using the above as the new Lagrangian, the new Feynman rules are



There are two new interaction vertices proportional to the counter terms. The two-loop contributions to the two-point Green's function are

$\Gamma^{(2)}\Big|_{\rm 2L} =$ 

In the third diagram contributes the C term, while in the last contribute the A and B terms. Similarly, the two-loop contributions to the four-point Green's function are

$\Gamma^{(4)}\Big|_{\rm 2L} =$ 

$+ \cdots$

To draw diagrams with counter terms is sufficient to use lower-loop diagrams and add a counter term (denoted by the box).

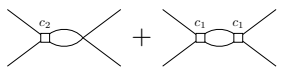
One may already see that the first diagram of the Green's function $\Gamma^{(2)}$ contains a one-loop divergence related to the upper loop. The last diagram exactly cancels this sub-divergence, but the first diagram is still divergent in its whole. Similarly for the others. When one sums genuine diagrams and counter term diagrams, one cancels some divergences, but the sum may still be divergent. If the total sum is still divergent, one may cure the remaining divergences by adding new counter terms at order λ^2

$$\mathcal{L}^{(2)}_{\rm ct} = \frac{1}{2}A(\partial_\mu\varphi)^2 - \frac{1}{2}m^2B\varphi^2 - \frac{\lambda}{4!}C\varphi^4\,, \quad A, B, C = O(\lambda^2)$$

- At three loops one uses $\mathcal{L} + \mathcal{L}^{(1)}_{\rm ct} + \mathcal{L}^{(2)}_{\rm ct}$. The Feynman rules have the same structure but with A, B and C given by expressions of the form

$$A = a_1\lambda + a_2\lambda^2\,, \quad B = b_1\lambda + b_2\lambda^2\,, \quad C = c_1\lambda + c_2\lambda^2$$

The four-point Green's function at three loops gets contributions from diagrams of the type

$\Gamma^{(4)}\Big|_{\rm 3L} =$ 

$+ \cdots$

- Repeating the steps, the total Lagrangian is the bare Lagrangian

$$\mathcal{L} + \sum_{n=1}^{\infty} \mathcal{L}_{\text{ct}}^{(n)} = \mathcal{L}_0$$

  The summation is not well-defined since the bare Lagrangian itself is not. For the theory $\lambda\varphi^4$, the divergences appear only for the Green's functions $\Gamma^{(2)}$ and $\Gamma^{(4)}$: one may cancel the divergences using counter terms that have the same form as the Lagrangian. The resummation above (called multiplicative renormalization) is equal to the bare Lagrangian because one has defined the field, mass and coupling renormalizations.

In a theory where the situation is more complicated, like $\lambda\varphi^5$ in $d = 4$ dimensions, one may draw a three-point diagram

 $\sim \int \frac{\mathrm{d}^4 q}{q^2 + m^2} \sim q^2\,, \quad D = 2$

Therefore, cancelling this three-point Green's function $\Gamma^{(3)}$ divergent term requires adding to the Lagrangian $\mathcal{L}$ a counter term of the type

$$\mathcal{L}_{\text{ct}} = \frac{\lambda}{3!} C\varphi^3 =$$ 

This term is not contained in the Lagrangian

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{5!}\varphi^5$$

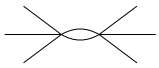
Therefore, the one-loop Lagrangian contains a new kind of vertex

$$\mathcal{L} + \mathcal{L}_{\text{ct}}^{(1)} = \mathcal{L} + \frac{\lambda}{3!} C\varphi^3$$

The multiplicative renormalization is not true for this theory. The theory can be either thrown away or modified by including in the original Lagrangian the three-point vertex

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{5!}\varphi^5 - \frac{\widetilde{\lambda}}{3!}\varphi^3$$

At one-loop, the counter term is related to the above three-point term and they combine to give a renormalization for the new coupling constant $\widetilde{\lambda}$. However, for the five-point vertex φ^5, there are also six-point Green's function diagrams which are logarithmically divergent

$\Gamma^{(6)} \to$ 

To cancel their divergence, one has to add a counter term at one-loop of the form

$$\frac{g}{6!}\varphi^6(\cdots)$$

If one wants to keep the theory, one has to add a six-point vertex to the original Lagrangian. By adding this vertex, one can build a divergent diagram for the eight-point Green's function $\Gamma^{(8)}$. To cancel this term one has to add an eight-point vertex to the original Lagrangian.

One may see that this process does not stop and one keeps adding vertices. In order to make renormalization consistent, one has to have an infinite number of vertices and as such an infinite number of coupling constants. A theory with an infinite number of coupling constants is not predictive at any energy scale (but it may work at some scale by truncating the number of interaction vertices in the Lagrangian).

In $d = 4$ dimensions, one concludes that $\lambda\varphi^4$ is a renormalizable theory, while $\lambda\varphi^5$ is not.

6 One-loop contributions — computation of Feynman integrals

See Cheng–Li, Ramond, Peskin. The Euclidean formalism is used. The prototype theory is still $\lambda\varphi^4$. For an ultraviolet divergent integral, one needs to regularize it. The regularization schemes used are the covariant regularization and the dimensional regularization.

6.1 Convenient tools

Feynman combining. A first tool is Feynman combining

$$\prod_{j=1}^{n}\frac{1}{a_j}=(n-1)!\int_0^1\prod_{j=1}^{n}\mathrm{d}z_j\,\frac{\delta(\sum_i z_i-1)}{(a_1z_1+\cdots+a_nz_n)^n}$$

Applying the derivative with respect to the i-th parameter, one obtains

$$\frac{1}{a_i}\prod_{j=1}^{n}\frac{1}{a_j}=n!\int_0^1\prod_{j=1}^{n}\mathrm{d}z_j\,\frac{\delta(\sum_k z_k-1)z_i}{(a_1z_1+\cdots+a_nz_n)^{n+1}}$$

One may prove the identity above by using the Schwinger representation of the propagator

$$\frac{1}{a_j}=\int_0^\infty\mathrm{d}\alpha_j\,\mathrm{e}^{-\alpha_j a_j}$$

One would obtain

$$\prod_{j=1}^{n}\frac{1}{a_j}=\int_0^\infty\prod_{j=1}^{n}\mathrm{d}\alpha_j\,\mathrm{e}^{-\sum_{j=1}^{n}\alpha_j a_j}$$

Applying the following change of variables

$$z=\sum_{j=1}^{n}\alpha_j\,,\quad z_j=\frac{\alpha_j}{z}\,,\quad j=1,\ldots,n-1$$

one gets

$$\sum_{j=1}^{n-1}z_j+\frac{\alpha_n}{z}=1\implies\alpha_n=z\Big[1-\sum_{j=1}^{n-1}z_j\Big]$$

From this, one has

$$\prod_{j=1}^{n}\frac{1}{a_j}=\int_0^1\prod_{j=1}^{n-1}\mathrm{d}z_j\int_0^\infty\mathrm{d}z\,z^{n-1}\exp\Big[-z\sum_{j=1}^{n-1}a_jz_j-z(1-\textstyle\sum_k z_k)a_n\Big]$$

Euler's Beta function. A second tool is Euler's Beta function

$$\mathrm{B}(z_1,z_2)\equiv\int_0^1\mathrm{d}t\,t^{z_1-1}(1-t)^{z_2-1}=\int_0^\infty\mathrm{d}t\,\frac{t^{z_1-1}}{(1+t)^{z_1+z_2}}=\frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}$$

where Γ is Euler's Gamma function. In particular

$$\frac{\mathrm{B}(m,n-m)}{(A^2)^{n-m}}=\int_0^\infty\mathrm{d}t\,\frac{t^{m-1}}{(t+A^2)^n}=\frac{1}{(A^2)^{n-m}}\frac{\Gamma(m)\Gamma(n-m)}{\Gamma(n)}$$

Spherical coordinates. A third tool is integration in spherical coordinates

$$\int\mathrm{d}^4q=\int_0^\infty\mathrm{d}q\,q^3\int_0^{2\pi}\mathrm{d}\theta_1\int_0^\pi\mathrm{d}\theta_2\,\sin\theta_2\int_0^\pi\mathrm{d}\theta_3\,\sin^2\theta_3$$

For a quantity independent of the angles, one obtains

$$\int\mathrm{d}^4q\,f(q^2)=2\pi^2\int_0^\infty\mathrm{d}q\,q^3f(q^2)=\pi^2\int_0^\infty2q\,\mathrm{d}q\,q^2f(q^2)=\pi^2\int_0^\infty\mathrm{d}q^2\,q^2f(q^2)$$

This identity can be generalized to n dimensions

$$\begin{aligned}\int\mathrm{d}^nq\,f(q^2)&=\int_0^\infty\mathrm{d}q\,q^{n-1}\int_0^{2\pi}\mathrm{d}\theta_1\Big[\prod_{j=2}^{n-1}\int_0^\pi\mathrm{d}\theta_j\,(\sin\theta_j)^{j-1}\Big]f(q^2)\\&=\frac{2\pi^{\frac{n}{2}}}{\Gamma(n/2)}\int_0^\infty\mathrm{d}q\,q^{n-1}f(q^2)=\frac{\pi^{\frac{n}{2}}}{\Gamma(n/2)}\int_0^\infty\mathrm{d}q^2\,(q^2)^{\frac{n}{2}-1}f(q^2)\end{aligned}$$

For dimensional regularization, one needs to analytically continue this expression to a non-integer number of dimensions.

6.2 Covariant regularization

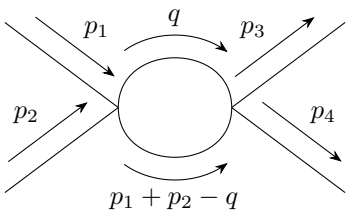
The covariant regularization scheme was introduce by Pauli and Villars. The simplest way to make a divergent integral finite is to integrate up to a cutoff Λ, however this breaks Lorentz invariance. The covariant regularization is a cutoff regularization that keeps Lorentz invariance. It is based on the idea to modify the propagator in such a way to end up with an integrand with more powers of the momentum at the denominator than the initial integrand so that the integral converges. One modifies the propagators inside an integral by adding extra terms

$$\frac{1}{q^2+m^2} \to \frac{1}{q^2+m^2} + \sum_i \frac{a_i}{q^2+\Lambda_i^2}$$

where $\Lambda_i \gg m$ is the cutoff. The regularized result is obtained in the limit $\Lambda \to \infty$.

One would like to compute one-loop contributions to the Green's functions $\Gamma^{(2)}$ and $\Gamma^{(4)}$.

One-loop contributions — four-point Green's function. [r] The one-loop contribution to the Green's function is the following



Let $p = p_1 + p_2$ and $p^2 = s$. The explicit algebraic expression of the vertex function is

$$\Gamma(p^2) = \frac{\lambda^2}{2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(q^2+m^2)[(p-q)^2+m^2]}$$

This contribution is logarithmically divergent $D = 0$. One cutoff is sufficient $\Lambda \gg m$ for the covariant regularization

$$\frac{1}{q^2+m^2} \to \frac{1}{q^2+m^2} - \frac{1}{q^2+\Lambda^2} = \frac{\Lambda^2-m^2}{(q^2+m^2)(q^2+\Lambda^2)} \sim \frac{\Lambda^2}{(q^2+m^2)(q^2+\Lambda^2)}$$

The vertex function is then

$$\Gamma(p^2) = \frac{\lambda^2}{2}\Lambda^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(q^2+m^2)(q^2+\Lambda^2)[(p-q)^2+m^2]}$$

Since the vertex function is logarithmically divergent, one may choose the subtraction point $p^2 = 0$ and Taylor expand around it

$$\Gamma(p^2) = \Gamma(0) + \widetilde{\Gamma}(p^2)\,, \quad \widetilde{\Gamma}(0) = 0$$

where the first addendum is logarithmically divergent and the second is finite.

One may split the computation. The divergent part is

$$\Gamma(0) = \frac{\lambda^2}{2}\Lambda^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(q^2+m^2)^2(q^2+\Lambda^2)}$$

The finite part is

$$\begin{aligned}
\widetilde{\Gamma}(p^2) &= \Gamma(p^2) - \Gamma(0) \\
&= \frac{\lambda^2}{2}\Lambda^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \left[\frac{1}{(q^2+m^2)(q^2+\Lambda^2)[(p-q)^2+m^2]} - \frac{1}{(q^2+m^2)^2(q^2+\Lambda^2)}\right] \\
&= \frac{\lambda^2}{2}\Lambda^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(q^2+m^2)(q^2+\Lambda^2)} \left[\frac{1}{(p-q)^2+m^2} - \frac{1}{(q^2+m^2)}\right] \\
&= \frac{\lambda^2}{2}\Lambda^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{2pq-p^2}{(q^2+m^2)^2[(p-q)^2+m^2](q^2+\Lambda^2)} \\
&\to \frac{\lambda^2}{2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{2pq-p^2}{(q^2+m^2)^2[(p-q)^2+m^2]}\,, \quad \Lambda \to \infty
\end{aligned}$$

One needs to compute the integrals in the two parts.

Lecture 13

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Finite part. Starting from

$$\widetilde{\Gamma}(p^2) = \frac{\lambda^2}{2}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{2pq-p^2}{(q^2+m^2)^2[(p-q)^2+m^2]} = \frac{\lambda^2}{2}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{2pq-p^2}{a_1^2 a_2}$$

where one sets

$$a_1 \equiv q^2+m^2\,, \quad a_2 \equiv (p-q)^2+m^2$$

one would like to have a non-factorized denominator. One applies Feynman combining to have

$$\begin{aligned}\widetilde{\Gamma}(p^2) &= \frac{\lambda^2}{2}\int \frac{\mathrm{d}^4q}{(2\pi)^4}(2pq-p^2)2\int_0^1 \mathrm{d}z_1\,\mathrm{d}z_2\frac{\delta(z_1+z_2-1)z_1}{(z_1a_1+z_2a_2)^3}\\ &= \lambda^2\int \frac{\mathrm{d}^4q}{(2\pi)^4}(2pq-p^2)\int_0^1 \mathrm{d}z\,\frac{z}{(za_1+(1-z)a_2)^3}\end{aligned}$$

At the second line one integrates in z_2 the Dirac delta. The denominator may be rewritten as

$$\begin{aligned}za_1+(1-z)a_2 &= z(q^2+m^2)+(1-z)[(q-p)^2+m^2] = q^2-2pq(1-z)+m^2+p^2(1-z)\\ &= [q-(1-z)p]^2+m^2+p^2(1-z)-p^2(1-z)^2\\ &= [q-(1-z)p]^2+m^2+p^2(1-z)z\end{aligned}$$

At the second line, one would like to remove the mixed products to have the integrand in terms of squared variables, so one may complete the square by adding and subtracting $p^2(1-z)^2$.

Since the integral is in the variable q, one can perform a change of variable by a constant shift

$$q' = q-(1-z)p\,, \quad q = q'+(1-z)p$$

Notice that in Feynman combining, the integration in the Feynman parameter is the last integral to be done. Swapping the order of integration, one obtains

$$\begin{aligned}\widetilde{\Gamma}(p^2) &= \lambda^2\int_0^1 \mathrm{d}z\,z\int \frac{\mathrm{d}^4q'}{(2\pi)^4}\frac{2p[q'+p(1-z)]-p^2}{[q'^2+m^2+p^2z(1-z)]^3}\\ &= \lambda^2\int_0^1 \mathrm{d}z'\,(1-z')\int \frac{\mathrm{d}^4q'}{(2\pi)^4}\frac{2pq'+(2z'-1)p^2}{[q'^2+m^2+p^2z'(1-z')]^3}\,, \quad z'=1-z\\ &= \lambda^2p^2\int_0^1 \mathrm{d}z'\,(1-z')(2z'-1)\int \frac{\mathrm{d}^4q'}{(2\pi)^4(q'^2+A^2)^3}\,, \quad A^2 = m^2+p^2z'(1-z')\\ &= \lambda^2p^2\int_0^1 \mathrm{d}z'\,(1-z')(2z'-1)2\pi^2\int_0^\infty \frac{\mathrm{d}q'}{(2\pi)^4}\,\frac{q'^3}{(q'^2+A^2)^3}\\ &= \lambda^2p^2\int_0^1 \mathrm{d}z'\,(1-z')(2z'-1)\pi^2\int_0^\infty \frac{\mathrm{d}q'^2}{(2\pi)^4}\,\frac{q'^2}{(q'^2+A^2)^3}\\ &= \lambda^2p^2\frac{\pi^2}{(2\pi)^4}\int_0^1 \mathrm{d}z'\,(1-z')(2z'-1)\int_0^\infty \mathrm{d}x\,\frac{x}{(x+A^2)^3}\,, \quad x=q'^2\\ &= \frac{\lambda^2p^2}{16\pi^2}\frac{\Gamma(2)\Gamma(1)}{\Gamma(3)}\int_0^1 \mathrm{d}z'\frac{(1-z')(2z'-1)}{A^2}\\ &= \frac{\lambda^2}{32\pi^2}p^2\int_0^1 \mathrm{d}z'\frac{(1-z')(2z'-1)}{p^2z'(1-z')+m^2}\end{aligned}$$

At the second line, the first addendum of the integrand gives a null contribution since it is an odd function. At the fourth line, one goes to spherical coordinates. At the penultimate line, one applies Euler's Beta function with $m=2$ and $n=3$. In the integral on the last line, one has to compute the discriminant of the denominator (in Minkowski space? [r])

$$\mathrm{D} = -(z'^2p^2-z'p^2+m^2) \implies \Delta = p^4-4m^2p^2 = p^2(p^2-4m^2)$$

and split the integration region based on the sign

$$\Delta>0 \iff p^2<0\vee p^2>4m^2\,, \qquad \Delta<0 \iff 0<p^2<4m^2\,,$$

One has

$$\begin{aligned}
\widetilde{\Gamma}(p^2) &= \frac{\lambda^2}{32\pi^2}\left[2+\left(\frac{4m^2-p^2}{|p^2|}\right)^{\frac{1}{2}}\ln\frac{(4m^2-p^2)^{\frac{1}{2}}-|p^2|^{\frac{1}{2}}}{(4m^2-p^2)^{\frac{1}{2}}+|p^2|^{\frac{1}{2}}}\right], \quad p^2<0\\
&= \frac{\lambda^2}{32\pi^2}\left[2-2\left(\frac{4m^2-p^2}{p^2}\right)^{\frac{1}{2}}\arctan\left(\frac{p^2}{4m^2-p^2}\right)^{\frac{1}{2}}\right], \quad 0<p^2<4m^2\\
&= \frac{\lambda^2}{32\pi^2}\left[2+\left(\frac{p^2-4m^2}{p^2}\right)^{\frac{1}{2}}\ln\frac{(p^2)^{\frac{1}{2}}-(p^2-4m^2)^{\frac{1}{2}}}{(p^2)^{\frac{1}{2}}+(p^2-4m^2)^{\frac{1}{2}}}\right], \quad p^2>4m^2
\end{aligned}$$

Divergent part. The divergent part is

$$\begin{aligned}
\Gamma(0) &= \frac{\lambda^2}{2}\Lambda^2\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{(q^2+m^2)^2(q^2+\Lambda^2)} = \frac{\lambda^2}{2}\Lambda^2\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{a_1^2a_2}\\
&= \lambda^2\Lambda^2\int_0^1\mathrm{d}z\,z\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{[q^2+m^2z+\Lambda^2(1-z)]^3}\\
&= \lambda^2\Lambda^2\int_0^1\mathrm{d}z\,z\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{(q^2+B^2)^3}, \quad B^2\equiv m^2z+\Lambda^2(1-z)\\
&= \frac{\lambda^2\Lambda^2}{16\pi^2}\int_0^1\mathrm{d}z\,z\int_0^\infty\mathrm{d}x\,\frac{x}{(x+B^2)^3}, \quad x\equiv q^2\\
&= \frac{\lambda^2\Lambda^2}{16\pi^2}\frac{1}{2}\int_0^1\mathrm{d}z\,\frac{z}{m^2z+\Lambda^2(1-z)} = \frac{\lambda^2}{32\pi^2}\int_0^1\mathrm{d}z\,\frac{z}{z(\frac{m^2}{\Lambda^2}-1)+1}\\
&= \frac{\lambda^2}{32\pi^2}\frac{1}{(\frac{m^2}{\Lambda^2}-1)^2}\int_0^{\frac{m^2}{\Lambda^2}-1}\mathrm{d}z'\,\frac{z'}{z'+1}, \quad z'\equiv z\left(\frac{m^2}{\Lambda^2}-1\right)\\
&= \frac{\lambda^2}{32\pi^2}\frac{1}{(\frac{m^2}{\Lambda^2}-1)^2}\int_0^{\frac{m^2}{\Lambda^2}-1}\mathrm{d}z'\,\frac{z'+1-1}{z'+1}\\
&= \frac{\lambda^2}{32\pi^2}\frac{1}{(\frac{m^2}{\Lambda^2}-1)^2}\left[\left(\frac{m^2}{\Lambda^2}-1\right)-\ln\frac{m^2}{\Lambda^2}\right]\\
&\sim \frac{\lambda^2}{32\pi^2}\ln\frac{\Lambda^2}{m^2}, \quad \Lambda\to\infty
\end{aligned}$$

At the second line one applies Feynman combining giving

$$a_1\equiv q^2+m^2, \quad a_2\equiv q^2+\Lambda^2 \implies [a_1z+a_2(1-z)]^3 = [q^2+m^2z+\Lambda^2(1-z)]^3$$

At the fourth line, one goes to spherical coordinates

$$\frac{\mathrm{d}^4q}{(2\pi)^4} = \frac{\mathrm{d}q^2}{(2\pi)^4}\pi^2q^2$$

At the fifth line, one applies Euler's Beta function. One may study the dependence on the cutoff parameter Λ. One expects the integral to depend on it. The denominator of the integral is

$$m^2z+\Lambda^2(1-z) = z(m^2-\Lambda^2)+\Lambda^2 = \Lambda^2\left[z\left(\frac{m^2}{\Lambda^2}-1\right)+1\right]$$

One sees that the divergence is logarithmic in accordance with the mass dimension of the divergent part.

Uniting the two results at one loop, the vertex function of the four-point Green's function is

$$\boxed{\Gamma(p^2)\Big|_{1\mathrm{L}} = \frac{\lambda^2}{32\pi^2}\ln\frac{\Lambda^2}{m^2}+\text{finite}}$$

Notice that the finite term is not only the finite part calculated above, but also the finite contributions arising from the divergent part.

One-loop contributions — two-point Green's function. The one-loop contribution to the two-point Green's function is

$$\Gamma^{(2)}(p^2)|_{1\mathrm{L}} = -\Sigma(p^2)\,, \quad \Sigma(p^2) = -\frac{\lambda}{2}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{q^2+m^2}$$

Since the integral is quadratically divergent, one must regularize more carefully

$$\begin{aligned}\frac{1}{q^2+m^2} &\to \frac{1}{q^2+m^2} + \frac{a_1}{q^2+\Lambda_1^2} + \frac{a_2}{q^2+\Lambda_2^2} \\ &= \frac{q^4(1+a_1+a_2) + q^2(\Lambda_1^2+\Lambda_2^2+a_1m^2+a_1\Lambda_2^2+a_2m^2+\Lambda_1^2a_2)}{(q^2+m^2)(q^2+\Lambda_1^2)(q^2+\Lambda_2^2)} \\ &\quad + \frac{\Lambda_1^2\Lambda_2^2+m^2a_1\Lambda_2^2+m^2a_2\Lambda_1^2}{(q^2+m^2)(q^2+\Lambda_1^2)(q^2+\Lambda_2^2)}\end{aligned}$$

One fixes the constants a_1 and a_2 such that the momentum q does not appear in the numerator. One sets

$$1+a_1+a_2 = 0\,, \quad \Lambda_1^2+\Lambda_2^2+m^2(a_1+a_2)+a_1\Lambda_2^2+a_2\Lambda_1^2 = 0$$

which are simultaneously solved by

$$a_1 = \frac{m^2-\Lambda_2^2}{\Lambda_2^2-\Lambda_1^2}\,, \quad a_2 = \frac{\Lambda_1^2-m^2}{\Lambda_2^2-\Lambda_1^2}$$

The integrand becomes

$$\frac{1}{q^2+m^2} \to \frac{(\Lambda_1^2-m^2)(\Lambda_2^2-m^2)}{(q^2+m^2)(q^2+\Lambda_1^2)(q^2+\Lambda_2^2)} \sim \frac{\Lambda^4}{(q^2+m^2)(q^2+\Lambda^2)^2}$$

where one remembers and notices

$$\Lambda_{1,2} \gg m\,, \quad \Lambda_1 \sim \Lambda_2 \equiv \Lambda$$

As such, the contribution is regularized

$$\Sigma(p^2) = -\frac{\lambda}{2}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\Lambda^4}{(q^2+m^2)(q^2+\Lambda^2)^2}$$

Proceeding as before, one has

$$\begin{aligned}\Sigma(p^2) &= -\frac{\lambda}{2}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\Lambda^4}{a_2a_1^2}\,, \quad a_1 \equiv q^2+\Lambda^2\,, \quad a_2 \equiv q^2+m^2 \\ &= -\frac{\lambda}{2}\Lambda^4\int_0^1 \mathrm{d}z\, z\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{2}{[z(q^2+\Lambda^2)+(1-z)(q^2+m^2)]^3} \\ &= -\frac{\lambda}{2}\Lambda^4\int_0^1 \mathrm{d}z\, z\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{2}{[q^2+z\Lambda^2+(1-z)m^2]^3} \\ &= -\frac{\lambda}{2}\Lambda^4\int_0^1 \mathrm{d}z\, z\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{2}{[q^2+A^2]^3}\,, \quad A^2 \equiv z\Lambda^2+(1-z)m^2 \\ &= -\lambda\Lambda^4\int_0^1 \mathrm{d}z\, z\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{(q^2+A^2)^3} \\ &= -\frac{\lambda\Lambda^4}{(2\pi)^4}\int_0^1 \mathrm{d}z\, z\pi^2\int \mathrm{d}q^2\,\frac{q^2}{(q^2+A^2)^3} \\ &= -\frac{\lambda\Lambda^4}{32\pi^2}\int_0^1 \mathrm{d}z\,\frac{z}{z\Lambda^2+(1-z)m^2} = -\frac{\lambda\Lambda^4}{32\pi^2}\int_0^1 \mathrm{d}z\,\frac{z}{z(\Lambda^2-m^2)+m^2} \\ &= -\frac{\lambda\Lambda^4}{32\pi^2}\frac{1}{(\Lambda^2-m^2)^2}\int_0^{\Lambda^2-m^2}\mathrm{d}z'\,\frac{z'}{z'+m^2}\,, \quad z' \equiv z(\Lambda^2-m^2) \\ &= -\frac{\lambda\Lambda^4}{32\pi^2}\frac{1}{(\Lambda^2-m^2)^2}\left[(\Lambda^2-m^2) - m^2\ln\frac{\Lambda^2}{m^2}\right] \\ &\sim -\frac{\lambda}{32\pi^2}\Lambda^2\,, \quad \Lambda\to\infty\end{aligned}$$

Therefore, the one-loop contribution to the two-point Green's function is

$$\boxed{\Sigma(p^2)\Big|_{1\mathrm{L}} = -\frac{\lambda}{32\pi^2}\Lambda^2 + \text{finite}}$$

6.3 Dimensional regularization

The dimensional regularization is simpler than covariant regularization. It is useful to calculate again the previous Green's functions to see how the divergences appear. Dimensional regularization is based on the observation that the superficial degree of divergence of a diagram is given by the difference between the number of momenta in the numerator and the number of momenta in the denominator. The numerator also includes the momenta coming from the measure. One observes that appropriately by decreasing the dimension of the momentum phase-space, one obtains converging integrals. One promotes the measure to n number of dimensions

$$\mathrm{d}^4q \to \mathrm{d}^nq\,, \qquad q = (q^0, \ldots, q^3) \to q = (q^0 \ldots, q^{n-1})$$

One may analytically continue the dimension in the complex plane and continue the measure. Integrals are of the type

$$I(p,n) \equiv \int \mathrm{d}^nq\, F(q,p)$$

The analytic continuation of this integral is meaningful for $\mathrm{Re}\, n < 4$. Increasing the number of dimensions means increasing the powers of momenta in the numerator, worsening the degree of divergence [r]. For $\mathrm{Re}\, n \geq 4$ singularities appear as simple? [r] poles at $n = 4$ (See Ramond for example for Euler's and Weyl's expansion of the Gamma function).

When one moves to n dimensions, the mass dimensions of the parameters φ, m and λ change since the action must remain dimensionless. The action is

$$S = \int \mathrm{d}^nx\, \left[\frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4\right]$$

The mass dimension of the Lagrangian has to be n. Therefore

$$\begin{aligned}
\dim[\partial_\mu\varphi\,\partial^\mu\varphi] = n \implies 2\dim\varphi + 2 = n \implies \dim\varphi = \frac{n-2}{2} \\
\dim[m^2\varphi^2] = n \implies 2\dim m + 2\frac{n-2}{2} = n \implies \dim m = 1 \\
\dim[\lambda\varphi^4] = n \implies \dim\lambda + 4\dim\varphi = n \implies \dim\lambda = 4 - n
\end{aligned}$$

One notices that the coupling constant is no longer dimensionless. To keep it such, one introduces a mass scale k and replaces

$$\lambda \to k^{4-n}\lambda = (k^2)^{2-\frac{n}{2}}\lambda$$

where now the right-hand side coupling constant λ is dimensionless. The action becomes

$$S = \int \mathrm{d}^nx\, \left[\frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda(k^2)^{2-\frac{n}{2}}}{4!}\varphi^4\right]$$

In the Feynman rules, every time a vertex appears, one has to put

$$\frac{\lambda}{4!}(k^2)^{2-\frac{n}{2}}$$

Remark. When using covariant regularization one introduces a mass parameter Λ. In dimensional regularization a mass parameter k is also introduced.

Two-point Green's function. One may keep n generic, but the convention is to set

$$n = 4 - 2\varepsilon$$

There is also the convention $n = 4-\varepsilon$ (for example Ramond?). In n dimensions, the contribution to the two-point Green's function is

$$\Sigma(p^2) = -\frac{\lambda}{2}(k^2)^{2-\frac{n}{2}} \int \frac{\mathrm{d}^n q}{(2\pi)^n} \frac{1}{q^2+m^2}$$

Following the previous procedure, Feynman combining is not needed, so one may go directly to spherical coordinates

$$\int \mathrm{d}^n q = \frac{2\pi^{\frac{n}{2}}}{\Gamma(n/2)} \int_0^\infty \mathrm{d}q\, q^{n-1} = \frac{\pi^{\frac{n}{2}}}{\Gamma(n/2)} \int_0^\infty \mathrm{d}q^2\, (q^2)^{\frac{n}{2}-1}$$

Therefore

$$\begin{aligned}
\Sigma(p^2) &= -\frac{\lambda}{2}(k^2)^{2-\frac{n}{2}} \frac{1}{(2\pi)^n} \frac{2\pi^{\frac{n}{2}}}{\Gamma(n/2)} \int_0^\infty \mathrm{d}q\, \frac{q^{n-1}}{q^2+m^2} \\
&= -\frac{\lambda}{2}(k^2)^{2-\frac{n}{2}} \frac{1}{(2\pi)^n} \frac{\pi^{\frac{n}{2}}}{\Gamma(n/2)} \int_0^\infty \mathrm{d}x\, \frac{x^{\frac{n}{2}-1}}{x+m^2}\,, \quad x \equiv q^2 \\
&= -\frac{\lambda}{2}(k^2)^{2-\frac{n}{2}} \frac{\pi^{\frac{n}{2}}}{(2\pi)^n \Gamma(n/2)} \frac{\Gamma(n/2)\Gamma(1-n/2)}{\Gamma(1)(m^2)^{1-\frac{n}{2}}} \\
&= -\frac{\lambda(k^2)^{2-\frac{n}{2}}}{2^{n+1}\pi^{\frac{n}{2}}} \frac{\Gamma(1-n/2)}{(m^2)^{1-\frac{n}{2}}} = -\frac{\lambda m^2}{32\pi^2} \left[\frac{4\pi k^2}{m^2}\right]^{2-\frac{n}{2}} \Gamma(1-n/2) \\
&= -\frac{\lambda m^2}{32\pi^2} \left[\frac{4\pi k^2}{m^2}\right]^{\varepsilon} \Gamma(-1+\varepsilon)\,, \quad n = 4-2\varepsilon \\
&\sim \frac{1}{\varepsilon}\frac{\lambda m^2}{32\pi^2}\,, \quad \varepsilon \to 0
\end{aligned}$$

At the third line, one applies Euler's integral with $m = \frac{n}{2}$ and $n_{\text{Euler}} = 1$. One may notice that $\Gamma(n/2)$ cancels and this always happens, if it does not, there has to be a mistake. At the fifth line, one notes that Euler's Gamma function $\Gamma(z)$ is defined for $\operatorname{Re} z > 0$ but can be analytically continued using the following identity

$$\Gamma(z+1) = z\Gamma(z) \implies \Gamma(z) = \frac{\Gamma(z+1)}{z}$$

Therefore

$$\Gamma(-1+\varepsilon) = \frac{\Gamma(\varepsilon)}{-1+\varepsilon} = \frac{\Gamma(1+\varepsilon)}{\varepsilon(-1+\varepsilon)} \sim -\frac{1}{\varepsilon}\,, \quad \varepsilon \to 0$$

The coefficient of the Gamma function is a pole ε^{-1} signal of the ultraviolet divergence.

Lecture 14

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See Ramond, §§4.3, 4.4. One would like to isolate the finite part. This can be done with a more careful expansion in the infinitesimal parameter ε. Consider the following general identity (see Ramond, eq. 4.3.17)

$$\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\varepsilon} + \psi(n+1) + \frac{\varepsilon}{2}\left(\frac{\pi^2}{3} + \psi^2(n+1) - \psi'(n+1)\right) + o(\varepsilon)\right]\,, \quad n \in \mathbb{N}$$

where ψ and ψ' are polygamma functions. In particular ψ is the digamma function

$$\psi(s) \equiv \mathrm{d}_s \ln\Gamma(s)\,, \quad \psi(n+1) = \sum_{k=1}^{n} \frac{1}{k} - \gamma\,, \quad \psi(1) = -\gamma$$

with γ the Euler–Mascheroni constant. The function ψ' is the trigamma function

$$\psi' = \mathrm{d}_s\psi = \mathrm{d}_z^2 \ln\Gamma(s)\,, \quad \psi'(n+1) = \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2}\,, \quad \psi'(1) = \frac{\pi^2}{6}$$

One may expand

$$\begin{aligned}
\Sigma(p^2) &= -\frac{\lambda m^2}{32\pi^2}\left[\frac{4\pi k^2}{m^2}\right]^{\varepsilon}\Gamma(-1+\varepsilon) = -\frac{\lambda m^2}{32\pi^2}\mathrm{e}^{\varepsilon\ln\frac{4\pi k^2}{m^2}}\Gamma(-1+\varepsilon) \\
&= -\frac{\lambda m^2}{32\pi^2}\left[1+\varepsilon\ln\frac{4\pi k^2}{m^2} + \frac{1}{2}\varepsilon^2\ln^2\frac{4\pi k^2}{m^2} + o(\varepsilon^2)\right] \\
&\quad \times(-1)\left[\frac{1}{\varepsilon} + \psi(2) + \frac{\varepsilon}{2}\left(\frac{\pi^2}{3} + \psi^2(2) - \psi'(2)\right) + o(\varepsilon)\right]\,, \quad n=1 \\
&= \frac{\lambda m^2}{32\pi^2}\left[\frac{1}{\varepsilon} + \psi(2) + \ln\frac{4\pi k^2}{m^2} + \varepsilon\left(\frac{\pi^2}{6} + \frac{1}{2}\psi^2(2) - \frac{1}{2}\psi'(2)\right.\right. \\
&\quad \left.\left. + \psi(2)\ln\frac{4\pi k^2}{m^2} + \frac{1}{2}\ln^2\frac{4\pi k^2}{m^2}\right) + o(\varepsilon)\right] \\
&\sim \boxed{\frac{\lambda m^2}{32\pi^2}\left[\frac{1}{\varepsilon} + \psi(2) + \ln\frac{4\pi k^2}{m^2}\right]}\,, \quad \varepsilon\to 0
\end{aligned}$$

Remark. At one loop, the ultraviolet divergences appear as simple poles in the infinitesimal parameter ε.

Remark. The finite part is an arbitrary contribution because it depends on the choice of the energy scale k^2. One may later observe that it is dependent on the regularization scheme.

Four-point Green's function at one-loop. Let p be the total momentum carried through the diagram. The divergent one-loop contribution to the vertex function is

$$\begin{aligned}
\Gamma(p^2) &= \frac{\lambda^2}{2}(k^2)^{4-n}\int\frac{\mathrm{d}^n q}{(2\pi)^n}\frac{1}{(q^2+m^2)[(p-q)^2+m^2]} \\
&= \frac{\lambda^2}{2}(k^2)^{4-n}\int\frac{\mathrm{d}^n q}{(2\pi)^n}\frac{1}{a_1 a_2}\,, \quad a_1 \equiv q^2+m^2\,, \quad a_2 \equiv (p-q)^2+m^2 \\
&= \frac{\lambda^2}{2}(k^2)^{4-n}\int_0^1\mathrm{d}z\int\frac{\mathrm{d}^n q}{(2\pi)^n}\frac{1}{[z(q^2+m^2)+(1-z)[(p-q)^2+m^2]]^2} \\
&= \frac{\lambda^2}{2}(k^2)^{4-n}\int_0^1\mathrm{d}z\int\frac{\mathrm{d}^n q'}{(2\pi)^n}\frac{1}{[q'^2+m^2+z(1-z)p^2]^2}\,, \quad q' \equiv q - p(1-z) \\
&= \frac{\lambda^2}{2}(k^2)^{4-n}\frac{2\pi^{\frac{n}{2}}}{\Gamma(n/2)}\frac{1}{(2\pi)^n}\frac{1}{2}\int_0^1\mathrm{d}z\int_0^\infty\mathrm{d}q'^2\,\frac{(q'^2)^{\frac{n}{2}-1}}{(q'^2+A^2)^2}\,, \quad A^2 \equiv m^2+z(1-z)p^2 \\
&= \frac{\lambda^2}{2}\frac{(k^2)^{4-n}}{(4\pi)^{\frac{n}{2}}}\frac{1}{\Gamma(n/2)}\frac{\Gamma(n/2)\Gamma(2-n/2)}{\Gamma(2)}\int_0^1\mathrm{d}z\,\frac{1}{(A^2)^{2-\frac{n}{2}}} \\
&= \frac{\lambda^2}{2}\frac{(k^2)^{2\varepsilon}}{(4\pi)^{2-\varepsilon}}\Gamma(\varepsilon)\int_0^1\mathrm{d}z\,\frac{1}{[m^2+z(1-z)p^2]^\varepsilon}\,, \quad n = 4-2\varepsilon \\
&= \frac{\lambda^2}{2}\frac{\Gamma(\varepsilon)}{(4\pi)^2}(k^2)^\varepsilon\frac{1}{(4\pi k^2)^{-\varepsilon}}\int_0^1\mathrm{d}z\,\frac{1}{[m^2+z(1-z)p^2]^\varepsilon} \\
&= \frac{\lambda^2}{32\pi^2}(k^2)^\varepsilon\Gamma(\varepsilon)\int_0^1\mathrm{d}z\,\frac{1}{[\frac{m^2}{4\pi k^2}+z(1-z)\frac{p^2}{4\pi k^2}]^\varepsilon} \\
&= \frac{\lambda^2}{32\pi^2}(k^2)^\varepsilon\left[\frac{1}{\varepsilon} - \gamma + \frac{\varepsilon}{2}\left(\frac{\pi^2}{3} + \gamma^2 - \frac{\pi^2}{6}\right) + o(\varepsilon)\right] \\
&\quad \times\int_0^1\mathrm{d}z\,\left[1-\varepsilon\ln\frac{m^2+z(1-z)p^2}{4\pi k^2} + o(\varepsilon)\right]
\end{aligned}$$

$$= \frac{\lambda^2}{32\pi^2}(k^2)^\varepsilon \left[\frac{1}{\varepsilon} - \gamma - \int_0^1 \mathrm{d}z\, \ln \frac{m^2 + z(1-z)p^2}{4\pi k^2} + o(\varepsilon^0)\right]$$
$$\sim \frac{\lambda^2}{32\pi^2}(k^2)^\varepsilon \left[\frac{1}{\varepsilon} - \gamma - \ln \frac{m^2}{4\pi k^2} - \int_0^1 \mathrm{d}z\, \ln\left(1 + \frac{z(1-z)p^2}{m^2}\right)\right], \quad \varepsilon \to 0$$

At the fourth and third to last lines, one expands the Gamma function and rewrites the fraction as an exponential and expands that too. At the last line one can apply

$$\int_0^1 \mathrm{d}z\, \ln\left[1 + \frac{4}{\alpha} z(1-z)\right] = -2 + \sqrt{1+a} \ln \frac{\sqrt{1+a}+1}{\sqrt{1+a}-1}, \quad a > 0$$

where one sets $a = \frac{4m^2}{p^2}$. Therefore

$$\boxed{\Gamma(p^2)\Big|_{1\mathrm{L}} = \frac{\lambda^2}{32\pi^2}(k^2)^\varepsilon \left[\frac{1}{\varepsilon} - \gamma - \ln \frac{m^2}{4\pi k^2} + 2 - \sqrt{1 + \frac{4m^2}{p^2}} \ln \frac{\sqrt{1+\frac{4m^2}{p^2}}+1}{\sqrt{1+\frac{4m^2}{p^2}}-1}\right]}$$

The factor $(k^2)^\varepsilon$ is kept in front instead of being expanded because the four-point function gives a contribution to the vertex which is associated to the coupling constant $(k^2)^\varepsilon \lambda$ in n dimensions. When calculating the divergent term, one may expand it by rewriting it as an exponential.

The total contribution to the four-point Green's function includes all vertex functions of three the Mandelstam variables

$$\Gamma^{(4)}(s,t,u) = \lambda_0 + [\Gamma(s) + \Gamma(t) + \Gamma(u)]_{1\mathrm{L}} = \lambda + \frac{3\lambda^2}{32\pi^2}\frac{1}{\varepsilon} + \text{finite}$$

Summary. A brief summary of the one-loop contributions follows

	Conventional	Dimensional
$\Sigma(0)$	$-\frac{\lambda}{32\pi^2}\Lambda^2$	$\frac{\lambda m^2}{32\pi^2}\frac{1}{\varepsilon}$
$\Gamma(0)$	$\frac{\lambda^2}{32\pi^2}\ln\frac{\Lambda^2}{m^2}$	$\frac{\lambda^2}{32\pi^2}\frac{1}{\varepsilon}$

All the one-loop divergences are simple poles. The mass dimensions between the two methods match: 2 for Σ and 0 for Γ. In this way one can subtract divergences [r].

6.4 BPHZ one-loop renormalization

See Ramond?. The Euclidean Lagrangian of the $\lambda\varphi^4$ theory is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi + \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4$$

One has to add a counter term Lagrangian that balances the divergences of the original Lagrangian

$$\mathcal{L}^{(1)}_{\mathrm{ct}} = \frac{1}{2}A\,\partial_\mu\varphi\,\partial^\mu\varphi + \frac{m^2}{2}B\varphi^2 + \frac{\lambda}{4!}C\varphi^4$$

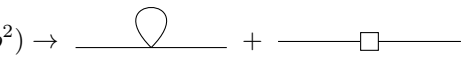
The parameters provide two extra interaction points. The first two give a two-point interaction $-A$ and $-Bm^2$, while the third gives a four-point interaction $-C\lambda$ (notice the minus since the formalism is Euclidean).

In fact, keeping in mind the minus sign in the exponential of the generating functional in the Euclidean signature coming from the minus sign of the action

$$W[J] = \int [\mathcal{D}\varphi]\, \mathrm{e}^{-(S+J\varphi)} = \int [\mathcal{D}\varphi]\, \exp\left[-S_0 - \int \mathrm{d}^4x \left(\frac{\lambda}{4!}\varphi^4 + \mathcal{L}^{(1)}_{\mathrm{ct}}\right) - \int \mathrm{d}^4x\, J\varphi\right]$$
$$= \int [\mathcal{D}\varphi]\, \exp\left[-\int \mathrm{d}^4x \left(\frac{\lambda}{4!}(-\delta_J)^4 + \mathcal{L}^{(1)}_{\mathrm{ct}}(-\delta_J)\right)\right] \mathrm{e}^{-S_0 - \int \mathrm{d}^4x\, J\varphi}$$
$$= \exp\left[-\int \mathrm{d}^4x \left(\frac{\lambda}{4!}(-\delta_J)^4 + \mathcal{L}^{(1)}_{\mathrm{ct}}(-\delta_J)\right)\right] W_0[J]$$

The contributions are given by $-A$, $-Bm^2$ and $-C\lambda$.

One-loop contribution to the two-point Green's function. At one-loop, the expression of $\Sigma(p^2)$ does not depend on p^2 so the choice of the subtraction point is arbitrary. This means that the parameter A does not contribute since its term in the Lagrangian is the Fourier transform of p^2 (to see this one may integrate by parts first to obtain a d'Alembertian). Therefore, the only contribution comes from $-Bm^2$

$\Sigma(p^2) \to$  $+$

In covariant regularization one has [r]

$$\Sigma(0) - m^2 B = -\frac{\lambda}{32\pi^2}\Lambda^2 - m^2 B \equiv \text{finite} \implies B = -\frac{\lambda}{32\pi^2}\frac{\Lambda^2}{m^2} + \text{finite}$$

Generally one has

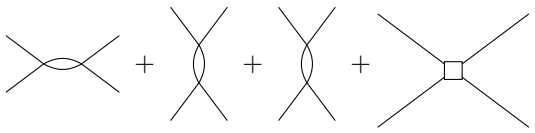
$$m^2 B = \Sigma(0) + \text{finite}$$

The finite part can be arbitrary since the parameter B can contain a part that needs to cancel the divergence and an arbitrary finite part. [r] The finite part in B defines a particular subtraction scheme: the arbitrariness is called scheme dependence. In general, finite parts depend on the choice of the subtraction scheme.

In dimensional regularization, the contribution is

$$\frac{\lambda m^2}{32\pi^2}\frac{1}{\varepsilon} - m^2 B \equiv \text{finite} \implies B = \frac{\lambda}{32\pi^2}\frac{1}{\varepsilon} + \text{finite}$$

One-loop contribution to the four-point Green's function. At one-loop, the Green's function receives contributions from the following diagrams



corresponding to s, t, u and $-\lambda C$.

In covariant regularization, one has

$$\frac{3\lambda^2}{32\pi^2}\ln\frac{\Lambda^2}{m^2} - \lambda C \equiv \text{finite} \implies \lambda C = \frac{3\lambda^2}{32\pi^2}\ln\frac{\Lambda^2}{m^2} + \text{finite} = 3\Gamma(0) + \text{finite}$$

In dimensional regularization, one has

$$\lambda C = 3\Gamma(0) + \text{finite} = \frac{3\lambda^2}{32\pi^2}\frac{1}{\varepsilon} + \text{finite}$$

Total renormalization. Substituting the above expressions into the counter term Lagrangian, one finds

$$\begin{aligned}
\mathcal{L} + \mathcal{L}^{(1)}_{\text{ct}} &= \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi + \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4 + \frac{1}{2}\Sigma(0)\varphi^2 + \frac{3}{4!}\Gamma(0)\varphi^4 \\
&= \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi + \frac{1}{2}[m^2 + \Sigma(0)]\varphi^2 + \frac{1}{4!}[\lambda + 3\Gamma(0)]\varphi^4 \\
&\equiv \frac{1}{2}Z_\varphi\,\partial_\mu\varphi\,\partial^\mu\varphi + \frac{1}{2}(m^2 + \delta m^2)\varphi^2 + \frac{\lambda}{4!}Z_\lambda\varphi^4
\end{aligned}$$

One has found the renormalization functions at one-loop

$$Z_\varphi = 1\,, \quad \delta m^2 = \Sigma(0)\,, \quad \lambda Z_\lambda = \lambda + 3\Gamma(0)$$

The divergences have been encoded into the renormalization functions above. The renormalized parameters are finite while the bare parameters are divergent because they are expressed in terms of the renormalized parameters times the renormalization constants

$$\varphi_0 = Z_\varphi^{\frac{1}{2}}\varphi\,, \quad m_0^2 = m^2 + \delta m^2\,, \quad \lambda_0 = Z_\lambda Z_\varphi^{-2}\lambda$$

[r]

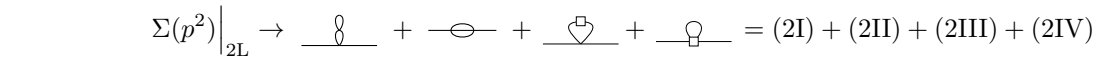
The next step is to calculate the renormalization of $\widetilde{\mathcal{L}} = \mathcal{L} + \mathcal{L}_{\text{ct}}^{(1)}$ at two loops. In fact, one would like to study if the renormalization function $\Sigma(0)$ does have a non-zero derivative; to study the power of the poles in dimensional regularization; and to study what is the role of the one-loop counter terms.

7 Two-loop contributions

See Ramond for two-point, Peskin for four-point.

7.1 Two-point Green's function

The diagrams at two-loop of the two-point Green's function are

$\Sigma(p^2)\Big|_{\text{2L}} \to$  $= (\text{2I}) + (\text{2II}) + (\text{2III}) + (\text{2IV})$

The second diagram is a genuine diagram, while the first is a product of two one-loop diagrams.

First diagram. Naming r the momentum in the upper loop and q the one in the lower loop, the first diagram corresponds to the integral

$$\Sigma(p^2)\Big|_{(\text{2I})} = \frac{\lambda^2}{4}(k^2)^{4-n}\int\frac{\mathrm{d}^n q}{(2\pi)^n}\frac{1}{(q^2+m^2)^2}\int\frac{\mathrm{d}^n r}{(2\pi)^n}\frac{1}{r^2+m^2}$$

The contribution is factorized into two disjointed integrals. The second integral has been already computed for the one-loop case

$$(k^2)^{2-\frac{n}{2}}\int\frac{\mathrm{d}^n r}{(2\pi)^n}\frac{1}{r^2+m^2} = -\frac{2}{\lambda}\Sigma(0)\Big|_{\text{1L}} = -\frac{m^2}{16\pi^2}\left[\frac{1}{\varepsilon}+\psi(2)+\ln\frac{m^2}{4\pi k^2}+o(\varepsilon^0)\right]$$

Exercise. Compute the q-integral in dimensional regularization. One may skip Feynman combining and start from spherical coordinates. One may obtain an intermediate result of

$$(k^2)^{2-\frac{n}{2}}\int\frac{\mathrm{d}^n q}{(2\pi)^n}\frac{1}{(q^2+m^2)^2} = \frac{1}{16\pi^2}\left[\frac{4\pi k^2}{m^2}\right]^\varepsilon\Gamma(\varepsilon)$$

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The lower loop integral is

$$\begin{aligned}
(k^2)^{2-\frac{n}{2}}\int\frac{\mathrm{d}^n q}{(2\pi)^n}\frac{1}{(q^2+m^2)^2} &= \frac{1}{16\pi^2}\left[\frac{4\pi k^2}{m^2}\right]^\varepsilon\Gamma(\varepsilon) = \frac{1}{16\pi^2}\mathrm{e}^{\varepsilon\ln\frac{4\pi k^2}{m^2}}\Gamma(\varepsilon)\\
&= \frac{1}{16\pi^2}\left[1+\varepsilon\ln\frac{4\pi k^2}{m^2}+\frac{1}{2}\ln^2\frac{4\pi k^2}{m^2}+o(\varepsilon^2)\right]\\
&\quad\times\left[\frac{1}{\varepsilon}+\psi(1)+\frac{\varepsilon}{2}\left(\frac{\pi^2}{3}+\psi^2(1)-\psi'(1)\right)+o(\varepsilon)\right]\\
&= \frac{1}{16\pi^2}\left[\frac{1}{\varepsilon}+\psi(1)+\ln\frac{4\pi k^2}{m^2}\right.\\
&\quad\left.+\frac{\varepsilon}{2}\left(\frac{\pi^2}{3}+\psi^2(1)-\psi'(1)+2\psi(1)\ln\frac{4\pi k^2}{m^2}+\ln^2\frac{4\pi k^2}{m^2}\right)+o(\varepsilon)\right]
\end{aligned}$$

The upper integral is

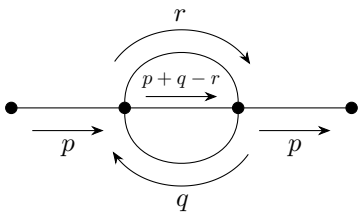
$$(k^2)^{2-\frac{n}{2}}\int\frac{\mathrm{d}^n r}{(2\pi)^n}\frac{1}{r^2+m^2}=-\frac{m^2}{16\pi^2}\left[\frac{1}{\varepsilon}+\psi(2)+\ln\frac{4\pi k^2}{m^2}\right.$$
$$\left.+\frac{\varepsilon}{2}\left(\frac{\pi^2}{3}+\psi^2(2)-\psi'(2)+2\psi(2)\ln\frac{4\pi k^2}{m^2}+\ln^2\frac{4\pi k^2}{m^2}\right)+o(\varepsilon)\right]$$

To find the whole integral, one has to multiply the two above integrals. Both of the factors have simple poles: therefore, at higher loops, the degree of the divergence increases. The final integral is then

$$\Sigma(p^2)\Big|_{(2\mathrm{I})}=-\frac{\lambda^2m^2}{2^{10}\pi^4}\left[\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\left(\psi(1)+\psi(2)+2\ln\frac{4\pi k^2}{m^2}\right)+\frac{\pi^2}{3}+\frac{1}{2}[\psi(1)+\psi(2)]^2\right.$$
$$\left.-\frac{1}{2}[\psi'(1)+\psi'(2)]+2[\psi(1)+\psi(2)]\ln\frac{4\pi k^2}{m^2}+2\ln^2\frac{4\pi k^2}{m^2}+o(\varepsilon^0)\right]$$

See Ramond, eq. (4.4.18). The pole ε^{-2} is independent of the energy scale k so it is universal, while the rest is scheme-dependent.

Second diagram. The second two-loop diagram of the two-point Green's function is a genuine diagram that does not factorize



$$\equiv\Sigma(p^2)\Big|_{(2\mathrm{II})}$$

The diagram is a function of p^2 since the final result has to be a scalar. Also, the diagram is symmetric by sending $p\to-p$, $q\to-q$, $r\to-r$, therefore there cannot be a preferred direction.

The associated integral is

$$\Sigma(p^2)\Big|_{(2\mathrm{II})}=\frac{\lambda^2}{6}(k^2)^{4-n}\int\frac{\mathrm{d}^n k}{(2\pi)^n}\frac{\mathrm{d}^n q}{(2\pi)^n}\frac{1}{(q^2+m^2)(r^2+m^2)[(q-r+p)^2+m^2]}$$

To improve the degree of divergence one may increase the number of propagators through the use of derivatives. One observes that

$$\partial_{r^\mu}r^\mu+\partial_{q^\mu}q^\mu=2n$$

Therefore, one may compute

$$2n\Sigma(p^2)\Big|_{(2\mathrm{II})}=\frac{\lambda^2}{6}(k^2)^{4-n}\int\frac{\mathrm{d}^n k}{(2\pi)^n}\frac{\mathrm{d}^n q}{(2\pi)^n}\frac{(\partial_{r^\mu}r^\mu+\partial_{q^\mu}q^\mu)}{(q^2+m^2)(r^2+m^2)[(q-r+p)^2+m^2]}$$
$$=-\frac{\lambda^2}{6}(k^2)^{4-n}\int\frac{\mathrm{d}^n k\,\mathrm{d}^n q}{(2\pi)^{2n}}(r^\mu\partial_{r^\mu}+q^\mu\partial_{q^\mu})\frac{1}{(q^2+m^2)(r^2+m^2)[(q-r+p)^2+m^2]}$$

At the second line one integrates by parts assuming suitable boundary conditions. Computing the derivatives, one has

$$q^\mu\,\partial_{q^\mu}\frac{1}{q^2+m^2}=-\frac{2q^2}{(q^2+m^2)^2}$$
$$q^\mu\,\partial_{q^\mu}\frac{1}{(q-r+p)^2+m^2}=-\frac{2q^\mu(q-r+p)_\mu}{[(q-r+p)^2+m^2]^2}$$

Similarly for r. Letting

$$F(q,r) \equiv \frac{1}{(q^2+m^2)(r^2+m^2)[(q-r+p)^2+m^2]}$$

one obtains

$$\begin{aligned} q^\mu \, \partial_{q^\mu} F &= -2F\left[\frac{q^2}{q^2+m^2} + \frac{q(q-r+p)}{(q-r+p)^2+m^2}\right] \\ r^\mu \, \partial_{r^\mu} F &= -2F\left[\frac{r^2}{r^2+m^2} - \frac{r(q-r+p)}{(q-r+p)^2+m^2}\right] \end{aligned}$$

Summing the two terms, one has

$$\begin{aligned} (r^\mu \, \partial_{r^\mu} + q^\mu \, \partial_{q^\mu})F &= -2F\left[\frac{q^2}{q^2+m^2} + \frac{r^2}{r^2+m^2} + \frac{(q-r)(q-r+p)}{(q-r+p)^2+m^2}\right] \\ &= -2F\left[2 - \frac{2m^2}{q^2+m^2} + \frac{(q-r+p)^2 - p(q-r+p)}{(q-r+p)^2+m^2}\right] \\ &= -2F\left[3 - \frac{3m^2}{(q^2+m^2)} - \frac{p^\mu (q-r+p)_\mu}{[(q-r+p)^2+m^2]}\right] \end{aligned}$$

At the first line, one may add and subtract the mass from the first two addenda, while completing the square in the last term

$$(q-r)(q-r+p) = (q-r+p)^2 - p(q-r+p)$$

This is done to trade internal momenta in the numerator with external momenta. At the second line, in the second addendum one swaps q with r since they are integration variables. In the third term one may add and subtract the mass, so one obtains a term that can be joined with the first and another that can be joined with the second by letting $q-r+p = -q'$ [r] why minus?.

Therefore, the total integral is

$$\begin{aligned} 2n\Sigma(p^2)\Big|_{(2\text{II})} &= -\frac{\lambda^2}{6}(k^2)^{4-n}\int \frac{\mathrm{d}^n k}{(2\pi)^n}\frac{\mathrm{d}^n q}{(2\pi)^n}\frac{1}{(q^2+m^2)(r^2+m^2)[(q-r+p)^2+m^2]} \\ &\quad \times \left[-6 + \frac{6m^2}{(q^2+m^2)} + \frac{2p^\mu (q-r+p)_\mu}{[(q-r+p)^2+m^2]}\right] \\ &= 6\Sigma(p^2)\Big|_{(2\text{II})} - \frac{\lambda^2}{6}(k^2)^{4-n}\int \frac{\mathrm{d}^n k}{(2\pi)^n}\frac{\mathrm{d}^n q}{(2\pi)^n}\frac{1}{(q^2+m^2)(r^2+m^2)[(q-r+p)^2+m^2]} \\ &\quad \times \left[\frac{6m^2}{(q^2+m^2)} + \frac{2p^\mu (q-r+p)_\mu}{[(q-r+p)^2+m^2]}\right] \end{aligned}$$

At the first line, the first addendum, together with the coefficient, is $6\Sigma(p^2)|_{(2\text{II})}$. Simplifying the present terms, one has

$$\begin{aligned} (n-3)\Sigma(p^2)\Big|_{(2\text{II})} &= -\frac{\lambda^2}{6}(k^2)^{4-n}\int \frac{\mathrm{d}^n k}{(2\pi)^n}\frac{\mathrm{d}^n q}{(2\pi)^n}\frac{1}{(q^2+m^2)(r^2+m^2)[(q-r+p)^2+m^2]} \\ &\quad \times \left[\frac{3m^2}{(q^2+m^2)} + \frac{p^\mu (q-r+p)_\mu}{[(q-r+p)^2+m^2]}\right] \\ &= -\frac{\lambda^2}{6}(k^2)^{4-n}[3m^2 K(p) + p^\mu K_\mu(p)] \end{aligned}$$

where one has

$$\begin{aligned} K(p) &\equiv \int \frac{\mathrm{d}^n r}{(2\pi)^n}\frac{\mathrm{d}^n q}{(2\pi)^n}\frac{1}{(q^2+m^2)^2(r^2+m^2)[(q-r+p)^2+m^2]} \\ K_\mu(p) &\equiv \int \frac{\mathrm{d}^n r}{(2\pi)^n}\frac{\mathrm{d}^n q}{(2\pi)^n}\frac{(q-r+p)_\mu}{(q^2+m^2)(r^2+m^2)[(q-r+p)^2+m^2]^2} \end{aligned}$$

To compute these integrals one may apply Feynman combining with two parameters. Remembering that this two-loop contribution $\Sigma|_{(2\mathrm{II})}$ is symmetric for $p \to -p$, then K is symmetric but K_μ is anti-symmetric.

In each term, the integrals do not factorize, but an order of integration has to be chosen. The first term is

$$\begin{aligned}
K(p) &= \int \frac{\mathrm{d}^n q}{(2\pi)^n} \frac{1}{(q^2+m^2)^2} \int \frac{\mathrm{d}^n r}{(2\pi)^n} \frac{1}{(r^2+m^2)[(r-(q+p))^2+m^2]} \\
&= \frac{\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}} \int_0^1 \mathrm{d}z \, \frac{1}{[z(1-z)]^\varepsilon} \int \frac{\mathrm{d}^n q}{(2\pi)^n} \frac{1}{(q^2+m^2)^2[(q+p)^2+\frac{m^2}{z(1-z)}]^\varepsilon} \\
&= \frac{\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}} \int_0^1 \mathrm{d}z \, \frac{1}{[z(1-z)]^\varepsilon} \int_0^1 \mathrm{d}y \, y^{\varepsilon-1}(1-y)^{2-1} \frac{\Gamma(2+\varepsilon)}{\Gamma(2)\Gamma(\varepsilon)} \\
&\quad \times \int \frac{\mathrm{d}^n q}{(2\pi)^n} \frac{1}{[q^2+y(1-y)p^2+m^2(1-y+\frac{y}{z(1-z)})]^{2+\varepsilon}} \\
&= \frac{\Gamma(2+\varepsilon)}{(4\pi)^{2-\varepsilon}} \int_0^1 \mathrm{d}z \, \frac{1}{[z(1-z)]^\varepsilon} \int_0^1 \mathrm{d}y \, y^{\varepsilon-1}(1-y)^{2-1} \\
&\quad \times \int \frac{\mathrm{d}^n q}{(2\pi)^n} \frac{1}{[q^2+y(1-y)p^2+m^2(1-y+\frac{y}{z(1-z)})]^{2+\varepsilon}} \\
&= \frac{\Gamma(2+\varepsilon)}{(4\pi)^{4-2\varepsilon}} \int_0^1 \mathrm{d}z \, \frac{1}{[z(1-z)]^\varepsilon} \int_0^1 \mathrm{d}y \, y^{\varepsilon-1}(1-y) \frac{1}{\Gamma(2-\varepsilon)} \int_0^\infty \mathrm{d}x \, \frac{x^{1-\varepsilon}}{(x+A^2)^{2+\varepsilon}} \\
&= \frac{\Gamma(2+\varepsilon)}{\Gamma(2-\varepsilon)} \frac{1}{(4\pi)^{4-2\varepsilon}} \frac{\Gamma(2-\varepsilon)\Gamma(2\varepsilon)}{\Gamma(2+\varepsilon)} \int_0^1 \mathrm{d}z \, \frac{1}{[z(1-z)]^\varepsilon} \int_0^1 \mathrm{d}y \, \frac{y^{\varepsilon-1}(1-y)}{(A^2)^{2\varepsilon}} \\
&= \frac{\Gamma(2\varepsilon)}{(4\pi)^{4-2\varepsilon}} \int_0^1 \mathrm{d}z \, \frac{1}{[z(1-z)]^\varepsilon} \int_0^1 \mathrm{d}y \, \frac{y^{\varepsilon-1}(1-y)}{[y(1-y)p^2+m^2(1-y+\frac{y}{z(1-z)})]^{2\varepsilon}} \\
&= -\frac{1}{\varepsilon} \frac{\Gamma(2\varepsilon)}{(4\pi)^{4-2\varepsilon}} \int_0^1 \mathrm{d}z \, \frac{1}{[z(1-z)]^\varepsilon} \int_0^1 \mathrm{d}y \, y^\varepsilon \, \partial_y \frac{(1-y)}{[y(1-y)p^2+m^2(1-y+\frac{y}{z(1-z)})]^{2\varepsilon}} \\
&= -\frac{1}{\varepsilon} \frac{\Gamma(2\varepsilon)}{(4\pi)^{4-2\varepsilon}} \frac{1}{(m^2)^{2\varepsilon}} \int_0^1 \mathrm{d}z \, \frac{1}{[z(1-z)]^\varepsilon} \int_0^1 \mathrm{d}y \, y^\varepsilon \, \partial_y \frac{(1-y)}{[y(1-y)\frac{p^2}{m^2}+(1-y+\frac{y}{z(1-z)})]^{2\varepsilon}} \\
&= -\frac{1}{\varepsilon} \frac{\Gamma(2\varepsilon)}{(4\pi)^{4-2\varepsilon}} \frac{1}{(m^2)^{2\varepsilon}} \int_0^1 \mathrm{d}z \, \frac{1}{[z(1-z)]^\varepsilon} \int_0^1 \mathrm{d}y \, y^\varepsilon \, \partial_y [(1-y)\mathrm{e}^{-2\varepsilon \ln \frac{A^2}{m^2}}] \\
&= \frac{1}{\varepsilon} \frac{\Gamma(2\varepsilon)}{(4\pi)^{4-2\varepsilon}} \frac{1}{(m^2)^{2\varepsilon}} \int_0^1 \mathrm{d}z \, \frac{1}{[z(1-z)]^\varepsilon} \int_0^1 \mathrm{d}y \, y^\varepsilon \left[1+2\varepsilon(1-y)\,\partial_y \ln \frac{A^2}{m^2}\right] \mathrm{e}^{-2\varepsilon \ln \frac{A^2}{m^2}} \\
&= \frac{1}{2\varepsilon^2} \frac{\Gamma(1+2\varepsilon)}{(4\pi)^{4-2\varepsilon}} \frac{1}{(m^2)^{2\varepsilon}} \int_0^1 \mathrm{d}z \, \frac{1}{[z(1-z)]^\varepsilon} \int_0^1 \mathrm{d}y \, y^\varepsilon \left[1+2\varepsilon(1-y)\,\partial_y \ln \frac{A^2}{m^2}\right] \mathrm{e}^{-2\varepsilon \ln \frac{A^2}{m^2}} \\
&\sim \frac{1}{2\varepsilon^2} \frac{1}{(4\pi)^4}, \quad \varepsilon \to 0
\end{aligned}$$

See Ramond, eqs. (4.4.33) and (4.4.35). At the first line, the second integral is a one-loop integral of a propagator with external momentum $q+p$. This integral has already been computed for the four-point Green's function

$$\int \frac{\mathrm{d}^n r}{(2\pi)^n} \frac{1}{(r^2+m^2)[(r-(q+p))^2+m^2]} = \frac{\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}} \int_0^1 \mathrm{d}z \, \frac{1}{[m^2+z(1-z)(q+p)^2]^\varepsilon}$$

for $n = 4-2\varepsilon$. At the second line, one may apply Feynman combining again using the more general identity

$$\prod_{j=1}^n \frac{1}{a_j^{D_j}} = \frac{\Gamma(D_1+D_2+\cdots+D_n)}{\Gamma(D_1)\Gamma(D_2)\cdots\Gamma(D_n)} \int_0^1 \prod_{j=1}^n \mathrm{d}x_j \, \delta(1-\textstyle\sum_{j=1}^n x_j) \Big[\sum_{j=1}^n x_j a_j\Big]^{-D_1-\cdots-D_n} \prod_{j=1}^n x_j^{D_j-1}$$

where D_j are numbers. In this case

$$a_1 \equiv (q+p)^2 + \frac{m^2}{z(1-z)}, \quad a_2 \equiv q^2+m^2, \quad D_1 = \varepsilon, \quad D_2 = 2$$

At the seventh line, one goes to spherical coordinates

$$\int \frac{\mathrm{d}^n q}{(2\pi)^n} f(q^2) = \frac{1}{(2\pi)^n} \frac{\pi^{\frac{n}{2}}}{\Gamma(n/2)} \int_0^\infty \mathrm{d}q^2 \, (q^2)^{\frac{n}{2}-1} f(q^2) = \frac{\pi^{2-\varepsilon}}{(2\pi)^n \Gamma(2-\varepsilon)} \int_0^\infty \mathrm{d}q^2 \, (q^2)^{1-\varepsilon} f(q^2)$$

and one sets

$$x \equiv q^2 \,, \quad A^2 \equiv y(1-y)p^2 + m^2 \left(1 - y + \frac{y}{z(1-z)}\right)$$

At the eight line one applies Euler's Beta function with $m = 2-\varepsilon$ and $n = 2+\varepsilon$. At the tenth line, the y-integral is ill-defined for $\varepsilon \to 0$ so one regularizes it by noting that

$$y^{\varepsilon-1} = \frac{1}{\varepsilon} \, \mathrm{d}_y y^\varepsilon$$

and integrating by parts: the resulting boundary terms are zero.

The leading divergence of the diagram is quadratic ε^{-2} which signal the presence of two poles.

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The second term K_μ has different powers of the propagators from the first term K. One may perform a change of variables to get back to the previous computation

$$q - r + p = -q' \implies q = -(q' - r + p)$$

Therefore

$$\begin{aligned}
p^\mu K_\mu(p) &= \int \frac{\mathrm{d}^n q'}{(2\pi)^n} \int \frac{\mathrm{d}^n r}{(2\pi)^n} \frac{-pq'}{(q'^2+m^2)^2 (r^2+m^2)[(q'-r+p)^2+m^2]} \\
&= -\int \frac{\mathrm{d}^n q'}{(2\pi)^n} \frac{pq'}{(q'^2+m^2)^2} \int \frac{\mathrm{d}^n r}{(2\pi)^n} \frac{1}{(r^2+m^2)[(r-(q'+p))^2+m^2]} \\
&= -\frac{\Gamma(\varepsilon)}{(4\pi)^{4-\varepsilon}} \int_0^1 \mathrm{d}z \int \frac{\mathrm{d}^n q'}{(2\pi)^n} \frac{pq'}{(q'^2+m^2)^2 [m^2 + z(1-z)(q'+p)^2]^\varepsilon} \\
&= -\frac{\Gamma(\varepsilon)}{(4\pi)^{4-\varepsilon}} \int_0^1 \frac{\mathrm{d}z}{[z(1-z)]^\varepsilon} \int \frac{\mathrm{d}^n q'}{(2\pi)^n} \frac{pq'}{(q'^2+m^2)^2 [\frac{m^2}{z(1-z)} + (q'+p)^2]^\varepsilon} \\
&= -\frac{\Gamma(2+\varepsilon)}{(4\pi)^{2-\varepsilon}} \int_0^1 \frac{\mathrm{d}z}{[z(1-z)]^\varepsilon} \int_0^1 \mathrm{d}y \, y^{\varepsilon-1}(1-y) \\
&\quad \times p^\mu \int \frac{\mathrm{d}^n q''}{(2\pi)^n} \frac{(q''-yp)_\mu}{[q''^2 + y(1-y)p^2 + m^2(1-y+\frac{y}{z(1-z)})]^{2+\varepsilon}} \\
&= \frac{\Gamma(2+\varepsilon)}{(4\pi)^{2-\varepsilon}} p^2 \int_0^1 \frac{\mathrm{d}z}{[z(1-z)]^\varepsilon} \int_0^1 \mathrm{d}y \, y^\varepsilon (1-y) \\
&\quad \times \int \frac{\mathrm{d}^n q''}{(2\pi)^n} \frac{1}{[q''^2 + y(1-y)p^2 + m^2(1-y+\frac{y}{z(1-z)})]^{2+\varepsilon}} \\
&= p^2 \frac{\Gamma(2\varepsilon)}{(4\pi)^{4-2\varepsilon}} \int_0^1 \frac{\mathrm{d}z}{[z(1-z)]^\varepsilon} \int_0^1 \mathrm{d}y \, \frac{y^\varepsilon (1-y)}{[q''^2 + y(1-y)p^2 + m^2(1-y+\frac{y}{z(1-z)})]^{2\varepsilon}} \\
&= \frac{p^2}{2\varepsilon(4\pi)^4} \int_0^1 \mathrm{d}y \, (1-y) + \text{finite} = \frac{p^2}{4\varepsilon(4\pi)^4} + \text{finite}
\end{aligned}$$

At the second line, the second integral is a loop integral with external momentum $q' + p$. At the third line, one applied Feynman combining with

$$a_1 = q'^2 + m^2 \,, \quad a_2 = m^2 + z(1-z)(q'+p)^2$$

At the fourth line, one needs to perform a change of variables

$$q' + yp = q''$$

At the sixth line, the first addendum is an odd function and gives a null integral. At the eighth line, the integral has already been done. At the tenth line, one computes the leading divergence. In this case, the y-integral is not divergent, so the final pole is simple.

Summing the two terms found, one obtains

$$\Sigma(p^2)\Big|_{(2\mathrm{II})} = -\frac{\lambda^2}{6}\frac{1}{(4\pi)^4}\left[\frac{3}{2}\frac{m^2}{\varepsilon^2} + \frac{3m^2}{\varepsilon}\left(\frac{1}{2} + \psi(1) + \ln\frac{4\pi k^2}{m^2}\right) + \frac{1}{4\varepsilon}p^2 + \text{finite}\right]$$

where $\psi(2)$ is obtained from expanding $\Gamma(2\varepsilon)$ and the logarithm comes from expanding

$$\left[\frac{4\pi k^2}{m^2}\right]^{2\varepsilon}$$

both of them in the K term.

Remark. The highest-order pole is universal, while other poles are scheme-dependent.

Remark. At one loop, the self-energy function Σ has a contribution only from one diagram and it does not depend on the momentum p^2: the first derivative is zero. In this case, at two loops, the self-energy function has a dependence on the momentum and the derivative is non-zero

$$\mathrm{d}_{p^2}\Sigma(p^2 = \mu^2)\Big|_{(2\mathrm{II})} \neq 0$$

Therefore, the self-energy function of the diagram (2II) is

$$\Sigma(p^2) = \Sigma(\mu^2) + \Sigma'(\mu^2)(p^2 - \mu^2) + \text{finite}$$

It may be that summing this contribution with the ones from the other diagrams, the derivative may cancel.

Third diagram. Consider Ramond conventions. The third diagram has a contribution from one-loop counter term. At one-loop, one has found that the counter terms that cancel the one-loop divergences are

$$\mathcal{L}|_{1\mathrm{L}} = \mathcal{L} + \mathcal{L}^{(1)}_{\mathrm{ct}}\,, \quad \mathcal{L}^{(1)}_{\mathrm{ct}} = \frac{1}{2}\Sigma(\mu^2)\Big|_{1\mathrm{L}}\varphi^2 + \frac{3}{4!}\Gamma(\mu^2)\Big|_{1\mathrm{L}}\varphi^4$$

where one has one-loop contributions to the self-energy and vertex functions given by

$$\Sigma(\mu^2)\Big|_{1\mathrm{L}} = \frac{\lambda}{(4\pi)^2}\frac{m^2}{2}\left[\frac{1}{\varepsilon} + F_1\right], \quad \Gamma(\mu^2)\Big|_{1\mathrm{L}} = \frac{\lambda}{2}\frac{\lambda}{(4\pi)^2}(k^2)^{\varepsilon}\left[\frac{1}{\varepsilon} + G_1\right], \quad \hat{\lambda} \equiv \frac{\lambda}{(4\pi)^2}$$

where F_1 and G_1 are finite terms.

Let q be the loop momentum. The diagram is obtain from joining the four-point vertex with a two-point counter term vertex given by $\Sigma_{1\mathrm{L}}$. In Euclidean, the perturbative expansion is of the exponential e^{-S} with the Lagrangian with all positive signs. Therefore

$$\int [\mathcal{D}\varphi]\,\mathrm{e}^{-S} = \int [\mathcal{D}\varphi]\,\exp\left[-\int \mathrm{d}^4x\,(\mathcal{L} + \mathcal{L}^{(1)}_{\mathrm{ct}})\right]$$

The exponential of the action has to be expanded up to second order. The combinatorial factor is

$$\frac{1}{2!}(-1)^2\frac{\lambda}{4!}\frac{1}{2}\Sigma(\mu^2)\Big|_{1\mathrm{L}} \cdot (4\cdot 3)\cdot 2\cdot 2 = \frac{\lambda}{2}\Sigma(\mu^2)\Big|_{1\mathrm{L}}$$

The second factor of 2 comes from the fact that one is expanding at second order an interaction of the type

$$\frac{(-1)^2}{2!}(\mathcal{L}_{\mathrm{int}} + \Sigma)^2$$

From the square, one has one ordinary vertex and one two-point vertex Σ; the factor comes from the mixed product.

The diagram's contribution to the self-energy function is

$$\begin{aligned}\Sigma(p^2)\Big|_{(2\text{III})} &= \frac{\lambda}{2}\frac{\hat{\lambda}}{2}m^2\left[\frac{1}{\varepsilon}+F_1\right]\int\frac{\mathrm{d}^n q}{(2\pi)^n}\frac{1}{(q^2+m^2)^2}\\ &= \frac{\lambda\hat{\lambda}}{4}m^2\left[\frac{1}{\varepsilon}+F_1\right]\frac{1}{(4\pi)^2}\left[\frac{1}{\varepsilon}+\psi(1)-\ln\frac{m^2}{4\pi k^2}+o(\varepsilon^0)\right]\\ &= \hat{\lambda}^2\frac{m^2}{4}\left[\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}(\psi(1)-\ln\hat{m}^2+F_1)+\text{finite}\right]\end{aligned}$$

where one sets

$$\hat{m}^2 \equiv \frac{m^2}{4\pi k^2}$$

Fourth diagram. The fourth diagram is obtained by joining two lines of the four-point counter term vertex, that is by performing a contraction to obtain a loop. The combinatorial factor is

$$(-1)\frac{3}{4!}\Gamma(\mu^2)\Big|_{1\text{L}}(4\cdot 3) = -\frac{3}{2}\Gamma(\mu^2)\Big|_{1\text{L}} = -\frac{3}{4}\lambda\hat{\lambda}(k^2)^\varepsilon\left[\frac{1}{\varepsilon}+G_1\right]$$

Therefore, the diagram's contribution is

$$\begin{aligned}\Sigma(p^2)\Big|_{(2\text{IV})} &= -\frac{3}{4}\lambda\hat{\lambda}(k^2)^\varepsilon\left[\frac{1}{\varepsilon}+G_1\right]\int\frac{\mathrm{d}^n q}{(2\pi)^n}\frac{1}{q^2+m^2}\\ &= -\frac{3}{4}\lambda\hat{\lambda}(k^2)^\varepsilon\left[\frac{1}{\varepsilon}+G_1\right]\left[-\frac{m^2}{16\pi^2}\right]\left[\frac{1}{\varepsilon}+\psi(2)-\ln\hat{m}^2+o(\varepsilon^0)\right]\\ &= \frac{3}{4}\hat{\lambda}^2m^2(k^2)^\varepsilon\left[\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}(\psi(2)-\ln\hat{m}^2+G_1)+\text{finite}\right]\end{aligned}$$

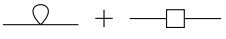
Two-loop contributions. The contributions are

$$\begin{aligned}\Sigma|_{(2\text{I})} &= -\frac{m^2}{4}\hat{\lambda}^2\left[\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}(\psi(1)+\psi(2)-2\ln\hat{m}^2)+\text{finite}\right]\\ \Sigma|_{(2\text{II})} &= -\hat{\lambda}^2\frac{m^2}{4}\left[\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}(1+2\psi(1)-2\ln\hat{m}^2)+\frac{1}{6}\frac{1}{\varepsilon}\frac{p^2}{m^2}+\text{finite}\right]\\ \Sigma|_{(2\text{III})} &= \hat{\lambda}^2\frac{m^2}{4}\left[\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}(\psi(1)-\ln\hat{m}^2+F_1)+\text{finite}\right]\\ \Sigma|_{(2\text{IV})} &= \frac{3}{4}\hat{\lambda}^2m^2\left[\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}(\psi(2)-\ln\hat{m}^2+G_1)+\text{finite}\right]\end{aligned}$$

where one has expanded $(k^2)^\varepsilon$ and has reabsorbed the terms inside the arbitrary term G_1 and the finite part. Intuitively, one expects to find related contributions from the first and third diagrams. Their sum is

$$\begin{aligned}\Sigma|_{(2\text{I})}+\Sigma|_{(2\text{III})} &= -\frac{m^2}{4}\hat{\lambda}^2\left[\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}(\psi(1)+\psi(2)-2\ln\hat{m}^2)+\text{finite}\right]\\ &\quad+\hat{\lambda}^2\frac{m^2}{4}\left[\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}(\psi(1)-\ln\hat{m}^2+F_1)+\text{finite}\right]\\ &= -\hat{\lambda}^2\frac{m^2}{4}\left[\frac{1}{\varepsilon}(\psi(2)-\ln\hat{m}^2-F_1)+\text{finite}\right]\end{aligned}$$

The highest pole disappears. This is expected: in the first diagram, the integrals are factorized and one of them is exactly the same present in the second diagram; their sum can be thought of as the insertion of the diagrams [r]



into the two-point one-loop diagram. This insertion is finite, therefore the divergence comes only from the q-integral and it is a simple pole. At higher loops one expected higher-order poles since there is a diagram that can be constructed as the product of one-loop diagrams. The diagram has to be combined with proper counter terms that cancel the divergences? [r].

Summing the other two diagrams, the poles do not simplify and one gets

$$\Sigma|_{(2\mathrm{II})} + \Sigma|_{(2\mathrm{IV})} = -\frac{\hat{\lambda}^2}{4} m^2 \left[-\frac{2}{\varepsilon^2} + \frac{1}{\varepsilon}(1 + 2\psi(1) - 3\psi(2) + \ln \hat{m}^2 - 3G_1) + \frac{1}{6\varepsilon}\frac{p^2}{m^2} + \text{finite} \right]$$

The total contribution at two-loop is

$$\begin{aligned}
\Sigma|_{2\mathrm{L}} &= -\frac{\hat{\lambda}^2}{24} p^2 \frac{1}{\varepsilon} + \hat{\lambda}^2 \frac{m^2}{2} \left[\frac{1}{\varepsilon^2} + \frac{1}{2\varepsilon}[F_1 + 3G_1 + 2\psi(2) - 2\psi(1) - 1] + \text{finite} \right] \\
&= -\frac{\hat{\lambda}^2}{24} p^2 \frac{1}{\varepsilon} + \hat{\lambda}^2 \frac{m^2}{2} \left[\frac{1}{\varepsilon^2} + \frac{1}{2\varepsilon}[F_1 + 3G_1 + 1] + \text{finite} \right] \\
&\equiv \Sigma'(0)\Big|_{2\mathrm{L}} p^2 + \Sigma(0)\Big|_{2\mathrm{L}}
\end{aligned}$$

The second line is obtained by noting that

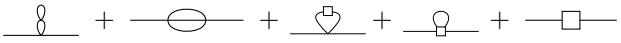
$$\psi(n+1) - \psi(n) = \frac{1}{n}$$

A few observations. At two-loops there are ε^2 poles and there is the p^2 term which is absent at one-loop. The leading pole is universal, while the other terms (F_1 and G_1 which are finite) depend on the regularization scheme. When one fixes these functions, one fixes the scheme. Lastly, the logarithms cancel.

At this point, one must add to the Lagrangian the counter terms to counter this two-loop divergence:

$$\mathcal{L}|_{2\mathrm{L}} = \mathcal{L} + \frac{1}{2}\Sigma(\mu^2)\Big|_{1\mathrm{L}} \varphi^2 + \frac{3}{4!}\Gamma(\mu^2)\Big|_{1\mathrm{L}} \varphi^4 + \mathcal{L}^{(2)}_{\mathrm{ct}}\,, \quad \mathcal{L}^{(2)}_{\mathrm{ct}} = \frac{1}{2}\Sigma(0)\Big|_{2\mathrm{L}} \varphi^2 + \frac{1}{2}\Sigma'(0)\Big|_{2\mathrm{L}} \partial_\mu\varphi\partial^\mu\varphi$$

The counter terms cancel the divergences only of the two-point Green's function? [r]. By correcting the Lagrangian in this way, one has to modify the Feynman rules using these two-loop counter terms



From the first four diagrams, which come from $\mathcal{L} + \mathcal{L}^{(1)}_{\mathrm{ct}}$, give $\Sigma|_{2\mathrm{L}}(0)$. The last diagram gives $-\Sigma|_{2\mathrm{L}}(0)$. The first four diagrams already include the minus sign of the action in the exponential

$$\int [\mathcal{D}\varphi] \, \exp\left[-\int \mathrm{d}^4x \, (\mathcal{L} + \mathcal{L}^{(1)}_{\mathrm{ct}} + \mathcal{L}^{(2)}_{\mathrm{ct}}) \right]$$

while this is not the case for the last diagram and one has to write it explicitly [r].

The second term in the counter term two-loop Lagrangian $\mathcal{L}^{(2)}_{\mathrm{ct}}$ can be rewritten as

$$\partial_\mu\varphi\,\partial^\mu\varphi \to -\varphi\,\Box\,\varphi \to \varphi p^2 \varphi$$

The one-loop Lagrangian gives $\Sigma'(0)p^2$ with the minus already included, the counter term Lagrangian gives $-\Sigma'(0)$ (the minus sign has to be written explicitly from the expansion of the exponential). Therefore, the final result is finite [r].

Summary. At two-loop, the general form of counter terms is

$$\mathcal{L}^{(2)}_{\mathrm{ct}} = \frac{1}{2}[\Sigma(0)|_{2\mathrm{L}} + F_2]\varphi^2 + \frac{1}{2}[\Sigma'(0)|_{2\mathrm{L}} + \hat{\lambda}^2 H_2]\partial_\mu\varphi\,\partial^\mu\varphi$$

where F_2 and H_2 are arbitrary finite functions. The complete Lagrangian is

$$\begin{aligned}
\mathcal{L}|_{2\mathrm{L}} &= \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi\,[1 + \Sigma'(0)|_{2\mathrm{L}} + \hat{\lambda}^2 H_2] + \frac{1}{2}\varphi^2[m^2 + \Sigma(0)|_{1\mathrm{L}} + \Sigma(0)|_{2\mathrm{L}} + F_2] + \text{interaction} \\
&\equiv \frac{1}{2} Z_\varphi\,\partial_\mu\varphi\,\partial^\mu\varphi + \frac{1}{2}(m^2 + \delta m^2) Z_\varphi \varphi^2 + \text{interaction}
\end{aligned}$$

One finds the explicit expression of the renormalization functions up to two loops

$$Z_\varphi = 1 + \Sigma'(0)|_{2\mathrm{L}} + \hat{\lambda}^2 H_2\,, \quad \delta m^2 = \Sigma(0)|_{1\mathrm{L}} + \Sigma(0)|_{2\mathrm{L}} + F_2$$

Lecture 17

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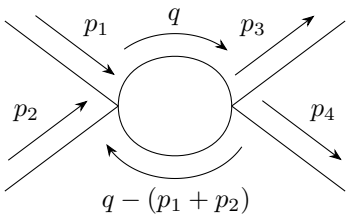
Any divergent term of the two-point function is either proportional to m^2 or to p^2 (due to the dimensions). The only terms one can add as counter terms are mass terms and kinetic terms (the d'Alembertian). This is a non-trivial result, the counter terms have the same form of the original Lagrangian. As such the multiplicative renormalization can be applied. The $\lambda\varphi^4$ theory is renormalizable.

7.2 Four-point Green's function

See Peskin, §10.5. One would like to compute the two-loop contributions to the vertex function Γ of the four-point Green's function.

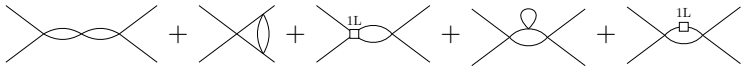
Tools. It is useful to utilize the one-loop integral $V(p^2)$ previously calculated associated to the diagram



where $p = p_1 + p_2 = p_3 + p_4$. The associated loop integral is

$$\begin{aligned} V(p^2) &\equiv \int \frac{\mathrm{d}^n q}{(2\pi)^n} \frac{1}{(q^2+m^2)[(q-p)^2+m^2]} \\ &= \frac{\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}} \int_0^1 \mathrm{d}z\, \frac{1}{[m^2 + z(1-z)p^2]^\varepsilon} \\ &= \frac{(k^2)^{-\varepsilon}}{(4\pi)^2} \left[\frac{1}{\varepsilon} + \psi(1) - \ln\frac{m^2}{4\pi k^2} - \int_0^1 \mathrm{d}z\, \ln\left(1 + \frac{z(1-z)p^2}{m^2}\right) + o(\varepsilon^0) \right] \end{aligned}$$

Contributions. The two-loop diagrams contributing to the vertex function are

$\Gamma|_{2\mathrm{L}} \to$ 

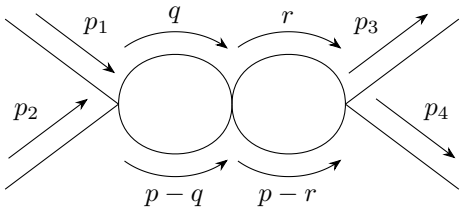
$= (4\mathrm{I}) + (4\mathrm{II}) + (4\mathrm{III}) + (4\mathrm{IV}) + (4\mathrm{V})$

The only new diagram is the second, the other are products of one-loop diagrams [r].

The third and fourth diagrams are an example of diagrams with one-loop corrections to a propagator which cancel each other due to one of them being a counter term. They are just one example, there are also other possible diagrams. However, they are not included since they are cancelled by the corresponding counter term.

For the other four diagrams, one must remember that the four-point Green's function has three channels, one for each Mandelstam variable. As such, the above diagrams should be drawn for each channel. However, due to crossing and reflection symmetry, only the four above in the s channel are independent. One may therefore consider only the s channel and obtain the other two by substituting s with t and u.

First diagram. Consider p the total momentum going through the diagram, and q and r the loop momenta



The combinatorial factor is given by expanding the exponential of the action up to third order (one order for each vertex), this gives $(-1)^3(3!)^{-1}$. Each vertex gives $\lambda(4!)^{-1}$. There are 12 ways to put p_1 around one of the three vertices, while 3 ways to put p_2 since it must neighbour p_1. Then p_3 has 8 ways and p_4 has 3 ways. The first loop line can be connected in 4 ways, while the second in 3 ways. The first loop line for the other loop has only 2 ways, while the last line has only 1 choice. The factor is

$$\frac{(-1)^3}{3!}\left(\frac{\lambda}{4!}\right)^3 (12\cdot 3)\cdot(8\cdot 3)\cdot(4\cdot 3)\cdot(2\cdot 1)$$

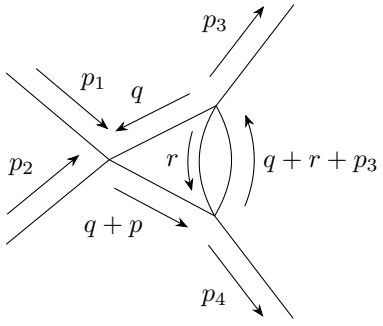
Therefore, in dimensional regularization, the diagram is given by

$$\begin{aligned}
\Gamma(p^2)\Big|_{(4\mathrm{I})} &= -\frac{\lambda^3}{4}(k^2)^{3\varepsilon}\left[\int \frac{\mathrm{d}^n q}{(2\pi)^n}\frac{1}{(q^2+m^2)[(p-q)^2+m^2)]}\right]^2 = -\frac{\lambda^3}{4!}(k^2)^{2\varepsilon}[V(p^2)]^2 \\
&= -\frac{\lambda^3}{4}(k^2)^{3\varepsilon}\left\{\frac{(k^2)^{-\varepsilon}}{(4\pi)^2}\left[\frac{1}{\varepsilon}+\psi(1)-\ln\frac{m^2}{4\pi k^2}-\int_0^1 \mathrm{d}z\,\ln\left(1+\frac{z(1-z)p^2}{m^2}\right)+o(\varepsilon^0)\right]\right\}^2 \\
&= -\frac{\lambda^3}{4}\frac{(k^2)^{\varepsilon}}{(4\pi)^4}\left\{\frac{1}{\varepsilon^2}+\frac{2}{\varepsilon}\left[\psi(1)-\ln\hat{m}^2-\int_0^1 \mathrm{d}z\,\ln\left(1+\frac{z(1-z)p^2}{m^2}\right)\right]+\text{finite}\right\}
\end{aligned}$$

In the first line, one renames $r \to q$ since it is an integration variable.

Exercise. Compute the combinatorial factor of the second diagram. It should be $-\lambda^3 2^{-1}$.

Second diagram. Let p be the total momentum going through the diagram and consider the s channel



where one remembers that

$$p = p_1 + p_2 = p_3 + p_4$$

The diagram gives a contribution of

$$\begin{aligned}
\Gamma(p^2)\Big|_{(4\mathrm{II})} &= -\frac{\lambda^3}{2}(k^2)^{3\varepsilon}\int\frac{\mathrm{d}^n q}{(2\pi)^n}\frac{1}{(q^2+m^2)[(q+p)^2+m^2]}V((q+p_3)^2)\\
&= -\frac{\lambda^3}{2}(k^2)^{3\varepsilon}\frac{\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}}\int_0^1\mathrm{d}z\int\frac{\mathrm{d}^n q}{(2\pi)^n}\frac{1}{(q^2+m^2)[(q+p)^2+m^2][m^2+z(1-z)p^2]^{\varepsilon}}\\
&= -\frac{\lambda^3}{2}(k^2)^{3\varepsilon}\frac{\Gamma(2+\varepsilon)}{(4\pi)^{2-\varepsilon}}\int_0^1\mathrm{d}z\int_0^1\mathrm{d}y\int_0^1\mathrm{d}w\,w^{\varepsilon-1}(1-w)\\
&\quad\times\int\frac{\mathrm{d}^n q'}{(2\pi)^n}\frac{1}{[m^2+(z(1-z)w+(1-w))q'^2+P^2]^{2+\varepsilon}}\\
&= -\frac{\lambda^3}{2}(k^2)^{3\varepsilon}\frac{\Gamma(2+\varepsilon)}{\Gamma(2-\varepsilon)}\frac{1}{(4\pi)^{4-2\varepsilon}}\int_0^1\mathrm{d}z\int_0^1\mathrm{d}y\int_0^1\mathrm{d}w\,\frac{w^{\varepsilon-1}(1-w)}{[z(1-z)w+(1-w)]^{2+\varepsilon}}\\
&\quad\times\int\frac{\mathrm{d}^n q'}{(2\pi)^n}\frac{(q'^2)^{1-\varepsilon}}{[q'^2+\frac{m^2+P^2}{z(1-z)w+(1-w)}]^{2+\varepsilon}}\\
&= -\frac{\lambda^3}{2}(k^2)^{3\varepsilon}\frac{\Gamma(2\varepsilon)}{(4\pi)^{4-2\varepsilon}}\int_0^1\mathrm{d}z\int_0^1\mathrm{d}y\int_0^1\mathrm{d}w\,\frac{1}{[P^2+m^2]^{2\varepsilon}}\frac{w^{\varepsilon-1}(1-w)}{[z(1-z)w+(1-w)]^{2-\varepsilon}}
\end{aligned}$$

At the first line, one may note that the one-loop integral V contains $r+q+p_3$ but may be expressed as $r-(q+p_3)$ by sending $r\to -r$ in the integral (and noting the presence of squares). At the second line one substituted the expression of the one-loop integral V before expanding in powers of the infinitesimal parameter ε. At the third and fourth lines, one applied Feynman combining where

$$\begin{aligned}
P^2(w,y,z) &= wz(1-z)p_3^2+(1-w)yp^2-\left[\frac{wz(1-z)p_3+y(1-w)p}{z(1-z)w+(1-w)}\right]^2\\
&= yp^2-y^2p^2+o(w^0)=y(1-y)p^2+o(w^0)\,,\quad w\to 0
\end{aligned}$$

At the fifth line, one applied spherical coordinates. At the last line, one applied Euler's Beta function with

$$A^2\equiv\frac{m^2+P^2}{z(1-z)w+(1-w)}\,,\quad m_{\mathrm{Euler}}=2-\varepsilon\,,\quad n_{\mathrm{Euler}}=2+\varepsilon$$

One notices that the w-integral is ill-defined in the limit $\varepsilon\to 0$. [r] To regularize it, one may write

$$w^{\varepsilon-1}=\frac{1}{\varepsilon}\,\mathrm{d}_w w^{\varepsilon}$$

and integrate by parts. However, there is another method. Consider

$$\int_0^1\mathrm{d}w\,w^{\varepsilon-1}f(w)=\int_0^1\mathrm{d}w\,w^{\varepsilon-1}f(w=0)+\int_0^1\mathrm{d}w\,w^{\varepsilon-1}[f(w)-f(0)]\equiv I_1+I_2$$

The second integral is well-defined, but not the first. In fact

$$\begin{aligned}
I_1 &= \int_0^1\mathrm{d}w\,w^{\varepsilon-1}f(w=0)\\
&= -\frac{\lambda^3}{2}(k^2)^{3\varepsilon}\frac{\Gamma(2\varepsilon)}{(4\pi)^{4-2\varepsilon}}\int_0^1\mathrm{d}w\,w^{\varepsilon-1}\int_0^1\mathrm{d}z\int_0^1\mathrm{d}y\,\frac{1}{[P^2(w=0)+m^2]^{2\varepsilon}}\\
&= -\frac{\lambda^3}{2}(k^2)^{3\varepsilon}\frac{\Gamma(2\varepsilon)}{(4\pi)^{4-2\varepsilon}}\int_0^1\mathrm{d}w\,w^{\varepsilon-1}\int_0^1\mathrm{d}z\int_0^1\mathrm{d}y\,\frac{1}{[y(1-y)p^2+m^2]^{2\varepsilon}}\\
&= -\frac{\lambda^3}{2}\frac{(k^2)^{3\varepsilon}}{(4\pi k^2)^{2\varepsilon}}\frac{\Gamma(2\varepsilon)}{(4\pi)^{4-2\varepsilon}}\int_0^1\mathrm{d}w\,w^{\varepsilon-1}\int_0^1\mathrm{d}z\int_0^1\mathrm{d}y\left[\frac{y(1-y)p^2+m^2}{4\pi k^2}\right]^{-2\varepsilon}\\
&= -\frac{\lambda^3}{2}(k^2)^{\varepsilon}\frac{\Gamma(2\varepsilon)}{(4\pi)^4}\frac{1}{\varepsilon}\int_0^1\mathrm{d}y\,\exp\left[-2\varepsilon\ln\frac{m^2+y(1-y)p^2}{4\pi k^2}\right]
\end{aligned}$$

$$= -\frac{\lambda^3}{2}\frac{(k^2)^\varepsilon}{(4\pi)^4}\left[\frac{1}{2\varepsilon}+\psi(1)+o(\varepsilon^0)\right]\frac{1}{\varepsilon}\left[1-2\varepsilon\int_0^1 \mathrm{d}y\,\ln\frac{m^2+y(1-y)p^2}{4\pi k^2}+o(\varepsilon)\right]$$
$$= -\frac{\lambda^3(k^2)^\varepsilon}{2(4\pi)^4}\left[\frac{1}{2\varepsilon^2}+\frac{1}{\varepsilon}\left(\psi(1)-\int_0^1 \mathrm{d}y\,\ln\frac{m^2+y(1-y)p^2}{4\pi k^2}\right)+\text{finite}\right]$$

At the fourth line, the w-integral gives ε^{-1} and the z-integral gives 1. At the penultimate line, one expands the Gamma function and the exponential. This integral has a quadratic pole.

The second integral has argument

$$\begin{aligned}
f(w)-f(0) &\propto \frac{1}{[P^2(w)+m^2]^{2\varepsilon}}\frac{1-w}{[z(1-z)w+(1-w)]^{2-\varepsilon}}-\frac{1}{[P^2(0)+m^2)]^{2\varepsilon}}\\
&= \left[P^2(0)+m^2+\mathrm{d}_w P^2(0)\,w+o(w)\right]^{-2\varepsilon}\frac{1-w}{[z(1-z)w+(1-w)]^{2-\varepsilon}}-[P^2(0)+m^2]^{-2\varepsilon}\\
&= [P^2(0)+m^2]^{-2\varepsilon}\left[\left(1-\frac{2\varepsilon\,\mathrm{d}_w P^2(0)\,w}{P^2(0)+m^2}+o(w,\varepsilon)\right)\frac{1-w}{[z(1-z)w+(1-w)]^{2-\varepsilon}}-1\right]\\
&= [P^2(0)+m^2]^{-2\varepsilon}\left[\left(1-\frac{2\varepsilon\,\mathrm{d}_w P^2(0)\,w}{P^2(0)+m^2}+o(w,\varepsilon)\right)\frac{(1-w)^{\varepsilon-1}}{[1+\frac{w}{1-w}z(1-z)]^{2-\varepsilon}}-1\right]\\
&= [P^2(0)+m^2]^{-2\varepsilon}\left\{\left[1-\frac{2\varepsilon\,\mathrm{d}_w P^2(0)\,w}{P^2(0)+m^2}+o(w,\varepsilon)\right](1-w)^{\varepsilon-1}\right.\\
&\quad\left.\times\left[1-(2-\varepsilon)z(1-z)\frac{w}{1-w}+o(\varepsilon)\right]-1\right\}\\
&\sim Cw\,,\quad w\to 0
\end{aligned}$$

where C is a constant [r]. Therefore, in the limit $w\to 0$, one has

$$w^{\varepsilon-1}[f(w)-f(0)]\sim w^\varepsilon$$

and there is no singularity in the integral I_2. Therefore

$$\begin{aligned}
I_2 &= -\frac{\lambda^3}{2}(k^2)^{3\varepsilon}\frac{\Gamma(2\varepsilon)}{(4\pi)^{4-2\varepsilon}}\int_0^1 \mathrm{d}z\int_0^1 \mathrm{d}y\int_0^1\frac{\mathrm{d}w}{w}\left[\frac{1-w}{[z(1-z)w+1-w]^2}-1\right]\\
&= \lambda^3\left[\frac{1}{2\varepsilon}+\psi(1)+\cdots\right]N
\end{aligned}$$

[r] steps? at first line, $\varepsilon\to 0$ in integral? where N is a number independent of the momentum p^2.

Summing up, the diagram's contribution to the vertex function is

$$\Gamma(p^2)\Big|_{(4\mathrm{II})} = -\frac{\lambda^3(k^2)^\varepsilon}{2(4\pi)^4}\left[\frac{1}{2\varepsilon^2}+\frac{1}{\varepsilon}\left(\psi(1)-\int_0^1 \mathrm{d}y\,\ln\frac{m^2+y(1-y)p^2}{4\pi k^2}\right)+\frac{1}{\varepsilon}\widetilde{N}\right]$$

Third diagram. The vertex one-loop counter term contains the one-loop vertex function $\Gamma|_{1\mathrm{L}}(\mu^2)$ for each channel (and this implies the coefficient 3 in the counter term Lagrangian) with subtraction point

$$s_0=t_0=u_0=\frac{4}{3}\mu^2$$

One would like to keep separate the three channel contributions. Therefore, when computing the diagram's contribution to the vertex function, there is a missing factor of 3. In fact, in order to see how partial cancellations occur with the genuine diagram, one has to combine the counter terms shrewdly and cleverly.

The diagram's contribution for only one channel is

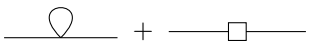
$$\begin{aligned}
\Gamma(p^2)\Big|_{(4\mathrm{III})} &= \frac{(-1)^2}{2!}\frac{\Gamma(\mu^2)|_{1\mathrm{L}}}{4!}\frac{\lambda}{4!}(k^2)^\varepsilon\cdot(8\cdot 3)\cdot(4\cdot 3)\cdot 2\cdot V(p^2)\\
&= \frac{\lambda^3}{4(4\pi)^4}(k^2)^\varepsilon\left\{\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\left[\psi(1)-\ln\hat{m}^2-\int_0^1 \mathrm{d}y\,\ln\left(1+\frac{y(1-y)p^2}{m^2}\right)\right]+o(\varepsilon^0)\right\}
\end{aligned}$$

[r] check 1st line. The second line is obtained by remembering that

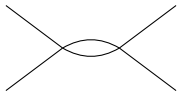
$$\Gamma(\mu^2)|_{1\mathrm{L}} = \frac{1}{2\varepsilon}\lambda\hat{\lambda}(k^2)^\varepsilon$$

and recalling the form of $V(p^2)$.

Fourth and fifth diagrams. One may rewrite their sum as the insertion of the diagrams



into a loop line of the diagram



One may recall that the sum of the two above diagrams is finite. Therefore [r], letting q be the loop momentum, the contribution of the fourth and fifth diagrams is

$$\Gamma(p^2)\Big|_{(4\mathrm{IV})} + \Gamma(p^2)\Big|_{(4\mathrm{V})} = \text{finite} \times \int \frac{\mathrm{d}^n q}{(2\pi)^n} \frac{1}{(q^2+m^2)^2[(p-q)^2+m^2]}$$

which is finite in $n = 4$, $D = -2$. The one-loop counter term cures the divergence.

Total contribution. In order to better see the partial cancellations, the sum of the diagrams is divided into three groups, remembering also that all the three channels have to be summed.

The first group contains (4I), $(4\mathrm{III})_s$ and its mirror copy. The second group contains (4II), $(4\mathrm{III})_t$, $(4\mathrm{III})_u$. The third group contains the mirrored diagrams of the previous group. The index specifies the channel of the one-loop counter term to the four-point vertex within the diagrams.

The reason of this grouping is the following. Consider the (4II) diagram. The integral of the logarithm could produce a function of the momentum p^2 that diverges. The above organization cancels this function of the momentum p^2. This is wanted since the four-point Green's function has mass dimension 0 so it has to be cured with a number instead of a dimensional quantity.

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First group. The first group contributes with

$$\begin{aligned}
&(4\mathrm{I}) + (4\mathrm{III})_s + (4\mathrm{III})'_s = \\
&\quad = -\frac{\lambda^3(k^2)^\varepsilon}{4(4\pi)^4}\left\{\frac{1}{\varepsilon^2} + \frac{2}{\varepsilon}\left[\psi(1) - \ln\hat{m}^2 - \int_0^1 \mathrm{d}z\, \ln\left(1 + \frac{z(1-z)p^2}{m^2}\right)\right] + \text{finite}\right\} \\
&\qquad + 2\frac{\lambda^3(k^2)^\varepsilon}{4(4\pi)^4}\left\{\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\left[\psi(1) - \ln\hat{m}^2 - \int_0^1 \mathrm{d}z \ln\left(1 + \frac{z(1-z)p^2}{m^2}\right)\right] + \text{finite}\right\} \\
&\quad = \frac{\lambda^3(k^2)^\varepsilon}{4(4\pi)^4}\left[\frac{1}{\varepsilon^2} + \text{finite}\right] = \frac{\lambda\hat{\lambda}^2(k^2)^\varepsilon}{4}\left[\frac{1}{\varepsilon^2} + \text{finite}\right]
\end{aligned}$$

At the first equality, the most concerning term is the $2\varepsilon^{-1}$ term with its parenthesis due to the logarithm within the integral, however it simplifies in the next equality.

Second group. The second group contributes with

$$\begin{aligned}
&(4\mathrm{II}) + (4\mathrm{III})_t + (4\mathrm{III})_u = \\
&\quad = -\frac{\lambda^3(k^2)^\varepsilon}{2(4\pi)^4}\left\{\frac{1}{2\varepsilon^2} + \frac{1}{\varepsilon}\left[\psi(1) - \ln\hat{m}^2 - \int_0^1 \mathrm{d}z\, \ln\left(1 + \frac{z(1-z)p^2}{m^2}\right)\right] + \frac{1}{\varepsilon}N + \text{finite}\right\} \\
&\qquad + 2\frac{\lambda^3(k^2)^\varepsilon}{4(4\pi)^4}\left\{\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\left[\psi(1) - \ln\hat{m}^2 - \int_0^1 \mathrm{d}z\, \ln\left(1 + \frac{z(1-z)p^2}{m^2}\right)\right] + \text{finite}\right\} \\
&\quad = \frac{\lambda^3(k^2)^\varepsilon}{4(4\pi)^4}\left[\frac{1}{\varepsilon^2} - \frac{2}{\varepsilon}N + \text{finite}\right] = \frac{\lambda\hat{\lambda}^2(k^2)^\varepsilon}{4}\left[\frac{1}{\varepsilon^2} - \frac{2}{\varepsilon}N + \text{finite}\right]
\end{aligned}$$

Total contribution. The contribution at two-loop to the vertex function is given by

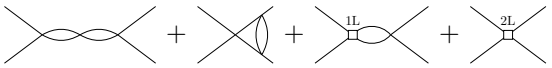
$$\Gamma(p^2)|_{2\mathrm{L}} = \frac{\lambda\hat{\lambda}^2}{4}(k^2)^{\varepsilon}\left[\frac{1}{\varepsilon^2} - \frac{2}{\varepsilon}N + \text{finite}\right]$$

This divergence can be cancelled by adding a two-loop counter term

$$\mathcal{L}^{(2)}_{\mathrm{ct}} \propto \frac{\lambda\hat{\lambda}^2}{4}(k^2)^{\varepsilon}\left[\frac{1}{\varepsilon^2} - \frac{2}{\varepsilon}N + \text{finite}\right]\frac{\varphi^4}{4!} \to$$

such that the two-loop contribution to the vertex function is finite

$\Gamma(p^2)|_{2\mathrm{L}} \to$



The finite part is an arbitrary contribution; when this is fixed, one fixes the scheme. The scheme where the finite part is null, is called minimal subtraction scheme.

Exercise. Find the right coefficient in $\mathcal{L}^{(2)}_{\mathrm{ct}}$ to cancel the divergence. Solution? Just 3? to have

$$\frac{3}{4!}\Gamma(\mu^2)|_{2\mathrm{L}}\varphi^4$$

like in one-loop?.

Higher loop. At three-loop, one has to consider Feynman rules coming from

$$\mathcal{L}|_{3\mathrm{L}} = \mathcal{L} + \mathcal{L}^{(1)}_{\mathrm{ct}} + \mathcal{L}^{(2)}_{\mathrm{ct}}$$

8 All loop renormalization

See Anselmi §2 for a general discussion on any order renormalization (for an advanced discussion see Itzykson–Zuber, where? [r]). From the particular case of $\lambda\varphi^4$ one may extrapolate a few properties.

A summary of what one has seen of the $\lambda\varphi^4$ theory in $d = 4$ dimensions up to two-loop, is the following:

1. power counting implies that only the two-point and four-point Green's functions are divergent;
2. at two-loop one has at most divergences in ε^{-2}, the appearance of higher order poles is a consequence of the presence of divergent sub-diagrams;
3. after subtracting non-leading divergences (that is combining the diagrams with the one-loop counter term ones), the divergences are ultra-local: they are not generic (analytic?) functions of the momentum p^2 and the mass m^2, but only polynomials.
4. Remembering the form of the renormalized parameters

$$\varphi = Z_{\varphi}^{-\frac{1}{2}}\varphi_0\,, \quad m^2 = m_0^2 + \delta m^2\,, \quad (k^2)^{\varepsilon}\lambda = Z_{\lambda}^{-1}Z_{\varphi}^2\lambda_0$$

then, when summing up the counter terms, one obtains the renormalized (classical) Lagrangian

$$\boxed{\mathcal{L} + \sum_{n=1}^{\infty}\mathcal{L}^{(n)}_{\mathrm{ct}} = \mathcal{L}_{\mathrm{R}}(\varphi, m, \lambda) = \mathcal{L}_0(\varphi_0, m_0, \lambda_0)}$$

where $\mathcal{L}_0$ is the classical Lagrangian expressed in terms of the bare quantities, while $\mathcal{L}$ is the renormalized classical Lagrangian at tree-level[3]. The renormalized classical Lagrangian

[3]The bare Lagrangian $\mathcal{L}_0$ produces divergent loop diagrams of the Green's functions. This Lagrangian is separated into two parts: the tree-level renormalized Lagrangian $\mathcal{L}$ and the counter terms $\mathcal{L}_{\mathrm{ct}}$. The tree-level renormalized Lagrangian still produces divergent loop diagrams, but these are cancelled exactly (by construction) by the loop diagrams containing the counter terms giving finite results. This division is useful since one may utilize the physical parameters which remain finite in the limit $\varepsilon \to 0$.

is made up of an infinite amount of addenda, while the bare classical Lagrangian is just one term. This equality is not trivial. One may notice that[4]

$$\mathcal{L}(\varphi, m, \lambda) = \mathcal{L}_0(\varphi, m, \lambda)$$

The first three results imply that the counter terms are proportional only to powers φ^2 and φ^4 since the only divergent Green's functions are the two-point and four-point. In particular, due to ultra-locality, these counter terms are polynomials in φ and its derivatives (of any order). Therefore, the counter terms have the same structure as the terms in the original Lagrangian.

Consequence of the fourth result. One may obtain a non-trivial consequence from the fourth result. [r] Consider the generating functional

$$W_0[J_0] = \int [\mathcal{D}\varphi_0] \, \exp\left[- \int \mathrm{d}^4x \, (\mathcal{L}_0 + J_0\varphi_0) \right]$$

One may define the source

$$J = Z_\varphi^{\frac{1}{2}} J_0 \quad \text{such that} \quad J_0\varphi_0 = J\varphi$$

Instead of integrating over the bare field φ_0, one integrates over the renormalized field φ and applies the above result

$$W_0[J_0] = \int [\mathcal{D}\varphi] \, \mathcal{J} \, \exp\left[- \int \mathrm{d}^4x \, (\mathcal{L}_\mathrm{R} + J\varphi) \right] \equiv W_\mathrm{R}[J]$$

where $\mathcal{J}$ is the Jacobian determinant and it is just $\mathcal{J} = 1$. This may be checked in dimensional regularization. Consider a generic transformation of the field

$$\varphi(x) \to \varphi'(x) \,, \quad [\mathcal{D}\varphi] = [\mathcal{D}\varphi'] \, \mathcal{J} \,, \quad \mathcal{J} = (\det \delta_\varphi \varphi')^{-1}$$

One shall consider only ultra-local transformations: the transformed field φ' is a polynomial in the field φ and its derivatives. The transformation is given by

$$\varphi'(x) = F(\varphi(x), \partial_\mu \varphi(x), \dots, \partial_{\mu_1,\dots,\mu_n} \varphi(x))$$

The Jacobian matrix is given by

$$\delta_{\varphi(y)} \varphi'(x) = \sum_{j=0}^{n} F^{\mu_1,\dots,\mu_j}(\varphi) \, \partial_{\mu_1,\dots,\mu_j} \delta^{(D)}(x - y)$$

where D is the dimension used in the regularization. The Jacobian determinant is then

$$\det \delta_\varphi \varphi' = \exp[\mathrm{Tr} \, \delta_\varphi \varphi'] = \exp\left[\sum_{j=0}^{n} \partial_{\mu_1,\dots,\mu_j} \delta^{(D)}(0) \int \mathrm{d}^D x \, F^{\mu_1,\dots,\mu_j} \right]$$

In dimensional regularization, one has

$$\partial_{\mu_1,\dots,\mu_j} \delta^{(D)}(0) = 0$$

In fact, by applying the Fourier transform, one has

$$\partial_{\mu_1,\dots,\mu_j} \delta^{(D)}(0) = \left[\int \frac{\mathrm{d}^D p}{(2\pi)^D} \, \mathrm{e}^{\mathrm{i}px} p_{\mu_1} \cdots p_{\mu_j} \right]_{x=0} = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \, p_{\mu_1} \cdots p_{\mu_j} = 0$$

For j odd, the integrand is odd. For j even, the integral is also zero (see Anselmi, eq. 2.13, idk [r]). Informally, this integral has mass dimension $D + j$, however it does not depend on any dimensional parameter and therefore can only be zero [r].

[r] From the above, one has found the generating functional

$$W_0[J_0] = W_\mathrm{R}[J]$$

Considering only connected diagrams, one finds

$$W[J] = \mathrm{e}^{Z[J]} \implies Z_0[J_0] = Z_\mathrm{R}[J]$$

[4] All of the divergent terms are inserted into the relations between the bare parameters and the physical parameters. Writing explicitly these relations reveals the counter terms.

Generic-point Green's function. The renormalized n-point Green's function is

$$G_{\mathrm{R}}^{(n)}(x_1,\dots,x_n;\lambda,m,k) = \left.\frac{\delta^n Z_{\mathrm{R}}[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right|_{J=0} = \frac{1}{(Z_\varphi^{\frac{1}{2}})^n}\left.\frac{\delta^n Z_0[J_0]}{\delta J_0(x_1)\cdots\delta J_0(x_n)}\right|_{J_0=0}$$
$$= Z_\varphi^{-\frac{n}{2}} G_0^{(n)}(x_1,\dots,x_n;\lambda_0,m_0)$$

The dependence of the energy scale k^2 comes from the dependence on the renormalized Lagrangian which contains the counter term Lagrangian.

For the amputated Green's function, one has

$$\Gamma_{\mathrm{R}}^{(n)}(x_1,\dots,x_n;\lambda,m,k) \equiv [G_{\mathrm{R}}^{(2)}]^{-n} G_{\mathrm{R}}^{(n)}(x_1,\dots,x_n;\lambda,m,k)$$
$$= [Z_\varphi^{-1} G_0^{(2)}]^{-n} Z_\varphi^{-\frac{n}{2}} G_0^{(n)}(x_1,\dots,x_n;\lambda_0,m_0)$$
$$= Z_\varphi^{\frac{n}{2}} \Gamma_0^{(n)}(x_1,\dots,x_n;\lambda_0,m_0)$$

Remark. When removing the regulator, $\varepsilon \to 0$, the renormalized Green's function is finite. These are the theoretical predictions that one may want to compare with experiments. Instead, the renormalized classical Lagrangian is divergent, but it is not observable while the renormalized generating functional, related to the observables, is finite.

Remark. The renormalized Green's function is directly related to physical observables, but it depends on the energy scale. One expects physical observables not to depend on the energy scale of the renormalization. The search for an answer leads to the renormalization group.

In order to generalize this framework to generic theories and to any number of loops, one shall prove that

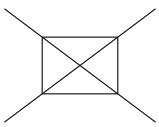
- ultra-locality is true at any loop order (see references for proof): after subtracting non-leading divergences, the remaining divergent terms are ultra-local functions of the momentum p^2 and the mass m^2;
- the counter term Lagrangian has the same form as the original Lagrangian at any loop; this prevents terms of the type

$$p^2 \left(\frac{p^2}{m^2}\right)^k \rightsquigarrow \Box^{k+1}$$

Qualitative properties. There are many diagrams that contain divergent sub-diagrams. For example

In particular, the top loop in the first diagram and the vertical loop in the second.

There are other diagrams that have convergent sub-diagrams, but still diverge. For example, the four-loop diagram appearing in the four-point Green's function



The superficial degree of divergence of the whole diagram is

$$D = 4\cdot 4 - (8\cdot 2) = 0$$

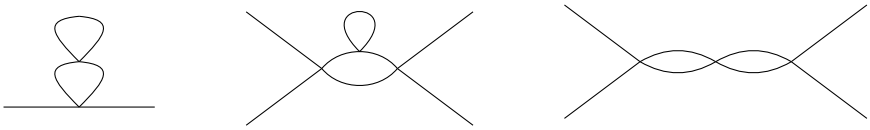
while the degree of every sub-diagram (a triangle) is

$$D = 4 - (3\cdot 2) = -2$$

Classification of diagrams. The number of diagrams increases very fast with each loop order. One may introduce a classification of diagrams. The primitively divergent diagrams are divergent diagrams without any divergent sub-diagram.

Classification of divergent sub-diagrams. Consider the $\lambda\varphi^4$ theory. The divergent sub-diagrams may be divided into three groups:

- disjoint, e.g.

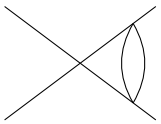


  Each loop is a disjoint divergent sub-diagram.

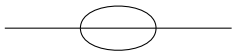
- nested, where the loop integrals are nested

$$I = \int \mathrm{d}^n q_1 \, f_1(q_1) \int \mathrm{d}^n q_2 \, f_2(q_1, q_2) \cdots \int \mathrm{d}^n q_k \, f_k(q_1, q_2, \ldots, q_k)$$

  e.g.



- overlapping, where the diagrams share one or more line, these are all the other sub-diagrams, e.g.



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One may cancel the disjoint divergent sub-diagrams through counter terms since the former factorize. For the other two cases, the counter terms only provide a partial cancellation.

After recognizing the possible sub-diagrams, one may conclude if the whole diagram is divergent through Weinberg's theorem: a diagram is finite if the superficial degree of diverge D of the whole diagram and of all the sub-diagrams is negative.

8.1 Ultra-locality

The following theorem proves that ultra-locality is true at any loop order.

Theorem. After subtracting non-leading divergences, the overall divergence of the diagram is local and polynomial in the momentum p^2 and the mass m^2. See Anselmi, p. 93 for proof.

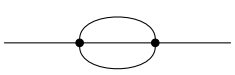
The leading divergence at a particular loop is given by the following theorem.

Theorem. For a massless theory, the leading divergence of a diagram with V vertices and L loops is at most

$$\frac{1}{\varepsilon^m}, \quad m = m(V, L) = \min(V - 1, L)$$

See Anselmi, p. 116 for proof.

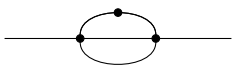
Example. At two-loop, consider the diagram



$$\sim \frac{m^2}{\varepsilon^2}$$

For $m = 0$ there are no ε^{-2} poles. There are two vertices $V = 2$ and two loops $L = 2$: the divergence is at most an ε^{-1} pole.

For massive theories, one may interpret the term $m^2\varphi^2$ as a two-point vertex rather than a part of the kinetic term. The example above may become



There are three vertices $V = 3$ and two loops $L = 2$, therefore

$$m = \min(3 - 1, 2) = 2$$

In fact one finds the pole $m^2\varepsilon^{-2}$.

At higher loops, both the number of vertices and loops increase: one expects that the leading divergence increases in power.

Exercise. For the $\lambda\varphi^4$ theory, draw all the three-loop connected diagrams for the two-point Green's function. Check that there are some diagrams with ε^{-3} poles by applying the theorem above.

8.2 Counter term form

[r] Consider a general scalar theory. One may generalize the power counting from the $\lambda\varphi^4$ theory. The Lagrangian is

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \sum_i \mathcal{L}_i^{\text{int}}(g_i)\,, \quad \mathcal{L}_i(g_i) = g_i\varphi^{p_i}(\partial_\mu\varphi\,\partial^\mu\varphi)^{\frac{d_i}{2}}$$

where g_i are the coupling constants and $p_i + d_i = b_i$ is fixed.

One would like to compute the superficial degree of divergence of a given diagram with

- n_i number of vertices of the type i,
- b_i number of fields lines in the vertex of type i,
- d_i number of derivatives in the vertex of type i,
- E number of external lines,
- and I number of internal lines.

The number of loops is fixed by

$$L = I - (V - 1) \rightsquigarrow L = I - \left[\sum_i n_i - 1\right]$$

There is constraint given by the number of lines

$$E + 2I = \sum_i n_i b_i$$

The superficial degree of divergence is

$$\begin{aligned} D &= 4L + \sum_i n_i d_i - 2I = 4\left[I - \sum_i n_i + 1\right] + \sum_i n_i d_i - 2I = 2I + 4 + \sum_i n_i(d_i - 4) \\ &= \left[\sum_i n_i b_i - E\right] + 4 + \sum_i n_i(d_i - 4) = 4 - E + \sum_i n_i(b_i + d_i - 4) = 4 - E + \sum_i n_i\delta_i \end{aligned}$$

where $\delta_i = b_i + d_i - 4$. Since this general theory has derivative vertices, these contribute with momenta at the numerator.

Remark. It holds

$$\delta_i = -\dim g_i$$

Consider the interaction Lagrangian

$$\mathcal{L}_i(g_i) = g_i \varphi^{p_i} (\partial_\mu \varphi \, \partial^\mu \varphi)^{\frac{d_i}{2}}$$

Its mass dimension is

$$4 = \dim \mathcal{L}_i = \dim g_i + (p_i + d_i) + 2\frac{d_i}{2} = \dim g_i + b_i + d_i \implies \dim g_i = 4 - b_i - d_i = -\delta_i$$

Example. For $\lambda\varphi^4$ theory, one has

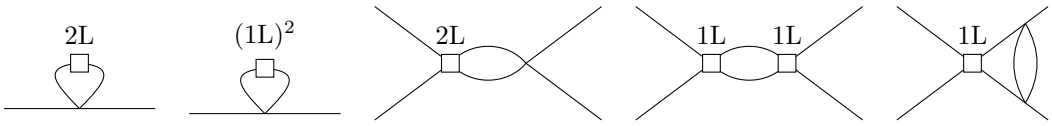
$$p_i = 4\,, \quad d_i = 0\,, \quad i = 1 \implies \delta_i = 0 \implies \dim \lambda = 0\,, \quad D = 4 - E$$

[r] In general,

- if the coupling constant has positive mass dimension, then $\delta_i < 0$, and the degree of divergence is eventually negative: the divergences appear only in a finite number of diagrams, in particular at low order. In this case, the theory is super-renormalizable since one has to add a finite number of counter terms. An example of scalar theory in $d = 4$ dimensions is $\lambda\varphi^3$.
- If the coupling constant has no mass dimension, then $\delta_i = 0$ and $D = 4 - E$. The degree of divergence depends only on the number of external lines and not on the internal structure. There are only two types of diagrams (with $E = 2$ and $E = 4$): the theory is renormalizable. An example is $\lambda\varphi^4$.
- If the coupling constant has negative mass dimension, then $\delta_i > 0$, and the degree of divergence is eventually positive: higher loops worsen the divergence, there are an infinite number of divergent diagrams. The theory is not renormalizable. Theories of these type can be used as effective field theories below a certain energy. An example is $\lambda\varphi^5$.

A Green's function has always mass dimension n since it is the product of n fields: every integral should give the same dimension. At higher loops, the coupling constant increases in powers, so the integral has to balance the dimensions. If the coupling's mass dimension is positive, then the integral has to converge more, and viceversa.

Exercise. Study the three-loop diagrams for the two-point and four-point Green's functions for $\lambda\varphi^4$. At three-loop one may use the one-loop and two-loop two-point propagators and four-point vertices. At three-loop there are genuine diagrams, but also combinations of loops and lower-loop counter terms. For example



The above are just products of separate diagrams: the calculations have already been done. One can utilize results at lower orders.

9 Renormalization of composite operators

See Cheng–Li, p. 62, Anselmi, p. 110.

Definition. A composite operator (or composite field) is a local product of fields and their derivatives.

"Operator" is a general term, since the theory is still classical and strictly speaking there are no operators. The term is used since one considers normal ordering: there are no equal-times contractions. This implies that the expectation value of the operators on the vacuum is zero.

Example. Consider $\lambda\varphi^4$. Examples of operators are

$$O_1 = \frac{1}{2}{:}\varphi^2{:}\,, \quad O_2 = \frac{1}{4!}{:}\varphi^4{:}\,, \quad O_3 = {:}\partial_\mu\varphi\,\partial^\mu\varphi{:}\,, \quad O_4 = \frac{1}{5!}{:}\varphi^5{:}\,, \quad O_5 = {:}\varphi^4\,\partial_\mu\varphi\,\partial^\mu\varphi{:}$$

These have mass dimensions

$$2\,, \quad 4\,, \quad 4\,, \quad 5\,, \quad 8$$

One would like to compute Green's functions with the insertion of composite operators. For example, one may consider an n-point Green's function with the insertion of field squared φ^2

$$G^{(n)}_{\varphi^2}(x_1,\ldots,x_n;x) = \frac{1}{2}\,\langle 0|\,\mathcal{T}\{{:}\varphi^2(x){:}\varphi(x_1)\cdots\varphi(x_n)\}\,|0\rangle$$

In momentum space, one has

$$G^{(n)}_{\varphi^2}(p_1,\ldots,p_n;p)(2\pi)^4\delta^{(4)}(\textstyle\sum_i p_i + p) \equiv \displaystyle\int \mathrm{d}^4x\,\mathrm{e}^{-\mathrm{i}px}\int\prod_{j=1}^{n}\mathrm{d}^4x_j\,\mathrm{e}^{-\mathrm{i}p_jx_j}G^{(n)}_{\varphi^2}(x_1,\ldots,x_n;x)$$

By making an insertion, there is an extra momentum in the Green's function and total momentum conservation law.

Insertion in the two-point Green's function. Let $n = 2$ and consider the two-point Green's function

$$\begin{aligned} G^{(2)}_{\varphi^2}(x_1,x_2;x) &= \frac{1}{2}\,\langle 0|\,\mathcal{T}\{{:}\varphi^2(x){:}\varphi(x_1)\varphi(x_2)\}\,|0\rangle \equiv N\int[\mathcal{D}\varphi]\,\mathrm{e}^{-S}\frac{1}{2}\varphi^2(x)\varphi(x_1)\varphi(x_2) \\ &= (-1)^3\frac{\delta^3}{\delta J(x_1)\delta J(x_2)\delta\chi(x)}N\int[\mathcal{D}\varphi]\,\exp\left[-S - \int\mathrm{d}^4x\,J\varphi - \int\mathrm{d}^4x\,\chi\frac{1}{2}\varphi^2\right] \end{aligned}$$

where χ is a source for the insertion.

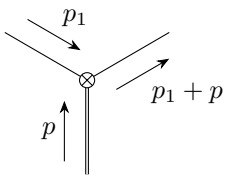
The functional integral can be calculated perturbatively. At tree-level, one may contract φ^2 only with $\varphi(x_1)$ and $\varphi(x_2)$. The Green's function is then

$$G^{(2)}_{\varphi^2}(x_1,x_2;x) = \Delta(x-x_1)\Delta(x-x_2)$$

due to Wick's theorem. In momentum space, it is

$$G^{(2)}_{\varphi^2}(p_1,p_2;p) = G^{(2)}_{\varphi^2}(p_1,-p_1-p;p) = \Delta(p_1)\Delta(p+p_1)\,, \quad p_1+p_2+p=0$$

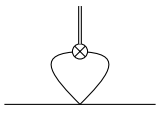
The corresponding Feynman diagram is



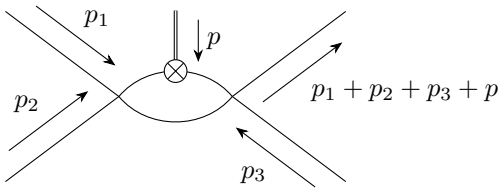
where the composite operator can be draw as a double line ending with a cross within a circle. By cutting the two upper lines, one obtains the amputated two-point Green's function

$$\Gamma^{(2)}_{\varphi}(p) = 1$$

This value is such since one is multiplying by the inverse of the propagator of the external lines. One may include a new Feynman rule: the insertion of a composite operator is a crossed dot carrying a momentum p and it links two momenta. One shall consider all the previous diagrams drawn for the two-point Green's function and add the composite operator. For example



The four-point Green's function with one insertion at one-loop is



In the outgoing momenta, one must also consider the momentum of the insertion entering the diagram.

Lecture 20

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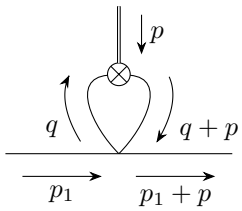
For an operator $O = \frac{1}{n!}\varphi^n(x)$, the corresponding vertex can be connected to n other lines.

One-loop contribution. One may study the one-loop contributions of the insertion in the two-point Green's function $G^{(2)}_{\varphi^2}(p_1, -p_1 - p; p)$. One inserts the operator into the one-loop diagram of the two-point Green's function



The composite operator carries momentum, so the outgoing momentum is different from the ingoing. Assigning internal momenta q and $q+p$, the corresponding algebraic expression is

$$\Gamma^{(2)}_{\varphi^2} = -\frac{\lambda}{2}\int \frac{\mathrm{d}^n q}{(2\pi)^n}\frac{1}{(q^2+m^2)[(q+p)^2+m^2]}$$

The combinatorial factor is

$$-\frac{\lambda}{4!}\cdot(4\cdot 3)\cdot(2\cdot 1)\frac{1}{2} = -\frac{\lambda}{2}$$

The first parenthesis is how many ways one can connect p_1 and p_1+p, while the second parenthesis is how many ways one may connect the insertion to the rest. The last factor of $\frac{1}{2}$ comes from the operator $O = \frac{1}{2}\varphi^2$.

The internal propagator is split into two propagators which aids with the divergence, however the integral is now logarithmically divergent, instead of quadratically. When inserting a composite operator, other kinds of divergences arise that are not cancelled by renormalizing the field.

The Green's function can be Taylor expanded around $(p_1, p) = (0, 0)$

$$\Gamma^{(2)}_{\varphi^2}(p_1, -p_1 - p; p) = \Gamma^{(2)}_{\varphi^2}(0, 0; 0) + \widetilde{\Gamma}^{(2)}_{\varphi^2}(p_1, -p_1 - p; p)\,, \quad \widetilde{\Gamma}^{(2)}_{\varphi^2}(p_1 = 0, p = 0) = 0$$

Only the first addendum is divergent. The Green's functions are generated by taking derivatives of a functional integral which contains a source term for the field φ and a source term for the composite operator O. The generating functional is

$$W_{\varphi^2}[J] = \int [\mathcal{D}\varphi]\, \exp\left[-\int \mathrm{d}^4 x\, \left(\mathcal{L} + J\varphi + \chi\frac{1}{2}\varphi^2\right)\right]$$

from which the two-point Green's function is

$$G^{(2)}(x_1, x_2; x) = (-1)^3 \frac{\delta^3 W_{\varphi^2}[J,\chi]}{\delta J(x_1)\delta J(x_2)\delta\chi(x)}\bigg|_{J=\chi=0}$$

[r] Letting the action be

$$S[J,\chi] \equiv \int \mathrm{d}^4x \, \left[\mathcal{L} + J\varphi + \frac{1}{2}\chi\varphi^2\right]$$

one adds the counter term

$$S^{(1)}_{\mathrm{ct}}[J,\chi] = \int \mathrm{d}^4x \, \chi\Gamma^{(2)}_{\varphi^2}(0,0;0)\frac{1}{2}\varphi^2$$

The sum of the one-loop integrals of these two terms is finite. In this way, the action becomes

$$S[J,\chi]|_{1\mathrm{L}} = S[J,\chi] + S^{(1)}_{\mathrm{ct}}[J,\chi] = \int \mathrm{d}^4x \, \left[\mathcal{L} + J\varphi + \chi\big(1+\Gamma^{(2)}_{\varphi^2}(0,0;0)\big)\frac{1}{2}\varphi^2\right]$$

The inner parenthesis is the renormalization function of the composite operator

$$Z_{\varphi^2} \equiv 1 + \Gamma^{(2)}_{\varphi^2}(0,0;0)$$

By renormalizing the operator in this way, the results at one-loop are finite. As such, starting from tree-level, one replaces

$$O = \frac{1}{2}\varphi^2 \to Z_O O = Z_{\varphi^2}\frac{1}{2}\varphi^2 = O_0$$

where O_0 is the bare operator. Therefore, composite operators are multiplicatively renormalized. The renormalization of a composite field is in general not related in an obvious way to the renormalization of its component fields, and has to be calculated anew. One must also distinguish bare and renormalized composite fields: the bare composite fields are just the products of the bare factors, while the renormalized composite fields are different from the products of the renormalized factors. With this in mind, the renormalized operator can be rewritten in terms of the renormalized fields

$$O = Z_O^{-1}O_0 = Z_{\varphi^2}^{-1}\frac{1}{2}\varphi_0^2 = Z_{\varphi^2}^{-1}\frac{1}{2}(Z_\varphi^{\frac{1}{2}}\varphi)^2 = Z_{\varphi^2}^{-1}Z_\varphi\frac{1}{2}\varphi^2$$

In conclusion, the Green's function of an arbitrary insertion operator is

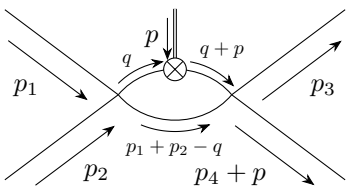
$$\begin{aligned} G^{(n)}_{O_0}(x_1,\dots,x_n;x) &= \langle 0|\,\mathcal{T}\{O_0(x)\varphi_0(x_1)\cdots\varphi_0(x_n)\}\,|0\rangle \\ &= Z_O Z_\varphi^{\frac{n}{2}}\,\langle 0|\,\mathcal{T}\{O(x)\varphi(x_1)\cdots\varphi(x_n)\}\,|0\rangle \\ &= Z_O Z_\varphi^{\frac{n}{2}}(G^{(n)}_O)_{\mathrm{R}}(x_1,\dots,x_n;x) \end{aligned}$$

One may generalize the insertion to consider multiple operators. An interesting quantity is

$$\langle 0|\,\mathcal{T}\{O(x_1)O(x_2)\}\,|0\rangle = \frac{1}{4}\,\langle 0|\,\mathcal{T}\{\varphi^2(x_1)\varphi^2(x_2)\}\,|0\rangle$$

This is used to compute the anomalous dimension operator. The anomalous dimension is the quantum dimension, which differs from the classical dimension (the one computed up until now).

Example. An example of convergence of a loop due to the insertion of an operator is the following. Consider the insertion of an operator in the four-point Green's function at one-loop. One shall draw the diagrams and put the insertion



Up to the combinatorial factor, the corresponding integral is

$$\Gamma^{(4)} \propto \int \frac{\mathrm{d}^n q}{(2\pi)^n}\frac{1}{(q^2+m^2)[(q+p)^2+m^2][(p_1+p_2-q)^2+m^2]}$$

When adding an insertion, a propagator is split into two propagators. This is good for convergences. In fact, the above integral converges.

Mixing. See Anselmi, Cheng–Li. In general, for higher mass dimensions, there are more operators: there is a degeneracy. The divergent part may be a mix of composite operators.

If there is a degeneracy of operators with a given classical dimension, under renormalization they can mix. For mass dimension $D = 2$ there is only one operator and there is no mixing

$$O_1 = \frac{1}{2} : \varphi^2 :$$

At $D = 4$ there are two operators that can mix

$$O_2 = \frac{1}{4!} : \varphi^4 : , \quad O_3 = : \partial_\mu \varphi \, \partial^\mu \varphi :$$

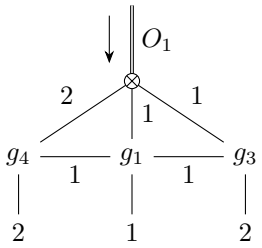
For example, consider a scalar theory of two fields with the following interaction Lagrangian

$$\mathcal{L}_{\rm I} = g_1 \varphi_1^4 + g_2 \varphi_2^4 + g_3 \varphi_1^2 \varphi_2 + g_4 \varphi_1 \varphi_2^2$$

Consider two composite operators

$$O_1 = : \varphi_1^2 \varphi_2 : , \quad O_2 = : \varphi_1 \varphi_2^2 :$$

At one-loop one may have the following diagram



From this one sees that from O_1 one has generated O_2: the three bottom lines can be connected to O_2. There is a non-trivial correlation function between the two operators. This is the mixing. At tree-level there are no contributions

$$\langle 0 | \, O_1 O_2 \, | 0 \rangle = \langle 0 | \, : \varphi_1^2 \varphi_2 :: \varphi_1 \varphi_2^2 : | 0 \rangle = 0$$

This means that the mixing is a purely quantum phenomenon.

If two operators mix, then at a given loop order the counter terms needed to cancel the divergences are

$$O_1^{\rm ct} = c_{11} O_1 + c_{12} O_2 \, , \quad O_2^{\rm ct} = c_{21} O_1 + c_{22} O_2$$

In the Lagrangian of the path integral, one has

$$\chi_1 O_1 + \chi_2 O_2 + \chi_1 O_1^{\rm ct} + \chi_2 O_2^{\rm ct} = \begin{bmatrix} \chi_1 & \chi_2 \end{bmatrix} \begin{bmatrix} 1 + c_{11} & c_{12} \\ c_{21} & 1 + c_{22} \end{bmatrix} \begin{bmatrix} O_1 \\ O_2 \end{bmatrix} = \boldsymbol{\chi}^\top C \mathbf{O}$$

The matrix may be diagonalized with a bi-unitary transformation

$$\boldsymbol{\chi}^\top C \mathbf{O} = (\boldsymbol{\chi}^\top U^\dagger)(U C V^\dagger)(V \mathbf{O}) = (\boldsymbol{\chi}')^\top D \mathbf{O}' = \chi_1' Z_{O_1'} O_1' + \chi_2' Z_{O_2'} O_2'$$

where one has

$$D = U C V^\dagger = {\rm diag}(Z_{O_1'}, Z_{O_2'}) \, , \quad (\boldsymbol{\chi}')^\top = \boldsymbol{\chi}^\top U^\dagger \, , \quad \mathbf{O}' = V \mathbf{O}$$

and U and V are unitary matrices.

10 Analytic continuation to Minkowski

See Ramond, p. 148. To go from Minkowski to Euclidean one sets and substitutes

$$\tau = -{\rm i} \tau_{\rm E} \, , \quad p^0 = {\rm i} p^0_{\rm E} \, , \quad (p^0_{\rm E}, \mathbf{p}) \to ({\rm i} p^0_{\rm E}, \mathbf{p})$$

One may compute the Feynman rules.

Propagator. The Euclidean and Minkowski propagators are

$$\frac{1}{p_{\mathrm{E}}^2+m^2}\,,\quad \frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon}$$

One may rewrite the first as

$$[(p_{\mathrm{E}}^0)^2+|\mathbf{p}|^2+m^2]^{-1}\to[-p_0^2+|\mathbf{p}|^2+m^2]^{-1}=-[p_0^2-|\mathbf{p}|^2-m^2]^{-1}=-[p^2-m^2+\mathrm{i}\varepsilon]^{-1}$$

To get to Minkowski one has to multiply by $-\mathrm{i}$. The rule is then

$$(p_{\mathrm{E}})^2\to -p^2\,,\quad m^2\to m^2-\mathrm{i}\varepsilon$$

and multiply by $-\mathrm{i}$

Vertex. The Euclidean and Minkowski vertices are

$$-\frac{\lambda}{4!}\,,\quad -\mathrm{i}\frac{\lambda}{4!}$$

One may transition from the first to the second by multiplying by i.

Loop integral. The Euclidean and Minkowski loop integral measures are

$$\frac{\mathrm{d}^4k_{\mathrm{E}}}{(2\pi)^4}\,,\quad \frac{\mathrm{d}^4k}{(2\pi)^4}=\mathrm{i}\frac{\mathrm{d}^4k_{\mathrm{E}}}{(2\pi)^4}$$

So one has to multiply by i.

Transition rules. In conclusion, going from Euclidean to Minkowski, one must multiply by

- a $-\mathrm{i}$ for each propagator,
- an i for each vertex,
- an i for each loop integral measure.

Green's function. The Green's function is

$$\begin{aligned}\Gamma^{(n)}(p_1,\dots,p_n;m^2)&=(-1)^I\mathrm{i}^{L+V+I}\Gamma_{\mathrm{E}}^{(n)}(p_1^{\mathrm{E}}=p_1,\dots,p_n^{\mathrm{E}}=p_n;m^2-\mathrm{i}\varepsilon)\\&=\mathrm{i}\Gamma_{\mathrm{E}}^{(n)}(p_1^{\mathrm{E}}=p_1,\dots,p_n^{\mathrm{E}}=p_n;m^2-\mathrm{i}\varepsilon)\end{aligned}$$

where L is the number of loops, V is the number of vertices, I is the number of internal lines and the momenta are replaced as

$$p^{\mathrm{E}}=(p_{\mathrm{E}}^0,\mathbf{p})\to p=(\mathrm{i}p_{\mathrm{E}}^0,\mathbf{p})$$

Knowing that

$$L=I-V+1\implies L+V+I=2I+1\implies(-1)^I\mathrm{i}^{2I+1}=(-1)^{2I}\mathrm{i}=\mathrm{i}$$

Therefore, the rule is [r]

$$G^{(n)}(p_1,\dots,p_n;m^2)=\mathrm{i}G_{\mathrm{E}}^{(n)}(p_1^{\mathrm{E}}=p_1,\dots,p_n^{\mathrm{E}}=p_n;m^2-\mathrm{i}\varepsilon)$$

For $V=0$ there appears a global minus sign.

The replacements introduce a different physical behaviour.

Lecture 21

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Change in physical behaviour. Consider the renormalized four-point Green's function $\Gamma^{(4)}_{\mathrm{R}}$ up to two-loops. In Minkowski it is

$$\Gamma^{(4)}_{\mathrm{M}}(p_1,p_2,p_3,p_4) = -\mathrm{i}\lambda - \mathrm{i}\frac{\lambda^2}{2(4\pi)^2}\int_0^1 \mathrm{d}z\, \ln\frac{m^2-\mathrm{i}\varepsilon - sz(1-z)}{m^2+M^2z(1-z)} + t,u \text{ channel}$$

where $s=p^2$ and M^2 is the subtraction point. The denominator of the logarithm lacks $\mathrm{i}\varepsilon$ because it is always positive. The numerator may be zero: the logarithm has a branch point at $z=0$. Knowing that

$$0<z<1 \implies 0<z(1-z)\leq\frac{1}{4}$$

and letting the numerator of the logarithm be

$$F(s,z)\equiv m^2 - sz(1-z) - \mathrm{i}\varepsilon$$

the numerator is zero when

$$F(s,z)=0 \implies m^2 = sz(1-z) \implies s=\frac{m^2}{z(1-z)}$$

The minimum value of the incoming energy s satisfying the above happens for

$$z(1-z)=\frac{1}{4} \implies s=4m^2$$

After taking the logarithm of the numerator $\ln F(s,z)$ and making an analytic continuation to the complex plane in s, one may put the branch cut on $[4m^2,\infty)$. The reason for its position is the following. [r] One may write

$$s=4m^2=(m+m)^2$$

[r] The four-point Green's function $\Gamma^{(4)}$ can be used to describe a two-by-two scattering. The minimum energy needed for the scattering is given by

$$p_1^2=m^2\,,\quad p_2^2=m^2$$

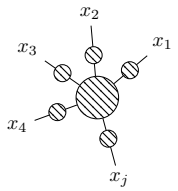
[r] The Green's function is real for $s<4m^2$ but develops an imaginary part above such threshold.

When promoting scattering amplitudes to complex quantities they become holomorphic functions and may have poles and branch cuts: the poles correspond to the physical masses and the branch cuts correspond to the scattering channels. The physical mass is the pole of the quantum propagator.

11 Effective action

See Weinberg §16.1 for physical explanation. The following is a general discussion that does not only apply to scalar theories.

Keeping in mind the perturbative evaluation of Green's functions, at any loop order, there is a way to summarize the structure of the diagrams

$G^{(n)}_{\mathrm{R}}(x_1,\dots,x_n) =$ 

[r] The n external lines receive contributions from the two-point Green's function $G^{(2)}$. The amputated Green's function has the external lines cut

$$\Gamma^{(n)}_{\mathrm{R}}(y_1,\dots,y_n) = \left[\prod_{j=1}^n G^{(2)}_{\mathrm{R}}(x_j,y_j)\right]^{-1} G^{(n)}_{\mathrm{R}}(x_1,\dots,x_n)$$

Consider the two-point Green's function (keeping in mind that the external lines are cut)

$$\Gamma^{(2)}_{\mathrm{R}}(p) = \quad = p^2 + m^2 + \widetilde{\Sigma}(p^2)$$

[r] It is the inverse of the quantum propagator, that is the kinetic term with the infinite series $\widetilde{\Sigma}$ of finite quantum correction. Therefore, the above is the quantum kinetic term.

For an n-point Green's function, one has the quantum n-point vertex

$$\Gamma^{(n)}_{\mathrm{R}}(p_1, \dots, p_n)(2\pi)^4 \delta^{(4)}(\textstyle\sum_i^n p_i) =$$

For $n = 4$, the four-point Green's function is the quantum four-point vertex which includes the classical contribution (for the $\lambda\varphi^4$ theory?)

$$= \lambda + \text{finite quantum corrections}$$

The Green's function is the Borel resummation of a perturbative series that is asymptotically divergent.

With these results, one may introduce the effective action

$$\begin{aligned}
\Gamma[\varphi] &= \frac{1}{2} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \hat{\varphi}(-p)[p^2 + m^2 + \widetilde{\Sigma}(p^2)]\hat{\varphi}(p) \\
&\quad + \sum_{n=3}^{\infty} \frac{1}{n!} \int \left[\prod_{j=1}^{n} \frac{\mathrm{d}^4 p_j}{(2\pi)^4} \right] (2\pi)^4 \delta^{(4)}(\textstyle\sum_j p_j) \Gamma^{(n)}_{\mathrm{R}}(p_1, \dots, p_n) \hat{\varphi}(p_1) \cdots \hat{\varphi}(p_n) \\
&= \frac{1}{2} \int \mathrm{d}^4 x_1 \int \mathrm{d}^4 x_2 \, \varphi(x_1)[\delta^{(4)}(x_1 - x_2)(\Box + m^2) + \widetilde{\Sigma}(x_1, x_2)]\varphi(x_2) \\
&\quad + \sum_{n=3}^{\infty} \frac{1}{n!} \int \left[\prod_{j=1}^{n} \mathrm{d}^4 x_j \right] \Gamma^{(n)}_{\mathrm{R}}(x_1, \dots, x_n) \varphi(x_1) \cdots \varphi(x_n)
\end{aligned}$$

where the hat denotes the Fourier transform. The form of the action is identical to the classical action but with quantum quantities. The bracket on the third line is the two-point Green's function $\Gamma^{(2)}$.

Remark. The effective action may be rewritten as

$$\Gamma[\varphi] = \sum_{n=2}^{\infty} \frac{1}{n!} \int \left[\prod_{j=1}^{n} \mathrm{d}^4 x_j \right] \Gamma^{(n)}_{\mathrm{R}}(x_1, \dots, x_n) \varphi(x_1) \cdots \varphi(x_n)$$

This is the functional version of the Taylor expansion. Therefore

$$\Gamma^{(n)}_{\mathrm{R}}(x_1, \dots, x_n) = \left. \frac{\delta^n \Gamma[\varphi]}{\delta\varphi(x_1) \cdots \delta\varphi(x_n)} \right|_{\varphi = 0}$$

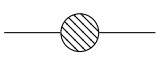
The effective action is the generating functional of the one-particle irreducible amputated connected Green's functions. The other generating functionals are $W[J]$ for the Green's function $G^{(n)}$ and $Z[J] = \ln W[J]$ for the connected Green's function $G^{(n)}_{\mathrm{c}}$. The effective action is related to the previous generating functionals. In particular, the effective action is the Legendre transform of the generating functional Z.

Remark. From the definition of the effective action, the first term is the classical kinetic term [r]. Therefore, one may write

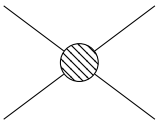
$$\Gamma[\varphi] = S[\varphi] + \text{quantum corrections}$$

Quantum corrections are classified in terms of the number of loops. The classical action is the tree-level action.

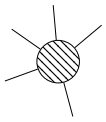
Remark. Instead of the classical action S, the effective action can be used to construct the (quantum) Feynman rules. The quantum propagator


$\to \dfrac{1}{p^2 + m^2 + \widetilde{\Sigma}(p^2)}$

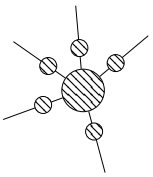
The quantum four-point vertex


$\to \lambda +$ quantum corrections

Arbitrary n-point Green's function


$\to \Gamma^{(n)}(x_1, \ldots, x_n)$

With these new rules, any contribution to the Green's function $G^{(n)}$ is only tree-level because the only diagram one may draw is the following

$G_{\mathrm{R}}^{(n)} =$ 

Therefore, one may only compute the effective action at a desired order.

The quantum Green's function is also called dressed Green's function and similarly for the vertex.

11.1 Relation between generating functionals

The effective action $\Gamma[\varphi]$ is the Legendre transform of the generating functional $Z[J]$.

Semi-classical approximation. The semi-classical approximation is also called stationary-phase approach. The following discussion is made in ordinary quantum mechanics and Minkowski space. Consider the kernel

$$K(q_f, t_f; q_i, t_i) = \int_{q_i, t_i}^{q_f, t_i} [\mathcal{D}q] \exp\left[\frac{\mathrm{i}}{\hbar} \int_{t_i}^{t_f} \mathrm{d}t\, L(q(t), \dot{q}(t))\right]$$

[r] One may find an approximation of the above path integral in the classical limit $\hbar \to 0$. This is same procedure is followed when studying classical or approximately classical limits. In the such limit, the exponential oscillates increasingly fast and, by the Riemann–Lebesgue lemma, the path integral is zero, unless the Lagrangian is stationary $\delta L = 0$. The only non-zero points are the ones that solve $\delta_q L = 0$, that is the classical equations of motion.

Let $\bar{q}(t)$ be the classical solution. Close to the classical trajectory, so trajectories around $\bar{q}$, one may set

$$q(t) = \bar{q}(t) + \sqrt{\hbar}\widetilde{q}$$

where the second addendum is a small fluctuation for $\hbar \to 0$. One may rewrite the Lagrangian as

$$\begin{aligned} L(q(t), \dot{q}(t)) = L(\bar{q} + \sqrt{\hbar}\widetilde{q}) &= L(\bar{q}) + \delta_q L\big|_{\bar{q}} \sqrt{\hbar}\widetilde{q} + \frac{1}{2}\delta_q^2 L\big|_{\bar{q}} (\sqrt{\hbar}\widetilde{q})^2 + o(\hbar) \\ &= L(\bar{q}) + \frac{1}{2}\delta_q^2 L\big|_{\bar{q}} (\sqrt{\hbar}\widetilde{q})^2 + o(\hbar) \end{aligned}$$

The kernel is approximately

$$\begin{aligned} K(q_t,t_f;q_i,t_i) &\approx \int [\mathcal{D}\widetilde{q}(t)]\, \exp\left[\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\, L(\bar{q}(t))\right] \exp\left[\frac{\mathrm{i}}{2}\int \mathrm{d}t\, \delta_q^2 L\big|_{\bar{q}}\, \widetilde{q}^2(t) + o(\hbar^0)\right] \\ &\sim \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[\bar{q}]} \int [\mathcal{D}\widetilde{q}]\, \exp\left[\frac{\mathrm{i}}{2}\int \mathrm{d}t\, \delta_q^2 L\big|_{\bar{q}}\, \widetilde{q}^2(t)\right], \quad \hbar \to 0 \\ &\approx \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[\bar{q}]} \left[\det \delta_q^2 L\big|_{\bar{q}}\right]^{-\frac{1}{2}} \end{aligned}$$

At the second line, the second integral is a Gaussian integral (where formula for path integrals? [r]). This is the semi-classical approximation.

Relation. Consider again Euclidean space. The generating functionals are

$$W[J] = \int [\mathcal{D}\varphi]\, \exp\left[-S + \int \mathrm{d}^4x\, J\varphi\right], \quad Z[J] = \ln W[J]$$

The connected Green's function is

$$G_{\mathrm{c}}^{(n)}(x_1,\ldots,x_n) = \left.\frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right|_{J=0}$$

[r] For $n=1$ one has

$$G_{\mathrm{c}}^{(1)}(x) = \left.\frac{\delta Z[J]}{\delta J(x)}\right|_{J=0} = \left.\frac{\langle 0|\,\varphi(x)\,|0\rangle_J}{\langle 0|0\rangle_J}\right|_{J=0} = \langle 0|\,\varphi(x)\,|0\rangle$$

One may define a new function

$$\varphi_J(x) \equiv \frac{\delta Z[J]}{\delta J(x)} = \frac{\langle 0|\,\varphi(x)\,|0\rangle_J}{\langle 0|0\rangle_J}$$

One may compute the generating functional $Z[J]$ in a perturbative manner. One has already computed the power in Planck's constant $\hbar$. The expansion in loops is equivalent to the expansion in powers of the Planck's constant.

Writing explicitly Planck's constant $\hbar$ in the generating functionals W and Z, one may write

$$Z_\hbar[J] = \ln \int [\mathcal{D}\varphi]\, \exp\left[-\frac{1}{\hbar}S + \frac{1}{\hbar}\int \mathrm{d}^4x\, J\varphi\right] = \sum_L \hbar^{L-1} Z_\hbar^{(L)}[J]$$

where L is the number of loops. One may define a new quantity by replacing the classical action with the effective action

$$\begin{aligned} Z_{\Gamma,\hbar}[J] &= \ln \int [\mathcal{D}\varphi]\, \exp\left[-\frac{1}{\hbar}\Gamma + \frac{1}{\hbar}\int \mathrm{d}^4x\, J\varphi\right] = \sum_L \hbar^{L-1} Z_{\Gamma,\hbar}^{(L)}[J] \\ &= \frac{1}{\hbar} Z_{\Gamma,\hbar}^{(0)}[J] + \sum_{L=1}^{\infty} \hbar^{L-1} Z_{\Gamma,\hbar}^{(L)}[J] \sim \frac{1}{\hbar} Z_{\Gamma,\hbar}^{(0)}[J], \quad \hbar \to 0 \end{aligned}$$

At the second line, the expression is expanded into the tree-level contribution and the rest. The limit $\hbar \to 0$ is equivalent to the tree-level approximation. In such approximation, the effective action becomes the classical action and

$$Z_{\Gamma,\hbar}^{(0)}[J] = Z[J]$$

Therefore

$$Z_{\Gamma,\hbar}[J] = \frac{1}{\hbar} Z[J] + o(\hbar^{-1})$$

In the limit $\hbar \to 0$, the integral of the generating functional $Z_{\Gamma,\hbar}$ can be computed using the semi-classical approximation. First one has to find the stationary configuration. It is given by solving the quantum equations of motion

$$\delta_{\varphi(x)}\left[-\Gamma[\varphi] + \int \mathrm{d}^4y\, J(y)\varphi(y)\right] = 0 \implies \delta_{\varphi(x)}\Gamma[\varphi] = J(x)$$

Let the solution be $\varphi_{\mathrm{c}}(x)$. Expanding around the classical solution, one has

$$\varphi(x) = \varphi_{\mathrm{c}}(x) + \sqrt{\hbar}\widetilde{\varphi}(x)$$

The generating functional is

$$\begin{aligned} Z_{\Gamma,\hbar}[J] &= \ln \exp\left[-\frac{1}{\hbar}\Gamma[\varphi_{\mathrm{c}}] + \frac{1}{\hbar}\int \mathrm{d}^4x\, J(x)\varphi_{\mathrm{c}}(x) + o(\hbar^{-1})\right] \\ &= -\frac{1}{\hbar}\Gamma[\varphi_{\mathrm{c}}] + \frac{1}{\hbar}\int \mathrm{d}^4x\, J(x)\varphi_{\mathrm{c}}(x) + o(\hbar^{-1}) \end{aligned}$$

Comparing this expression with the above, one obtains

$$\begin{aligned} \frac{1}{\hbar}Z[J] + o(\hbar^{-1}) &= -\frac{1}{\hbar}\Gamma[\varphi_{\mathrm{c}}] + \frac{1}{\hbar}\int \mathrm{d}^4x\, J(x)\varphi_{\mathrm{c}}(x) + o(\hbar^{-1}) \\ \Gamma[\varphi_{\mathrm{c}}] &= \left[\int \mathrm{d}^4x\, J\varphi_{\mathrm{c}} - Z[J]\right]_{J=J(\varphi_c)} \end{aligned}$$

This is similar to the Legendre transform.

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The effective action $\Gamma[\varphi_{\mathrm{c}}]$ is the analogue of the Hamiltonian and generating functional Z is the analogue of the Lagrangian with

$$\dot{q} \leftrightarrow J\,, \quad p \leftrightarrow \varphi_{\mathrm{c}}$$

In fact

$$p = \partial_{\dot{q}}\mathcal{L} \leftrightarrow \varphi_{\mathrm{c}}(x) = \delta_{J(x)}Z[J]$$

This may be seen as follows. Recalling

$$\varphi_J(x) = \delta_{J(x)}Z[J]$$

one shall prove that

$$\varphi_{\mathrm{c}}(x) = \varphi_J(x)$$

In fact

$$\begin{aligned} \varphi_J(x) = \delta_{J(x)}Z[J] &= \delta_{J(x)}\left[-\Gamma[\varphi_{\mathrm{c}}] + \int \mathrm{d}^4y\, J\varphi_{\mathrm{c}}\right] \\ &= -\int \mathrm{d}^4y\, \delta_{\varphi(y)}\Gamma[\varphi_{\mathrm{c}}]\, \delta_{J(x)}\varphi(y) + \varphi_{\mathrm{c}}(x) + \int \mathrm{d}^4y\, J(y)\, \delta_{J(x)}\varphi_{\mathrm{c}}(y) \\ &= \varphi_{\mathrm{c}}(x) + \int \mathrm{d}^4y\, \left[J(y) - \delta_{\varphi(y)}\Gamma[\varphi_{\mathrm{c}}]\right]\delta_{J(x)}\varphi_{\mathrm{c}}(y) \\ &= \varphi_{\mathrm{c}}(x) + 0 \end{aligned}$$

At the third line, the integrand is the quantum equations of motions. Therefore

$$\boxed{\varphi_{\mathrm{c}}(x) = \delta_{J(x)}Z[J]}$$

When computing the effective action in momentum space, one computes the Green's functions which contain powers of the momentum. In configuration space, this corresponds to derivatives. One may obtain an expansion of the effective action $\Gamma[\varphi]$ (the index c is always understood)

$$\Gamma[\varphi] = \int \mathrm{d}^4x\, \left[-V(\varphi) + \frac{1}{2}Z(\varphi)\, \partial_\mu\varphi\, \partial^\mu\varphi + \text{higher derivatives}\right]$$

where $V(\varphi)$ is the quantum potential called effective potential.

[r] One may reconcile the above expression with the definition of the effective action in configuration space. From momentum space, one has [r]

$$\begin{aligned}\Gamma[\varphi] &= \sum_{n=2}^{\infty} \frac{1}{n!} \int \left[\prod_{j=1}^{n} \frac{\mathrm{d}^4 p_j}{(2\pi)^4}\right] (2\pi)^4 \delta^{(4)}(\textstyle\sum_j p_j) \Gamma^{(n)}(p_1, \dots, p_n) \hat{\varphi}(p_1) \cdots \hat{\varphi}(p_n) \\ &= \sum_{n=2}^{\infty} \frac{1}{n!} \int \left[\prod_{j=1}^{n} \frac{\mathrm{d}^4 p_j}{(2\pi)^4}\right] \int \mathrm{d}^4 x \, \mathrm{e}^{\mathrm{i}x(p_1+p_2+\cdots+p_n)} \Gamma^{(n)}(p_1, \dots, p_n) \int \left[\prod_{j=1}^{n} \mathrm{d}^4 x_j \, \mathrm{e}^{-\mathrm{i}p_j x_j} \varphi(x_j)\right] \\ &= \sum_{n=2}^{\infty} \frac{1}{n!} \int \mathrm{d}^4 x \left[\prod_{j=1}^{n} \mathrm{d}^4 x_j\right] \left[\prod_{k=1}^{n} \frac{\mathrm{d}^4 p_k}{(2\pi)^4} \mathrm{e}^{\mathrm{i}p_k(x-x_k)} \varphi(x_k)\right] \Gamma^{(n)}(p_1, \dots, p_n)\end{aligned}$$

Expanding around zero

$$\Gamma^{(n)}(p_1, \dots, p_n) = \Gamma^{(n)}(0, \dots, 0) + \partial_{p_j} \Gamma^{(n)}(0, \dots, 0) p_j + o(p_j)$$

[r] one has

$$\Gamma[\varphi] \sim \sum_{n=2}^{\infty} \frac{1}{n!} \Gamma^{(n)}(0, \dots, 0) \int \mathrm{d}^4 x \, \varphi^n(x)$$

Comparing this with the derivative expansion, one obtains

$$\boxed{V(\varphi) = -\sum_{n=1}^{\infty} \frac{1}{n!} \Gamma^{(n)}(0, \dots, 0) \varphi^n(x)}$$

In order to compute the effective potential $V(\varphi)$ one may compute the Feynman diagrams in momentum space with all external momenta set to zero.

Example. Consider the four-point Green's function at one-loop $\Gamma^{(n)}(p_1, p_2, p_3, p_4)$. When setting the external momenta to zero, the three channels give equal contributions to the effective potential. The integral contributing to the Green's function is

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(q^2 + m^2)[(q-p)^2 + m^2]} \rightsquigarrow \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(q^2 + m^2)^2}$$

which is easier than the original one.

The effective potential is computed in order to study spontaneous symmetry breaking (SSB) at a quantum level. Classically, a spontaneous symmetry breaking happens when the solution to the classical equations of motion breaks some symmetry of the Lagrangian. For the quantum case, the non-vanishing solutions to the quantum equations of motions (obtained from the effective action) are non-vanishing one-point functions [r]

$$\langle 0 | \, \varphi(x) \, | 0 \rangle \neq 0$$

even for $J = 0$. In fact, the quantum equations of motion are

$$\delta_{\varphi(x)} \Gamma[\varphi] = J(x)$$

and one has

$$J = 0 \implies \delta_{\varphi(x)} \Gamma[\varphi] = 0$$

The translational invariant vacuum expectation values are solutions of

$$\delta_{\varphi(x)} V(\varphi) = 0$$

12 Renormalization group

See Cheng–Li §3, Ramond, Anselmi, Peskin. The renormalization group is general framework, valid for any renormalizable theory. The $\lambda\varphi^4$ theory is used as an example.

The definitions of the renormalized parameters present an arbitrariness since one arbitrarily fixes the subtraction point.

Previously, there different renormalization (or subtraction) schemes used are

- intermediate scheme, expand around $p^2 = 0$,
- on-shell scheme, $p^2 = m^2$ where m^2 is the physical mass,
- generic scheme, $p^2 = M^2$,
- minimal subtraction (MS) scheme (also called mass-independent).

Consider the first three schemes, also called momentum renormalization schemes.

Intermediate renormalization scheme. In the $\lambda\varphi^4$ theory, the divergent Green's functions are the two-point and four-point. One has computed the self-energy and expanded it around zero

$$\Sigma(p^2) = \Sigma(0) + \Sigma'(0)p^2 + \widetilde{\Sigma}(p^2)\,, \quad \widetilde{\Sigma}(0) = 0\,, \quad \partial_{p^2}\widetilde{\Sigma}(0) = 0$$

The last two equations are called normalization conditions. The fact that these two are zero is related to the fact that the subtraction point is $p^2 = 0$.

For the renormalized two-point Green's function

$$\Gamma^{(2)}_{\mathrm{R}}(p^2) = p^2 + m^2 + \widetilde{\Sigma}(p^2)$$

the normalization conditions are

$$\Gamma^{(2)}_{\mathrm{R}}(p^2) = m^2\,, \quad \mathrm{d}_{p^2}\Gamma^{(2)}_{\mathrm{R}}(0) = 1$$

The same can be done for the four-point Green's function

$$\Gamma^{(4)}(p_1, p_2, p_3, p_4) = \Gamma^{(4)}(0,0,0,0) + \widetilde{\Gamma}^{(4)}(p_1, p_2, p_3, p_4)\,, \quad \widetilde{\Gamma}^{(4)}(0,0,0,0) = 0$$

[r] While the renormalized four-point Green's function is

$$\Gamma^{(4)}_{\mathrm{R}}(p_1, p_2, p_3, p_4) = \lambda + \widetilde{\Gamma}^{(4)}(p_1, p_2, p_3, p_4)$$

with normalization condition

$$\Gamma_{\mathrm{R}}(0,0,0,0) = \lambda$$

The parameter m is not the physical mass, since $p^2 = 0$ is not physical for a massive particle. Similarly, λ is not the physical coupling constant. The physical quantities are related to them, but not equal.

The physical mass $\bar{m}$ is the pole of the propagator, so that one has

$$\Gamma^{(2)}_{\mathrm{R}}(p^2 = \bar{m}^2) = 0$$

Since the two-point Green's function is a function of the parameters m and λ, the above is an algebraic expression connecting them to the physical mass $\bar{m}$ and physical coupling constant.

On-shell renormalization scheme. Consider the subtraction point $p^2 = m^2$ where m is the physical mass. Therefore, the self-energy is

$$\Sigma(p^2) = \Sigma(m^2) + \Sigma'(m^2)(p^2 - m^2) + \widetilde{\Sigma}(p^2, m^2)\,, \quad \widetilde{\Sigma}(p^2 = m^2) = 0\,, \quad \mathrm{d}_{p^2}\widetilde{\Sigma}(p^2 = m^2) = 0$$

In Minkowski, the renormalized two-point Green's function is

$$-\Gamma^{(2)}_{\mathrm{R}}(p^2) = p^2 - m^2 + \widetilde{\Sigma}(p^2, m^2)$$

The normalization conditions are

$$\Gamma^{(2)}_{\mathrm{R}}(p^2 = m^2) = 0\,, \quad \mathrm{d}_{p^2}\Gamma^{(2)}_{\mathrm{R}}(p^2 = m^2) = -1$$

From which m is the physical mass.

For the four-point Green's function, one considers the symmetrical subtraction point

$$s = t = u = \frac{4}{3}m^2$$

Therefore

$$\Gamma^{(4)}_{\mathrm{R}}(p_1, p_2, p_3, p_4) = -\mathrm{i}\lambda + \widetilde{\Gamma}^{(4)}(p_1, p_2, p_3, p_4)\,, \quad \widetilde{\Gamma}^{(4)}(s = t = u = {}^4\!/_3\, m^2) = 0$$

The normalization condition is

$$\Gamma^{(4)}_{\mathrm{R}}(s = t = u = {}^4\!/_3\, m^2) = -\mathrm{i}\lambda$$

Generic renormalization scheme. Going back to Euclidean, consider a generic subtraction point $p^2 = M^2$. The normalization conditions for the two-point Green's function are

$$\Gamma^{(2)}_{\mathrm{R}}(p^2 = M^2) = M^2 + m^2\,, \quad \mathrm{d}_{p^2}\Gamma^{(2)}_{\mathrm{R}}(p^2 = M^2) = 1$$

For the four-point Green's function, the condition is

$$\Gamma^{(4)}_{\mathrm{R}}(p_j^2 = M_j^2) = \lambda$$

Remark. The normalization conditions contain a lot of information: the number of divergent Green's functions, their degrees of divergence, the subtraction point, how the physical parameters are uniquely defined in terms of the renormalized ones.

Remark. The normalization conditions depend on the renormalization scheme. There is still an arbitrariness in the choice of the scheme [r].

Group. One may wonder what is the meaning of "group" in renormalization group and how one may relate the renormalized quantities in different renormalization schemes. Consider two different schemes R and R'. For the first scheme, one has

$$\varphi_R = Z_\varphi(R)^{-\frac{1}{2}}\varphi_0\,, \quad m_R^2 = m_0^2 - \delta m^2(R)\,, \quad \lambda_R = Z_\lambda(R)^{-1} Z_\varphi^2(R)\lambda_0$$

For the second scheme, one has

$$\varphi_{R'} = Z_\varphi(R')^{-\frac{1}{2}}\varphi_0\,, \quad m_{R'}^2 = m_0^2 - \delta m^2(R')\,, \quad \lambda_{R'} = Z_\lambda(R')^{-1} Z_\varphi^2(R')\lambda_0$$

[r] mass, minus?

The renormalized Lagrangian is

$$\mathcal{L}_R(\varphi_R, m_R, \lambda_R; M_R^2) = \mathcal{L}_0(\varphi_0, m_0, \lambda_0) = \mathcal{L}_{R'}(\varphi_{R'}, m_{R'}, \lambda_{R'}; M_{R'}^2)$$

where M_R^2 is the subtraction point. This relation must hold for both schemes. There must be a relation between the set of counter terms in the first Lagrangian $\mathcal{L}_R$ and in the second Lagrangian $\mathcal{L}_{R'}$.

The relation between the counter terms may be found by looking at the renormalization functions and substituting the bare parameters

$$\varphi_{R'} = Z_\varphi(R')^{-\frac{1}{2}}\varphi_0 = Z_\varphi(R')^{-\frac{1}{2}} Z_\varphi^{\frac{1}{2}}(R)\varphi_R = \left[\frac{Z_\varphi(R)}{Z_\varphi(R')}\right]^{\frac{1}{2}} \varphi_R \equiv Z_\varphi^{-\frac{1}{2}}(R', R)\varphi_R$$

Since the fields are finite and the renormalization functions diverge, then the bracket must be a finite quantity.

Similarly, for the mass, one has

$$m_{R'}^2 = m_R^2 + [\delta m^2(R) - \delta m^2(R')] \equiv m_R^2 + \delta m^2(R', R)$$

The bracket must be a finite term. For the coupling constant, one has

$$\lambda_{R'} = [Z_\lambda^{-1}(R') Z_\lambda(R) Z_\varphi^2(R', R)]\lambda_R \equiv Z_\lambda^{-1}(R', R) Z_\varphi^2(R', R)\lambda_R$$

The bracket must be finite.

The above three relations constitute an algebraic group.

Lecture 23

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12.1 Callan–Symanzik equation

When on fixes the subtraction point $p^2 = M^2$, one fixes the energy scale (or mass scale) of the renormalization of the theory and its Green's functions

$$\Gamma^{(n)}_{\mathrm{R}}(p_1, \dots, p_n; m_{\mathrm{R}}, \lambda_{\mathrm{R}})$$

The n-point Green's functions are related the scattering amplitudes and cross-sections at a particular energy scale. It is interesting to understand how the renormalized quantities depend on the energy scale. To this end, one shall introduce the Callan–Symanzik equation.

Covariant regularization. Historically, this equation was found in covariant regularization. Its dimensional regularization counter part is easier to find.

One introduces a cutoff Λ on which the renormalization functions depend. For example, in the $\lambda\varphi^4$ theory, at one-loop one has found

$$Z_\varphi = 1\,, \quad Z_\lambda = 1 + \frac{3\lambda}{32\pi^2} \ln \frac{\Lambda^2}{m^2}\,, \quad \delta m^2 = \frac{\lambda}{32\pi^2}\Lambda^2$$

One may observe that the derivative of the bare propagator is

$$\partial_{m_0^2} \frac{1}{p^2 + m_0^2} = -\frac{1}{(p^2 + m_0^2)^2} = \frac{1}{p^2 + m_0^2}(-1)\frac{1}{p^2 + m_0^2}$$

This means that a propagator is split in two propagators, connected with an insertion of zero momentum. The insertion involves a composite operator that does not carry any momentum, in particular

$$O = -\frac{1}{2}\varphi_0^2$$

The situation can be generalized to an n-point Green's function

$$\begin{aligned}
\partial_{m_0^2} G^{(n)}(x_1, \dots, x_n) &= \partial_{m_0^2} \langle 0| \mathcal{T}\{\varphi_0(x_1) \cdots \varphi_0(x_n)\} |0\rangle = \partial_{m_0^2} \int [\mathcal{D}\varphi]\, \mathrm{e}^{-S_0} \varphi_0(x_1) \cdots \varphi_0(x_n) \\
&= \int [\mathcal{D}\varphi]\, \mathrm{e}^{-S_0} \int \mathrm{d}^4x \left[-\frac{1}{2}\varphi_0^2(x)\right] \varphi_0(x_1) \cdots \varphi_0(x_n) \\
&= -\int \mathrm{d}^4x \, \langle 0| \mathcal{T}\{{}^1\!/_2\, \varphi_0^2(x) \varphi_0(x_1) \cdots \varphi_0(x_n)\} |0\rangle \\
&\equiv -\int \mathrm{d}^4x \, G^{(n)}_{\varphi^2}(x_1, \dots, x_n; x)
\end{aligned}$$

where the bare action is

$$S_0 = \int \mathrm{d}^4x \left[\frac{1}{2}\, \partial_\mu \varphi_0\, \partial^\mu \varphi_0 + \frac{1}{2} m_0^2 \varphi_0^2 + \text{interactions}\right]$$

In momentum space, one has

$$\partial_{m_0^2} \Gamma^{(n)}(p_1, \dots, p_n) = -\Gamma^{(n)}_{\varphi^2}(p_1, \dots, p_n; 0)$$

All the quantities are bare and not renormalized. The renormalized quantities are

$$\Gamma^{(n)}_{\mathrm{R}}(p_i; m, \lambda) = Z_\varphi^{\frac{n}{2}} \Gamma^{(n)}(p_i; m_0, \lambda_0)\,, \quad (\Gamma^{(n)}_{\varphi^2})_{\mathrm{R}}(p_i; p; m, \lambda) = Z_{\varphi^2}^{-1} Z_\varphi^{\frac{n}{2}} \Gamma^{(n)}_{\varphi^2}(p_i; p; m_0, \lambda_0)$$

One may consider the finite parameters m and λ functions of their bare counter parts (which are treated as independent parameters).

The derivative of the Green's function is (swapping the left- and right-hand sides)

$$\begin{aligned}
-Z_{\varphi^2} Z_\varphi^{-\frac{n}{2}} (\Gamma^{(n)}_{\varphi^2})_{\mathrm{R}}(p_i;0;m,\lambda) &= \partial_{m_0^2}[Z_\varphi^{-\frac{n}{2}} \Gamma^{(n)}_{\mathrm{R}}(p_i;m,\lambda)] \\
-Z_{\varphi^2} Z_\varphi^{-\frac{n}{2}} (\Gamma^{(n)}_{\varphi^2})_{\mathrm{R}}(p_i;0;m,\lambda) &= -\frac{n}{2} Z_\varphi^{-\frac{n}{2}-1} (\partial_{m_0^2} Z_\varphi) \Gamma^{(n)}_{\mathrm{R}} + Z_\varphi^{-\frac{n}{2}} \partial_{m^2} \Gamma^{(n)}_{\mathrm{R}} \partial_{m_0^2} m^2 \\
&\quad + Z_\varphi^{-\frac{n}{2}} \partial_\lambda \Gamma^{(n)}_{\mathrm{R}} \partial_{m_0^2} \lambda \\
-Z_{\varphi^2} (\Gamma^{(n)}_{\varphi^2})_{\mathrm{R}} &= \partial_{m_0^2} m^2 \, \partial_{m^2} \Gamma^{(n)}_{\mathrm{R}} + \partial_{m_0^2} \lambda \, \partial_\lambda \Gamma^{(n)}_{\mathrm{R}} - \frac{n}{2} (\partial_{m_0^2} \ln Z_\varphi) \Gamma^{(n)}_{\mathrm{R}} \\
-Z_{\varphi^2} 2m^2 [\partial_{m_0^2} m^2]^{-1} (\Gamma^{(n)}_{\varphi^2})_{\mathrm{R}} &= 2m^2 \, \partial_{m^2} \Gamma^{(n)}_{\mathrm{R}} + 2m^2 \, \partial_{m_0^2} \lambda \, [\partial_{m_0^2} m^2]^{-1} \, \partial_\lambda \Gamma^{(n)}_{\mathrm{R}} \\
&\quad - \frac{n}{2} 2m^2 \, (\partial_{m_0^2} \ln Z_\varphi) [\partial_{m_0^2} m^2]^{-1} \Gamma^{(n)}_{\mathrm{R}}
\end{aligned}$$

At the fifth line, one has divided the whole expression by

$$\frac{1}{2m^2} \, \partial_{m_0^2} m^2$$

The coefficients are the renormalization group functions

$$\alpha \equiv 2 Z_{\varphi^2} [\partial_{m_0^2} m^2]^{-1} \,, \quad \beta \equiv 2m^2 \, \partial_{m_0^2} \lambda \, [\partial_{m_0^2} m^2]^{-1} \,, \quad \gamma \equiv m^2 \, (\partial_{m_0^2} \ln Z_\varphi) [\partial_{m_0^2} m^2]^{-1}$$

The β parameter is called beta function. The γ parameter is the anomalous dimension. These have deep physical meanings. Knowing that

$$2m^2 \, \partial_{m^2} = m \, \partial_m$$

One obtains the Callan–Symanzik equation in covariant regularization (for the $\lambda\varphi^4$ theory)

$$\boxed{(m \, \partial_m + \beta \, \partial_\lambda - n\gamma) \Gamma^{(n)}_{\mathrm{R}}(p_i;m,\lambda) = -m^2 \alpha (\Gamma^{(n)}_{\varphi^2})_{\mathrm{R}}(p_i;0;m,\lambda)}$$

The role of the cutoff is not yet manifest.

Remark. The three renormalization group functions are not independent: in particular the first, α, and the third, γ. Consider the case of the two-point Green's function $n = 2$

$$(m \, \partial_m + \beta \, \partial_\lambda - 2\gamma) \Gamma^{(2)}_{\mathrm{R}}(p_i;m,\lambda) = -m^2 \alpha (\Gamma^{(2)}_{\varphi^2})_{\mathrm{R}}(p_i;0;m,\lambda)$$

The differential operator on the left-hand side (the parentheses) does not act on the momentum p^2. Therefore, one may evaluate the entire equation at some fixed value of the momentum. One may choose $p^2 = 0$ using the intermediate renormalization. The two-point function is

$$\Gamma^{(2)}_{\mathrm{R}}(0;m,\lambda) = m^2 \,, \quad (\Gamma^{(2)}_{\varphi^2})_{\mathrm{R}}(0;0;m,\lambda) = 1$$

The Callan–Symanzik equation becomes

$$(m \, \partial_m + \beta \, \partial_\lambda - 2\gamma) m^2 = -m^2 \alpha \implies 2m^2 - 2\gamma m^2 = -\alpha m^2 \implies \alpha = 2(\gamma - 1)$$

Remark. The renormalized Green's functions $\Gamma^{(n)}_{\mathrm{R}}$ and $(\Gamma^{(n)}_{\varphi^2})_{\mathrm{R}}$ do not depend on the cutoff Λ. As such, in the Callan–Symanzik equation, one expects no dependence on the cutoff parameter. This implies that the renormalization group functions do not depend on it.

Let $n = 2$. One applies the derivative with respect to the momentum p^2 on both sides of the Callan–Symanzik equation to get

$$(m \, \partial_m + \beta \, \partial_\lambda - 2\gamma) \, \partial_{p^2} \Gamma^{(2)}_{\mathrm{R}} \big|_{p^2=0} = -2(\gamma - 1) m^2 \, \partial_{p^2} (\Gamma^{(2)}_{\varphi^2})_{\mathrm{R}} \big|_{p^2=0}$$

Recalling the renormalization condition

$$\partial_{p^2} \Gamma^{(2)}_{\mathrm{R}} \big|_{p^2=0} = 1$$

one has

$$-2\gamma = -2(\gamma - 1) m^2 \, \partial_{p^2} (\Gamma^{(2)}_{\varphi^2})_{\mathrm{R}} \big|_{p^2=0}$$

The right-hand side does not depend on the cutoff parameter Λ. Therefore, the function γ is also independent of the parameter. Going back to the Callan–Symanzik equation, one may see that β must also be independent. This is in agreement with the fact that the two functions β and γ have physical meaning.

Remark. From the definitions of the renormalization group functions, and recalling that the mass dimension of the coupling constant is zero for the $\lambda\varphi^4$ theory, the functions have all mass dimension equal to zero.

For the $\lambda\varphi^4$ theory, the only possible dependence is on the coupling constant

$$\beta = \beta(\lambda)\,, \quad \gamma = \gamma(\lambda)$$

Simpler expressions. One may look for simpler expressions defining the renormalization group functions. Considering the bare parameters as independent parameters on which the finite ones depend, then

$$m = m(m_0, \lambda_0, \Lambda)\,, \quad \lambda = \lambda(\lambda_0, {}^{\Lambda}\!/_{m_0}) = \lambda(\lambda_0, {}^{\Lambda}\!/_{m})\,, \quad Z_\varphi = Z_\varphi(\lambda_0, {}^{\Lambda}\!/_{m})$$

recalling that Z_φ has no mass dimension. In general, for a function of the ratio $\frac{\Lambda}{m}$, it must hold

$$(m\,\partial_m + \Lambda\,\partial_\Lambda) f({}^{\Lambda}\!/_{m}) = 0$$

Therefore

$$\begin{aligned}
\beta &= 2m^2\,\partial_{m_0^2}\lambda\,[\partial_{m_0^2} m^2]^{-1} = 2m^2\,\partial_{m^2}\lambda = m\,\partial_m\lambda = -\Lambda\,\partial_\Lambda\lambda = -\partial_{\ln\Lambda}\lambda \\
&= -\partial_{\ln\Lambda}(\lambda_0 Z_\lambda^{-1} Z_\varphi^2) = -\lambda_0\,\partial_{\ln\Lambda}(Z_\lambda^{-1} Z_\varphi^2)\,, \quad \bar{Z} \equiv Z_\lambda^{-1} Z_\varphi^2 \\
&= -\lambda_0\,\partial_{\ln\Lambda}\bar{Z} = -\lambda_0 \frac{\bar{Z}}{\bar{Z}}\,\partial_{\ln\Lambda}\bar{Z} = -\lambda\frac{1}{\bar{Z}}\,\partial_{\ln\Lambda}\bar{Z} = \boxed{-\lambda\,\partial_{\ln\Lambda}\ln\bar{Z}}
\end{aligned}$$

In the second line, the bare parameter λ_0 is considered an independent parameter so it can be brought out of the derivative. The beta function is related to the renormalization of the coupling constant: it describes the behaviour of the renormalization function of the coupling constant according to the position of the cutoff on the energy axis.

For the anomalous dimension, one has

$$\begin{aligned}
\gamma &= m^2\,(\partial_{m_0^2}\ln Z_\varphi)[\partial_{m_0^2} m^2]^{-1} = m^2\,\partial_{m^2}\ln Z_\varphi = \frac{1}{2} m\,\partial_m \ln Z_\varphi \\
&= -\frac{1}{2}\Lambda\,\partial_\Lambda \ln Z_\varphi = \boxed{-\frac{1}{2}\,\partial_{\ln\Lambda}\ln Z_\varphi}
\end{aligned}$$

The anomalous dimension is related to the renormalization function of the field [r]: it describes the behaviour of the [r] according to the position of the cutoff.

[r] Since the theory is studied through a perturbative expansion, then the renormalization functions are given through the loop expansion.

Example. Consider the $\lambda\varphi^4$ theory at one-loop. The renormalization functions are

$$Z_\varphi = 1\,, \quad Z_\lambda = 1 + \frac{3\lambda}{32\pi^2}\ln\frac{\Lambda^2}{m^2}\,, \quad \delta m^2 = \frac{\lambda}{32\pi^2}\Lambda^2$$

Considering

$$\bar{Z} = Z_\lambda^{-1} = 1 - \frac{3\lambda}{32\pi^2}\ln\frac{\Lambda^2}{m^2} + o(\lambda) = 1 - \frac{3\lambda}{16\pi^2}\ln\frac{\Lambda}{m} + o(\lambda)$$

the beta function at one-loop is

$$\beta = -\lambda\,\partial_{\ln\Lambda}\ln\left[1 - \frac{3\lambda}{16\pi^2}\ln\frac{\Lambda}{m} + o(\lambda)\right] = \lambda\,\partial_{\ln\Lambda}\left[\frac{3\lambda}{16\pi^2}\ln\frac{\Lambda}{m} + o(\lambda)\right] = \frac{3\lambda^2}{16\pi^2} + o(\lambda^2)$$

The anomalous dimension is

$$\gamma = 0 + o(\lambda)$$

Generalized formula. The Callan–Symanzik equation is found by differentiating with respect to the bare mass. This procedure can be generalized to other Green's function with several composite operators. Consider the composite operators $A, B, C, \ldots$. These are related to the bare ones through their own renormalization functions

$$O = Z_O^{-1} O_0$$

The renormalized Green's function is

$$(\Gamma^{(n)}_{AB\ldots})_{\mathrm{R}} = Z_A^{-1} Z_B^{-1} \cdots Z_\varphi^{\frac{n}{2}} (\Gamma^{(n)}_{AB\ldots})_0$$

The Callan–Symanzik equation becomes

$$(m\,\partial_m + \beta\,\partial_\lambda - n\gamma + \gamma_{AB\ldots})(\Gamma^{(n)}_{AB\ldots})_{\mathrm{R}} = -\alpha m^2 (\Gamma^{(n)}_{\varphi^2 AB\ldots})_{\mathrm{R}}$$

with

$$\gamma_{AB\ldots} = -\frac{1}{2}\,\partial_{\ln\Lambda} \ln(Z_A Z_B \cdots)$$

Exercise. Prove the generalized Callan–Symanzik equation.

12.2 Asymptotic behaviour

To understand how the renormalized Green's functions rescale when moving along the energy axis, one shall apply the Callan–Symanzik equation. This objective can be rephrased. Suppose to rescale the Euclidean momenta

$$p_i \to \sigma p_i\,, \quad \Gamma^{(n)}_{\mathrm{R}}(p_i) \to \Gamma^{(n)}_{\mathrm{R}}(\sigma p_i)$$

One would like to study how the renormalized Green's function behaves when the parameter σ is changed. In particular, how it behaves in the deep ultraviolet region, $\sigma \to \infty$.

Definition (Non-exceptional momenta)**.** Given a set of Euclidean momenta p_i with $i = 1, \ldots, n$ (recalling $p_i^2 < 0$), the momenta are non-exceptional if there is no non-trivial subset $\{p_{i_1}, \ldots, p_{i_k}\}$ with

$$\sum_{j=1}^{k} p_{i_j} = 0$$

This means that only

$$\sum_{i=1}^{n} p_i = 0$$

is acceptable.

Theorem (Weinberg)**.** Rescaling the momenta of the renormalized Green's function $\Gamma^{(n)}_{\mathrm{R}}(p_1, \ldots, p_n)$ with non-exceptional momenta implies that the Green's function $\Gamma^{(n)}_{\mathrm{R}}$ behaves as

$$\Gamma^{(n)}_{\mathrm{R}}(\sigma p_i) \sim \sigma^{4-n}[a_0(\ln\sigma)^{b_0} + \lambda a_1(\ln\sigma)^{b_1} + o(\lambda)]\,, \quad \sigma \to \infty$$

in the deep ultraviolet region. Similarly

$$(\Gamma^{(n)}_{\varphi^2})_{\mathrm{R}}(\sigma p_i) \sim \sigma^{2-n}[c_0(\ln\sigma)^{(d_0)} + \lambda c_1(\ln\sigma)^{(d_1)} + o(\lambda)]\,, \quad \sigma \to \infty$$

Proposition. The power of the coefficient σ is the (naive) mass dimension of the Green's function.

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Proof. The mass dimension of the scalar field is $\dim\varphi = 1$. From this, the n-point Green's function has mass dimension $\dim G^{(n)}(x_i) = n$. In momentum space, one has

$$G^{(n)}(p_i)(2\pi)^4\delta^{(4)}(\textstyle\sum_i p_i) = \int\left[\prod_i \mathrm{d}^4x_i\,\mathrm{e}^{\mathrm{i}p_ix_i}\right]G^{(n)}(x_j)$$

the mass dimensions are then

$$\dim G^{(n)}(p_i) + \dim\delta^{(4)} = -4n + n \implies \dim G^{(n)}(p_i) - 4 = -3n \implies \dim G^{(n)}(p_i) = 4 - 3n$$

For the two-point Green's function, one has

$$\dim G^{(2)}(p_i) = 2$$

For the amputated Green's function, one obtains

$$\dim\Gamma^{(n)}(p_i) = \dim\left[G^{(n)}(p_i)\left(\prod_j G^{(2)}(p_j)\right)^{-1}\right] = 4 - 3n + 2n = 4 - n$$

Similarly, one finds that

$$\dim\Gamma^{(n)}_{\varphi^2}(p_i;p) = 2 - n$$

□

Exercise. Prove the proposition for Γ_{φ^2}.

Example. An example of the validity of Weinberg theorem is the following. Consider the four-point Green's function $\Gamma^{(4)}$ at one-loop in the $\lambda\varphi^4$ theory. The integral associated to the one-loop diagram of the vertex function is

$$\Gamma(p)|_{1\mathrm{L}} \propto \int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{(q^2+m^2)[(q-p)^2+m^2]}$$

Recall that the regularization used is the covariant regularization. One shall compute the integral in three complementary regions.

Let $|q| \gg |p|$. One has

$$\Gamma(p)|_{1\mathrm{L}} \sim \int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{(q^2+m^2)^2} \sim \ln\frac{\Lambda}{m}$$

The mass dimension is equivalent to the superficial degree of divergence and is zero.

Let $|q| \ll |p|$. One has

$$\Gamma(p)|_{1\mathrm{L}} \sim \frac{1}{p^2+m^2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{q^2+m^2} \sim \frac{1}{p^2}\Lambda^2\,, \quad |p| \gg 1$$

The mass dimension is 2.

Let $|q| \sim |p|$. One has

$$\Gamma(p)|_{1\mathrm{L}} \sim \ln\frac{\Lambda}{m} + \ln\frac{|p|}{m} + \cdots$$

After renormalization, the cutoff contributions do not appear. For a rescale $p \to \sigma p$ and in the deep ultraviolet, $\sigma \to \infty$, the dominant term is the last of the three above

$$\ln(\sigma|p|)$$

This confirms that, up to one-loop, the four-point Green's function is

$$\Gamma^{(4)} \sim \sigma^{4-4}[\lambda + a_2(\ln\sigma)\lambda^2 + o(\lambda^2, \ln\sigma)]$$

[r]

Anomalous dimension. If the series of polynomials of logarithms in the renormalized Green's function can be resummed to an exponential, one obtains

$$\mathrm{e}^{C\log\sigma} = \sigma^C$$

[r] and this exponent of σ modifies the naive mass dimension of the Green's function. In this case, one has

$$\Gamma^{(n)}_{\mathrm{R}}(\sigma p_i) \sim \sigma^{4-n+C}$$

where C is the quantum correction to the classical mass dimension.

This see whether this happens, one shall exploit the asymptotic form of the Callan–Symanzik equation. Starting from the original equation

$$(m\,\partial_m + \beta\,\partial_\lambda - n\gamma)\Gamma^{(n)}_{\mathrm{R}}(p_i; m, \lambda) = -m^2\alpha(\Gamma^{(n)}_{\varphi^2})_{\mathrm{R}}(p_i; 0; m, \lambda)$$

one rescales $p_i \to \sigma p_i$. From Weinberg theorem one may see that $\Gamma^{(n)}_{\mathrm{R}} \gg \Gamma^{(n)}_{\mathrm{R}}$ to any finite order of the coupling constant λ in the deep ultraviolet. As such one may neglect the right-hand side of the above equation

$$(m\,\partial_m + \beta\,\partial_\lambda - n\gamma)\Gamma^{(n)}_{\mathrm{as}}(p_i; m, \lambda) = 0\,, \quad \sigma \to \infty$$

where Γ_{as} is the asymptotic form of Γ_{R}. This is the asymptotic form of the Callan–Symanzik equation.

From dimensional analysis [r] the n-point Green's function may be rewritten as

$$\Gamma^{(n)}_{\mathrm{as}}(\sigma p_i; m, \lambda) = m^{4-n}\bar{\Gamma}^{(n)}_{\mathrm{R}}({}^{\sigma p_i}\!/_m; \lambda)$$

where $\bar{\Gamma}^{(n)}_{\mathrm{R}}$ is dimensionless and satisfies

$$(\sigma\,\partial_\sigma + m\,\partial_m)\bar{\Gamma}^{(n)}_{\mathrm{R}}({}^{\sigma p_i}\!/_m; \lambda) = 0$$

The first term in the left-hand side of the asymptotic Callan–Symanzik equation is

$$\begin{aligned} m\,\partial_m[m^{4-n}\bar{\Gamma}^{(n)}({}^{\sigma p_i}\!/_m; \lambda)] &= (4-n)m^{4-n}\bar{\Gamma}^{(n)}_{\mathrm{R}} + m^{4-n}m\,\partial_m\bar{\Gamma}^{(n)}_{\mathrm{R}} \\ &= (4-n)\Gamma^{(n)}_{\mathrm{as}}(\sigma p_i; m, \lambda) - m^{4-n}\sigma\,\partial_\sigma\bar{\Gamma}^{(n)}_{\mathrm{R}} \\ &= [(4-n) - \sigma\,\partial_\sigma]\Gamma^{(n)}_{\mathrm{as}}(\sigma p_i; m, \lambda) \end{aligned}$$

At the second and third lines, one has used the two equations above. Inserting this expression back into the asymptotic Callan–Symanzik equation, one finds

$$\boxed{[\sigma\,\partial_\sigma - \beta(\lambda)\,\partial_\lambda + n\gamma(\lambda) + (n-4)]\Gamma^{(n)}_{\mathrm{as}}(\sigma p_i; m, \lambda) = 0\,, \quad \sigma \to \infty}$$

To solve the equation, the first step is to remove the non-derivative terms $n\gamma + n - 4$. To this end, one looks for solutions of the form

$$\Gamma^{(n)}_{\mathrm{as}}(\sigma p_i; m, \lambda) = \sigma^{4-n}\exp\left[n\int_0^\lambda \mathrm{d}x\,\frac{\gamma(x)}{\beta(x)}\right]F^{(n)}(\sigma p_i; m, \lambda)$$

Thus the equation reduces to

$$(\sigma\,\partial_\sigma - \beta(\lambda)\,\partial_\lambda)F^{(n)}(\sigma p_i; m, \lambda) = 0$$

The second step is setting

$$\sigma = \mathrm{e}^t \iff t = \ln\sigma$$

In this way, one has

$$\sigma\,\partial_\sigma = \partial_t$$

The equation then becomes

$$(\partial_t - \beta(\lambda)\,\partial_\lambda)F^{(n)}(\mathrm{e}^t p_i; m, \lambda) = 0$$

The third step is defining the running coupling constant $\bar{\lambda}(\lambda, t)$ as the solution to

$$\beta(\bar{\lambda}(\lambda,t)) = \mathrm{d}_t \bar{\lambda}(\lambda,t)\,, \quad \bar{\lambda}(\lambda,0) = \lambda$$

The first equation is the beta function equation or renormalization group flow equation. The second equation is the boundary condition. Rewriting the equation above as

$$1 = \frac{1}{\beta(\bar{\lambda})}\,\partial_t \bar{\lambda}\,, \quad x \equiv \bar{\lambda}(t)$$

and integrating, one has

$$\int_0^t \mathrm{d}t = \int_\lambda^{\bar{\lambda}(\lambda,t)} \frac{\mathrm{d}x}{\beta(x)} \implies t = \int_\lambda^{\bar{\lambda}(\lambda,t)} \frac{\mathrm{d}x}{\beta(x)}$$

Differentiating with respect to λ, one applies the Leibniz integral rule and obtains

$$0 = \partial_\lambda \int_\lambda^{\bar{\lambda}(\lambda,t)} \frac{\mathrm{d}x}{\beta(x)} = \frac{1}{\beta(\bar{\lambda})}\partial_\lambda \bar{\lambda} - \frac{1}{\beta(\lambda)} \implies \beta(\lambda)\,\partial_\lambda \bar{\lambda} = \beta(\bar{\lambda}) = \partial_t \bar{\lambda}$$

or equivalently

$$(\partial_t - \beta(\lambda)\partial_\lambda)\bar{\lambda}(\lambda,t) = 0$$

Therefore, if $F^{(n)}$ depends on t and λ through the combination $\bar{\lambda}(\lambda,t)$, then it satisfies the equation of interest. The solution to the above is

$$F^{(n)}(p_i; m, \bar{\lambda}(\lambda,t))$$

The most general solution to the asymptotic Callan–Symanzik equation is

$$\Gamma^{(n)}_{\mathrm{as}}(\mathrm{e}^t p_i; m, \lambda) = \sigma^{4-n} \exp\left[n \int_0^\lambda \mathrm{d}x\, \frac{\gamma(x)}{\beta(x)}\right] F^{(n)}(p_i; m, \bar{\lambda}(\lambda,t))$$

One may rewrite this expression. The exponent is

$$\begin{aligned} n\int_0^\lambda \mathrm{d}x\,\frac{\gamma(x)}{\beta(x)} &= n\int_0^\lambda \mathrm{d}x\,\frac{\gamma(x)}{\beta(x)} + n\int_\lambda^{\bar{\lambda}} \mathrm{d}x\,\frac{\gamma(x)}{\beta(x)} - n\int_\lambda^{\bar{\lambda}} \mathrm{d}x\,\frac{\gamma(x)}{\beta(x)} \\ &= n\int_0^{\bar{\lambda}} \mathrm{d}x\,\frac{\gamma(x)}{\beta(x)} - n\int_\lambda^{\bar{\lambda}} \mathrm{d}x\,\frac{\gamma(x)}{\beta(x)} \\ &= n\int_0^{\bar{\lambda}} \mathrm{d}x\,\frac{\gamma(x)}{\beta(x)} - n\int_0^t \mathrm{d}t'\,\gamma(\bar{\lambda}(\lambda,t')) \end{aligned}$$

From which

$$\begin{aligned} \exp\left[n\int_0^\lambda \mathrm{d}x\,\frac{\gamma(x)}{\beta(x)}\right] &= \exp\left[n\int_0^{\bar{\lambda}} \mathrm{d}x\,\frac{\gamma(x)}{\beta(x)}\right] \exp\left[-n\int_0^t \mathrm{d}t'\,\gamma(\bar{\lambda}(\lambda,t'))\right] \\ &= H(\bar{\lambda}) \exp\left[-n\int_0^t \mathrm{d}t'\,\gamma(\bar{\lambda}(\lambda,t'))\right] \end{aligned}$$

[r] The solution is then

$$\Gamma^{(n)}_{\mathrm{as}}(\mathrm{e}^t p_i; m, \lambda) = \sigma^{4-n} \exp\left[-n\int_0^t \gamma(\bar{\lambda}(\lambda,t'))\,\mathrm{d}t'\right] H(\bar{\lambda}(\lambda,t)) F^{(n)}(p_i; m, \bar{\lambda}(\lambda,t))$$

Computing this expression for $t = 0$ (equivalent to $\sigma = 1$, this is the boundary condition) one has

$$\Gamma^{(n)}_{\mathrm{as}}(p_i; m, \lambda) = H(\lambda) F^{(n)}(p_i; \lambda, m)$$

From which one may substitute $\lambda \to \bar{\lambda}$ and obtain

$$H(\bar{\lambda}) F^{(n)}(p_i; \bar{\lambda}, m) = \Gamma^{(n)}_{\mathrm{R}}(p_i; m, \bar{\lambda})$$

Therefore, the general solution to the asymptotic Callan–Symanzik equation is

$$\Gamma^{(n)}_{\text{as}}(\mathrm{e}^t p_i; m, \lambda) = \exp\left[t(4-n) - n\int_0^t \gamma(\bar\lambda(\lambda, t'))\,\mathrm{d}t'\right]\Gamma^{(n)}_{\text{as}}(p_i; m, \bar\lambda(\lambda, t))$$

The mass dimension receives a correction from the anomalous dimension γ. The way the system reacts to the rescaling of the momentum is to adapt the coupling constant. The rescaling of the momentum $p_i \to \mathrm{e}^t p_i$ has two effects:

- the naive dimension of $4-n$ is corrected by γ which is interpreted as quantum corrections. This is the reason why γ is called the anomalous dimension.
- The Green's function with a rescaled momentum implies the rescaling of the coupling constant. The system adjusts the coupling constant when moving along the energy axis.

The coupling constants run. The QED coupling constant increases with energy and this can be intuitively explained: the more energy is input into the system, the more channels are possible for both real and virtual particles [r].

In the deep ultraviolet region, the momentum is great. At a certain point it is much bigger than any mass parameter. One would be tempted to neglect the masses and one would expect to find a conformal field theory. However, the asymptotic Callan–Symanzik equation implies a non-trivial rescaling of the n-point function which would not be present if the theory tended to a CFT. The answer is that in the deep ultraviolet, even if the masses can be neglected and the classical theory is conformally invariant, renormalization introduces a subtraction point, which is a mass parameter, and breaks the CFT [r].

12.3 Minimal subtraction scheme

See Ramond (where?), Cheng–Li, §3.2. The renormalization group equation (called Callan–Symanzik equation in covariant regularization) has other forms, for example in dimensional regularization. In this regularization scheme, the number D of space-time dimensions is analytically continued to the reals and one introduces a massive parameter to keep the coupling constant dimensionless

$$D = 4 - 2\varepsilon\,, \quad \lambda k^{2\varepsilon}$$

The bare n-point Green's function is

$$\Gamma^{(n)}(p_i; m_0, \lambda_0) = Z_\varphi^{-\frac{n}{2}}\Gamma^{(n)}_{\text{R}}(p_i; m, \lambda, k)$$

If one treats the bare parameters as the independent parameters, then only the right-hand side depends on the energy scale k, both explicitly and implicitly through the mass m and the coupling constant λ. One may apply the derivative $k\,\mathrm{d}_k$ to have

$$\begin{aligned}
0 &= k\,\mathrm{d}_k[Z_\varphi^{-\frac{n}{2}}\Gamma^{(n)}_{\text{R}}(p_i; m, \lambda, k)] \\
&= -\frac{n}{2}(\frac{1}{Z_\varphi}k\,\mathrm{d}_k Z_\varphi)Z_\varphi^{-\frac{n}{2}}\Gamma^{(n)}_{\text{R}} + Z_\varphi^{-\frac{n}{2}}[k\,\partial_k\lambda\,\partial_\lambda + k\,\partial_k m\,\partial_m + k\,\partial_k]\Gamma^{(n)}_{\text{R}} \\
&= -\frac{n}{2}(k\,\mathrm{d}_k \ln Z_\varphi)Z_\varphi^{-\frac{n}{2}}\Gamma^{(n)}_{\text{R}} + Z_\varphi^{-\frac{n}{2}}[k\,\partial_k\lambda\,\partial_\lambda + k\,\partial_k m\,\partial_m + k\,\partial_k]\Gamma^{(n)}_{\text{R}}
\end{aligned}$$

From this one obtains the renormalization group equation

$$\boxed{[k\,\partial_k + \beta\,\partial_\lambda + \gamma_m m\,\partial_m - n\gamma]\Gamma^{(n)}_{\text{R}}(p_i; m, \lambda, k) = 0}$$

where one has the renormalization group functions

$$\begin{aligned}
\beta(\lambda) &\equiv k\,\partial_k\lambda = \beta(\lambda, {}^m\!/_k) \\
\gamma_m &\equiv \frac{1}{m}k\,\partial_k m = k\,\partial_k \ln m = \gamma_m(\lambda, {}^m\!/_k) \\
\gamma &\equiv \frac{1}{2}k\,\partial_k \ln Z_\varphi = \gamma(\lambda, {}^m\!/_k)
\end{aligned}$$

that are all dimensionless quantities.

There are few differences with the Callan–Symanzik equation:

- the right-hand side is zero, so one need not take the asymptotic limit;
- there is an extra renormalization group function γ_m.
- Since the renormalization group functions are dimensionless, they can only be functions of dimensionless quantities. In this case, the scale k is not a regulator and does not have to be removed. Therefore, the renormalization group functions are also functions of the ratio $\frac{m}{k}$.

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The minimal subtraction scheme is well suited for dimensional regularization where the divergences are poles in ε. The minimal subtraction scheme adds counter terms to cancel these poles and have no finite parts. [r] This implies that the counter terms do not depend on the mass and that the renormalization group functions depend only on the coupling constant λ.

Example. Consider the one-loop contribution to the self-energy function Σ in the $\lambda\varphi^4$ theory

$$\Sigma(p^2)|_{1\mathrm{L}} = \frac{\lambda m^2}{32\pi^2}\left[\frac{1}{\varepsilon} + \psi(2) + \ln\frac{4\pi k^2}{m^2} + o(\varepsilon^0)\right]$$

[r] do computations, see Cheng p 78, notes lec 13-14
The finite part is constant and does not depend on [r]. In the intermediate subtraction scheme, $p^2 = 0$, one would write

$$\Sigma(p^2) = \Sigma(0) + \text{finite}$$

since at one-loop the derivative is zero. [r] The counter term to add to the Lagrangian would be

$$\mathcal{L}^{(1)}_{\mathrm{ct}} = \frac{1}{2}\Sigma(0)\varphi^2$$

However, in the minimal subtraction scheme, the counter term removes only the divergent contributions while keeping the finite terms

$$\mathcal{L}^{(1)}_{\mathrm{ct}} = \frac{1}{2}\frac{\lambda m^2}{32\pi^2}\frac{1}{\varepsilon}\varphi^2$$

so the whole Lagrangian is then

$$\mathcal{L} + \mathcal{L}^{(1)}_{\mathrm{ct}} = \frac{1}{2}m^2\left[1 + \frac{\lambda}{32\pi^2}\frac{1}{\varepsilon}\right]\varphi^2$$

The bracket is the renormalization function of the mass

$$m_0^2 = m^2 + \delta m^2 = Z_m m^2$$

[r] The coefficient of the pole is independent of the mass.

This statement is true at any loop. For this reason the minimal subtraction scheme is also called mass-independent renormalization scheme. Therefore, one shall omit the mass dependence

$$\lambda_0 = k^{2\varepsilon}\left[\lambda + \sum_{r=1}^{\infty}\frac{a_r(\lambda)}{\varepsilon^r}\right] \equiv k^{2\varepsilon} Z_\lambda Z_\varphi^{-2}\lambda$$

$$m_0 = m\left[1 + \sum_{r=1}^{\infty}\frac{b_r(\lambda)}{\varepsilon^r}\right] \equiv Z_m m$$

$$\varphi_0 = \varphi\left[1 + \sum_{r=1}^{\infty}\frac{c_r(\lambda)}{\varepsilon^r}\right] \equiv Z_\varphi^{\frac{1}{2}}\varphi$$

One would like to find a prescription to easily compute the renormalization group functions. The beta function and the anomalous dimension are found as the coefficients of the simple pole in ε for the coupling constant and the field.

Beta function. Consider the expression of the coupling constant [r] and apply $k\,\partial_k$ to obtain

$$0 = k\,\partial_k\left[k^{2\varepsilon}\left(\lambda + \sum_{r=1}^{\infty}\frac{a_r(\lambda)}{\varepsilon^r}\right)\right] = 2\varepsilon k^{2\varepsilon}\left[\lambda + \sum_{r=1}^{\infty}\frac{a_r(\lambda)}{\varepsilon^r}\right] + k^{2\varepsilon}\left[k\,\partial_k\lambda + \sum_{r=1}^{\infty}\frac{1}{\varepsilon^r}\,\partial_\lambda a_r(\lambda)\,k\,\partial_k\lambda\right]$$

Reorganizing the expression, one has

$$2\varepsilon\lambda + [k\,\partial_k\lambda + 2a_1] + \sum_{r=1}^{\infty}\frac{1}{\varepsilon^r}[\partial_\lambda a_r(\lambda)\,k\,\partial_k\lambda + 2a_{r+1}(\lambda)] = 0$$

Knowing that the beta function is finite, it does not contain poles in ε. Therefore, it may be rewritten as

$$\beta = k\,\partial_k\lambda = \sum_{r=0}^{\infty} d_r\varepsilon^r$$

[r] One may substitute this expression back into the equation above and solve order by order in ε

$$\begin{aligned}
0 &= 2\varepsilon\lambda + \left[\sum_{r=0}^{\infty} d_r\varepsilon^r + 2a_1\right] + \sum_{r=1}^{\infty}\frac{1}{\varepsilon^r}\left[\partial_\lambda a_r(\lambda)\sum_{s=0}^{\infty} d_s\varepsilon^s + 2a_{r+1}(\lambda)\right] \\
&= \varepsilon(2\lambda + d_1) + (d_0 + 2a_1 + d_1\,\partial_\lambda a_1) + \sum_{r=1}^{\infty}\frac{1}{\varepsilon^r}[(2a_{r+1} + d_0\,\partial_\lambda a_r) + d_1\,\partial_\lambda a_{r+1}] \\
&\quad + \sum_{j=2}^{\infty}\left[d_j\varepsilon^j + d_j\sum_{r=1}^{\infty}\frac{1}{\varepsilon^{r-j}}\,\partial_\lambda a_r\right]
\end{aligned}$$

The d_j terms are multiplied by an infinite number of poles. For the equation to be zero at every order, these terms have to be zero

$$d_j = 0\,, \quad j \geq 2$$

Therefore, the beta function is not a series, but a polynomial. The equations for the remaining orders are

$$\begin{aligned}
&0 = 2\lambda + d_1 \implies d_1 = -2\lambda \\
&0 = d_0 + 2a_1 + d_1\,\partial_\lambda a_1 \implies d_0 = -2a_1 + 2\lambda\,\partial_\lambda a_1 \\
&0 = 2a_{r+1} + d_0\,\partial_\lambda a_r + d_1\,\partial_\lambda a_{r+1}
\end{aligned}$$

The beta function is then

$$\beta(\lambda) = k\,\partial_k\lambda = d_0 + d_1\varepsilon = -2a_1 + 2\lambda\,\partial_\lambda a_1 - 2\lambda\varepsilon$$

from which, in the limit $\varepsilon \to 0$,

$$\boxed{\beta(\lambda) = 2(\lambda\,\partial_\lambda - 1)a_1(\lambda)}$$

The beta function depends only on the coefficient of the simple pole in ε at any order of the expansion of the bare coupling constant λ_0 (or equivalently loops). The beta function is given by the residue of the simple pole of such expansion. The beta function measures how the coupling constant depends on the energy scale.

Anomalous dimension. Applying the same procedure for the mass m_0 and the field φ_0, one finds

$$\boxed{\gamma_m \equiv k\,\partial_k \ln m = 2\lambda\,\partial_\lambda b_1(\lambda)}\,, \quad \boxed{\gamma \equiv \frac{1}{2}k\,\partial_k \ln Z_\varphi = \lambda\,\partial_\lambda c_1(\lambda)}$$

Recursive relations. [r] Considering the equations for the surviving d_n coefficients and subtracting the second from the third, one has

$$2(a_{r+1} - a_1) + d_1(\partial_\lambda a_{r+1} - \partial_\lambda a_1) + d_0(\partial_\lambda a_r - 1) = 0$$

Applying

$$d_1 = -2\lambda\,, \quad d_0 = -2a_1 + 2\lambda\,\partial_\lambda a_1$$

one may reorganize the above as

$$2(1 - \lambda\,\partial_\lambda)(a_{r+1} - a_1) = 2(1 - \lambda\,\partial_\lambda)a_1(\partial_\lambda a_r - 1)$$

$$\boxed{(1 - \lambda\,\partial_\lambda)a_{r+1}(\lambda) = \partial_\lambda a_r\,(1 - \lambda\,\partial_\lambda)a_1}$$

This is a recursive relation for computing the residues of higher poles from the lower ones. This relation can be used to check the computations, not to find the results.

For the anomalous dimension, one may find analogous relations

$$\boxed{\lambda\,\partial_\lambda c_{r+1} = c_r\lambda\,\partial_\lambda c_1 - \partial_\lambda c_r\,(1 - \lambda\,\partial_\lambda)a_1}$$

Example. One may apply the previous prescriptions to the $\lambda\varphi^4$ theory at one-loop

$$Z_\varphi = 1\,, \quad Z_\lambda = 1 + \frac{3\lambda^2}{32\pi^2}\frac{1}{\varepsilon}\,, \quad m_0^2 = m^2\left(1 + \frac{\lambda}{32\pi^2}\frac{1}{\varepsilon}\right) = Z_m m^2$$

The coupling constant series at one-loop is

$$\lambda_0 = k^{2\varepsilon}\left[\lambda + \frac{3\lambda^2}{32\pi^2}\frac{1}{\varepsilon}\right] \implies a_1(\lambda) = \frac{3\lambda^2}{32\pi^2}$$

The beta function is then

$$\beta(\lambda) = 2(\lambda\,\partial_\lambda - 1)a_1 = 2(2a_1 - a_1) = 2a_1 = \frac{3\lambda^2}{16\pi^2}$$

This agrees with the result found in covariant regularization.

Similarly, for the anomalous dimension, one has

$$\gamma_m = k\,\partial_k \ln m = 2\lambda\,\partial_\lambda b_1 = \frac{\lambda}{16\pi^2}\,, \quad b_1 = \frac{\lambda}{32\pi^2}$$

while $\gamma = 0$.

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Solution. [r] One would like to solve the renormalization group equation [r]. Through dimensional analysis, one finds

$$\Gamma_{\rm R}^{(n)}(\sigma p_i; m, \lambda, k) = k^{4-n}\bar\Gamma_{\rm R}^{(n)}(\sigma p_i/k; m/k, \lambda)$$

and

$$(\sigma\,\partial_\sigma + k\,\partial_k + m\,\partial_m)\bar\Gamma_{\rm R}^{(n)} = 0$$

[r] where σ can be any number, it does not need to be arbitrarily large. The first term of the renormalization group equation is

$$\begin{aligned} k\,\partial_k\Gamma_{\rm R}^{(n)}(\sigma p_i; m, \lambda, k) &= k\,\partial_k[k^{4-n}\bar\Gamma_{\rm R}^{(n)}(\sigma p_i/k; m/k, \lambda)] \\ &= (4-n)\Gamma_{\rm R}^{(n)} + k^{4-n}[-\sigma\,\partial_\sigma\bar\Gamma^{(n)} - m\,\partial_m\bar\Gamma^{(n)}] \\ &= (4-n)\Gamma_{\rm R}^{(n)} - \sigma\,\partial_\sigma\Gamma_{\rm R}^{(n)} - m\,\partial_m\Gamma_{\rm R}^{(n)} \end{aligned}$$

At the second line one has applied the relation above for $\bar\Gamma$.

Substituting into the renormalization group equation, one finds

$$[\sigma\,\partial_\sigma - \beta(\lambda)\,\partial_\lambda - (\gamma_m - 1)m\,\partial_m + n\gamma + (n-4)]\Gamma_{\mathrm{R}}^{(n)}(\sigma p_i; m, \lambda, k) = 0$$

One introduces two running coupling constants $\bar{\lambda}$ and $\bar{m}$ solutions to the equations

$$\beta(\bar{\lambda}(\lambda,t)) = \mathrm{d}_t\bar{\lambda}(\lambda,t)\,, \quad [\gamma_m(\bar{\lambda}) - 1]\bar{m}(t) = \mathrm{d}_t\bar{m}(t)$$

where $t = \ln\sigma$ and with boundary conditions

$$\bar{\lambda}(t=0) = \lambda\,, \quad \bar{m}(t=0) = m$$

Proceeding as in the covariant regularization scheme, one finds the solution to be

$$\boxed{\Gamma_{\mathrm{R}}^{(n)}(\sigma p_i; m, \lambda, k) = \sigma^{4-n}\exp\left[-n\int_0^t \mathrm{d}t'\,\gamma(\bar{\lambda}(t'))\right]\Gamma_{\mathrm{R}}(p_i; \bar{m}(t), \bar{\lambda}(t), k)}$$

This solution has the same interpretation. The n-point Green's function at scaled momenta is equivalent to the n-point Green's function at non-scaled momenta, but evaluated with different parameters up to some factors giving the quantum dimension: the classical dimension corrected by the anomalous dimension.

In this scheme, when one scales the physical quantities, the value of σ can be any number, not necessarily $\sigma \to \infty$. When changing the energy scale, the theory behaves as being at one particular energy, but with different couplings.

12.4 Effective coupling constants

One would like to study the behaviour of the running coupling constants (also called effective coupling constants). One may recall that the renormalized n-point Green's functions are related to the physical quantities. In fact, using the LSZ reduction formula, the differential cross-section for a two-by-two scattering is

$$\mathrm{d}_\Omega\sigma = \frac{1}{64\pi^2}\frac{1}{s}|\Gamma_{\mathrm{R}}^{(4)}|^2$$

The four-point Green's function changes under the rescaling of momenta (equivalently under change of energy scale) since it depends on the running coupling constants. This implies that the running coupling constants have physical meaning. This is discussed in the following. There is a connection between the choice of the subtraction point and the running of the coupling constants. In general, a running coupling constant is a function of the physical coupling constant. Only in the on-shell renormalization scheme the two coincide [r].

The running of the coupling constant with the energy scale is related to the renormalization procedure which requires performing the subtraction of divergences at a specific scale given by the subtraction point. Different subtraction points are equivalent to different renormalization schemes. The coupling constants in various schemes can be related by

$$\lambda_{R'} = Z_\lambda^{-1}(R', R)Z_\varphi^2(R', R)\lambda_R$$

where the Z factors are finite and the two coupling constants correspond to two energy scales (so to subtraction points). The above equation is equivalent to the running of the coupling. More explicitly, when working with a particular subtraction point in the scheme R, one has

$$\Gamma_R^{(4)}(s,t,u) = \lambda_R + \widetilde{\Gamma}(s,\lambda_R) + \widetilde{\Gamma}(t,\lambda_R) + \widetilde{\Gamma}(u,\lambda_R)$$

where the last three addenda are finite contributions that arise after subtracting the divergences. Also

$$\widetilde{\Gamma}(s = t = u = \text{sub. pt.}) = 0$$

Therefore, the Green's function at the subtraction point (equivalently evaluated at a given scale) is

$$\Gamma_R^{(4)}(s = t = u = \text{sub. pt.}) = \lambda_R$$

Going to the scheme R', one has

$$\Gamma^{(4)}_{R'}(s,t,u) = \lambda_{R'} + \widetilde{\Gamma}(s,\lambda_{R'}) + \widetilde{\Gamma}(t,\lambda_{R'}) + \widetilde{\Gamma}(u,\lambda_{R'})\,, \quad \Gamma^{(4)}_{R'}(s=t=u=\text{sub. pt.}) = \lambda_{R'}$$

Since the Green's function appears in the cross-sections, one expects it not to depend on the renormalization scheme

$$\Gamma^{(4)}_{R}(s,t,u;\lambda_R) = \Gamma^{(4)}_{R'}(s,t,u;\lambda_{R'})$$

from which

$$\lambda_R + \widetilde{\Gamma}(s,\lambda_R) + \widetilde{\Gamma}(t,\lambda_R) + \widetilde{\Gamma}(u,\lambda_R) = \lambda_{R'} + \widetilde{\Gamma}(s,\lambda_{R'}) + \widetilde{\Gamma}(t,\lambda_{R'}) + \widetilde{\Gamma}(u,\lambda_{R'})$$

Evaluating this expression at the subtraction point of the scheme R gives

$$\lambda_R = \lambda_{R'} + [\widetilde{\Gamma}(s,\lambda_{R'}) + \widetilde{\Gamma}(t,\lambda_{R'}) + \widetilde{\Gamma}(u,\lambda_{R'})]|_R$$

This is a non-trivial relation between the coupling constants at different scales. This relation can be compared with the one involving the functions Z. Since $\lambda_R \neq \lambda_{R'}$, the coupling constant runs: the coupling constant in any renormalization scheme can be regarded as a function of the subtraction point.

Beta function equation. The beta function equation is

$$\beta(\bar{\lambda}) = \mathrm{d}_t\bar{\lambda}\,, \quad \bar{\lambda}(t=0) = \lambda$$

This is equivalent to

$$(\partial_t - \beta\,\partial_\lambda)\bar{\lambda}(\lambda,t) = 0$$

The beta function measures how the coupling constant runs with the energy. From the above equation, one sees how the change in the energy scale (given by t) is related to the change in the coupling constant.

One would like to find the physical information contained within.

Remark. If, in a given range of the coupling constant, the beta function is positive $\beta(\lambda) > 0$, then the coupling constant $\bar{\lambda}(t)$ increases with the energy. Lower energies correspond to weaker coupling constants, and viceversa. Low energies mean large distances, while high energies mean smaller distances. An example of this is QED.

If the beta function is negative $\beta(\lambda) < 0$, then the coupling constant decreases with the energy. An example of this is QCD. The low energies correspond to confinement and high energies correspond to asymptotic freedom. One needs to be careful treating confinement because, for lower energies, the coupling constant increases, but for the formalism developed so far to be predictive, the coupling must allow a perturbative expansion. To study confinement one may use the lattice or the AdS/CFT correspondence.

Remark. Fixed points are the zeros of the beta function $\beta(\lambda) = 0$. At a fixed point, the coupling does not depend on the scale and it is no longer running. Theory is conformally invariant. Notice that $\lambda = 0$ is a trivial fixed point. One may analyze a theory around a fixed point to study how the coupling constant $\bar{\lambda}$ behaves. Being close to a fixed point λ_{F} allows an expansion of the beta function to be written

$$\beta(\lambda) = \beta(\lambda_{\mathrm{F}}) + (\lambda - \lambda_{\mathrm{F}})\beta'(\lambda_{\mathrm{F}}) + o(\lambda - \lambda_{\mathrm{F}}) = (\lambda - \lambda_{\mathrm{F}})\beta'(\lambda_{\mathrm{F}}) + o(\lambda - \lambda_{\mathrm{F}})$$

The beta function equation becomes

$$\mathrm{d}_t\bar{\lambda} = (\bar{\lambda} - \lambda_{\mathrm{F}})\beta'(\lambda_{\mathrm{F}})$$

There are two distinct behaviours depending on the sign of the derivative of the beta function at the fixed point. Consider $\beta'(\lambda_{\mathrm{F}}) < 0$. The right-hand side gives

$$(\bar{\lambda} - \lambda_{\mathrm{F}})\beta'(\lambda_{\mathrm{F}}) > 0 \iff \bar{\lambda} < \lambda_{\mathrm{F}}$$

For $\bar{\lambda} < \lambda_{\mathrm{F}}$ the coupling constant is increasing, for $\bar{\lambda} > \lambda_{\mathrm{F}}$ the coupling constant is decreasing. The fixed point is stable: in the deep ultraviolet, $t \to \infty$, the coupling constant tends to the fixed point. The fixed point is called ultraviolet stable fixed point.

Consider $\beta'(\lambda_{\mathrm{F}}) > 0$. The right-hand side gives

$$(\bar{\lambda} - \lambda_{\mathrm{F}})\beta'(\lambda_{\mathrm{F}}) > 0 \iff \bar{\lambda} > \lambda_{\mathrm{F}}$$

For $\bar{\lambda} > \lambda_{\mathrm{F}}$ the coupling constant increases, while for $\bar{\lambda} < \lambda_{\mathrm{F}}$ the coupling constant decreases. This is an infrared stable fixed point (or ultraviolet unstable fixed point).

Remark. The beta function may have a Landau singularity (also called Landau pole)

$$\lim_{\lambda \to \lambda_s} \beta(\lambda) = \infty \iff \lim_{t \to t_s} \bar{\lambda}(t) = \infty$$

Example. Consider the $\lambda\varphi^4$ theory. The beta function up to one-loop is

$$\beta(\lambda) = \frac{3}{16\pi^2}\lambda^2$$

The equation to solve is

$$\mathrm{d}_t\bar{\lambda} = \frac{3}{16\pi^2}\bar{\lambda}^2\,, \quad \bar{\lambda}(t=0) = \lambda$$

Integrating in t gives

$$\bar{\lambda}(\lambda, t) = \frac{\lambda}{1 - \frac{3\lambda}{16\pi^2}t}$$

The beta function has only the trivial fixed point. This result holds only for small values of the coupling constant since only the one-loop correction has been considered. For large values, the above predicts a Landau pole at

$$t_s = \frac{16\pi^2}{3\lambda}$$

Asymptotic solution. Consider covariant regularization. One would like to find the behaviour of the solution of the asymptotic Callan–Symanzik equation close to a fixed point. One has found

$$\Gamma^{(n)}_{\mathrm{as}}(\sigma p_i; m, \lambda) = \sigma^{4-n} \exp\left[- n \int_0^{\ln\sigma} \mathrm{d}t'\, \gamma(\bar{\lambda}(t', \lambda))\right] \Gamma^{(n)}_{\mathrm{as}}(p_i; m, \bar{\lambda}(\lambda, t))$$

Let $\beta' < 0$. One may solve

$$\mathrm{d}_t\bar{\lambda} = -\beta'(\lambda_{\mathrm{F}})(\lambda_{\mathrm{F}} - \bar{\lambda}) = a(\lambda_{\mathrm{F}} - \bar{\lambda})\,, \quad \bar{\lambda}(t=0) = \lambda\,, \quad a > 0$$

The solution is

$$\bar{\lambda}(t) = \lambda_{\mathrm{F}} + (\lambda - \lambda_{\mathrm{F}})\mathrm{e}^{-at}$$

Up to higher orders in $\bar{\lambda} - \lambda_{\mathrm{F}}$, the integral in the exponential is

$$\int_0^t \mathrm{d}t'\, \gamma(\bar{\lambda}(t', \lambda)) \sim \gamma(\lambda_{\mathrm{F}})t = \gamma(\lambda_{\mathrm{F}}) \ln\sigma$$

where one Taylor expands the anomalous dimension. Therefore, the solution to the asymptotic Callan–Symanzik equation is

$$\Gamma^{(n)}_{\mathrm{as}}(\sigma p_i; m, \lambda) = \sigma^{4-n-n\gamma(\lambda_{\mathrm{F}})}\Gamma^{(n)}_{\mathrm{as}}(p_i; m, \lambda_{\mathrm{F}} + (\lambda - \lambda_{\mathrm{F}})\mathrm{e}^{-at})\,, \quad \sigma \to \infty$$

The exponent can be rewritten as

$$4 - n - n\gamma(\lambda_{\mathrm{F}}) = 4 - n[1 + \gamma(\lambda_{\mathrm{F}})]$$

Recalling that n comes from the dimension of the n fields, then every field gets a contribution to its classical dimension from the anomalous dimension.

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13 Wilsonian effective field theory

See Srednicki §29 (different notation, in particular Minkowski), Peskin §12 and paper by Wilson[5].

Wilson proposes a different interpretation of the renormalization procedure that allows to extend the procedure to non-renormalizable field theories which can be used as effective field theories.

In order to discuss the Wilsonian perspective, one may adopt a cutoff regularization: the momentum is integrated up to some cutoff Λ. Previously, divergent integrals depend on the cutoff in such a way that when removing the regulator, the integrals diverge. Now the cutoff is kept and not removed through renormalization.

The Wilsonian interpretation is the following: introducing the cutoff for the momentum, the higher energies are removed and so the shorter distances (by considering the Fourier transform) beyond Λ^{-1}. This is equivalent to considering a lattice with spacing proportional to Λ^{-1}. Space-time is discretized in a lattice. Changing the cutoff changes the lattice spacing.

Consider three cutoffs $\Lambda_1 > \Lambda_2 > \Lambda_3$. The lattice spacing increases with each cutoff. From a physical point of view, when going from Λ_1 to Λ_3, the degrees of freedom of the first lattice are denser and are substituted with effective degrees of freedom of the second lattice which are less dense. A square (or collection of them) connecting four neighbouring points on the first lattice is substituted with a single effective lattice point on the second lattice. Higher energies are integrated out. The degrees of freedom of the last lattice are called effective degrees of freedom.

The change in the cutoff is a rescaling. Wilson sets a cutoff and moves it along the energy axis [r] without sending it to infinity. The cutoff is of the same order of magnitude as the energy of the system [r]. The effective degrees of freedom are described by an effective field theory. When keeping the cutoff finite, renormalization is not needed. The following holds for renormalizable and non-renormalizable theories.

When using a cutoff, there are various ways to define the renormalized parameters [r]. Consider the intermediate renormalization scheme, $p^2 = 0$, in the $\lambda\varphi^4$ theory. The only divergent quantities are the two-point and four-point Green's functions. Omitting the index R to denote renormalized quantities, the normalization conditions are

$$\Gamma^{(2)}(p^2 = 0) = m^2\,, \quad \Gamma^{(4)}(s = t = u = 0) = \lambda$$

[r] Using the counter term renormalization, the renormalized Lagrangian is

$$\mathcal{L} + \mathcal{L}_{\text{ct}} = \frac{1}{2} Z_\varphi\, \partial_\mu\varphi\, \partial^\mu\varphi + \frac{1}{2} m^2 Z_m \varphi^2 + \frac{\lambda}{4!} Z_\lambda \varphi^4\,, \quad Z_m = \left[1 + \frac{\delta m^2}{m^2}\right] Z_\varphi$$

The action is

$$\begin{aligned} S &= \int \mathrm{d}^4x\, (\mathcal{L} + \mathcal{L}_{\text{ct}}) \\ &= \frac{1}{2} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \varphi(-k)(Z_\varphi k^2 + Z_m m^2)\varphi(k) \\ &\quad + \frac{\lambda}{4!} Z_\lambda \int \frac{\mathrm{d}^4k_1\, \mathrm{d}^4k_2\, \mathrm{d}^4k_3\, \mathrm{d}^4k_4}{[(2\pi)^4]^4} (2\pi)^4 \delta^{(4)}(k_1 + k_2 + k_3 + k_4)\varphi(k_1)\varphi(k_2)\varphi(k_3)\varphi(k_4) \end{aligned}$$

where one has

$$\varphi(k) = \int \mathrm{d}^4x\, \mathrm{e}^{-\mathrm{i}kx}\varphi(x)\,, \quad k \in \mathbb{R}^4$$

At the second equality one has applied the Fourier transform.

One would like to split the integration of the magnitude of the momentum according to the cutoff

$$0 < |k| < \Lambda\,, \quad |k| > \Lambda$$

[5] Kenneth G. Wilson, J. Kogut, The renormalization group and the ϵ expansion, Physics Reports, Volume 12, Issue 2, 1974, Pages 75-199, ISSN 0370-1573, https://doi.org/10.1016/0370-1573(74)90023-4.

To this end, the Fourier transform of the field is split

$$\varphi(k) = \begin{cases} \varphi_1(k)\,, & |k| < \Lambda \\ \varphi_2(k)\,, & |k| > \Lambda \end{cases}$$

In configuration space, one has

$$\varphi_j(x) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{e}^{\mathrm{i}kx} \varphi_j(k)$$

Instead of functionally integrating over all possible degrees of freedom of the field φ, one integrates only the high-energy degrees of freedom corresponding to φ_2

$$\int [\mathcal{D}\varphi_2]\, \mathrm{e}^{-S[\varphi_1,\varphi_2]} \equiv \mathrm{e}^{-S_{\mathrm{eff}}[\varphi_1]}$$

In this way one defines the (Wilsonian) effective action S_{eff}. This action describes the degrees of freedom over a lattice of spacing Λ^{-1}.

If the classical Lagrangian presents a $\mathbb{Z}_2$ symmetry, the effective Lagrangian is

$$S_{\mathrm{eff}} = \int \mathrm{d}^4 x\, \mathcal{L}_{\mathrm{eff}}[\varphi, m, \lambda, \Lambda]$$

also exhibits such symmetry and contains even powers of the field. Since every term must have mass dimension of four, then one may write

$$\mathcal{L}_{\mathrm{eff}}[\varphi, m, \lambda, \Lambda] = \frac{1}{2} Z(\Lambda)\, \partial_\mu \varphi\, \partial^\mu \varphi + \frac{1}{2} m^2(\Lambda) \varphi^2 + \frac{\lambda(\Lambda)}{4!} \varphi^4 + \sum_{d \geq 6} \sum_i c_{d,i}(\Lambda) O_{d,i}$$

where d labels the mass dimension of the operator O and i labels the degeneracy. One can consider higher powers of the field and its derivatives that have coupling constants c with negative mass dimension. For example, at a fixed dimension, the operators O can be

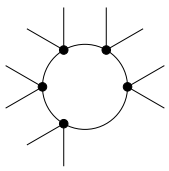
$$O_{d,i} \rightsquigarrow \varphi^d\,, \varphi^{d-4} (\partial_\mu \varphi)^2\,, \ldots$$

Integrating out degrees of freedom. The meaning of integrating a subset of degrees of freedom shall be discussed. The configurations which are integrated in the functional integral are not present in the final result. When doing a perturbative calculation, these configurations are the ones contributing to the loops, they are the quantum configurations. By definition, the integrated configurations of a perturbative functional integral are the quantum fluctuations [r]

$$\int [\mathcal{D}\varphi] = \int [\mathcal{D}\delta\varphi]\,, \quad \varphi = \varphi_{\mathrm{cl}} + \delta\varphi$$

The remaining configurations are the ones present on the external lines of a diagram.

Example. At one-loop, the contribution to the effective Lagrangian for the operator $O_{2n,1} \equiv \varphi^{2n}$ is given by



The loop contains n vertices. The external lines are φ_1 while the internal lines are φ_2 because they are integrated. One knows how to write one-loop integral [r]: one integrates n propagators.

Consider the Fourier transform of the field with zero external momentum $\varphi(p_i = 0)$. The contribution to the effective coefficient is

$$c_{2n,1} = -\frac{(-\lambda)^n}{2^n 2n} \int_{|k|=\Lambda}^{\infty} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{(k^2 + m^2)^n} + o(\lambda^n) = \frac{\lambda^n}{2^n} \frac{(-1)^n}{32\pi^2 n(n-2)} \frac{1}{\Lambda^{2n-4}} + o(\lambda^n)$$

where $\Lambda \gg m$. This is true as long as one considers operators with $2n \geq 6$ since there are no divergences.

Four-point Green's function. Consider $2n = 4$. To compute the contributions, one has to consider the tree-level contribution given by λZ_λ. One finds

$$\lambda(\Lambda) = Z_\lambda \lambda + \frac{3}{2}\lambda^2 \int_{|k|=\Lambda}^{\infty} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{(k^2+m^2)^2} + o(\lambda^2)$$

The integral diverges at infinity. One knows that

$$\lambda = \Gamma^{(4)}(0,0,0) = \lambda Z_\lambda + \frac{3}{2}\lambda^2 \int_{|k|=0}^{\infty} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{(k^2+m^2)^2} + o(\lambda^2)$$

[r] from which one sees that

$$\lambda(\Lambda) = \lambda - \frac{3}{2}\lambda^2 \int_{|k|=0}^{\Lambda} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{(k^2+m^2)^2} = \lambda - \frac{3}{16\pi^2}\lambda^2 \left[\ln \frac{\Lambda}{m} - \frac{1}{2}\right] + o(\lambda^2)$$

Two-point Green's function. Let $2n = 2$. [r] One obtains

$$m^2(\Lambda) = m^2 + \frac{\lambda}{16\pi^2}\left[\Lambda^2 - m^2 \ln \frac{\Lambda^2}{m^2}\right] + o(\lambda)$$

[r] sign

Large logarithm problem. For $2n \geq 6$, the contribution from $(\Lambda^{2n-4})^{-1}$ is less the more the cutoff Λ increases. However, for the two cases above, the loop contributions to the coupling constant λ and the mass m^2 may grow bigger than the tree-level contributions which is incompatible with a perturbative description.

The solution to this is to change the renormalization scheme and consider the effective action at a given cutoff $S_{\text{eff}}[\Lambda_0]$. From the contributions above, one may read

$$c_{2n,i}(\Lambda_0)\,, \quad \lambda(\Lambda_0)\,, \quad m^2(\Lambda_0)$$

The effective action $S_{\text{eff}}[\varphi(|k|<\Lambda_0), \Lambda_0,]$ is used as the starting action from which one reads the Feynman rules. In particular, let $Z[\Lambda_0] = 1$ recalling that Z is always a finite number and can be reabsorbed into the field by rescaling (which can be seen from the explicit expression of the Lagrangian); one assumes

$$\lambda(\Lambda_0) \ll 1\,, \quad \left|m^2(\Lambda_0)\right| \ll \Lambda_0^2\,, \quad c_{d,i}(\Lambda_0) \ll \Lambda_0^{-(d-4)}$$

The Feynman rules contain an infinite number of interaction vertices.

One considers a second cutoff $\Lambda < \Lambda_0$ and splits the degrees of freedom according to this cutoff

$$0 < |k| < \Lambda\,, \quad \Lambda < |k| < \Lambda_0$$

One defines a new effective action by integrating out the second set of momenta

$$\mathrm{e}^{-S_{\text{eff}}[\varphi(|k|<\Lambda),\Lambda,\Lambda_0]} \equiv \int [\mathcal{D}\varphi(\Lambda < |k| < \Lambda_0)]\, \mathrm{e}^{-S_{\text{eff}}[\varphi(|k|<\Lambda_0),\Lambda_0]}$$

The momenta integrated out correspond to lattice spacings in the range

$$\frac{1}{\Lambda_0} < a < \frac{1}{\Lambda}$$

The action on the left-hand side is the Wilsonian effective action [r].

By starting from the cutoff Λ_0 and computing again the two-point and four-point Green's functions, one has

$$\begin{aligned}
m^2(\Lambda) &= m^2(\Lambda_0) + \frac{\lambda(\Lambda_0)}{16\pi^2}[\Lambda_0^2 - \Lambda^2] + o(\lambda) \\
\lambda(\Lambda) &= \lambda(\Lambda_0) - \frac{3}{16\pi^2}\lambda^2(\Lambda_0) \ln \frac{\Lambda_0}{\Lambda} + o(\lambda^2) \\
c_{2n,1} &= -\frac{(-1)^n}{2^n 32\pi^2 n(n-2)}[\lambda(\Lambda_0)]^n \left[\frac{1}{\Lambda^{2n-4}} - \frac{1}{\Lambda_0^{2n-4}}\right] + o(\lambda^n)
\end{aligned}$$

The dominant term for the mass is Λ_0^2, for the coefficient $c_{2n,1}$ is $\Lambda^{-(2n-4)}$ while for the coupling constant both Λ_0 and Λ compete equally.

One may apply again the procedure: take the action $S_{\text{eff}}(\Lambda)$ as the starting point, introduce a cutoff $b\Lambda < \Lambda$, split the degrees of freedom according to $b\Lambda$ and find

$$m^2(b\Lambda) = m^2(\Lambda) + \frac{\lambda(\Lambda)}{16\pi^2}[\Lambda^2 - (b\Lambda)^2] + o(\lambda)$$

The dominant term is given by $\Lambda^2 < \Lambda_0^2$. One has managed to define an effective action where the corrections are smaller than he previous step.

By iterating the procedure, one may get to a correction of the order

$$\lambda(b\Lambda)(b\Lambda)^2 \sim m^2$$

This iteration is called fine-tuning.

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The parameters $Z(\Lambda)$, $m^2(\Lambda)$, $\lambda(\Lambda)$ and $c_{d,i}(\Lambda)$ in the effective action all depend on the cutoff Λ. If an operator $O_{d,i}$ has mass dimension d, then the coupling constant $c_{d,i}$ has dimension $4-d$. For negative-mass-dimension coupling constants, the theory is not renormalizable, but in this case the cutoff is not sent to infinity and, as such, renormalization is not needed.

The Wilsonian effective field theory is a reinterpretation of the running of the coupling constant. When integrating out high-energy degrees of freedom, one may utilize any theory, even non-renormalizable, as an effective field theory. An example of (yet) non-renormalizable theory is gravity which is used as effective field theory.

Integrating over a momentum shell. See also Peskin. The set of momenta that is integrated out $b\Lambda < |k| < \Lambda$ is called momentum shell. The way to explicitly integrate out a set of degrees of freedom can be seen as follows. Consider

$$\varphi(k) = \begin{cases} \varphi(k)\,, & |k| < b\Lambda \\ 0\,, & |k| > b\Lambda \end{cases}, \qquad \hat{\varphi}(k) = \begin{cases} \varphi(k)\,, & b\Lambda < |k| < \Lambda \\ 0\,, & |k| < b\Lambda \vee |k| > \Lambda \end{cases}$$

The original degrees of freedom of the field can be split into

$$\varphi(k) \rightsquigarrow \varphi(k) + \hat{\varphi}(k)$$

Similarly, in configuration space, the field can be split into two parts

$$\varphi(x) \rightsquigarrow \varphi(x) + \hat{\varphi}(x)$$

Setting $J = 0$ for simplicity, the generating functional is

$$\begin{aligned} W &= \int [\mathcal{D}\varphi\,\mathcal{D}\hat{\varphi}]\, \exp\left\{ -\int \mathrm{d}^4x \left[\frac{1}{2}(\partial_\mu\varphi + \partial_\mu\hat{\varphi})^2 + \frac{1}{2}m^2(\varphi+\hat{\varphi})^2 + \frac{\lambda}{4!}(\varphi+\hat{\varphi})^4 \right] \right\} \\ &= \int [\mathcal{D}\varphi]\, \mathrm{e}^{-\int \mathrm{d}^4x\, \mathcal{L}(\varphi)} \int [\mathcal{D}\hat{\varphi}] \exp\left\{ -\int \mathrm{d}^4x \left[\mathcal{L}(\hat{\varphi}) + \partial_\mu\varphi\,\partial^\mu\hat{\varphi} + m^2\varphi\hat{\varphi} \right.\right. \\ &\quad \left.\left. + \lambda\left(\frac{1}{6}\varphi^3\hat{\varphi} + \frac{1}{4}\varphi^2\hat{\varphi}^2 + \frac{1}{6}\varphi\hat{\varphi}^3 \right) \right] \right\} \end{aligned}$$

From the mass term one obtains $m^2\varphi\hat{\varphi}$, but it does not contribute to the action since the fields have non-overlapping domains

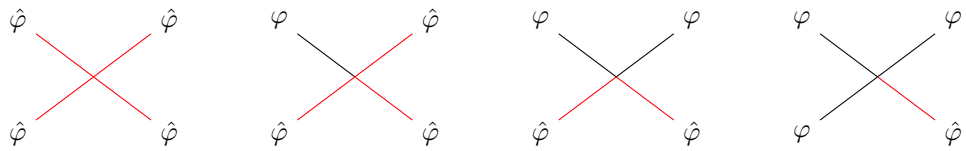
$$\begin{aligned} \int \mathrm{d}^4x\, \varphi(x)\hat{\varphi}(x) &= \int \mathrm{d}^4x \int \mathrm{d}^4k\, \mathrm{e}^{\mathrm{i}kx}\varphi(k) \int \mathrm{d}^4k'\, \mathrm{e}^{\mathrm{i}k'x}\hat{\varphi}(k') \\ &= \int \mathrm{d}^4k \int \mathrm{d}^4k'\, \varphi(k)\hat{\varphi}(k') \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}(k+k')x} \\ &= \int \mathrm{d}^4k \int \mathrm{d}^4k'\, \varphi(k)\hat{\varphi}(k')(2\pi)^4\delta^{(4)}(k+k') \\ &= (2\pi)^4 \int \mathrm{d}^4k\, \varphi(k)\varphi(-k) \\ &= 0 \end{aligned}$$

The two fields are non-zero for complementary regions of the momentum $|k|$. Similarly for the kinetic term $\partial_\mu\varphi\,\partial^\mu\hat{\varphi}$.

In the generating functional, one may first perturbatively integrate $\hat{\varphi}$ taking into account that the corresponding Feynman rules include the propagator

$$\longrightarrow \frac{1}{p^2+m^2}$$

as well as four vertices



One expects to find corrections to the Lagrangian $\mathcal{L}(\varphi)$ [r] given by the removal of the high-energy degrees of freedom $\hat{\varphi}$.

Remark. Consider the expression for the coupling constant

$$\lambda(\Lambda)=\lambda(\Lambda_0)-\frac{3}{16\pi^2}\lambda^2(\Lambda_0)\ln\frac{\Lambda_0}{\Lambda}+o(\lambda^2)$$

It describes the running of the coupling. By applying the derivative $\Lambda\,\partial_\Lambda$ and evaluating at $\Lambda_0=\Lambda$, one has

$$\Lambda\,\partial_\Lambda\lambda(\Lambda)\big|_{\Lambda_0=\Lambda}=\frac{3}{16\pi^2}\lambda^2(\Lambda)=\beta(\lambda(\Lambda))$$

Letting $t=\ln\Lambda$, this is the beta function equation. This gives a new interpretation of the beta function: it is the rate of change in the coefficient of the φ^4 term in the effective action as the ultraviolet cutoff in that action varies.

Remark. Perturbative QFT and renormalization group is a big topic and many results may be interpreted in different ways. In effective field theory, one may analyze the meaning of a Landau pole in the beta function for the $\lambda\varphi^4$ theory. Recall that, at one-loop, the beta function exhibits a Landau pole at

$$t=\ln\Lambda=\frac{16\pi^2}{3\lambda}\iff\Lambda\sim\mathrm{e}^{\frac{16\pi^2}{3\lambda}}$$

[r] From the beta function equation, one has

$$\mathrm{d}_{\ln\Lambda}\lambda=\beta(\lambda)\implies\int_{\lambda(\Lambda=m)}^{\lambda(\Lambda_0)}\frac{\mathrm{d}\lambda}{\beta(\lambda)}=\int_{\Lambda=m}^{\Lambda_0}\mathrm{d}\ln\Lambda=\ln\frac{\Lambda_0}{m}$$

In the limit $\Lambda_0\to\infty$, the right-hand side diverges, while the integral on the left-hand side is finite since $\beta\sim\lambda^2$. This is a contradiction: one cannot remove the cutoff Λ_0. The left-hand side implies a maximal value

$$\ln\frac{\Lambda_{\max}}{m}\equiv\int_{\lambda(m)}^{\infty}\frac{\mathrm{d}\lambda}{\beta(\lambda)}$$

This is the manifestation of the Landau pole. The presence of a maximum value implies that one must use an effective action with a cutoff as a starting point.

The way to resolve the contradiction is to go beyond the Landau pole and have the coupling constant approach zero in the limit $\Lambda_0\to\infty$. In this case, the theory is said to be asymptotically trivial.

Remark. Consider the subset of couplings of the higher mass dimension operators in the Wilsonian effective action with $2n$ powers of the field φ and $i-1$ derivatives denoted by $c_{2n,i}$. One has seen that the coupling constant $c_{2n,1}$ has dominant term given by $\Lambda^{-(2n-4)}$.

Consider starting from an effective action $S_{\text{eff}}(\Lambda)$ and construct another effective action $S'_{\text{eff}}(b\Lambda)$ with $b<1$. When rescaling the momentum to $k=bk'$ in the second action, one

obtains $|k'| < \Lambda$. Therefore, the second action S'_{eff} has the same formal expression as the first S_{eff}. In configuration space, the rescaling is

$$x = \frac{1}{b} x'$$

The Lagrangian $\mathcal{L}'_{\text{eff}}$ contained in the second action is changed. In fact

$$\int \mathrm{d}^d x \, (\partial_\mu \varphi)^2 = \int \mathrm{d}^d x' \, b^{-d+2} (\partial'_\mu \varphi)^2 \equiv \int \mathrm{d}^d x' (\partial'_\mu \varphi')^2 \implies \varphi'^2 \equiv b^{-d+2} \varphi^2$$

also

$$\int \mathrm{d}^d x \, c_{2n,1} \varphi^{2n} = \int \mathrm{d}^d x' \, b^{-d} b^{n(d-2)} \varphi'^{2n} c_{2n,1} \equiv \int \mathrm{d}^d x' \, c'_{2n,1} \varphi'^{2n} \implies c'_{2n,1} \equiv b^{n(d-2)-d} c_{2n,1}$$

The parameters are formally the same, but are related through a rescaling. The power the couplings $c_{2n,1}$ is equal to

$$n(d-2) - d = 2n \left(\frac{d}{2} - 1 \right) - d = 2n \dim \varphi - d = \dim O_{2n,1} - d$$

where one recognizes the parentheses to be the dimension of the field φ.

In general, one has

$$c'_{2n,i} = b^{\dim O_{2n,i} - d} c_{2n,i} \,, \quad b < 1$$

From this equation, it is apparent the behaviour of the coupling constants in the infrared limit. In such limit, the energy decreases and b gets closer to zero. The rescaling depends on the sign of the exponent. For a negative sign, the coupling constant gets stronger. For positive sign, the coupling constant gets weaker. For a null exponent, the coupling constant does not rescale.

Operator classification. According to the exponent, one may classify the operators:

- relevant operators, $\dim O_{2n,i} < d$; in the infrared, the coupling constant gets stronger; these operators drive the system out of an ultraviolet fixed point [r].
- irrelevant, $\dim O_{2n,i} > d$; in the infrared, the coupling constant gets weaker;
- marginal, $\dim O_{2n,i} = d$; the coupling constant is the same at any energy scale; the operators are scale invariant and do not disturb the system, they are present regardless.

This classification is made for classical dimension. Composite operator may get corrections from the anomalous dimension. According to the sign of the anomalous dimension, the marginal operators may become:

- marginally relevant $\gamma < 0$,
- marginally irrelevant $\gamma > 0$,
- exactly marginal $\gamma = 0$, an operator does not get a correction (as in the case of being protected by symmetry).