Quantum Field Theory II

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Contents

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Lecture 1

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Topics. Generalize the formalism for complex scalar fields. Introduce fermions and their functional quantization. Discrete symmetries and PCT theorem. Gauge theories and their functional quantization. Anomalies. Possibly instantons, applications of RG flows, etc.

1 Complex scalar boson fields

[r] sources?

1.1 Generalization

Consider a complex scalar field φ. It has two degrees of freedom: either the fields φ and φ^* or the real and imaginary parts of the field φ. Both are useful for computation

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + \mathrm{i}\varphi_2)\,, \quad \varphi^* = \frac{1}{\sqrt{2}}(\varphi_1 - \mathrm{i}\varphi_2)$$

The action of the $\lambda\varphi^4$ theory is

$$\begin{aligned} S &= \int \mathrm{d}^4x\, [\partial_\mu\varphi\, \partial^\mu\varphi^* - m^2\varphi\varphi^* - \lambda(\varphi\varphi^*)^2] \\ &= \int \mathrm{d}^4x\, \left[\frac{1}{2}\,\partial_\mu\varphi_1\,\partial^\mu\varphi_1 - \frac{1}{2}m^2\varphi_1^2 + \frac{1}{2}\,\partial_\mu\varphi_2\,\partial^\mu\varphi_2 - \frac{1}{2}m^2\varphi_2^2 - \frac{\lambda}{4}(\varphi_1^2+\varphi_2^2)^2\right] \end{aligned}$$

Each degree of freedom has an equation of motion. Varying the action with respect to φ gives the equations of motion for φ^* and vice versa. For $\lambda = 0$, the fields components satisfy the Klein–Gordon equation.

Since the interaction term is real, it corresponds to a quartic vertex $\varphi\varphi^*\varphi\varphi^*$. For $\lambda = 0$, one can rewrite the action in terms of real φ_1 and imaginary φ_2 parts and see that the two degrees of freedom decouple since there is no mixed term $\varphi_1^2\varphi_2^2$.

With two degrees of freedom, the Euclidean generating functional depends on two source terms

$$\begin{aligned} W[J,J^*] &= \int [\mathcal{D}\varphi\,\mathcal{D}\varphi^*]\,\exp\left[-\int \mathrm{d}^4x\,[\mathcal{L}(\varphi,\varphi^*) - J\varphi - J^*\varphi^*]\right] \\ W[J_1,J_2] &= \int [\mathcal{D}\varphi_1\,\mathcal{D}\varphi_2]\,\exp\left[-\int \mathrm{d}^4x\,[\mathcal{L}(\varphi_1,\varphi_2) - J_1\varphi_1 - J_2\varphi_2]\right] \end{aligned}$$

The Euclidean Green's functions are

$$\begin{aligned} &\langle 0|\,\mathcal{T}\{\varphi(x_1)\cdots\varphi(x_l)\varphi^*(x_{l+1})\cdots\varphi^*(x_n)\}\,|0\rangle = \\ &\qquad = \left.\frac{\delta^n W[J,J^*]}{\delta J(x_1)\cdots\delta J(x_l)\,\delta J^*(x_{l+1})\cdots\delta J^*(x_n)}\right|_{J=J^*=0} \end{aligned}$$

Similarly

$$\langle 0|\,\mathcal{T}\{\varphi_1(x_1)\cdots\varphi_1(x_l)\varphi_2(x_{l+1})\cdots\varphi_2(x_n)\}\,|0\rangle = \\ = \left.\frac{\delta^n W[J_1, J_2]}{\delta J_1(x_1)\cdots\delta J_1(x_l)\,\delta J_2(x_{l+1})\cdots\delta J_2(x_n)}\right|_{J_1=J_2=0}$$

Free theory. For a free real scalar theory, i.e. setting $\lambda = 0$, one can compute the free propagator exactly. In this case the situation is slightly different? [r]. In the complex fields formulation, one has to compute a complex Gaussian integral of the form

$$\int\left[\prod_{j=1}^{n}\mathrm{d}z_j\,\mathrm{d}z_j^*\right]\mathrm{e}^{-z^* A z} = \frac{\pi^n}{\det A}$$

where $z \equiv (z_1, \ldots, z_n)$ and A is a square matrix of dimension n (with hermitian part positive-definite). The Euclidean generating functional is then

$$\begin{aligned} W[J_1, J_2] &= \int[\mathcal{D}\varphi_1\,\mathcal{D}\varphi_2]\,\mathrm{e}^{-S_0[\varphi_1]-S_0[\varphi_2]}\exp\left[\int\mathrm{d}^4x\,(J_1\varphi_1 + J_2\varphi_2)\right] \\ &= \int[\mathcal{D}\varphi_1]\,\mathrm{e}^{-S[\varphi_1]+\int\mathrm{d}^4x\,J_1\varphi_1}\int[\mathcal{D}\varphi_2]\,\mathrm{e}^{-S[\varphi_2]+\int\mathrm{d}^4x\,J_2\varphi_2} \\ &= W_1[J_1]W_2[J_2] \end{aligned}$$

The theory factorizes.

The Lagrangian above has a U(1) symmetry of the fields

$$\varphi' = \mathrm{e}^{\mathrm{i}\alpha}\varphi\,, \quad \varphi'^* = \mathrm{e}^{-\mathrm{i}\alpha}\varphi^*\,, \quad \alpha \in \mathbb{R}$$

Equivalently, the Lagrangian is invariant under SO(2) of the components φ_1 and φ_2.

Exercise. Write the most general SO(2) transformation.

Exercise. See Peskin, Problem 12.3, p. 428. The interaction in the action $S[\varphi_1, \varphi_2]$ can be split into self-interaction and coupling between the fields. This form can be generalized

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi_1\,\partial^\mu\varphi_1 + \frac{1}{2}\,\partial_\mu\varphi_2\,\partial^\mu\varphi_2 - \frac{\lambda}{4!}(\varphi_1^4 + \varphi_2^4) - \frac{2\rho}{4!}\varphi_1^2\varphi_2^2$$

Do the following:

- Write the Euclidean Feynman rules.
- Compute all one-loop corrections to the λ-vertex and the ρ-vertex in dimensional regularization $D = 4 - 2\varepsilon$, use massless integrals (discussed in the following).
- Impose the normalization conditions at $s = t = u = \Lambda^2$ such that

  $$\Gamma^{(4)}(s = t = u = \Lambda^2) = -\lambda\,, \quad \Gamma^{(4)}_{\text{mixed}}(s = t = u = \Lambda^2) = -2\rho$$

  [r] minus?
  Notice that this implies that there are no one-loop finite corrections to the couplings.
- Compute the beta functions β_λ, β_ρ and

  $$\beta_{\frac{\lambda}{\rho}} = \mathrm{d}_t\frac{\lambda}{\rho} = \frac{1}{\rho}\,\mathrm{d}_t\lambda - \frac{1}{\rho^2}\lambda\,\mathrm{d}_t\rho = \frac{1}{\rho}\beta_\lambda - \frac{\lambda}{\rho^2}\beta_\rho$$

- Find the fixed points and describe the RG flow.
- Explain what happens for

  $$\frac{1}{3} < \frac{\lambda}{\rho} < 1$$

  and for $\lambda = \rho$.

1.2 Massless integrals in dimensional regularization

Proposition. In dimensional regularization, it holds

$$I_\alpha = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{(p^2)^\alpha} = 0\,, \quad \alpha \in \mathbb{C}$$

Notice that it is null also for $\alpha = 0$. This integral describes tadpoles for $\alpha = 1$: the two-point Green's function receives no correction at one-loop if the fields are massless.

The intuitive argument of why this is true can be seen for $D \neq 2\alpha$ where the integral is dimensionful: the result has to be dimensionful as well and must be written in terms of dimensionful parameters, however the integral does not depend on any parameter and so the result must be zero.

Proof. Consider the theorem below and the Euclidean formalism. One may rewrite the integrand

$$\frac{1}{(p^2)^\alpha} = \frac{1}{(p^2)^\alpha} \frac{p^2 + m^2}{p^2 + m^2} = \frac{1}{(p^2)^\alpha} \frac{m^2}{p^2 + m^2} + \frac{1}{(p^2)^{\alpha - 1}} \frac{1}{p^2 + m^2}$$

The region of convergence for the integral

$$\int \mathrm{d}^D p \, \frac{1}{(p^2)^\alpha} \frac{m^2}{p^2 + m^2}$$

is given by the following limits. At infinity, one has

$$|p| \to \infty\,, \quad I \sim \frac{1}{p^{2\alpha + 2 - D}}$$

There is no singularity for $2\alpha + 2 - \operatorname{Re} D > 0$. At the origin, one has

$$|p| \to 0\,, \quad I \sim p^{D - 2\alpha}$$

There is no singularity for $\operatorname{Re} D - 2\alpha > 0$. The integral of the first addendum is well-defined for

$$2\alpha < \operatorname{Re} D < 2\alpha + 2$$

Similarly, the region of convergence of the integral of the second addendum is

$$2\alpha - 2 < \operatorname{Re} D < 2\alpha$$

The two regions do not overlap and, by the theorem below, one has

$$I_\alpha = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{m^2}{(p^2)^\alpha (p^2 + m^2)} + \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{(p^2)^{\alpha - 1} (p^2 + m^2)} \equiv I_\alpha^{(1)} + I_\alpha^{(2)}$$

Looking at tables of integrals [r], one has a general formula

$$\int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{(p^2)^a (p^2 + m^2)^b} = (m^2)^{\frac{D}{2} - a - b} \frac{\Gamma(D/2 - a)\Gamma(a + b - D/2)}{(4\pi)^{\frac{D}{2}} \Gamma(b) \Gamma(D/2)}$$

For the first integral, one has $a = \alpha$, $b = 1$ and for the second $a = \alpha - 1$ and $b = 1$. Therefore

$$\begin{aligned} I_\alpha &= m^{D - 2\alpha} \frac{\Gamma(D/2 - \alpha)\Gamma(1 + \alpha - D/2)}{(4\pi)^{\frac{D}{2}} \Gamma(D/2)} + m^{D - 2\alpha} \frac{\Gamma(D/2 + 1 - \alpha)\Gamma(\alpha - D/2)}{(4\pi)^{\frac{D}{2}} \Gamma(D/2)} \\ &= \frac{m^{D - 2\alpha}}{(4\pi)^{\frac{D}{2}} \Gamma(D/2)} \Gamma(\alpha - D/2) \Gamma(D/2 - \alpha) \left[\alpha - \frac{D}{2} + \frac{D}{2} - \alpha \right] = 0 \end{aligned}$$

where one applies $\Gamma(z + 1) = z\Gamma(z)$ to the Gamma function with three addenda in its argument.

$\square$

Theorem. If an integrand can be written as a sum of terms with non-overlapping regions of convergence (for their integrals), then the integral is a sum of their integrals.

One a-posteriori argument to see why the integral is null follows. One may rescale $p = sp'$ to have

$$I_\alpha = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{(p^2)^\alpha} = s^{D-2\alpha} \int \frac{\mathrm{d}^D p'}{(2\pi)^D} \frac{1}{(p'^2)^\alpha} \implies I_\alpha = s^{D-2\alpha} I_\alpha$$

For $D \neq 2\alpha$ then $I_\alpha = 0$. For the case $D = 2\alpha$, one can argue that there is a fine-tuning between the ultraviolet and the infrared divergences. In this case, there divergences are both present and give zero when summed. Consider the particular example of $D = 2 - \varepsilon$ and $\alpha = 1$. Then, the integral becomes

$$\begin{aligned} I_1 &= \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{p^2} = -\int \frac{\mathrm{d}^D p}{(2\pi)^D} \int_0^\infty \mathrm{d}\lambda \, \frac{1}{(p^2-\lambda)^2} = -\int_0^\infty \mathrm{d}\lambda \int \frac{\mathrm{d}^{2-\varepsilon} p}{(2\pi)^{2-\varepsilon}} \frac{1}{(p^2-\lambda)^2} \\ &= -\int_0^\infty \mathrm{d}\lambda \, \frac{\Gamma(1+\varepsilon/2)}{(4\pi)^{1-\frac{\varepsilon}{2}}\Gamma(2)} (-\lambda)^{-\frac{\varepsilon}{2}-1} = (-1)^{\frac{\varepsilon}{2}} \frac{\Gamma(1+\varepsilon/2)}{(4\pi)^{1-\frac{\varepsilon}{2}}} \int_0^\infty \frac{\mathrm{d}\lambda}{\lambda^{1+\frac{\varepsilon}{2}}} \end{aligned}$$

Ignoring the coefficients, the integral is

$$\begin{aligned} I_1 &\propto \int_{a_{\mathrm{IR}}}^1 \frac{\mathrm{d}\lambda}{\lambda^{1+\frac{\varepsilon}{2}}} + \int_1^{a_{\mathrm{UV}}} \frac{\mathrm{d}\lambda}{\lambda^{1+\frac{\varepsilon}{2}}} = -\frac{2}{\varepsilon}\lambda^{-\frac{\varepsilon}{2}}\Big|_{a_{\mathrm{IR}}}^1 - \frac{2}{\varepsilon}\lambda^{-\frac{\varepsilon}{2}}\Big|_1^{a_{\mathrm{UV}}} = -\frac{2}{\varepsilon}[-a_{\mathrm{IR}}^{-\frac{\varepsilon}{2}} + a_{\mathrm{UV}}^{-\frac{\varepsilon}{2}}] \\ &= -\frac{2}{\varepsilon}\left[-\frac{\varepsilon}{2}(-\ln a_{\mathrm{IR}} + \ln a_{\mathrm{UV}}) + o(\varepsilon)\right] = \ln a_{\mathrm{UV}} - \ln a_{\mathrm{IR}} + o(\varepsilon^0) = 0 \end{aligned}$$

At the second line, one has integrated up to a cutoff [r]. From this one sees that $a_{\mathrm{IR}} = a_{\mathrm{UV}}$ and $I_\alpha = 0$ is the product of a cancellation between the ultraviolet and infrared divergences.

When dealing with massless theories in two dimensions $D = 2$, the cancellation of the tadpole is due to a balance between infrared and ultraviolet divergences. When interested in either divergence, one has to remove the other divergence in order to renormalize the one of interest. For example, one replaces

$$I_1 = \int \frac{\mathrm{d}^{2-\varepsilon} p}{(2\pi)^{2-\varepsilon}} \frac{1}{p^2} \to \int \frac{\mathrm{d}^{2-\varepsilon} p}{(2\pi)^{2-\varepsilon}} \frac{1}{p^2+\mu^2} = \frac{\Gamma(\varepsilon/2)}{(4\pi)^{1-\frac{\varepsilon}{2}}}\mu^{-\varepsilon} \sim \frac{2}{\varepsilon}\mathrm{e}^{-\varepsilon \ln \mu + \cdots} \sim \frac{2}{\varepsilon}\,, \quad \varepsilon \to 0$$

where μ^2 is an infrared regulator. This is the ultraviolet divergence of the tadpole in two dimensions.

Therefore, in dimensional regularization, massless tadpoles can be ignored when $D \neq 2$. In $D = 2$ the tadpole is dimensionless and gives a contribution when removing one divergence.

Part I
Spin-half fermion fields

2 Introduction

Review – classical fields. See Srednicki. A fermion field is a field describing Dirac spinors. A Dirac spinor in the Weyl basis is comprised of two fixed-chirality Weyl spinors

$$\psi = \begin{bmatrix} \chi_{\mathrm{L}} \\ \chi_{\mathrm{R}} \end{bmatrix}$$

where the left-chiral spinor belongs to the $(\frac{1}{2}, 0)$ representation of the Lorentz group $\mathrm{SO}(1,3)$, while the right-chiral spinor belongs to $(0, \frac{1}{2})$.

The equation of motion of the Dirac field is the Dirac equation

$$(\mathrm{i}\,\partial\!\!\!/ - m)\psi(x) = 0$$

where the Dirac matrices are four square matrices of dimension four defining the Dirac algebra $\mathrm{Cl}_{1,3}(\mathbb{C})$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

The equation of motion can be obtained from the action principle with the Lagrangian

$$\mathcal{L} = \mathrm{i}\bar{\psi}\,\partial\!\!\!/\psi - m\bar{\psi}\psi = \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi\,, \quad \bar{\psi} = \psi^\dagger\gamma^0$$

If the field ψ satisfies the Dirac equation of motion, then each of its four components (each two of the Weyl spinors) satisfies the Klein–Gordon equation

$$(\Box + m^2)\psi_j(x) = 0$$

In momentum space, the above is an algebraic equation that gives the dispersion relation

$$(-p^2 + m^2)\psi(p) = 0 \implies p^2 = m^2 \iff p^0 = \sqrt{|\mathbf{p}|^2 + m^2} \equiv \omega$$

The most general solution of the Klein–Gordon equation has the form[1]

$$\psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[u(\mathbf{p})\mathrm{e}^{-\mathrm{i}px} + v(\mathbf{p})\mathrm{e}^{\mathrm{i}px}]_{p^0=\omega}$$

Imposing the Dirac equation one finds

$$(p\!\!\!/ - m)u(\mathbf{p}) = 0\,, \quad (p\!\!\!/ + m)v(\mathbf{p}) = 0$$

One solves these equation by going to the rest frame $\mathbf{p} = 0$ for which

$$p\!\!\!/ = \gamma^0 p_0 = \gamma^0 m \implies (\gamma^0 - 1)u(0) = 0\,, \quad (\gamma^0 + 1)v(0) = 0$$

Each equation has two independent solutions $u_\pm$ and $v_\pm$. To get the solution for an arbitrary momentum one has to perform a boost. Therefore, the general solution to the Dirac equation is

$$\psi(x) = \sum_{s=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[b_s(\mathbf{p})u_s(\mathbf{p})\mathrm{e}^{-\mathrm{i}px} + d_s^\dagger v_s(\mathbf{p})\mathrm{e}^{\mathrm{i}px}]_{p^0=\omega}$$

where b and d are numbers.

The two-spinors u_s and v_s satisfy several identities (see Peskin, p. 48). The normalization is chosen to be

$$\bar{u}_r(\mathbf{p})u_s(\mathbf{p}) = 2m\delta_{rs}\,, \quad \bar{v}_r(\mathbf{p})v_s(\mathbf{p}) = -2m\delta_{rs}$$

Equivalent to

$$u_r^\dagger(\mathbf{p})u_s(\mathbf{p}) = 2p^0\delta_{rs}\,, \quad v_r^\dagger(\mathbf{p})v_s(\mathbf{p}) = 2p^0\delta_{rs}$$

The spinors are orthogonal

$$\bar{u}_r(\mathbf{p})v_s(\mathbf{p}) = \bar{v}_r(\mathbf{p})u_s(\mathbf{p}) = 0\,, \quad u_r^\dagger(\mathbf{p})v_s(-\mathbf{p}) = v_r^\dagger(-\mathbf{p})u_s(\mathbf{p}) = 0$$

The last two relations are not zero for both momenta being $+\mathbf{p}$. With these, one can obtain

$$b_s(\mathbf{p}) = \int \mathrm{d}^3x\,\mathrm{e}^{\mathrm{i}px}u_s^\dagger(\mathbf{p})\psi(x)\,, \quad d_s(\mathbf{p}) = \int \mathrm{d}^3x\,\mathrm{e}^{\mathrm{i}px}\psi^\dagger(x)v_s(\mathbf{p})$$

To see this, it is sufficient to Fourier transform $u_s^\dagger(\mathbf{p})\psi(x)$, substitute $\psi(x)$ and apply the above rules. Notice that

$$\int \mathrm{d}^3x\,\mathrm{e}^{\mathrm{i}x(p-q)} = \mathrm{e}^{\mathrm{i}x^0(p^0-q^0)}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})$$

and that

$$|\mathbf{p}| = |\mathbf{q}| \implies p^0 = q^0$$

Lecture 2

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Canonical quantization. See Srednicki, §39. The classical fields are promoted to operator fields by imposing a suitable set of equal-time anti-commutation rules (ACR)

$$\{\psi_\alpha(x), \psi_\beta(y)\} = 0\,, \quad \{\psi_\alpha(x), \psi^\dagger_\beta(y)\} = \delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{y})$$

From these, one obtains the rules for the Fourier coefficients which are promoted to operators

$$\{b_r(\mathbf{p}), b_s(\mathbf{q})\} = \{d_r(\mathbf{p}), d_s(\mathbf{q})\} = 2\omega(2\pi)^3\delta_{rs}\delta^{(3)}(\mathbf{p}-\mathbf{q})$$

One defines the vacuum state as the state destroyed by every annihilation operator

$$b_s(\mathbf{p})\,|0\rangle = d_s(\mathbf{p})\,|0\rangle = 0\,, \quad \forall b, d$$

The excited states are obtained by acting on the vacuum with the creation operators $b^\dagger_s$ and $d^\dagger_s$. One interprets $b^\dagger_s(\mathbf{p})\,|0\rangle$ as a single-particle state with momentum $\mathbf{p}$, energy $\omega = \sqrt{|\mathbf{p}|^2 + m^2}$ and projection s (in units) of the spin along the z-direction, $S_z = \frac{1}{2}s$. The difference between the two creation operators is not yet clear.

The use of anti-commutation rules implies the Pauli exclusion principle: there cannot be two fermions with the same quantum numbers. This is reflected in the spectrum of the number operator and the relation

$$\{b^\dagger_s(\mathbf{p}), b^\dagger_s(\mathbf{p})\} = 0 \implies [b^\dagger_s(\mathbf{p})]^2 = 0$$

The Hamiltonian density is obtained by Legendre transforming the Lagrangian density. By substituting the fields ψ and $\bar{\psi}$, one obtains the energy (Hamiltonian) of the system

$$H = \sum_{s=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}\omega[N^b_s(\mathbf{p}) + N^d_s(\mathbf{p})]$$

where N^a_s is the number operator of spin s for the ladder operator a

$$N^a_s(\mathbf{p}) \equiv a^\dagger_s(\mathbf{p})a_s(\mathbf{p})$$

The anti-commutation relations imply a finite spectrum

$$[N^a_s(\mathbf{p})]^2 = N^a_s(\mathbf{p}) \implies n = 0, 1$$

The Dirac Lagrangian exhibits a global U(1) symmetry

$$\psi' = \mathrm{e}^{\mathrm{i}\alpha}\psi\,, \quad \bar{\psi} = \mathrm{e}^{-\mathrm{i}\alpha}\bar{\psi}\,, \quad \alpha \in \mathbb{R}$$

Noether's theorem implies a conserved current

$$\partial_\mu J^\mu = 0\,, \quad J^\mu = \bar{\psi}\gamma^\mu\psi$$

The conserved charge is

$$Q = \int \mathrm{d}^3x\, J^0(x) = \int \mathrm{d}^3x\,\bar{\psi}\gamma^0\psi = \sum_{s=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[N^b_s(\mathbf{p}) - N^d_s(\mathbf{p})]$$

The global symmetry gives a minus sign for the operator d: the creation operators have different meanings. The operator d treats particles with opposite U(1) charge of the operator b. The b-particles, with charge $Q = 1$, are simply called particles; while the d-particles, with $Q = -1$, are called anti-particles.

Free fermion propagator. See Srednicki, §42. The free fermion propagator is the inverse of the kinetic term in the Lagrangian

$$S_{\alpha\beta}(x-y) \equiv -\mathrm{i}\,\langle 0|\,\mathcal{T}\{\psi_\alpha(x)\bar{\psi}_\beta(y)\}\,|0\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{(\not{p}+m)_{\alpha\beta}}{p^2 - m^2 + \mathrm{i}\varepsilon}\,\mathrm{e}^{\mathrm{i}p(x-y)}$$

Exercise. Check that the propagator is the Green's function of the Dirac operator

$$(\mathrm{i}\,\not{\partial} - m)_{\alpha\beta}S_{\beta\gamma}(x) = \delta_{\alpha\gamma}\delta^{(4)}(x)$$

[1] Understood as a group of four components that solve the equation.

3 LSZ reduction formula

See Srednicki, §41. The physical observables are the cross-sections which are expressed in terms of scattering amplitudes $\langle f|i\rangle$. The initial and final states are defined asymptotically in the distant past and future, $t \to \mp\infty$. One utilizes the adiabatic hypothesis: the region where the interaction is non-trivial is finite and, beyond, the theory is essentially free. The hypothesis has to be supported by the ansatz that the free regions are reached in the distant past and future.

One-particle states can be constructed by applying one a creation operator

$$a_s^\dagger(\mathbf{p})\,|0\rangle = |\mathbf{p}, s\rangle$$

Due to the Heisenberg uncertainty principle, a more physical particle is the one comprised of a wave-packet of momenta. The physical creation operator is a Gaussian distribution centered around a central momentum $\mathbf{q}$

$$a_s^\dagger = \int \mathrm{d}^3p\, f(\mathbf{p}) a_s^\dagger(\mathbf{p})\,, \quad f(\mathbf{p}) \propto \mathrm{e}^{-\frac{(\mathbf{p}-\mathbf{q})^2}{4\sigma^2}}$$

The operation of applying the creation operator to the vacuum to obtain a particle is valid and meaningful in the asymptotic limit, where the states and theory are free. This holds since the vacuum defined by the annihilation operators is the free vacuum. By introducing an interaction, the vacuum is interacting. One assumes to be able to define a vacuum in the interacting theory and that the particles can be obtained in the same way as the free theory. For instance, a one-particle initial and final state are taken to be

$$|i\rangle = \lim_{t\to-\infty} b_r^\dagger(t)\,|0\rangle\ , \quad |f\rangle = \lim_{t\to\infty} b_s^\dagger(t)\,|0\rangle$$

In the interacting theory, the ladder operators depend on time.

Time dependence. To obtain a useful expression for the scattering amplitude, one needs to study the time dependence of the ladder orpeators. As with the scalar fields, consider

$$\begin{aligned}
b_s(+\infty) - b_s(-\infty) &= \int_{-\infty}^{\infty} \mathrm{d}t\,\partial_0 b_s(t) = \int \mathrm{d}^3p\, f(\mathbf{p}) \int_{-\infty}^{\infty} \mathrm{d}t\,\partial_0 b_s(\mathbf{p}) \\
&= \int \mathrm{d}^3p\, f(\mathbf{p}) \int_{-\infty}^{\infty} \mathrm{d}t \int \mathrm{d}^3x\,\partial_0[\mathrm{e}^{\mathrm{i}px} u_s^\dagger(\mathbf{p})\psi(x)] \\
&= \int \mathrm{d}^3p\, f(\mathbf{p}) \int \mathrm{d}^4x\,\partial_0[\mathrm{e}^{\mathrm{i}px} \bar{u}_s(\mathbf{p})\gamma^0\psi(x)] \\
&= \int \mathrm{d}^3p\, f(\mathbf{p}) \int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}px} \bar{u}_s(\mathbf{p})[\mathrm{i}p_0\gamma^0 + \gamma^0\,\partial_0]\psi(x) \\
&= \int \mathrm{d}^3p\, f(\mathbf{p}) \int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}px} \bar{u}_s(\mathbf{p})[-\mathrm{i}p_j\gamma^j + \mathrm{i}m + \gamma^0\,\partial_0]\psi(x) \\
&= \int \mathrm{d}^3p\, f(\mathbf{p}) \int \mathrm{d}^4x\,\bar{u}_s(\mathbf{p})[-\gamma^j\,\partial_j \mathrm{e}^{\mathrm{i}px} + \mathrm{e}^{\mathrm{i}px}(\mathrm{i}m + \gamma^0\,\partial_0)]\psi(x) \\
&= \int \mathrm{d}^3p\, f(\mathbf{p}) \int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}px} \bar{u}_s(\mathbf{p})[\gamma^j\,\partial_j + \mathrm{i}m + \gamma^0\,\partial_0]\psi(x) \\
&= -\mathrm{i} \int \mathrm{d}^3p\, f(\mathbf{p}) \int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}px} \bar{u}_s(\mathbf{p})[\mathrm{i}\,\partial\!\!\!/ - m]\psi(x)
\end{aligned}$$

At the first line, one inserts the expression of the wave-packet operator b_1. At the second line, one inserts the expression of the annihilation operator $b_s(\mathbf{p})$. At the fifth line, one has remembered that

$$\bar{u}(\mathbf{p})(p\!\!\!/ - m) = 0 \implies \bar{u}(\mathbf{p})(p_0\gamma^0 + p_j\gamma^j - m) = 0 \implies \bar{u}(\mathbf{p})p_0\gamma^0 = \bar{u}(\mathbf{p})(-p_j\gamma^j + m)$$

At the penultimate line, one has integrated by parts the first addendum.

The integrand is zero if the theory is free, but the theory is interacting and so

$$(\mathrm{i}\,\partial\!\!\!/ - m)\psi(x) = -\delta_{\bar{\psi}(x)}\mathcal{L}_{\mathrm{int}} \neq 0$$

and the ladder operators are functions of time. For the creation operators, one has

$$b_s^\dagger(+\infty) - b_s^\dagger(-\infty) = -\mathrm{i}\int \mathrm{d}^3p\, f(\mathbf{p}) \int \mathrm{d}^4x\, \bar{\psi}(x)(\mathrm{i}\overleftarrow{\not{\partial}} + m)u_s(\mathbf{p})\mathrm{e}^{-\mathrm{i}px}$$

[r]

Two-by-two scattering. Consider a two-by-two scattering $p_1p_2 \to p_{1'}p_{2'}$. The scattering amplitude is

$$\langle f|i\rangle = \langle 0|\, b_{2'}(+\infty)b_{1'}(+\infty)b_1^\dagger(-\infty)b_2^\dagger(-\infty)\,|0\rangle$$

where the index of the spin projection is dropped and $1, 2$ or $1', 2'$ indicates the particle. Thanks to the ordering of the operators, the above is equal to

$$\langle f|i\rangle = \langle 0|\, \mathcal{T}\{b_{2'}(+\infty)b_{1'}(+\infty)b_1^\dagger(-\infty)b_2^\dagger(-\infty)\}\,|0\rangle$$

In the limit $\sigma \to 0$, the wave-packet is a Dirac delta giving

$$b_s^\dagger(+\infty) - b_s^\dagger(-\infty) = -\mathrm{i}\int \mathrm{d}^4x\, \mathrm{e}^{-\mathrm{i}px}\bar{\psi}(x)(\mathrm{i}\overleftarrow{\not{\partial}} + m)u_s(\mathbf{p})$$

One would like to replace $b_j^\dagger(-\infty)$ with $b_j^\dagger(+\infty)$ and the integral above. Due to the time-ordered product, the operator $b_j^\dagger(+\infty)$ is moved to the left and gives zero when acting on the bra [r]. Similarly happens for the annihilation operators. Therefore the scattering amplitude is

$$\begin{aligned}
\langle f|i\rangle &= (-1)^2\mathrm{i}^4\,\langle 0|\,\mathcal{T}\bigg\{\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\mathrm{d}^4x_{1'}\,\mathrm{d}^4x_{2'} \\
&\quad\times \mathrm{e}^{\mathrm{i}p_{2'}x_{2'}}\,\bar{u}_{s_{2'}}(\mathbf{p}_{2'})(\mathrm{i}\not{\partial} - m)\psi(x_{2'})\,\mathrm{e}^{\mathrm{i}p_{1'}x_{1'}}\,\bar{u}_{s_{1'}}(\mathbf{p}_{1'})(\mathrm{i}\not{\partial} - m)\psi(x_{1'}) \\
&\quad\times \bar{\psi}(x_1)(\mathrm{i}\overleftarrow{\not{\partial}} + m)u_{s_1}(\mathbf{p}_1)\mathrm{e}^{-\mathrm{i}p_1x_1}\,\bar{\psi}(x_2)(\mathrm{i}\overleftarrow{\not{\partial}} + m)u_{s_2}(\mathbf{p}_2)\mathrm{e}^{-\mathrm{i}p_2x_2}\bigg\}\,|0\rangle \\
&= \int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\mathrm{d}^4x_{1'}\,\mathrm{d}^4x_{2'}\,\mathrm{e}^{\mathrm{i}p_{2'}x_{2'}}\,\bar{u}_{s_{2'}}(\mathbf{p}_{2'})(\mathrm{i}\not{\partial} - m)_{2'}\mathrm{e}^{\mathrm{i}p_{1'}x_{1'}}\,\bar{u}_{s_{1'}}(\mathbf{p}_{1'})(\mathrm{i}\not{\partial} - m)_{1'} \\
&\quad\times \langle 0|\,\mathcal{T}\{\psi(x_{2'})\psi(x_{1'})\bar{\psi}(x_1)\bar{\psi}(x_2)\}\,|0\rangle\,(\mathrm{i}\overleftarrow{\not{\partial}} + m)_1u_{s_1}\mathrm{e}^{\mathrm{i}p_1x_1}(\mathrm{i}\overleftarrow{\not{\partial}} + m)_2u_{s_2}\mathrm{e}^{\mathrm{i}p_2x_2}\}
\end{aligned}$$

[r] This is the LSZ reduction formula for fermions. In general, the computation of a scattering amplitude can be expressed as a computation of correlation functions

$$G^{(2n)}(x_1, \ldots, x_n, x_1', \ldots x_n') = \langle 0|\,\mathcal{T}\{\psi(x_1')\cdots\psi(x_n')\bar{\psi}(x_1)\cdots\bar{\psi}(x_n)\}\,|0\rangle$$

[r] One needs to develop a functional approach to compute the above Green's functions.

4 Functional quantization

See Srednicki, §43. Consider an interacting fermionic theory

$$\mathcal{L}(\psi, \bar{\psi}) = \bar{\psi}(\mathrm{i}\not{\partial} - m)\psi + \mathcal{L}_{\mathrm{int}}(\psi, \bar{\psi})$$

One needs to generalize the generating functional for fermions. One introduces the spinorial source terms η and $\bar{\eta}$ to formally write

$$W[\eta, \bar{\eta}] = \int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\exp\left[\mathrm{i}\int \mathrm{d}^4x\,[\mathcal{L}(x) + \bar{\psi}\eta + \bar{\eta}\psi]\right]$$

where the spinorial product is defined using the van der Waerden notation[2]. The integrand must be a real quantity. In fact

$$(\bar{\psi}\eta)^\dagger = (\psi^\dagger\gamma^0\eta)^\dagger = \eta^\dagger\gamma^0\psi = \bar{\eta}\psi$$

[2] See also eigenchris, Spinors for Beginners 9, `https://youtu.be/4NJBvkjpC3E?t=2340`, at minute 39:00 as well as A. Steane, An Introduction to Spinors, `https://arxiv.org/abs/1312.3824`.

which implies that $\bar{\psi}\eta + \bar{\eta}\psi$ is real. Since spinors anti-commute, then

$$\bar{\psi}\eta = -\eta\bar{\psi}$$

Therefore, when differentiating one has to be careful about the signs

$$\delta_{\eta(x)} \int \mathrm{d}^4 y\, \bar{\psi}(y)\eta(y) = -\bar{\psi}(x)$$

The Green's function is

$$\begin{aligned} G^{(2n)}(x_1, \dots, x_n, y_1, \dots, y_n) &= \langle 0| \mathcal{T}\{\psi_{\alpha_1}(x_1) \cdots \psi_{\alpha_n}(x_n)\bar{\psi}_{\beta_1}(y_1) \cdots \bar{\psi}_{\beta_n}(y_n)\} |0\rangle \\ &= (-1)^n \mathrm{i}^{2n} \frac{\delta^{2n} W[\eta, \bar{\eta}]}{\delta\bar{\eta}_{\alpha_1}(x_1) \cdots \delta\bar{\eta}_{\alpha_n}(x_n)\delta\eta_{\beta_1}(y_1) \cdots \delta\eta_{\beta_n}(y_n)}\bigg|_{\eta=\bar{\eta}=0} \\ &= \frac{\delta^{2n} W[\eta, \bar{\eta}]}{\delta\bar{\eta}_{\alpha_1}(x_1) \cdots \delta\bar{\eta}_{\alpha_n}(x_n)\delta\eta_{\beta_1}(y_1) \cdots \delta\eta_{\beta_n}(y_n)}\bigg|_{\eta=\bar{\eta}=0} \end{aligned}$$

Each derivative with respect to $\bar{\eta}$ brings a $-\mathrm{i}$ while each derivative with respect to η brings an i.

The Euclidean functional integral is

$$W_{\mathrm{E}}[\eta, \bar{\eta}] = \int [\mathcal{D}\psi\, \mathcal{D}\bar{\psi}] \exp\left[-\int \mathrm{d}^4 x\, [\mathcal{L}(x) - \bar{\psi}\eta - \bar{\eta}\psi]\right]$$

and the Euclidean Green's function is [r]

$$G_{\mathrm{E}}^{(2n)}(x_1, \dots, x_n, y_1, \dots, y_n) = (-1)^n \frac{\delta^{2n} W_{\mathrm{E}}[\eta, \bar{\eta}]}{\delta\bar{\eta}_{\alpha_1}(x_1) \cdots \delta\bar{\eta}_{\alpha_n}(x_n)\delta\eta_{\beta_1}(y_1) \cdots \delta\eta_{\beta_n}(y_n)}\bigg|_{\eta=\bar{\eta}=0}$$

The functional integral involves spinorial variables. This kind of integration has to be properly defined.

Exercise. Let $n = 1$, then

$$\begin{aligned} G^{(2)}(x, y) &= (-1) \frac{\delta^2 W_{\mathrm{E}}[\eta, \bar{\eta}]}{\delta\bar{\eta}_\alpha(x)\delta\eta_\beta(y)}\bigg|_{\eta=\bar{\eta}=0} \\ &= (-1) \frac{\delta}{\delta\bar{\eta}_\alpha(x)} \int [\mathcal{D}\psi\, \mathcal{D}\bar{\psi}] \exp\left[-\int \mathrm{d}^4 x'\, [\mathcal{L} - \bar{\psi}\eta - \bar{\eta}\psi]\right] [-\bar{\psi}_\beta(y)]\bigg|_{\eta=\bar{\eta}=0} \\ &= \int [\mathcal{D}\psi\, \mathcal{D}\bar{\psi}] \exp\left[-\int \mathrm{d}^4 x'\, [\mathcal{L} - \bar{\psi}\eta - \bar{\eta}\psi]\right] \psi_\alpha(x)\bar{\psi}_\beta(y)\bigg|_{\eta=\bar{\eta}=0} \\ &= \langle 0| \mathcal{T}\{\psi_\alpha(x)\bar{\psi}_\beta(y)\} |0\rangle \end{aligned}$$

Lecture 3

4.1 Grassmann algebra

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See Cheng, §1.3. See also DeWitt, Supermanifolds. The fields ψ and $\bar{\psi}$ are classical spinor fields with four components in Dirac's notation

$$\psi_\alpha\,, \quad \alpha = 1, 2, 3, 4$$

Each component is a Grassmann-odd number[3]: it anti-commutes with itself and other Grassmann-odd numbers. For example

$$\psi_1\psi_2 = -\psi_2\psi_1$$

[3] Odd or even refers to the number of Grassmann variables θ_i in the expansion of a Grassmann number z in terms of such Grassmann variables θ_i. A Grassmann number may not have definite Grassmann parity, but can be separated into odd and even parts. Odd Grassmann numbers anti-commute between themselves and even Grassmann numbers commute with every Grassmann number.

Grassmann numbers are needed when taking the classical limit of the quantized spinor fields which are anti-commuting. When using a functional definition of quantum field theory, the quantities appearing inside the path integral are all classical, not operators, so they have to be Grassmann numbers.

The path integral is the continuum limit of an ordinary integral on a lattice. A field is evaluated only at the sites of the lattice

$$\psi_\alpha(x^i) \equiv \psi_\alpha^i$$

The path integral on the lattice is an ordinary integral but on Grassmann-odd variables[4]

$$\int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}] \leftrightarrow \int \prod_{\alpha,i} \mathrm{d}\psi_\alpha^i \prod_{\alpha,i} \mathrm{d}\bar{\psi}_\alpha^i$$

One needs to define analysis for Grassmann numbers.

Algebra. An n-dimensional Grassmann algebra is generated by n Grassmann variables θ_i that anti-commute

$$\{\theta_i, \theta_j\} = 0 \implies \theta_i^2 = 0$$

A generic element of the algebra can be expanded in a finite Taylor series

$$f(\theta_1, \dots, \theta_n) = f_0 + f_{1,i}\theta^i + f_{2,ij}\theta^i\theta^j + \cdots + f_{n,i_1\cdots i_n}\theta^{i_1}\cdots\theta^{i_n}$$

This is because if there is any repeated variable, then it is zero $\theta_j^2 = 0$. The coefficients of the expansion are complex numbers that are completely anti-symmetric in the indices i_k.

Consider the expansion of an element of the algebra by writing explicitly the dependence on one particular[5] Grassmann variable θ. In such variable, the expansion is at most linear

$$f(\theta) = f_0 + f_1\theta$$

The coefficients f_0 and f_1 are independent of θ and depend on the other $n-1$ Grassmann variables.

In general, one has to consider a function also of space-time

$$f(x, \theta) = f_0(x) + f_1(x)\theta$$

The coefficient of the θ-expansion are space-time fields. If $f(x,\theta)$ is a Grassmann-even field, then the Grassmann parity must be the same on either side of the equation, so $f_0(x)$ is a Grassmann-even field and $f_1(x)$ is a Grassmann-odd field.

The addition of Grassmann variables to space-time gives superspace[6]. A field on superspace is a superfield and contains different ordinary fields. In supersymmetry, a superfield is a representation of the supersymmetry algebra.

Differentiation. The left and right derivatives are defined as

$$\mathrm{d}_\theta\theta = \theta\,\overleftarrow{\mathrm{d}_\theta} = 1$$

Therefore, the left derivative of the above element of the algebra is

$$\mathrm{d}_\theta f(\theta) = \begin{cases} +f_1\,, & f \text{ odd} \\ -f_1\,, & f \text{ even} \end{cases}$$

while the right derivative is simpler

$$f(\theta)\,\overleftarrow{\mathrm{d}_\theta} = f_1$$

[4]Grassmann numbers are individual elements of the exterior algebra generated by a set of n Grassmann variables.

[5]This means that the dependence on the other $n-1$ variables is hidden inside the expansion coefficients which are no longer complex numbers, but Grassmann numbers.

[6]In particular Minkowski superspace: Minkowski space is extended with anti-commuting fermionic degrees of freedom, taken to be anti-commuting Weyl spinors from the Clifford algebra associated to the Lorentz group.

Integration. The integration over Grassmann-odd variables is called Berenzin integration[7] which differs from Lebesgue's.

Typically, the integral is the inverse operator of the derivative. Since the partial derivative with respect to a Grassmann variable is a Grassmann-odd operator then the second derivative is zero[8]. The derivative is not an invertible operator. The integral must be defined in another way

$$\int \mathrm{d}\theta\, f(\theta)$$

One imposes two properties: linearity

$$\int \mathrm{d}\theta\,(f+\alpha g) = \int \mathrm{d}\theta\, f + \alpha \int \mathrm{d}\theta\, g\,, \quad \alpha \in \mathbb{C}$$

and translational invariance

$$\int \mathrm{d}\theta\, f(\theta) \equiv \int \mathrm{d}(\theta+\eta)\, f(\theta+\eta)$$

where η is Grassmann-odd number independent of the particular variable θ.

The second property can be expanded to have

$$\int \mathrm{d}\theta\, f(\theta) = \int \mathrm{d}(\theta+\eta)\, f(\theta+\eta) = \int \mathrm{d}\theta\, f(\theta+\eta) \implies \int \mathrm{d}\theta\,(f_0 - \theta f_1) = \int \mathrm{d}\theta\,[f_0 - (\theta+\eta) f_1]$$

Applying linearity, one finds

$$\int \mathrm{d}\theta\, \eta f_1 = 0 \implies \eta f_1 \int \mathrm{d}\theta = 0 \implies \boxed{\int \mathrm{d}\theta = 0}$$

since the above has to hold for every f_1 and η (which do not depend on θ). Using this result, one obtains

$$\int \mathrm{d}\theta\, f(\theta) = \int \mathrm{d}\theta\, f_0 + f_1\theta = \int \mathrm{d}\theta\, f_1\theta = \pm f_1 \int \mathrm{d}\theta\, \theta \equiv \pm f_1 \implies \boxed{\int \mathrm{d}\theta\, \theta = 1}$$

where $+$ is for f odd and $-$ for f even. One may notice that Berenzin integration is equivalent to differentiation

$$\int \mathrm{d}\theta\, f(\theta) = \pm f_1 = \mathrm{d}_\theta f$$

Therefore, the general definition of integration in a Grassmann-odd variable is

$$\boxed{\int \mathrm{d}\theta\, f(\theta) \equiv \mathrm{d}_\theta f|_{\theta=0}}$$

[r] why evaluated at $\theta = 0$? for defining the Berenzin integral also for ordinary functions?

Change of variables. When performing a change of variables, one has to account for the Jacobian. For the Berenzin integral, the inverse Jacobian is produced instead.

For ordinary (Grassmann-even, Riemann or Lebesgue) integration, a change of variable produces

$$\int \mathrm{d}y\, f(y)\,, \quad y = g(x) \implies \int \mathrm{d}y\, f(y) = \int \mathrm{d}x\, |\partial_x y| f(g(x))$$

For Grassmann-odd integration, a change of variables is

$$\theta' = g(\theta) = a + b\theta\,, \quad \mathrm{d}_\theta \theta' = b$$

[7]See Berezin, F. A. (1966). The Method of Second Quantization. Pure and Applied Physics. Vol. 24. New York. ISSN 0079-8193. `https://www.sciencedirect.com/bookseries/pure-and-applied-physics/vol/24`.

[8]This is a consequence of $\partial_z z = 1$.

Notice that a is odd and b is even. Knowing that

$$\int d\theta' \, f(\theta') = \int d\theta' \, (f_0 + f_1 \theta') = \pm f_1$$

one may consider the integration without Jacobian

$$\int d\theta \, f(\theta'(\theta)) = \int d\theta \, f(a + b\theta) = \int d\theta \, [f_0 + f_1(a + b\theta)] = \pm f_1 b = \pm f_1 \, d_\theta \theta'$$

Therefore, when performing the change of variables, one has to use the inverse Jacobian

$$\int d\theta' \, f(\theta') = \int d\theta \, |d_\theta \theta'|^{-1} f(\theta'(\theta))$$

Generalization to more variables. The differentiation and integration results above can be generalized to more Grassmann variables.

Differentiation. The left and right derivatives are

$$d_{\theta_i} \theta_j = \theta_j \overleftarrow{d}_{\theta_i} = \delta_{ij}$$

The derivative are Grassmann-odd operators

$$\{d_{\theta_i}, d_{\theta_j}\} = 0 \,, \quad \{d_{\theta_i}, \theta_j\} = 0$$

where, in the second equality, the operator is understood as not acting on θ_j: when computing the derivative of a product, a minus sign appears every time the derivative goes though an odd variable. As such, the derivative of a product is

$$d_{\theta_i}(\theta_1 \cdots \theta_n) = \delta_{i1} \theta_2 \cdots \theta_n - \theta_1 \delta_{i2} \theta_3 \cdots \theta_n + \cdots + (-1)^{n-1} \theta_1 \cdots \theta_{n-1} \delta_{in} = \sum_{j=1}^n (-1)^{j-1} \delta_{ij} \prod_{k \neq j}^n \theta_k$$

while the right derivative is similar

$$(\theta_1 \cdots \theta_n) \overleftarrow{d}_{\theta_i} = \theta_1 \cdots \theta_{n-1} \delta_{in} - \theta_1 \cdots \delta_{i,n-1} \theta_n + \cdots + (-1)^{n-1} \delta_{i1} \theta_2 \cdots \theta_n = \sum_{j=1}^n (-1)^{n-j} \delta_{ij} \prod_{k \neq j}^n \theta_k$$

Integration. For integration, the measures anti-commute

$$\{d\theta_i, d\theta_j\} = 0$$

The fundamental integral properties are

$$\int d\theta_i \, 1 = 0 \,, \quad \int d\theta_i \, \theta_j = \delta_{ij}$$

The integral in one variable of an element of the algebra is

$$\int d\theta_i \, f(\theta_1, \dots, \theta_n) = d_{\theta_i} f(\theta_1, \dots, \theta_n)|_{\theta_i = 0}$$

The integral in multiple variables is

$$\int d\theta_n \, \cdots \, d\theta_1 \, f(\theta_1, \dots, \theta_n) = d_{\theta_n} \cdots \, d_{\theta_1} f|_{\theta_1 = \cdots = \theta_n = 0}$$

Change of variables. For a change of variables $\theta'_i = b_{ij} \theta_j$, one has

$$\int d\theta'_n \cdots d\theta'_1 f(\theta'_1, \dots, \theta'_n) = \int d\theta_n \cdots d\theta_1 \, (\det b)^{-1} f(\theta'_1(\theta), \dots, \theta'_n(\theta))$$

Gaussian integrals. The Gaussian integral is

$$G(A) = \int \mathrm{d}\theta_n \cdots \mathrm{d}\theta_1 \, \mathrm{e}^{\frac{1}{2}\theta_i A_{ij} \theta_j}$$

where A is an anti-symmetric square matrix of size n, $A_{ij} = -A_{ji}$.

Let $n = 2$. The exponent is

$$\theta_i A_{ij} \theta_j = \begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix} \begin{bmatrix} 0 & A_{12} \\ -A_{12} & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

The integral is

$$\begin{aligned} G(A) &= \int \mathrm{d}\theta_2 \, \mathrm{d}\theta_1 \, \exp\left[\frac{1}{2}(\theta_1 A_{12} \theta_2 - \theta_2 A_{12} \theta_1)\right] = \int \mathrm{d}\theta_2 \, \mathrm{d}\theta_1 \mathrm{e}^{\theta A_{12} \theta_2} \\ &= \int \mathrm{d}\theta_2 \, \mathrm{d}\theta_1 \, (1 + \theta_1 A_{12} \theta_2) = \int \mathrm{d}\theta_2 \, \mathrm{d}\theta_1 \, \theta_1 A_{12} \theta_2 = A_{12} \int \mathrm{d}\theta_2 \, \theta_2 = A_{12} = \sqrt{\det A} \end{aligned}$$

This is true for any n. For Berenzin integrals, the determinant is in the numerator of the result and not the denominator.

Table of Gaussian integrals. For real Grassmann-even variables

$$\int \mathrm{d}x_1 \cdots \mathrm{d}x_n \, \mathrm{e}^{-\frac{1}{2} x^\top A x} = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}}$$

For complex Grassmann-even variables

$$\int \left[\prod_{j=1}^{n} \mathrm{d}z_j \, \mathrm{d}\bar{z}_j\right] \mathrm{e}^{-z^\dagger A z} = \frac{\pi^{\frac{n}{2}}}{\det A}$$

[r] For real Grassmann-odd variables

$$\int \mathrm{d}\theta_n \cdots \mathrm{d}\theta_1 \, \mathrm{e}^{\frac{1}{2}\theta A \theta} = \sqrt{\det A}$$

For complex Grassmann-odd variables

$$\int \left[\prod_{j=n}^{1} \mathrm{d}\theta_j \, \mathrm{d}\bar{\theta}_j\right] \mathrm{e}^{\bar{\theta}^\top A \theta} = \det A$$

Functional integrals. The generalization to functional integration can be defined on the lattice. The continuum theory is retrieved in the continuum limit of the lattice.

Example. Consider the free generating functional of a scalar theory

$$W_0[0] = \int [\mathcal{D}\varphi] \, \exp\left[-\int \mathrm{d}^4 x \, \frac{1}{2} \varphi(\Box + m^2) \varphi\right] \propto [\det(\Box + m^2)]^{-\frac{1}{2}}$$

where the determinant of an operator is the product of its eigenvalues. For a spinor theory, one has

$$W_0[0] = \int [\mathcal{D}\psi \, \mathcal{D}\bar{\psi}] \, \exp\left[-\int \mathrm{d}^4 x \, \frac{1}{2} \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi\right] \propto \det(\mathrm{i}\,\partial\!\!\!/ - m)$$

4.2 Free field theory

For a free theory, the functional integral can be computed exactly. For a weak interacting theory, one may applie perturbation theory.

Through Grassmann numbers, one may give meaning to the path integral for fermions

$$W[\eta, \bar{\eta}] = \int [\mathcal{D}\psi \, \mathcal{D}\bar{\psi}] \exp\left[\mathrm{i} \int \mathrm{d}^4 x \, [\mathcal{L}(x) + \bar{\psi}\eta + \bar{\eta}\psi]\right]$$

Consider the free Lagrangian

$$\mathcal{L}_0(\psi,\bar\psi) = \bar\psi(\mathrm{i}\,\partial\!\!\!/ - m)\psi$$

The generating functional is

$$W_0[\eta,\bar\eta] = N\int [\mathcal{D}\psi\,\mathcal{D}\bar\psi]\exp\left[\mathrm{i}\int \mathrm{d}^4x\,[\bar\psi(\mathrm{i}\,\partial\!\!\!/ - m)\psi + \bar\psi\eta + \bar\eta\psi]\right]$$

The propagator is the inverse of the kinetic term. Using the fact that the propagator is Green's function of the Dirac operator, one gets

$$\int \mathrm{d}^4x\,(\mathrm{i}\,\partial\!\!\!/ - m)_{\alpha\beta}S_{\beta\gamma}(x) = \delta_{\alpha\gamma} \implies S = (\mathrm{i}\,\partial\!\!\!/ - m)^{-1}$$

One may complete the square in the exponent of the generating functional

$$\begin{aligned}
-\mathrm{i}E &= \int \mathrm{d}^4x\,[\bar\psi(\mathrm{i}\,\partial\!\!\!/ - m)\psi + \bar\psi\eta + \bar\eta\psi] \\
&= \int \mathrm{d}^4x\left\{\left[\bar\psi + \int \mathrm{d}^4y\,\bar\eta(y)(\mathrm{i}\,\partial\!\!\!/ - m)^{-1}_y\right](\mathrm{i}\,\partial\!\!\!/ - m)_x\left[\psi + \int \mathrm{d}^4y'\,(\mathrm{i}\,\partial\!\!\!/ - m)^{-1}_{y'}\eta(y')\right]\right. \\
&\quad \left. - \int \mathrm{d}^4y\,\mathrm{d}^4y'\,\bar\eta(y)(\mathrm{i}\,\partial\!\!\!/ - m)^{-1}_y(\mathrm{i}\,\partial\!\!\!/ - m)_x(\mathrm{i}\,\partial\!\!\!/ - m)^{-1}_{y'}\eta(y')\right\} \\
&= \int \mathrm{d}^4x\,\bar\chi(\mathrm{i}\,\partial\!\!\!/ - m)_x\chi - \int \mathrm{d}^4x\,\mathrm{d}^4y'\,\bar\eta(x)(\mathrm{i}\,\partial\!\!\!/ - m)^{-1}_{y'}\eta(y')
\end{aligned}$$

[r] where

$$\chi \equiv \psi + \int \mathrm{d}^4y'\,(\mathrm{i}\,\partial\!\!\!/ - m)^{-1}_{y'}\eta(y')\,, \quad (\mathrm{i}\,\partial\!\!\!/ - m)^{-1}_y(\mathrm{i}\,\partial\!\!\!/ - m)_x = \delta^{(4)}(x-y)$$

The generating functional becomes

$$\begin{aligned}
W_0[\eta,\bar\eta] &= N\int [\mathcal{D}\chi\,\mathcal{D}\bar\chi]\exp\left[\mathrm{i}\int \mathrm{d}^4x\,\bar\chi(\mathrm{i}\,\partial\!\!\!/ - m)_x\chi - \mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y'\,\bar\eta(x)(\mathrm{i}\,\partial\!\!\!/ - m)^{-1}_{y'}\eta(y')\right] \\
&= \exp\left[-\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y'\,\bar\eta(x)(\mathrm{i}\,\partial\!\!\!/ - m)^{-1}_{y'}\eta(y')\right]W_0[0,0] \\
&= \exp\left[-\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y'\,\bar\eta_\alpha(x)S_{\alpha\beta}(x-y')\eta_\beta(y')\right]
\end{aligned}$$

At the first line, the path integral of the first addendum of the exponent is a Gaussian integral and can be absorbed into the normalization constant N to give a normalization of $W_0[\eta=\bar\eta=0]=1$ [r].

Remark. The two-point Green's function for the free theory is

$$\begin{aligned}
G^{(2)}(x_1,x_2) &= \left.\frac{\delta^2 W_0[\eta,\bar\eta]}{\delta\bar\eta_\gamma(x_1)\delta\eta_\delta(x_2)}\right|_{\eta=\bar\eta=0} = \frac{\delta}{\delta\bar\eta_\gamma(x_1)}\left.\left[\mathrm{i}\int \mathrm{d}^4x\,\bar\eta_\alpha(x)S_{\alpha\delta}(x-x_2)W_0\right]\right|_{\eta=\bar\eta=0} \\
&= \mathrm{i}S_{\gamma\delta}(x_1-x_2) \equiv \langle 0|\,\mathcal{T}\{\psi_\gamma(x_1)\bar\psi_\delta(x_2)\}\,|0\rangle
\end{aligned}$$

4.3 Interacting field theory

The Lagrangian is

$$\mathcal{L}(\psi,\bar\psi) = \bar\psi(\mathrm{i}\,\partial\!\!\!/ - m)\psi + \mathcal{L}_{\mathrm{int}}(\psi,\bar\psi)$$

Since the Lagrangian is a scalar, it has to be a function only of bilinears of spinors. The generating functional is

$$W[\eta,\bar\eta] = \int [\mathcal{D}\psi\,\mathcal{D}\bar\psi]\exp\left[\mathrm{i}\int \mathrm{d}^4x\,[\mathcal{L}_0(x) + \mathcal{L}_{\mathrm{int}}(\psi,\bar\psi) + \bar\psi\eta + \bar\eta\psi]\right]$$

Noting that

$$\delta_{\eta(y)} \int \mathrm{d}^4x\, \bar{\psi}(x)\eta(x) = -\bar{\psi}(x)\,, \quad \delta_{\bar{\eta}(x)} \int \mathrm{d}^4x\, \bar{\eta}(x)\psi(x) = \psi(x)$$

one may apply the useful property of functional integrals[9] and rewrite the fields in terms of derivatives

$$\mathcal{L}_{\mathrm{int}}(\psi, \bar{\psi}) \exp\left[\mathrm{i} \int \mathrm{d}^4x\, (\bar{\psi}\eta + \bar{\eta}\psi)\right] = \mathcal{L}_{\mathrm{int}}(-\mathrm{i}\,\delta_{\bar{\eta}(x)}, \mathrm{i}\,\delta_{\eta(x)}) \exp\left[\mathrm{i} \int \mathrm{d}^4x\, (\bar{\psi}\eta + \bar{\eta}\psi)\right]$$

Therefore the generating functional is

$$\begin{aligned} W[\eta, \bar{\eta}] &= \int [\mathcal{D}\psi\, \mathcal{D}\bar{\psi}]\, \exp\left[\mathrm{i} \int \mathrm{d}^4x\, \mathcal{L}_{\mathrm{int}}(-\mathrm{i}\,\delta_{\bar{\eta}}, \mathrm{i}\,\delta_{\eta})\right] \exp\left[\mathrm{i} \int \mathrm{d}^4x\, [\mathcal{L}_0(x) + \bar{\psi}\eta + \bar{\eta}\psi]\right] \\ &= \exp\left[\mathrm{i} \int \mathrm{d}^4x\, \mathcal{L}_{\mathrm{int}}(-\mathrm{i}\,\delta_{\bar{\eta}}, \mathrm{i}\,\delta_{\eta})\right] W_0[\eta, \bar{\eta}] \end{aligned}$$

If the coupling constant is weak, then one may expand the exponential of the interaction in a perturbative series in powers of the coupling constant.

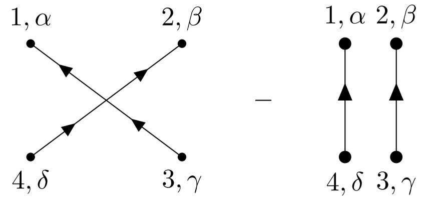
Wick's theorem. To see the effect of Wick's theorem, one may compute the four-point Green's function in the free theory

$$\begin{aligned} G^{(4)}_{\alpha\beta\gamma\delta}(x_1, x_2, x_3, x_4) &= \langle 0|\, \mathcal{T}\{\psi_\alpha(x_1)\psi_\beta(x_2)\bar{\psi}_\gamma(x_3)\bar{\psi}_\delta(x_4)\}\, |0\rangle \\ &= \left.\frac{\delta^4 W_0[\eta, \bar{\eta}]}{\delta\bar{\eta}_\alpha(x_1)\delta\bar{\eta}_\beta(x_2)\delta\eta_\gamma(x_3)\delta\eta_\delta(x_4)}\right|_{\eta=\bar{\eta}=0} \\ &= \frac{\delta^3}{\delta\bar{\eta}_\alpha(x_1)\delta\bar{\eta}_\beta(x_2)\delta\eta_\gamma(x_3)} \left\{ \exp\left[-\mathrm{i} \int \mathrm{d}^4x\, \mathrm{d}^4y\, \bar{\eta}_\varepsilon(x) S_{\varepsilon\zeta}(x-y)\eta_\zeta(y)\right] \right. \\ &\quad \left.\left. \times\, \mathrm{i} \int \mathrm{d}^4x\, \bar{\eta}_\varepsilon S_{\varepsilon\delta}(x - x_4) \right\}\right|_{\eta=\bar{\eta}=0} \\ &= \frac{\delta^2}{\delta\bar{\eta}_\alpha(x_1)\delta\bar{\eta}_\beta(x_2)} \left[W_0[\eta, \bar{\eta}]\mathrm{i} \int \mathrm{d}^4y\, \bar{\eta}_\iota S_{\iota\gamma}(y - x_3) \right. \\ &\quad \left.\left. \times\, \mathrm{i} \int \mathrm{d}^4x\, \bar{\eta}_\varepsilon S_{\varepsilon\delta}(x - x_4) \right]\right|_{\eta=\bar{\eta}=0} \\ &= \frac{\delta}{\delta\bar{\eta}_\alpha(x_1)} \left[\frac{\delta W_0[\eta, \bar{\eta}]}{\delta\bar{\eta}_\beta(x_2)}\, \mathrm{i} \int \mathrm{d}^4y\, \bar{\eta}_\iota S_{\iota\gamma}(y - x_3)\mathrm{i} \int \mathrm{d}^4x\, \bar{\eta}_\varepsilon S_{\varepsilon\delta}(x - x_4) \right. \\ &\quad + W_0[\eta, \bar{\eta}]\mathrm{i}S_{\beta\gamma}(x_2 - x_3)\mathrm{i} \int \mathrm{d}^4x\, \bar{\eta}_\varepsilon S_{\varepsilon\delta}(x - x_4) \\ &\quad \left.\left. - W_0[\eta, \bar{\eta}]\mathrm{i} \int \mathrm{d}^4y\, \bar{\eta}_\iota S_{\iota\gamma}(y - x_3)\mathrm{i}S_{\beta\delta}(x_2 - x_4) \right]\right|_{\eta=\bar{\eta}=0} \\ &= \left.\frac{\delta}{\delta\bar{\eta}_\alpha(x_1)} \left[\frac{\delta W_0[\eta, \bar{\eta}]}{\delta\bar{\eta}_\beta(x_2)}\, \mathrm{i} \int \mathrm{d}^4y\, \bar{\eta}_\iota S_{\iota\gamma}(y - x_3)\mathrm{i} \int \mathrm{d}^4x\, \bar{\eta}_\varepsilon S_{\varepsilon\delta}(x - x_4) \right]\right|_{\eta=\bar{\eta}=0} \\ &\quad + \left[\frac{\delta W_0[\eta, \bar{\eta}]}{\delta\bar{\eta}_\alpha(x_1)}\mathrm{i}S_{\beta\gamma}(x_2 - x_3)\mathrm{i} \int \mathrm{d}^4x\, \bar{\eta}_\varepsilon S_{\varepsilon\delta}(x - x_4) \right. \\ &\quad \left.\left. - \frac{\delta W_0[\eta, \bar{\eta}]}{\delta\bar{\eta}_\alpha(x_1)}\mathrm{i} \int \mathrm{d}^4y\, \bar{\eta}_\iota S_{\iota\gamma}(y - x_3)\mathrm{i}S_{\beta\delta}(x_2 - x_4) \right]\right|_{\eta=\bar{\eta}=0} \\ &\quad + \mathrm{i}^2 \left[W_0[\eta, \bar{\eta}] S_{\beta\gamma}(x_2 - x_3) S_{\alpha\delta}(x_1 - x_4) \right. \\ &\quad \left.\left. - W_0[\eta, \bar{\eta}] S_{\alpha\gamma}(x_1 - x_3) S_{\beta\delta}(x_2 - x_4) \right]\right|_{\eta=\bar{\eta}=0} \\ &= S_{\alpha\gamma}(x_1 - x_3)S_{\beta\delta}(x_2 - x_4) - S_{\beta\gamma}(x_2 - x_3)S_{\alpha\delta}(x_1 - x_4) \end{aligned}$$

[9] See QFT I.

In terms of diagrams, one has

$$G^{(4)}_{\alpha\beta\gamma\delta}(x_1, x_2, x_3, x_4) =$$



where every directed line is a propagator. In the Feynman diagrams, scalars are represented as undirected dashed lines, while fermions are represented as directed solid lines.

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The different sign can be interpreted using Wick's theorem: the only non-vanishing contributions are the ones where the fields are contracted completely. The minus sign appears in the second diagram since one has to contract the fields when they are not next to each other

$$\langle 0|\, \mathcal{T}\{\psi_1\psi_2\bar{\psi}_3\bar{\psi}_4\}\, |0\rangle = :\psi_1\psi_2\bar{\psi}_3\bar{\psi}_4: + :\psi_1\psi_2\bar{\psi}_3\bar{\psi}_4:$$

This gives a general rule: when performing contractions of spinor fields one has to be careful about the signs.

4.3.1 Yukawa theory

See Srednicki, §§45, 51. The Yukawa theory treats the coupling between a massive real scalar field φ and a massive spinor field ψ. This theory is the analogue of the $\lambda\varphi^4$ theory for scalar fields.

Lagrangian. Let M be the mass of the boson field and m the mass of the fermion field. The Yukawa interaction Lagrangian is

$$\mathcal{L}_{\mathrm{Y}} = g\varphi\bar{\psi}\psi$$

[r] One would like to write the simplest Lagrangian. To this end, one may impose a U(1) symmetry on the spinor field along with Lorentz symmetry and the three discrete symmetries of parity, time reversal and charge conjugation. The scalar terms allowed are all the powers of field with non-negative coupling constant: φ, φ^2, φ^3 and φ^4. While no further spinor terms are allowed other than the kinetic term, the three scalar interaction terms above are too many (notice that φ^2 is a mass term). One may modify the Yukawa interaction

$$\mathcal{L}_{\mathrm{Y}} = \mathrm{i}g\varphi\bar{\psi}\gamma_5\psi$$

The imaginary unit makes the interaction Lagrangian real. In fact, consider

$$(\bar{\psi}\gamma_5\psi)^\dagger = (\psi^\dagger\gamma_0\gamma_5\psi)^\dagger = \psi^\dagger\gamma_5^\dagger\gamma_0^\dagger\psi = \psi^\dagger\gamma_5\gamma_0\psi = -\bar{\psi}\gamma_5\psi$$

This term is anti-hermitian and therefore is purely imaginary. Since the scalar field is real, then an imaginary unit is needed to keep the Lagrangian real.

One notices that under a parity transformation, the interaction term changes sign

$$P(\bar{\psi}\gamma_5\psi) = -\bar{\psi}\gamma_5\psi$$

while the kinetic fermionic term is invariant. For the Lagrangian to conserve parity and in particular $\varphi \to -\varphi$, the scalar field φ must be a pseudo-scalar field and the linear and cubic scalar interactions along with their ultraviolet divergences cannot appear:

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}M^2\varphi^2 + \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi - \frac{\lambda}{4!}\varphi^4 + \mathrm{i}g\varphi\bar{\psi}\gamma_5\psi$$

Perturbation theory and Feynman rules. For a weak interacting theory, $\lambda, g \ll 1$, one may apply perturbation theory. Trading

$$\bar{\psi} \to \mathrm{i}\,\delta_\eta\,, \quad \psi \to -\mathrm{i}\,\delta_{\bar{\eta}}\,, \quad \varphi \to -\mathrm{i}\,\delta_J$$

one can write the generating functional

$$W[J,\eta,\bar{\eta}] = \exp\left\{\mathrm{i}\int \mathrm{d}^4x\, \left[-\frac{\lambda}{4!}(-\mathrm{i}\,\delta_{J(x)})^4 + \mathrm{i}g(-\mathrm{i}\,\delta_{J(x)})(\mathrm{i}\,\delta_{\eta(x)})\gamma_5(-\mathrm{i}\,\delta_{\bar{\eta}(x)})\right]\right\} W_0[J,\eta,\bar{\eta}]$$

where the free generating functional is

$$\begin{aligned} W_0[J,\eta,\bar{\eta}] &= \int [\mathcal{D}\varphi]\, \exp\left[\mathrm{i}\int \mathrm{d}^4x\, \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}M^2\varphi^2 + J\varphi\right] \\ &\quad \times \int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\, \exp\left[\mathrm{i}\int \mathrm{d}^4x\, \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi + \bar{\psi}\eta + \bar{\eta}\psi\right] \\ &= \exp\left[\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y\, J(x)\Delta(x-y)J(y)\right] \\ &\quad \times \exp\left[-\,\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y\, \bar{\eta}_\alpha(x) S_{\alpha\beta}(x-y)\eta_\beta(y)\right] \end{aligned}$$

and the propagators are

$$\Delta(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}k(x-y)}}{k^2 - M^2 + \mathrm{i}\varepsilon}\,, \quad S_{\alpha\beta}(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(k\!\!\!/ + m)_{\alpha\beta}}{k^2 - m^2 + \mathrm{i}\varepsilon}\mathrm{e}^{-\mathrm{i}k(x-y)}$$

By computing the generating functional at one-loop, one finds

$$W[J,\eta,\bar{\eta}] = W_0[J,\eta,\bar{\eta}] + \mathrm{i}\int \mathrm{d}^4x\, \left[-\frac{\lambda}{4!}(-\mathrm{i}\,\delta_{J(x)})^4 + \mathrm{i}g(-\mathrm{i}\,\delta_{J(x)})(\mathrm{i}\,\delta_{\eta(x)})\gamma_5(-\mathrm{i}\,\delta_{\bar{\eta}(x)})\right] W_0 + o(\lambda, g)$$

Proceeding as the $\lambda\varphi^4$ theory, the expression above can be expressed in terms of Feynman diagrams according to the following rules:

- Scalar propagator

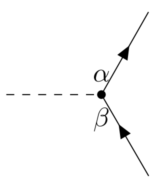
  $\Delta(x-y) \equiv$ x y

- Fermion propagator

  $S_{\alpha\beta}(x-y) \equiv$ $\psi_\alpha(x)$  $\bar{\psi}_\beta(y)$

  Notice the addition of an arrow to denote direction and distinguish between the field ψ and its Dirac adjoint $\bar{\psi}$.

- Internal point

  $\mathrm{i}g\int \mathrm{d}^4x\,(\gamma_5)_{\alpha\beta} \equiv$ 

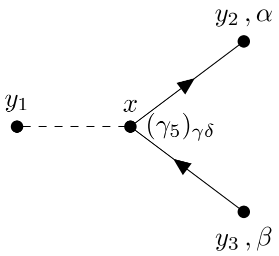
- External scalar point

  $\mathrm{i}\int \mathrm{d}^4y_j\, J(y_j) \to$

- External fermionic point

  $\mathrm{i}\int \mathrm{d}^4y_j\, \eta_\alpha(y_j) \to$, $\qquad$ $\mathrm{i}\int \mathrm{d}^4y_j\, \bar{\eta}_\alpha(y_j) \to$

Example. The tree-level diagram for the Yukawa interaction is

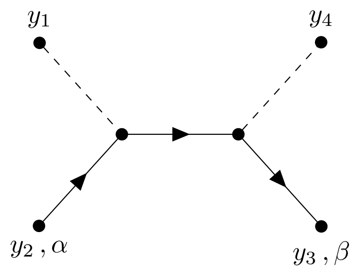


which corresponds to

$$\mathrm{i}^3 \int \mathrm{d}^4y_1\,\mathrm{d}^4y_2\,\mathrm{d}^4y_3\,\mathrm{d}^4x\,\bar{\eta}_\alpha(y_2)S_{\alpha\gamma}(y_2-x)(\mathrm{i}g)(\gamma_5)_{\gamma\delta}S_{\delta\beta}(x-y_3)\eta_\beta(y_3)\Delta(x-y_1)J(y_1)$$

[r] Since order matters for spinors, to write the integral without indices one has to go from right to left and follow against the direction of the arrows.

Example. With two Yukawa vertices, one can draw



[r] When writing amputated Green's function, the external lines are cut and only the internal part of the diagram is computed.

Remark. One neglects vacuum diagrams by choosing a suitable normalization

$$W[J=\eta=\bar{\eta}=0]=1$$

Remark. One is interested in connected Green's functions because the non-connected diagrams can be written as products of connected ones. The generating functional for connected Green's functions is

$$Z[J,\eta,\bar{\eta}]=\ln W[J,\eta,\bar{\eta}]$$

Remark. In the interest of renormalization, one computes amputated Green's functions $\Gamma^{(n)}$ which are the quantum vertices in the effective action and generated by it.

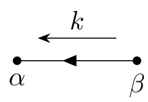
Remark. Computations are done in momentum space.

Feynman rules in momentum space. The rules are

- Scalar propagator

$$\frac{1}{k^2-M^2+\mathrm{i}\varepsilon} \equiv$$

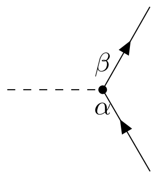
- Fermion propagator

$$\frac{(\not{k}+m)_{\alpha\beta}}{k^2-m^2+\mathrm{i}\varepsilon} \equiv$$ 

Notice that the momentum arrow follows the line's arrow for particles, but it is in the opposite direction for anti-particles.

- Yukawa vertex

$$\mathrm{i}g(\gamma_5)_{\alpha\beta} \equiv$$



[r]

Symmetries. The Lagrangian is invariant under a global U(1) transformation of the fermionic fields

$$\psi' = \mathrm{e}^{\mathrm{i}q}\psi\,, \quad \bar{\psi} = \mathrm{e}^{-\mathrm{i}q}\bar{\psi}\,, \quad \varphi' = \varphi$$

Renormalizability. To study renormalizability one has to look at the mass dimension of the coupling constants. One already knows that λ is dimensionless and $\dim\varphi = 1$. From the fermionic mass term, one obtains

$$\dim(\bar{\psi}m\psi) = 4 \implies \dim\psi = \frac{3}{2}$$

The Yukawa coupling is

$$\dim(g\varphi\bar{\psi}\gamma_5\psi) = 4 \implies \dim g = 0$$

Therefore the theory is renormalizable.

By applying the BPHZ renormalization, one finds that the fields are renormalized as

$$\varphi_0 = Z_\varphi^{\frac{1}{2}}\varphi\,, \quad \psi_0 = Z_\psi^{\frac{1}{2}}\psi\,, \quad \bar{\psi}_0 = Z_\psi^{\frac{1}{2}}\bar{\psi}$$

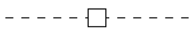
The parameters are renormalized as

$$\lambda_0 = Z_\lambda Z_\varphi^{-2}\lambda\,, \quad g_0 = Z_g Z_\varphi^{-\frac{1}{2}} Z_\psi^{-1} g\,, \quad M_0^2 = Z_M Z_\varphi^{-1} M^2\,, \quad m_0^2 = Z_m Z_\psi^{-1} m^2$$

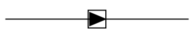
The bare Lagrangian can be split into a renormalized Lagrangian and the counter terms

$$\begin{aligned}\mathcal{L}_0 = {} & \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}M^2\varphi^2 - \frac{\lambda}{4!}\varphi^4 + \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi + \mathrm{i}g\varphi\bar{\psi}\gamma_5\psi \\ & + (Z_\varphi - 1)\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}(Z_M - 1)M^2\varphi^2 + (Z_\psi - 1)\bar{\psi}\mathrm{i}\,\partial\!\!\!/\psi - (Z_m - 1)m\bar{\psi}\psi \\ & - (Z_\lambda - 1)\frac{\lambda}{4!}\varphi^4 - (Z_g - 1)\mathrm{i}g\varphi\bar{\psi}\gamma_5\psi\end{aligned}$$

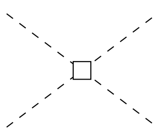
The counter terms (second and third line) produce divergent contributions that cancel the divergent contributions given by the renormalized Lagrangian. Therefore, one has to add the counter term vertices to the one already present. The first two addenda of the second line are a propagator

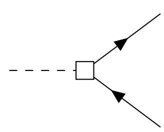


The last two addenda on the second line are another propagator



The third line corresponds to two vertices





The counter terms are fixed in order to cancel the divergences up to a finite part (which is scheme dependent).

5 One-loop contributions

One applies power counting to find the divergent contributions. From the $\lambda\varphi^4$ theory, one already knows that the two-point and four-point scalar Green's function, $\Gamma^{(2)}_\varphi$ and $\Gamma^{(4)}_\varphi$, are divergent. One expects a divergence in the two-point fermionic Green's function $\Gamma^{(2)}_\psi$ and the three-point Yukawa Green's function $\Gamma^{(3)}_{\mathrm{Y}}$.

Exercise. Find the superficial degree of divergence for a generic diagram at L loops with n_φ external scalar lines, $2n_\psi$ external fermion lines, I_φ internal scalar propagators, I_ψ internal fermion propagators, V_{Y} Yukawa vertices, V_4 scalar λ vertices.

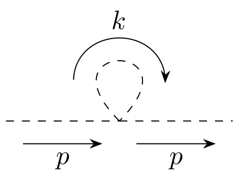
From this one finds that only a particular topology of diagrams is divergent, which are the ones above.

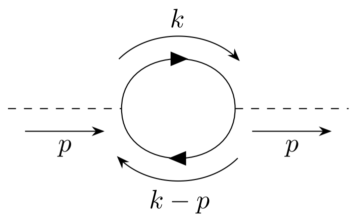
5.1 Two-point scalar Green's function

Recall that the two-point scalar Green's function up to one-loop is

$$\Gamma^{(2)}_\varphi = p^2 + m^2 - \Sigma_\varphi\,, \quad \Gamma^{(2)}_\varphi|_{1\mathrm{L}} = -\Sigma_\varphi$$

The one-loop contributions are given by the diagrams





First diagram. For the first diagram, one has found

$$\Sigma_\varphi(\mathrm{I}) = -\frac{\lambda M^2}{32\pi^2}\left[\frac{1}{\varepsilon} + 1 - \gamma - \ln\frac{M^2}{4\pi k^2} + o(\varepsilon^0)\right]$$

[r] where the overall minus is due to working in Minkowski (see Srednicki, eq. 51.21, but remember that their conventions are a spacelike metric, $D = 4 - \varepsilon$ and $4\pi k^2 = \mu^2$).

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Second diagram. From two Yukawa vertices one has to produce two fermionic propagators through contractions. In one of them, the spinor fields have to be swapped and one gets a negative sign. This is a general rule: alla fermionic loops bring a -1. Therefore, the one-loop contribution to the two-point scalar Green's function is

$$\begin{aligned}
\mathrm{i}\Sigma_\varphi(\mathrm{II}) &= \frac{\mathrm{i}^2}{2!}(\mathrm{i}g)^2(-1)\cdot 2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(\not{k}+m)_{\delta\alpha}}{k^2-m^2+\mathrm{i}\varepsilon}(\gamma_5)_{\alpha\beta}\frac{(\not{k}-\not{p})_{\beta\gamma}+m}{(k-p)^2-m^2+\mathrm{i}\varepsilon}(\gamma_5)_{\gamma\delta} \\
&= -g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}[(\not{k}+m)\gamma_5(\not{k}-\not{p}+m)\gamma_5]}{(k^2-m^2+\mathrm{i}\varepsilon)[(k-p)^2-m^2+\mathrm{i}\varepsilon]} \\
&= -g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}[(\not{k}+m)(\not{p}-\not{k}+m)]}{(k^2-m^2+\mathrm{i}\varepsilon)[(k-p)^2-m^2+\mathrm{i}\varepsilon]} \\
&= -g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[\gamma^\mu\gamma^\nu k_\mu(p-k)_\nu + m^2\right]}{(k^2-m^2+\mathrm{i}\varepsilon)[(k-p)^2-m^2+\mathrm{i}\varepsilon]} \\
&= -4g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k(p-k)+m^2}{(k^2-m^2+\mathrm{i}\varepsilon)[(k-p)^2-m^2+\mathrm{i}\varepsilon]}
\end{aligned}$$

[r] why 2 in 1st line?

At the penultimate line, one applies

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4\eta^{\mu\nu}\,, \quad \mathrm{Tr}\,I = 4$$

Since the Dirac matrices disappeared, one can operate a Wick rotation to have

$$k_0 = \mathrm{i}k_0^{\mathrm{E}}\,, \quad k^2 = -k_{\mathrm{E}}^2\,, \quad \mathrm{d}^4k = \mathrm{i}\,\mathrm{d}^4k_{\mathrm{E}}$$

Therefore

$$k(k-p) = k_0(k-p)^0 - \mathbf{k}\cdot(\mathbf{k}-\mathbf{p}) \to -k(k-p)$$

One could begin from Euclidean space instead and rotate the Dirac matrices and the Clifford algebra.

The one-loop contribution is

$$\begin{aligned}
\Sigma_\varphi(\mathrm{II}) &= -4g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-k(p-k)+m^2}{(k^2+m^2)[(p-k)^2+m^2]} \\
&= 4g^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{(q+xp)(p-q-xp)-m^2}{(q^2+D)^2} \\
&= 4g^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{-q^2+(1-2x)qp+x(1-x)p^2-m^2}{(q^2+D)^2} \\
&= 4g^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{-q^2+x(1-x)p^2-m^2+D-D}{(q^2+D)^2} \\
&= 4g^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{x(1-x)p^2-m^2+D}{(q^2+D)^2}-\frac{1}{q^2+D} \\
&= 4g^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{2x(1-x)p^2}{(q^2+D)^2}-\frac{1}{q^2+D} \\
&\equiv \Sigma_2(\mathrm{II})+\Sigma_1(\mathrm{II})
\end{aligned}$$

At the second line, one has applied Feynman combining

$$x[(p-k)^2+m^2]+(1-x)(k^2+m^2) = k^2+xp^2-2xpk+m^2 = (k-xp)^2+m^2+x(1-x)p^2$$

and one lets

$$q\equiv k-xp\,, \quad D\equiv m^2+x(1-x)p^2$$

At the third line, one notices that linear terms in q vanish due to parity.

The two integral are divergent since their dimensions are 2 and 0. Using dimensional regularization, the first integral is

$$\begin{aligned}
\Sigma_1(\mathrm{II}) &= -4g^2k^{2\varepsilon}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^nq}{(2\pi)^n}\frac{1}{q^2+D} = -4g^2\int_0^1\mathrm{d}x\,\frac{\Gamma(-1+\varepsilon)}{(4\pi)^{2-\varepsilon}\Gamma(1)}\frac{1}{D^{-1+\varepsilon}} \\
&= -\frac{4g^2k^{2\varepsilon}}{(4\pi)^{2-\varepsilon}}\Gamma(-1+\varepsilon)\int_0^1\frac{\mathrm{d}x}{[x(1-x)p^2+m^2]^{-1+\varepsilon}} \\
&= -\frac{4g^2}{(4\pi)^2}(4\pi)^\varepsilon\frac{\Gamma(1+\varepsilon)}{(-1+\varepsilon)}\frac{k^{2\varepsilon}}{\varepsilon}\int_0^1\frac{\mathrm{d}x}{[x(1-x)p^2+m^2]^{-1+\varepsilon}} \\
&= \frac{4g^2}{(4\pi)^2}(4\pi k^2)^\varepsilon\frac{\Gamma(1+\varepsilon)}{\varepsilon(1-\varepsilon)}\int_0^1\mathrm{d}x\,\frac{x(1-x)p^2+m^2}{[x(1-x)p^2+m^2]^\varepsilon} \\
&= \frac{4g^2}{(4\pi)^2}\frac{\Gamma(1+\varepsilon)}{\varepsilon(1-\varepsilon)}\int_0^1\mathrm{d}x\,[x(1-x)p^2+m^2]\exp\left[-\varepsilon\ln\frac{x(1-x)p^2+m^2}{4\pi k^2}\right] \\
&= \frac{g^2}{4\pi^2}\frac{1}{\varepsilon}\int_0^1\mathrm{d}x\,[x(1-x)p^2+m^2]+o(\varepsilon^{-1}) \\
&= \frac{g^2}{4\pi^2}\frac{1}{\varepsilon}\left[\frac{p^2}{6}+m^2\right]+o(\varepsilon^{-1})
\end{aligned}$$

At the first line, one has applied

$$\int\frac{\mathrm{d}^nq}{(2\pi)^n}\frac{(q^2)^a}{(q^2+D)^b} = \frac{1}{D^{b-a-\frac{n}{2}}}\frac{\Gamma(b-a-{}^n\!/_2)\Gamma(a+{}^n\!/_2)}{(4\pi)^{\frac{n}{2}}\Gamma(b)\Gamma({}^n\!/_2)}$$

with $a=0$, $b=1$ and $\frac{n}{2}=2-\varepsilon$. At the third line one has applied $\Gamma(z+1)=z\Gamma(z)$.

The second integral is

$$\begin{aligned}\Sigma_2(\mathrm{II}) &= 8g^2\int_0^1 \mathrm{d}x\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{x(1-x)p^2}{(q^2+D)^2}\\ &= 8g^2\frac{k^{2\varepsilon}}{(4\pi)^{2-\varepsilon}}\Gamma(\varepsilon)\int_0^1\mathrm{d}x\,\frac{x(1-x)p^2}{[x(1-x)p^2+m^2]^\varepsilon}\\ &= \frac{2g^2}{4\pi^2}\frac{1}{\varepsilon}\int_0^1\mathrm{d}x\,x(1-x)p^2+o(\varepsilon^{-1})\\ &= \frac{2g^2}{4\pi^2}\frac{1}{\varepsilon}\frac{p^2}{6}+o(\varepsilon^{-1})=\frac{g^2}{4\pi^2}\frac{1}{\varepsilon}\frac{p^2}{3}+o(\varepsilon^{-1})\end{aligned}$$

At the second line, one has applied the integral formula above with $a=1$, $b=2$ and $\frac{n}{2}=2-\varepsilon$.

The total one-loop contribution to the two-point scalar Green's function coming from the second diagram is

$$\begin{aligned}\Sigma_\varphi(\mathrm{II})=\Sigma_1(\mathrm{II})+\Sigma_2(\mathrm{II}) &= \frac{g^2}{4\pi^2}\frac{1}{\varepsilon}\left[\frac{p^2}{6}+m^2\right]+\frac{g^2}{4\pi^2}\frac{1}{\varepsilon}\frac{p^2}{3}+\text{finite}\\ &= \frac{g^2}{4\pi^2}\frac{1}{\varepsilon}\left[\frac{p^2}{2}+m^2\right]\end{aligned}$$

Notice that, at one-loop, there is a dependence on the momentum as opposed to the $\lambda\varphi^4$ theory. The first addendum of the above contribution is cancelled by a counter term of the kinetic term, while the second addendum is added to the first diagram's contribution and is cancelled by a counter term of the mass term.

Total contribution. The total contribution at one-loop is

$$\Sigma_\varphi|_{1\mathrm{L}}=\Sigma_\varphi(\mathrm{I})+\Sigma_\varphi(\mathrm{II})=\frac{g^2}{8\pi^2}\frac{1}{\varepsilon}p^2+\frac{1}{\varepsilon}\left[\frac{g^2}{4\pi^2}m^2-\frac{\lambda M^2}{32\pi^2}+\text{finite}\right]$$

This total contributing divergence can be cancelled by a counter term of the kinetic term and mass term that give contributions

$$(Z_\varphi-1)\,\partial_\mu\varphi\,\partial^\mu\varphi-(Z_M-1)M^2\varphi^2$$

This counter term is computed in configuration space and not momentum space, like the one-loop contribution above. Integrating by parts, one has

$$(Z_\varphi-1)\,\partial_\mu\varphi\,\partial^\mu\varphi=-(Z_\varphi-1)\varphi\,\Box\,\varphi$$

Therefore, the counter term contribution in momentum space is

$$(Z_\varphi-1)p^2=-(Z_\varphi-1)p_\mathrm{E}^2$$

[r] Wick rotation of the field, do computations? Since one is expanding e^{-S} then the counter term gives

$$(Z_\varphi-1)\varphi p_\mathrm{E}^2\varphi$$

Therefore

$$\frac{g^2}{8\pi^2}\frac{1}{\varepsilon}p_\mathrm{E}^2+(Z_\varphi-1)p_\mathrm{E}^2=\text{finite}\implies\boxed{Z_\varphi|_{1\mathrm{L}}=1-\frac{g^2}{8\pi^2}\frac{1}{\varepsilon}+\text{finite}}$$

The mass term for the scalar field is

$$\left[\frac{g^2}{4\pi^2}m^2+\frac{\lambda M^2}{32\pi^2}+\text{finite}\right]\frac{1}{\varepsilon}-(Z_M-1)M^2\equiv\text{finite}$$

from which one has

$$\boxed{Z_M|_{1\mathrm{L}}=1+\left[\frac{\lambda}{32\pi^2}+\frac{g^2}{4\pi^2}\frac{m^2}{M^2}\right]\frac{1}{\varepsilon}+\text{finite}}$$

Lecture 6

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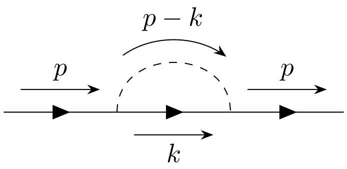
5.2 Two-point spinor Green's function

Consider the diagram



One has to contract to Yukawa vertices

$$\mathrm{i}g\varphi\bar{\psi}_\gamma(\gamma_5)_{\gamma\delta}\psi_\delta\,,\quad \mathrm{i}g\varphi\bar{\psi}_\alpha(\gamma_5)_{\alpha\beta}\psi_\beta$$

to produce a fermion propagator and a scalar propagator (recall that the diagram has to be read in the opposite direction when writing its associated integral). The one-loop contribution to the two-point spinor Green's function is

$$\begin{aligned}[\mathrm{i}\Sigma^{(2)}_\psi]_{\alpha\delta} &= \frac{\mathrm{i}^2}{2!}(\mathrm{i}g)^2 2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(\gamma_5)_{\alpha\beta}(\not{k}+m)_{\beta\gamma}(\gamma_5)_{\gamma\delta}}{[(p-k)^2-M^2+\mathrm{i}\varepsilon](k^2-m^2+\mathrm{i}\varepsilon)}\\ &= g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{[\gamma_5(\not{k}+m)\gamma_5]_{\alpha\delta}}{[(p-k)^2-M^2+\mathrm{i}\varepsilon](k^2-m^2+\mathrm{i}\varepsilon)}\end{aligned}$$

One applies dimensional regularization to have

$$\begin{aligned}[\mathrm{i}\Sigma^{(2)}_\psi]_{\alpha\delta} &= g^2k^{2\varepsilon}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(-\not{k}+m)_{\alpha\delta}}{[(p-k)^2-M^2](k^2-m^2)}\\ &= g^2k^{2\varepsilon}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^nq}{(2\pi)^n}\frac{(-\not{q}-x\not{p}+m)_{\alpha\delta}}{(q^2+D)^2}\\ &= g^2k^{2\varepsilon}\int_0^1\mathrm{d}x\,(m-x\not{p})_{\alpha\delta}\int\frac{\mathrm{d}^nq}{(2\pi)^n}\frac{1}{(q^2+D)^2}\\ &= g^2k^{2\varepsilon}\frac{1}{(4\pi)^{2-\varepsilon}}\Gamma(\varepsilon)\int_0^1\mathrm{d}x\,\frac{(m-x\not{p})_{\alpha\delta}}{[x(1-x)p^2-m^2(1-x)-M^2x]^\varepsilon}\\ &= \frac{g^2}{16\pi^2}\frac{\Gamma(1+\varepsilon)}{\varepsilon}\int_0^1\mathrm{d}x\,\frac{(m-x\not{p})_{\alpha\delta}}{[\frac{x(1-x)p^2-m^2(1-x)-M^2x}{4\pi k^2}]^\varepsilon}\\ &= \frac{g^2}{16\pi^2}\frac{1}{\varepsilon}\int_0^1\mathrm{d}x\,(m-x\not{p})_{\alpha\delta}+o(\varepsilon^{-1})\\ &= \frac{g^2}{16\pi^2}\frac{1}{\varepsilon}\left[m-\frac{1}{2}\not{p}\right]_{\alpha\delta}\end{aligned}$$

[r] check. At the second line, one has applied Feynman combining

$$\mathrm{Den} = k^2-2xkp+xp^2-m^2(1-x)-M^2x = (k-xp)^2+x(1-x)p^2-m^2(1-x)-M^2x$$

and one lets

$$q\equiv k-xp\,,\quad D\equiv x(1-x)p^2-m^2(1-x)-M^2x$$

Renormalization. The mass contribution is [r]

$$[\Sigma^{(2)}_\psi]_m = -\mathrm{i}\frac{g^2}{16\pi^2}\frac{m}{\varepsilon}$$

From the counter term Lagrangian

$$\exp[\mathrm{i}S_{\mathrm{ct}}]\to\exp\left[\mathrm{i}\int\mathrm{d}^4x\,[-(Z_m-1)m\bar{\psi}\psi]\right]$$

one has

$$-\mathrm{i}\frac{g^2}{16\pi^2}\frac{1}{\varepsilon}-\mathrm{i}(Z_m-1)\equiv\text{finite}\implies \boxed{Z_m=1-\frac{g^2}{16\pi^2}\frac{1}{\varepsilon}+\text{finite}}$$

The momentum contribution is

$$[\Sigma^{(2)}_\psi]_p=\mathrm{i}\frac{g^2}{32\pi^2}\frac{1}{\varepsilon}\not{p}$$

Therefore

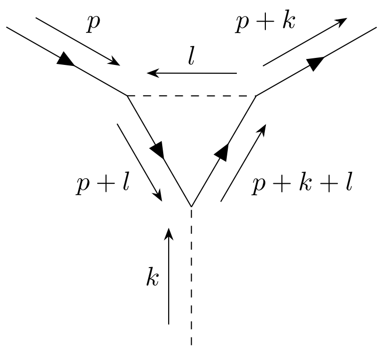
$$e^{\mathrm{i}S}\to\exp\left[\mathrm{i}\int\mathrm{d}^4x\,\bar{\psi}\mathrm{i}\not{\partial}\psi(Z_\psi-1)\right]\to\exp\left[-\mathrm{i}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\bar{\psi}(-p)\not{p}\psi(p)(Z_\psi-1)\right]$$

[r] $\not{p}=-\mathrm{i}\not{\partial}$?? check Fourier convention
from which

$$\mathrm{i}\frac{g^2}{32\pi^2}\frac{1}{\varepsilon}\not{p}-\mathrm{i}(Z_\psi-1)\not{p}=\text{finite}\implies\boxed{Z_\psi=1+\frac{g^2}{32\pi^2}\frac{1}{\varepsilon}+\text{finite}}$$

5.3 Three-point Yukawa Green's function

One would like to compute the one-loop correction to the Yukawa vertex $\Gamma^{(3)}_{\mathrm{Y}}$. Consider the following diagram



Let $p'=p+k$, the one-loop contribution to the three-point Green's function is

$$\begin{aligned}
\Gamma^{(3)}_{\mathrm{Y}}|_{1\mathrm{L}}&=\frac{\mathrm{i}^3}{3!}(\mathrm{i}g)^3(3\cdot 2)\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{(\gamma_5)_{\alpha\beta}(\not{p}'+\not{l}+m)_{\beta\gamma}(\gamma_5)_{\gamma\delta}(\not{p}+\not{l}+m)_{\delta\eta}(\gamma_5)_{\eta\rho}}{(l^2-M^2+\mathrm{i}\varepsilon)[(p+l)^2-m^2+\mathrm{i}\varepsilon][(p'+l)^2-m^2+\mathrm{i}\varepsilon]}\\
&=-g^3\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{[(\not{p}'+\not{l}+m)(-\not{p}-\not{l}+m)\gamma_5]_{\alpha\rho}}{(l^2-M^2+\mathrm{i}\varepsilon)[(p+l)^2-m^2+\mathrm{i}\varepsilon][(p'+l)^2-m^2+\mathrm{i}\varepsilon]}\\
&=-g^3k^{3\varepsilon}\int\frac{\mathrm{d}^nl}{(2\pi)^n}\frac{[(\not{p}'+\not{l}+m)(-\not{p}-\not{l}+m)\gamma_5]_{\alpha\rho}}{(l^2-M^2+\mathrm{i}\varepsilon)[(p+l)^2-m^2+\mathrm{i}\varepsilon][(p'+l)^2-m^2]}\\
&=-g^3k^{3\varepsilon}\int_0^1\mathrm{d}x_1\,\mathrm{d}x_2\int\frac{\mathrm{d}^nq}{(2\pi)^n}\frac{(-q^2\gamma_5+\widetilde{N}+\text{linear in }q)_{\alpha\rho}}{(q^2+D)^3}\\
&=g^3k^{3\varepsilon}\int_0^1\mathrm{d}x_1\,\mathrm{d}x_2\int\frac{\mathrm{d}^nq}{(2\pi)^n}\frac{q^2}{(q^2+D)^3}(\gamma_5)_{\alpha\rho}+\text{finite}\\
&=g^3k^{3\varepsilon}\frac{1}{(4\pi)^{2-\varepsilon}}\frac{\Gamma(\varepsilon)\Gamma(3-\varepsilon)}{\Gamma(3)\Gamma(2-\varepsilon)}\int_0^1\mathrm{d}x_1\,\mathrm{d}x_2\,D^{-\varepsilon}(\gamma_5)_{\alpha\rho}+\text{finite}\\
&=\frac{g^3k^{\varepsilon}}{16\pi^2}(4\pi k^2)^{\varepsilon}\frac{1}{\varepsilon}\frac{\Gamma(1-\varepsilon)\Gamma(3-\varepsilon)}{\Gamma(3)\Gamma(2-\varepsilon)}\int_0^1\mathrm{d}x_1\,\mathrm{d}x_2\,D^{-\varepsilon}(\gamma_5)_{\alpha\rho}+\text{finite}\\
&=\frac{g^3}{16\pi^2}\frac{1}{\varepsilon}(\gamma_5)_{\alpha\rho}+o(\varepsilon^{-1})
\end{aligned}$$

[r] At the fourth line, one has applied Feynman combining. The denominator is

$$\mathrm{Den}=(1-x_1x_2)(l^2-M^2)+x_1[(l+p)^2-m^2)]+x_2[(l+p')^2-m^2]\equiv q^2+D$$

where one has

$$\begin{aligned} q &\equiv l + x_1 p + x_2 p' \\ D &\equiv -(1-x_1-x_2)M^2 - (x_1+x_2)m^2 + x_1(1-x_1)p^2 + x_2(1-x_2)p'^2 - 2x_1x_2pp' \end{aligned}$$

The numerator is

$$\begin{aligned} \text{Num} &= (-\not{q} + x_1\not{p} + x_2\not{p}' - \not{p} + m)(\not{q} - x_1\not{p} - x_2\not{p}' + \not{p}' + m)\gamma_5 \\ &= -\not{q}\not{q}\gamma_5 + [x_2\not{p}' - (1-x_1)\not{p} + m)][(1-x_2)\not{p}' - x_1\not{p} + m]\gamma_5 + (\text{linear in } q) \\ &= -q^2\gamma_5 + \widetilde{N} + (\text{linear in } q) \end{aligned}$$

where one has

$$\not{q}\not{q} = \gamma^\mu q_\mu \gamma^\nu q_\nu = \frac{1}{2} q_\mu q_\nu \{\gamma^\mu, \gamma^\nu\} = \frac{1}{2} q_\mu q_\nu 2\eta^{\mu\nu} = q^2$$

Renormalization. From the exponential, one gets the counter term

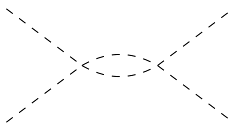
$$e^{iS} \to \exp\left[i\int d^4x\, ig(Z_g - 1)\varphi\bar{\psi}\gamma_5\psi\right] \to -g(Z_g-1)\varphi\bar{\psi}\gamma_5\psi$$

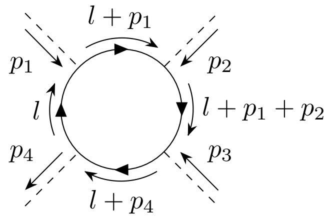
Therefore

$$\frac{g^3}{16\pi^2}\frac{1}{\varepsilon} - g(Z_g - 1) \equiv \text{finite} \implies \boxed{Z_g = 1 + \frac{g^2}{16\pi^2}\frac{1}{\varepsilon} + \text{finite}}$$

5.4 Four-point Green's function

The vertex function for the λ-vertex gets two contributions:





In principle, the Yukawa theory contains the coupling g, but the theory is renormalizable only if one includes the $\lambda\varphi^4$ term. [r] In fact, the second diagram produces a divergent term proportional to λ^4.

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Second diagram. The integral associated with the diagram is

$$\begin{aligned} i\Gamma^{(4)}_\varphi(\text{II}) = &\frac{i^4}{4!}(ig)^4 4!(-1) \\ &\times \int \frac{d^4l}{(2\pi)^4} \frac{\text{Tr}[(\not{l}+m)\gamma_5(\not{l}+\not{p}_4+m)\gamma_5(\not{l}+\not{p}_1+\not{p}_2+m)\gamma_5(\not{l}+\not{p}_1+m)\gamma_5]}{(l^2-m^2)[(l+p_4)^2-m^2][(l+p_1+p_2)^2-m^2][(l+p_1)^2-m^2]} \\ &+ 5 \text{ permutations of } (k_2, k_3, k_4) \end{aligned}$$

where $i\varepsilon$ is understood in all propagators. Since one is interested only in the divergent part, one selects only the l^4 term in the numerator. One may set all external momenta to zero $p_1 = p_2 = p_3 = p_4 = 0$ since the integral has superficial degree of divergence equal to zero and so it does not depend on momentum. The trace is

$$\text{Tr}[\not{l}\not{l}\not{l}\not{l}] = l_\mu l_\nu l_\rho l_\sigma \,\text{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4[l^4 - l^4 + l^4] = 4l^4$$

Therefore

$$i\Gamma^{(4)}_\varphi(\text{II}) = -4g^4 \cdot 6 \int \frac{d^4l}{(2\pi)^4}\frac{l^4}{(l^2-m^2)^4} + \text{finite} = -\frac{3g^4}{2\pi^2}\frac{1}{\varepsilon} + \text{finite}$$

The coefficient 6 comes from the inequivalent permutations of the momenta of the diagram.

Total contribution. Recalling that the first diagram gives

$$i\Gamma_\varphi^{(4)}(\mathrm{I}) = \frac{3\lambda^2}{32\pi^2}\frac{1}{\varepsilon} + \text{finite}$$

[r] then one finds

$$\boxed{Z_\lambda = 1 + \left[\frac{3\lambda}{32\pi^2} - \frac{3}{2\pi^2}\frac{g^4}{\lambda}\right]\frac{1}{\varepsilon} + \text{finite}}$$

6 Renormalization group

See Srednicki, §52.

Renormalization functions. The renormalization functions at one-loop are

$$\begin{aligned}
Z_\varphi &= 1 - \frac{g^2}{8\pi^2}\frac{1}{\varepsilon} + \text{finite} \\
Z_\psi &= 1 - \frac{g^2}{32\pi^2}\frac{1}{\varepsilon} + \text{finite} \\
Z_g &= 1 + \frac{g^2}{16\pi^2}\frac{1}{\varepsilon} + \text{finite} \\
Z_\lambda &= 1 + \left[\frac{3\lambda}{32\pi^2} - \frac{3}{2\pi^2}\frac{g^4}{\lambda}\right]\frac{1}{\varepsilon} + \text{finite}
\end{aligned}$$

[r] wrong sign Z_ψ, affects following argument
The bare coupling constant are

$$g_0 = Z_g Z_\varphi^{-\frac{1}{2}} Z_\psi^{-1} g k^\varepsilon\,, \quad \lambda_0 = Z_\lambda Z_\varphi^{-2}\lambda k^{2\varepsilon}$$

Using the minimal subtraction scheme, one writes the renormalization functions as

$$Z = 1 + \sum_n \frac{a_n(g,\lambda)}{\varepsilon^n}$$

Therefore, the coupling constants are

$$g_0 = g k^\varepsilon\left[1 + \sum_{n=1}^\infty \frac{\widetilde{G}_n(g,\lambda)}{\varepsilon^n}\right], \quad \lambda_0 = \lambda k^{2\varepsilon}\left[1 + \sum_{n=1}^\infty \frac{\widetilde{L}_n(g,\lambda)}{\varepsilon^n}\right]$$

Comparing these series with the above expressions for the coupling constants, one sees that the brackets are the products of the renormalization functions

$$\begin{aligned}
1 + \sum_{n=1}^\infty \frac{\widetilde{G}_n(g,\lambda)}{\varepsilon^n} &= Z_g Z_\varphi^{-\frac{1}{2}} Z_\psi^{-1} \\
&= \left[1 + \frac{g^2}{16\pi^2}\frac{1}{\varepsilon} + \cdots\right]\left[1 + \frac{g^2}{16\pi^2}\frac{1}{\varepsilon} + \cdots\right]\left[1 + \frac{g^2}{32\pi^2}\frac{1}{\varepsilon} + \cdots\right] \\
&= 1 + \frac{5g^2}{32\pi^2}\frac{1}{\varepsilon} + \cdots
\end{aligned}$$

where the ellipses contain higher-order corrections in g and λ (and thus higher powers of ε^{-1}). Therefore

$$\widetilde{G}_1(g,\lambda)|_{1\mathrm{L}} = \frac{5g^2}{32\pi^2}$$

Similarly

$$\widetilde{L}_1(g,\lambda)|_{1\mathrm{L}} = \frac{3\lambda}{32\pi^2} + \frac{g^2}{4\pi^2} - \frac{3}{2\pi^2}\frac{g^4}{\lambda}$$

One notices that $\widetilde{G}_1$ depends only on one coupling constant.

Beta functions. Recall that the beta functions are

$$\beta_g = k\,\mathrm{d}_k g\,,\quad \beta_\lambda = k\,\mathrm{d}_k\lambda$$

The bare Yukawa coupling constant can be rewritten as

$$\begin{aligned}
g_0 &= gk^\varepsilon\left[1+\sum_{n=1}^\infty \frac{\widetilde{G}_n(g,\lambda)}{\varepsilon^n}\right] = g\mathrm{e}^{\varepsilon\ln k}\exp\left[\sum_{n=1}^\infty \frac{G_n(g,\lambda)}{\varepsilon^n}\right] \\
\lambda_0 &= \lambda k^{2\varepsilon}\left[1+\sum_{n=1}^\infty \frac{\widetilde{L}_n(g,\lambda)}{\varepsilon^n}\right] = \lambda\mathrm{e}^{2\varepsilon\ln k}\exp\left[\sum_{n=1}^\infty \frac{L_n(g,\lambda)}{\varepsilon^n}\right]
\end{aligned}$$

Taking the logarithm, one has

$$\begin{aligned}
\ln g_0 &= \ln g + \varepsilon\ln k + \sum_{n=1}^\infty \frac{G_n(g,\lambda)}{\varepsilon^n} \\
\ln\lambda_0 &= \ln\lambda + 2\varepsilon\ln k + \sum_{n=1}^\infty \frac{L_n(g,\lambda)}{\varepsilon^n}
\end{aligned}$$

The right-hand side depends on the mass scale k, but not the left-hand side. Applying $k\,\mathrm{d}_k$, one has

$$0 = k\,\mathrm{d}_k\ln g + \varepsilon + \sum_{n=1}^\infty \frac{1}{\varepsilon^n}\left[\partial_g G_n\,k\,\partial_k g + \partial_\lambda G_n\,k\,\partial_k\lambda\right]$$

Multiplying by the coupling constant g, one has

$$0 = g\varepsilon + k\,\mathrm{d}_k g + \sum_{n=1}^\infty \frac{1}{\varepsilon^n}[g\,\partial_g G_n\,k\,\partial_k g + g\,\partial_\lambda G_n\,k\,\partial_k\lambda]$$

This expression contains the beta function. Since the beta functions are finite in the limit $\varepsilon\to 0$, one may write

$$k\,\mathrm{d}_k g = \beta_g(g,\lambda) - \varepsilon g\,,\quad k\,\mathrm{d}_k\lambda = \beta_\lambda(g,\lambda) - 2\varepsilon\lambda$$

[r] Substituting this inside the expression above, one obtains

$$\begin{aligned}
0 &= \beta_g + \sum_{n=1}^\infty \frac{1}{\varepsilon^n}[g\,\partial_g G_n\,(-g\varepsilon+\beta_g) + g\,\partial_\lambda G_n\,(-2\lambda\varepsilon+\beta_\lambda)] \\
&= \beta_g - g^2\,\partial_g G_1 - 2g\lambda\,\partial_\lambda G_1 + 0\cdot\text{poles in }\frac{1}{\varepsilon}
\end{aligned}$$

From this one has

$$\boxed{\beta_g(g,\lambda) = g[g\,\partial_g G_1 + 2\lambda\,\partial_\lambda G_1]}$$

Similarly, one has

$$\boxed{\beta_\lambda = \lambda[g\,\partial_g L_1 + 2\lambda\,\partial_\lambda L_1]}$$

Since $\widetilde{G}_1 = G_1$ and $\widetilde{L}_1 = L_1$, then the one-loop beta functions are

$$\beta_g|_{1\mathrm{L}} = \frac{5}{16\pi^2}g^3\,,\quad \beta_\lambda|_{1\mathrm{L}} = \frac{1}{16\pi^2}(3\lambda^2 + 8g^2\lambda - 48g^4)$$

6.1 Fixed points and RG flow

One introduces a new coupling ρ such that $\lambda = \rho g^2$ and studies the flow of g and ρ. The beta function is

$$\beta_\rho = k\,\mathrm{d}_k\rho = k\,\mathrm{d}_k\frac{\lambda}{g^2} = \frac{1}{g^2}\beta_\lambda - \frac{2\lambda}{g^3}\beta_g$$

At one-loop, one finds

$$\beta_\rho|_{1\mathrm{L}} = \frac{g^2}{16\pi^2}(3\rho^2 - 2\rho - 48)$$

This expression is a product of polynomials of the coupling constants as opposed to the beta function of λ that is a sum of polynomials.

The only fixed point of the flow of g is

$$\beta_g = 0 \implies g = 0$$

In order to study the flow, one considers small perturbations away from the fixed point, $|g| \ll 1$. The fixed points for ρ are

$$\beta_\rho = 0 \implies \rho_{1,2} = \frac{1 \pm \sqrt{145}}{3}$$

The first is positive and the second is negative. The small perturbations around the solutions ρ_j can be studied by Taylor expanding

$$k\, \mathrm{d}_k \rho = \beta_\rho = \beta_\rho(\rho_1) + \beta'_\rho(\rho_1)(\rho - \rho_1) + o(\Delta\rho) = \beta'_\rho(\rho_1)(\rho - \rho_1) + o(\Delta\rho)$$

From the sign of the derivative, one obtains the flow. If the derivative is positive, then for $\rho > \rho_j$ the derivative of the coupling with respect to the energy is positive, so the energy is increasing, while for $\rho < \rho_j$, the derivative of the coupling is negative so the energy decreases; vice versa, if the derivative is negative, then for $\rho > \rho_j$, the energy decreases, while for $\rho < \rho_j$ it increases. The derivative is

$$\beta'_\rho = \frac{g^2}{16\pi^2}(6\rho - 2)$$

from which

$$\beta'(\rho_1) = \frac{g^2}{8\pi^2}\sqrt{145} > 0\,, \quad \beta'(\rho_2) = -\frac{g^2}{8\pi^2}\sqrt{145} < 0$$

[r] The flow is away from ρ_1 and towards ρ_2. The first is an infrared-stable fixed point while the second is an ultraviolet-stable fixed point.

Trajectories. One may study the trajectory of the theory in the (ρ, g) plane. To this end, one needs to find $g(\rho)$. One has to solve

$$\mathrm{d}_\rho g = \mathrm{d}_{\ln k} g \, \mathrm{d}_\rho \ln k = \frac{\beta_g}{\beta_\rho} = \frac{5}{16\pi^2} g^3 \left[\frac{g^2}{16\pi^2} 3(\rho - \rho_1)(\rho - \rho_2)\right]^{-1} = \frac{5g}{3(\rho - \rho_1)(\rho - \rho_2)}$$

Therefore

$$\frac{1}{g}\, \mathrm{d}_\rho g = \mathrm{d}_\rho \ln g = \frac{5}{3(\rho - \rho_1)(\rho - \rho_2)}$$

after integrating one gets

$$\ln \frac{g(\rho)}{g(0)} = \frac{5}{3}\int_0^\rho \frac{\mathrm{d}x}{(x - \rho_1)(x - \rho_2)} = \frac{5}{3(\rho_1 - \rho_2)} \ln\left|\frac{\rho - \rho_1}{\rho - \rho_2}\frac{\rho_2}{\rho_1}\right|$$

from which

$$g(\rho) = g(0)\left|\frac{\rho - \rho_1}{\rho - \rho_2}\right|^{\frac{5}{3(\rho_1 - \rho_2)}} = g_0\left|\frac{\rho - \rho_1}{\rho - \rho_2}\right|^{\frac{5}{3(\rho_1 - \rho_2)}}$$

Due to the absolute value, the constant-sign domains are

$$\rho < \rho_2\,, \quad \rho_2 < \rho < \rho_1\,, \quad \rho > \rho_1$$

[r] diagr. Let

$$\nu = \frac{5}{3(\rho_1 - \rho_2)} > 0$$

then the derivatives are

$$g'(\rho) = g_0 \nu \left|\frac{\rho - \rho_1}{\rho - \rho_2}\right|^{\nu - 1} \frac{1}{(\rho - \rho_2)^2} \begin{cases} \rho_1 - \rho_2 > 0\,, & \rho < \rho_2 \lor \rho > \rho_1 \\ \rho_2 - \rho_1 < 0\,, & \rho_2 < \rho < \rho_1 \end{cases}$$

[r] With two coupling constants, it is interesting to study how the coupling constants affect each other.

Exercise. Compute the anomalous dimensions. Recall that

$$\gamma_\varphi = \frac{1}{2} k \, \mathrm{d}_k \ln Z_\varphi \,, \quad \gamma_\psi = \frac{1}{2} k \, \mathrm{d}_k \ln Z_\psi$$

Solution. Recalling that up to one-loop one has

$$Z_\varphi = 1 - \frac{g^2}{8\pi^2} \frac{1}{\varepsilon} \implies \ln Z_\varphi = \sum_{n=1}^{\infty} \frac{F_n(g,\lambda)}{\varepsilon^n} \implies F_1(g,\lambda) = -\frac{g^2}{8\pi^2}$$

for the scalar field follows

$$\begin{aligned}
\gamma_\varphi &= \frac{1}{2} k \, \mathrm{d}_k \ln Z_\varphi = \frac{1}{2}[(\mathrm{d}_g \ln Z_\varphi)(\mathrm{d}_{\ln k} g) + (\mathrm{d}_\lambda \ln Z_\varphi)(\mathrm{d}_{\ln k} \lambda)] \\
&= \frac{1}{2}[(\mathrm{d}_g \ln Z_\varphi)(\beta_g - \varepsilon g) + (\mathrm{d}_\lambda \ln Z_\varphi)(\beta_\lambda - 2\varepsilon\lambda)] \\
&= \frac{1}{2}(\mathrm{d}_g \ln Z_\varphi)(\beta_g - \varepsilon g) + 0 = \frac{1}{2} \mathrm{d}_g \left[-\frac{g^2}{8\pi^2} \frac{1}{\varepsilon} + \cdots \right] (\beta_g - \varepsilon g) \\
&= \frac{g^2}{8\pi^2} + \cdots
\end{aligned}$$

where the ellipses are higher powers of ε^{-1}.

For the spinor field, one has

$$Z_\psi = 1 - \frac{g^2}{32\pi^2} \frac{1}{\varepsilon} \implies P_1(g,\lambda) = -\frac{g^2}{32\pi^2}$$

The procedure is exactly the same

$$\begin{aligned}
\gamma_\psi &= \frac{1}{2} k \, \mathrm{d}_k \ln Z_\psi = \frac{1}{2}[(\mathrm{d}_g \ln Z_\psi)(\mathrm{d}_{\ln k} g) + (\mathrm{d}_\lambda \ln Z_\psi)(\mathrm{d}_{\ln k} \lambda)] \\
&= \frac{1}{2}[(\mathrm{d}_g \ln Z_\psi)(\beta_g - \varepsilon g) + (\mathrm{d}_\lambda \ln Z_\varphi)(\beta_\lambda - 2\varepsilon\lambda)] \\
&= \frac{1}{2}(\mathrm{d}_g \ln Z_\psi)(\beta_g - \varepsilon g) + 0 = \frac{1}{2} \mathrm{d}_g \left[-\frac{g^2}{32\pi^2} \frac{1}{\varepsilon} + \cdots \right] (\beta_g - \varepsilon g) \\
&= \frac{g^2}{32\pi^2} + \cdots
\end{aligned}$$

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7 Functional determinants

See Srednicki, §53. Functional determinants are useful to motivate the use of an infinite sum of Feynman diagrams.

Example. Consider a complex scalar field χ coupled to a background real scalar field φ

$$\mathcal{L} = \partial_\mu \bar{\chi} \, \partial^\mu \chi - m^2 \bar{\chi}\chi + g\varphi\bar{\chi}\chi$$

The generating functional with no source is

$$e^{i\Gamma[\varphi]} = \int [\mathcal{D}\chi \, \mathcal{D}\bar{\chi}] \, e^{i \int \mathrm{d}^4 x \, \mathcal{L}}$$

where $\Gamma[\varphi]$ is the effective action for the field φ. The action is quadratic in the complex field and one may rewrite

$$\begin{aligned}
\int \mathrm{d}^4 x \, \mathcal{L} &= \int \mathrm{d}^4 x \, [-\bar{\chi}(\Box + m^2)\chi + g\varphi\bar{\chi}\chi] \\
&= -\int \mathrm{d}^4 x \, \mathrm{d}^4 y \, \bar{\chi}(x)[(\Box + m^2) - g\varphi(x)]\delta^{(4)}(x-y)\chi(y) \\
&= -\int \mathrm{d}^4 x \, \mathrm{d}^4 y \, \bar{\chi}(x) M(x,y) \chi(y)
\end{aligned}$$

where the differential operator is

$$M(x,y) \equiv [(\Box + m^2) - g\varphi(x)]\delta^{(4)}(x-y)$$

The generating functional is

$$\mathrm{e}^{\mathrm{i}\Gamma[\varphi]} = \int [\mathcal{D}\chi\,\mathcal{D}\bar{\chi}]\,\exp\left[-\mathrm{i}\int \mathrm{d}^4x \int \mathrm{d}^4y\,\bar{\chi}(x)M(x,y)\chi(y)\right]$$

One recognizes the functional generalization of the Gaussian integral

$$\int \mathrm{d}^n z\,\mathrm{d}^n\bar{z}\,\mathrm{e}^{-\mathrm{i}\bar{z}_i M_{ij} z_j} \propto \frac{1}{\det M}$$

One may rewrite the differential operator as

$$M(x,y) = \int \mathrm{d}^4z\,[(\Box + m^2)\delta^{(4)}(x-z)][\delta^{(4)}(z-y) - g\Delta(z-y)\varphi(y)] = \int \mathrm{d}^4z\,M_0(x,z)\tilde{M}(z,y)$$

where

$$M_0(x,z) \equiv (\Box + m^2)\delta^{(4)}(x-z)\,, \quad \tilde{M}(z,y) \equiv \delta^{(4)}(z-y) - g\Delta(z-y)\varphi(y)$$

and

$$\Delta(x) = \frac{1}{\Box + m^2}$$

One may notice that the operator M_0 is the operator M with no background, $\varphi = 0$, and the structure of the operator $\tilde{M}$ is $\tilde{M} = I - G$.

In general, the determinant of a linear operator is the product of its eigenvalues. The determinant also satisfies the typical matrix relations [r]

$$\det M = \det(M_0\tilde{M}) = (\det M_0)(\det \tilde{M})$$

Therefore, the generating functional is

$$\mathrm{e}^{\mathrm{i}\Gamma[\varphi]} \propto \frac{1}{(\det M_0)(\det \tilde{M})} \propto \frac{1}{\det \tilde{M}}$$

Since the operator M_0 does not depend on the background field, then its determinant $\det M_0$ is a constant that can be absorbed into the normalization constant. Knowing that

$$\det \tilde{M} = \mathrm{e}^{\mathrm{Tr}\ln \tilde{M}}$$

the generating functional is

$$\mathrm{e}^{\mathrm{i}\Gamma[\varphi]} \propto \mathrm{e}^{-\mathrm{Tr}\ln \tilde{M}} = \mathrm{e}^{-\mathrm{Tr}\ln(I-G)} = \exp\left[\mathrm{Tr}\sum_{n=1}^{\infty}\frac{G^n}{n}\right]$$

where one Taylor-expands the logarithm assuming that the theory is weakly interacting, $g \ll 1$. Recalling that $\mathrm{Tr}(A+B) = \mathrm{Tr}\,A + \mathrm{Tr}\,B$, then one may write

$$\mathrm{Tr}\,G^n = g^n \int \left[\prod_{j=1}^{n} \mathrm{d}^4x_j\right] \Delta(x_1 - x_2)\varphi(x_2)\Delta(x_2 - x_3)\varphi(x_3)\cdots\Delta(x_n - x_1)\varphi(x_1)$$

This trace is represented as a loop with n external φ lines and n internal χ propagators. Therefore the effective action is

$$\mathrm{i}\Gamma[\varphi] \propto \sum_{n=1}^{\infty}\frac{\mathrm{Tr}\,G^n}{n}$$

This is the typical expansion in powers of the coupling constant g of Feynman diagrams.

Example. Consider a Dirac spinor field ψ with a Yukawa interaction and a background real scalar field φ

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi + g\varphi\bar{\psi}\psi$$

The generating functional is

$$\mathrm{e}^{\mathrm{i}\Gamma[\varphi]} = \int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\,\mathrm{e}^{\mathrm{i}\int\,\mathrm{d}^4x\,\mathcal{L}}$$

The action can rewritten as

$$\begin{aligned}\int \mathrm{d}^4x\,\mathcal{L} &= \int \mathrm{d}^4x\,[\bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi + g\varphi\bar{\psi}\psi] \\ &= -\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{\psi}_\alpha(x)[(-\mathrm{i}\,\partial\!\!\!/ + m)_{\alpha\beta} - g\varphi(x)\delta_{\alpha\beta}]\delta^{(4)}(x-y)\psi_\beta(y) \\ &= -\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{\psi}_\alpha(x)M_{\alpha\beta}(x,y)\psi_\beta(y)\end{aligned}$$

where one has

$$M_{\alpha\beta}(x,y) \equiv (-\mathrm{i}\,\partial\!\!\!/ + m)_{\alpha\beta} - g\varphi(x)\delta_{\alpha\beta}$$

The generating functional becomes

$$\mathrm{e}^{\mathrm{i}\Gamma[\varphi]} = \int [\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\,\exp\left[-\mathrm{i}\int \mathrm{d}^4x\;\mathrm{d}^4y\,\bar{\psi}M\psi\right] \propto \det M$$

recalling that a Grassmann-odd Gaussian integral is proportional to the determinant and not its inverse.

The differential operator may be rewritten as

$$\begin{aligned}M_{\alpha\beta}(x,y) &= \int \mathrm{d}^4z\,[(-\mathrm{i}\,\partial\!\!\!/ + m)_{\alpha\gamma}\delta^{(4)}(x-y)][\delta_{\gamma\beta}\delta^{(4)}(y-z) - g\varphi(z)S_{\gamma\beta}(y-z)] \\ &= \int \mathrm{d}^4z\,M_0(x,z)_{\alpha\gamma}\tilde{M}(z,y)_{\gamma\beta}\end{aligned}$$

where one defines

$$\begin{aligned}M_0(x,z)_{\alpha\gamma} &\equiv (-\mathrm{i}\,\partial\!\!\!/ + m)_{\alpha\gamma}\delta^{(4)}(x-y) \\ \tilde{M}(z,y)_{\gamma\beta} &\equiv \delta_{\gamma\beta}\delta^{(4)}(y-z) - g\varphi(z)S_{\gamma\beta}(y-z) = I - \tilde{G}\end{aligned}$$

Like the previous example, the generating functional is

$$\mathrm{e}^{\mathrm{i}\Gamma[\varphi]} \propto \det M_0\,\det\tilde{M} \propto \det\tilde{M} = \mathrm{e}^{\mathrm{Tr}\ln\tilde{M}} = \mathrm{e}^{\mathrm{Tr}\ln(1-\tilde{G})} = \exp\left[-\sum_{n=1}^{\infty}\frac{\mathrm{Tr}\,\tilde{G}^n}{n}\right]$$

where one has

$$\mathrm{Tr}\,\tilde{G}^n = g^n\int\left[\prod_{j=1}^{n}\mathrm{d}^nx_j\right]S_{\alpha_1\alpha_2}(x_1-x_2)\varphi(x_2)S_{\alpha_2\alpha_3}(x_2-x_3)\varphi(x_3)\cdots S_{\alpha_n\alpha_1}(x_n-x_1)\varphi(x_1)$$

The corresponding diagram is again a loop with n external scalar lines and n internal spinor propagators. Notice that the negative sign associated with fermion loops is the one present in the exponential.

The effective action is

$$\mathrm{i}\Gamma[\varphi] \propto -\sum_{n=1}^{\infty}\frac{\mathrm{Tr}\,\tilde{G}^n}{n}$$

Conclusion. The scalar effective action $\Gamma[\varphi]$ is given by an infinite sum of one-loop diagrams with external scalar lines.

8 Discrete symmetries

See Srednicki, §§2, 23, 40. There are many theories forbidden by parity, time reversal and charge conjugation.

Lorentz group. The Lorentz group is

$$\mathrm{O}(1,3) = \{\Lambda \in \mathrm{GL}(4,\mathbb{R}) \mid \Lambda^\top \eta \Lambda = \eta\} = \mathcal{L}$$

where the Minkowski metric η is timelike. The above condition is

$$\Lambda^\mu{}_\nu \eta_{\mu\rho} \Lambda^\rho{}_\sigma = \eta_{\nu\sigma}$$

Relativistic field theories have to be invariant under Lorentz changes of coordinates

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

From the definition of the group, it follows

$$\det \Lambda = \pm 1$$

and

$$\Lambda^\mu{}_0 \eta_{\mu\nu} \Lambda^\nu{}_0 = 1 \implies (\Lambda^0{}_0)^2 \geq 1$$

In this way, the Lorentz group can be split into four disjoint subsets, but only one is a subgroup[10]

$$\begin{aligned}
\mathcal{L}_+^\uparrow &= \{\Lambda \in \mathrm{O}(1,3) \mid \det\Lambda = 1\,, \Lambda^0{}_0 \geq 1\} \\
\mathcal{L}_+^\downarrow &= \{\Lambda \in \mathrm{O}(1,3) \mid \det\Lambda = 1\,, \Lambda^0{}_0 \leq -1\} \\
\mathcal{L}_-^\uparrow &= \{\Lambda \in \mathrm{O}(1,3) \mid \det\Lambda = -1\,, \Lambda^0{}_0 \geq 1\} \\
\mathcal{L}_-^\downarrow &= \{\Lambda \in \mathrm{O}(1,3) \mid \det\Lambda = -1\,, \Lambda^0{}_0 \leq -1\}
\end{aligned}$$

The first one is a group and it is generated by proper rotations and Lorentz boosts. Since it is a subgroup it contains the identity. Furthermore, it is continuously connected to it: transformations can be written infinitesimally close to the identity.

The other subsets contain discrete transformations. One such transformation is parity

$$\Lambda_P = \mathrm{diag}(1,-1,-1,-1) \in \mathcal{L}_-^\uparrow$$

It mirrors all spatial coordinates. Another is time reversal

$$\Lambda_T = \mathrm{diag}(-1,1,1,1) \in \mathcal{L}_-^\downarrow$$

It mirrors time. The combination of these two gives

$$\Lambda_{PT} = \mathrm{diag}(-1,-1,-1,-1) \in \mathcal{L}_+^\downarrow$$

The set of the transformations

$$I = \{1, \Lambda_P, \Lambda_T, \Lambda_{PT}\}$$

is a subgroup of the Lorentz group.

Any transformation in the three non-group subsets of the Lorentz group can be obtained by combining a proper Lorentz transformation with a transformation in group I.

One would like to study how the fields (which are finite-dimensional, non-unitary irreducible representations of the proper Lorentz group) transform under discrete symmetries.

8.1 Representation on scalar fields

Scalar fields are the representation with spin $s = 0$. The transformation of a scalar field under a proper[11] Lorentz transformation is

$$U^{-1}(\Lambda)\varphi(x)U(\Lambda) = \varphi(\Lambda^{-1}x)\,, \quad U(\Lambda) = \mathrm{e}^{\frac{\mathrm{i}}{2}\varepsilon^{\mu\nu}L_{\mu\nu}}$$

where $L_{\mu\nu}$ are the generators Lorentz group.

Similarly, a discrete transformation has a representation in terms of the field.

[10]Called restricted proper Lorentz group, cf. Mathematical Methods for Physics.

[11]For this section, the word "proper" includes the adjective restricted when referring to Lorentz transformations.

Parity. For parity, one has

$$U^{-1}(\Lambda_P)\varphi(x)U(\Lambda_P) = \eta_P\varphi(\Lambda_P x)$$

where η_P is a constant. Applying twice a parity transformation, one gets unity

$$U^{-2}(\Lambda_P)\varphi(x)U^2(\Lambda_P) = \eta_P U^{-1}(\Lambda_P)\varphi(\Lambda_P x)U(\Lambda_P x) = \eta_P^2\varphi(\Lambda_P^2 x) = \eta_P^2\varphi(x) \equiv \varphi(x)$$

Therefore, one obtains

$$\eta_P^2 = 1 \implies \eta_P = \pm 1$$

For $\eta_P = 1$, the field is parity-even and is a scalar field[12], while $\eta_P = -1$ the field is parity-odd and is a pseudo-scalar field.

Time reversal. For time reversal, one has

$$U^{-1}(\Lambda_T)\varphi(x)U(\Lambda_T) = \eta_T\varphi(\Lambda_T x)$$

where η_T is a constant. Like before, applying twice the transformation, one gets unity. Thus

$$\eta_T = \pm 1$$

The field can be time reversal-even or time reversal-odd.

Physical constraints. To decide whether the scalar fields are even or odd under a discrete symmetry, one utilizes the physical input: the Lagrangian of a scalar field is parity- and time reversal-even.

8.1.1 Unitarity

The operator associated to parity is unitary

$$U^\dagger(\Lambda_P)U(\Lambda_P) = U(\Lambda_P)U^\dagger(\Lambda_P) = 1$$

while the operator associated to time reversal is anti-unitary

$$\langle Ux|Uy\rangle = \langle x|\, U^\dagger U\, |y\rangle \equiv \langle x|y\rangle^*$$

or equivalently

$$U^{-1}(\Lambda_T)\mathrm{i}U(\Lambda_T) = -\mathrm{i}$$

Motivation. Consider a proper Lorentz transformation, $\Lambda \in \mathcal{L}_+^\uparrow$, and a four-momentum p^μ. Under such transformation, one has

$$U^{-1}(\Lambda)p^\mu U(\Lambda) = \Lambda^\mu{}_\nu p^\nu$$

Under a parity transformation and a time reversal transformation, one obtains

$$\begin{aligned} U^{-1}(\Lambda_P)p^\mu U(\Lambda_P) &= (\Lambda_P)^\mu{}_\nu p^\nu \implies p^\mu = (p^0, p^i) \to (p^0, -p^i) \\ U^{-1}(\Lambda_T)p^\mu U(\Lambda_T) &= (\Lambda_T)^\mu{}_\nu p^\nu \implies p^\mu = (p^0, p^i) \to (-p^0, p^i) \end{aligned}$$

Since p^0 is the Hamiltonian density, then time reversal can be a symmetry if and only if $H = 0$. One then must have

$$U^{-1}(\Lambda_T)p^\mu U(\Lambda_T) = -(\Lambda_T)^\mu{}_\nu p^\nu$$

and as such one must consider an anti-unitary operator.

[12] Notice that pseudo-tensors transform like tensors under proper rotations, but additionally change sign under improper rotations. For example, under improper rotations, scalars do not change sign, while pseudo-scalars do; vectors change sign, while pseudo-vectors do not.

Proof of anti-unitarity. The existence of anti-unitarity is proven as follows. Consider the space-time translation operator

$$T(a) = \mathrm{e}^{-\mathrm{i}a_\mu p^\mu} = 1 - \mathrm{i}a_\mu p^\mu + o(a_\mu)\,, \quad a_\mu \ll 1$$

A translation of the field is

$$T^{-1}(a)\varphi(x)T(a) = \varphi(x-a)$$

Applying a proper Lorentz transformation gives

$$\begin{aligned}
U^{-1}(\Lambda)\varphi(x-a)U(\Lambda) &= U^{-1}(\Lambda)T^{-1}(a)\varphi(x)T(a)U(\Lambda) \\
&= (U^{-1}T^{-1}U)(U^{-1}\varphi U)(U^{-1}TU) \\
\varphi(\Lambda^{-1}(x-a)) &= (U^{-1}T^{-1}U)\varphi(\Lambda^{-1}x)(U^{-1}TU) \\
\varphi(\Lambda^{-1}x - \Lambda^{-1}a) &= (U^{-1}TU)^{-1}\varphi(\Lambda^{-1}x)(U^{-1}TU) \\
\varphi(y - \Lambda^{-1}a) &= (U^{-1}TU)^{-1}\varphi(y)(U^{-1}TU)
\end{aligned}$$

Therefore, the operator that translates by $\Lambda^{-1}a$ is

$$T(\Lambda^{-1}a) = U^{-1}(\Lambda)T(a)U(\Lambda)$$

which can be written as an infinitesimal transformation

$$U^{-1}(\Lambda)(I - \mathrm{i}a_\mu p^\mu)U(\Lambda) = I - \mathrm{i}(\Lambda^{-1}a)_\mu p^\mu = I - \mathrm{i}(\Lambda^{-1})_\mu{}^\nu a_\nu p^\mu$$

Therefore, for a generic Lorentz transformation, one finds

$$\boxed{U^{-1}(\Lambda)(\mathrm{i}a_\mu p^\mu)U(\Lambda) = \mathrm{i}(\Lambda^{-1})_\mu{}^\nu a_\nu p^\mu}$$

Consider a proper Lorentz transformation and the fact that

$$(\Lambda^{-1})_\mu{}^\nu = (\Lambda^\top)_\mu{}^\nu = \Lambda^\nu{}_\mu$$

then one has

$$U^{-1}(\Lambda)(\mathrm{i}a_\mu p^\mu)U(\Lambda) = \mathrm{i}a_\nu \Lambda^\nu{}_\mu p^\mu$$

Knowing that

$$\Lambda^\nu{}_\mu p^\mu = U^{-1}(\Lambda)p^\nu U(\Lambda)$$

one finds

$$U^{-1}(\Lambda)\mathrm{i}U(\Lambda) = \mathrm{i}$$

Consider a time reversal transformation

$$U^{-1}(\Lambda_T)(\mathrm{i}a_\mu p^\mu)U(\Lambda_T) = \mathrm{i}a_\nu(\Lambda_T)^\nu{}_\mu p^\mu$$

Assuming that

$$U^{-1}(\Lambda_T)\mathrm{i}U(\Lambda_T) = \mathrm{i}\eta\,, \quad \eta = \pm 1$$

one has

$$\mathrm{i}a_\nu \eta U^{-1}(\Lambda_T)p^\nu U(\Lambda_T) = \mathrm{i}a_\nu(\Lambda_T)^\nu{}_\mu p^\mu$$

One requires

$$U^{-1}(\Lambda_T)p^\nu U(\Lambda_T) = -(\Lambda_T)^\nu{}_\mu p^\mu$$

and there one obtains

$$\eta = -1$$

from which $U(\Lambda_T)$ is anti-unitary.

Lecture 9

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8.1.2 Charge conjugation

Charge conjugating is a $\mathbb{Z}_2$ symmetry:

$$Z^{-1}\varphi(x)Z = \pm\varphi(x)\,, \quad Z^2 = 1$$

For real scalar fields there is no charge and the discussion ends here. For complex scalar fields, a Lagrangian may be

$$\begin{aligned}\mathcal{L} &= \partial_\mu\varphi^\dagger\,\partial^\mu\varphi - m^2\varphi^\dagger\varphi - \frac{\lambda}{4!}(\varphi^\dagger\varphi)^2 \\ &= \frac{1}{2}\,\partial_\mu\varphi_1\,\partial^\mu\varphi_1 - \frac{1}{2}m^2\varphi_1^2 + \frac{1}{2}\,\partial_\mu\varphi_2\,\partial^\mu\varphi_2 - \frac{1}{2}m^2\varphi_2^2 - \frac{\lambda}{4!}(\varphi_1+\varphi_2)^4\end{aligned}$$

where one has

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + \mathrm{i}\varphi_2)\,, \quad \varphi^\dagger = \frac{1}{\sqrt{2}}(\varphi_1 - \mathrm{i}\varphi_2)$$

The Lagrangian written with complex fields has a global U(1) symmetry

$$\varphi' = \mathrm{e}^{\mathrm{i}q}\varphi\,, \quad \varphi'^\dagger = \mathrm{e}^{-\mathrm{i}q}\varphi^\dagger$$

The field has charge q while the conjugate field has charge $-q$. They are associated to particles and anti-particles. The Lagrangian written in terms of real fields has a global SO(2) symmetry.

Along the previous symmetry, the complex-field Lagrangian exhibits also a $\mathbb{Z}_2$ symmetry, $\varphi \leftrightarrow \varphi^\dagger$. The generator of this symmetry is given by the operator C:

$$C^{-1}\varphi(x)C = \varphi^\dagger(x)\,, \quad C\varphi^\dagger(x)C^{-1} = \varphi(x) \implies C^{-1}\mathcal{L}C = \mathcal{L}$$

For the real-field Lagrangian, one has

$$C^{-1}\varphi_1(x)C = \varphi_1(x)\,, \quad C^{-1}\varphi_2(x)C = -\varphi_2(x)$$

In matrix form, one has

$$C = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\,, \quad \det C = -1$$

Since this operator has negative determinant, one has to enlarge the symmetry group to O(2).

The operator C is responsible for charge conjugation since it exchanges particles of charge q with anti-particles of charge $-q$.

8.2 Representation on spinor fields

Consider spinor fields, that is spin-half fermionic fields, $s = \frac{1}{2}$. Recall that, in canonical quantization, the most general solution to the free Dirac equation is

$$\psi_\alpha(x) = \sum_{s=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[b_s(\mathbf{p})u_{s\alpha}(\mathbf{p})\mathrm{e}^{-\mathrm{i}px} + d_s^\dagger(\mathbf{p})v_{s\alpha}(\mathbf{p})\mathrm{e}^{\mathrm{i}px}]\,, \quad p_0 \equiv \omega = \sqrt{|\mathbf{p}|^2 + m^2}$$

In the Weyl basis, the Dirac matrices are

$$\gamma^0 = \begin{bmatrix}0 & I_2\\ I_2 & 0\end{bmatrix}\,, \quad \gamma^k = \begin{bmatrix}0 & \sigma^k\\ -\sigma^k & 0\end{bmatrix}\,, \quad \gamma^5 = \begin{bmatrix}-I_2 & 0\\ 0 & I_2\end{bmatrix}$$

where σ^k are the Pauli matrices. One introduces the charge conjugation matrix[13]

$$C = \mathrm{i}\gamma^0\gamma^2 = \begin{bmatrix}-\mathrm{i}\sigma^2 & 0\\ 0 & \mathrm{i}\sigma^2\end{bmatrix} = \begin{bmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{bmatrix}$$

[13] Typically the convention is to define $C = \mathrm{i}\gamma^2\gamma_0$. In this convention the following identities change too.

Identities. For the plane-wave spinor solutions (see Peskin, eqs. 3.59, 3.62)

$$u_s(\mathbf{p}) = \begin{bmatrix} \sqrt{\mathbf{p}\cdot\boldsymbol{\sigma}}\xi_s \\ \sqrt{\mathbf{p}\cdot\bar{\boldsymbol{\sigma}}}\xi_s \end{bmatrix} , \quad v_s(\mathbf{p}) = \begin{bmatrix} \sqrt{\mathbf{p}\cdot\boldsymbol{\sigma}}\eta_s \\ -\sqrt{\mathbf{p}\cdot\bar{\boldsymbol{\sigma}}}\eta_s \end{bmatrix}$$

it holds (see Srednicki, §38, eqs. 38.32, 38.40)

$$u_s(-\mathbf{p}) = \gamma^0 u_s(\mathbf{p}) , \quad v_s(-\mathbf{p}) = -\gamma^0 v_s(\mathbf{p})$$

also

$$u^*_{-s}(-\mathbf{p}) = -sC\gamma^5 u_s(\mathbf{p}) , \quad v^*_{-s}(-\mathbf{p}) = -sC\gamma^5 v_s(\mathbf{p})$$

Proper Lorentz transformation. Under a proper Lorentz transformation, $\Lambda \in \mathcal{L}_+^\uparrow$, a spinor transforms as

$$U^{-1}(\Lambda)\psi(x)U(\Lambda) = D(\Lambda)\psi(\Lambda^{-1}x)$$

with

$$D(\Lambda) = \mathrm{e}^{\frac{\mathrm{i}}{2}\omega_{\mu\nu}S^{\mu\nu}} , \quad S^{\mu\nu} = \frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu]$$

where $S^{\mu\nu}$ is the spin representation of the Lorentz generators.

8.2.1 Parity

One makes the ansatz that

$$U^{-1}(\Lambda_P)\psi(x)U(\Lambda_P) = D(P)\psi(\Lambda_P x)$$

where $D(P)$ is an operator to be determined in the following. By applying the relation above twice, one obtains $-\psi(x)$. One does not require unity because spinors are not observables, only bilinear functions made from them are

$$U^{-2}(\Lambda_P)\psi(x)U^2(\Lambda_P) = D^2(P)\psi(\Lambda_P x) \equiv \pm\psi(x) \implies D^2(P) = \pm 1$$

Remembering that, under parity, the momentum $\mathbf{p}$ changes sign while the angular momentum $\mathbf{J}$ does not, then one expects that

$$U^{-1}(\Lambda_P)b^\dagger_s(\mathbf{p})U(\Lambda_P) = \eta b^\dagger_s(-\mathbf{p}) , \quad U^{-1}(\Lambda_P)d^\dagger_s(\mathbf{p})U(\Lambda_P) = \eta d^\dagger_s(-\mathbf{p})$$

with $\eta^2 = \pm 1$. One introduces the same constant η for b and d to be able to relate the operators when working with Majorana fermions.

[r] Therefore, substituting the wave expansion — and recalling that the dispersion relation holds —, one has

$$\begin{aligned} U^{-1}(\Lambda_P)\psi(x)U(\Lambda_P) &= \sum_{s=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[\eta^* b_s(-\mathbf{p})u_s(\mathbf{p})\mathrm{e}^{\mathrm{i}px} + \eta d^\dagger_s(-\mathbf{p})v_s(\mathbf{p})\mathrm{e}^{\mathrm{i}px}]_{p^0=\omega} \\ &= \sum_{s=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[\eta^* b_s(\mathbf{p})u_s(-\mathbf{p})\mathrm{e}^{-\mathrm{i}p\Lambda_P x} + \eta d^\dagger_s(\mathbf{p})v_s(-\mathbf{p})\mathrm{e}^{\mathrm{i}p\Lambda_P x}] \\ &= \sum_{s=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[\eta^* b_s(\mathbf{p})\gamma^0 u_s(\mathbf{p})\mathrm{e}^{-\mathrm{i}p\Lambda_P x} - \eta d^\dagger_s(\mathbf{p})\gamma^0 v_s(\mathbf{p})\mathrm{e}^{\mathrm{i}p\Lambda_P x}] \\ &= \mathrm{i}\gamma^0\sum_{s=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[b_s(\mathbf{p})u_s(\mathbf{p})\mathrm{e}^{-\mathrm{i}p\Lambda_P x} + \eta d^\dagger_s(\mathbf{p})v_s(\mathbf{p})\mathrm{e}^{\mathrm{i}p\Lambda_P x}] \\ &= \mathrm{i}\gamma^0\psi(\Lambda_P x) \end{aligned}$$

At the second line, one has sent $\mathbf{p} \to -\mathbf{p}$ in the integral. The spatial components of the momentum in the exponential gain an extra minus sign which can be put into the position $\mathbf{x}$ to have

$$\mathrm{e}^{-\mathrm{i}px} \to \mathrm{e}^{-\mathrm{i}p\Lambda_P x}$$

Notice that there is no overall minus sign since the integration limits are mapped into one another and the sign from the measure is absorbed to swap them. At the third line, one has applied the identities above. At the fourth line, one has required that

$$\eta^* = -\eta \implies \eta \in \mathrm{i}\mathbb{R}$$

so that one obtains a spinor field again. One chooses $\eta = -\mathrm{i}$.

From this, one sees that

$$\boxed{D(P) = \mathrm{i}\gamma^0} \implies D^2(P) = -I$$

Spinor transformation. Consider a spinor[14]

$$\psi = \begin{bmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{bmatrix} = \begin{bmatrix} \chi_\alpha \\ \bar{\xi}^{\dot{\alpha}} \end{bmatrix}$$

The spinor transforms as

$$D(P)\psi = \mathrm{i}\gamma^0\psi = \mathrm{i}\begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}\begin{bmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{bmatrix} = \mathrm{i}\begin{bmatrix} \psi_{\mathrm{R}} \\ \psi_{\mathrm{L}} \end{bmatrix}$$

The chiral components are swapped.

8.2.2 Time reversal

One assumes that under a time reversal transformation, the spinor field changes as

$$U^{-1}(\Lambda_T)\psi(x)U(\Lambda_T) = D(T)\psi(\Lambda_T x)$$

As before, one must have $D^2(T) = \pm 1$.

Noting that, under time reversal, the momentum $\mathbf{p}$ and the angular momentum $\mathbf{J}$ both change sign, one has

$$U^{-1}(\Lambda_T)b_s^\dagger(\mathbf{p})U(\Lambda_T) = \zeta_s b_{-s}^\dagger(-\mathbf{p}), \quad U^{-1}(\Lambda_T)d_s^\dagger(\mathbf{p})U(\Lambda_T) = \zeta_s d_{-s}^\dagger(-\mathbf{p})$$

with $\zeta_s^2 = \pm 1$. When one applies the time reversal transformation to the spinor field, one produces the term

$$U^{-1}(\Lambda_T)\psi(x)(\mathbf{p})U(\Lambda_T) \propto U^{-1}(\Lambda_T)b_s(\mathbf{p})u_s(\mathbf{p})\mathrm{e}^{-\mathrm{i}px}U(\Lambda_T) = U^{-1}(\Lambda_T)b_s(\mathbf{p})U(\Lambda_T)u_s^*(\mathbf{p})\mathrm{e}^{\mathrm{i}px}$$

where one inserts UU^{-1} between bu and recalls that $U(\Lambda_T)$ is anti-unitary. Therefore

$$\begin{aligned} U^{-1}(\Lambda_T)\psi(x)U(\Lambda_T) &= \sum_{s=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[\zeta_s^* b_{-s}(-\mathbf{p})u_s^*(\mathbf{p})\mathrm{e}^{\mathrm{i}px} + \zeta_s d_{-s}^\dagger(-\mathbf{p})v_s^*(\mathbf{p})\mathrm{e}^{-\mathrm{i}px}]_{p^0=\omega} \\ &= \sum_{s=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[\zeta_{-s}^* b_s(\mathbf{p})u_{-s}^*(-\mathbf{p})\mathrm{e}^{-\mathrm{i}p\Lambda_T x} + \zeta_{-s} d_s^\dagger(\mathbf{p})v_{-s}^*(-\mathbf{p})\mathrm{e}^{\mathrm{i}p\Lambda_T x}] \\ &= \sum_{s=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2\omega}[\zeta_{-s}^* b_s(\mathbf{p})(-sC\gamma^5 u_s(\mathbf{p}))\mathrm{e}^{-\mathrm{i}p\Lambda_T x} \\ &\quad + \zeta_{-s} d_s^\dagger(\mathbf{p})(-sC\gamma^5 v_s(\mathbf{p}))\mathrm{e}^{\mathrm{i}p\Lambda_T x}] \\ &= C\gamma^5\psi(\Lambda_T x) \end{aligned}$$

At the second line, one has interchanged $\mathbf{p} \to -\mathbf{p}$ and $s \to -s$. Similar to before, the spatial components of the momentum in the exponential gain a minus sign which is put in front along with the minus sign of the time-like component coming from $\Lambda_T x$. At the third line, one has applied the identities above. At the fourth line, to obtain a spinor field, one has to choose

$$\zeta_s = s \implies \zeta_{-s} = -s \implies (-s)^2 = 1$$

Therefore

$$\boxed{D(T) = C\gamma^5} \implies D^2(T) = -I$$

[14]The second equality is in the van der Waerden notation.

Spinor transformation. Noting that

$$C = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -\varepsilon^{\alpha\beta} & 0 \\ 0 & -\varepsilon_{\dot{\alpha}\dot{\beta}} \end{bmatrix} , \quad \gamma^5 = \begin{bmatrix} -I_2 & 0 \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} -\delta_{\beta}{}^{\gamma} & 0 \\ 0 & \delta^{\dot{\beta}}{}_{\dot{\gamma}} \end{bmatrix}$$

one finds

$$C\gamma^5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} \varepsilon^{\alpha\gamma} & 0 \\ 0 & -\varepsilon_{\dot{\alpha}\dot{\gamma}} \end{bmatrix}$$

from which

$$C\gamma^5 \begin{bmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{bmatrix} = \begin{bmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{bmatrix} \begin{bmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{bmatrix} = \begin{bmatrix} \varepsilon\psi_{\mathrm{L}} \\ -\varepsilon\psi_{\mathrm{R}} \end{bmatrix}$$

Time reversal does not exchange chiral components, but changes each component into its dual. [r]

8.2.3 Charge conjugation

Let the charge conjugation matrix be C and the charge conjugation generator on the field ψ be $\mathcal{C}$. Then a charge conjugation transformation is

$$\mathcal{C}^{-1}\psi(x)\mathcal{C} = C\bar{\psi}^\top(x) = C(\psi^\dagger\gamma^0)^\dagger = C\gamma^0\psi^*$$

Similarly

$$\mathcal{C}^{-1}\bar{\psi}(x)\mathcal{C} = \psi^\top(x)C$$

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The charge conjugation matrix satisfies

$$C^\top = C^\dagger = C^{-1} = -C\,, \quad \{\gamma^0, C\} = 0\,, \quad C^{-1}\gamma^\mu C = -(\gamma^\mu)^\top\,, \quad C^{-1}\gamma^5 C = \gamma^5$$

8.2.4 Majorana spinors

Definition. A Majorana spinor is a bi-spinor satisfying

$$\psi \equiv \psi^c = C\bar{\psi}^\top = C\gamma^0\psi^*$$

Recalling that

$$\gamma^0 = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix} \implies C\gamma^0 = \begin{bmatrix} 0 & -\mathrm{i}\sigma^2 \\ \mathrm{i}\sigma^2 & 0 \end{bmatrix}$$

The Majorana condition is equivalent to

$$C\gamma^0\psi^* = \psi \iff \begin{bmatrix} 0 & -\mathrm{i}\sigma^2 \\ \mathrm{i}\sigma^2 & 0 \end{bmatrix} \begin{bmatrix} \psi_{\mathrm{L}}^* \\ \psi_{\mathrm{R}}^* \end{bmatrix} = \begin{bmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{bmatrix} \implies \begin{cases} -\mathrm{i}\sigma^2\psi_{\mathrm{L}}^* = \psi_{\mathrm{L}} \\ \mathrm{i}\sigma^2\psi_{\mathrm{R}}^* = \psi_{\mathrm{R}} \end{cases}$$

The degrees of freedom are half the ones for a Dirac spinor.

Summary. A brief summary of parity, time reversal and charge conjugation follows

$$\begin{aligned} U^{-1}(\Lambda_P)\psi(x)U(\Lambda_P) &= \mathrm{i}\gamma^0\psi(\Lambda_P x) \\ U^{-1}(\Lambda_T)\psi(x)U(\Lambda_T) &= C\gamma^5\psi(\Lambda_T x) \\ \mathcal{C}^{-1}\psi(x)\mathcal{C} &= C\gamma^0\psi^*(x) \end{aligned}$$

One may consider the transformations for the Dirac adjoint field $\bar{\psi}$. Parity gives

$$\begin{aligned}U^{-1}(\Lambda_P)\bar{\psi}(x)U(\Lambda_P) &= U^{-1}(\Lambda_P)\psi^\dagger(x)\gamma^0 U(\Lambda_P) = U^{-1}(\Lambda_P)\psi^\dagger(x)\gamma^0 U(\Lambda_P)\\ &= -\mathrm{i}\psi^\dagger(\Lambda_P x)\gamma^0\gamma^0 = -\mathrm{i}\bar{\psi}(\Lambda_P x)\gamma^0\end{aligned}$$

For time reversal, one finds

$$U^{-1}(\Lambda_T)\bar{\psi}(x)U(\Lambda_T) = \bar{\psi}(\Lambda_T x)\gamma^5 C^{-1}$$

Therefore, the summary for the Dirac adjoint is

$$\begin{aligned}U^{-1}(\Lambda_P)\bar{\psi}(x)U(\Lambda_P) &= -\mathrm{i}\bar{\psi}(\Lambda_P x)\gamma^0\\ U^{-1}(\Lambda_T)\bar{\psi}(x)U(\Lambda_T) &= \bar{\psi}(\Lambda_T x)\gamma^5 C^{-1}\\ \mathcal{C}^{-1}\bar{\psi}(x)\mathcal{C} &= \psi^\top(x)C\end{aligned}$$

8.3 Spinor bilinears

A bilinear in the spinor field is a real expression of the type $\bar{\psi}A\psi$ where A is a product of Dirac matrices such that

$$(\bar{\psi}A\psi)^\dagger = \bar{\psi}A\psi \implies \bar{A} = \gamma^0 A^\dagger\gamma^0 = A$$

where the bar denotes the Dirac adjoint of a matrix.

Parity. A parity transformation of a bilinear is

$$\begin{aligned}U^{-1}(\Lambda_P)(\bar{\psi}A\psi)U(\Lambda_P) &= [U^{-1}(\Lambda_P)\bar{\psi}U(\Lambda_P)][U^{-1}(\Lambda_P)AU(\Lambda_P)][U^{-1}(\Lambda_P)\psi U(\Lambda_P)]\\ &= [-\mathrm{i}\bar{\psi}\gamma^0 A\mathrm{i}\gamma^0\psi](\Lambda_P x) = \bar{\psi}\gamma^0 A\gamma^0\psi = \bar{\psi}\bar{A}\psi\end{aligned}$$

To go further, one needs to specify the bilinear and in particular A.

Let $A = I$, one has a scalar

$$U^{-1}(\Lambda_P)\bar{\psi}\psi U(\Lambda_P) = \bar{\psi}\psi$$

Let $A = \mathrm{i}\gamma^5$, one has a pseudo-scalar

$$U^{-1}(\Lambda_P)\mathrm{i}\bar{\psi}\gamma^5\psi U(\Lambda_P) = -\mathrm{i}\bar{\psi}\gamma^5\psi$$

Let $A = \gamma^\mu$, one has a vector (also called polar vector)

$$U^{-1}(\Lambda_P)\bar{\psi}\gamma^\mu\psi U(\Lambda_P) = (\Lambda_P)^\mu{}_\nu(\bar{\psi}\gamma^\nu\psi)$$

where the parity transformation Λ_P encodes the following relations

$$\bar{\psi}\gamma^0\gamma^\mu\gamma^0\psi = \begin{cases}\bar{\psi}\gamma^0\psi\,, & \mu = 0\\ -\bar{\psi}\gamma^i\psi\,, & \mu = i\end{cases}$$

Let $A = \gamma^\mu\gamma^5$, one has a pseudo-vector (also called axial vector)

$$U^{-1}(\Lambda_P)\bar{\psi}\gamma^\mu\gamma^5\psi U(\Lambda_P) = -(\Lambda_P)^\mu{}_\nu(\bar{\psi}\gamma^\nu\gamma^5\psi)$$

Time reversal. The transformation of a bilinear is

$$\begin{aligned}U^{-1}(\Lambda_T)(\bar{\psi}A\psi)U(\Lambda_T) &= [U^{-1}(\Lambda_T)\bar{\psi}U(\Lambda_T)][U^{-1}(\Lambda_T)AU(\Lambda_T)][U^{-1}(\Lambda_T)\psi U(\Lambda_T)]\\ &= \bar{\psi}\gamma^5 C^{-1}A^* C\gamma^5\psi\end{aligned}$$

The second bracket evaluates to A^* since the operator $U(\Lambda_T)$ is anti-unitary. For the following it is useful to remember that

$$\gamma_{0,5} = \gamma_{0,5}^\dagger\,, \quad \gamma_i = -\gamma_i^\dagger$$

which can be shortened to

$$\gamma_5 = \gamma_5^\dagger\,, \quad (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$$

Let $A = I$, the bilinear is T-even

$$U^{-1}(\Lambda_T)(\bar\psi\psi)U(\Lambda_T) = \bar\psi\gamma^5C^{-1}C\gamma^5\psi = \bar\psi\psi$$

Let $A = \mathrm{i}\gamma^5$, the bilinear is T-odd

$$U^{-1}(\Lambda_T)(\bar\psi\mathrm{i}\gamma^5\psi)U(\Lambda_T) = \bar\psi\gamma^5C^{-1}(-\mathrm{i}\gamma^5)C\gamma^5\psi = -\mathrm{i}\bar\psi(\gamma^5)^3\psi = -\mathrm{i}\bar\psi\gamma^5\psi$$

where one applies the properties of the charge conjugation matrix. Let $A = \gamma^\mu$, the bilinear is T-odd

$$U^{-1}(\Lambda_T)(\bar\psi\gamma^\mu\psi)U(\Lambda_T) = \bar\psi\gamma^5C^{-1}(\gamma^\mu)^*C\gamma^5\psi = -\bar\psi\gamma^5(\gamma^\mu)^\dagger\gamma^5\psi = \bar\psi(\gamma^\mu)^\dagger\psi = -(\Lambda_T)^\mu{}_\nu\bar\psi\gamma^\nu\psi$$

The sign difference is encoded into the time reversal transformation Λ_T.

Exercise. Check that $\bar\psi\gamma^\mu\gamma^5\psi$ is also T-odd.

Charge conjugation. The transformation of a bilinear is

$$\begin{aligned}\mathcal{C}^{-1}(\bar\psi A\psi)\mathcal{C} &= [\mathcal{C}^{-1}\bar\psi\mathcal{C}][\mathcal{C}^{-1}A\mathcal{C}][\mathcal{C}^{-1}\psi\mathcal{C}] = \psi^\top CAC\bar\psi^\top \\ &= (\psi^\top CAC\bar\psi^\top)^\top = -\bar\psi(CAC)^\top\psi = \bar\psi C^{-1}A^\top C\psi\end{aligned}$$

at the second line one has noticed that the transpose of a bilinear gains a minus sign due to interchanging the spinor fields[15].

Let $A = I$, the bilinear is C-even

$$\mathcal{C}^{-1}\bar\psi\psi\mathcal{C} = \bar\psi\psi$$

Let $A = \mathrm{i}\gamma^5$, the bilinear is C-even

$$\mathcal{C}^{-1}(\bar\psi\mathrm{i}\gamma^5\psi)\mathcal{C} = \bar\psi\mathrm{i}\gamma^5\psi$$

Let $A = \gamma^\mu$, the bilinear is C-odd

$$\mathcal{C}^{-1}(\bar\psi\gamma^\mu\psi)\mathcal{C} = -\bar\psi\gamma^\mu\psi$$

Let $A = \gamma^\mu\gamma^5$, the bilinear is C-even

$$\mathcal{C}^{-1}(\bar\psi\gamma^\mu\gamma^5\psi)\mathcal{C} = \bar\psi\gamma^\mu\gamma^5\psi$$

Remark. If the spinor field is Majorana, then

$$\psi^c = \psi \iff \mathcal{C}^{-1}\psi\mathcal{C} = \psi\,, \quad \mathcal{C}^{-1}\bar\psi\mathcal{C} = \bar\psi$$

Therefore, one has [r]

$$\mathcal{C}^{-1}\bar\psi A\psi\mathcal{C} = \bar\psi A\psi$$

This means that the odd bilinears have to be zero. In particular

$$\bar\psi\gamma^\mu\psi = 0$$

8.4 Representation on vector fields

[r] Source?. Each component of a vector field $V^\mu(x)$ behaves as a scalar field under charge conjugation.

[15] See `https://physics.stackexchange.com/q/458451`.

Parity. Under parity, there are vectors and pseudo-vectors

$$U^{-1}(\Lambda_P)\mathbf{V}(x)U(\Lambda_P) = -\mathbf{V}(\Lambda_P x)\,, \quad U^{-1}(\Lambda_P)\mathbf{A}(x)U(\Lambda_P) = \mathbf{A}(\Lambda_P x)$$

[r]

Time reversal. For time reversal one has T-even and T-odd vectors, similar to vector bilinears.

Theorem (Furry's in QED)**.** In QED, the interaction term is

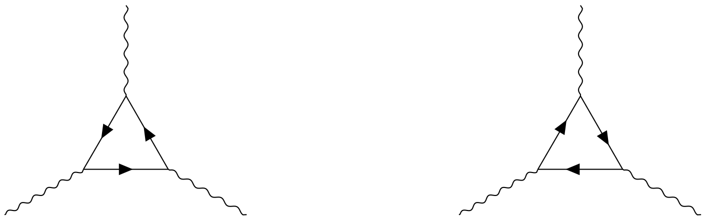
$$\mathcal{L}_{\text{int}} = J^\mu A_\mu\,, \quad J^\mu \propto \bar{\psi}\gamma^\mu\psi$$

A Lagrangian invariant under charge conjugation implies that the four-potential A^μ has to be odd

$$\mathcal{C}^{-1}A^\mu\mathcal{C} = -A^\mu$$

since the current J^μ is odd. This implies that the correlation functions of an odd number of gauge fields A^μ are zero.

Example. For example, the triangle diagram has two configurations depending on the orientation of the internal fermionic propagators. These two configurations sum up to zero.



8.5 CPT theorem

See Weinberg, vol. 1, §5.8, Srednicki, §40.

Theorem. The action of a local and hermitian[16] Lagrangian is symmetric under a simultaneous transformation of charge conjugation, parity and time reversal

$$(CPT)^{-1}\mathcal{L}(x)(CPT) = \mathcal{L}(-x) \implies S = \int \mathrm{d}^4x\,\mathcal{L}(-x) = \int \mathrm{d}^4x\,\mathcal{L}(x)$$

Proof. For scalar fields, one has

$$P^{-1}\varphi P = \pm\varphi = \eta\varphi\,, \quad T^{-1}\varphi T = \pm\varphi = \zeta^*\varphi\,, \quad \mathcal{C}^{-1}\varphi\mathcal{C} = \pm\varphi^\dagger = \xi\varphi^\dagger$$

Therefore

$$(CPT)^{-1}\varphi(CPT) = \eta\zeta^*\xi\varphi^\dagger(-x) = \varphi^\dagger(-x)$$

where one chooses the inversion phases such that

$$\eta\zeta^*\xi = 1$$

For spinor fields, one has

$$\begin{aligned}(CPT)^{-1}\bar{\psi}\psi(CPT) &= \bar{\psi}\psi\\ (CPT)^{-1}\bar{\psi}\mathrm{i}\gamma^5\psi(CPT) &= \bar{\psi}\mathrm{i}\gamma^5\psi\\ (CPT)^{-1}\bar{\psi}\gamma^\mu\psi(CPT) &= -\bar{\psi}\gamma^\mu\psi\\ (CPT)^{-1}\bar{\psi}\gamma^\mu\gamma^5\psi(CPT) &= -\bar{\psi}\gamma^\mu\gamma^5\psi\end{aligned}$$

[16]Hermitian so that it guarantees unitarity.

where the first two are CPT-even and the other two are CPT-odd. This is true for any bilinear. In general, bilinears with an even number of vector indices are even under CPT; while for an odd number, they are for CPT-odd. This is true also for four-derivatives. In fact

$$(CPT)^{-1}\partial_\mu\varphi(x)(CPT) = -\partial_\mu\varphi^\dagger(-x)$$

On to the actual proof. Since a Lagrangian is a Lorentz scalar, it is CPT-invariant [r]. Since the Lagrangian is hermitian, it contains the same number of fields and their conjugates, therefore the Lagrangian is CPT-even. This means that the Lagrangian is CPT-invariant even if $\varphi \leftrightarrow \varphi^\dagger$. □

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Part II
Gauge fields

Review of classical fields – abelian gauge theory. See Cheng, §8.1. Consider QED as an abelian gauge theory. The free Lagrangian is

$$\mathcal{L} = \bar\psi(\mathrm{i}\,\partial\!\!\!/ - m)\psi$$

The Lagrangian is invariant under a global U(1) transformation

$$\psi' = \mathrm{e}^{\mathrm{i}\alpha}\psi\,, \quad \bar\psi' = \bar\psi\mathrm{e}^{-\mathrm{i}\alpha}$$

The gauge principle involves promoting the global parameter α to a function of space-time $\alpha(x)$. By requiring a local U(1) invariance, one introduces a gauge covariant derivative[17] and a gauge field

$$D_\mu = \partial_\mu + \mathrm{i}qA_\mu\,, \quad A'_\mu = A_\mu - \frac{1}{q}\,\partial_\mu\alpha$$

where q is the charge (with sign) of the bispinor field. The gauge-invariant Lagrangian has an interaction term

$$\mathcal{L} = \bar\psi(\mathrm{i}\,D\!\!\!\!/ - m)\psi = \bar\psi(\mathrm{i}\,\partial\!\!\!/ - m)\psi - qA_\mu\bar\psi\gamma^\mu\psi$$

If there is no kinetic term for the gauge field, then the field is a background field. To make it dynamical, one has to insert such kinetic term

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

where the field strength tensor is defined by

$$\mathrm{i}qF_{\mu\nu} = [D_\mu, D_\nu]$$

and it is invariant under a gauge transformation.

One observes that gauge invariance forbids any mass term for the gauge field. The gauge Lagrangian term is a free Lagrangian — meaning it does not contain any interaction vertex — since the gauge field A_μ is neutral (because it is real) and cannot couple to itself.

Review of classical fields – non-abelian gauge theory. An example of non-abelian gauge theory is QCD. Consider the gauge group $G = \mathrm{SU}(n)$ (so the theory is a Yang–Mills theory). The gauge group has n^2-1 generators that satisfy the (fundamental representation of the) $\mathfrak{su}(n)$ Lie algebra

$$[T^a, T^b] = \mathrm{i}c^{abc}T^c$$

[17] Note that Cheng uses $q = e$ where, for an electron, $e < 0$.

where c^{abc} are the structure constants, abc are indices labelling the dimensions of the algebra and the generators are traceless Hermitian complex $n \times n$ matrices[18]. Consider a set of Dirac spinors in some representation of the algebra $\mathfrak{su}(n)$. The Lagrangian is

$$\mathcal{L} = \bar{\psi}_j(\mathrm{i}\,\partial\!\!\!/ - m)\psi_j$$

It has a global $\mathrm{SU}(n)$ symmetry

$$\psi'_j = U_{jk}(\theta)\psi_k\,, \quad \bar{\psi}'_j = \bar{\psi}_k U^\dagger_{kj}(\theta)$$

where the transformation matrix is

$$U_{jk}(\theta) = [\mathrm{e}^{-\mathrm{i}\theta^a T^a}]_{jk}\,, \quad UU^\dagger = U^\dagger U = I$$

By applying the gauge principle, one obtains $\theta = \theta(x)$. The requirement of local $\mathrm{SU}(n)$ symmetry implies

$$D_\mu = \partial_\mu - \mathrm{i}gT^a A^a_\mu\,, \quad (A^a_\mu)' = A^a_\mu + c^{abc}\theta^b A^c_\mu - \frac{1}{g}\,\partial_\mu\theta^a$$

Let $A_\mu = T^a A^a_\mu$, then the infinitesimal transformation is

$$A'_\mu = A_\mu - \mathrm{i}[\theta, A_\mu] - \frac{1}{g}\,\partial_\mu\theta$$

To make the gauge fields dynamical, one introduces the field strengths

$$[D_\mu, D_\nu] = -\mathrm{i}gT^a F^a_{\mu\nu}\,, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gc^{abc}A^b_\mu A^c_\nu$$

The field strength is not gauge-invariant, but transforms infinitesimally as

$$F'^a_{\mu\nu} = F^a_{\mu\nu} + c^{abc}\theta^b F^c_{\mu\nu}$$

For a finite transformation, it is a covariant tensor

$$F'_{\mu\nu} = U^\dagger(\theta)F_{\mu\nu}U(\theta)$$

The gauge kinetic term is then

$$\begin{aligned}\mathcal{L}_{\mathrm{gauge}} &= -\frac{1}{4}\operatorname{Tr}(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{4}F^a_{\mu\nu}F^{b\mu\nu}\operatorname{Tr}\big(T^aT^b\big) = -\frac{1}{4}F^a_{\mu\nu}F^{b\mu\nu}\delta^{ab} \\ &= -\frac{1}{2}\,\partial_\mu A^a_\nu(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) - gc^{abc}\,(\partial_\mu A^a_\nu)A^{b\mu}A^{c\nu} - \frac{g^2}{4}c^{abc}c^{adf}A^b_\mu A^c_\nu A^{d\mu}A^{f\nu}\end{aligned}$$

The convention of the trace may be different. When inserting the explicit expression for the field strength one finds the following general structure

$$\text{kinetic} + g\partial A\,A^2 + g^2A^4$$

There is derivative three-point vertex and a four-point vertex. The gauge Lagrangian term is not free Lagrangian since it contains two vertices.

One observes that the covariant derivative introduces an interaction term

$$gA^a_\mu\bar{\psi}_j T^a_{jk}\gamma^\mu\psi_k = gA^a_\mu J^{a\mu}$$

where jk label the bispinor fields of the theory and a is an $\mathfrak{su}(n)$ index. [r]

While in abelian gauge theories, one may have different matter sectors coupled to the same gauge field with different coupling constants, in this case the coupling constant g is fixed when choosing the gauge group and it has to be the same for all matter fields since it appears in the field strength.

For each simple gauge group (e.g. $\mathrm{SU}(n)$), there is a corresponding single coupling constant.

[18] In the physicists' convention. In the mathematicians' convention they are traceless anti-Hermitian matrices. The two conventions differ by a factor of i.

Canonical quantization. Consider QED. Out of the four components of a gauge field, only two are physical degrees of freedom, typically A^1 and A^2. This is due to gauge invariance which can be used to eliminate non-physical degrees of freedom.

One may use gauge invariance to fix the Lorenz gauge. If there are no sources in the free theory, one may perform a further gauge transformation while being in the Lorenz gauge to set $A_3 = 0$. In fact, consider a generic gauge

$$\partial_\mu A^\mu \neq 0$$

One may transform the gauge field to go to the Lorenz gauge

$$A'_\mu(x) = A_\mu(x) - \partial_\mu \Lambda(x) , \quad 0 = \partial_\mu A'^\mu = \partial_\mu A^\mu - \Box \Lambda \implies \Box \Lambda = \partial_\mu A^\mu$$

From gauge field A', one can always choose another gauge field within the Lorenz gauge

$$A''_\mu(x) = A'_\mu(x) - \partial_\mu \lambda(x) , \quad 0 = \partial^\mu A''_\mu = \partial_\mu A'^\mu - \Box \lambda \implies \Box \lambda = 0$$

Since λ is an arbitrary superposition of plane waves, one may choose

$$A''_3 = A'_3 - \partial_3 \lambda = 0 \implies \partial_3 \lambda = A'_3$$

There are two ways to perform canonical quantization.

1. One eliminates the non-physical degrees of freedom and quantizes only the physical ones. This method is not Lorentz-invariant because one fixes a component, $A_3 = 0$.
2. The other possibility follows the approach by Gupta–Bleuler[19] in quantizing all four degrees of freedom and removing the non-physical ones at the end. This approach produces negative-norm states that can be eliminated from the spectrum of physical states by defining the physical states as
$$(\partial_\mu A^\mu)^{(+)} |\text{phys}\rangle = 0$$
where the superscript $(+)$ denotes the positive frequencies of the gauge field, those corresponding to the annihilation operators, $a_r(\mathbf{k})$.

9 Functional quantization

See Cheng, §9. The method of functional quantization is an alternative to canonical quantization. The LSZ reduction formula can also be formulated for vector fields. The scattering amplitudes are directly related to the computation of correlation functions.

For a pure gauge theory (i.e. no matter fields), one would like to construct a generating functional that provides all the possible correlation functions

$$W[J] = \int [\mathcal{D}A_\mu] \exp\left[\mathrm{i} \int \mathrm{d}^4 x \, (\mathcal{L}_\mathrm{g} + J^a_\mu A^{a\mu})\right]$$

The Green's functions are

$$\langle 0| \mathcal{T}\{A^{a_1}_{\mu_1}(x) \cdots A^{a_n}_{\mu_n}(x_n)\} |0\rangle = (-\mathrm{i})^n \frac{\delta^n W[J]}{\delta J_{a_1\mu_1}(x_1) \cdots \delta J_{a_n\mu_n}(x_n)}\bigg|_{J=0}$$

9.1 Free theory

The free Lagrangian (which does not contain the self-interactions) is

$$\mathcal{L}_0 = -\frac{1}{2} \partial_\mu A^a_\nu (\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu})$$

The associated generating functional is

$$W_0[J] = \int [\mathcal{D}A_\mu] \exp\left[\mathrm{i} \int \mathrm{d}^4 x \, (\mathcal{L}_0 + J^a_\mu A^\mu_a)\right]$$

[19] See Theoretical Physics II for a covariant quantization of the electromagnetic field.

[r] The free action is

$$S_0 = -\frac{1}{2}\int \mathrm{d}^4x\,\partial_\mu A^a_\nu(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) = \frac{1}{2}\int \mathrm{d}^4x\,A^a_\mu(\eta^{\mu\nu}\,\Box - \partial^\mu\partial^\nu)A^a_\nu = \frac{1}{2}\int \mathrm{d}^4x\,AKA$$

The differential operator is the kinetic operator

$$K^{\mu\nu} \equiv \eta^{\mu\nu}\,\Box - \partial^\mu\partial^\nu$$

This operator is not invertible. If it could be inverted, one would complete the square in the exponent of the generating functional

$$\int \mathrm{d}^4x\,(AKA + 2JA) = \int \mathrm{d}^4x\,[(A+JK^{-1})K(A+K^{-1}J) - JK^{-1}J] = \int \mathrm{d}^4x\,(\tilde{A}K\tilde{A} - JK^{-1}J)$$

therefore the generating functional would be

$$\begin{aligned} W_0[J] &= \exp\left[-\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\,JK^{-1}J\right]\int[\mathcal{D}\tilde{A}_\mu]\,\exp\left[\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\,\tilde{A}K\tilde{A}\right] \\ &\propto \frac{1}{\sqrt{\det K}}\exp\left[-\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\,JK^{-1}J\right] \end{aligned}$$

However, the inverse of the kinetic operator does not exist. Consider

$$K_{\mu\nu}K^\nu{}_\rho = (\eta_{\mu\nu}\,\Box - \partial_\mu\partial_\nu)(\delta^\nu_\rho\,\Box - \partial^\nu\partial_\rho) = \eta_{\mu\rho}\,\Box^2 - \Box\,\partial_\mu\partial_\rho = \Box(\eta_{\mu\rho}\,\Box - \partial_\mu\partial_\rho) = \Box\,K_{\mu\rho}$$

From this one sees that the operator K behaves like a projection operator and as such is not invertible (because it is not injective).

One looks for the Green's function G of the kinetic operator

$$K_{\mu\nu}G^{\nu\rho} = \delta^\rho_\mu \iff (\eta_{\mu\nu}\,\Box - \partial_\mu\partial_\nu)G^{\nu\rho}(x-y) = \delta^\rho_\mu\delta^{(4)}(x-y)$$

By Fourier transformation, one has

$$G^{\nu\rho}(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{\mathrm{i}k(x-y)}\widetilde{G}^{\nu\rho}(k)$$

from which the equation above becomes

$$(-k^2\eta_{\mu\nu} + k_\mu k_\nu)\widetilde{G}^{\nu\rho}(k) = \delta^\rho_\mu$$

The most general type-(2,0) tensor field, function of the momentum k has the form

$$\widetilde{G}^{\nu\rho}(k) = A(k^2)\eta^{\mu\nu} + B(k^2)k^\nu k^\rho$$

Inserting this expression above, one finds no possible functions A and B.

Remark. Since the kinetic term is not invertible, then at least one eigenvalue is zero. The operator has a zero mode. Its determinant is zero and the generating functional is ill-defined. This is because the functional integral goes over all configurations of the gauge field. Recalling that the field is gauge invariant, the exponential in the generating functional is the same for each field connected by a gauge transformation. One is summing the same phase.

Remark. If one were to fix the gauge, one could modify the kinetic term to obtain an invertible operator.

Lecture 12

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9.1.1 Faddeev–Popov prescription

All the field configurations linked by a gauge transformation constitute a gauge orbit. An orbit has infinite volume which makes the generating functional ill-defined. The problem can be solved by employing the Faddeev–Popov prescription. One rewrites the generating functional as

$$W[J] \sim \mathrm{e}^{\mathrm{i}S} \times \text{(volume gauge orbit)}$$

and removes the infinite volume of the gauge orbits. One declares this as the well-defined prescription.

Example. An illustrative example with ordinary integrals is the following. Consider a Lebesgue bidimensional integral

$$W = \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{\mathrm{i}S(x,y)}$$

Going to polar coordinates (r, θ)

$$x = r\cos\theta\,, \quad y = r\sin\theta\,, \quad \mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$$

one has

$$W = \int r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{e}^{\mathrm{i}S(r,\theta)}$$

To each fixed radius corresponds a circle. The prescription is read as integrating along a circle and then integrating over all circles. Let S be a function invariant under rotation $\theta \to \theta+\varphi$. This invariance is the analogue of gauge invariance in QFT. This implies that the function depends only on the radius

$$S(r,\theta) = S(r)$$

The integral is

$$W = \int r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{e}^{\mathrm{i}S(r)} = 2\pi \int r\,\mathrm{d}r\,\mathrm{e}^{\mathrm{i}S(r)}$$

The factor 2π is the volume of the circle, which corresponds to the volume of the orbit. In this case, the volume of the orbit is finite, but in QFT it is infinite.

One may compute the integral in an alternative way. One inserts unity into the initial integral

$$1 = \int_0^{2\pi} \mathrm{d}\varphi\,\delta(\theta - \varphi)$$

therefore

$$W = \int_0^{2\pi} \mathrm{d}\varphi \int r\,\mathrm{d}r\,\mathrm{d}\theta\,\delta(\theta-\varphi)\mathrm{e}^{\mathrm{i}S(r)} \equiv \int_0^{2\pi} \mathrm{d}\varphi\,W_\varphi$$

This corresponds to fixing an angle $\theta = \varphi$, integrating over the radius and then sum over all angles. The integral W_φ is invariant under a rotation by an angle φ (because the integrand is invariant). Therefore W_φ does not depend on such angle φ. Thus, the whole integral may be written as

$$W = \int_0^{2\pi} \mathrm{d}\varphi\,W_\varphi = W_\varphi \int_0^{2\pi} \mathrm{d}\varphi = 2\pi W_\varphi$$

The φ-integral is the integral on the orbit and the original integral W can be normalized by the volume

$$\frac{W}{\text{volume orbit}} = W_\varphi$$

and one considers this renormalized function.

The insertion of the delta function means integrating first over the radius and then over the angle. One can generalize the approach by integrating first over an arbitrary line (i.e. not straight) that intersects each orbit only once and then integrate over the angle. This line is given by a gauge function $g(r,\theta)$ and is such that, for any r,

$$g(r, \theta+\varphi) = 0$$

has a unique solution. One generalizes unity as

$$1 = \Delta_g(\mathbf{r})\Delta_g^{-1}(\mathbf{r})\,, \quad \mathbf{r}_\varphi = (r, \theta+\varphi)$$

where one has

$$\Delta_g^{-1}(\mathbf{r}) = \int_0^{2\pi} \mathrm{d}\varphi\,\delta(g(\mathbf{r}_\varphi)) = \int \mathrm{d}g\,\partial_g\varphi\,\delta(g) = \partial_g\varphi|_{g=0} \implies \Delta_g(\mathbf{r}) = \partial_\varphi g|_{g=0}$$

A property of this object is gauge invariance under rotations

$$\Delta_g^{-1}(\mathbf{r}_{\varphi'}) = \int_0^{2\pi} \mathrm{d}\varphi\,\delta(g(\mathbf{r}_{\varphi+\varphi'})) = \int_0^{2\pi} \mathrm{d}(\varphi+\varphi')\,\delta(g(\mathbf{r}_{\varphi'+\varphi})) = \int_0^{2\pi} \mathrm{d}\varphi''\,\delta(g(\mathbf{r}_{\varphi''})) = \Delta_g^{-1}(\mathbf{r})$$

where one sets $\varphi'' = \varphi + \varphi'$.

Inserting this tool inside the integral W, one finds

$$\begin{aligned}
W &= \int r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{e}^{\mathrm{i}S(r)} = \int r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{e}^{\mathrm{i}S(r)}\Delta_g(\mathbf{r})\Delta_g^{-1}(\mathbf{r}) = \\
&= \int_0^{2\pi} \mathrm{d}\varphi \int r\,\mathrm{d}r\,\mathrm{d}\theta\,\Delta_g(\mathbf{r})\delta(g(\mathbf{r}_\varphi))\mathrm{e}^{\mathrm{i}S(r)} \\
&= \int_0^{2\pi} \mathrm{d}\varphi \int r\,\mathrm{d}r\,\mathrm{d}(\theta+\varphi)\,\Delta_g(\mathbf{r}_\varphi)\delta(g(\mathbf{r}_\varphi))\mathrm{e}^{\mathrm{i}S(\mathbf{r}_\varphi)} \\
&= \int_0^{2\pi} \mathrm{d}\varphi \int r\,\mathrm{d}r\,\mathrm{d}\theta'\,\Delta_g(\mathbf{r}')\delta(g(\mathbf{r}'))\mathrm{e}^{\mathrm{i}S(\mathbf{r}')} \\
&\equiv \int_0^{2\pi} \mathrm{d}\varphi\, W_\varphi
\end{aligned}$$

At the third line, one has applied

$$\Delta_g(\mathbf{r}) = \Delta(\mathbf{r}_\varphi)\,, \quad S(r) = S(\mathbf{r}) = S(\mathbf{r}_\varphi)\,, \quad r\,\mathrm{d}r\,\mathrm{d}\theta = r\,\mathrm{d}r\,\mathrm{d}(\theta+\varphi)$$

At the fourth line one renames

$$\mathbf{r}_\varphi = (r, \theta+\varphi) \to \mathbf{r}' = (r, \theta')$$

The quantity W_φ is independent of the angle φ (due to the integral over θ'?), therefore one can normalize the integral and consider this normalized quantity

$$W \to \frac{W}{\int_0^{2\pi} \mathrm{d}\varphi} = W_\varphi = \int r\,\mathrm{d}r\,\mathrm{d}\theta\,\Delta_g(\mathbf{r})\delta(g(\mathbf{r}))\mathrm{e}^{\mathrm{i}S(r)}$$

The reason why one is allowed to normalize by the orbit volume is because the path integral of the generating functional counts infinitely many times the same phase. The normalization removes this over-counting.

Functional quantization. The discussion proceeds by analogy to the previous example. Consider gauge invariance under finite gauge transformations

$$A_\mu^\theta = U(\theta)A_\mu U^{-1}(\theta) - \frac{\mathrm{i}}{g}[\partial_\mu U(\theta)]U^{-1}(\theta)\,, \quad U(\theta) = \mathrm{e}^{-\mathrm{i}\theta^a T^a}$$

The generating functional with no source is

$$W = \int [\mathcal{D}A_\mu]\,\mathrm{e}^{\mathrm{i}S_\mathrm{g}(A_\mu)}\,, \quad S_\mathrm{g}(A_\mu) = S_\mathrm{g}(A_\mu^\theta)\,, \quad \forall\theta$$

where the index g stands for "gauge". Notice that the action is gauge-invariant.

One chooses a set of $\dim G$ gauge functions and imposes the gauge-fixing condition

$$f_a(A_\mu) = 0$$

These functions are such that, for each A_μ,

$$f_a(A_\mu^\theta) = 0$$

has a unique solution $\theta(x)$. The gauge functions intersect each orbit only once.

The operator providing unity is

$$\Delta_f^{-1}[A_\mu] = \int [\mathrm{d}\theta(x)] \prod_a \delta(f_a(A_\mu^\theta(x)))\,, \quad [\mathrm{d}\theta(x)] = \prod_a \mathrm{d}\theta_a(x)$$

[r] One requires the measure $[\mathrm{d}\theta(x)]$ to respect the group's composition law

$$U(\theta)U(\theta') = U(\theta\theta') \implies [\mathrm{d}\theta] = [\mathrm{d}(\theta\theta')]$$

[r] Therefore

$$\Delta_f^{-1}[A_\mu] = \int \prod_a [\mathrm{d}f_a] |\det \partial_{f_a}\theta_b| \prod_a \delta(f_a) \implies \Delta_f^{-1} = \det M_{ab}^{-1}$$

from which

$$\Delta_f = \left|\det \partial_{\theta_b} f_a(A_\mu^\theta)\right|_{f_a=0} = \det M_{ab}$$

An important property is gauge invariance

$$\Delta_f[A_\mu] = \Delta_f[A_\mu^\theta]\,, \quad \forall\theta$$

This can be checked as follows

$$\begin{aligned}\Delta_f^{-1}[A_\mu^\theta] &= \int [\mathrm{d}\theta'(x)] \prod_a \delta(f_a(A_\mu^{\theta\theta'})) = \int [\mathrm{d}\theta\theta'(x)] \prod_a \delta(f_a(A_\mu^{\theta\theta'})) \\ &= \int [\mathrm{d}\theta''(x)] \prod_a \delta(f_a(A_\mu^{\theta''})) = \Delta_f^{-1}[A_\mu]\end{aligned}$$

where $\mathrm{d}\theta' = \mathrm{d}(\theta\theta')$ and $\theta'' = \theta\theta'$. Therefore, the generating functional is

$$\begin{aligned}W &= \int [\mathcal{D}A_\mu]\, \mathrm{e}^{\mathrm{i}S[A_\mu]} \Delta_f[A_\mu] \int [\mathrm{d}\theta(x)] \prod_a \delta(f_a(A_\mu^\theta)) \\ &= \int [\mathrm{d}\theta(x)] \int [\mathcal{D}A_\mu]\, \mathrm{e}^{\mathrm{i}S[A_\mu]} \Delta_f[A_\mu] \prod_a \delta(f_a(A_\mu^\theta(x))) \\ &= \int [\mathrm{d}\theta(x)] \int [\mathcal{D}A_\mu^\theta]\, \mathrm{e}^{\mathrm{i}S[A_\mu^\theta]} \Delta_f[A_\mu^\theta] \prod_a \delta(f_a(A_\mu^\theta(x))) \\ &= \int [\mathrm{d}\theta(x)] \int [\mathcal{D}A'_\mu]\, \mathrm{e}^{\mathrm{i}S[A'_\mu]} \Delta_f[A'_\mu] \prod_a \delta(f_a(A'(x))) \\ &= \infty \times (\text{smth})\end{aligned}$$

at the third line, one notices that the action is gauge invariant [r]. At the fourth line, one has renamed the field $A_\mu^\theta = A'_\mu$.

One notices that the second integral does not depend on the parameter θ and the first integral is the infinite volume of the gauge orbits. One normalizes the generating functional by the orbits' volume and considers the result as the well-defined generating functional for gauge theories

$$\boxed{W_f = \int [\mathcal{D}A_\mu]\, \Delta_f[A_\mu] \prod_a \delta(f_a(A(x)))\mathrm{e}^{\mathrm{i}S[A_\mu]} = \int [\mathcal{D}A_\mu]\, (\det M_f) \prod_a \delta(f_a(A(x)))\mathrm{e}^{\mathrm{i}S[A_\mu]}}$$

where

$$(M_f)_{ab} = \left.\frac{\delta f_a(A_\mu^\theta)}{\delta\theta_b}\right|_{f_a=0}$$

[r] where absolute value?
This is the Faddeev–Popov prescription. One is not integrating over every field configuration due to the presence of the delta function: one is summing only one single representative of each orbit.

Determinant. One may rewrite the coefficients of the exponential as phases. Recall

$$\int \left[\prod_{j=n}^{1} \mathrm{d}\theta_j\, \mathrm{d}\bar\theta_j\right] \mathrm{e}^{\mathrm{i}\bar\theta_a M_{ab}\theta_b} \propto \det M$$

One introduces a set of Grassmann-odd fields

$$c_a(x)\,, \quad \bar c_a(x) = c_a^\dagger(x)\,, \quad a = 1, \dots, \dim G$$

so the determinant is

$$\det M_f \propto \int [\mathrm{d}c\, \mathrm{d}\bar c] \exp\left[\mathrm{i} \int \mathrm{d}^4x\, \mathrm{d}^4y\, \bar c_a(x)[M_f(x,y)]_{ab} c_b(y)\right]$$

These new fields are non-physical degrees of freedom called Faddeev–Popov ghosts.

Delta function. One generalizes the gauge-fixing condition from $f_a(A_\mu) = 0$ to

$$f_a(A_\mu(x)) = B_a(x)$$

where $B_a(x)$ is a smooth function independent of the gauge field A_μ. The delta function is then

$$\delta(f_a(A_\mu) - B_a)$$

Considering that the original generating functional W is independent of B_a due to the dependence being inside $\Delta\Delta^{-1}$, one may change the normalization by a constant

$$\text{const} \equiv \int [\mathcal{D}B_a] \exp\left[-\frac{\mathrm{i}}{2\xi}\int \mathrm{d}^4x\, B_a(x)B_a(x)\right]$$

Therefore

$$\begin{aligned}
W[J] &= \int [\mathcal{D}B_a]\,[\mathcal{D}A_\mu]\,(\det M_f)\prod_a \delta(f_a(A_\mu) - B_a)\exp\left[-\frac{\mathrm{i}}{2\xi}\int \mathrm{d}^4x\, B_a(x)B_a(x)\right]\mathrm{e}^{\mathrm{i}S_\mathrm{g}[A_\mu]} \\
&= \int [\mathcal{D}A_\mu]\,(\det M_f)\exp\left[-\frac{\mathrm{i}}{2\xi}\int \mathrm{d}^4x\,[f_a(A_\mu)]^2\right]\mathrm{e}^{\mathrm{i}S_\mathrm{g}[A_\mu]} \\
&= \int [\mathcal{D}A_\mu\,\mathcal{D}c_a\,\mathcal{D}\bar{c}_a]\,\mathrm{e}^{\mathrm{i}S_\mathrm{g}[A_\mu]} \\
&\quad\times \exp\left[-\frac{\mathrm{i}}{2\xi}\int \mathrm{d}^4x\,[f_a(A_\mu)]^2 + \mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{c}_a(x)(M_f)_{ab}(x,y)c_b(y)\right]
\end{aligned}$$

At the second line, one integrates using the delta function. From the third line, one sees that ghosts are propagating fields, but just quantum [r].

The total action is therefore

$$S = S_\text{gauge} + S_\text{gauge-fixing} + S_\text{ghost}$$

where one has

$$\begin{aligned}
S_\mathrm{g}(A^\mu) &= -\frac{1}{4}\int \mathrm{d}^4x\ \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \\
S_\mathrm{gf}(A^\mu) &= -\frac{1}{2\xi}\int \mathrm{d}^4x\,[f_a(A^\mu)]^2 \\
S_\mathrm{gh}(A^\mu, c, \bar{c}) &= \int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{c}_a(x)(M_f)_{ab}(x,y)c_b(y)
\end{aligned}$$

with ξ an arbitrary gauge parameter and

$$f_a(A^\mu) = B_a(x)\,,\quad (M_f)_{ab}(x,y) = \left.\frac{\delta f_a(A^\theta_\mu(x))}{\delta\theta_b(y)}\right|_{f^a=0}$$

Remark. The total action is not gauge-invariant due to the presence of the gauge-fixing term.

Lecture 13

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Example (Axial gauge)**.** Consider the axial gauge, also called Arnowitt–Fickler gauge[20],

$$f_a(A^\mu) = A^3_a$$

An infinitesimal gauge transformation by $\theta_a(x)$ is

$$f^a(A^\theta_\mu) = A^a_3 + c^{abc}\theta^b A^c_3 - \frac{1}{g}\,\partial_3\theta^a \implies f_a(A^\theta_\mu) = -\frac{1}{g}\,\partial_3\theta_a$$

[20] Generally it is written as $n^\mu A_\mu$ with n^μ a unit four-vector specifying the direction.

[r] where one applies the gauge-fixing condition $f^a = 0$. The matrix M is

$$M_{ab}(x,y) = \left.\frac{\delta f_a(A_\mu(x))}{\delta\theta_b(y)}\right|_{f_a=0} = -\frac{1}{g}\,\partial_3[\delta^{(4)}(x-y)\,\delta_{ab}]$$

Notice how it does not depend on the gauge field A^a_μ. This implies that the ghost Lagrangian depends only on the ghosts: the gauge field decouples from the ghosts. The functional integral of the ghosts in the generating functional can be performed and absorbed into the normalization constant, so that the ghosts do not appear anymore.

From this follows that the ghost action describes non-propagating degrees of freedom[21]

$$\begin{aligned} S_{\text{gh}} &= \int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{c}_a(x)(M_f)_{ab}(x,y)c_b(y) = -\frac{1}{g}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{c}_a\,\partial_3[\delta^{(4)}(x-y)\delta_{ab}]c_b(y) \\ &= \frac{1}{g}\int \mathrm{d}^4x\,\partial_3\bar{c}_a(x)\,c_a(x) \end{aligned}$$

where one applies the properties of the Dirac delta function under differentiation noting that ∂_3 is done with respect to x. Since the ghost fields are non-propagating, they can be ignored. In the axial gauge there are no ghosts. This result may be generalized: a gauge-fixing condition that is linear in the gauge field implies that M_{ab} is independent of the field and the ghosts decouple from the it.

Example (Abelian theory)**.** Consider an abelian theory. The gauge transformation is

$$A^\theta_\mu = A_\mu - \frac{1}{g}\,\partial_\mu\theta$$

Taking a gauge-fixing condition linear in the gauge field permits to decouple the ghosts. This is why QED can be a theory without ghosts. An example of non-linear condition is

$$A_\mu A^\mu = 0$$

Example (Abelian Lorenz gauge)**.** Consider the Lorenz gauge

$$f(A_\mu) = \partial_\mu A^\mu = 0$$

The gauge transformation is

$$f_a(A^\theta_\mu) = \partial_\mu A^\mu_\theta = \partial_\mu A^\mu - \frac{1}{g}\,\Box\,\theta = -\frac{1}{g}\,\Box\,\theta$$

The matrix is

$$M(x,y) = -\frac{1}{g}\partial_{\theta(x)}[\Box\,\theta(y)] = -\frac{1}{g}\,\Box[\delta^{(4)}(x-y)]$$

The ghost action is

$$S_{\text{gh}} = -\frac{1}{g}\int \mathrm{d}^4x\,\bar{c}(x)\,\Box\,c(x)$$

This is a free action, describing propagating but non-interacting fields.

Example (Non-abelian Lorenz gauge)**.** Consider the Lorenz gauge like before

$$f_a(A_\mu) = \partial_\mu A^\mu_a = 0$$

The gauge function for the transformed field is

$$\begin{aligned} f^a(A^\theta_\mu) &= \partial_\mu\left[A^{a\mu} + c^{abc}\theta^b A^{c\mu} - \frac{1}{g}\,\partial^\mu\theta^a\right] = -\frac{1}{g}\,\partial_\mu(\partial^\mu\theta^a - gc^{abc}\theta^b A^{c\mu}) \\ &= -\frac{1}{g}\,\partial_\mu(\partial^\mu\theta^a + gc^{abc}A^{b\mu}\theta^c) = -\frac{1}{g}\,\partial_\mu(D^\mu\theta)^a \end{aligned}$$

[21] Notice how there is only one derivative. The equations of motion imply no wave equation: $\partial_3\bar{c} = \partial_3 c = 0$. Any function independent of x^3 is a solution, and there is no sense that the past determines future.

where the covariant derivative D_μ acts on a field $\theta_a(x)$ in the adjoint representation[22]. In fact, recalling that

$$D_\mu = \partial_\mu - \mathrm{i}gA^a_\mu T^a$$

the covariant derivative acts on the adjoint representation of a field φ as

$$[D_\mu, \varphi] = [\partial_\mu - \mathrm{i}gA^a_\mu T^a, \varphi^b T^b] = (\partial_\mu \theta^a + gc^{abc} A^b_\mu \theta^c) T^a$$

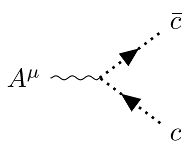
One obtains

$$M_{ab}(x,y) = -\frac{1}{g}\,\partial_{\theta^b(y)}[\partial_\mu(D^\mu\theta(x))_a]|_{f_a=0} = -\frac{1}{g}\,\partial^\mu (D_\mu)_{ab}\delta^{(4)}(x-y)$$

The ghost action is

$$\begin{aligned} S_{\mathrm{gh}} &= -\frac{1}{g}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{c}_a(x)[\partial^\mu D^{ab}_\mu \delta^{(4)}(x-y)]c_b(y) = \frac{1}{g}\int \mathrm{d}^4x\,\partial^\mu \bar{c}_a(x)\,D^{ab}_\mu\,c_b(x) \\ &= \frac{1}{g}\int \mathrm{d}^4x\,[\partial^\mu \bar{c}_a(x)\,\partial_\mu c_a(x) + gc^{acb}\,\partial_\mu \bar{c}_a(x) A^{c\mu}(x) c^b(x)] \end{aligned}$$

where one integrates by parts. The second addendum is present only for non-abelian theories: it is a three-point vertex between the gauge field and two ghosts:



The complete action is then

$$S = S_{\mathrm{g}} + S_{\mathrm{gf}} + S_{\mathrm{gh}} = S_0 + S_{\mathrm{I}}$$

where the free and interacting parts are

$$\begin{aligned} S_0 &= \int \mathrm{d}^4x\,\left[-\frac{1}{2}\,\partial_\mu A^a_\nu(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 - \bar{c}_a\,\Box\,c_a\right] \\ S_{\mathrm{I}} &= \int \mathrm{d}^4x\,\left[-c^{abc}\,\partial_\mu A^a_\nu\,A^{b\mu}A^{c\nu} - \frac{g^2}{4}c^{abc}c^{ade}A^b_\mu A^c_\nu A^{d\mu}A^{e\nu} - gc^{abc}\,(\partial_\mu \bar{c}^a)c^b A^{c\mu}\right] \end{aligned}$$

where the fields c and $\bar{c}$ have been redefined to

$$c_a \to \sqrt{g}c_a$$

The interacting part includes a three-point and a four-point self-coupling vertices and a three-point ghost-field vertex.

9.2 Perturbation theory

[r] The kinetic term found using ghosts is invertible. The generating functional is

$$W[J,\eta,\bar{\eta}] = \int [\mathcal{D}A_\mu\,\mathcal{D}c\,\mathcal{D}\bar{c}]\,\exp\left[\mathrm{i}S + \mathrm{i}\int \mathrm{d}^4x\,(J^a_\mu A^\mu_a + \bar{\eta}^a c^a + \bar{c}^a \eta^a)\right]$$

As with scalar and spinor fields, one may replace

$$A^a_\mu(x) \to -\mathrm{i}\,\frac{\delta}{\delta J^\mu_a(x)}\,, \quad c^a \to -\mathrm{i}\,\frac{\delta}{\delta \bar{\eta}^a}\,, \quad \bar{c}^a \to \mathrm{i}\,\frac{\delta}{\delta \eta^a}$$

to have

$$W[J,\eta,\bar{\eta}] = \exp\left[\mathrm{i}S_{\mathrm{I}}(-\mathrm{i}\,\partial_J, -\mathrm{i}\,\delta_{\bar{\eta}}, \mathrm{i}\,\delta_\eta)\right] W_0[J,\eta,\bar{\eta}]$$

[22] For the meaning of a field to be in a representation, see `https://physics.stackexchange.com/q/412232`.

noting that the free generating functionals split

$$W_0[J,\eta,\bar\eta] = W_0[J]W_0[\eta,\bar\eta]$$

with

$$\begin{aligned}
W_0[J] &= \int [\mathcal{D}A_\mu]\, \exp\left[\mathrm{i}(S_\mathrm{g}^{(0)} + S_\mathrm{gf}) + \mathrm{i}\int \mathrm{d}^4x\, J_a^\mu A_\mu^a\right] \\
W_0[\eta,\bar\eta] &= \int [\mathcal{D}c\,\mathcal{D}\bar c]\, \exp\left[\mathrm{i}S_\mathrm{gh}^{(0)} + \mathrm{i}\int \mathrm{d}^4x\,(\bar\eta c + \bar c\eta)\right]
\end{aligned}$$

where the superscript (0) indicates the free part.

Gauge field propagator. Consider the free action of the gauge boson field

$$\begin{aligned}
S_\mathrm{g}^{(0)} + S_\mathrm{gf} &= \int \mathrm{d}^4x\, \left[-\frac{1}{2}\,\partial_\mu A_\nu^a(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) - \frac{1}{2\xi}(\partial_\mu A^{a\mu})(\partial_\nu A^{a\nu})\right] \\
&= \int \mathrm{d}^4x\, \frac{1}{2} A_\nu^a \left[\eta^{\mu\nu}\,\Box - \left(1 - \frac{1}{\xi}\right)\partial^\mu\,\partial^\nu\right] A_\mu^a \\
&= \int \mathrm{d}^4x\,\mathrm{d}^4y\, \frac{1}{2} A_\nu^a(x) \left[\eta^{\mu\nu}\,\Box - \left(1 - \frac{1}{\xi}\right)\partial^\mu\,\partial^\nu\right] \delta^{(4)}(x-y)\delta_b^a A_\mu^b(y) \\
&= \int \mathrm{d}^4x\,\mathrm{d}^4y\, \frac{1}{2} A_\nu^a(x) K_{ab}^{\mu\nu}(x,y) A_\mu^b(y)
\end{aligned}$$

At the second line one has integrated by parts.

The kinetic term is an invertible operator

$$K_{ab}^{\mu\nu}(x-y) = \left[\eta^{\mu\nu}\,\Box - \left(1 - \frac{1}{\xi}\right)\partial^\mu\,\partial^\nu\right]\delta^{(4)}(x-y)\delta_{ab}$$

[r] In fact, its Green's function $G_{\mu\nu}(x-y)$ that satisfies

$$K^{\mu\nu}(x-y)G_{\nu\rho}(y-z) = \delta_\rho^\mu \delta^{(4)}(x-z)$$

Applying the Fourier transform

$$G_{\nu\rho}(y-z) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \mathrm{e}^{\mathrm{i}k(y-z)} G_{\nu\rho}(k)$$

gives

$$\left[-k^2\eta^{\mu\nu} + \left(1 - \frac{1}{\xi}\right)k^\mu k^\nu\right] G_{\nu\rho}(k) = \delta_\rho^\mu$$

which has solution

$$G_{\nu\rho}^{ab}(k) = -\left[\eta_{\nu\rho} - (1-\xi)\frac{k_\nu k_\rho}{k^2}\right]\frac{\delta^{ab}}{k^2 + \mathrm{i}\varepsilon}$$

Completing the square of the generating functional $W_0[J]$ gives

$$W_0[J] = \exp\left[-\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\, J_\mu^a(x) G_{ab}^{\mu\nu}(x,y) G_\nu^b(y)\right]$$

The two-point Green's function is

$$G_{ab}^{\mu\nu}(x-y) = \langle 0|\,\mathcal{T}\{A_a^\mu(x)A_b^\nu(y)\}\,|0\rangle = (-\mathrm{i})^2 \left.\frac{\delta^2 W_0[J]}{\delta J_\mu^a(x)\delta J_\nu^b(y)}\right|_{J=0} = \mathrm{i}\Delta_{ab}^{\mu\nu}(x-y)$$

and in Fourier space it is

$$G_{\mu\nu}^{ab}(k) = \langle 0|\,\mathcal{T}\{A_\mu^a(k)A_\nu^b(-k)\}\,|0\rangle = -\mathrm{i}\left[\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]\frac{\delta^{ab}}{k^2 + \mathrm{i}\varepsilon} = \mathrm{i}\Delta_{\mu\nu}^{ab}(k)$$

Ghost propagator. The free generating functional is

$$W_0[\eta,\bar\eta] = \int [\mathcal{D}c\,\mathcal{D}\bar c]\,\exp\left[-\mathrm{i}\int \mathrm{d}^4x\,[\bar c_a\,\Box\,c_a - \bar\eta_a c_a - \bar c_a\eta_a\right]$$

Knowing that the kinetic term in momentum space is

$$\delta_{ab}\,\Box \implies -\delta_{ab}k^2$$

one obtains

$$G_{ab}(x-y) = -\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}k(x-y)}}{k^2+\mathrm{i}\varepsilon}\delta_{ab}$$

[r] Completing the quare in the exponent gives

$$W_0[\eta,\bar\eta] = \exp\left[-\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar\eta_a G_{ab}(x-y)\eta_b(y)\right]$$

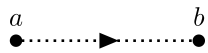
The two-point Green's function is

$$G^{ab}(x-y) = \langle 0|\,\mathcal{T}\{\bar c^a(x)c^b(y)\}\,|0\rangle = \left.\frac{\delta^2 W_0[\eta,\bar\eta]}{\delta\bar\eta_a(x)\delta\eta_b(y)}\right|_{\eta=\bar\eta=0} = \mathrm{i}\Delta^{ab}(x-y)$$

and in momentum space, one has

$$G_{ab}(k) = -\frac{\mathrm{i}\delta_{ab}}{k^2+\mathrm{i}\varepsilon} = \mathrm{i}\Delta_{ab}(k)$$

which corresponds to the line



10 Becchi–Rouet–Stora–Tyutin quantization

[r] See Cheng, §9.3. Adding a gauge-fixing Lagrangian implies that the action is no longer gauge-invariant. One would like to find another symmetry (and the corresponding transformations) for the gauge-fixed action. This symmetry is the Becchi–Rouet–Stora–Tyutin (BRST) symmetry.

Consider a gauge field minimally coupled to some spinor fields also called matter fields. The most general Yang–Mills Lagrangian (gauge group $\mathrm{SU}(n)$) is

$$\mathcal{L} = -\frac{1}{4}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \bar\psi(\mathrm{i}\,\not{D} - m)\psi - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 + \bar c_a\,\partial^\mu(D_\mu)_{ab}c_b$$

where

$$F^a_{\mu\nu} = \partial_{[\mu}A^a_{\nu]} + gc^{abc}A^b_\mu A^c_\nu\,,\quad D_\mu\psi = (\partial_\mu - \mathrm{i}gA^a_\mu T^a)\psi\,,\quad (D_\mu)_{ab}c_b = \partial_\mu c_a + gc^{acb}A^c_\mu c^b$$

It is convenient to express the complex Grassmann-odd ghost fields as real Grassmann-odd fields

$$c_a = \frac{1}{\sqrt{2}}(\rho_a + \mathrm{i}\sigma_a)\,,\quad \bar c_a = \frac{1}{\sqrt{2}}(\rho_a - \mathrm{i}\sigma_a)$$

This implies that the ghost Lagrangian is

$$\mathcal{L}_{\mathrm{gh}} = -\mathrm{i}\,\partial^\mu\rho^a\,(D_\mu)_{ab}\sigma^b$$

which can be obtained noting that

$$\rho^a\,\partial^\mu(D_\mu)_{ab}\rho^b = \sigma^a\,\partial^\mu(D_\mu)_{ab}\sigma^b = 0$$

Proposition. The action is invariant under the BRST transformation

$$\begin{aligned}\delta A^a_\mu &= \omega D^{ab}_\mu \sigma_b\\ \delta\rho^a &= -\frac{\mathrm{i}}{\xi}\omega\,\partial^\mu A^a_\mu\\ \delta\sigma^a &= -\frac{g}{2}\omega c^{abc}\sigma^b\sigma^c\\ \delta\psi &= \mathrm{i}g\omega(T^a\sigma^a)\psi\end{aligned}$$

where ω is a constant Grassmann-odd number and ξ is the gauge parameter.

In general, infinitesimal transformations can be written as the commutator of a parameter ε and the generator Q of a symmetry

$$\delta\chi = [\varepsilon Q, \chi]$$

In this case, the parameter $\varepsilon = \omega$ and the generator are Grassmann-odd quantities. This also implies $Q^2 = 0$ and that applying twice the transformations gives zero.

Proof. The action is

$$S = (S_\mathrm{g} + S_\psi) + S_\mathrm{gf} + S_\mathrm{gh}$$

The first two addenda are gauge-invariant. The transformations of the gauge field and the matter fields are the typical (infinitesimal) gauge transformations with parameter $\theta^a = -g\omega\sigma^a$. Thus one has

$$\delta_\mathrm{BRST}(S_\mathrm{g} + S_\psi) = 0$$

The gauge-fixing Lagrangian transforms as

$$\delta L_\mathrm{gf} = -\frac{1}{2\xi}\,\delta(\partial_\mu A^{a\mu})^2 = -\frac{1}{\xi}\,\partial_\mu A^{a\mu}\,\partial_\nu(\delta A^{a\nu}) = -\frac{1}{\xi}\,\partial_\mu A^{a\mu}\,\partial_\nu[\omega(D^\nu)^{ab}\sigma^b]$$

The ghost Lagrangian transforms as

$$\begin{aligned}\delta\mathcal{L}_\mathrm{gh} &= \delta[-\mathrm{i}\,\partial^\mu\rho_a\,D^{ab}_\mu\sigma_b] = -\mathrm{i}\,\partial^\mu(\delta\rho_a)\,D^{ab}_\mu\sigma_b - \mathrm{i}(\partial^\mu\rho_a)\,\delta[D^{ab}_\mu\sigma_b]\\ &= -\frac{1}{\xi}\omega\,\partial^\mu(\partial^\nu A^a_\nu)\,D^{ab}_\mu\sigma_b - \mathrm{i}(\partial^\mu\rho_a)\,\delta[D^{ab}_\mu\sigma_b]\end{aligned}$$

At the second line, one may integrate by parts the first addendum and see that it cancels the variation of the gauge-fixing Lagrangian.

The variation of the total Lagrangian (up to four-divergences) is

$$\delta\mathcal{L} = -\mathrm{i}(\partial^\mu\rho_a)\,\delta[D^{ab}_\mu\sigma_b]$$

One needs to show that the last variation is zero

$$\begin{aligned}\delta[D^{ab}_\mu\sigma_b] &= \delta[\partial_\mu\sigma^a - gc^{abc}\sigma^b A^c_\mu]\\ &= \partial_\mu(\delta\sigma^a) - gc^{abc}(\delta\sigma^b)A^c_\mu - gc^{abc}\sigma^b(\delta A^c_\mu)\\ &= -\frac{g}{2}\omega c^{abc}\,\partial_\mu(\sigma^b\sigma^c) + gc^{abc}\frac{g}{2}\omega c^{bef}\sigma^e\sigma^f A^c_\mu - gc^{abc}\sigma^b\omega D^{cd}_\mu\sigma^d\\ &= -\frac{g}{2}\omega c^{abc}\,\partial_\mu(\sigma^b\sigma^c) + \frac{g^2}{2}\omega c^{abc}c^{bef}\sigma^e\sigma^f A^c_\mu + g\omega c^{abc}\sigma^b(\partial_\mu\sigma^c - gc^{cdf}\sigma^d A^f_\mu)\\ &= \frac{g^2}{2}\omega c^{abc}c^{bef}\sigma^e\sigma^f A^c_\mu - g^2\omega c^{abc}c^{cef}\sigma^b\,\sigma^e A^f_\mu\\ &= \frac{g^2}{2}\omega c^{abe}c^{bcf}\sigma^c\sigma^f A^e_\mu - g^2\omega c^{acb}c^{bfe}\sigma^c\,\sigma^f A^e_\mu\\ &= \frac{g^2}{2}\omega(c^{abe}c^{bcf} + c^{abc}c^{bfe} - c^{abf}c^{bce})\sigma^c\sigma^f A^e_\mu\\ &= 0\end{aligned}$$

[r] At the fourth line, the last addendum has a positive sign since ω and σ are both Grassmann-odd. At the fifth line, one has considered the two terms without A_μ and has recalled that the

structure constants c^{abc} are totally anti-symmetric. At the sixth line, one renames $c \leftrightarrow e$ in the first addendum, and $c \leftrightarrow b$ and $f \leftrightarrow e$ in the second addendum. At the penultimate line, the third addendum one exchanges the two σs and renames $c \leftrightarrow f$.

The result is zero due to the Jacobi identity. This means that the variation of the Lagrangian is zero under BRST transformations. □

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Remark. The above variation can be written as

$$\delta_{\mathrm{BRST}}(D_\mu\sigma)^a = 0$$

10.1 Slavnov–Taylor identities

The Slavnov–Taylor identities are the generalization of the Ward–Takahashi identity (originally derived in QED) to non-abelian gauge theories.

Review – Ward identity in QED. Quantum electrodynamics is a renormalizable theory. Its ultraviolet divergences can be taken care of using the renormalization functions

$$\psi_0 = Z_2^{\frac{1}{2}}\psi\,,\quad A_0^\mu = Z_3^{\frac{1}{2}}A^\mu\,,\quad q_0 = k^\varepsilon Z_1 Z_2^{-1} Z_3^{-\frac{1}{3}} q\,,\quad m_0 = Z_m Z_2^{-1} m$$

Gauge invariance imposes the use of covariant derivatives

$$D_\mu = \partial_\mu + \mathrm{i}qA_\mu\,,\quad g = |q|$$

where q is the charge (with sign) of the matter field. Assuming that gauge invariance is preserved after quantization implies that the covariant derivative of a matter field $D_\mu\psi$ has to be protected against quantum corrections

$$D_{\mathrm{R}}^\mu \equiv D_{\mathrm{bare}}^\mu \iff (D_\mu\psi)_{\mathrm{R}} \equiv Z_2^{-\frac{1}{2}}(D_\mu\psi)_{\mathrm{bare}}$$

This implies that

$$(qA_\mu)_{\mathrm{R}} = (qA_\mu)_{\mathrm{bare}} \implies (qA_\mu)_{\mathrm{bare}} = q_0 A_{0\mu} = k^\varepsilon \frac{Z_1}{Z_2} qA_\mu \equiv qA_\mu$$

from which one obtains the Ward–Takahashi identity in QED

$$\boxed{Z_1 = Z_2}$$

Non-abelian theories. One assumes that the BRST symmetry is preserved after quantization

$$\delta S = 0 \implies \delta W = 0$$

If the classical action S is invariant, then so is the effective action (equivalently the generating functional W). Since the generating functional contains the classical action, then the phase $e^{\mathrm{i}S}$, is invariant under BRST transformations. The above statement implies that the functional measure must be invariant.

Consider a generating functional containing the sources for the physical fields, the ghost fields and the composite fields that appear in the BRST transformations

$$W[J,\alpha,\beta,\chi,\bar\chi,\kappa,\nu,\lambda,\bar\lambda] = \int [\mathcal{D}A_\mu\,\mathcal{D}\psi\,\mathcal{D}\bar\psi\,\mathcal{D}\rho\,\mathcal{D}\sigma]\,\exp\left[\mathrm{i}\int \mathrm{d}^4x\,(\mathcal{L}+\Sigma)\right]$$

where the source term is

$$\Sigma = J_\mu^a A_a^\mu + \alpha_a\rho_a + \beta_a\sigma_a + \bar\chi\psi + \bar\psi\chi + \kappa_\mu^a\,(D^\mu\sigma)^a + \frac{1}{2}\nu_a c^{abc}\sigma_b\sigma_c + \bar\lambda T^a\sigma^a\psi + \bar\psi T^a\sigma^a\lambda$$

notice that the last four addenda involve composite fields and, within their sources, only κ is Grassmann-odd [r]. The assumption that the generating functional is invariant implies that

$$\delta W = \int \mathrm{d}^4x \int [\mathcal{D}A_\mu\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \mathcal{D}\rho\, \mathcal{D}\sigma]\, \delta\Sigma \,\exp\left[\mathrm{i}\int \mathrm{d}^4x\, (\mathcal{L}+\Sigma)\right] = 0 \implies \delta\Sigma = 0$$

If the transformation of the source term Σ is non-trivial, then the equation above gives useful relations between composite fields. The variation is

$$\begin{aligned}\delta\Sigma &= J^a_\mu\, \delta A^\mu_a + \alpha_a\, \delta\rho_a + \beta_a\, \delta\sigma_a + \bar{\chi}\, \delta\psi + \delta\bar{\psi}\, \chi + \kappa^a_\mu\, \delta(D^\mu\sigma)^a \\ &+ \frac{1}{2}\nu_a \delta(c^{abc}\sigma_b\sigma_c) + \bar{\lambda}T^a\delta(\sigma^a\psi) + \delta(\bar{\psi}T^a\sigma^a)\lambda\end{aligned}$$

From the proof of invariance the gauge-fixed action one has $\delta(D_\mu\sigma)^a = 0$. The variation of all the composite operators is zero

$$\delta(c^{abc}\sigma_b\sigma_c) = T^a\delta(\sigma^a\psi) = \delta(\bar{\psi}T^a\sigma^a) = 0$$

See Cheng, p. 276 for computations. Inserting the BRST transformations, one finds

$$\delta\Sigma = \omega\left[J^a_\mu\,(D^\mu\sigma)^a + \frac{\mathrm{i}}{\xi}\alpha_a\, \partial_\mu A^\mu_a + \frac{g}{2}\beta_a c^{abc}\sigma_b\sigma_c - \mathrm{i}g\bar{\chi}(T^a\sigma^a)\psi - \mathrm{i}g\bar{\psi}(T^a\sigma^a)\chi\right]$$

In the functional integral, the fields can be replaced with the derivative with respect to the corresponding source. In this way, the source term Σ is no longer function of the integrated fields and can be brought outside the functional integral. The variation of the generating functional is then

$$\delta W = \int \mathrm{d}^4x\, \delta\Sigma\, W[J,\alpha,\beta,\chi,\bar{\chi},\kappa,\nu,\lambda,\bar{\lambda}] = 0$$

which implies[23]

$$\boxed{\int \mathrm{d}^4x\, \omega\left[J^a_\mu\frac{\delta}{\delta\kappa^a_\mu} + \frac{\mathrm{i}}{\xi}\alpha_a\,\partial_\mu\frac{\delta}{\delta J^a_\mu} + g\beta_a\frac{\delta}{\delta\nu_a} - \mathrm{i}g\bar{\chi}\frac{\delta}{\delta\bar{\lambda}} - \mathrm{i}g\frac{\delta}{\delta\lambda}\chi\right] W[J,\alpha,\beta,\chi,\bar{\chi},\kappa,\nu,\lambda,\bar{\lambda}] = 0}$$

This is the Slavnov–Taylor identity.

From the identity one may obtain relations between different types of Green's functions by applying derivatives with respect to the sources and setting them to zero afterwards. This procedure is equivalent to the implication that, since the generating functional W is invariant under BRST transformations, so are the Green's function obtained from it

$$\delta W = 0 \implies \delta G^{(n)} = 0$$

Example. Consider a two-point Green's function for the gauge field

$$(G^{ab}_{\mu\nu})^{(2)}(x) = \langle 0|\, \mathcal{T}\{A^a_\mu(x)A^b_\nu(0)\}\, |0\rangle$$

If the BRST symmetry holds after quantization, one has to find

$$\delta G^{(2)}(x) = 0$$

Explicitly calculating the variation gives

$$\begin{aligned}\delta(G^{ab}_{\mu\nu})^{(2)}(x) &= \delta\,\langle 0|\, \mathcal{T}\{A^a_\mu(x)A^b_\nu(0)\}\, |0\rangle \\ &= \langle 0|\, \mathcal{T}\{\delta A^a_\mu(x)\, A^b_\nu(0)\}\, |0\rangle + \langle 0|\, \mathcal{T}\{A^a_\mu(x)\, \delta A^b_\nu(0)\}\, |0\rangle \\ &= \langle 0|\, \mathcal{T}\{\omega(D_\mu\sigma)^a(x)\, A^b_\nu(0)\}\, |0\rangle + \langle 0|\, \mathcal{T}\{A^a_\mu(x)\omega(D_\nu\sigma)^b(0)\}\, |0\rangle \\ &= 0\end{aligned}$$

This implies that the two addenda are related.

[23] The following integral is the variation δW when the integral is multiplied by $-\mathrm{i}$ which cancels the imaginary unit coming from differentiating the exponential.

Example. Consider a four-point Green's function

$$G^{(4)} = \langle 0| \mathcal{T}\{\rho^a A^a_\mu \bar{\psi}\psi\} |0\rangle$$

The variation is

$$\begin{aligned}\delta G^{(4)} &= -\frac{\mathrm{i}}{\xi}\omega\big\langle \mathcal{T}\{\partial_\nu A^\nu_a\, A^a_\mu \bar{\psi}\psi\}\big\rangle - \omega\big\langle \mathcal{T}\{\rho^a (D_\mu\sigma)^a \bar{\psi}\psi\}\big\rangle \\ &\quad + \mathrm{i}g\omega\big\langle \mathcal{T}\{\rho^a A^a_\mu \bar{\psi}(T^b\sigma^b)\psi\}\big\rangle + \mathrm{i}g\omega\big\langle \mathcal{T}\{\rho^a A^a_\mu \bar{\psi}(T^b\sigma^b)\psi\}\big\rangle \\ &= 0\end{aligned}$$

This linear combination of different Green's function has to be zero.

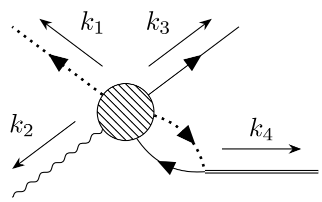
10.2 Unitarity

From the relation in the above example, one may derive physically important identities[24]. Thanks to the LSZ reduction formula, one may compute scattering amplitudes with an integral expression of kinetic operators applied to a Green's function (which can be a linear combination of different kinds of Green's functions). The LSZ formula gives a non-trivial expression for a scattering amplitude when the Green's function contains propagators that simplify the kinetic terms. This is necessary since external particles are on-shell: the kinetic terms are zero, while the propagators diverge. One then would like to understand which terms in a Green's function contribute.

Consider the above example. When setting the external particles on-shell, the terms in the second line do not contribute. The reasoning is the following. Consider the Green's function

$$\big\langle \mathcal{T}\{\rho(x_1)A_\mu(x_2)\bar{\psi}(x_3)T^a(\sigma^a\psi)(x_4)\big\rangle$$

Notice that $\sigma^a\psi$ is composite field. In momentum space, there are four momenta associated to the above Green's function



Since the momentum k_4 splits into the sum of two momenta, one never gets a propagator like

$$\frac{1}{\not{k}_4 - m}$$

which cancels the associated kinetic term of the form $\not{k}_4 - m$ in the LSZ reduction formula. Since, on-shell it holds $\not{k}_4 - m = 0$, then this Green's function does not contribute on-shell to the scattering amplitude. In general, only the terms linear in each field survive.

Therefore, on-shell the four-point Green's function above gives

$$\frac{\mathrm{i}}{\xi}\big\langle \mathcal{T}\{\partial_\nu A^\nu_a\, A^a_\mu \bar{\psi}\psi\}\big\rangle + \big\langle \mathcal{T}\{\rho_a (D_\mu\sigma)_a \bar{\psi}\psi\}\big\rangle = 0$$

Since the above discussion also applies for the composite operator σA_μ, then the covariant derivative can be traded for the ordinary derivative

$$\frac{\mathrm{i}}{\xi}\big\langle \mathcal{T}\{\partial_\nu A^\nu_a\, A^a_\mu \bar{\psi}\psi\}\big\rangle + \big\langle \mathcal{T}\{\rho_a\, \partial_\mu\sigma_a\, \bar{\psi}\psi\}\big\rangle = 0$$

In momentum space, the derivatives become momenta. In the 't Hooft–Feynman gauge, $\xi = 1$, one may define

$$\mathrm{i}T^{ab}_{\nu\mu} \equiv \langle 0| \mathcal{T}\{A^a_\nu A^b_\mu \bar{\psi}\psi\} |0\rangle \ , \quad \mathrm{i}S^{ab} \equiv \langle 0| \mathcal{T}\{\rho^a\sigma^b \bar{\psi}\psi\} |0\rangle$$

[24]Since they come from the Slavnov–Taylor identity they are referred to as (being part of the) Slavnov–Taylor identities.

[r] source?
With these definitions, the Ward identity in momentum space becomes

$$\boxed{k_1^\nu T^{ab}_{\nu\mu} = \mathrm{i}S^{ab}(k_2)_\mu}$$

Multiplying by k_2^μ gives

$$\boxed{k_1^\nu T^{ab}_{\nu\mu} k_2^\mu = 0}$$

because $(k_2)^2 = 0$ on-shell.

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From these two identities, one may show that unitarity is preserved. Unitarity is equivalent to the conservation of probability which implies that the S-matrix is unitary. The scattering matrix enables one to write

$$|f\rangle = S\,|i\rangle$$

The unitarity of the S-matrix can be written as

$$SS^\dagger = I \implies S_{ac}S^*_{bc} = \delta_{ab}$$

The scattering matrix can be written as

$$S_{ab} = \delta_{ab} + \mathrm{i}(2\pi)^4\delta^{(4)}(p_a - p_b)T_{ab}$$

where T_{ab} is the transition amplitude. Inserting this expression into the unitarity constraint, one obtains the optical theorem

$$\boxed{\operatorname{Im} T_{ab} = \frac{1}{2}T_{ac}T^*_{bc}(2\pi)^4\delta^{(4)}(p_a - p_c)}$$

The imaginary part of a transition amplitude is the product of the transition amplitudes associated to the transitions to all possible intermediate physical states.

Optical theorem. The optical theorem is satisfied through the two Slavnov–Taylor identities above. Consider a fermion-fermion $\bar\psi\psi \to \bar\psi\psi$ scattering in a Yang-Mills theory. To compute the imaginary part of the transition amplitude, one considers the spinor, vector and ghost propagators. These last two are

$$\Delta^{ab}_{\mu\nu}(k) = -\frac{\delta^{ab}\eta_{\mu\nu}}{k^2 + \mathrm{i}\varepsilon}\,, \quad \Delta^{ab}(k) = -\frac{\delta^{ab}}{k^2 + \mathrm{i}\varepsilon}$$

One employs the Sokhotski–Plemelj theorem

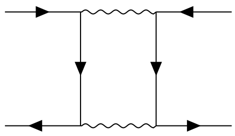
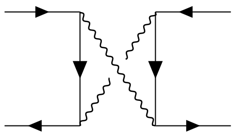
$$\lim_{\varepsilon\to 0^+}\frac{1}{k^2 + \mathrm{i}\varepsilon} = P\frac{1}{k^2} - \mathrm{i}\pi\delta(k^2)$$

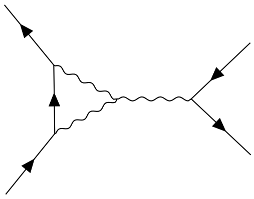
where P is the Cauchy principal value. Close to the k^0 singularity, one has

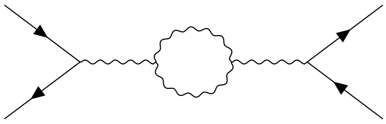
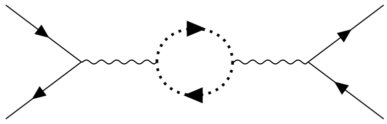
$$\operatorname{Im}\Delta^{ab}_{\mu\nu} = \pi\eta_{\mu\nu}\delta^{ab}\delta(k^2)\theta(k^0)\,, \quad \operatorname{Im}\Delta^{ab} = \pi\delta^{ab}\delta(k^2)\theta(k^0)$$

To compute the imaginary part of a scattering amplitude, one replaces the propagators in the intermediate states by their imaginary parts and multiplies them by the on-shell scattering amplitudes. This is the Cutkowski rule. Concretely, the imaginary part of an amplitude is obtained by cutting the internal propagators of a diagram and then multiply together the on-shell amplitudes obtained from the cuts.

Example. The one-loop contributions to the fermion-fermion scattering are

$T_{\bar{\psi}\psi}|_{1\mathrm{L}} =$  $+$ 

$+$  $+$ mirror copy

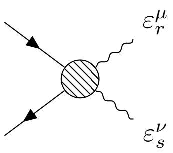
$+$  $+$ 

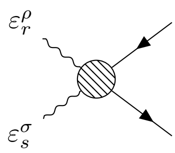
To compute only the imaginary part, one has to cut the internal propagators. The imaginary part is the product of tree-level amplitudes (see Peskin for proof). In the first five diagrams, there are products of amplitudes of the type $\bar{\psi}\psi AA$ which corresponds to the transition amplitude T, while the last diagram involves the amplitude $\bar{\psi}\psi\rho\sigma$ which corresponds to the scattering amplitude S. Therefore, the left-hand side of the optical theorem is

$$\mathrm{Im}\, T_{\bar{\psi}\psi} = \int \mathrm{d}\phi_2 \left[\frac{1}{2} T^{ab}_{\mu\nu}(T^*)^{ab}_{\rho\sigma}\eta^{\mu\rho}\eta^{\nu\sigma} - S^{ab}(S^*)^{ab}\right]$$

where the measure $\mathrm{d}\phi_2$ denotes the massless two-particle phase space. The minus comes from cutting a fermionic loop.

For the right-hand side of the optical theorem, the intermediate states cannot be ghosts, but only vector bosons. One builds the diagrams with these amplitudes





and sums over all the intermediate physical states. The two diagrams correspond both to the transition amplitude T. In the sum, one has to consider the polarization of the vector bosons. The general classical solution to the massless Klein–Gordon equation for vector bosons is

$$A^\mu(x) = \sum_{r=0}^{3} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} [\varepsilon^\mu_r(\mathbf{k}) a_r(\mathbf{k}) \mathrm{e}^{-\mathrm{i}kx} + \varepsilon^\mu_r(\mathbf{k}) a^\dagger_r(\mathbf{k}) \mathrm{e}^{\mathrm{i}kx}]_{k^0=\omega}$$

where ε_r are the polarization four-vectors. The physical polarization are ε_1 and ε_2. Therefore, the sum over the intermediate physical states involves only on the physical polarizations. In the sum there is a product of the polarization vectors. In the end, one obtains a right-hand side equal to

$$\mathrm{RHS} = \frac{1}{2} \int \mathrm{d}\phi_2\, T^{ab}_{\mu\nu}(T^*)^{ab}_{\rho\sigma} P^{\mu\rho}(k_1) P^{\nu\sigma}(k_2)$$

where one has

$$P^{\mu\rho}(k_j) = \sum_{r=1,2} \varepsilon^\mu_r(\mathbf{k}_j)\varepsilon^\rho_r(\mathbf{k}_j)$$

One may check the validity of the optical theorem for this particular example

$$\mathrm{Im}\, T \stackrel{?}{=} \mathrm{RHS}$$

A useful choice of polarization basis is

$$\varepsilon^\mu_0 = (1,0,0,0) = n^\mu\,, \quad \varepsilon^\mu_i = (0, \boldsymbol{\varepsilon}_i)$$

where

$$\boldsymbol{\varepsilon}_i \cdot \boldsymbol{\varepsilon}_j = \delta_{ij} \,, \quad \boldsymbol{\varepsilon}_{1,2} \cdot \mathbf{k} = 0$$

One sets the longitudinal polarization vector to be

$$\varepsilon_3^\mu = \frac{k^\mu - (nk)n^\mu}{[(nk)^2 - k^2]^{\frac{1}{2}}}$$

The first vector ε_0 is the scalar polarization, while the transverse polarizations are $\varepsilon_{1,2}$. The polarization vectors satisfy

$$\sum_{r=0}^{3} \zeta_r \varepsilon_r^\mu(\mathbf{k}) \varepsilon_r^\nu(\mathbf{k}) = -\eta^{\mu\nu} \,, \quad \zeta_r = \begin{cases} -1 \,, & r = 0 \\ 1 \,, & r = 1,2,3 \end{cases}$$

Therefore, one has

$$\begin{aligned} P^{\mu\rho}(k_1) &= \sum_{r=1,2} \varepsilon_r^\mu(\mathbf{k}_1) \varepsilon_r^\rho(\mathbf{k}_1) = \sum_{r=0}^{3} \zeta_r \varepsilon_r^\mu(\mathbf{k}_1) \varepsilon_r^\rho(\mathbf{k}_1) + \varepsilon_0^\mu(\mathbf{k}_1) \varepsilon_0^\rho(\mathbf{k}_1) - \varepsilon_3^\mu(\mathbf{k}_1) \varepsilon_3^\rho(\mathbf{k}_1) \\ &= -\eta^{\mu\rho} + \varepsilon_0^\mu(\mathbf{k}_1) \varepsilon_0^\rho(\mathbf{k}_1) - \varepsilon_3^\mu(\mathbf{k}_1) \varepsilon_3^\rho(\mathbf{k}_1) = -\eta^{\mu\rho} + \frac{(nk)(k^\mu n^\rho + n^\mu k^\rho) - k^\mu k^\rho}{(kn)^2} \end{aligned}$$

The second addendum is the contribution of non-physical polarizations. Inserting the above expression into the right-hand side of the optical theorem, and applying the Ward identities, one finds the left-hand side, $\operatorname{Im} T$.

Remark. The optical theorem is holds thanks to the Ward identities which arise from the BRST symmetry being preserved in the quantum theory.

Remark. Non-physical degrees of freedom are exactly compensated by ghosts. The term SS present in the imaginary part $\operatorname{Im} T$ can be rewritten on the right-hand side to see that it balances the non-physical polarizations.

11 Quantum electrodynamics

See Ramond, §8.2 and consider the Euclidean formalism.

11.1 One-loop structure

Consider an abelian gauge theory based on U(1) with one massive Dirac spinor field ψ minimally coupled to the gauge field A^μ. The covariant derivative is[25]

$$D_\mu = \partial_\mu + \mathrm{i} q A_\mu$$

The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(\mathrm{i} \not{D} - m)\psi - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

The generating functional is

$$W[J, \eta, \bar{\eta}] = \int [\mathcal{D}A_\mu \, \mathcal{D}\psi \, \mathcal{D}\bar{\psi}] \exp\left[\mathrm{i} \int \mathrm{d}^4 x \, (\mathcal{L} + J_\mu A^\mu + \bar{\psi}\eta + \bar{\eta}\psi)\right]$$

Wick rotation. See Ramond, §5.2. When going to Euclidean space, one would like to keep the covariant derivative

$$x^0 \equiv -\mathrm{i} x_{\mathrm{E}}^0 \,, \quad \partial_0 \equiv \mathrm{i} \, \partial_{0\mathrm{E}} \,, \quad A_0 \equiv \mathrm{i} A_{0\mathrm{E}}$$

The Dirac matrices are rotated as

$$\gamma_0 \equiv \mathrm{i} \gamma_{0\mathrm{E}} \implies \{\gamma_{\mathrm{E}}^\mu, \gamma_{\mathrm{E}}^\nu\} = -2\delta^{\mu\nu}$$

[25] Note that Ramond uses $q = -e$ for electrons, with $e > 0$.

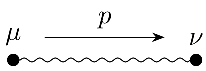
The Euclidean generating functional becomes

$$W_{\mathrm{E}}[J,\eta,\bar{\eta}] = \int [\mathcal{D}A_\mu\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}]\,\exp\left[-\int \mathrm{d}^4x\,(\mathcal{L}_{\mathrm{E}} + J_\mu A^\mu - \mathrm{i}\bar{\eta}\psi - \mathrm{i}\bar{\psi}\eta)\right]$$

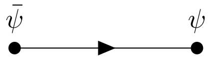
where the Euclidean Lagrangian is[26]

$$\mathcal{L}_{\mathrm{E}} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \bar{\psi}(\not{\partial} + \mathrm{i}m)\psi - \mathrm{i}qA_\mu(\bar{\psi}\gamma^\mu\psi)$$

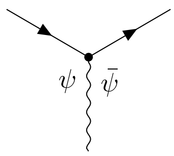
Euclidean Feynman rules. The gauge field propagator in the 't Hooft–Feynman gauge, $\xi = 1$, is

$$\frac{\delta_{\mu\nu}}{p^2} =$$ 

The spinor propagator is

$$-\frac{\mathrm{i}}{\not{p} + m} =$$ 

The vertex is

$$\mathrm{i}qk^\varepsilon\gamma^\mu =$$ 

One has to use Furry's theorem also: all odd-fermion loops are zero.

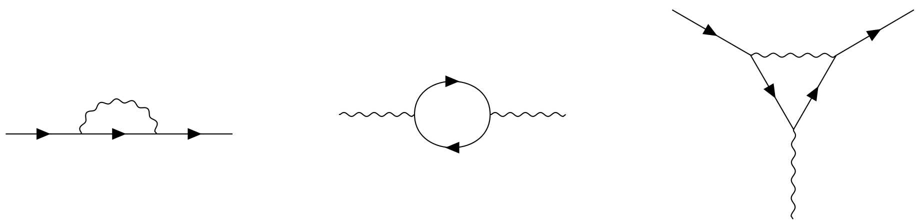
Power counting. The superficial degree of divergence obtained through power counting is

$$D = 4 - E_{\mathrm{B}} - \frac{3}{2}E_{\mathrm{F}}$$

where E_{B} is the number of external boson lines while E_{F} is the number of external fermion lines. Pure vector diagrams, $E_{\mathrm{F}} = 0$, are divergent for $E_{\mathrm{B}} = 2$ and $E_{\mathrm{B}} = 4$: the self-energy of the gauge field and the light-light diagram. For two external fermion lines, $E_{\mathrm{F}} = 2$, the only divergent diagrams have $E_{\mathrm{B}} = 1$ and $E_{\mathrm{B}} = 0$: the vertex correction and the self-energy of the fermion field.

11.1.1 Regularization

At one-loop, one has to evaluate the diagrams



To perform the computations, one has to

- compute the combinatorial factor,
- apply Feynman combining,
- move to spherical coordinates,

[26]Notice that the partial derivative and the interaction term have different signs. The Euclidean Lagrangian is defined starting from the Euclidean action where one sends $x_{\mathrm{E}} \to x_{\mathrm{E}}$ in the integral for the derivative. See notes last year, Lecture 14, p. 18.

- evaluate momentum integrals of the form

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{(l^2)^m}{(l^2+A^2)^n}$$

See Ramond, appendix B.

- consider the simple poles of the regularization parameter ε.

One-loops contributions. The fermion self-energy is

$$\Sigma(p) = -\frac{\mathrm{i}}{\varepsilon}\frac{q^2}{16\pi^2}(\not{p}+4m) + \text{finite}$$

The gauge field self-energy is

$$\Pi_{\mu\nu}(p) = \frac{q^2}{12\pi^2}(p_\mu p_\nu - p^2\delta_{\mu\nu})\frac{1}{\varepsilon} + \text{finite}$$

The vertex function is

$$\Gamma_\mu(p,q) = \mathrm{i}\frac{q^3}{16\pi^2}k^\varepsilon\gamma_\mu\frac{1}{\varepsilon} + \text{finite}$$

11.1.2 Renormalization

One may apply the BPHZ renormalization. Let the bare quantities be

$$\psi_0 = Z_2^{\frac{1}{2}}\psi\,, \quad A_0^\mu = Z_3^{\frac{1}{2}}A^\mu\,, \quad q_0 = Z_1 Z_2^{-1} Z_3^{-\frac{1}{2}} q k^\varepsilon\,, \quad m_0 = Z_m Z_2^{-1} m\,, \quad \xi_0^{-1} = Z_\xi Z_3^{-1}\xi^{-1}$$

The bare Lagrangian is

$$\begin{aligned}\mathcal{L}_0 &= \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \bar\psi\gamma^\mu\,\partial_\mu\psi + \mathrm{i}m\bar\psi\psi - \mathrm{i}qA_\mu\bar\psi\gamma^\mu\psi + \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \\ &\quad + (Z_3-1)(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + (Z_2-1)\bar\psi\gamma^\mu\,\partial_\mu\psi + (Z_m-1)\mathrm{i}m\bar\psi\psi \\ &\quad - (Z_1-1)\mathrm{i}qA_\mu\bar\psi\gamma^\mu\psi + (Z_\xi-1)\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \\ &= \mathcal{L}_\mathrm{R} + \mathcal{L}_\mathrm{ct}\end{aligned}$$

In the following, let $K_j = Z_j - 1$.

Fermion field: mass term. The divergence of the mass term from the fermion self-energy is cancelled by the mass counter term $\mathrm{i}mK_m$ to have

$$-\frac{\mathrm{i}}{\varepsilon}\frac{q^2}{4\pi^2} - \mathrm{i}mK_m = \text{finite} \implies k_m = -\frac{q^2}{4\pi^2}\frac{1}{\varepsilon} + \text{finite} \implies \boxed{Z_m = 1 - \frac{q^2}{4\pi^2}\frac{1}{\varepsilon} + \text{finite}}$$

Remember that the minus sign of the counter term comes from the minus sign of the action in the exponential of the generating functional.

Fermion field: kinetic term. From the self-energy one has

$$-\frac{\mathrm{i}}{\varepsilon}\frac{q^2}{16\pi^2}\not{p} - \mathrm{i}K_2\not{p} = \text{finite} \implies k_2 = -\frac{q^2}{16\pi^2}\frac{1}{\varepsilon} + \text{finite} \implies \boxed{Z_2 = 1 - \frac{q^2}{16\pi^2}\frac{1}{\varepsilon} + \text{finite}}$$

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Gauge field. [r] From the counter term Lagrangian, one reads

$$\begin{aligned}\mathcal{L}_{\text{ct}} &\propto (Z_3-1)\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{1}{2\xi}(Z_\xi - 1)(\partial_\mu A^\mu)^2 \\ &= \frac{1}{2}K_3\,\partial_\mu A_\nu\,\partial^\mu A^\nu - \frac{1}{2}K_3\,\partial_\mu A_\nu\,\partial^\nu A^\mu + \frac{1}{2\xi}K_\xi\,\partial_\mu A^\mu\,\partial_\nu A^\nu \\ &= -\frac{1}{2}K_3 A^\nu\,\Box A_\nu - \frac{1}{2}\left[\frac{K_\xi}{\xi} - K_3\right] A^\nu\,\partial_\mu\partial_\nu A^\mu + \text{4-div}\end{aligned}$$

At the third line, one has integrated all terms by parts. Remembering the minus sign coming from the action in the exponential of the generating functional

$$-S \propto \frac{1}{2}K_3 A^\nu\,\Box A_\nu + \frac{1}{2}\left[\frac{K_\xi}{\xi} - K_3\right] A^\nu\,\partial_\mu\partial_\nu A^\mu$$

In momentum space, this becomes

$$\frac{1}{2}K_3 A_\nu\,(-p^2)A^\nu + \frac{1}{2}\left[\frac{K_\xi}{\xi} - K_3\right] A^\nu(-p_\mu p_\nu)A^\mu$$

The counter term contribution is then

$$-\frac{1}{2}2\left[\delta_{\mu\nu}K_3 p^2 + \left(\frac{K_\xi}{\xi} - K_3\right)p_\mu p_\nu\right]$$

where the 2 is a symmetry factor [r]. Therefore, the total contribution is

$$\frac{q^2}{12\pi^2}(p_\mu p_\nu - p^2\delta_{\mu\nu})\frac{1}{\varepsilon} - \left[\delta_{\mu\nu}K_3 p^2 + \left(\frac{K_\xi}{\xi} - K_3\right)p_\mu p_\nu\right] = \text{finite}$$

The terms proportional to p^2 give

$$-p^2\delta_{\mu\nu}\left[\frac{q^2}{12\pi^2}\frac{1}{\varepsilon} + K_3\right] \implies K_3 = -\frac{q^2}{12\pi^2}\frac{1}{\varepsilon} \implies \boxed{Z_3 = 1 - \frac{q^2}{12\pi^2}\frac{1}{\varepsilon} + \text{finite}}$$

while the ones proportional to $p_\mu p_\nu$ give

$$\left[\frac{q^2}{12\pi^2}\frac{1}{\varepsilon} - \left(\frac{K_\xi}{\xi} - K_3\right)\right]p_\mu p_\nu = \text{finite} \implies \frac{q^2}{12\pi^2}\frac{1}{\varepsilon} + K_3 - K_\xi = \text{finite}\,, \quad \xi = 1$$

In the 't Hooft–Feynman gauge, the above relation implies no gauge parameter renormalization

$$K_\xi = \text{finite} = 0 \implies Z_\xi = 1$$

The divergences of the two terms of self-energy $\Pi_{\mu\nu}$ are both cancelled by K_3 in the 't Hooft–Feynman gauge.

Vertex function. The vertex counter term is

$$-S \propto K_1 \mathrm{i}qk^\varepsilon A_\mu\bar{\psi}\gamma^\mu\psi$$

which leads to a counter term

$$K_1 \mathrm{i}qk^\varepsilon\gamma^\mu$$

From this one finds

$$K_1 = -\frac{q^2}{16\pi^2}\frac{1}{\varepsilon} + \text{finite} \implies \boxed{Z_1 = 1 - \frac{q^2}{16\pi^2}\frac{1}{\varepsilon} + \text{finite}}$$

Remark. Notice that at one-loop, the Ward identity of QED holds: $Z_1 = Z_2$.

11.1.3 Beta function

The coupling constant is the elementary charge $g = |q|$. In the minimal subtraction scheme (i.e. no finite parts), one has

$$g_0 = Z_1 Z_2^{-1} Z_3^{-\frac{1}{2}} g k^\varepsilon = Z_3^{-\frac{1}{2}} g k^\varepsilon = \left[1 - \frac{g^2}{12\pi^2}\frac{1}{\varepsilon}\right]^{-\frac{1}{2}} g k^\varepsilon = \left[1 + \frac{g^2}{24\pi^2}\frac{1}{\varepsilon} + o(g^2)\right] g k^\varepsilon$$

From this, one notices the coefficient of the simple pole

$$a_1 = \frac{g^3}{24\pi^2}$$

Recall that for the $\lambda\varphi^4$ theory it holds

$$\lambda_0 = k^{2\varepsilon}\left[\lambda + \sum_{r=1}^{\infty}\frac{a_r(\lambda)}{\varepsilon^r}\right] \implies \beta(\lambda) = 2\left[\lambda \, \mathrm{d}_\lambda - 1\right] a_1$$

where the factor of 2 comes from the 2 in the exponent of the energy scale. For QED, one has

$$\beta(g) = \left[g \, \mathrm{d}_g - 1\right] a_1 = 2a_1 = \frac{g^3}{12\pi^2}$$

The beta function enters in the renormalization group flow equation

$$k \, \mathrm{d}_k g = \beta(g)$$

Since the coupling constant is positive, so is the beta function and as such at higher energies the coupling constant gets stronger.

Landau pole. One may solve the above equation. One considers $q^2(k)$. The equation is

$$k \, \mathrm{d}_k g^2 = 2gk \, \mathrm{d}_k g = \frac{g^4}{6\pi^2} \implies \frac{1}{g^4} \, \mathrm{d}_k g^2 = \frac{1}{6\pi^2 k}$$

Integrating from $g(k = k_0) = g_0$ to g gives

$$\boxed{g^2(k) = \frac{g_0^2}{1 - \frac{g_0^2}{6\pi^2}\ln\frac{k}{k_0}}}$$

The running coupling constant of QED has a Landau pole at

$$k = k_0 \mathrm{e}^{\frac{6\pi^2}{g_0^2}}$$

When getting close to the pole, the coupling constant gets bigger than 1 and perturbation theory cannot be applied, so this result should not be trusted in such region.

There is no asymptotic freedom in QED because the coupling constant grows stronger with the energy scale. This lack of asymptotic freedom implies a well-defined notion of asymptotic states (of the S-matrix): at large distances, which correspond to low energies, the coupling constant is weak [r].

11.2 Ward identities

See Ramond, §8.3. Since QED is an abelian theory, one may use the results for non-abelian theories and set the structure constants to zero, $c^{abc} = 0$. The generating functional is

$$\begin{aligned} W[J, \eta, \bar\eta] = \int [\mathcal{D}A_\mu \, \mathcal{D}\psi \, \mathcal{D}\bar\psi] \exp\left[-\int \mathrm{d}^4x \, (\mathcal{L} + J_\mu A^\mu - \mathrm{i}\bar\eta\psi - \mathrm{i}\bar\psi\eta)\right] \\ \times \int [\mathcal{D}\rho \, \mathcal{D}\sigma] \exp\left[-\int \mathrm{d}^4x \, (\mathcal{L}_{\mathrm{gh}} - \alpha\rho - \beta\sigma)\right] \end{aligned}$$

Since ghosts decouple, one introduces the ghost fields with Lagrangian[27]

$$\mathcal{L}_{\text{gh}} = \rho \,\square\, \sigma$$

The total Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{g}} + \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{gh}} + \mathcal{L}_{\psi}$$

is invariant under BRST transformations[28]

$$\begin{aligned}
\delta A_\mu &= \omega\, \partial_\mu \sigma \\
\delta\rho &= -\frac{\omega}{\xi}(\partial_\mu A^\mu) \\
\delta\sigma &= 0 \\
\delta\psi &= \mathrm{i} g\omega\sigma\psi \\
\delta\bar{\psi} &= -\mathrm{i} g\omega\sigma\bar{\psi}
\end{aligned}$$

Recall that $\mathcal{L}_{\text{g}} + \mathcal{L}_{\psi}$ is invariant thanks to gauge invariance. The remaining terms are also invariant

$$\delta(\mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{gh}}) = 0$$

With these transformations, one may obtain the Ward identity imposing that

$$\delta W[J, \eta, \bar{\eta}, \alpha, \beta] = 0$$

where one has

$$W[J, \eta, \bar{\eta}, \alpha, \beta] = \int [\mathcal{D}A_\mu\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \mathcal{D}\rho\, \mathcal{D}\sigma] \mathrm{e}^{-S} \exp\left[-\int \mathrm{d}^4 x\, (J_\mu A^\mu - \mathrm{i}\bar{\eta}\psi - \mathrm{i}\bar{\psi}\eta - \alpha\rho - \beta\sigma) \right]$$

The variation of the generating functional gives

$$\begin{aligned}
0 &= -J_\mu \omega\, \partial_\mu \sigma + \mathrm{i}\bar{\eta}(\mathrm{i} g\omega\sigma\psi) + \mathrm{i}(-\mathrm{i} g\omega\sigma\bar{\psi})\eta - \alpha\frac{\omega}{\xi}(\partial_\mu A^\mu) \\
&= -J_\mu \omega\, \partial_\mu \sigma - g\omega\sigma\bar{\eta}\psi + g\omega\sigma\bar{\psi}\eta - \alpha\frac{\omega}{\xi}(\partial_\mu A^\mu)
\end{aligned}$$

It is useful to rewrite this Ward identity in terms of the effective action Γ. In the $\lambda\varphi^4$ theory, one has

$$\Gamma[\varphi_c] = -Z[J] + \int \mathrm{d}^4 x\, J\varphi_c\,, \quad J(x) = \delta_{\varphi(x)}\Gamma[\varphi]$$

Doing the same for QED, one has

$$\Gamma[A_\mu, \psi, \bar{\psi}, \rho, \sigma] = -Z[J, \eta, \bar{\eta}, \alpha, \beta] + \int \mathrm{d}^4 x\, [-J_\mu A^\mu + \mathrm{i}\bar{\eta}\psi + \mathrm{i}\bar{\psi}\eta + \alpha\rho + \beta\sigma]$$

where all the fields are classical and a subscript c is understood. One has

$$J_\mu = -\delta_{A^\mu}\Gamma\,, \quad \mathrm{i}\eta = \delta_{\bar{\psi}}\Gamma\,, \quad \mathrm{i}\bar{\eta} = -\delta_{\psi}\Gamma\,, \quad \alpha = -\delta_\rho\Gamma\,, \quad \beta = -\delta_\sigma\Gamma$$

[r] Imposing the invariance under the BRST transformations of the effective action Γ above gives the Ward identity

$$\boxed{\delta_{A_\mu}\Gamma\, \partial_\mu\sigma + \mathrm{i} g\, \delta_\psi\Gamma\, \sigma\psi - \mathrm{i} g\sigma\bar{\psi}\, \delta_{\bar{\psi}}\Gamma - \frac{1}{\xi}(\partial_\mu A^\mu)\, \delta_\rho\Gamma = 0}$$

where one recalls that $\delta Z = 0$.

[27]Where one has renamed $\mathrm{i}\rho \to \rho$.

[28]See previous note. Notice a missing factor of i in the transformation of ρ and that $(\partial_\mu A^\mu)_{\text{M}} \to -(\partial_\mu A_\mu)_{\text{E}}$ which preserves the minus sign when performing a Wick rotation.

Effective action of QED. Recall that

$$\mathrm{e}^{-\Gamma} = \int [\mathcal{D}A_\mu \, \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, \mathcal{D}\sigma \, \mathcal{D}\rho] \exp\left[- \int \mathrm{d}^4x \, (\mathcal{L}_{\mathrm{QED}} + \mathcal{L}_{\mathrm{gh}}) \right]$$

The effective action is

$$\begin{aligned}\Gamma[A_\mu, \psi, \bar{\psi}, \rho, \sigma] = \int \mathrm{d}^4x \, \mathrm{d}^4y \bigg[& \rho(x)\Delta^{-1}(x-y)\sigma(y) \\ & + \frac{1}{2} A_\mu(x)\Delta^{-1}_{\mu\nu}(x-y)A_\nu(y) + \bar{\psi}(x)S^{-1}(x-y)\psi(y) \\ & + g \int \mathrm{d}^4z \, \bar{\psi}(x)A^\rho(y)\Gamma_\rho(x,y,z)\psi(z) + o(g) \bigg]\end{aligned}$$

Since the ghosts do not interact with the fields, then their inverse propagator $\Delta^{-1} = \Box$ gets no loop corrections. The other propagators and the vertex function are all corrected.

Kinetic term of the gauge field. One may use the Ward identity to find the general expression of the kinetic term of the gauge field $\Delta^{-1}_{\mu\nu}$. One may set the fermion fields to zero. After an integration by parts of the first term, the Ward identity becomes

$$0 = [\partial_\mu \, \delta_{A_\mu} \Gamma] \, \sigma + \frac{1}{\xi}(\partial_\mu A^\mu) \, \delta_\rho \Gamma = \partial_\mu (\Delta^{-1}_{\mu\nu} A_\nu)\sigma + \frac{1}{\xi}(\partial_\mu A^\mu) \, \Delta^{-1}\sigma$$

Recalling that $\Delta^{-1} = \Box$, in momentum space one has

$$\mathrm{i}k_\mu \Delta^{-1}_{\mu\nu} A_\nu \sigma + \frac{1}{\xi} \mathrm{i}k_\mu A^\mu (-k^2)\sigma = 0 \implies \boxed{k^\mu \Delta^{-1}_{\mu\nu}(k) - \frac{1}{\xi} k_\nu k^2 = 0}$$

This constraint must hold at every loop order.

One may obtain the general form of the kinetic term. Starting from the ansatz

$$\Delta^{-1}_{\mu\nu}(k) = A(k^2)\delta_{\mu\nu} + B(k^2)k_\mu k_\nu$$

From the constraint above, one obtains

$$A(k^2) = [1 - B(k^2)]k^2 \,, \quad \xi = 1$$

Therefore

$$\boxed{\Delta^{-1}_{\mu\nu}(k) = k^2\delta_{\mu\nu} - (\delta_{\mu\nu}k^2 - k_\mu k_\nu)B(k^2)}$$

When computing quantum corrections to the kinetic term, one has to determine $B(k^2)$ which is always multiplied by the projection operator

$$\Pi_{\mu\nu} = \delta_{\mu\nu}k^2 - k_\mu k_\nu$$

that satisfies

$$\Pi_{\mu\nu}\Pi_{\nu\rho} = \Pi_{\mu\rho}k^2$$

The projector is not invertible, but the first addendum in the kinetic term Δ^{-1} comes from gauge-fixing and makes the kinetic term invertible.

Renormalization functions. Another application of the Ward identity is the following. Let $A_\mu = 0$. Then the Ward identity is

$$\delta_{A_\mu} \Gamma|_{A_\mu = 0} \, \partial_\mu \sigma + \mathrm{i}q \, \delta_\psi \Gamma \, \sigma\psi + \mathrm{i}q\bar{\psi}\sigma \, \delta_{\bar{\psi}} \Gamma = 0$$

One gets a non-trivial contribution from the vertex function. Inserting the effective action gives

$$\begin{aligned}0 = g \int \mathrm{d}^4x \, \mathrm{d}^4z \, \bar{\psi}(x)\Gamma_\rho(x,w,z)\psi(z) \, \partial_\rho \sigma(w) - \mathrm{i}g \int \mathrm{d}^4x \, \bar{\psi}(x)S^{-1}(x-w)\sigma(w)\psi(w) \\ + \mathrm{i}g \int \mathrm{d}^4y \, \bar{\psi}(w)\sigma(w)S^{-1}(w-y)\psi(y)\end{aligned}$$

In momentum space, one has

$$\boxed{S^{-1}(p) - S^{-1}(q) = (q-p)^\rho \Gamma_\rho(p, q-p, q)}$$

[r] check. Knowing that S^{-1} is renormalized by Z_2 and Γ is renormalized by Z_1, this Ward identity gives

$$\boxed{Z_1 = Z_2}$$