General Relativity

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Contents

Lecture 1

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Exam: three easy questions seen during the course, three questions somewhat new, two harder questions that require critical thinking.

1 Introduction

General relativity replaces the Newtonian description of gravity. The force of Newtonian gravity is proportional to the reciprocal of the distance squared between two masses. Changing the distance changes the force immediately, but this is not consistent with Special Relativity. To describe gravity, one can utilize field theory to be in agreement with Special Relativity.

Einstein took a different route. He identified something overlooked: the equivalence principle. [r] In its basic form, the equivalence principle states that gravitational mass — that is the mass that appears in Newton's law of gravitation — is equal to inertial mass — that is the mass that appears in Newton's second law of motion. A priori, these two masses can be different, but the equivalence principle postulates their equality. One can study the implications of the principle. Two thought experiments (gedankenexperimente) can help understand such implications.

First thought experiment. Consider a mass m inside a box with two propellers at the bottom. The box is accelerated upwards at a constant acceleration g equal to the sea-level Earth's acceleration. In the reference frame of the mass, there is a downward force pushing the mass against the floor of the box. This behaviour is the exact same experienced on the surface of the Earth: holding a mass and letting it go, it experiences a downward force towards the floor. One can locally mimic gravity with an apparent force. The principle works locally: because of tidal forces — the force of gravity varies with distance — one can distinguish rocket-powered acceleration from a mass' gravitational field.

Second thought experiment. Consider a free-falling box towards Earth. A mass inside the box experiences only gravity, but falling together with the box, one may not distinguish the situation from the one where gravity is absent in the first place. An example of a free-falling experiment is the International Space Station (ISS): its altitude from the surface is just 400 km, so the gravity is about 90% the one on the surface, but the feeling is that of weightlessness: the ISS is constantly falling, though it has enough lateral velocity that the Earth below moves away faster than the ISS can fall.

Einstein's equivalence principle. The equivalence principle is still true for electromagnetism and all of physics. For a small enough box, one may not tell whether gravity is acting or not.

Consequences. Consider a laser shining a beam of light from left to right across a box. If the box is propelled upwards, the laser hits the right wall lower (relative to the floor) than it was shot from the left side. From the reference frame of the box, the laser is bending downwards. By the equivalence principle, the same should apply to a box immersed in a gravitational field. This prediction has been experimentally verified by gravitational lensing.

Curvature. Space has always been thought of as Euclidean space. However, if light curves, then the definition of straight line requires more caution. On a sphere, the sum of internal angles of a triangle is no longer π. After observing light curving, the geometry of space can no longer be Euclidean. On a sphere, the minimum distance between two points is given by the arc of a great circle passing between the two points. A straight line is then defined as the line that minimizes distance.

Since light is described differently by the equivalence principle, then massive objects need new equations also. These need to also explain the motion of planets: the precession of Mercury was not explained by Newtonian gravity. A free particle in space-time maximizes its proper time which is proportional to the relativistic action. Objects that are only subject to gravity, either massive or massless, — that is, objects in free fall — follow geodesics: trajectories that maximize proper time. The trajectory of objects in free fall is described purely by geometrical ideas: there is no force of gravity. [r]

Curvature is described in a way that matter bends space-time and space-time tells matter how to move.

Lecture 2

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2 Special relativity

Special relativity is a theory obtain from uniting Galilean principle of relativity and the idea that the speed of light is the same in every frame of reference. From the postulates one can derive time dilation, length contraction, relativity of simultaneity, etc. Lorentz transformations are used to go from one frame of reference to another. The Lorentz transformation Λ for a boost in the x direction is given by

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} , \quad \beta = \frac{v}{c} , \quad \gamma = (1-\beta^2)^{-\frac{1}{2}}$$

In this course, the natural unit $c = 1$ will be use. One can also express Lorentz transformations in terms of rapidity λ by setting

$$\gamma = \cosh\lambda , \quad \beta\gamma = \sinh\lambda$$

The Lorentz transformation above becomes

$$\Lambda = \begin{bmatrix} \cosh\lambda & -\sinh\lambda & 0 & 0 \\ -\sinh\lambda & \cosh\lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is a hyperbolic rotation. Ordinary three-dimensional rotation matrices are orthogonal matrices $R^\top R = RR^\top = I$ and leave the norm of $\mathbb{R}^3$ unchanged. Similarly, Lorentz transformations preserve the Minkowski metric $\tau^2 = t^2 - |\vec{x}|^2$. This invariant is the proper time.

Points in Minkowski space are called events to highlight the fact that time is also a coordinate. Proper time of a trajectory is the time measured by an observer going along that trajectory. [r] The proper time between two timelike events is the shortest time one can hope to measure between those two events.

Length of a curve. In Euclidean space, the length of a curve γ is given by the integral

$$\int_\gamma \mathrm{d}l = \int_\gamma \sqrt{\mathrm{d}x_i\,\mathrm{d}x^i} = \int_\gamma \mathrm{d}x\,\sqrt{\partial_\lambda x_i\,\partial_\lambda x^i}$$

In Minkowski space, the time of a trajectory measured by an observer is

$$\tau(\gamma) = \int_\gamma \sqrt{(\mathrm{d}t)^2 - \mathrm{d}x_i\,\mathrm{d}x_i} = \int_\gamma \mathrm{d}t\sqrt{1 - \partial_t x_i\,\partial_t x_i} = \int_\gamma \frac{\mathrm{d}t}{\gamma} = \int_\gamma \mathrm{d}\tau$$

Because of the minus sign, proper time is maximized by straight paths in spacetime. Proper time can be made arbitrarily short by approaching the speed of light. Proper time is proportional to the action of a relativistic free particle

$$S = -m\int \mathrm{d}\tau = -m\int \mathrm{d}t\,\sqrt{1-v^2} \sim -m\int \mathrm{d}t\,\left[1 - \frac{1}{2}v^2 + o(v^2)\right] \approx \int \mathrm{d}t\,\left[-m + \frac{1}{2}mv^2\right]$$

By varying the action one can obtain the equations of motion $\ddot{x}^j = 0$.

Four-momentum. Energy and momentum can be combined into a four-vector: $P^\mu = (E, \vec{p})$. The invariant associated with the norm of the vector is mass $m^2 = P_\mu P^\mu$. For a body at rest, $\vec{p} = 0$, one obtains $E = mc^2$. To get the explicit expression for the four-momentum, one can boost a stationary object:

$$\begin{bmatrix} E' \\ p'_x \\ p'_y \\ p'_z \end{bmatrix} = \begin{bmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \gamma m \\ -\gamma m v \\ 0 \\ 0 \end{bmatrix}$$

More generally, it follows $E = \gamma m$ and $\vec{p} = \gamma m\vec{v}$. In the limit of low speeds, one recovers the classical relations

$$E = \frac{m}{\sqrt{1-v^2}} = m\left(1 + \frac{1}{2}v^2 + o(v^2)\right) \approx m + \frac{1}{2}mv^2$$

2.1 Covariant notation

The norm of a four-vector is a quadratic form. For the proper time, one has

$$\tau^2 = -\begin{pmatrix} t & x & y & z \end{pmatrix} \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = -x^\top \eta x$$

The metric follows a spacelike notation typical of General Relativity opposite to the one in QFT. In matrix notation, Lorentz transformations are given by

$$x' = \Lambda x$$

Proper time is an invariant and, calculated in two reference frames, one gets

$$\tau^2 = -x'^\top \eta x' = -(\Lambda x)^\top \eta (\Lambda x) = -x^\top \Lambda^\top \eta \Lambda x \equiv -x^\top \eta x \implies \eta = \Lambda^\top \eta \Lambda$$

This means that inner products in Minkowski space are invariant under change of basis. Lorentz transformations are elements of the Lorentz group $\mathrm{O}(3,1)$. The most general transformation depends on six parameters.

The inner product between two four-vectors $x_1^\top x_2$ is not invariant. Though, by defining a four-vector that transforms as

$$y' = \eta\Lambda\eta y$$

One can obtain an invariant inner product $y'^\top x' = y^\top x$. Vectors transforming as the position vector x are called contravariant, while the vectors defined above are their dual and called covariant. From a contravariant vector x, one can derive a covariant one by ηx. From this their inner product is invariant.

The nature of a vector under transformations is an important property and is made apparent with notation. Upper indices denote contravariant components x^μ, while lower indices denote covariant components y_μ. Greek indices include all four coordinates, while Latin indices only spatial indices. The inner product in Minkowski space becomes $x^\mu y_\mu$, using Einstein's summation convention: upper and lower indices appearing once are summed. Free indices should appear on both sides of an equations in the same position. If these are not the cases, the equation written has not the same for in every reference frame.