Quantum Field Theory I

Maso*

March 23, 2024

Contents

Lecture 1

lun 04 mar 2024 10:30

Exam. The exam can be done whenever, just send an email at least ten days before. Half an hour to calculate something like in class, or generalize something, just technical stuff. It is possible to do QFT 1 and 2 together.

Course structure. In order to go towards increasing complexity, one starts from real scalar boson fields, then fermions, then gauge fields and gauge theories. QFT I is about scalar bosons, while QFT II starts from fermions. Between the two parts there is a bit of conformal field theory.

*https://github.com/M-a-s-o/notes

1 Introduction

The aim is to develop an alternative description of elementary particles and fundamental interactions using functional methods instead of canonical quantization. The path integral formulation of quantum mechanics is an example of functional method. Canonical quantization works by taking classical observable quantities and promoting them to operators obeying certain commutator relations. This was done in the relativistic formulation of quantum field theory in Theoretical Physics I.

The alternative, albeit equivalent, formulation of quantum mechanics is based on Feynman path integrals: the propagators are written in term of path integrals. Quantum Field Theory I and II reformulates quantum field theory in terms of a generalization of the quantum mechanical path integral to relativistic field theories. The content of Theoretical Physics I is studied using a functional approach.

One needs to formulate quantum field theories for particles of different spins: scalar bosons, spinor fermions and vector bosons. Gauge theories are quantized using functional methods. One does not look for spin $\frac{3}{2}$ and 2 fields, because quantum field theory is not suitable for their description: the theory is inconsistent because it is not renormalizable at every energy scale, in particular in the ultraviolet. For higher spins, there are problems in the propagation of particles in ordinary quantum field theory and one needs a more general approach.

The functional approach lets one study phenomena for which canonical quantization is not suitable.

1.1 Prerequisites

The prerequisites are the following.

Real Gaussian integral. Gaussian integrals are useful for many computations. In one dimension, one has

$$\int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-\frac{a}{2}x^2+bx} = \sqrt{\frac{2\pi}{a}}\mathrm{e}^{\frac{b^2}{2a}}$$

In n dimensions (considering only diagonal matrices) one has

$$\int_{\mathbb{R}^n} \mathrm{d}x_1 \cdots \mathrm{d}x_n\, \mathrm{e}^{-\frac{a_1}{2}x_1^2-\cdots-\frac{a_n}{2}x_n^2+b_1x_1+\cdots+b_nx_n} = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{a_1\cdots a_n}}\mathrm{e}^{\frac{b_1^2}{2a_1}+\cdots+\frac{b_n^2}{2a_n}}$$

Introducing the matrix and vectors

$$A = \operatorname{diag}(a_1,\dots,a_n)\,, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ a_n \end{bmatrix}\,, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

one obtains

$$\int_{\mathbb{R}^n} \mathrm{d}x_1 \cdots \mathrm{d}x_n\, \mathrm{e}^{-\frac{1}{2}x^\top Ax+b^\top x} = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}}\mathrm{e}^{\frac{1}{2}b^\top A^{-1}b}$$

which more generally holds for positive-definite symmetric matrices[1] A. This result is analytically continued to the cases where the exponent in the integrand has arbitrary signs or imaginary numbers while assuming the right-hand side keeps its validity.

Complex Gaussian integral. The complex Gaussian integral is the following

$$\int \prod_{j=1}^{n} \mathrm{d}z_j\, \mathrm{d}\bar{z}_j\, \mathrm{e}^{-\bar{z}^\top Az+\bar{b}^\top z+b^\top \bar{z}} = \frac{(2\pi)^n}{\det A}\mathrm{e}^{\bar{b}^\top A^{-1}b}$$

where A is a positive-definite hermitian matrix, b is a complex column vector and the bar is complex conjugation. Since $\mathbb{C} \simeq \mathbb{R}^2$, and the dimension is still n, one expects double the integrals.

There is also the generalization of Gaussian integrals for Grassmann odd variables suitable for spinors.

[1] See `https://en.wikipedia.org/wiki/Gaussian_integral`.

Average of a polynomial with Gaussian distribution. Consider the mean value of Gaussian-distributed variables (see Weinberg, eq. 9.A.10)

$$\begin{aligned}
\langle x_{k_1} \cdots x_{k_l} \rangle &\equiv \int \prod_{j=1}^{n} \mathrm{d}x_j \, x_{k_1} \cdots x_{k_l} \mathrm{e}^{-\frac{1}{2} x^\top A x + b^\top x} \Big|_{b=0} \\
&= \int \prod_{j=1}^{n} \mathrm{d}x_j \, \frac{\partial^l}{\partial b_{k_1} \cdots \partial b_{k_l}} \mathrm{e}^{-\frac{1}{2} x^\top A x + b^\top x} \Big|_{b=0} \\
&= \frac{\partial^l}{\partial b_{k_1} \cdots \partial b_{k_l}} \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}} \mathrm{e}^{\frac{1}{2} b^\top A^{-1} b} \Big|_{b=0} \\
&= \begin{cases} 0 \,, & l \text{ odd} \\ \dfrac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}} \displaystyle\sum_{\substack{\text{pairings} \\ \text{of } \{k_i\}}} \prod_{\{k_i k_j\}} A^{-1}_{k_i k_j} \,, & l \text{ even} \end{cases}
\end{aligned}$$

The last sum is over the pairings of $(k_1 \cdots k_l)$ — two pairings are the same if they differ only by the order of the pairs, or by the order of indices within a pair — and the product is over all such pairs.

One can compute averages through the derivatives of Gaussian integrals. From the first line (or the third), one may notice that if l is odd then the integral is null.

Functional calculus. Informally, functional calculus is need for infinite (non-countable) dimensional vectors (for example functions). From a discrete set of quantities q_i for which one knows how to take the derivative, one moves to the case of continuous variables where one works with functions $q(x)$.

Inner product. The inner product of two finite dimensional vector fields is

$$u \cdot v \equiv \sum_{i=1}^{n} u_i v_i$$

while for infinite dimensional vector fields one has

$$u \cdot v \equiv \int \mathrm{d}x \, u(x) v(x)$$

The inner product with respect to a matrix is

$$u^\top M v = \sum_{ij=1}^{n} u_i M_{ij} v_j \rightsquigarrow u^\top M v = \int \mathrm{d}x \, \mathrm{d}y \, u(x) M(x,y) v(y)$$

Identity operator. The identity operator is

$$q = Iq \iff q_i = \sum_j \delta_{ij} q_j \rightsquigarrow q(x) = \int \mathrm{d}y \, \delta(x-y) q(y)$$

The identity operator is the Dirac delta function.

Functional derivatives. The notation for functional derivatives is $\frac{\delta}{\delta q(x)}$. The concept of functional derivative can be defined by imposing a set of ansätze (pl. of ansatz):

- Linearity
$$\delta_q [F_1(q) + F_2(q)] = \delta_q F_1 + \delta_q F_2$$
- Leibniz rule
$$\delta_q [F_1(q) F_2(q)] = (\delta_q F_1) F_2 + F_1 \delta_q F_2$$

- It must hold
$$\delta_{q(x)} q(y) = \delta(y - x)$$

With these ansätze one can prove that the functional derivative enjoys the same properties as the regular derivative. For example

$$\delta_{q(x)} \int \mathrm{d}y\, q^P(y) = \int \mathrm{d}y\, P q^{P-1}(y)\, \delta_{q(x)} q(y) = P q^{P-1}(x)$$

Likewise

$$\delta_{q(x)} \int \mathrm{d}y\, f(y)\, \partial_y q(y) = \int \mathrm{d}y\, f(y)\, \partial_y \delta(x - y) = -\partial_x f(x)$$

where the result is obtained remembering that the Dirac delta is a distribution.

Generalization of Gaussian integrals to functional integrals. The Gaussian integral becomes

$$\int \prod_{j=1}^{n} \mathrm{d}q_j\, \mathrm{e}^{-\frac{1}{2} q^\top A q} = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}} \rightsquigarrow \int [\mathcal{D}q]\, \mathrm{e}^{-\frac{1}{2}\int \mathrm{d}x\, \mathrm{d}y\, q(x) A(x,y) q(y)} = \frac{\text{const.}}{\sqrt{\text{“}\det A\text{”}}}$$

where $\mathcal{D}q$ is the functional measure that, along with the determinant, needs to be properly defined.

Useful property. Starting from

$$\mathrm{e}^{\mathrm{i} q \cdot J} \equiv \mathrm{e}^{\mathrm{i}\int \mathrm{d}y\, q(y) J(y)}$$

one may find its derivative

$$\begin{aligned}\delta_{J(x)} \mathrm{e}^{\mathrm{i} q \cdot J}|_{J=0} &= \delta_{J(x)} \mathrm{e}^{\mathrm{i}\int \mathrm{d}y\, q(y) J(y)}|_{J=0} = \delta_{J(x)} \left[\mathrm{i} \int \mathrm{d}y\, q(y) J(y)\right] \mathrm{e}^{\mathrm{i} q \cdot J}|_{J=0} \\ &= \mathrm{i} q(x) \mathrm{e}^{\mathrm{i} q \cdot J}|_{J=0} = \mathrm{i} q(x)\end{aligned}$$

A function can be expressed as the derivative of an exponential with respect to a parameter. This parameter J is called the source of q. Since

$$q(x) = -\mathrm{i}\, \delta_{J(x)} \mathrm{e}^{\mathrm{i} q \cdot J}|_{J=0}$$

then one may write any function of $q(x)$ as

$$G(q(x)) = G(-\mathrm{i}\, \delta_{J(x)}) \mathrm{e}^{\mathrm{i} q \cdot J}|_{J=0}$$

To make sense of the right-hand expression, one may expand the function $G(x)$ in a Taylor series.

2 Feynman path integral in quantum mechanics

See Feynman–Hibbs, Srednicki, Anselmi, Cheng–Li. Quantum mechanics is a non-relativistic theory developed through the canonical quantization of observables (also called first quantization) $[\hat{x}, \hat{p}] = i\hbar$, but it can also be equivalently formulated through Feynman's path integrals. In quantum field theory one can use the canonical quantization of fields (also called second quantization): expanding the free field in terms of plane waves, one promotes the Fourier coefficients to operators obeying (anti-)commutation relations

$$[a(\mathbf{k}), a^\dagger(\mathbf{k})] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}')$$

Though one may also use functional quantization which is developed in the following.

A massive particle. See Anselmi, Cheng–Li and Srednicki (for philosophy, application etc see Feynman–Hibbs). A simple case is the one of one massive particle in one dimension subject to a potential V. The Hamiltonian of the system is

$$H = \frac{p^2}{2m} + V(q)$$

where q is the position of the particle. The observables are promoted to operators

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}) \,, \quad [\hat{q}, \hat{p}] = \mathrm{i}\hbar$$

The propagation of the particle is described by Schrödinger's equation

$$\hat{H} \, |\psi(t)\rangle = \mathrm{i}\hbar \, \partial_t \, |\psi(t)\rangle$$

where $|\psi(t)\rangle$ is the state of the particle. Starting from the Hamiltonian, the corresponding Lagrangian is obtained by the Legendre transform

$$L(q, \dot{q}) = [p\dot{q} - H(p, q)]_{p=p(q,\dot{q})}$$

The expression of the momentum p in terms of q and $\dot{q}$ is obtain by solving Hamilton's equations

$$\dot{q} = \partial_p H \,, \quad \dot{p} = -\partial_q H$$

To describe the propagation of a particle one needs to know the probability of measuring such particle after some time. One would like an alternative mathematical formulation to ordinary quantum mechanics. Consider a particle at some initial point q_i and at some initial time t_i. Letting it propagate up to a final time t_f, one would like to compute the probability that the particle is at a final position q_f. Between the two measurements $t_i < t < t_f$, the particle is not being observed and it propagates. One can at most compute a probability amplitude given by

$${}_{\mathrm{S}}\langle q_f | \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} H(t_f - t_i)} \, |q_i\rangle_{\mathrm{S}} \equiv {}_{\mathrm{H}}\langle q_f, t_f | q_i, t_i \rangle_{\mathrm{H}}$$

where $|q, t\rangle_{\mathrm{H}} = \mathrm{e}^{\frac{\mathrm{i}}{\hbar} H t} \, |q\rangle_{\mathrm{S}}$ is an instantaneous (and time independent) eigenvector of the time dependent position operator $Q(t)$ (see Srednicki, p. 43). Feynman observes that one does not know the precise path of the particle during its propagation, so the particle could take any path between the start and the end points. The only condition is that when $\hbar \to 0$, the classical limit, among the infinite number of possible paths between the start and the end, the only path available is the classical one, which is taken from the variational principle (called action principle in quantum field theory)

$$\delta S = 0 \,, \quad \forall q \mid \delta q(t_i) = \delta q(t_f) = 0$$

Feynman proposes that the probability amplitude should be the sum over all possible paths, each weighed by the probability of the particle to travel along such path: when taking the classical limit, the most probable path is the classical one. They weight of each path is

$$\mathrm{e}^{\frac{\mathrm{i}}{\hbar} S(q_f, t_f, q_i, t_i)}$$

where the action is the classical one. The reason the weight is the above can be seen as follows. Interpreting the sum as an average, in the classical limit the weight oscillates rapidly and its average is zero. The classical path is the only one that can survive because the action is at a minimum, so minimal oscillation. Therefore, the path integral formulation is

$${}_{\mathrm{S}}\langle q_f | \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} H(t_f - t_i)} \, |q_i\rangle_{\mathrm{S}} = \int [\mathcal{D}q] \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S(q_f, t_f, q_i, t_i)}$$

The right-hand side is just a formal definition of the idea of path integral.

Lecture 2

mar 05 mar 2024 10:30

One would like to give an explicit prescription for the sum of all possible paths between two points

$$\int [\mathcal{D}q]$$

Since one does not know how to deal with a continuum of paths, one can discretize the theory and then take the continuum limit. Time and space are discretized in equal-length steps

$$\delta t = \frac{t_f - t_f}{N+1}$$

The position of the particle is no longer $q(t)$ but $q(t_j)$. The smaller is the step $\delta t \ll 1$, the better the discrete path approximates the Feynman path. Consecutive discrete points can be joined by straight segments giving the discrete path as

$$q(t) = \frac{q_j - q_{j-1}}{\delta t}(t - t_{j-1}) + q_{j-1}\,, \quad j = 1, \dots, N+1$$

where $q_0 \equiv q_i$ and $q_{N+1} \equiv q_f$. From the equation above, the time interval is

$$t_f - t_i = (N+1)\delta t$$

so the probability amplitude becomes

$$\begin{aligned} A &= {}_{\mathrm{S}}\langle q_f | \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H(t_f - t_i)} \, |q_i\rangle_{\mathrm{S}} = {}_{\mathrm{S}}\langle q_f | \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\delta t\,(N+1)} \, |q_i\rangle_{\mathrm{S}} = {}_{\mathrm{S}}\langle q_f | \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\delta t} \cdots \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\delta t} \, |q_i\rangle_{\mathrm{S}} \\ &= \int \prod_{j=1}^{N} \mathrm{d}q_j \; \langle q_f | \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\,\delta t} \, |q_N\rangle\langle q_N| \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\,\delta t} \, |q_{N-1}\rangle \cdots \langle q_1| \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\,\delta t} \, |q_i\rangle \end{aligned}$$

at the second line one has inserted N identities as the completeness relation in the position eigenstates

$$\int \mathrm{d}q \; |q\rangle\langle q| = I$$

Considering a generic factor, one notices

$$\langle q_{j+1}| \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\,\delta t} \, |q_j\rangle = \langle q_{j+1}| \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}[\frac{p^2}{2m} + V(q)]\,\delta t} \, |q_j\rangle$$

Since q and p do not compute, one cannot split the exponential the same way one does with numbers, but one has to apply the Zassenhaus formula (related to the Baker–Campbell–Hausdorff formula):

$$\mathrm{e}^{t(x+y)} = \mathrm{e}^{tx}\mathrm{e}^{ty}\mathrm{e}^{-\frac{1}{2}t^2[x,y] + o(t^2)}$$

The first order in t is just $\mathrm{e}^{t(x+y)} \approx \mathrm{e}^{tx}\mathrm{e}^{ty}$. Therefore, one has

$$\langle q_{j+1}| \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\,\delta t} \, |q_j\rangle = \langle q_{j+1}| \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{p^2}{2m}\,\delta t}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}V(q)\,\delta t} \, |q_j\rangle$$

The position states are eigenstates of the potential, but not of the momentum. One can insert a completeness relation in the momentum eigenstates between the two exponentials

$$\begin{aligned} \langle q_{j+1}| \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\,\delta t} \, |q_j\rangle &= \int \mathrm{d}p_j \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{p_j^2}{2m}\,\delta t}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}V(q_j)\,\delta t} \, \langle q_{j+1}|p_j\rangle \, \langle p_j|q_j\rangle \\ &= \int \frac{\mathrm{d}p_j}{2\pi\hbar} \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{p_j^2}{2m}\,\delta t - \frac{\mathrm{i}}{\hbar}V(q_j)\,\delta t}\mathrm{e}^{\mathrm{i}p_j(q_{j+1} - q_j)} \end{aligned}$$

the operators can then act on their eigenstates producing the associated eigenvalue. At the second line one remembers

$$\langle q|p\rangle = \frac{\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}}{\sqrt{2\pi\hbar}}$$

Since δt is infinitesimal, then one may substitute the potential with its value at the midpoint between two coordinates

$$V(q_j) \to V(\bar{q}_j)\,, \quad \bar{q}_j = \frac{q_{j+1}+q_j}{2}$$

Replacing every factor inside the probability amplitude, one gets

$$A = \int \prod_{j=1}^{N} \mathrm{d}q_j \prod_{k=0}^{N} \frac{\mathrm{d}p_k}{2\pi\hbar}\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}p_k(q_{k+1}-q_k)-\frac{\mathrm{i}}{\hbar}H(p_k,\bar{q}_k)\,\delta t}$$

In the limit $N \to \infty$, equivalent to $\delta t \to 0$, one has

$$\begin{aligned} A &= \lim_{N\to\infty} \int \prod_{j=1}^{N} \mathrm{d}q_j \prod_{k=0}^{N} \frac{\mathrm{d}p_k}{2\pi\hbar}\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}p_k \frac{q_{k+1}-q_k}{\delta t}\,\delta t-\frac{\mathrm{i}}{\hbar}H(p_k,\bar{q}_k)\,\delta t} \\ &= \lim_{N\to\infty} \int \prod_{j=1}^{N} \mathrm{d}q_j \prod_{k=0}^{N} \frac{\mathrm{d}p_k}{2\pi\hbar}\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}p_k\dot{q}_k\,\delta t-\frac{\mathrm{i}}{\hbar}H(p_k,\bar{q}_k)\,\delta t} \\ &\equiv \int [\mathcal{D}q\,\mathcal{D}p] \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_i}^{t_f} \mathrm{d}t\,\big[p(t)\dot{q}(t)-H(p(t),q(t))\big]\right] \end{aligned}$$

The last line is the definition of the path integral.

Since the explicit formula of the Hamiltonian is known

$$H = \frac{p^2}{2m} + V(q)$$

one can integrate over a momenta p_k to obtain

$$\begin{aligned} \int \mathrm{d}p_k\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p_k\dot{q}_k-\frac{p_k^2}{2m})\,\delta t} &= \frac{1}{2\pi\hbar}\sqrt{\frac{2\pi\hbar m}{\mathrm{i}\,\delta t}} \exp\left[-\frac{\dot{q}_k^2\,(\delta t)^2}{\hbar^2}\left(\frac{2\mathrm{i}}{\hbar}\frac{\delta t}{m}\right)^{-1}\right] \\ &= \left[\frac{m}{2\pi\hbar\mathrm{i}\,\delta t}\right]^{\frac{1}{2}} \mathrm{e}^{\frac{\mathrm{i}m}{2\hbar}(\frac{q_{k+1}-q_k}{\delta t})^2\,\delta t} \end{aligned}$$

where one applies the analytic continuation of the Gaussian integral

$$\int \mathrm{d}x\, \mathrm{e}^{-\frac{a}{2}x^2+bx} = \sqrt{\frac{2\pi}{a}}\mathrm{e}^{\frac{b^2}{2a}}\,, \quad a = \frac{\mathrm{i}}{\hbar}\frac{\delta t}{m}\,, \quad b = \frac{\mathrm{i}}{\hbar}\dot{q}_k\,\delta t$$

At the second line, one replaces

$$\dot{q}_k \equiv \frac{q_{k+1}-q_k}{\delta t}$$

Performing the integration for every $k = 0, \ldots, N$, the probability amplitude is then

$$A = \lim_{N\to\infty}\left[\frac{m}{2\pi\hbar\mathrm{i}\,\delta t}\right]^{\frac{N+1}{2}} \int \prod_{j=1}^{N} \mathrm{d}q_j\, \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{j=0}^{N}\delta t\left(\frac{m}{2}\left(\frac{q_{j+1}-q_j}{\delta t}\right)^2 - V\right)\right]$$

in the limit, the exponent is the classical Lagrangian (i.e. there are no operators)

$$\frac{\mathrm{i}}{\hbar}\sum_{j=0}^{N}\delta t\left[\frac{m}{2}\left(\frac{q_{j+1}-q_j}{\delta t}\right)^2 - V\right] = \frac{\mathrm{i}}{\hbar}\int_{t_i}^{t_f} \mathrm{d}t\, L(q,\dot{q})$$

The coefficient in square bracket of the amplitude is divergent but it compensates the infinitesimal nature of the integration measure. The path integral is then defined as

$$\int [\mathcal{D}q] \equiv \lim_{N\to\infty}\left[\frac{m}{2\pi\hbar\mathrm{i}\,\delta t}\right]^{\frac{N}{2}} \int \prod_{j=1}^{N-1} \mathrm{d}q_j$$

mind the rescaling $N+1 \to N$. To give a meaningful definition, time has to be discretized: continuous functions become discrete (like functions on a lattice).

Remark. See Feynman–Hibbs. If the path integral formulation is a consistent alternative prescription, then it concerns the computation of probability amplitudes, called kernels or Feynman propagators

$$K(q_f, t_f; q_i, t_i) \equiv \langle q_f | \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} H(t_f - t_i)} \, | q_i \rangle$$

The definition is independent of ordinary quantum mechanics and their operators. Probability amplitudes are weighed with the exponential of the classical action, the only sign of quantum mechanics is the appearance of Planck's constant $\hbar$.

One may notice that

- The weight $\mathrm{e}^{\frac{\mathrm{i}}{\hbar} S}$ is given in terms of the classical action

$$S = \int_{t_i}^{t_f} \mathrm{d}t \, L(q(t), \dot{q}(t))$$

  The path integral is then referred to as the quantum integral.

- One may find a first equivalence with ordinary quantum mechanics. Consider the quantum mechanical wave function

$$\begin{aligned} \psi(q,t) = {}_{\mathrm{H}}\langle q, t | \psi \rangle = {}_{\mathrm{S}}\langle q | \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} H t} \, | \psi \rangle = \int \mathrm{d}q' \; {}_{\mathrm{S}}\langle q | \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} H (t - t')} \, | q' \rangle \, \langle q', t' | \psi \rangle \\ = \int \mathrm{d}q' \, K(q, t; q', t') \psi(q', t') \end{aligned}$$

  The kernel has the meaning of evolution operator. One may check that the wave function $\psi(q,t)$ satisfies Schrödinger's equation (see Anselmi, Feynman–Hibbs) thanks to the properties of the kernel.

Kernel of a free particle. See Anselmi. The potential of a free particle is identically zero $V(q) = 0$. The kernel is

$$K_0(q_f, t_f; q_i, t_i) = \lim_{N \to \infty} \left[\frac{m}{2\pi\hbar \mathrm{i} \, \delta t} \right]^{\frac{N}{2}} \int \prod_{j=1}^{N-1} \mathrm{d}q_j \, \exp\left[\frac{\mathrm{i}m}{2\hbar \delta t} \sum_{j=1}^{N} (q_j - q_{j-1})^2 \right]$$

Its form is that of many Gaussian integrals, but they are not factorized, so a change of variables is necessary

$$\widetilde{q}_j = q_j - q_f \,, \quad \widetilde{q}_0 = q_i - q_f \,, \quad \widetilde{q}_N = 0 \,, \quad \mathrm{d}q_j = \mathrm{d}\widetilde{q}_j$$

remembering that after rescaling? [r] one has

$$q_0 = q_i \,, \quad q_N = q_f$$

The sum in the exponent is then

$$\begin{aligned} \sum_{j=1}^{N} (q_j - q_{j-1})^2 &= (q_1 - q_i)^2 + (q_2 - q_1)^2 + \cdots + (q_N - q_{N-1})^2 \\ &= [\widetilde{q}_1 - (q_i - q_f)]^2 + (\widetilde{q}_2 - \widetilde{q}_1)^2 + \cdots + (\cancel{\widetilde{q}_N} - \widetilde{q}_{N-1})^2 \\ &= (q_i - q_f)^2 - 2\widetilde{q}_1 (q_i - q_f) + 2\widetilde{q}_1^2 + 2\widetilde{q}_2^2 + \cdots + 2\widetilde{q}_{N-1}^2 \\ &\quad -2\widetilde{q}_1 \widetilde{q}_2 - 2\widetilde{q}_3 \widetilde{q}_2 - \cdots - 2\widetilde{q}_{N-2} \widetilde{q}_{N-1} \\ &= (q_i - q_f)^2 + 2\widetilde{q}_1 (q_f - q_i) + \widetilde{q}^{\top} \widetilde{M} \widetilde{q} \end{aligned}$$

where one has a $N-1$ dimensional vector and square matrix

$$q = \begin{bmatrix} q \\ \vdots \\ q_{N-1} \end{bmatrix} \,, \quad \widetilde{M} = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots \\ -1 & 2 & -1 & 0 & \cdots \\ 0 & -1 & 2 & -1 & \cdots \\ 0 & 0 & -1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The kernel is then

$$K_0(q_f,t_f;q_i,t_i) = \lim_{N\to\infty}\left[\frac{m}{2\pi\hbar \mathrm{i}\,\delta t}\right]^{\frac{N}{2}} \mathrm{e}^{\frac{\mathrm{i}m}{2\hbar\,\delta t}(q_f-q_i)^2}\int\prod_{j=1}^{N-1}\mathrm{d}\widetilde{q}_j \exp\left[\frac{\mathrm{i}m}{2\hbar\,\delta t}\left[\widetilde{q}^{\top}\widetilde{M}\widetilde{q}+2\widetilde{q}_1(q_f-q_i)\right]\right]$$
$$= \lim_{N\to\infty}\left[\frac{m}{2\pi\hbar \mathrm{i}\,\delta t\,N}\right]^{\frac{1}{2}} \mathrm{e}^{\frac{\mathrm{i}m}{2\hbar\,\delta t}(q_f-q_i)^2}\mathrm{e}^{-\frac{\mathrm{i}m}{2\hbar\,\delta t}(q_f-q_i)^2\frac{N-1}{N}}$$
$$= \lim_{N\to\infty}\left[\frac{m}{2\pi\hbar \mathrm{i}\,\delta t\,N}\right]^{\frac{1}{2}} \mathrm{e}^{\frac{\mathrm{i}m}{2\hbar\,\delta t\,N}(q_f-q_i)^2} = \left[\frac{m}{2\pi\hbar \mathrm{i}\,\Delta t}\right]^{\frac{1}{2}} \mathrm{e}^{\frac{\mathrm{i}m}{2\hbar\,\Delta t}(q_f-q_i)^2}$$

at the second line one remembers that

$$\int\prod_{i=1}^{N-1}\mathrm{d}x_i\,\mathrm{e}^{-\frac{1}{2}x^{\top}Mx+b^{\top}x} = \frac{(2\pi)^{\frac{N-1}{2}}}{\sqrt{\det M}}\mathrm{e}^{\frac{1}{2}b^{\top}M^{-1}b}$$

where

$$M = -\frac{\mathrm{i}m}{\hbar\,\delta t}\widetilde{M}\,,\quad \det\widetilde{M} = N\,,\quad (\widetilde{M}^{-1})_{11} = \frac{N-1}{N}\,,\quad b_1 = \frac{\mathrm{i}m}{\delta t}(q_f-q_i)$$

At the last line, one notices that

$$\delta t = \frac{t_f-t_i}{N} = \frac{\Delta t}{N}$$

With this specific example, one may notice that the divergence of δt is compensated to give a finite result.

Properties of the kernel. A few properties:

- The equal-time limit is
$$\lim_{t_f\to t_i} K_0(q_f,t_f;q_i,t_i) = \delta(q_f-q_i)$$
equivalent to $q_f\to q_i$. This is consistent with an expression of the Dirac delta given by
$$\lim_{\varepsilon\to 0}\frac{1}{\sqrt{2\pi\varepsilon}}\mathrm{e}^{-\frac{q^2}{2\varepsilon}} = \delta(q)$$
[r]

- The kernel satisfies Schrödinger's equation
$$\mathrm{i}\hbar\,\partial_t K_0 = -\frac{\hbar^2}{2m}\,\partial_q^2 K_0$$
This can be checked for
$$K_0(q,t;0,0) = \sqrt{\frac{m}{2\pi\hbar i t}}\mathrm{e}^{\frac{\mathrm{i}m}{2\hbar t}q^2}$$
It follows that the wave function
$$\psi(q,t) = \int \mathrm{d}q'\,K_0(q,t;q',t')\psi(q',t')$$
satisfies Schrödinger's equation because its dependence on time comes from the kernel.

- Composition law
$$K_0(q,t;q_0,t_0) = \int \mathrm{d}q'\,K_0(q,t;q',t')K_0(q',t';q_0,t_0)$$
This can be seen from
$$K_0(q,t,;q_0,t_0) = \left[\frac{m}{2\pi\hbar \mathrm{i}(t-t_0)}\right]^{\frac{1}{2}} \mathrm{e}^{\frac{\mathrm{i}m}{2\hbar}\frac{(q-q_0)^2}{t-t_0}}$$

but working backwards: integrating to obtain such expression. The integral is not trivial since in the exponents one has

$$\begin{aligned}\frac{\mathrm{i}m}{2}\left[\frac{(q-q')^2}{t-t'}+\frac{(q'-q_0)^2}{t'-t_0}\right] &= \frac{\mathrm{i}m}{2}\left[\frac{[(t'-t_0)q+(t-t')q_0+(t_0-t)q']^2}{(t-t_0)(t-t')(t'-t_0)}+\frac{(q-q_0)^2}{t-t_0}\right] \\ &= \frac{\mathrm{i}m}{2}\frac{(q-q_0)^2}{t-t_0}+\frac{\mathrm{i}m}{2}\left[-\frac{q'}{(t-t')(t'-t_0)}+\delta q\right]\end{aligned}$$

[r] idk if true
where δq is some shift and one can substitute the bracket with another variable.

Exercise. Finish the calculation.

Lecture 3

mer 06 mar 2024 10:30

2.1 Quadratic potential

See Srednicki, Schulman cap 6. From now on one sets $\hbar = 1$. Consider a one dimensional massive particle in a quadratic potential

$$V(q) = \frac{1}{2}c(t)q^2$$

The Lagrangian is

$$L(q,\dot{q}) = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}c(t)q^2$$

One would like to compute the kernel, the Feynman propagator

$$K(q_b,t_b;q_a,t_a) = \int_{q_a,t_a}^{q_b,t_b}[\mathcal{D}q]\,\mathrm{e}^{\mathrm{i}S}$$

The following procedure works well for quadratic potentials. One can expand a generic trajectory $q(t)$ around a classical trajectory $\bar{q}(t)$ (that is, solution of the classical equations of motion obtained from the variational principle)

$$q(t) = \bar{q}(t) + \delta q(t)$$

This fluctuation does not apply to the end points $\delta q(t_a) = \delta q(t_b) = 0$. The variation of the action is

$$\delta S = \int \mathrm{d}t\,\frac{\delta S}{\delta q(t)}\,\delta q(t) = 0\,, \quad \forall \delta q(t) \implies \frac{\delta S}{\delta q(t)} = 0$$

For the quadratic potential, the equations of motion are

$$m\ddot{\bar{q}} + c(t)\bar{q} = 0$$

The path integral sums all the fluctuations δq from the classical path. To find the kernel, one needs to compute the action

$$S[\bar{q}+\delta q, \dot{\bar{q}}+\delta\dot{q}]$$

First method. There are two methods to compute the kernel. The first one is

$$S = \int \mathrm{d}t\,\left[\frac{1}{2}m\dot{q}^2 - \frac{1}{2}c(t)q^2\right] = \int \mathrm{d}t\,\left[\frac{1}{2}m(\dot{\bar{q}}+\delta\dot{q})^2 - \frac{1}{2}c(t)(\bar{q}+\delta q)^2\right] = \cdots = S[\bar{q}] + S[\delta q]$$

[r] This result is particular to quadratic potentials.

Second method. The second method utilizes functional derivatives and the equivalent of the Taylor series

$$S[\bar{q}+\delta q] = S[\bar{q}] + \delta_q S|_{\bar{q}}\,\delta q + \frac{1}{2}\,\delta_q^2 S|_{\bar{q}}\,(\delta q)^2$$

The expansion ends at second order because the potential is quadratic. [r] The linear term is zero because $\bar{q}$ is the solution to the classical equations of motion, thus extremizing the action. Only the last term has to be computed

$$\delta_q^2 S\,(\delta q)^2 = \delta_q(\delta_q S\,\delta q)\,\delta q = \int \mathrm{d}t''\,\frac{\delta}{\delta q(t'')}\left[\int \mathrm{d}t'\,\frac{\delta S}{\delta q(t')}\delta q(t')\right]\delta q(t'')$$

The inner integral is

$$\begin{aligned}\int \mathrm{d}t'\,\frac{\delta S}{\delta q(t')}\,\delta q(t') &= \int \mathrm{d}t'\int \mathrm{d}t\,\frac{\delta L(q(t))}{\delta q(t')}\,\delta q(t') \\ &= \int \mathrm{d}t'\,\mathrm{d}t\,[m\dot{q}(t)\,\partial_t\delta(t-t') - c(t)q(t)\delta(t-t')]\,\delta q(t') \\ &= \int \mathrm{d}t\,[-m\dot{q}(t)\delta\dot{q}(t) - c(t)q(t)\delta q(t)]\end{aligned}$$

where one remembers

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}c(t)q^2\,, \quad \frac{\delta q(t)}{\delta q(t')} = \delta(t-t')$$

in particular

$$\frac{\delta\dot{q}^2(t)}{\delta q(t')} = \frac{\delta\dot{q}^2(t)}{\delta\dot{q}(t)}\frac{\delta\dot{q}(t)}{\delta q(t')} = 2\dot{q}(t)\,\partial_t\frac{\delta q(t)}{\delta q(t')} = 2\dot{q}(t)\,\partial_t\delta(t-t')$$

The second derivative is then

$$\begin{aligned}\delta_q^2 S\,(\delta q)^2 &= \int \mathrm{d}t''\,\frac{\delta}{\delta q(t'')}\left[\int \mathrm{d}t\,[-m\dot{q}(t)\delta\dot{q}(t) - c(t)q(t)\delta q(t)]\right]\delta q(t'') \\ &= \int \mathrm{d}t''\,\mathrm{d}t\,[-m\,\partial_t\delta(t-t'')\,\delta\dot{q}(t)\,\delta q(t'') - c(t)\delta(t-t'')\,\delta q(t)\,\delta q(t'')] \\ &= \int \mathrm{d}t\,\left[m(\delta\dot{q})^2 - c(t)[\delta q(t)]^2\right]\end{aligned}$$

This is exactly the original action but with δq instead of q (and a missing factor of $1/2$). Therefore

$$S[\bar{q}+\delta q] = S[\bar{q}] + \frac{1}{2}\,\delta_q^2 S|_{\bar{q}}\,(\delta q)^2 = S[\bar{q}] + S[\delta q]$$

Remark. This method of the Taylor expansion is more general and can be applied to arbitrary potentials of the form $V(q,g)$ where $g \ll 1$ is a coupling constant. Starting from the cubic term of the Taylor series, one has contributions only from the potential since the kinetic term is quadratic.

Kernel. The kernel becomes

$$K(q_b,t_b;q_a,t_a) = \mathrm{e}^{\mathrm{i}S[\bar{q}]}\int_{0,t_a}^{0,t_b}[\mathcal{D}\delta q]\,\mathrm{e}^{\mathrm{i}S[\delta q]} \equiv \mathrm{e}^{\mathrm{i}S[\bar{q}]}K(0,t_b;0,t_a)$$

To compute the kernel, one has to discretize time

$$\frac{t_b-t_a}{N+1} = \delta t \implies K(0,t_b;0,t_a) = \lim_{N\to\infty}\left[\frac{m}{2\pi\mathrm{i}\,\delta t}\right]^{\frac{1}{2}}\int\prod_{j=1}^{N}\mathrm{d}(\delta q_j)\,\mathrm{e}^{\mathrm{i}S[\delta q]}$$

where the action is discretized also

$$\mathrm{i}S[\delta q] = \sum_{j=0}^{N}\mathrm{i}\left[\frac{m}{2\,\delta t}(\delta q_{j+1}-\delta q_j)^2 - \frac{1}{2}\delta t\,c_j(\delta q_j)^2\right], \quad c_j = c(t_a+\delta t\,j)\,, \quad \delta q_j = \delta q\,(t_a+\delta t\,j)$$

keeping in mind that the extremes have to be constant

$$\delta q_0 = \delta q(t_a) = 0\,, \quad \delta q_{N+1} = \delta q(t_b) = 0$$

[r] Considering a vector

$$\eta = \begin{bmatrix} \delta q_1 \\ \vdots \\ \delta q_N \end{bmatrix}$$

by completing the square? [r], the action can be rewritten as

$$\mathrm{i}S[\delta q] = -\eta^\top A \eta\,, \quad A = \frac{m}{2\pi \mathrm{i}\,\delta t} \begin{bmatrix} 2 - \frac{(\delta t^2)}{m} c_1 & -1 & 0 & \cdots \\ -1 & 2 - \frac{(\delta t^2)}{m} c_2 & -1 & \cdots \\ 0 & -1 & 2 - \frac{(\delta t)^2}{m} c_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The kernel is then

$$\begin{aligned} K(0,t_b;0,t_a) &= \lim_{N\to\infty} \left[\frac{m}{2\pi\mathrm{i}\,\delta t}\right]^{\frac{N+1}{2}} \int \mathrm{d}^N \eta\, \mathrm{e}^{-\eta^\top A \eta} = \lim_{N\to\infty} \left[\frac{m}{2\pi\mathrm{i}\,\delta t}\right]^{\frac{N+1}{2}} \frac{\pi^{\frac{N}{2}}}{\sqrt{\det A}} \\ &= \lim_{N\to\infty} \left[\frac{m}{2\pi\mathrm{i}\,\delta t} \frac{\pi^N}{(\frac{2\pi\mathrm{i}\,\delta t}{m})^N \det A}\right]^{\frac{1}{2}} = \lim_{N\to\infty} \left[\frac{m}{2\pi\mathrm{i}} \frac{1}{\delta t (\frac{2\pi\mathrm{i}\,\delta t}{m})^N \det A}\right]^{\frac{1}{2}} \end{aligned}$$

At the second equality of the first line one applies the Gaussian integral (notice there is no $1/2$). Let the denominator be

$$F_N(t_b,t_a) \equiv \delta t \left(\frac{2\pi\mathrm{i}\,\delta t}{m}\right)^N \det A = \delta t\, P_N$$

and

$$P_N \equiv \left(\frac{2\pi\mathrm{i}\,\delta t}{m}\right)^N \det A = \det \begin{bmatrix} 2 - \frac{(\delta t)^2}{m} c_1 & -1 & 0 & \cdots \\ -1 & 2 - \frac{(\delta t)^2}{m} c_2 & -1 & \cdots \\ 0 & -1 & 2 - \frac{(\delta t)^2}{m} c_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

One can find a recursion relation for a generic P_j given by

$$P_{j+1} = \left(2 - \frac{(\delta t)^2}{m} c_{j+1}\right) P_j - P_{j-1}\,, \quad j = 1, \ldots, N\,, \quad P_0 = 1\,, \quad P_1 = 2 - \frac{(\delta t)^2}{m} c_1$$

This relation can be rewritten as

$$\delta t\, \frac{P_{j+1} - 2P_j + P_{j-1}}{\delta t^2} = -\frac{c_{j+1}}{m} P_j\, \delta t$$

Letting

$$F_j = \delta t\, P_j \equiv \varphi_j = \varphi(t_a + \varepsilon j)$$

the previous equation becomes

$$\frac{\varphi_{j+1} - 2\varphi_j + \varphi_{j-1}}{(\delta t)^2} = -\frac{c_{j+1}}{m} \varphi_j$$

For $\delta t \to 0$ the identity above becomes a second order differential equation

$$\mathrm{d}_t^2 \varphi(t) = -\frac{c(t)}{m} \varphi(t)$$

The initial value problem is

$$\varphi(t = t_a) = \delta t\, P_0 \to 0\,, \quad \mathrm{d}_t \varphi(t = t_a) = \delta t\, \frac{P_1 - P_0}{\delta t} = 2 - \frac{\delta t^2}{m} c_1 - 1 \to 1$$

The quantity needed to compute in the kernel $K(0,t_b;0,t_a)$ is $F(t_b,t_a)=\varphi(t=t_b)$. The full kernel is thus

$$K(q_b,t_b;q_a,t_a)=\mathrm{e}^{\mathrm{i}S[\bar{q}]}K(0,t_b;0,t_a)=\mathrm{e}^{\mathrm{i}S[\bar{q}]}\left[\frac{m}{2\pi\mathrm{i}F(t_b,t_a)}\right]^{\frac{1}{2}}$$

where one has

$$F(t,t_a)=\varphi(t)$$

which satisfies

$$m\,\mathrm{d}_t^2F(t,t_a)+c(t)F(t,t_a)=0\,,\quad F(t_a,t_a)=0\,,\quad \mathrm{d}_tF(t_a,t_a)=1$$

Simple harmonic oscillator. The parameter is

$$c(t)=m\omega^2$$

The differential equation above becomes

$$\ddot{F}+\omega^2F=0\,,\quad F(t_a)=0\,,\quad \dot{F}(t_a)=1$$

whose solution is

$$F(t,t_a)=\frac{1}{\omega}\sin[\omega(t-t_a)]$$

The desired quantity is then

$$F(t_b,t_a)=\frac{1}{\omega}\sin[\omega(t_b-t_a)]=\frac{1}{\omega}\sin\omega T\,,\quad T\equiv t_b-t_a$$

To compute the kernel one has to find the action and path for the classical equations of motion. The Lagrangian is

$$L=\frac{1}{2}m\dot{q}^2-\frac{1}{2}m\omega^2q^2$$

The equations of motion are

$$m\ddot{q}+m\omega^2q=0\,,\quad q(t_a)=q_a\,,\quad q(t_b)=q_b$$

The solution is therefore

$$q(t)=\frac{1}{\sin\omega T}[q_b\sin\omega(t-t_a)+q_a\sin\omega(t_b-t)]$$

Its derivative is

$$\dot{q}(t)=\frac{\omega}{\sin\omega T}[q_b\cos\omega(t-t_a)-q_a\cos\omega(t_b-t)]$$

The action is

$$\begin{aligned}S&=\int\mathrm{d}t\left[\frac{1}{2}m\dot{q}^2-\frac{1}{2}m\omega^2q^2\right]=\frac{1}{2}mq\dot{q}\Big|_{t_a}^{t_b}-\int\mathrm{d}t\left[\frac{1}{2}mq\ddot{q}+\frac{1}{2}m\omega^2q^2\right]\\&=\frac{1}{2}m\bar{q}(t_b)\dot{\bar{q}}(t_b)-\frac{1}{2}m\bar{q}(t_a)\dot{\bar{q}}(t_a)=\cdots\\&=\frac{m\omega}{2\sin\omega T}[(q_a^2+q_b^2)\cos\omega T-2q_aq_b]\end{aligned}$$

At the first equality one integrates by parts. The last bracket is zero

$$\frac{1}{2}mq(\ddot{q}+\omega^2q)|_{\bar{q}}=0$$

on the classical path since the equations of motion are

$$\ddot{\bar{q}}+\omega^2\bar{q}=0$$

Finally, the kernel is

$$K(q_b,t_b;q_a,t_a)=\left[\frac{m\omega}{2\pi\mathrm{i}\sin\omega T}\right]^{\frac{1}{2}}\exp\left[\frac{\mathrm{i}m\omega}{2\sin\omega T}\big((q_a^2+q_b^2)\cos\omega T-2q_aq_b\big)\right]$$

For any quadratic potential, the kernel can be computed by solving the differential equation of $F(t,t_a)$. For other potentials, one has to use perturbative approaches.

2.2 Partition function

The kernel is defined as

$$K(q_f,t_f;q_i,0) = {}_{\mathrm{S}}\langle q_f|\mathrm{e}^{-\mathrm{i}Ht_f}\,|q_i\rangle_{\mathrm{S}} = \int [\mathcal{D}q]\,\mathrm{e}^{\mathrm{i}S[q]}$$

The partition function is the integral of a periodic kernel (i.e. its initial and final points coincide)

$$Z(t) = \int \mathrm{d}q\, K(q,t;q,0)$$

Relation to statistical mechanics. This definition coincides with the statistical mechanical definition

$$Z(t) = \mathrm{Tr}\,\mathrm{e}^{-\mathrm{i}\hat{H}t}$$

Setting the system in a box, one has a discrete energy spectrum E_n with corresponding energy states $|n\rangle$. The identity can be decomposed as

$$I = \sum_n |n\rangle\langle n|$$

The kernel is then

$$\begin{aligned} K(q_f,t_f;q_i,0) &= \langle q_f|\,\mathrm{e}^{-\mathrm{i}Ht_f}\,|q_i\rangle = \sum_n \langle q_f|\,\mathrm{e}^{-\mathrm{i}Ht_f}\,|n\rangle\,\langle n|q_i\rangle = \sum_n \mathrm{e}^{-\mathrm{i}E_nt_f}\,\langle q_f|n\rangle\,\langle n|q_i\rangle \\ &= \sum_n \mathrm{e}^{-\mathrm{i}E_nt_f}\,\psi_n^\dagger(q_f)\psi_n(q_i) \end{aligned}$$

The partition function is then

$$\begin{aligned} Z(t) &= \int \mathrm{d}q\, K(q,t;q,0) = \int \mathrm{d}q \sum_n \mathrm{e}^{-\mathrm{i}E_nt}\psi_n^\dagger(q)\psi_n(q) \\ &= \sum_n \mathrm{e}^{-\mathrm{i}E_nt}\int \mathrm{d}q\,|\psi_n(q)|^2 = \sum_n \mathrm{e}^{-\mathrm{i}E_nt} \end{aligned}$$

remembering that the wave function is normalized such that the above integral (i.e. the probability) is unity. Finally one notices that

$$Z(t) = \mathrm{Tr}\,\mathrm{e}^{-\mathrm{i}\hat{H}t} = \sum_n \mathrm{Tr}\Big[\mathrm{e}^{-\mathrm{i}\hat{H}t}\,|n\rangle\langle n|\Big] = \sum_n \langle n|\,\mathrm{e}^{-\mathrm{i}\hat{H}t}\,|n\rangle = \sum_n \mathrm{e}^{-\mathrm{i}E_nt}$$

at the second equality one inserts the identity inside the trace (remembering that $\mathrm{Tr}(A+B) = \mathrm{Tr}\,A + \mathrm{Tr}\,B$) and the cyclic property of the trace noting that the Hamiltonian is diagonal on the energy eigenstates $|n\rangle$.

Exercise. .[r] Compute the partition function of the harmonic oscillator. Use Taylor expansion. Prove that the harmonic oscillator energy spectrum is

$$E_n = \hbar\omega\left[n+\frac{1}{2}\right]$$

Solution. Starting from the kernel

$$K(q_b,t_b;q_a,t_a) = \Big[\frac{m\omega}{2\pi\mathrm{i}\sin\omega T}\Big]^{\frac{1}{2}}\exp\Big[\frac{\mathrm{i}m\omega}{2\sin\omega T}\big((q_a^2+q_b^2)\cos\omega T - 2q_aq_b\big)\Big]$$

The relevant terms are

$$K(q,t;q,0) = \sqrt{\frac{A}{2\pi\mathrm{i}}}\exp\big[\mathrm{i}Aq^2(\cos\omega t-1)\big]\,,\quad A\equiv\frac{m\omega}{\sin\omega t}$$

Therefore

$$\begin{aligned}
Z(t) &= \int \mathrm{d}q\, K(q,t;q,0) = \sqrt{\frac{A}{2\pi\mathrm{i}}} \int \mathrm{d}q\, \exp\left[\mathrm{i}Aq^2(\cos\omega t - 1)\right], \quad B \equiv A(\cos\omega t - 1) \\
&= \frac{1}{\sqrt{2\pi\mathrm{i}}} \frac{1}{\sqrt{\cos\omega t - 1}} \int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}x^2}, \quad \sqrt{B}q \equiv x\,, \quad \mathrm{d}q = \frac{\mathrm{d}x}{\sqrt{B}} \\
&= \frac{1}{\sqrt{2\pi\mathrm{i}(\cos\omega t - 1)}} \sqrt{\frac{2\pi}{-2\mathrm{i}}} = \frac{1}{\sqrt{2(\cos\omega t - 1)}} = \left[\mathrm{e}^{\mathrm{i}\omega t} - 2 + \mathrm{e}^{-\mathrm{i}\omega t}\right]^{-\frac{1}{2}} \\
&= \left[\mathrm{e}^{\frac{\mathrm{i}}{2}\omega t} - \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t}\right]^{-1} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t} \frac{1}{1 - \mathrm{e}^{-\mathrm{i}\omega t}} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t} \sum_{n\geq 0} \mathrm{e}^{-\mathrm{i}n\omega t} \\
&= \sum_{n\geq 0} \exp\left[-\mathrm{i}\omega \left(n + \frac{1}{2}\right) t\right] = \sum_{n\geq 0} \mathrm{e}^{-\mathrm{i}E_n t} \implies E_n = \omega\left(n + \frac{1}{2}\right)
\end{aligned}$$

At the second line, one applies the Gaussian integral

$$\int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-\frac{a}{2}x^2 + bx} = \sqrt{\frac{2\pi}{a}} \mathrm{e}^{\frac{b^2}{2a}}, \quad a = -2\mathrm{i}$$

At the third line, one expresses the cosine in terms of complex exponentials. At the fourth line, one uses the analytic continuation of the geometric series.

The same calculation can be done in Euclidean time (and considering hyperbolic functions).

2.3 Correlation function

Two-point correlation function. A two-point correlation function for an operator O_{H} in the Heisenberg picture

$$O_{\mathrm{H}}(t) = \mathrm{e}^{\mathrm{i}Ht} O_{\mathrm{S}} \mathrm{e}^{-\mathrm{i}Ht}$$

is given by

$$C(t_1, t_2, t_f, t_i) = {}_{\mathrm{H}}\langle q_f, t_f | \mathcal{T}\{O_{\mathrm{H}}(t_1) O_{\mathrm{H}}(t_2)\} | q_i, t_i \rangle_{\mathrm{H}}$$

Considering $t_i < t_2 < t_1 < t_f$ one has

$$C(t_1, t_2, t_f, t_i) = \langle q_f, t_f | O_{\mathrm{H}}(t_1) O_{\mathrm{H}}(t_2) | q_i, t_i \rangle = \langle q_f | \mathrm{e}^{-\mathrm{i}H(t_f - t_1)} O_{\mathrm{S}} \mathrm{e}^{-\mathrm{i}H(t_1 - t_2)} O_{\mathrm{S}} \mathrm{e}^{-\mathrm{i}H(t_2 - t_i)} | q_i \rangle$$

Supposing that O_{S} is a function of $\hat{q}$ only, then one has

$$\begin{aligned}
&C(t_1, t_2, t_f, t_i) = \\
&\quad = \int \mathrm{d}q_1\, \mathrm{d}q_2\, \langle q_f | \mathrm{e}^{-\mathrm{i}H(t_f - t_1)} O_{\mathrm{S}}(q_1) | q_1 \rangle\langle q_1 | \mathrm{e}^{-\mathrm{i}H(t_1 - t_2)} O_{\mathrm{S}}(q_2) | q_2 \rangle\langle q_2 | \mathrm{e}^{-\mathrm{i}H(t_2 - t_i)} | q_i \rangle
\end{aligned}$$

where one has substituted the operators O with their eigenvalues, so the ones above are functions (as hinted by the presence of an explicit function argument). One obtains

$$C(t_1, t_2, t_f, t_i) = \int \mathrm{d}q_1\, \mathrm{d}q_2\, K(q_f, t_f; q_1, t_1) O(q_1) K(q_1, t_1; q_2, t_2) O(q_2) K(q_2, t_2; q_i, t_i)$$

This corresponds to a path integral where one integrates all fluctuations of the classical path γ_1 between q_f and q_1, then measures with the operator, integrates the fluctuations of the classical path γ_2 between q_1 and q_2, measures with other operator and finally integrates the fluctuations to the beginning of the classical path γ_3 between q_2 and q_i.

Lecture 4

gio 07 mar 2024 10:30

Inserting the definition of the kernels, one obtains

$$C(t_1, t_2) = \int \mathrm{d}q_1\, \mathrm{d}q_2\, O(q_1) O(q_2) \int [\mathcal{D}q^{\gamma_1}]\, \mathrm{e}^{\mathrm{i}S_{\gamma_1}} \int [\mathcal{D}q^{\gamma_2}]\, \mathrm{e}^{\mathrm{i}S_{\gamma_2}} \int [\mathcal{D}q^{\gamma_3}]\, \mathrm{e}^{\mathrm{i}S_{\gamma_3}}$$

Discretizing the path integrals

$$\int [\mathcal{D}q]\, \mathrm{e}^{\mathrm{i}S[q]} \to \frac{1}{A} \int \prod_{j=1}^{N} \frac{\mathrm{d}q_j}{A}\, \mathrm{e}^{\mathrm{i}S}\,, \quad A = \left[\frac{2\pi \mathrm{i}\, \delta t}{m}\right]^{\frac{1}{2}}$$

gives

$$\begin{aligned} C &= \frac{1}{A^3} \int \left[\prod_{j=1}^{N_1} \frac{\mathrm{d}q_j}{A}\right] \mathrm{d}q_1 \left[\prod_{k=1}^{N_2} \frac{\mathrm{d}q_k}{A}\right] \mathrm{d}q_2 \left[\prod_{l=1}^{N_3} \frac{\mathrm{d}q_l}{A}\right] O(q_1) O(q_2) \mathrm{e}^{\mathrm{i}S_\Gamma} \\ &= \frac{1}{A} \int \prod_{j=1}^{N_1} \frac{\mathrm{d}q_j}{A} \frac{\mathrm{d}q_1}{A} \prod_{k=1}^{N_2} \frac{\mathrm{d}q_k}{A} \frac{\mathrm{d}q_2}{A} \prod_{l=1}^{N_3} \frac{\mathrm{d}q_l}{A} O(q_1) O(q_2) \mathrm{e}^{\mathrm{i}S_\Gamma} \end{aligned}$$

where $\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$. Keep in mind that the product (capital pi) refers only to the fraction immediately after.

The first integral is for the first path $\gamma_1 : q_1 < q_j < q_f$, the second is for the path $\gamma_2 : q_2 < q_k < q_1$ and the third is for $\gamma_3 : q_i < q_l < q_2$. In the limit $N_i \to \infty$, the measure is

$$[\mathcal{D}q]_\Gamma = \frac{1}{A} \int \prod_{j=1}^{N_1} \frac{\mathrm{d}q_j}{A} \frac{\mathrm{d}q_1}{A} \prod_{k=1}^{N_2} \frac{\mathrm{d}q_k}{A} \frac{\mathrm{d}q_2}{A} \prod_{l=1}^{N_3} \frac{\mathrm{d}q_l}{A}$$

Therefore, the two-point correlation function is

$$C(t_1, t_2) = \langle q_f, t_f | \, \mathcal{T}\{O_{\mathrm{H}}(t_1) O_{\mathrm{H}}(t_2)\} \, | q_i, t_i \rangle = \int [\mathcal{D}q]\, O(q(t_1)) O(q(t_2)) \mathrm{e}^{\mathrm{i}S}$$

The correlation function is an intermediate step. The actual interesting quantity is the Green's function.

2.4 Green's function

See Cheng–Li. The Green's function is a particular correlation function where the initial and final states are the ground state $|0\rangle$

$$G(t_1, \ldots, t_n) = \langle 0 | \, \mathcal{T}\{O_{\mathrm{H}}(t_1) \cdots O_{\mathrm{H}}(t_n)\} \, | 0 \rangle$$

The definition can be generalized to different operators within the same time-ordered product, but, for simplicity, the following uses the same operator.

Two-point Green's function. One would like to compute the Green's function with the path integral. The two-point Green's function is given by

$$G(t_1, t_2) = \langle 0 | \, \mathcal{T}\{O_{\mathrm{H}}(t_1) O_{\mathrm{H}}(t_2)\} \, | 0 \rangle = \int \mathrm{d}q\, \mathrm{d}q' \; \langle 0 | q' t' \rangle \, \langle q' t' | \, \mathcal{T}\{O_{\mathrm{H}}(t_1) O_{\mathrm{H}}(t_2)\} \, | q t \rangle \, \langle q t | 0 \rangle$$

The expectation value in the middle can be computed through the correlation function above. The bra-kets are the wave function

$$\langle q t | 0 \rangle = \langle q | \, \mathrm{e}^{-\mathrm{i}Ht} \, | 0 \rangle = \langle q | \, \mathrm{e}^{-\mathrm{i}E_0 t} \, | 0 \rangle = \mathrm{e}^{-\mathrm{i}E_0 t} \varphi_0(q)$$

In quantum field theory, the energy levels can be arbitrarily shifted so one sets $E_0 = 0$ (as opposed to general relativity). Therefore

$$\langle q t | 0 \rangle = \varphi_0(q)\,, \quad \langle 0 | q' t' \rangle = \varphi_0^*(q')$$

The Green's function is then

$$\begin{aligned} G(t_1, t_2) &= \int \mathrm{d}q\, \mathrm{d}q'\, \varphi_0^*(q') \varphi'(q) \int [\mathcal{D}q]\, O(q(t_1)) O(q(t_2)) \mathrm{e}^{\mathrm{i}S} \\ &= \int [\mathcal{D}q]\, \varphi_0^*(q') \varphi_0(q) O(q(t_1)) O(q(t_2)) \mathrm{e}^{\mathrm{i}S} \end{aligned}$$

at the second line one incorporates the measures $\mathrm{d}q\,\mathrm{d}q'$ into the path integral's.

One would like to elaborate the above prescription in order to eliminate the wave functions $\varphi_0^*(q')\varphi_0(q)$ because they make calculations cumbersome. [r] The correlation function is

$$\begin{aligned} C(t_1,t_2) &= \langle q't'|\,\mathcal{T}\{O_{\mathrm{H}}(t_1)O_{\mathrm{H}}(t_2)\}\,|qt\rangle \\ &= \int \mathrm{d}Q\,\mathrm{d}Q'\;\langle q't'|Q'T'\rangle\,\langle Q'T'|\,\mathcal{T}\{O_{\mathrm{H}}(t_1)O_{\mathrm{H}}(t_2)\}\,|QT\rangle\,\langle QT|qt\rangle \end{aligned}$$

where one has inserted two completeness relations. [r] Letting E_n be the energy eigenvalue with corresponding eigenstate $|n\rangle$, the wave function of the eigenvector is

$$\langle q|n\rangle = \varphi_n(q)$$

[r] Therefore

$$\begin{aligned} \langle q't'|Q'T'\rangle &= \langle q'|\,\mathrm{e}^{-\mathrm{i}H(t'-T')}\,|Q'\rangle = \sum_n \langle q'|\,\mathrm{e}^{-\mathrm{i}H(t'-T')}\,|n\rangle\,\langle n|Q'\rangle = \sum_n \mathrm{e}^{-\mathrm{i}E_n(t'-T')}\varphi_n(q')\varphi_n^*(Q') \\ &= \varphi_0(q')\varphi_0^*(Q') + \sum_{n>0} \mathrm{e}^{-\mathrm{i}E_n(t'-T')}\varphi_n(q')\varphi_n^*(Q') \end{aligned}$$

where one has separated the contribution of the ground state from the rest. The times t and t' can (almost) be freely moved around the number line since they are not integrated. One observes that in the limit $t' \to -\mathrm{i}\infty$ (similar to a Wick's rotation), the exponential tends to zero (remember that the energy is positive since $E_0 = 0$). Therefore

$$\lim_{t'\to-\mathrm{i}\infty} \langle q't'|Q'T'\rangle = \varphi_0(q')\varphi_0^*(Q')$$

while for the non-primed bra-ket

$$\lim_{t\to+\mathrm{i}\infty} \langle QT|qt\rangle = \varphi_0(Q)\varphi_0^*(q)$$

Performing a similar calculation one has

$$\lim_{\substack{t'\to-\mathrm{i}\infty \\ t\to+\mathrm{i}\infty}} \langle q't'|qt\rangle = \varphi_0(q')\varphi_0^*(q)$$

Taking the above limit for the correlator, one has

$$\begin{aligned} \lim_{\substack{t'\to-\mathrm{i}\infty \\ t\to+\mathrm{i}\infty}} C(t_1,t_2) &= \lim_{\substack{t'\to-\mathrm{i}\infty \\ t\to+\mathrm{i}\infty}} \langle q't'|\,\mathcal{T}\{O_{\mathrm{H}}(t_1)O_{\mathrm{H}}(t_2)\}\,|qt\rangle \\ &= \int \mathrm{d}Q\,\mathrm{d}Q'\,\varphi_0(q')\varphi_0^*(Q')\,\langle Q'T'|\,\mathcal{T}\{O_{\mathrm{H}}(t_1)O_{\mathrm{H}}(t_2)\}\,|QT\rangle\,\varphi_0(Q)\varphi_0^*(q) \\ &= \varphi_0(q')\varphi_0^*(q)\int \mathrm{d}Q\,\mathrm{d}Q'\,\varphi_0^*(Q')\varphi_0(Q)\,\langle Q'T'|\,\mathcal{T}\{O_{\mathrm{H}}(t_1)O_{\mathrm{H}}(t_2)\}\,|QT\rangle \\ &= \lim_{\substack{t'\to-\mathrm{i}\infty \\ t\to+\mathrm{i}\infty}} \langle q't'|qt\rangle\,G(t_1,t_2) \end{aligned}$$

in the last line, the integral is the definition of Green's function but with Q, Q'. [r] Therefore, reorganizing the above equation, the Green's function is

$$G(t_1,t_2) = \lim_{\substack{t'\to-\mathrm{i}\infty \\ t\to+\mathrm{i}\infty}} \frac{\langle q't'|\,\mathcal{T}\{O_{\mathrm{H}}(t_1)O_{\mathrm{H}}(t_2)\}\,|qt\rangle}{\langle q't'|qt\rangle}$$

This operation is similar to a normalization: the Green's function is a correlation function normalized by the product of the two states.

This definition of two-point Green's function can be generalized to n points

$$G(t_1,\ldots,t_n) = \lim_{\substack{t'\to-\mathrm{i}\infty \\ t\to+\mathrm{i}\infty}} \frac{\langle q't'|\,\mathcal{T}\{O_{\mathrm{H}}(t_1)\cdots O_{\mathrm{H}}(t_n)\}\,|qt\rangle}{\langle q't'|qt\rangle}$$

In this way, the Green's function does not explicitly depend on the ground state.

Generating functional. The generating functional for the Green's function is

$$W[J] = \lim_{\substack{t' \to -\mathrm{i}\infty \\ t \to +\mathrm{i}\infty}} \frac{1}{\langle q't'|qt\rangle} \int [\mathcal{D}q] \, \exp\left[\mathrm{i}\int_t^{t'} \mathrm{d}\tau \, [L(\tau) + J(\tau) O_{\mathrm{H}}(q(\tau))]\right]$$

Applying an arbitrary number of derivatives with respect to J brings the operator O_{H} in front of the exponential. In this way one obtains the desired Green's function

$$G(t_1, \ldots, t_n) = \left.\frac{(-\mathrm{i})^n \delta^n W[J]}{\delta J(t_1) \cdots \delta J(t_n)}\right|_{J=0}$$

The function J is called the source of the operator O_{H}.

The generating functional is a modification of a path integral that one would get without any insertion of operators and with a source different from zero. The functional is the transition amplitude from the ground state at time t to the ground state at time t' in the presence of an external source $J(\tau)$

$$W[J] = \langle 0|0\rangle_J$$

The normalization is given by the absence of the source

$$W[0] = \langle 0|0\rangle = 1$$

This is in accordance with the prescription of the Green's function above from the state $|qt\rangle$ to the state $|q't'\rangle$ but without any insertion of operators: the ratio is unity.

Euclidean Green's function. Working in Euclidean space is useful to make sense of the limits of the times t, t'. By performing a Wick's rotation, the time coordinate becomes $\tau = -\mathrm{i}\tau_{\mathrm{E}}$ while the limits are

$$t' \to -\mathrm{i}\infty \rightsquigarrow t'_{\mathrm{E}} \to +\infty$$
$$t \to +\mathrm{i}\infty \rightsquigarrow t_{\mathrm{E}} \to -\infty$$

The exponential of the Lagrangian becomes

$$\exp\left[\mathrm{i}\int \mathrm{d}\tau \, L(\tau)\right] \rightsquigarrow \exp\left[\int \mathrm{d}\tau_{\mathrm{E}} \, L(-\mathrm{i}\tau_{\mathrm{E}})\right]$$

Considering the Lagrangian of a one-dimensional particle in a potential

$$L = \frac{1}{2}m\dot{q}^2 - V(q) = \frac{1}{2}m(\mathrm{d}_\tau q)^2 - V(q) \rightsquigarrow L(-\mathrm{i}\tau_{\mathrm{E}}) = -\frac{1}{2}m\dot{q}_{\mathrm{E}}^2 - V(q) \equiv -L_{\mathrm{E}}$$

the exponential is then

$$\exp\left[\mathrm{i}\int \mathrm{d}\tau \, L(\tau)\right] \rightsquigarrow \exp\left[-\int \mathrm{d}\tau_{\mathrm{E}} \, L_{\mathrm{E}}\right]$$

The ground energy can always be set to zero, so the energies are positive and the exponential is no longer a phase but tends to zero. The integral becomes well-defined.

Euclidean generating functional. The generating functional becomes

$$W_{\mathrm{E}}[J] = \lim_{\substack{t' \to \infty \\ t \to -\infty}} \frac{1}{\langle q't'|qt\rangle} \int [\mathcal{D}q] \, \exp\left[-\int \mathrm{d}\tau_{\mathrm{E}} [L_{\mathrm{E}} + J O_{\mathrm{H}}(\tau_{\mathrm{E}})]\right]$$

2.5 Arbitrary dimensions

The formalism developed can be generalized to D dimensions. If a particle propagates in D dimensions, the configuration point is

$$\mathbf{q} = (q^1, \ldots, q^D)$$

The kernel is

$$K(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i) = \int [\mathcal{D}\mathbf{q}]\, \mathrm{e}^{\mathrm{i}S[\mathbf{q}]} = \lim_{N\to\infty} \frac{1}{A} \int \prod_{j=1}^{N} \frac{\mathrm{d}\mathbf{q}_j}{A^D}\, \mathrm{e}^{\mathrm{i}S[\mathbf{q}_j]}$$
$$= \lim_{N\to\infty} \left[\frac{m}{2\pi \mathrm{i}\,\delta t}\right]^{\frac{ND+1}{2}} \int \prod_{j=1}^{N} \mathrm{d}\mathbf{q}_j\, \mathrm{e}^{\mathrm{i}S[\mathbf{q}_j]}$$

Conclusion. Some phenomena are easier described through the path integral formulation of quantum mechanics, like tunnelling and scattering.

3 Scalar boson fields

In quantum field theory, canonical quantization is limited and cumbersome. One uses the functional approach which generalizes the quantum mechanical path integral.

Spin zero particles are bosons and are described by scalar fields $\varphi(x)$. Neutral particles are real scalar fields.

3.1 Canonical quantization

One may start from neutral particles. For free massive scalar fields, the Lagrangian density and action are given by

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\,\partial_\mu\varphi\,\partial_\nu\varphi - \frac{1}{2}m^2\varphi^2\,, \quad S = \int \mathrm{d}^4x\,\mathcal{L} = \int \mathrm{d}^4x\,\left[\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\nabla\varphi)^2 - \frac{1}{2}m^2\varphi^2\right]$$

where the Minkowski metric η convention is timelike, mostly minus. Applying the action principle $\delta S = 0$ one obtains the classical equations of motion with some boundary conditions: when integrating over all space-time, one imposes that the fields go to zero sufficiently fast. For a scalar field, the equations of motion are Klein–Gordon's

$$(\Box + m^2)\varphi(x) = 0\,, \quad \Box \equiv \partial_\mu\partial^\mu = \partial_0^2 - \nabla^2$$

The canonical momentum of the field is

$$\pi(x) = \frac{\delta\mathcal{L}}{\delta\dot{\varphi}(x)} = \dot{\varphi}(x)$$

The most general solution for the Klein–Gordon equation is

$$\varphi(x) = \varphi_+(x) + \varphi_-(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2k^0}}[a_+(\mathbf{k})\mathrm{e}^{\mathrm{i}k^\mu x_\mu} + a_-(\mathbf{k})\mathrm{e}^{-\mathrm{i}k^\mu x_\mu}]\,, \quad k^0 = \sqrt{|\mathbf{k}|^2 + m^2}$$

where $\pm$ is not related to positive and negative frequency components, it just labels the sign of the exponent. The reality condition given by the neutral nature of the boson field implies

$$[a_-(\mathbf{k})]^\dagger = a_+(\mathbf{k})$$

The Fourier coefficients can be expressed in terms of the field as

$$a_\pm(\mathbf{k}) = \mp\frac{\mathrm{i}}{(2\pi)^{\frac{3}{2}}\sqrt{2k^0}} \int \mathrm{d}^3x\,\mathrm{e}^{\mp\mathrm{i}kx} \overset{\leftrightarrow}{\partial_0}\varphi(x)\,, \quad f \overset{\leftrightarrow}{\partial_0} g \equiv f\,\partial_0 g - (\partial_0 f)g$$

One can prove that, since the field $\varphi(x)$ is a solution of the free Klein–Gordon equation, the annihilation and creation operators are time-independent. For the interacting theory, the Klein–Gordon equation of motion is not zero. In fact, the Lagrangian has a potential

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - V(\varphi)$$

from which the equations of motion are

$$(\Box + m^2)\varphi(x) = -\partial_\varphi V \equiv J(x)$$

and in general the annihilation and destruction operators are functions of time.

Exercise. Check that the previous expressions give back the field φ. Check that the operators of the free theory are time-independent

$$\partial^0 a(\mathbf{k}) = \frac{\mathrm{i}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}} \int \mathrm{d}^3x\, \partial^0[(\partial_0 \mathrm{e}^{\mathrm{i}kx})\varphi(x) - \mathrm{e}^{\mathrm{i}kx}\, \partial_0\varphi] = \cdots = 0$$

To get zero one has to apply the Klein–Gordon equation.

Canonical quantization. The canonical quantization of the fields (called second quantization) is implemented by promoting the Fourier coefficients $a_\pm(\mathbf{k})$ to be operators obeying the canonical commutation relations (CCR)

$$[a_-(\mathbf{k}), a_+(\mathbf{k}')] = \delta^{(3)}(\mathbf{k}-\mathbf{k}') \implies [a(\mathbf{k}), a^\dagger(\mathbf{k})] = \delta^{(3)}(\mathbf{k}-\mathbf{k}')$$

and all others zero. The number operator and the total number are

$$N(\mathbf{k}) \equiv a^\dagger(\mathbf{k})a(\mathbf{k})\,, \quad N = \int \mathrm{d}^3k\, N(\mathbf{k})$$

Lecture 5

lun 11 mar 2024 10:30

One defines a vacuum state as the state annihilated by all destruction operators

$$a(\mathbf{k})\,|0\rangle = 0\,, \quad \forall \mathbf{k}$$

The excited states are constructed by acting on the vacuum with a creation operator

$$a^\dagger(\mathbf{k})\,|0\rangle = |\psi(\mathbf{k})\rangle$$

The excited states are interpreted as particle with momentum $\mathbf{k}$ and on-shell dispersion relation $E = \sqrt{|\mathbf{k}|^2 + m^2}$. Though it has no physical sense that a particle has a definite momentum $\mathbf{k}$, but it is more physical to consider a wave packet with a distribution of momenta centered around a momentum $\mathbf{k}$

$$|\psi(\mathbf{k})\rangle = \int \mathrm{d}^3k'\, f(\mathbf{k},\mathbf{k}')a^\dagger(\mathbf{k})\,|0\rangle$$

where f is a function that gives the distribution of the plane waves inside the wave packet, typically a Gaussian (see Srednicki, §5)

$$f(\mathbf{k},\mathbf{k}') \propto \exp\left[-\frac{1}{4\sigma^2}(\mathbf{k}'-\mathbf{k})^2\right], \quad f(\mathbf{k},\mathbf{k}') \to \delta^{(3)}(\mathbf{k}'-\mathbf{k})\,, \quad \sigma \to 0$$

Commutators. The commutators between the fields can be obtained by applying the canonical commutation relations of the annihilation and creation operators. The relations for the separate parts of the field are

$$[\varphi_\mp(x), \varphi_\pm(y)] = \pm \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0} \mathrm{e}^{\mp \mathrm{i}k(x-y)}$$

From these, the relations for the field are

$$[\varphi(x), \varphi(y)] = [\varphi_+(x), \varphi_-(y)] + [\varphi_-(x), \varphi_+(y)] = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}[\mathrm{e}^{\mathrm{i}k(x-y)} - \mathrm{e}^{-\mathrm{i}k(x-y)}]$$

The canonical commutations relations are then

$$[\varphi(x), \dot\varphi(y)]_{x^0=y^0} = [\varphi(x^0,\mathbf{x}), \pi(x^0,\mathbf{y})] = \mathrm{i}\delta^{(3)}(x-y)$$

because $\pi = \dot\varphi$ is the canonical momentum of the field.

The commutators for space-like separations are null

$$[\varphi(x), \varphi(y)] = 0\,, \quad |x-y|^2 < 0$$

The above commutators are the Pauli–Jordan functions, which are related to the Klein–Gordon propagator. The meaning of the above commutator is that two particles separated by arbitrary space-like distances cannot influence each other: it is the condition of micro-causality.

The (Feynman) propagator for a free scalar field is the following two-point function

$$\begin{aligned}
\mathrm{i}\Delta_0(x-y) &\equiv \langle 0|\,\mathcal{T}\{\varphi(x)\varphi(y)\}\,|0\rangle = \theta(x^0-y^0)\,\langle 0|\,\varphi(x)\varphi(y)\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,\varphi(y)\varphi(x)\,|0\rangle \\
&= \theta(x^0-y^0)\,\langle 0|\,\varphi_-(x)\varphi_+(y)\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,\varphi_-(y)\varphi_+(x)\,|0\rangle \\
&= \theta(x^0-y^0)\,\langle 0|\,[\varphi_-(x),\varphi_+(y)]\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,[\varphi_-(y),\varphi_+(x)]\,|0\rangle \\
&= \theta(x^0-y^0)\,\langle 0|\,[\varphi_-(x),\varphi_+(y)]\,|0\rangle - \theta(y^0-x^0)\,\langle 0|\,[\varphi_+(x),\varphi_-(y)]\,|0\rangle \\
&= \theta(x^0-y^0)\int\frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}\mathrm{e}^{-\mathrm{i}k(x-y)} + \theta(y^0-x^0)\int\frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}\mathrm{e}^{\mathrm{i}k(x-y)} \\
&= \mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}k(x-y)}}{k^2-m^2+\mathrm{i}\varepsilon}
\end{aligned}$$

At the second line only the non-zero contributions have been explicitly written; at the last line one has made the analytic continuation of k^0 in the complex plane and has integrated using Feynman's prescription and Jordan's lemmas.

Once quantum field theory is reformulated through functional methods, one finds the same expression of the propagator above using functional integrals.

The propagator is the probability amplitude of creating a particle at y, letting it propagate to x and destroying it there. The propagator above is a two-point function and it is the Green's function of the Klein–Gordon operator

$$(\Box + m^2)\Delta_0(x) = -\delta^{(4)}(x-y)$$

[r]

The propagator (which is related to the Green's function) is needed to justify the fact that the evaluation of a scattering amplitude is related to the computation of the Green's function. The scattering amplitude can be expressed in terms of the Green's function thanks to the Lehmann–Symanzik–Zimmermann (LSZ) reduction formula.

3.2 Lehmann–Symanzik–Zimmermann reduction formula

See Srednicki §5, Weinberg, Itzykson–Zuber. One would like to compute the scattering amplitude between an initial and a final state $\langle f|i\rangle$ which is related to the probability that a system prepared in the state $|i\rangle$ at $t\to-\infty$ transitions to the state $|f\rangle$ at $t\to+\infty$.

Adiabatic hypothesis. The interaction occurs only in a finite volume of space. At infinity the interaction is negligible and there are only the free asymptotic states $|i\rangle$ and $|f\rangle$ given by

$$|i\rangle = \lim_{t\to-\infty} a_1^\dagger(t)a_2^\dagger(t)\cdots|0\rangle\ , \quad |f\rangle = \lim_{t\to\infty} a_{1'}^\dagger(t)a_{2'}^\dagger(t)\cdots|0\rangle$$

where

$$a_1^\dagger(t)\,|0\rangle = \int \mathrm{d}^3k\, f_1(\mathbf{k})a^\dagger(\mathbf{k},t)\,|0\rangle$$

remembering that the operators depend on time because the theory has interaction and is not free. Therefore it is useful to compute $a_j^\dagger(+\infty) - a_j^\dagger(-\infty)$. Starting from the first one, it follows

$$\begin{aligned}
a_1^\dagger(+\infty) - a_1^\dagger(-\infty) &= \int_{\mathbb{R}} \mathrm{d}t\, \partial_0 a_1(t) = \int_{\mathbb{R}} \mathrm{d}t\, \partial_0 \int \mathrm{d}^3k\, f_1(\mathbf{k}) a^\dagger(\mathbf{k}, t) \\
&= -\mathrm{i} \int \mathrm{d}^3k\, f_1(\mathbf{k}) \int_{\mathbb{R}} \mathrm{d}t\, \partial_0 \int \frac{\mathrm{d}^3x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}} \mathrm{e}^{-\mathrm{i}kx} \overset{\leftrightarrow}{\partial_0} \varphi(x) \\
&= -\mathrm{i} \int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}}\, \partial_0 [\mathrm{e}^{-\mathrm{i}kx} \overset{\leftrightarrow}{\partial_0} \varphi(x)] \\
&= -\mathrm{i} \int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}}\, [\mathrm{e}^{-\mathrm{i}kx}\, \partial_0^2 \varphi + k_0^2 \mathrm{e}^{-\mathrm{i}kx} \varphi] \\
&= -\mathrm{i} \int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}}\, [\mathrm{e}^{-\mathrm{i}kx}\, \partial_0^2 \varphi + (|\mathbf{k}|^2 + m^2) \mathrm{e}^{-\mathrm{i}kx} \varphi] \\
&= -\mathrm{i} \int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}} \mathrm{e}^{-\mathrm{i}kx} (\partial_0^2 - \nabla^2 + m^2) \varphi(x) \\
&= -\mathrm{i} \int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}} \mathrm{e}^{-\mathrm{i}kx} (\Box + m^2) \varphi(x)
\end{aligned}$$

at the second line one expresses the creation operator with its expansions in terms of the fields; at the second to last line, one notices

$$|\mathbf{k}|^2 \mathrm{e}^{-\mathrm{i}kx} = -\nabla^2 \mathrm{e}^{-\mathrm{i}kx}$$

and integrates by parts supposing that at infinity the fields to go zero fast enough to produce no boundary terms. At the last line, in the free theory, the field is a solution of the Klein–Gordon equation and the desired difference is zero which is consistent with the fact that the two operators at infinity are the same; however, in the interacting theory this is not true because the Klein–Gordon operator does not necessarily give zero.

Similarly, for the destruction operators

$$a_1(+\infty) - a_1(-\infty) = \mathrm{i} \int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}} \mathrm{e}^{\mathrm{i}kx} (\Box + m^2) \varphi(x)$$

Scattering amplitude. The following considers a probability amplitude for a $2 \to 2$ scattering process. The initial and final states are

$$|i\rangle = a_1^\dagger(-\infty) a_2^\dagger(-\infty) |0\rangle \ , \quad |f\rangle = a_{1'}^\dagger(+\infty) a_{2'}^\dagger(+\infty) |0\rangle$$

The scattering amplitude is then

$$\begin{aligned}
\langle f|i\rangle &= \langle 0| a_{1'}(+\infty) a_{2'}(+\infty) a_1^\dagger(-\infty) a_2^\dagger(-\infty) |0\rangle \\
&= \langle 0| \mathcal{T}\{a_{1'}(+\infty) a_{2'}(+\infty) a_1^\dagger(-\infty) a_2^\dagger(-\infty)\} |0\rangle
\end{aligned}$$

At the second line, one notices that the product is already time-ordered. One may insert the expression of the operators in terms of the fields

$$\begin{aligned}
a_j^\dagger(-\infty) &= a_j^\dagger(+\infty) + \mathrm{i} \int \mathrm{d}^3k\, f_j(\mathbf{k}) \int \frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}} \mathrm{e}^{-\mathrm{i}kx} (\Box + m^2) \varphi(x) \\
a_{j'}(+\infty) &= a_{j'}(-\infty) + \mathrm{i} \int \mathrm{d}^3k\, f_{j'}(\mathbf{k}) \int \frac{\mathrm{d}^4x}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}} \mathrm{e}^{\mathrm{i}kx} (\Box + m^2) \varphi(x)
\end{aligned}$$

Looking at the second expression, the time-ordered product above pushes the primed destruction operators to the right and they annihilate the vacuum, so only the integral remains. Therefore

$$\begin{aligned}
\langle f|i\rangle = &\int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \mathrm{d}^3k'\, f_2(\mathbf{k}') \int \mathrm{d}^3k''\, f_{1'}(\mathbf{k}'') \int \mathrm{d}^3k'''\, f_{2'}(\mathbf{k}''') \\
&\times \int \frac{\mathrm{d}^4x_1\, \mathrm{e}^{-\mathrm{i}kx_1}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_1^0}} \int \frac{\mathrm{d}^4x_2\, \mathrm{e}^{-\mathrm{i}k'x_2}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_2^0}} \int \frac{\mathrm{d}^4x_3\, \mathrm{e}^{\mathrm{i}k''x_3}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_{1'}^0}} \int \frac{\mathrm{d}^4x_4\, \mathrm{e}^{\mathrm{i}k'''x_4}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_{2'}^0}} \\
&\times \langle 0|\, \mathcal{T}\{J(x_1)J(x_2)J(x_3)J(x_4)\}\, |0\rangle \\
= &\int \frac{\mathrm{d}^4x_1\, \mathrm{e}^{-\mathrm{i}k_1x_1}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_1^0}} \int \frac{\mathrm{d}^4x_2\, \mathrm{e}^{-\mathrm{i}k_2x_2}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_2^0}} \int \frac{\mathrm{d}^4x_{1'}\, \mathrm{e}^{\mathrm{i}k_{1'}x_{1'}}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_{1'}^0}} \int \frac{\mathrm{d}^4x_{2'}\, \mathrm{e}^{\mathrm{i}k_{2'}x_{2'}}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_{2'}^0}} \\
&\times \langle 0|\, \mathcal{T}\{J(x_1)J(x_2)J(x_{1'})J(x_{2'})\}\, |0\rangle \\
= &\int \frac{\mathrm{d}^4x_1\, \mathrm{e}^{-\mathrm{i}k_1x_1}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_1^0}} \int \frac{\mathrm{d}^4x_2\, \mathrm{e}^{-\mathrm{i}k_2x_2}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_2^0}} \int \frac{\mathrm{d}^4x_{1'}\, \mathrm{e}^{\mathrm{i}k_{1'}x_{1'}}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_{1'}^0}} \int \frac{\mathrm{d}^4x_{2'}\, \mathrm{e}^{\mathrm{i}k_{2'}x_{2'}}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_{2'}^0}} \\
&\times (\Box_1 + m_1^2)(\Box_2 + m_2^2)(\Box_{1'} + m_{1'}^2)(\Box_{2'} + m_{2'}^2)\, \langle 0|\, \mathcal{T}\{\varphi(x_1)\varphi(x_2)\varphi(x_{1'})\varphi(x_{2'})\}\, |0\rangle
\end{aligned}$$

At the second equality, one takes the limit where the distributions tend to the Dirac delta function

$$f_j(\mathbf{k}^{(l)}) \to \delta^{(3)}(\mathbf{k}^{(l)} - \mathbf{k}_j)$$

At the third equality, source is replaced by using the equations of motion and the d'Alembertian is brought outside the expectation value.

This last expectation value is the four-point Green's function. The generalization of the scattering amplitude to N particles is straightforward

$$\begin{aligned}
\langle f_{n'}|i_n\rangle = \mathrm{i}^{n+n'} \prod_{j=1}^{n} \int \frac{\mathrm{d}^4x_j\, \mathrm{e}^{-\mathrm{i}k_jx_j}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_j^0}} \prod_{l=1}^{n'} \int \frac{\mathrm{d}^4x_l'\, \mathrm{e}^{-\mathrm{i}k_l'x_l'}}{(2\pi)^{\frac{3}{2}}\sqrt{2k_{l'}^0}} \\
\times \prod_j (\Box_j + m_j^2) \prod_l (\Box_l + m_l'^2)\, \langle 0|\, \mathcal{T}\{\varphi(x_j)\varphi(x_l')\}\, |0\rangle
\end{aligned}$$

The last expectation value is the $(n+n')$-point Green's function. This is the LSZ reduction formula: the scattering amplitudes are related to the computation of Green's functions.

Caveat. The calculation was based on the assumption that the particles are always created from time-dependent operators $a^\dagger(t)$ acting on the unique vacuum $|0\rangle$. This is true for the free theory, but in the interacting theory it may not be. The following discussion is not a proof, but a motivation of the validity of the assumption.

If the assumption is true for the interacting theory, then one should have

$$\langle 0|\, \varphi(x)\, |0\rangle = 0$$

because one creates a particle and projects it onto the vacuum. One may obtain the above by writing the field in terms of the field in the origin

$$\langle 0|\, \varphi(x)\, |0\rangle = \langle 0|\, \mathrm{e}^{\mathrm{i}Px}\varphi(x)\mathrm{e}^{-\mathrm{i}Px}\, |0\rangle = \langle 0|\, \mathrm{e}^{0}\varphi(0)\mathrm{e}^{-0}\, |0\rangle = \langle 0|\, \varphi(0)\, |0\rangle = v\,, \quad P^\mu\, |0\rangle = 0$$

where P is the translation generator (the momentum) and the vacuum is assumed to be unique; at the second equality, the momentum of the vacuum is $p = 0$ which then implies its translational invariance. It may be that $v \neq 0$, but one can shift the field φ to obtain a zero expectation value

$$\varphi'(x) = \varphi(x) - v \implies \langle 0|\, \varphi'(x)\, |0\rangle = \langle 0|\, \varphi'(0)\, |0\rangle = 0$$

This is justified because the physics of the Lagrangian does not change: it is just a renaming of the operator of interest.

Consider now a one particle state. One expects to find

$$\langle p|\, \varphi(x)\, |0\rangle = \mathrm{e}^{\mathrm{i}px}$$

like in the free case. In fact

$$\langle p|\,\varphi(x)\,|0\rangle = \langle p|\,\mathrm{e}^{\mathrm{i}Px}\varphi(0)\mathrm{e}^{-\mathrm{i}Px}\,|0\rangle = \mathrm{e}^{\mathrm{i}px}\,\langle p|\,\varphi(x)\,|0\rangle = A\mathrm{e}^{\mathrm{i}px}$$

By rescaling the field, one has

$$\langle p|\,\frac{1}{A}\varphi(x)\,|0\rangle = \langle p|\,\varphi'(x)\,|0\rangle = \mathrm{e}^{\mathrm{i}px}$$

Consider now a multi-particle state $|p,n\rangle$ where p is the total momentum and n are the other relevant quantum numbers. One expects to find

$$\langle p,n|\,\varphi(x)\,|0\rangle = 0$$

To this end, one may write

$$\langle p,n|\,\varphi(x)\,|0\rangle = \mathrm{e}^{\mathrm{i}px}\,\langle p,n|\,\varphi(0)\,|0\rangle = \mathrm{e}^{\mathrm{i}px}A_n(p)$$

where $A_n(p)$ is a function of products of Lorentz-invariants. For more than one particle, the total energy is

$$E_{\mathrm{tot}} = p_0 = \sqrt{p^2+M^2} = \sum_j \sqrt{p_j^2+m_j^2}$$

with momenta p_j and masses m_j. Considering two particles with equal mass, one has

$$p = p_1 + p_2\,, \quad E_{\mathrm{tot}} = \sqrt{p_1^2+m^2} + \sqrt{p_2^2+m^2}$$

Therefore, the minimum energy is $E_{\mathrm{tot}} = 2m$ and above there is a continuum of hyperbolae of energies (given by $\sqrt{E^2-p^2} = m = \text{const.}$, see Srednicki Fig. 5.1). On the other hand, the energy of a one-particle state is completely determined by its momentum and it lies on an isolated hyperbola. Instead of

$$\langle p,n|\,\varphi(0)\,|0\rangle$$

being zero, one should strictly consider

$$\langle p,n|\,a^\dagger(\pm\infty)\,|0\rangle$$

Considering only renormalizable states $|\psi\rangle$

$$|\psi\rangle = \sum_n \int \mathrm{d}^3p\,\psi_n(p)\,|p,n\rangle$$

[r] it follows

$$\langle\psi|\,a^\dagger(t)\,|0\rangle = \sum_n \int \mathrm{d}^3p\,\psi_n^*(f)\,\langle p,n|\,a^\dagger(t)\,|0\rangle$$

Inserting the expression of the operator in terms of the fields, one may finally get

$$\langle\psi|\,a^\dagger(t)\,|0\rangle = \sum_n \int \mathrm{d}^3p\,\psi_n^*(p)\frac{(2\pi)^3}{(2\pi)^{\frac{3}{2}}\sqrt{2k_0}}f_1(\mathbf{p})(p_0+k_0)A_n(\mathbf{p})\mathrm{e}^{-\mathrm{i}(k_0-p_0)t}$$

where

$$p_0 = \sqrt{p^2+M^2}\,, \quad k_0 = \sqrt{p^2+m^2}\,, \quad M \geq 2m\,, \quad p_0 > k_0$$

When taking the limit $t \to \pm\infty$, the exponential becomes a very fast oscillating function: integrating over momenta gives zero (due to the Riemann–Lebesgue lemma).

Conclusion. The LSZ reduction formula works under the assumptions that

$$\langle 0|\,\varphi(0)\,|0\rangle = 0\,, \quad \langle p|\,\varphi(x)\,|0\rangle = \mathrm{e}^{\mathrm{i}px}$$

which are related to translating and scaling the field.

Lecture 6

mar 12 mar 2024 10:30

3.3 Free field theory

Comparison with quantum mechanics. In quantum mechanics one finds similar observables to the ones present in the LSZ reduction formula? [r]

$$G(t_1,\dots,t_n)=\langle 0|\,\mathcal{T}\{O(t_1)\cdots O(t_n)\}\,|0\rangle=(-\mathrm{i})^n\frac{\delta^n W[J]}{\delta J(t_1)\cdots\delta J(t_n)}\bigg|_{J=0}$$

where $|0\rangle$ is the fundamental state (there is no notion of Fock vacuum in ordinary quantum mechanics) and the generating functional is

$$W[J]=\lim_{\substack{t'\to-\mathrm{i}\infty\\ t\to+\mathrm{i}\infty}}\frac{1}{\langle q't'|qt\rangle}\int[\mathcal{D}q]\,\exp\left[\mathrm{i}\int\mathrm{d}\tau\,[L(\tau)+J(\tau)O(\tau)]\right]$$

One would like to generalize the above prescription to quantum field theory: the operators must be functions of position too because time and space must be treated on the same level.

The discrete positions $q_i(t)$ become a field $q_x(t)=\varphi(\mathbf{x},t)$ with continuous index $i\to x$. The ordinary operators $O(t)=O(q_i(t))$ becomes a field operators $O(\varphi(x))$. The Lagrangian $L(q_i,\dot{q}_i)$ becomes a Lagrangian density $\mathcal{L}(\varphi(x),\partial_\mu\varphi)$. The path integral in quantum mechanics has a well-defined prescription. The functional measure before and after integrating the momenta is

$$\int[\mathrm{d}q\,\mathrm{d}p]\equiv\lim_{N\to\infty}\int\prod_{j=1}^{N}\mathrm{d}q_j\prod_{k=1}^{N+1}\frac{\mathrm{d}p_k}{2\pi}\implies\int[\mathrm{d}q]=\lim_{N\to\infty}\frac{1}{A}\int\prod_{j=1}^{N}\frac{\mathrm{d}q_j}{A},\quad A=\frac{2\pi\mathrm{i}\,\delta t}{m}$$

which becomes

$$[\mathcal{D}\varphi(x)\,\mathcal{D}\pi(x)]\implies[\mathcal{D}\varphi]$$

whose rigorous definition has not been presented. To define the measure in quantum field theory, one can discretize the theory on a lattice to obtain ordinary quantum mechanics with the identity

$$\varphi(x_i)=\varphi_i$$

This way can be efficient for some problems. Though the approach taken next is different: one leaves the above just as a formal definition of the measure since one almost never explicitly computes functional integrals.

Generalization. The definitions of the Green's function and the generating functional can be generalized. [r] The first one is

$$G^{(n)}(x_1,\dots,x_n)=\langle 0|\,\mathcal{T}\{\varphi(x_1)\cdots\varphi(x_n)\}\,|0\rangle=(-\mathrm{i})^n\frac{\delta^n W[J]}{\delta J(t_1)\cdots\delta J(t_n)}\bigg|_{J=0}$$

where the generating functional for Green's functions is

$$W[J]=N\int[\mathcal{D}\varphi]\exp\left[\mathrm{i}\int\mathrm{d}^4x\,[\mathcal{L}(\varphi(x),\partial_\mu\varphi(x))+J(x)\varphi(x)]\right]$$

where N is a suitable normalization. If one is able to evaluate the generating functional W, then one may compute the Green's function of an arbitrary number of observables. The normalization N is typically chosen such that

$$W[J=0]\equiv\langle 0|0\rangle=1$$

One would like to study how to compute the generating functional $W[J]$ for a real scalar field.

Generating functional. The simplest case is the free field theory. The Lagrangian density is

$$\mathcal{L}(\varphi) = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2$$

The generating functional is

$$W_0[J] = N\int[\mathcal{D}\varphi]\,\exp\left[\mathrm{i}\int \mathrm{d}^4x\,(\mathcal{L}+J\varphi)\right]$$

To study the above one may use the momentum space. The Fourier conventions are

$$\widetilde{\varphi}(k) = \int \mathrm{d}^4x\,\mathrm{e}^{-\mathrm{i}kx}\varphi(x)\,,\quad \varphi(x) = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{\mathrm{i}kx}\widetilde{\varphi}(k)$$

For the source, one has

$$\begin{aligned}\int \mathrm{d}^4x\,J(x)\varphi(x) &= \int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\,\mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}(k_1+k_2)x}\widetilde{J}(k_1)\widetilde{\varphi}(k_2)\\ &= \int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}(2\pi)^4\delta^{(4)}(k_1+k_2)\widetilde{J}(k_1)\widetilde{\varphi}(k_2)\\ &= \int\frac{\mathrm{d}^4k}{(2\pi)^4}\widetilde{J}(k)\widetilde{\varphi}(-k) = \frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}[\widetilde{J}(k)\widetilde{\varphi}(-k)+\widetilde{J}(-k)\widetilde{\varphi}(k)]\end{aligned}$$

The same can be done for the mass term, while for the kinetic term one should integrate by parts first to get

$$\int \mathrm{d}^4x\,\frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi = -\int\mathrm{d}^4x\,\frac{1}{2}\varphi\,\Box\,\varphi$$

To quickly transform, one substitutes the function with its Fourier transform and absorbs the Jacobian into the normalization constant [r].

The generating functional is

$$W_0[J] = N\int[\mathcal{D}\widetilde{\varphi}]\,\exp\left[\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}[\widetilde{\varphi}(k)(k^2-m^2)\widetilde{\varphi}(-k)+\widetilde{J}(k)\widetilde{\varphi}(-k)+\widetilde{J}(-k)\widetilde{\varphi}(k)]\right]$$

Without formal development, one uses the Gaussian integrals for the functional integral above. One completes the square in the exponent

$$\begin{aligned}E &= \widetilde{\varphi}(k)(k^2-m^2)\widetilde{\varphi}(-k)+\widetilde{J}(k)\widetilde{\varphi}(-k)+\widetilde{J}(-k)\widetilde{\varphi}(k)\\ &= \left[\widetilde{\varphi}(k)+\widetilde{J}(k)\frac{1}{k^2-m^2}\right](k^2-m^2)\left[\widetilde{\varphi}(-k)+\frac{1}{k^2-m^2}\widetilde{J}(-k)\right]-\widetilde{J}(k)\frac{1}{k^2-m^2}\widetilde{J}(-k)\end{aligned}$$

By making a change of variables

$$\widetilde{\varphi}'(-k) \equiv \widetilde{\varphi}(-k)+\frac{1}{k^2-m^2}\widetilde{J}(-k)$$

the generating functional becomes

$$\begin{aligned}W_0[J] &= N\int[\mathcal{D}\widetilde{\varphi}']\,\exp\left[\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\widetilde{\varphi}'(k^2-m^2)\widetilde{\varphi}'\right]\exp\left[-\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\widetilde{J}(k)\frac{1}{k^2-m^2}\widetilde{J}(-k)\right]\\ &= \exp\left[-\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\widetilde{J}(k)\frac{1}{k^2-m^2}\widetilde{J}(-k)\right]N\int[\mathcal{D}\widetilde{\varphi}']\,\exp\left[\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\widetilde{\varphi}'(k^2-m^2)\widetilde{\varphi}'\right]\\ &= \exp\left[-\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\widetilde{J}(k)\frac{1}{k^2-m^2}\widetilde{J}(-k)\right]W_0[J=0]\,,\quad W_0[J=0]=1\\ &= \exp\left[-\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\widetilde{J}(k)\frac{1}{k^2-m^2}\widetilde{J}(-k)\right]\end{aligned}$$

Going back to the configuration space, one notices that $(k^2-m^2)^{-1}$ is the propagator. Recalling that it is

$$\Delta_0(x-x') = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}k(x-x')}}{k^2-m^2+\mathrm{i}\varepsilon}$$

then the generating functional is

$$W_0[J] = \exp\left[-\frac{\mathrm{i}}{2}\int\mathrm{d}^4x\,\mathrm{d}^4x'\,J(x)\Delta_0(x-x')J(x')\right]$$

Two-point function. The two-point Green's function is

$$\begin{aligned}\langle 0|\,\mathcal{T}\{\varphi(x_1)\varphi(x_2)\}\,|0\rangle &= (-\mathrm{i})^2\frac{\delta^2 W_0[J]}{\delta J(x_1)\,\delta J(x_2)}\bigg|_{J=0}\\ &= (-\mathrm{i})^2\frac{\delta}{\delta J(x_1)}\bigg[-\mathrm{i}\int \mathrm{d}^4x\, J(x)\Delta(x-x_2)\,W_0[J]\bigg]_{J=0}\\ &= [(-\mathrm{i})^3\Delta_0(x_1-x_2)+(\text{terms with } J\text{s})]W_0[J]\Big|_{J=0}\\ &= \mathrm{i}\Delta_0(x_1-x_2)\end{aligned}$$

at the second line one notices

$$\frac{\delta J(x')}{\delta J(x_2)} = \delta^{(4)}(x'-x_2)$$

Also there is a factor of 2 that is simplified with the fraction because the derivative can act on both $J(x)$ and $J(x')$? [r].

Euclidean space. The Lagrangian is

$$\mathcal{L} = -\frac{1}{2}\varphi\,\Box\,\varphi - \frac{1}{2}m^2\varphi^2$$

Applying Wick's rotation $x^0 = -\mathrm{i}x^0_{\mathrm{E}}$, the d'Alembertian operator is [r]

$$\Box = \partial_\mu\partial^\mu = \partial_0^2 - \nabla^2 = -\partial_{0\mathrm{E}}^2 - \nabla^2 = \Box_{\mathrm{E}}$$

In this way, in the Fourier transform one has

$$\Box \rightsquigarrow -k^2\,, \quad \Box_{\mathrm{E}} \rightsquigarrow k_{\mathrm{E}}^2$$

The generating function is

$$\begin{aligned}W_0^{\mathrm{E}}[0] &= N\int[\mathcal{D}\varphi]\,\exp\bigg[-\frac{1}{2}\int \mathrm{d}^4x_{\mathrm{E}}\,\varphi(\Box_{\mathrm{E}}+m^2)\varphi\bigg]\\ &= N\int[\mathcal{D}\widetilde{\varphi}]\,\exp\bigg[-\frac{1}{2}\int\frac{\mathrm{d}^4k_{\mathrm{E}}}{(2\pi)^4}\widetilde{\varphi}(k_{\mathrm{E}})(k_{\mathrm{E}}^2+m^2)\widetilde{\varphi}(-k_{\mathrm{E}})\bigg] = 1\end{aligned}$$

[r] The formulation above is well-defined since the exponential is decaying. With the source, one has

$$\begin{aligned}W_0^{\mathrm{E}}[J] &= N\int[\mathcal{D}\widetilde{\varphi}]\,\exp\bigg[-\frac{1}{2}\int\frac{\mathrm{d}^4k_{\mathrm{E}}}{(2\pi)^4}[\widetilde{\varphi}(k_{\mathrm{E}})(k_{\mathrm{E}}^2+m^2)\widetilde{\varphi}(-k_{\mathrm{E}})+2\widetilde{J}(k_{\mathrm{E}})\widetilde{\varphi}(-k_{\mathrm{E}})]\bigg]\\ &= \exp\bigg[\frac{1}{2}\int\frac{\mathrm{d}^4k_{\mathrm{E}}}{(2\pi)^4}\widetilde{J}(k_{\mathrm{E}})\frac{1}{k_{\mathrm{E}}^2+m^2}\widetilde{J}(-k_{\mathrm{E}})\bigg]\\ &= \exp\bigg[\frac{1}{2}\int\mathrm{d}^4x\,\mathrm{d}^4x'\,J(x)\Delta_0(x-x')J(x')\bigg]\end{aligned}$$

The Euclidean Green's function is

$$G_{\mathrm{E}}^{(n)}(x_1,\dots,x_n) = \frac{\delta^n W_0^{\mathrm{E}}[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0}$$

The propagator is the two-point function

$$\Delta_0(x_1-x_2) = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}k(x_1-x_2)}}{k^2+m^2} = \langle 0|\,\mathcal{T}\{\varphi(x_1)\varphi(x_2)\}\,|0\rangle$$

where all variables are in Euclidean space (see Cheng–Li, eq. 1.84).

Exercise. Check that the propagator is the two-point function.

Green's function. [r] The Green's function is

$$G_{\mathrm{E}}^{(n)}(x_1,\dots,x_n) = \langle 0|\,\mathcal{T}\{\varphi(x_1)\cdots\varphi(x_n)\}\,|0\rangle = \frac{\delta^n W_0^{\mathrm{E}}[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0}$$

The Lagrangian has a $\mathbb{Z}_2$ symmetry and the Green's function must have that symmetry too. So only even-point functions are non-zero

$$G^{(2k+1)} = 0\,, \quad k \in \mathbb{N}_0$$

The four-point function is

$$\begin{aligned} G_{\mathrm{E}}^{(4)}(x_1,\dots,x_4) &= \langle 0|\,\mathcal{T}\{\varphi(x_1)\cdots\varphi(x_4)\}\,|0\rangle \\ &= \frac{\delta^4}{\delta J(x_1)\cdots\delta J(x_4)} \exp\left[\frac{1}{2}\int \mathrm{d}^4y\,\mathrm{d}^4y'\,J(y)\Delta_0(y-y')J(y')\right]_{J=0} \end{aligned}$$

One may use the fact that

$$\frac{\delta}{\delta J(x_j)}\int \mathrm{d}^4y\,\mathrm{d}^4y'\,J(y)\Delta_0(y-y')J(y') = 2\int \mathrm{d}^4y\,J(y)\Delta(y-x_j)$$

and the only terms that survive after plugging in $J=0$ are the ones without the factor J. The Green's function is

$$\begin{aligned} G_{\mathrm{E}}^{(4)}(x_1,\dots,x_4) &= \frac{\delta^2}{\delta J(x_1)J(x_2)}\bigg[\Delta(x_3-x_4)W_0^{\mathrm{E}}[J] \\ &\quad + \int \mathrm{d}^4y\,J(y')\Delta(y-x_3)\int \mathrm{d}^4y'\,J(y')\Delta(y-x_4)W_0^{\mathrm{E}}[J]\bigg] \\ &= \Delta(x_3-x_4)\Delta(x_1-x_2) + \Delta(x_2-x_3)\Delta(x_1-x_4) + \Delta(x_1-x_3)\Delta(x_2-x_4) \end{aligned}$$

At the last line [r]. The last line are all the combinations of products of propagators. This is Wick's theorem: all the possible two contractions of four terms.

3.4 Interacting field theory

Generating functional. A simple interacting theory is the $\lambda\varphi^4$ theory. Its Lagrangian is

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4 = \mathcal{L}_0 + \mathcal{L}_1$$

where the first two terms are the free part and the last term is the interacting part. The Euclidean time Lagrangian is

$$\mathcal{L}_{\mathrm{E}} = \frac{1}{2}\varphi(\Box_{\mathrm{E}} + m^2)\varphi + \frac{\lambda}{4!}\varphi^4$$

The generating functional is

$$\begin{aligned}
W^{\mathrm{E}}[J] &= N\int[\mathcal{D}\varphi]\,\exp\left[-\int \mathrm{d}^4x\,\left(\frac{1}{2}\varphi(\Box_{\mathrm{E}}+m^2)\varphi+\frac{\lambda}{4!}\varphi^4+J\varphi\right)\right]\\
&= N\int[\mathcal{D}\varphi]\,\exp\left[-\int \mathrm{d}^4x\,\left(\frac{1}{2}\varphi(\Box_{\mathrm{E}}+m^2)\varphi+J\varphi\right)\right]\exp\left[-\int \mathrm{d}^4x\,\frac{\lambda}{4!}\varphi^4\right]\\
&= N\int[\mathcal{D}\varphi]\sum_{n=0}^{\infty}\frac{1}{n!}\left[-\int \mathrm{d}^4x\,\frac{\lambda}{4!}\varphi^4\right]^n\exp\left[-\int \mathrm{d}^4x\,\left(\frac{1}{2}\varphi(\Box_{\mathrm{E}}+m^2)\varphi+J\varphi\right)\right]\\
&= N\int[\mathcal{D}\varphi]\sum_{n=0}^{\infty}\frac{1}{n!}\left[-\int \mathrm{d}^4x\,\frac{\lambda}{4!}\left(-\frac{\delta}{\delta J(x)}\right)^4\right]^n\\
&\quad\times\exp\left[-\int \mathrm{d}^4x\,\left(\frac{1}{2}\varphi(\Box_{\mathrm{E}}+m^2)\varphi+J\varphi\right)\right]\\
&= \exp\left[-\int \mathrm{d}^4x\,\frac{\lambda}{4!}(-\delta_{J(x)})^4\right]N\int[\mathcal{D}\varphi]\,\exp\left[-\int \mathrm{d}^4x\,\left(\frac{1}{2}\varphi(\Box_{\mathrm{E}}+m^2)\varphi+J\varphi\right)\right]\\
&= \exp\left[-\int \mathrm{d}^4x\,\mathcal{L}_1(-\delta_{J(x)})\right]N\int[\mathcal{D}\varphi]\,\exp\left[-\int \mathrm{d}^4x\,\left(\frac{1}{2}\varphi(\Box_{\mathrm{E}}+m^2)\varphi+J\varphi\right)\right]\\
&= \exp\left[-\int \mathrm{d}^4x\,\mathcal{L}_1(-\delta_{J(x)})\right]W_0[J]
\end{aligned}$$

where the variables are all in Euclidean time, the index is dropped from now on. [r]

3.5 Perturbation theory

See Cheng–Li, Sdrenicki. For small a parameter λ one can Taylor expand the exponential in the generating functional up to a desired order. The generating functional is

$$\begin{aligned}
W[J] &= \exp\left[-\frac{\lambda}{4!}\int \mathrm{d}^4x\,(-\delta_{J(x)})^4\right]W_0[J]\\
&= \left[1-\frac{\lambda}{4!}\int \mathrm{d}^4x\,(\delta_{J(x)})^4+\frac{1}{2!}\frac{\lambda^2}{(4!)^2}\int \mathrm{d}^4x\,(\delta_{J(x)})^4\int \mathrm{d}^4x'\,(\delta_{J(x')})^4+o(\lambda^2)\right]W_0[J]\\
&\equiv W_0[J](1+\lambda\omega_1[J]+\lambda^2\omega_2[J]+\cdots)
\end{aligned}$$

where

$$\begin{aligned}
\omega_1[J] &= -\frac{1}{4!}W_0^{-1}[J]\int \mathrm{d}^4x\,(\delta_{J(x)})^4W_0[J]\\
\omega_2[J] &= \frac{1}{2!}\frac{1}{(4!)^2}W_0^{-1}[J]\int \mathrm{d}^4x\,(\delta_{J(x)})^4\int \mathrm{d}^4x'\,(\delta_{J(x')})^4W_0[J]\\
W_0[J] &= \exp\left[\frac{1}{2}\int \mathrm{d}^4y\,\mathrm{d}^4y'\,J(y)\Delta(y-y')J(y')\right]\equiv \mathrm{e}^{k[J]}
\end{aligned}$$

and so on. The derivatives of the exponent of the generating functional are

$$\frac{\delta k[J]}{\delta J(x)}=\int \mathrm{d}^4y\,J(y)\Delta(y-x)\,,\quad \frac{\delta^2k[J]}{\delta J(x)^2}=\Delta_0(x-x)=\Delta_0(0)\,,\quad \frac{\delta^n k[J]}{\delta J(x)^n}=0\,,\quad n\geq 3$$

also

$$\frac{\delta W_0[J]}{\delta J(x)}=\frac{\delta k[J]}{\delta J(x)}\,W_0[J]\iff \frac{\delta \mathrm{e}^{k[J]}}{\delta J(x)}=\frac{\delta k[J]}{\delta J(x)}\,\mathrm{e}^{k[J]}$$

Lecture 7

mer 13 mar 2024 10:30

3.5.1 First-order expansion coefficient

One would like to compute the first-order expansion coefficient

$$\omega_1[J]=-\frac{1}{4!}W_0^{-1}[J]\int \mathrm{d}^4x\,(\delta_{J(x)})^4W_0[J]$$

Omitting the argument x and compacting the derivative notation, the integrand is

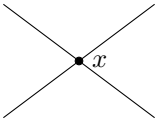
$$\begin{aligned}\delta_J^4 \mathrm{e}^{k[J]} &= \delta_J^3(\delta_J k\, \mathrm{e}^k) = \delta_J^2[\delta_J^2 k\, \mathrm{e}^k + (\delta_J k)^2 \mathrm{e}^k] = \delta_J[3\,\delta_J^2 k\, \delta_J k\, \mathrm{e}^k + (\delta_J k)^3 \mathrm{e}^k] \\ &= 3(\delta_J^2 k)^2 \mathrm{e}^k + 6\,\delta_J^2 k\,(\delta_J k)^2 \mathrm{e}^k + (\delta_J k)^4 \mathrm{e}^k \\ &= W_0[J][3(\delta_J^2 k)^2 + 6\,\delta_J^2 k\,(\delta_J k)^2 + (\delta_J k)^4]\end{aligned}$$

At the first two lines one applies one derivative at a time remembering that derivatives of order higher than two are zero. The coefficient is then

$$\begin{aligned}\omega_1[J] = -\frac{1}{4!}\int \mathrm{d}^4x \Big[&3\Delta(x-x)\Delta(x-x) \\ &+ 6\Delta(x-x)\int \mathrm{d}^4y_1\, \mathrm{d}^4y_2\, J(y_1)\Delta(x-y_1)J(y_2)\Delta(x-y_2) \\ &+ \int \mathrm{d}^4y_1\, \mathrm{d}^4y_2\, \mathrm{d}^4y_3\, \mathrm{d}^4y_4\, J(y_1)\Delta(x-y_1)J(y_2)\Delta(x-y_2)J(y_3)\Delta(x-y_3)J(y_4)\Delta(x-y_4)\Big]\end{aligned}$$

Feynman diagrams may be used to quickly obtain the expression above. [r] One may interpret the sum of the terms above in a convenient diagrammatic way:

- The internal point x is associated to

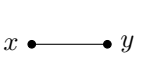


$$-\frac{1}{4!}\int \mathrm{d}^4x$$

- The external points y_i are associated to

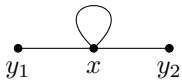
$$\int \mathrm{d}^4y_i\, J(y_i)$$

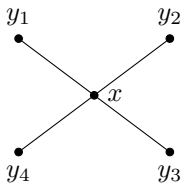
- The propagators $\Delta(x-y)$ are internal lines associated to



$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{\mathrm{e}^{\mathrm{i}k(x-y)}}{k^2+m^2}$$

In the first addendum of the coefficient ω_1 there is a single internal point with two propagators. The second addendum has one internal point and two external points with three propagators. The third addendum has one internal point and four external points with four propagators.





Notice that there are no odd numbers of external points: this is again related to the $\mathbb{Z}_2$ symmetry of the Lagrangian.

At fixed order in the λ expansion, one has a corresponding number of internal vertices: first order, one internal vertex; second order, two vertices, etc.

One observes:

- The number of powers of λ is equal to the number of internal vertices.
- The internal vertex is connected to four lines which corresponds to the four derivatives of the source J, which is related to the study of the $\lambda\varphi^4$ theory. The number of lines coming out of an internal vertex is determined by the choice of the interaction $\mathcal{L}_1$.

- The way one represents the coefficient ω_1 in terms of diagrams — which is related to the algebraic structure of the theory [r] — is a linear combination of diagrams of increasing number of external points. At a given order in the parameter λ, the corresponding coefficient ω is a linear combination of diagrams with increasing number of external points.
- The diagrams without any external points survive when setting $J = 0$: these are vacuum diagrams. They are the only ones contributing to $W[J = 0]$. The normalization with $W[0] = 1$ is equivalent to dividing the integral within $W[J]$ by $W[0]$:

$$W[J] = \frac{1}{W[J=0]} \int [\mathcal{D}\varphi] \exp\left[-\frac{1}{2}\int \mathcal{L}_{\mathrm{E}} + J\varphi\right]$$

  In this way one cancels all vacuum diagrams, so one may forget about them during calculations:

$$\begin{aligned} W[J] &= \frac{1 + 3D_1 + 3!D_2 + D_3 + o(\lambda)}{1 + 3D_1 + o(\lambda)_{\text{vacuum}}} \\ &= [1 + 3D_1 + 3!D_2 + D_3 + o(\lambda)][1 - 3D_1 + o(\lambda)_{\text{vacuum}}] \\ &= 1 + 3D_1 - 3D_1 + \cdots \end{aligned}$$

  where D_j are the algebraic expressions related to the diagrams. [r]
- Each diagram has an associated combinatorial factor. To find the factor one must find all the ways one can connect an internal point with four lines to n external points through one internal line each. The number obtained must be divided by the symmetry factor of the diagram — it is the number of equivalent ways one can draw the diagram. This factor can be computed in a more efficient way.

One would like to use the diagrams to compute the expansion terms [r].

For the coefficient ω_1, the factor $6 = 3!$ of the second addendum is obtained by the $12 = 4 \cdot 3$ ways one can connect an internal point to two external points divided by a factor of 2 because the external points can be swapped while giving the same diagram. For the third addendum, the internal point can be connected in 4! ways to the four external points. The symmetry factor comes from the ways in which one can permute the indices of the external line which is 4!. The factor of the addendum is then just 1. In both of these examples, the common factor of $(4!)^{-1}$ has been omitted, but it should always be included.

In general, for the order λ^p, the expansion coefficient is

$$\omega_p[J] = \frac{W_0^{-1}[J]}{p!}\left[-\frac{1}{4!}\right]^p \left[\int \mathrm{d}^4x \, (\delta_{J(x)})^4\right]^p W_0[J]$$

The diagrams have p internal points. The number of external points is even. The combinatorial factor is

$$s = \frac{1}{p!}\left[-\frac{1}{4!}\right]^p \frac{N}{D}$$

where N is the number of equivalent ways to connect the points with propagators and D is the symmetry factor of the diagram (which is the number of equivalent topologies).

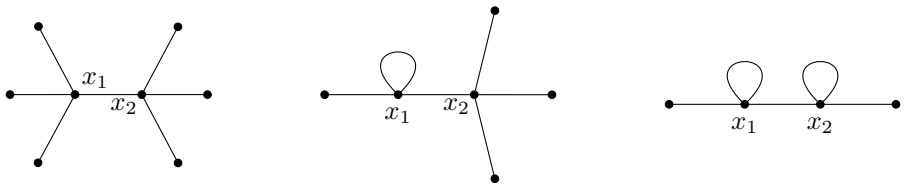
3.5.2 Second-order expansion coefficient

The coefficient contains two internal points x_1 and x_2. One may first draw the diagram and derive from it the algebraic expression.

Disconnected diagrams. The disconnected diagrams are

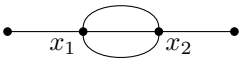
These sum up to $\frac{1}{2}\omega_1^2$.

Connected diagrams. The connected diagrams have lines connecting the internal points. With one line, one has



With two lines, one has

With three lines, one has



The vacuum diagrams have not been considered.

The diagrams can be translated into algebraic expressions. Consider the second connected diagram:

$$\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2 \int \mathrm{d}^4y_1\,\mathrm{d}^4y_2\,\mathrm{d}^4y_3\,\mathrm{d}^4y_4 J(y_1)J(y_2)J(y_3)J(y_4)\Delta(x_1-x_2)\Delta(x_2-x_2)$$
$$\times\,\Delta(x_1-y_1)\Delta(x_1-y_2)\Delta(x_1-y_3)\Delta(x_2-y_4)$$

One still needs the combinatorial factor. All diagrams have a common factor of

$$\frac{1}{2!}\frac{1}{(4!)^2}$$

One has to compute N and D for each diagram.

First diagram. The first line can be connected to the first internal point in 8 possible ways. The second line must connect to the same point as the first line, so there are 3 ways. Similarly, the third line has 2 ways. The first line on the second internal point can be put in 4 ways, the second in 3 and the third in 2. The last line of each internal point can be connected in one way only. Therefore

$$N = 8 \cdot 3! \cdot 4!$$

The symmetries are:

- exchanging the internal points, so two configurations;
- permutations of the three line of each internal vertex, 3! configurations each.

Therefore

$$D = 2 \cdot 3! \cdot 3!$$

The combinatorial factor is

$$S = \frac{1}{2!}\frac{1}{(4!)^2}\frac{N}{D} = \frac{1}{2(3!)^2}$$

The diagram translates to

$$\frac{1}{2(3!)^2}\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\prod_{j=1}^{6}\mathrm{d}y_j\,\Delta(x_1-y_1)\Delta(x_1-y_2)\Delta(x_1-y_3)$$
$$\times\,\Delta(x_2-y_4)\Delta(x_2-y_5)\Delta(x_2-y_6)J(y_1)J(y_2)J(y_3)J(y_4)J(y_5)J(y_6)$$

Second diagram. One has

$$N=8\cdot(4!)\cdot 3\,,\quad D=2\cdot(3!)\,,\quad s=\frac{1}{4\cdot 3!}=\frac{1}{4!}$$

Third diagram. One has

$$N=8\cdot(4)\cdot(9)\,,\quad D=2\,,\quad s=\frac{1}{8}$$

Fourth diagram. The fourth diagram has

$$N=8\cdot(3)\cdot(4\cdot 3)\cdot 2$$

and the symmetry factor is

$$D=2\cdot(2!)\cdot(2!)$$

The combinatorial factor is [r]

$$s=\frac{1}{16}$$

Fifth diagram. One may compute the factor for two copies of the diagram with the internal points swapped and sum their contributions. The first one gives

$$N=4\cdot 3\cdot(4\cdot 3)\,,\quad D=2\,,\quad s=\frac{1}{16}$$

Summing up with the second, one obtains

$$s=\frac{1}{8}$$

Sixth diagram. One has

$$N=8\cdot 4\cdot(3!)\,,\quad D=2\,,\quad s=\frac{1}{12}$$

Exercise. Translate each diagram into an algebraic expression.

Lecture 8

gio 14 mar 2024 10:30

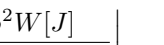
Remark. The generating functional is given by

$$W[J]=W_0[J]\Big[1+\lambda\Big(\quad+\quad\Big)+\lambda^2\Big(\quad+\quad+\quad+\quad+\quad$$
$$+\quad+\quad+\quad+\quad+\quad+\quad\Big)+o(\lambda^2)\Big]$$

The number of the external points in a diagram corresponds to the number of sources J appearing in the associated algebraic expression. The Green's function is

$$G^{(n)}(z_1,\dots,z_n)=\langle 0|\,\mathcal{T}\{\varphi(z_1)\cdots\varphi(z_n)\}\,|0\rangle=\left.\frac{\delta^n W[J]}{\delta J(z_1)\cdots\delta J(z_n)}\right|_{J=0}$$

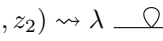
The two-point function is

$$G^{(2)}(z_1,z_2)=\frac{\delta^2 W[J]}{\delta J(z_1)\,\delta J(z_2)}\bigg|_{J=0}=\lambda$$  $$+\lambda^2\Big($$ $$+$$  $$+$$  $$\Big)+o(\lambda^2)$$

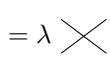
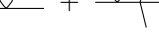
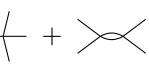
When evaluating at $J=0$, the non-vanishing terms are the ones that have exactly two external points because the derivatives remove precisely that many sources from the generating functional. When taking the derivative, one has

$$\delta_{J(z_1)}\int \mathrm{d}^4y_1\,J(y_1)\Delta(x-y_1)=\int \mathrm{d}^4y_1\,\delta^{(4)}(y_1-z_1)\Delta(x-y_1)=\Delta(x-z_1)$$

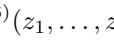
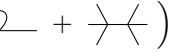
[r] The contributions to the Green's function are the propagators coming from the external points [r]

$$G^{(2)}(z_1,z_2)\rightsquigarrow\lambda$$  $$\rightsquigarrow\int \mathrm{d}^4x\,\Delta(x-z_1)\Delta(x-z_2)\Delta(x-x)$$

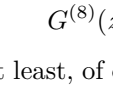
The four-point function is

$$G^{(4)}(z_1,\dots,z_4)=\lambda$$  $$+\lambda^2\Big($$ $$+$$  $$+$$  $$+$$ $$\Big)$$

The six-point function is

$$G^{(6)}(z_1,\dots,z_6)=\lambda^2\Big($$  $$+$$  $$\Big)$$

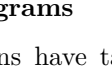
The eight-point function is

$$G^{(8)}(z_1,\dots,z_8)=\lambda^2$$ 

The ten-point function is, at least, of order λ^3. Higher point Green's functions get contributions only from higher order terms in the coupling constant λ.

3.5.3 Disconnected diagrams

The above Green's functions have taken into account disconnected diagrams. Consider the eight-point function

$$G^{(8)}(z_1,\dots,z_8)=\langle 0|\,\mathcal{T}\{\varphi(z_1)\cdots\varphi(z_8)\}\,|0\rangle=\lambda^2$$  $$+\text{permutations}$$

Up to combinatorial contributions, the Green's function is

$$\begin{aligned}G^{(8)}(z_1,\dots,z_8)&=\lambda^2\int \mathrm{d}^4x_1\,\Delta(x_1-z_1)\Delta(x_1-z_2)\Delta(x_1-z_3)\Delta(x_1-z_4)\\&\quad\times\int \mathrm{d}^4x_2\,\Delta(x_2-z_5)\Delta(x_2-z_6)\Delta(x_2-z_7)\Delta(x_2-z_8)\\&\sim\lambda^2\,\langle 0|\,\mathcal{T}\{\varphi(z_1)\cdots\varphi(z_4)\}\,|0\rangle\,\langle 0|\,\mathcal{T}\{\varphi(z_5)\cdots\varphi(z_8)\}\,|0\rangle\end{aligned}$$

From this one notices that disconnected Feynman diagrams contribute to disconnected Green's functions. These can be built from lower-order connected Green's functions.

To eliminate the disconnected contributions both in the generating functional and the Green's function, one notices that the connected Green's functions are given by

$$G_{\mathrm{c}}^{(n)}(z_1,\dots,z_n)=\frac{\delta^n\ln W[J]}{\delta J(z_1)\cdots\delta J(z_n)}\bigg|_{J=0}$$

One may check the above for the order λ^2. Rewriting the generating functional as

$$W[J]=W_0[J]\Big[1+W_0^{-1}[J](W[J]-W_0[J])\Big]$$

One sees that the second addendum is the perturbative expansion

$$W_0^{-1}[J](W[J]-W_0[J])=\lambda\omega_1[J]+\lambda^2\omega_2[J]+o(\lambda^2)$$

[r] Taking the logarithm, one has

$$\begin{aligned}\ln W[J] &= \ln W_0[J] + \ln[1 + W_0^{-1}[J](W[J] - W_0[J])] \\ &= \ln W_0 + W_0^{-1}(W - W_0) - \frac{1}{2}[W_0^{-1}(W - W_0)]^2 + o(\lambda^2) \\ &= \ln W_0 + \lambda\omega_1 + \lambda^2\omega_2 + o(\lambda^2) - \frac{1}{2}[\lambda\omega_1 + \lambda^2\omega_2 + o(\lambda^2)]^2 + o(\lambda^2) \\ &= \ln W_0 + \lambda\omega_1 + \lambda^2\left[\omega_2 - \frac{1}{2}\omega_1^2\right] + o(\lambda^2)\end{aligned}$$

At the first line, the perturbative expansion is of order λ, so one Taylor expands

$$\ln[1 + \varepsilon] = 1 + \varepsilon - \frac{1}{2}\varepsilon^2 + o(\varepsilon^2)$$

The second addendum in the bracket is the negative sum of the disconnected diagrams, therefore the whole bracket is the sum of only the connected diagrams at order λ^2.

One defines the generating functional of connected Green's functions $Z[J]$ as

$$W[J] \equiv \mathrm{e}^{Z[J]} \implies Z[J] \equiv \ln W[J]$$

Therefore, the connected Green's function prescription is

$$G_{\mathrm{c}}^{(n)}(z_1, \ldots, z_n) = \left.\frac{\delta^n Z[J]}{\delta J(z_1) \cdots \delta J(z_n)}\right|_{J=0}$$

The normalization follows from the one of the generating functional W

$$W[J = 0] = 1 \implies Z[J = 0] = 0$$

From the expansion above of $\ln W[J]$ one can see that

$$Z[J] = Z_0[J] + \lambda(\cdots) + \lambda^2(\cdots) + o(\lambda^2)$$

At the present moment, one has eliminated vacuum diagrams and disconnected diagrams.

3.6 Green's functions in momentum space

The integrals of Green's functions are easier to compute in momentum space rather than configuration space. One needs to reformulate the Feynman rules after applying the Fourier transform.

Definition. The n-point Green's function in momentum space is

$$\widetilde{G}_{\mathrm{c}}^{(n)}(p_1, \ldots, p_n)(2\pi)^4\delta^{(4)}(p_1 + p_2 + \cdots + p_n) = \int \prod_{j=1}^{n} \mathrm{d}^4 x_j \, \mathrm{e}^{-\mathrm{i}p_j x_j} \, G_{\mathrm{c}}^{(n)}(x_1, \ldots, x_n)$$

One may apply this prescription to the two-point Green's function. In configuration space, the free Green's function is

$$G_0^{(2)}(x_1, x_2) = \Delta(x_1 - x_2) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}k(x_1 - x_2)}}{k^2 + m^2}$$

In momentum space one has

$$\begin{aligned}\widetilde{G}_0^{(2)}(p_1, p_2)(2\pi)^4\delta^{(4)}(p_1 + p_2) &= \int \mathrm{d}^4 x_1 \, \mathrm{d}^4 x_2 \, \mathrm{e}^{-\mathrm{i}(p_1 x_1 + p_2 x_2)} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}k(x_1 - x_2)}}{k^2 + m^2} \\ &= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} (2\pi)^4\delta^{(4)}(k - p_1)(2\pi)^4\delta^{(4)}(k + p_2) \frac{1}{k^2 + m^2} \\ &= (2\pi)^4\delta^{(4)}(p_1 + p_2) \frac{1}{p_1^2 + m^2}\end{aligned}$$

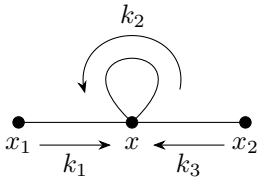
from which

$$\widetilde{G}_0^{(2)}(p_1,p_2)=\frac{1}{p_1^2+m^2}\,,\quad p_1=-p_2\iff\boxed{\widetilde{G}_0^{(2)}(p,-p)=\frac{1}{p^2+m^2}}$$

One would like to find the momentum space version of Feynman rules. Consider the correction of order λ to the two-point Green's function $G^{(2)}$. There is only one contributing diagram [r]. One has

$$\begin{aligned}G^{(2)}(x_1,x_2)\Big|_\lambda&=-\frac{\lambda}{2}\int\mathrm{d}^4x\,\Delta(x-x_1)\Delta(x-x_2)\Delta(x-x)\\&=-\frac{\lambda}{2}\int\mathrm{d}^4x\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}k_1(x-x_1)}}{k_1^2+m^2}\int\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{1}{k_2^2+m^2}\int\frac{\mathrm{d}^4k_3}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}k_3(x-x_2)}}{k_3^2+m^2}\end{aligned}$$

Since one can always send $k\to-k$ in the integral, one has to choose in which way to assign momentum and has to be consistent with the assignment. For example, in the above, the momentum k_1 goes from an external point x_1 towards an internal point x. The momentum in the loop can be put in either direction.



Performing the x-integral, one obtains

$$\int\mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}x(k_1+k_3)}=(2\pi)^4\delta^{(4)}(k_1+k_3)$$

Therefore, the Green's function is

$$G^{(2)}(x_1,x_2)\Big|_\lambda=-\frac{\lambda}{2}\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}k_1(x_1-x_2)}}{(k_1^2+m^2)^2(k_2^2+m^2)}$$

The Fourier transform is then

$$\begin{aligned}&\widetilde{G}^{(2)}(p_1,p_2)\Big|_\lambda(2\pi)^4\delta^{(4)}(p_1+p_2)=\\&\qquad=-\frac{\lambda}{2}\int\mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\mathrm{e}^{-\mathrm{i}p_1x_1-\mathrm{i}p_2x_2}\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}k_1(x_1-x_2)}}{(k_1^2+m^2)^2(k_2^2+m^2)}\\&\qquad=-\frac{\lambda}{2}\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{(2\pi)^4\delta^{(4)}(k_1-p_2)(2\pi)^4\delta^{(4)}(p_1+k_1)}{(k_1^2+m^2)^2(k_2^2+m^2)}\\&\qquad=-\frac{\lambda}{2}(2\pi)^4\delta^{(4)}(p_1+p_2)\int\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{1}{k_2^2+m^2}\frac{1}{(p_2^2+m^2)^2}\end{aligned}$$

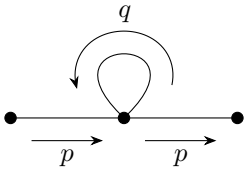
Keeping in mind the momentum conservation, one has

$$\widetilde{G}^{(2)}(p_1,p_2)\Big|_\lambda=-\frac{\lambda}{2}\frac{1}{(p_2^2+m^2)^2}\int\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{1}{k_2^2+m^2}\,,\quad p_1=-p_2$$

Tidying up the expression, one can write

$$\widetilde{G}^{(2)}(p,-p)=-\frac{\lambda}{2}\frac{1}{(p^2+m^2)^2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{q^2+m^2}$$

The corresponding diagram becomes



[r] In this way one guarantees momentum conservation at the internal vertex.

Feynman rules. One has to assign a set of momenta to guarantee momentum conservation at each vertex and total momentum conservation using the direction of the arrows. [r] Every time a loop appears, there is an integral on the loop momentum.

Therefore

- an internal vertex corresponds to

$$-\frac{\lambda}{4!}(2\pi)^4\delta^{(4)}(p_1+p_2+p_3+p_4)$$

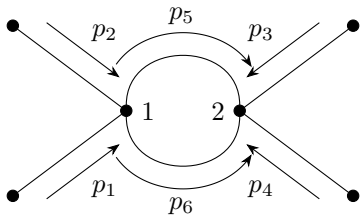
- a propagator is an internal line corresponding to

$$\frac{1}{p^2+m^2}$$

- a loop corresponds to an integral

$$\int\frac{\mathrm{d}^4q}{(2\pi)^4}$$

Exercise. Write the momentum integral corresponding to the following diagram



One should obtain a λ^2 contribution to the Green's function $\widetilde{G}^{(4)}(p_1,\dots,p_4)$.

One should assign a momentum to every external line and internal line. One imposes momentum conservation at each internal vertex. At the first vertex, one has

$$p_1+p_2=p_5+p_6$$

At the second vertex, one has

$$p_5+p_6=-p_3-p_4$$

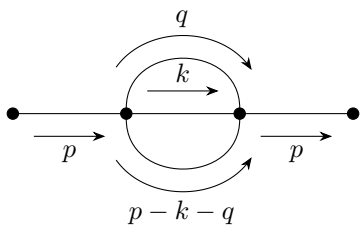
From these two conditions, one has the total momentum conservation and the dependence of one momentum on the others

$$p_1+p_2+p_3+p_4=0\,,\quad p_6=p_1+p_2-p_5$$

There is only a single integral in p_5.

Briefly, one draws the diagram, assigns the momenta remembering total momentum conservation; there is only one loop which corresponds to an integral, the other line is given by momentum conservation.

Example. Consider the following two-loop diagram



The algebraic expression without the combinatorial factor is

$$\widetilde{G}^{(2)}(p,-p)\Big|_{\lambda^2}\rightsquigarrow\frac{1}{(p^2+m^2)^2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(q^2+m^2)(k^2+m^2)[(p-k-q)^2+m^2]}$$

The main point is to compute the integrals. One does not care about external propagators because they are spectators. One moves towards the evaluation of cut diagrams where one focuses only on the integrals of the diagram.

3.7 Loop expansion

Previously, the generating functional has been perturbatively expanded in powers of the coupling constant λ. One may reshuffle the terms and organize them by how may loops are present. In this way one can obtain an expansion in powers of the reduced Planck's constant $\hbar$ [r].

One should first fix the Green's function, so fix the number of external points. Writing explicitly the Planck's constant, the generating functional is

$$\begin{aligned} W[J] &= N \int [\mathcal{D}\varphi] \exp\left[-\int \mathrm{d}^4x \left(\frac{1}{\hbar}(\mathcal{L}_0 + \mathcal{L}_1) + J\varphi\right)\right] \\ &= N \int [\mathcal{D}\varphi] \exp\left[-\frac{1}{\hbar}\int \mathrm{d}^4x \,(\mathcal{L}_0 + \mathcal{L}_1 + \hbar J\varphi)\right] \\ &= \exp\left[-\frac{1}{\hbar}\int \mathrm{d}^4x \,\mathcal{L}_1(-\delta_{J(x)})\right] N \int [\mathcal{D}\varphi] \exp\left[-\frac{1}{\hbar}\int \mathrm{d}^4x \,(\mathcal{L}_0 + \hbar J\varphi)\right] \\ &= \exp\left[-\frac{1}{\hbar}\int \mathrm{d}^4x \,\mathcal{L}_1(-\delta_{J(x)})\right] W_0[J] \end{aligned}$$

The propagator is the inverse of the kinetic term, [r] therefore

$$W_0[J] = \exp\left[\frac{\hbar}{2}\int \mathrm{d}^4x \,\mathrm{d}^4x' \, J(x)\Delta(x-x')J(x')\right]$$

The Feynman rules have to be modified. The propagator is $\hbar\Delta(x-y)$ and so it carries one Planck's constant $\hbar$. Each internal point brings one inverse of the Planck's constant $\hbar^{-1}$ since the internal points come from the expansion of the exponential of the integral of the interaction Lagrangian $\mathcal{L}_1$.

Considering a diagram with V internal vertices and I internal lines. Its order is

$$\hbar^{I-V} = \frac{1}{\hbar}\hbar^{I-V+1} = \frac{1}{\hbar}\hbar^{L}$$

where $L = I - V + 1$ is the number of loops. In this way one can reorganize the perturbative expansion in powers of the Planck's constant and as such in terms of the number of loops.

Lecture 9

lun 18 mar 2024 10:30

One would like to show that for a theory with only one coupling constant, the loop expansion is equivalent to the expansion in powers of the coupling constant λ. In the following, one analyzes the $\lambda\varphi^4$ theory. One considers a contribution to the Green's function of order λ^V where V is the number of internal vertices.

A relation particular of this theory is the total number of lines exiting from the vertices

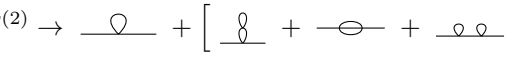
$$4V = E + 2I$$

where E is the number of external lines and I is the number of internal lines. Combining this equation with the formula for the number of loops, one finds

$$4V = E + 2(L + V - 1) \implies V = \frac{1}{2}E + L - 1$$

One may remember that the number of external lines E characterizes the Green's functions, so one may keep it fixed. Therefore, from λ^V one can obtain the order in powers of the Planck's constant $\lambda^{L-1} \sim \hbar^{L-1}$.

Example. Considering the two-point Green's function, it has contribution from diagrams of order λ and λ^2

$G^{(2)} \to$  $+\cdots$

Since $E = 2$ then one finds $\hbar$ and $\hbar^2$ respectively.

Remark. The Taylor expansion of a Green's function $G^{(n)}$ organized in powers of Planck's constant $\hbar^L$ contains terms of order $\hbar^0$ which are the classical contributions. These correspond to diagrams without any loop. These diagrams are called tree diagrams (tree in the sense of graph theory).

Example. The tree diagram for the two-point Green's function is the classical propagator, also called free propagator. For the four-point Green's function, the tree diagram is the cross diagram. In fact, one has $E = 4$ for which

$$G^{(4)} \to \times + \text{(cross)} + \text{permutations} \sim \lambda + \lambda^2 \sim \hbar^0 + \hbar$$

The Taylor expansion is done in powers of Planck's constant keeping in mind the correspondence with the loop expansion.

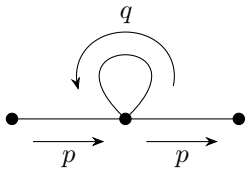
3.8 Ultraviolet divergences

[r] write discussion lec 08.

The functional measure is not well-defined for high energies. [r] For example, one finds ill-defined integrals associated to the two-point Green's function. Letting

$$\widetilde{G}^{(2)}(p, -p) \equiv \widetilde{G}^{(2)}(p)$$

one finds 1-loop contributions given by



$$\to \quad \frac{1}{(p^2+m^2)^2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{q^2+m^2}$$

This integral is divergent for large momenta q. The superficial degree of divergence D is the difference between the number of momenta in the numerator and the number of momenta in the denominator. For the above integral one has

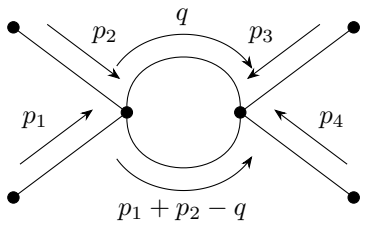
$$D = 4 - 2 = 2$$

When $D \geq 0$, ultraviolet divergences occur. The integral diverges quadratically and its dimension is D.

The 1-loop correction to the four-point Green's function

$$\widetilde{G}^{(4)}(p_1, p_2, p_3, -p_1 - p_2 - p_3)$$

is given by the diagram



+ permutations

whose algebraic expression is proportional to

$$\prod_{i=1}^{4} \frac{1}{p_i^2+m^2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(q^2+m^2)[(p_1+p_2-q)^2+m^2]}$$

The superficial degree of divergence is

$$D = 4 - 4 = 0$$

The integral diverges logarithmically.

One would like to study ultraviolet divergences through a general method instead of computing the superficial degree of freedom for each diagram. Given a certain theory, one would like to know how many kinds of diagrams are divergent and how many diagrams are divergent. Given the infinite Taylor expansion, there are three scenarios: the theory contains a finite number of ultraviolet divergent diagrams, there is a finite number of topologies of divergent diagrams, there is an infinite number of different topologies of divergent diagrams. The nature of the theory depends on these three possibilities. One needs a way to compute how many diagrams and topologies are divergent given a theory, an interaction and a dimension.

Number of divergent diagrams. Consider a theory in d space-time dimensions with a vertex connected to N lines (for example $\lambda\varphi^N$). Given a diagram with V_N internal vertices, E external lines, I internal lines and L loops, the superficial degree of divergence is

$$D = dL - 2I$$

The factor 2 depends on the spin of the field: scalar fields have 2 while fermion fields have 1. Noting that

$$L = I - V_N + 1\,, \quad NV_N = E + 2I$$

one obtains the superficial degree of divergence of a scalar $\lambda\varphi^N$ theory

$$\boxed{D = d - \frac{1}{2}(d-2)E + V_N\left[\frac{N-2}{2}d - N\right]}$$

Divergent diagrams correspond to $D \geq 0$.

Fixing $d = 4$, one finds that

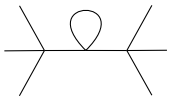
$$D = 4 - E + (N-4)V_N$$

The last term is negative only for $N = 3$ because for $N = 2$ one has a mass term, not a vertex. In this case, higher order terms are more convergent because the number of internal points increases. For $N = 4$, the superficial degree of divergence

$$D = 4 - E$$

does not depend on the number of internal vertices: at every order, the degree of divergence is the same. Also, the degree is $D \geq 0$ only for $E = 0, 2, 4$. For $E = 0$ one has vacuum diagrams, for $E = 2$ one has the two-point Green's function $\widetilde{G}^{(2)}$ and for $E = 3$ one has the four-point $\widetilde{G}^{(4)}$. Therefore, for the $\lambda\varphi^4$ theory, there are only two divergent topologies each with an infinite number of diagrams. There are ultraviolet divergences at any loop and they are of the type

However the analysis is not conclusive. Consider a six-point Green's function. A 1-loop correction is



$$\to \quad \text{external propagators} \times \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{1}{q^2 + m^2}$$

Its superficial degree of divergence is $D = -2$, but the diagram is divergent as it has a divergent sub-diagram. If the superficial degree of divergence is negative $D < 0$, it is not enough to conclude that the diagram is convergent.

Theorem (Weinberg)**.** A Feynman diagram is ultraviolet finite if

- the superficial degree of divergence is negative $D < 0$,
- the degree of every sub-diagram is negative $D_i < 0$.

[r] One may split a diagram into sub-diagrams by cutting free propagators.

One-particle irreducible diagrams. Considering again the six-point Green's function, one may see that the loop sub-diagram can isolated by cutting two free propagators: the whole diagram is 2-particle reducible.

Generally, consider a loop diagram with a given number of external lines connected to another loop diagram by n free propagators. The diagram is then n-particle reducible: it can be split into two sub-diagrams by cutting n free propagators. The superficial degree of divergence of the diagram is

$$D = D_1 + D_2 + 4(n-1) - 2n$$

[r]

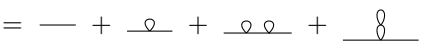
For example, if $D_1 = D_2 = 0$, then the sub-diagrams are logarithmically divergent, but the total degree is

$$D = 2n - 4$$

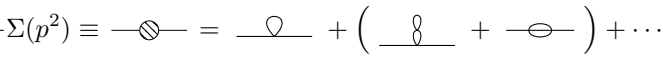
which is negative for $n = 1$. So the total diagram seems to be convergent, but the sub-diagrams diverge.

One would like to focus only on diagrams that cannot be reduced by cutting free propagators: these are called one-particle irreducible (1PI) diagrams. To compute divergences, it suffices to compute one-particle irreducible diagrams because the reducible diagrams can be constructed from the irreducible ones by using free propagators.

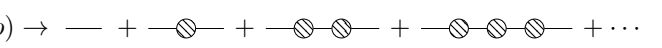
Consider the corrections to the two-point Green's function

$\widetilde{G}^{(2)} =$  $+ o(\hbar^2)$

The third diagram is the only one reducible diagram given by two one-loop contributions connected by a free propagator. One defines the sum of all irreducible diagrams also called self-energy

$1\mathrm{PI} = -\Sigma(p^2) \equiv$  $+ \cdots$

from which one can see that [r]

$\widetilde{G}^{(2)}(p) \to$  $+ \cdots$

The associated algebraic expression is

$$\begin{aligned}\widetilde{G}^{(2)}(p) &= \frac{1}{p^2+m^2} + \frac{1}{p^2+m^2}[-\Sigma(p^2)]\frac{1}{p^2+m^2} \\ &\quad + \frac{1}{p^2+m^2}[-\Sigma(p^2)]\frac{1}{p^2+m^2}[-\Sigma(p^2)]\frac{1}{p^2+m^2} + \cdots \\ &= \frac{1}{p^2+m^2}\left[1 + \frac{\Sigma(p^2)}{p^2+m^2}\right]^{-1} = \frac{1}{p^2+m^2+\Sigma(p^2)}\end{aligned}$$

At the second line, the addenda contain a resummation of higher order loops given by the resummation $\Sigma(p^2)$. At the second equality one applies the geometric series. The resummation is a function of p^2 due to Lorentz invariance. The Green's function contains all the quantum corrections, so it becomes the quantum propagator.

From this discussion, one sees that one can study only the one-particle irreducible diagrams. From all diagrams, one has restricted oneself to the study of non-vacuum diagrams, connected diagrams and one-particle irreducible diagrams. One also notices that the external propagators can be omitted since they are recovered in the final resummation: one computes amputated one-particle irreducible diagrams. The resummation of these amputated diagrams is

$$\Gamma^{(n)}_{\mathrm{c}}(p_1,\ldots,p_n) = G^{(n)}_{\mathrm{c}}(p_1,\ldots,p_n)_{1\mathrm{PI}}\left[\prod_{k=1}^{n} G^{(2)}(p_k)\right]^{-1}$$

One may now drop the tilde to denote the Fourier transform of a Green's function and use the argument as a marker. One notices that

$$\Gamma^{(2)}(p^2) = -\Sigma(p^2)$$

In the $\lambda\varphi^4$ theory, the only ultraviolet divergent diagrams are the ones in $\Gamma^{(2)}$ and $\Gamma^{(4)}$.

3.8.1 Excursus: superficial degree of divergence

Two dimensions. The general case for the superficial degree of divergence is

$$D = d - \frac{1}{2}(d-2)E + V_N \left[\frac{N-2}{2}d - N\right]$$

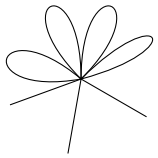
Consider two dimensions $d = 2$. The degree becomes

$$D = 2(1 - V_N)$$

It does not depend on the number of external lines. Therefore the divergences appear in the same way in any Green's function. Also, the degree does not depend on the number N of lines of the interaction vertex. The divergences appear for

$$2(1 - V_N) \geq 0 \implies V_N = 0, 1$$

Only the n-point Green's function with only one vertex is divergent

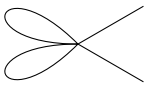


For a vertex $\lambda\varphi^N$, one connects n of lines to other points and contracts the remaining $N - n$ lines into loops. These diagrams are divergent.

These diagrams can be avoided by regularizing the theory with normal ordering (see Theoretical Physics I)

$$\lambda\varphi^N \rightsquigarrow :\lambda\varphi^N:$$

Example. For $\lambda\varphi^6$, an example of two-point function diagram is the following



Four dimensions. For four dimensions $d = 4$, the degree is

$$D = 4 - E + (N-4)V_N$$

For $\lambda\varphi^3$ theory, $N = 3$, one has

$$D = 4 - E - V_3$$

Fixing the number of external lines E, that is fixing the Green's function, then

$$D \geq 0 \implies 4 - E \geq V_3 \implies E \leq 4 - V_3 < 4$$

The divergences appear only for $E = 1, 2, 3$ and the degree is positive only for a restricted number of vertices

$$\begin{aligned} &E = 1\,, \quad D = 3 - V_3 \geq 0\,, \quad V_3 \leq 3 \\ &E = 2\,, \quad D = 2 - V_3 \geq 0\,, \quad V_3 \leq 2 \\ &E = 3\,, \quad D = 1 - V_3 \geq 0\,, \quad V_3 \leq 1 \end{aligned}$$

The number of topologies is finite and, in each, the number of divergent diagrams is finite.

Six dimensions. In six dimensions $d = 6$, one has

$$D = 6 - 2E + 2V_N(N - 3)$$

Setting $N = 3$, one finds

$$D = 6 - 2E$$

This is a similar situation situation as $\lambda\varphi^4$ in $d = 4$: no dependence on internal vertices. In fact

$$D \geq 0 \implies E = 1, 2, 3$$

So only the Green's functions $G^{(1)}$, $G^{(2)}$ and $G^{(3)}$ are divergent with an infinite number of divergent diagrams each.

Exercise. Study $d = 3$ and $d = 5$.

Lecture 10

mar 19 mar 2024 10:30

3.9 Conventional renormalization

See Cheng–Li, §2.1. Before evaluating the integrals, one would like to isolate the divergent part.

Historically, conventional renormalization was the first method of renormalization, however it is not suitable for higher orders: beyond two loops, the counter terms renormalization is more efficient.

As a prototype, one considers the $\lambda\varphi^4$ theory in $d = 4$ dimensions. The divergences appear only for the Green's functions $G^{(2)}$ and $G^{(4)}$ since the superficial degree of divergence is

$$D = 4 - E$$

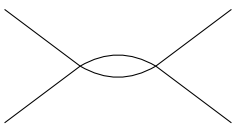
3.9.1 Regularization

In order to compute the integrals one employs regularization by using a cutoff or through a dimensional parameter: one introduces a parameter that makes the integral finite. The divergent integrals depend on some parameters in such a way that when one removes these regulators, the integrals diverge again.

Example. The following one-loop diagram is divergent since its degree is $D = 2$

One integrates up to a finite cutoff Λ. This regularization parameter has mass dimension 1. The integral associated to the diagram is quadratically divergent and asymptotic to $c_1\Lambda^2 \to \infty$.

Example. The following two-loop diagram [r] has degree $D = 0$



It is asymptotic to

$$c_2 \log \frac{\Lambda^2}{m^2} \to \infty$$

One may use effective field theory [r] like in phenomenology. However, this method does not permit to work at any energy scale.

3.9.2 Renormalization

Before removing the regulator, one has to find a consistent way to remove the divergent part of ultraviolet divergent integrals. For example, the previous integrals are a sum of a divergent part and a finite part. One would like to remove the divergent part and keep the finite one.

Isolating the divergence — four-point connected amputated Green's function. The first step is understand how to isolate the divergent part in the result of a given integral. Consider the diagram in the second example above. It is given by

$$\Gamma^{(4)}(s) \propto \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(q^2+m^2)[(p-q)^2+m^2]}\,, \quad s \equiv p^2 = (p_1+p_2)^2$$

where s is the first Mandelstam variable. The integral depends on the external momenta p_1 and p_2. Applying any number of derivatives with respect to the external momenta improves the convergence of the integral

$$\begin{aligned}
\partial_{p^2}\Gamma^{(4)}(p^2) = \partial_{p^2} p^\mu\, \partial_{p^\mu}\Gamma^{(4)}(p^2) &= \frac{p^\mu}{2p^2}\,\partial_{p^\mu}\Gamma^{(4)}(p^2) \\
&\propto \frac{p^\mu}{2p^2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\partial_{p^\mu}\frac{1}{(q^2+m^2)[(p-q)^2+m^2]} \\
&= -\frac{p^\mu}{2p^2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{2(p_\mu - q_\mu)}{(q^2+m^2)[(p-q)^2+m^2]^2} \\
&= -\frac{1}{p^2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{p(p-q)}{(q^2+m^2)[(p-q)^2+m^2]^2}
\end{aligned}$$

at the first line, one as

$$\partial_{p^\mu} p^2 = \partial_{p^\mu}(p^\nu p_\nu) = 2p_\mu \implies \partial_{p^2} p^\mu = \frac{1}{2p^2}p^\mu$$

The degree of divergence in the integration variable q is

$$D = 4 + 1 - 6 = -1$$

However, one would like to separate the various contribution to the integral [r]. One may note that

$$\begin{aligned}
p(p-q) &= -\frac{1}{2}[(p-(p-q))^2 - p^2 - (p-q)^2] = -\frac{1}{2}[q^2 - p^2 - (p-q)^2] \\
&= -\frac{1}{2}[-p^2 + (q^2+m^2) - ((p-q)^2+m^2)]
\end{aligned}$$

The idea is to either get a term depending only on the external momenta or get something resembling the denominator. Therefore

$$\begin{aligned}
\partial_{p^2}\Gamma^{(4)}(p^2) \propto &-\frac{1}{2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{1}{(q^2+m^2)[(p-q)^2+m^2]^2} + \frac{1}{2p^2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{1}{[(p-q)^2+m^2]^2} \\
&- \frac{1}{2p^2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{1}{(q^2+m^2)[(p-q)^2+m^2]}
\end{aligned}$$

The respective degrees of divergence of these three integrals are

$$D = -2\,, \quad D = 0\,, \quad D = 0$$

The first integral is finite. The last two are separately ultraviolet divergent, however their difference is finite. Taking the derivative with respect to the external momentum p^2 lowers the degree of divergence by 2 and gives a convergent integral.

To separate the divergent part from the finite part, one may Taylor expand the amputated function $\Gamma^{(4)}$ around a point. Choosing the origin, $p^2 = 0$, one obtains

$$\Gamma^{(4)}(p^2) = \Gamma^{(4)}(0) + \mathrm{d}_{p^2}\Gamma^{(4)}\Big|_{p^2=0} p^2 + o(p^2) \equiv \Gamma^{(4)}(0) + \widetilde{\Gamma}^{(4)}(p^2)$$

The first addendum is logarithmically divergent while the other terms are finite (denoted by the tilde). One may notice that

$$\widetilde{\Gamma}^{(4)}(p^2 = 0) = 0$$

To isolate the finite part of the function $\Gamma^{(4)}$ around $p^2 = 0$ one needs to evaluate

$$\widetilde{\Gamma}^{(4)}(p^2) = \Gamma^{(4)}(p^2) - \Gamma^{(4)}(0)$$

The choice of the point $p^2 = 0$ is arbitrary. One may take any other point $p^2 = \mu^2$ and obtain

$$\Gamma^{(4)}(p^2) = \Gamma^{(4)}(\mu^2) + \mathrm{d}_{p^2}\Gamma^{(4)}\Big|_{p^2=\mu^2}(p^2 - \mu^2) + o(p^2 - \mu^2) \equiv \Gamma^{(4)}(\mu) + \widetilde{\Gamma}^{(4)}(p^2, \mu^2)$$

where, once again, the first addendum is logarithmically divergent while the other terms are finite. One similarly has

$$\widetilde{\Gamma}^{(4)}(p^2 = \mu^2, \mu^2) = 0$$

Though, the finite part is different based on the chosen point. One may want to understand how to relate these various finite parts (without explicitly setting $\mu^2 = 0$).

The chosen expansion point μ^2 is called subtraction point: a point in momentum space around which one performs the Taylor expansion. When one chooses a subtraction point, one subsequently fixes an energy scale.

Isolating the divergence — two-point connected amputated Green's function. Since the function $\Gamma^{(2)}$ has a degree $D = 2$ one needs to differentiate at least twice to obtain a finite result.

Consider the first example above. One needs two derivatives to reach a finite integral

$$\Gamma^{(2)}(p^2) = \Gamma^{(2)}(\mu^2) + \partial_{p^2}\Gamma^{(2)}\Big|_{p^2=\mu^2}(p^2 - \mu^2) + \frac{1}{2}\partial^2_{p^2}\Gamma^{(2)}\Big|_{p^2=\mu^2}(p^2 - \mu^2)^2 + o(\cdots)$$

The first two addenda are divergent, while the rest are finite. Recalling that $\Gamma^{(2)} = -\Sigma$ [r] one then has

$$\Sigma(p^2) = \Sigma(\mu^2) + \Sigma'(\mu^2)(p^2 - m^2) + \widetilde{\Sigma}(p^2, \mu^2)$$

with

$$\widetilde{\Sigma}(p^2 = \mu^2, \mu^2) = 0\,, \quad \widetilde{\Sigma}'(p^2 = \mu^2, \mu^2) = 0$$

One may notice that, considering only one-loop correction to Σ, one has

$$\Sigma^{(1)} \propto \int \frac{\mathrm{d}^4 q}{q^2 + m^2}$$

So it does not depend on p^2 and it follows

$$[\Sigma^{(1)}]'(p^2 = \mu^2) = 0$$

Removal of divergences. The second step is remove the divergent parts of the integrals by redefining the parameters of the theory: the field φ, the mass m and the coupling constant λ. This is the renormalization step.

In the conventional normalization, one expresses the Lagrangian in terms of non-renormalized quantities (called bare) φ_0, m_0 and λ_0; the Feynman rules are given in terms of these bare parameters. On the other hand, in the counter terms renormalization, one uses renormalized quantities to begin with.

Renormalization — two-point Green's function. Following Cheng–Li, one may rotate back to Minkowski. The sum of one-particle irreducible corrections is $-\mathrm{i}\Sigma(p^2)$ and the two-point Green's function is

$$\mathrm{i}G^{(2)}(p) = \frac{\mathrm{i}}{p^2 - m_0^2 - \Sigma(p^2) + \mathrm{i}\varepsilon}$$

One may choose a subtraction point $p^2 = \mu^2$. The corrections are divergent

$$\Sigma(p^2) = \Sigma(\mu^2) + \Sigma'(\mu^2)(p^2 - \mu^2) + \widetilde{\Sigma}(p^2, \mu^2)$$

The first two addenda are ultraviolet divergent with degrees $D = 2$ and $D = 0$ (which correspond to their mass dimensions). Expanding the corrections, the Green's function is

$$\mathrm{i}G^{(2)}(p) = \frac{\mathrm{i}}{p^2 - m_0^2 - \Sigma(\mu^2) - (p^2 - \mu^2)\Sigma'(\mu^2) - \widetilde{\Sigma}(p^2) + \mathrm{i}\varepsilon}$$

Looking at the dimensions, one redefines the mass parameter

$$\boxed{m_0^2 + \Sigma(\mu^2) = \mu^2}$$

On the right-hand side, one could choose an arbitrary parameter since one can always add finite terms that are reabsorbed into the finite term $\widetilde{\Sigma}$.

In the expression above, there is a sum between a parameter and a divergent quantity that gives a finite term: one assumes that the bare parameter m_0 is itself divergent in a way exactly opposite to the corrections Σ. The arbitrariness above can also be seen as a consequence of the subtraction of two infinite quantities.

The employment of the subtraction point μ^2 as the mass parameter is purely utilitarian: one could choose any other parameter. In this way one obtains

$$\begin{aligned}\mathrm{i}G^{(2)}(p) &= \frac{\mathrm{i}}{(p^2 - \mu^2)[1 - \Sigma'(\mu^2)] - \widetilde{\Sigma}(p^2) + \mathrm{i}\varepsilon} = \frac{\mathrm{i}}{[1 - \Sigma'(\mu^2)][p^2 - \mu^2 - \widetilde{\Sigma}(p^2)] + \mathrm{i}\varepsilon} \\ &\equiv \frac{\mathrm{i}Z_\varphi}{p^2 - \mu^2 - \widetilde{\Sigma}(p^2) + \mathrm{i}\varepsilon}\end{aligned}$$

One would like to isolate the divergent bracket in the first equality. The second equality holds at a given order in the coupling constant λ_0, because higher order terms appear and they are neglected. One defines the field renormalization function

$$Z_\varphi = [1 - \Sigma'(\mu^2)]^{-1} \sim 1 + \Sigma'(\mu^2)$$

This ultraviolet divergent term can be removed by renormalizing the bare field

$$\boxed{\varphi = Z_\varphi^{-\frac{1}{2}}\varphi_0}$$

One may study how this renormalization works. Consider the renormalized Green's function

$$\begin{aligned}\mathrm{i}G_{\mathrm{R}}^{(2)}(p) &\equiv \int \mathrm{d}^4x\, \mathrm{e}^{-\mathrm{i}px} \langle 0| \mathcal{T}\{\varphi(x)\varphi(0)\} |0\rangle = Z_\varphi^{-1} \int \mathrm{d}^4x\, \mathrm{e}^{-\mathrm{i}px} \langle 0| \mathcal{T}\{\varphi_0(x)\varphi_0(0)\} |0\rangle \\ &= Z_\varphi^{-1}\mathrm{i}G^{(2)}(p) = \frac{\mathrm{i}}{p^2 - \mu^2 - \widetilde{\Sigma}(p^2) + \mathrm{i}\varepsilon}\end{aligned}$$

where one omits the Dirac delta function of conservation.

For the two-point Green's function, one has removed the divergences by performing mass and field renormalization.

Remark. Remembering that

$$\widetilde{\Sigma}(p^2 = \mu^2) = 0$$

the denominator of the two-point Green's function is

$$[p^2 - \mu^2 - \widetilde{\Sigma}(p^2)]_{p^2=\mu^2} = 0$$

Therefore $p^2 = \mu^2$ a pole of the propagator and the parameter μ is the physical mass. This may not be true depending on the choice of the subtraction point.

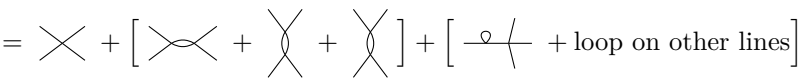
This procedure also applies to higher-point Green's functions. The renormalized Green's function is

$$G_{\mathrm{R}}^{(n)}(p_1, \dots, p_n) = Z_\varphi^{-\frac{n}{2}} G^{(n)}(p_1, \dots, p_n)$$

The amputated Green's function is

$$\begin{aligned}\Gamma_{\mathrm{R}}^{(n)}(p_1,\dots,p_n) &= G_{\mathrm{R}}^{(n)}(p_1,\dots,p_n)\left[\prod_{j=1}^{n} G_{\mathrm{R}}^{(2)}(p_j)\right]^{-1} \\ &= Z_\varphi^{-\frac{n}{2}} G^{(n)}(p_1,\dots,p_n)\left[\prod_{j=1}^{n} Z_\varphi^{-1} G^{(2)}(p_j)\right]^{-1} \\ &= Z_\varphi^{\frac{n}{2}} G^{(n)}(p_1,\dots,p_n)\left[\prod_{j=1}^{n} G^{(2)}(p_j)\right]^{-1} = Z_\varphi^{\frac{n}{2}}\Gamma^{(n)}(p_1,\dots,p_n)\end{aligned}$$

Renormalization — four-point Green's function. The following discussion deals mainly with one-loop contributions. The four-point Green's function up to one-loop contributions is

$G^{(4)}(p_1,\dots,p_n)\Big|_{\text{1-loop}} =$ 

The diagrams in the first bracket are one-particle irreducible, while the diagrams in the second bracket are one-particle reducible. The amputated Green's function $\Gamma^{(4)}$ is obtained by removing the external lines — including their corrections (by multiplying by $[G^{(2)}]^{-1}$) — of the one-particle irreducible diagrams.

In Minkowski, one has

$$\Gamma^{(4)}(s,t,u)\Big|_{\text{1-loop}} = -\mathrm{i}\lambda_0 + \Gamma(s) + \Gamma(t) + \Gamma(u)$$

where the first addendum corresponds to the four-point vertex and s, t and u are the three Mandelstam variables. The corresponding diagrams diverge logarithmically and their derivatives are convergent. One has to choose a subtraction point for each diagram: s_0, t_0, u_0. To make it easier, on-shell one knows that

$$s + t + u = 4\mu^2$$

where μ is a mass; so one may choose

$$s_0 = t_0 = u_0 = \frac{4}{3}\mu^2$$

Therefore, the Taylor expansion of each term gives

$$\begin{aligned}\Gamma^{(4)}(s,t,u)\Big|_{\text{1L}} &= -\mathrm{i}\lambda_0 + [\Gamma(s_0) + \widetilde{\Gamma}(s)] + [\Gamma(t_0) + \widetilde{\Gamma}(t)] + [\Gamma(u_0) + \widetilde{\Gamma}(u)] \\ &= -\mathrm{i}\lambda_0 + 3\Gamma(s_0) + \widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u)\end{aligned}$$

The first addendum in each bracket is divergent since the four-point function is logarithmically divergent $D = 0$. Also it holds

$$\widetilde{\Gamma}(s = s_0) = 0\,, \quad \widetilde{\Gamma}(t = t_0) = 0\,, \quad \widetilde{\Gamma}(u = u_0) = 0$$

The divergent terms can be absorbed into a redefinition of the coupling constant since their mass dimensions coincide.

Lecture 11

mer 20 mar 2024 10:30

Since the amputated Green's function is dimensionless, its divergent terms may be reabsorbed inside the coupling constant

$$-\mathrm{i}\lambda_0 + 3\Gamma(s_0) = -\mathrm{i}\lambda_0 Z_\lambda^{-1}$$

from which one has

$$-\mathrm{i}\lambda_0\left[1 + \frac{3\mathrm{i}}{\lambda_0}\Gamma(s_0)\right] = -\mathrm{i}\lambda_0 Z_\lambda^{-1} \implies Z_\lambda = \left[1 + \frac{3\mathrm{i}}{\lambda_0}\Gamma(s_0)\right]^{-1} \sim 1 - \frac{3\mathrm{i}}{\lambda_0}\Gamma(s_0)$$

The renormalized amputated Green's function is

$$\begin{aligned}\Gamma^{(4)}_{\mathrm{R}}(s,t,u) &= Z_\varphi^2\Gamma^{(4)}(s,t,u) = Z_\varphi^2[-\mathrm{i}\lambda_0 Z_\lambda^{-1} + \widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u)] \\ &= -\mathrm{i}\lambda_0 Z_\lambda^{-1} Z_\varphi^2 + Z_\varphi^2[\widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u)] \\ &= -\mathrm{i}\lambda_0 Z_\lambda^{-1} Z_\varphi^2 + [\widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u)] + o(\lambda_0^2)\end{aligned}$$

At the second line, the bracket is of order λ_0^2 but the field renormalization function is $Z_\varphi = 1 + O(\lambda_0)$, so, at one loop, one may consider only the unity.

The first addendum contains two divergent terms (the renormalization functions Zs). The renormalized coupling constant is defined as

$$\boxed{\lambda \equiv \lambda_0 Z_\lambda^{-1} Z_\varphi^2}$$

Like for the mass, the bare coupling constant is ultraviolet divergent in a way that exactly cancels the other terms' divergence giving a finite renormalized coupling. Therefore, the renormalized amputated Green's function up to one-loop contributions is

$$\Gamma^{(4)}_{\mathrm{R}}(s,t,u) = -\mathrm{i}\lambda + \widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u) + o(\lambda^2)$$

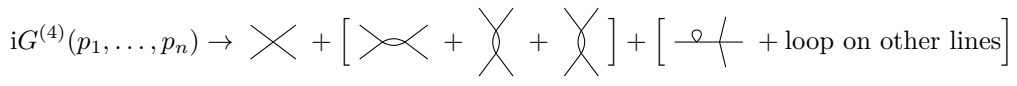
which is finite.

[r] At this point one has a list of renormalization prescriptions

$$\varphi_0 = Z_\varphi^{\frac{1}{2}}\varphi\,, \quad m_0^2 = \mu^2 - \Sigma(\mu^2)\,, \quad \lambda_0 = Z_\lambda Z_\varphi^{-2}\lambda$$

which make the one-particle irreducible contributions to the Green's function $G^{(2)}$ and $G^{(4)}$ finite. The Z and Σ terms are called renormalization functions [r].

Correctness check. One may check that the above prescription is sufficient to make one-particle reducible diagrams finite. The discussion is done for one-loop. The four-point Green's function contains diagrams of the type

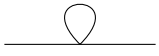
$\mathrm{i}G^{(4)}(p_1,\dots,p_n) \to$  + loop on other lines

$$\begin{aligned}&= \left[\prod_{j=1}^{4}\frac{\mathrm{i}}{p_j^2 - m_0^2 + \mathrm{i}\varepsilon}\right]\Big[-\mathrm{i}\lambda_0 + [3\Gamma(s_0) + \widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u)] \\ &\quad + (-\mathrm{i}\lambda_0)\sum_{k=1}^{4}(-\mathrm{i}\Sigma(p_k^2))\frac{\mathrm{i}}{p_k^2 - m_0^2 + \mathrm{i}\varepsilon}\Big]\end{aligned}$$

Performing a partial resummation of the tree-level diagram and the one-particle reducible diagrams, one obtains

$$\begin{aligned}\mathrm{T} + 1\mathrm{PR} &= -\mathrm{i}\lambda_0\left[\prod_{j=1}^{4}\frac{\mathrm{i}}{p_j^2 - m_0^2 + \mathrm{i}\varepsilon}\right]\left[1 + \sum_{k=1}^{4}(-\mathrm{i}\Sigma(p_k^2))\frac{\mathrm{i}}{p_k^2 - m_0^2 + \mathrm{i}\varepsilon}\right] \\ &= -\mathrm{i}\lambda\prod_{j=1}^{4}\frac{\mathrm{i}}{p_j^2 - m_0^2 - \Sigma(p_j^2) + \mathrm{i}\varepsilon} + o(\lambda_0^2)\end{aligned}$$

The second line is obtained by noting that the renormalization function Σ at one loop is given by the diagram



and then applying the geometric series.

The contribution from the one-particle irreducible diagrams is

$$1\mathrm{PI} = \left[\prod_{j=1}^{4} \frac{\mathrm{i}}{p_j^2 - m_0^2 - \Sigma(p_j^2) + \mathrm{i}\varepsilon}\right][3\Gamma(s_0) + \widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u)] + o(\lambda^2)$$

since $\Gamma \sim \lambda_0^2$ and $\widetilde{\Gamma} \sim \lambda_0^2$: one adds Σ and applies the geometric series.

Therefore, the four-point Green's function is

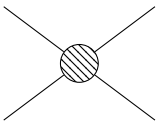
$$\begin{aligned}\mathrm{i}G^{(4)} &= \left[\prod_{j=1}^{4} \frac{\mathrm{i}}{p_j^2 - m_0^2 - \Sigma(p_j^2) + \mathrm{i}\varepsilon}\right]\Big[-\mathrm{i}\lambda_0 + [3\Gamma(s_0) + \widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u)]\Big] \\ &= \left[\prod_{j=1}^{4} \mathrm{i}G^{(2)}(p_j)\right]\Gamma^{(4)}(s,t,u)\end{aligned}$$

The one-particle reducible diagrams contain a (quantum) two-point propagator at one-loop order λ_0, while one-particle irreducible diagrams do not, since otherwise their order would be higher.

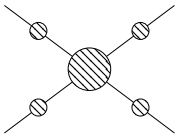
Now one may renormalize the above quantities

$$\begin{aligned}G_{\mathrm{R}}^{(4)} &= Z_\varphi^{-2} G^{(4)} = Z_\varphi^{-2}\left[\prod_{j=1}^{4} G^{(2)}(p_j)\right]\Gamma^{(4)}(s,t,u) = Z_\varphi^{-2}\left[\prod_{j=1}^{4} Z_\varphi G_{\mathrm{R}}^{(2)}(p_j)\right] Z_\varphi^{-2}\Gamma_{\mathrm{R}}^{(4)}(s,t,u) \\ &= Z_\varphi^{-4} Z_\varphi^4\left[\prod_{j=1}^{4} G_{\mathrm{R}}^{(2)}(p_j)\right]\Gamma_{\mathrm{R}}^{(4)}(s,t,u) = \left[\prod_{j=1}^{4} G_{\mathrm{R}}^{(2)}(p_j)\right]\Gamma_{\mathrm{R}}^{(4)}(s,t,u)\end{aligned}$$

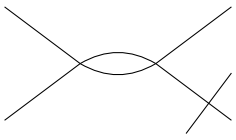
The diagram corresponding to the above expression is equivalent to computing the renormalized Green's function using renormalized Feynman rules: the propagator $G_{\mathrm{R}}^{(2)}$ and the vertex $\Gamma_{\mathrm{R}}^{(4)}$ are quantum because they contain all the corrections

$G_{\mathrm{R}}^{(4)}(p) =$ $\qquad\qquad \Gamma_{\mathrm{R}}^{(4)} =$ 

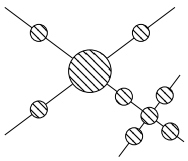
The four-point Green's function is then

$G_{\mathrm{R}}^{(4)} =$ 

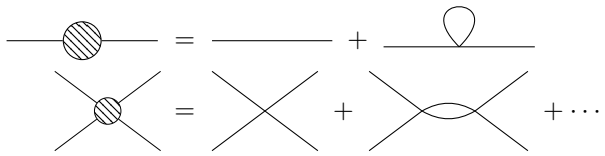
Higher-point Green's functions. The six-point Green's function at one loop gives only one-particle reducible diagrams, for example



The total degree of divergence is negative $D < 0$, but the loop sub-diagram is divergent. The Green's function has to be renormalized by adding the loop corrections (the blobs) on every line and vertex.

$G_{\mathrm{R}}^{(6)} =$ 

The renormalized vertex contains the loop. This Green's function is finite. This diagram is the sum of many diagrams each with one loop correction on one line (or vertex) only (otherwise the order increases). At one-loop, the corrected propagator and vertex are



The same is true for higher-point Green's functions. The procedure involves starting from the tree-level diagram and correct every line and vertex.

Exercise. Expand the above corrected $G_{\mathrm{R}}^{(6)}$ diagram in terms of all possible one-loop contributions. Do the same for $G^{(8)}$.

Higher loops. At one loop, for the $\lambda\varphi^4$ theory there is no Σ' term in the renormalization function Σ and as such no field renormalization. At one loop, the first term $\Sigma(\mu^2)$ of the renormalization function is proportional to the coupling constant λ_0. The renormalization of the mass $\delta\mu^2$ contains both λ_0 and m_0. One has also found that the amputated four-point Green's function is $\Gamma(s) \sim \lambda_0^2$ from which $Z_\lambda \sim \lambda_0, m_0$.

At one loop, the renormalization functions depend on the bare parameters. However, at this order, one can replace the bare parameters with the physical ones and ignore the higher orders.

At two loops one has to consider

- genuine two-loop diagrams contributing to the Green's functions $G^{(2)}$ and $G^{(4)}$;
- revisit the lower-loop calculations and replace the bare parameters up to the desired loop order.

An L-loop calculation requires revisiting all lower-loop results.

3.10 Bogoliubov–Parasiuk–Hepp–Zimmermann renormalization

See Cheng–Li, Ramond, Anselmi. The Bogoliubov–Parasiuk–Hepp–Zimmermann (BPHZ) renormalization is a more efficient procedure from a more technical point of view, but it is equivalent to the conventional renormalization. It is also called counter terms renormalization.

The procedure is the same as before: regularize the ultraviolet divergences, choose a subtraction point, perform the Taylor expansion and remove the divergences by renormalizing the parameters

$$\varphi_0 = Z_\varphi^{\frac{1}{2}}\varphi\,, \quad m_0^2 = m^2 - \delta m^2\,, \quad \lambda_0 = Z_\lambda Z_\varphi^{-2}\lambda$$

The counter terms renormalization is based on a simple manipulation of the original Lagrangian. The bare Lagrangian is

$$\mathcal{L}_0 = \frac{1}{2}\,\partial_\mu\varphi_0\,\partial^\mu\varphi_0 - \frac{m_0^2}{2}\varphi_0^2 - \frac{\lambda_0}{4!}\varphi_0^4$$

One expresses the bare parameters in terms of the physical ones

$$\begin{aligned}
\mathcal{L}_0 &= \frac{1}{2}Z_\varphi\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}(m^2-\delta m^2)Z_\varphi\varphi^2 - \frac{\lambda}{4!}Z_\lambda\varphi^4 \\
&= \frac{1}{2}Z_\varphi\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}(m^2-\delta m^2)Z_\varphi\varphi^2 - \frac{\lambda}{4!}Z_\lambda\varphi^4 + \mathcal{L} - \mathcal{L} \\
&= \mathcal{L} + \frac{1}{2}(Z_\varphi - 1)[\,\partial_\mu\varphi\,\partial^\mu\varphi - m^2\varphi^2] + \frac{1}{2}\,\delta m^2\,Z_\varphi\varphi^2 - \frac{\lambda}{4!}(Z_\lambda - 1)\varphi^4 \\
&= \mathcal{L} + \frac{1}{2}A\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}Bm^2\varphi^2 - \frac{\lambda}{4!}C\varphi^4
\end{aligned}$$

where

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4\,, \quad A = Z_\varphi - 1\,, \quad B = A - \frac{\delta m^2}{m^2}Z_\varphi\,, \quad C = Z_\lambda - 1$$

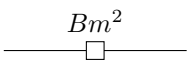
[r] check

The renormalized parameters are the finite parameters in terms of which one can write the physical parameters (since they do not necessarily coincide). The Lagrangian $\mathcal{L}$ is welcome, but not the remaining terms. These are still local terms called counter terms

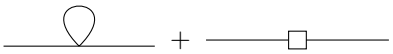
$$\mathcal{L}_0 = \mathcal{L} + \mathcal{L}_{\text{ct}}$$

The Feynman rules are read from the renormalized Lagrangian $\mathcal{L}$ and the integrals of the loops are still (ultraviolet) divergent but written in terms of renormalized parameters. One adds new Feynman rules for the counter terms so that they cancel the previous divergences. Therefore one may

- write a set of Feynman rules from the renormalized Lagrangian; the propagator and the vertex are
$$\frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon}, \quad -\mathrm{i}\lambda$$
- perform the one-loop calculations of the Green's functions $\Gamma^{(2)}$ and $\Gamma^{(4)}$; for the former one has
$$\Gamma^{(2)} \to \lambda I_{\text{div}} = \lambda \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathrm{i}}{q^2 - m^2 + \mathrm{i}\varepsilon}$$
[r] The divergent terms are $-\mathrm{i}\Sigma(\mu^2)$ with dimension 2. One may add a counter term to the renormalized Lagrangian $\mathcal{L}$ of the form
$$\mathcal{L}^{(1)} = \frac{1}{2} B m^2 \varphi^2$$
This modifies the Feynman rules by adding a two-point vertex indicated as follows

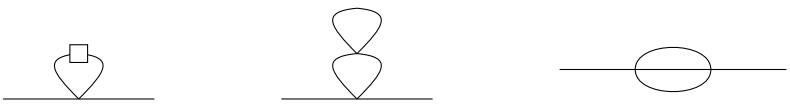


and one fixes B at one-loop such that the following sum is finite



This new (box) vertex is of order one-loop.
- At two loops one must construct all two-point diagrams also using the new vertex

For example



Lecture 12

gio 21 mar 2024 10:30

One may notice that the two-point amputated Green's function is the complete quantum kinetic term

$$\Gamma^{(2)} = G^{(2)}[G^{(2)}G^{(2)}]^{-1} = [G^{(2)}]^{-1}$$

While the four-point amputated Green's function is the complete quantum vertex. [r] Therefore, to balance the divergences, one has to add terms that counter the two previous Green's function. In fact, the counter term Lagrangian contains a classical kinetic part (the first two terms) and a classical vertex.

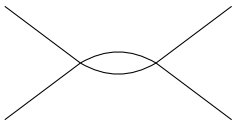
Perturbative procedure. The perturbative procedure is the following.

- Starting from the renormalized Lagrangian $\mathcal{L}$, one computes the one-loop corrections. The two-point Green's function $\Gamma^{(2)}$ has contributions from the diagram of order λ

because it has one vertex, and is therefore balanced by the kinetic counter term

$$(\mathcal{L}^{(1)}_{\text{ct}})_{\text{I}} = \frac{1}{2}A(\partial_\mu\varphi)^2 - \frac{1}{2}m^2B\varphi^2$$

where the index (1) indicates the order of the coupling constant λ. The terms A and B are both order λ. The Green's function $\Gamma^{(4)}$ is of order λ^2



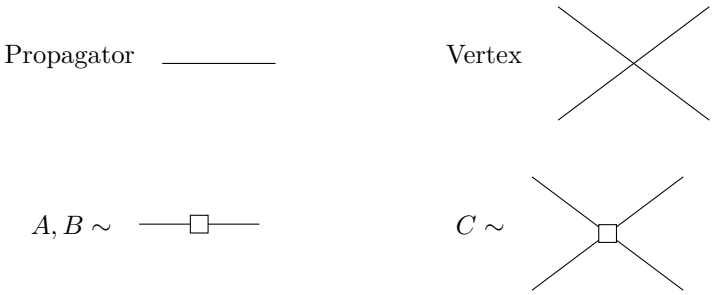
and one adds a vertex counter term

$$(\mathcal{L}^{(2)}_{\text{ct}})_{\text{II}} = -\frac{\lambda}{4!}C\varphi^4$$

where the term C is order λ. The appropriate Lagrangian for one-loop calculation is then

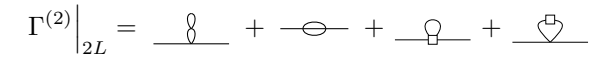
$$\mathcal{L} + \mathcal{L}^{(1)}_{\text{ct}}$$

- At two loops, using the above as the new Lagrangian, the new Feynman rules are



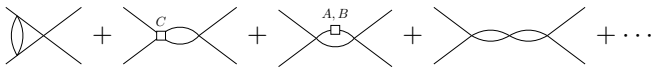
There are two new interaction vertices proportional to the counter terms. The two-point Green's function is

$$\Gamma^{(2)}\Big|_{2L} =$$



In the third diagram contributes the C term, while in the last contribute the A and B terms. Similarly, the four-point Green's function is

$$\Gamma^{(4)}\Big|_{2L} =$$



To draw diagrams with counter terms is sufficient to use lower-loop diagrams and add a counter term (denoted by the box).

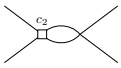
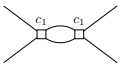
One may already see that the first diagram of the Green's function $\Gamma^{(2)}$ contains a one-loop divergence related to the upper loop. The last diagram exactly cancels this sub-divergence, but the first diagram is still divergent in its whole. Similarly for the others. When one sums genuine diagrams and counter term diagrams, one cancels some divergences, but the sum may still be divergent. If the total sum is still divergent, one may cure the remaining divergences by adding new counter terms at order λ^2

$$\mathcal{L}^{(2)}_{\text{ct}} = \frac{1}{2}A(\partial_\mu\varphi)^2 - \frac{1}{2}m^2B\varphi^2 - \frac{\lambda}{4!}C\varphi^4\,, \quad A, B, C = O(\lambda^2)$$

- At three loops one uses $\mathcal{L} + \mathcal{L}_{\rm ct}^{(1)} + \mathcal{L}_{\rm ct}^{(2)}$. The Feynman rules have the same structure but with A, B and C given by expressions of the form

$$A = a_1\lambda + a_2\lambda^2\,, \quad B = b_1\lambda + b_2\lambda^2\,, \quad C = c_1\lambda + c_2\lambda^2$$

  The four-point Green's function at three loops gets contributions from diagrams of the type

$$\Gamma^{(4)}\Big|_{\rm 3L} =$$  $+$  $+\cdots$

- Repeating the steps, the total Lagrangian is the bare Lagrangian

$$\mathcal{L} + \sum_{n=1}^{\infty} \mathcal{L}_{\rm ct}^{(n)} = \mathcal{L}_0$$

  The summation is not well-defined since the bare Lagrangian itself is not. For the theory $\lambda\varphi^4$, the divergences appear only for the Green's functions $\Gamma^{(2)}$ and $\Gamma^{(4)}$: one may cancel the divergences using counter terms that have the same form as the Lagrangian. The resummation above (called multiplicative renormalization) is equal to the bare Lagrangian because one has defined the field, mass and coupling renormalizations.

In a theory where the situation is more complicated, like $\lambda\varphi^5$ in $d = 4$ dimensions, one may draw a three-point diagram [r] diagr

$$\sim \int \frac{\mathrm{d}^4 q}{q^2 + m^2} \sim q^2$$

Therefore, cancelling this $\Gamma^{(3)}$ divergent term requires adding to the Lagrangian $\mathcal{L}$ a counter term of the type

$$\mathcal{L}_{\rm ct} = \frac{\lambda}{3!} C\varphi^3$$

This term is not contained in the Lagrangian

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{5!}\varphi^5$$

Therefore, the one-loop Lagrangian contains a new kind of vertex

$$\mathcal{L} + \mathcal{L}_{\rm ct}^{(1)} = \mathcal{L} + \frac{\lambda}{3!} C\varphi^3$$

The multiplicative renormalization is not true for this theory. The theory can be either thrown away or modified by including in the original Lagrangian the cubed vertex

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{5!}\varphi^5 - \frac{\widetilde{\lambda}}{3!}\varphi^3$$

At one-loop calculation, the counter term is related to the above cubed term and they combine to give a renormalization for $\widetilde{\lambda}$. However, for the φ^5 vertex, there are also six-point Green's function diagrams which are divergent. To cancel their divergence, one has to add a counter term at one-loop of the form

$$\frac{g}{6!}\varphi^6(\cdots)$$

If one wants to keep the theory, one has to add a six-point vertex to the original Lagrangian. By adding this vertex, one can build a divergent diagram for $\Gamma^{(8)}$, to cancel this term one has to add an eight-point vertex to the original Lagrangian.

One may see that this process does not stop and one keeps adding vertices. In order to make renormalization consistent, one has to have an infinite number of vertices and as such an infinite number of coupling constants. A theory with a theory with an infinite number of coupling constants is not predictive at any energy scale (but it may work at some by cutting short the sum).

In $d = 4$ dimensions, one concludes that $\lambda\varphi^4$ is a renormalizable theory, while $\lambda\varphi^5$ is not renormalizable.

3.11 Computation of Feynman integrals

See Cheng–Li, Ramond, Peskin. The Euclidean formalism is used. The prototype theory is still $\lambda\varphi^4$. For an ultraviolet divergent integral, one needs to regularize it. The regularization schemes used are the covariant regularization and the dimensional regularization.

3.11.1 Covariant regularization

This regularization scheme was introduce by Pauli and Villars. The simplest way to make a divergent integral finite is to integrate up to a cutoff Λ. Inserting a cutoff breaks Lorentz invariance. The covariant regularization is a cutoff regularization but keeping Lorentz invariance.

One modifies the propagators inside an integral by adding extra terms

$$\frac{1}{q^2+m^2} \to \frac{1}{q^2+m^2} + \sum_i \frac{a_i}{q^2+\Lambda_i^2}$$

where $\Lambda_i \gg m$ is the cutoff. The regularized result is obtained in the limit $\Lambda \to \infty$.

The idea is to modify the propagator in such a way to end up with an integrand with more powers of the momentum than the initial integrand so that the integral converges.

One would like to compute one-loop contributions to the Green's functions $\Gamma^{(4)}$ and $\Gamma^{(4)}$.

Four-point Green's function. [r] The one-loop contribution to the Green's function is the following [r] diagr. Let $p = p_1 + p_2$ and $p^2 = s$. The explicit algebraic expression of the Green's function is

$$\Gamma(p) = \frac{\lambda^2}{2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(q^2+m^2)[(p-q)^2+m^2]}$$

This Green's function is logarithmically divergent $D = 0$. For this Green's function, one cutoff is sufficient $\Lambda \gg m$ for the covariant regularization

$$\frac{1}{q^2+m^2} \to \frac{1}{q^2+m^2} - \frac{1}{q^2-\Lambda^2} = \frac{\Lambda^2-m^2}{(q^2+m^2)(q^2+\Lambda^2)} \approx \frac{\Lambda^2}{(q^2+m^2)(q^2+\Lambda^2)}$$

The Green's function is then

$$\Gamma(p) = \frac{\lambda^2}{2}\Lambda^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(q^2+m^2)(q^2+\Lambda^2)[(p-q)^2+m^2]}$$

Since the Green's function is logarithmically divergent, one may choose the subtraction point $p^2 = 0$ and Taylor expand around it

$$\Gamma(p) = \Gamma(0) + \widetilde{\Gamma}(p^2)\,, \quad \widetilde{\Gamma}(0) = 0$$

where the first addendum is logarithmically divergent and the second is finite.

One may split the computation. The divergent part is

$$\Gamma(0) = \frac{\lambda^2}{2}\Lambda^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(q^2+m^2)^2(q^2+\Lambda^2)}$$

The finite part is

$$\begin{aligned}
\widetilde{\Gamma}(p) = \Gamma(p) - \Gamma(0) &= \frac{\lambda^2}{2}\Lambda^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \left[\frac{1}{(q^2+m^2)(q^2+\Lambda^2)[(p-q)^2+m^2]} \right. \\
&\qquad \left. - \frac{1}{(q^2+m^2)^2(q^2+\Lambda^2)} \right] \\
&= \frac{\lambda^2}{2}\Lambda^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{2pq-p^2}{(q^2+m^2)^2[(p-q)^2+m^2)(q^2+\Lambda^2)} \\
&\to \frac{\lambda^2}{2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{2pq-p^2}{(q^2+m^2)^2[(p-q)^2+m^2]}\,, \quad \Lambda \to \infty
\end{aligned}$$

One needs to compute the integrals in the two parts.

Convenient tools. A first tool is Feynman combining

$$\prod_{j=1}^{n}\frac{1}{a_j}=(n-1)!\int_0^1\prod_{j=1}^{n}\mathrm{d}z_j\,\frac{\delta(\sum_i z_i-1)}{(a_1z_1+\cdots+a_nz_n)^n}$$

Applying the derivative with respect to one parameter, one obtains

$$\frac{1}{a_1}\prod_{j=1}^{n}\frac{1}{a_j}=n!\int_0^1\prod_{j=1}^{n}\mathrm{d}z_j\,\frac{\delta(\sum_i z_i-1)z_1}{(a_1z_1+\cdots+a_nz_n)^{n+1}}$$

One may prove the identity above by using Schwinger representation of the propagator

$$\frac{1}{a_j}=\int_0^\infty\mathrm{d}\alpha_j\,\mathrm{e}^{-\alpha_j a_j}$$

One would obtain

$$\prod_{j=1}^{n}\frac{1}{a_j}=\int_0^\infty\prod_{j=1}^{n}\mathrm{d}\alpha_j\,\mathrm{e}^{-\sum_{j=1}^{n}\alpha_j a_j}$$

Applying the following change of variables

$$z=\sum_{j=1}^{n}\alpha_j\,,\quad z_j=\frac{\alpha_j}{z}\,,\quad j=1,\ldots,n-1$$

one gets

$$\sum_{j=1}^{n-1}z_j+\frac{\alpha_n}{z}=1\implies\alpha_n=z\Big[1-\sum_{j=1}^{n-1}z_j\Big]$$

From this, one has

$$\prod_{j=1}^{n}\frac{1}{a_j}=\int_0^1\prod_{j=1}^{n-1}\mathrm{d}z_j\int_0^\infty\mathrm{d}z\,z^{n-1}\exp\Big[-z\sum_{j=1}^{n-1}a_jz_j-z(1-\sum_k z_k)a_n\Big]$$

A second tool is

$$\int_0^\infty\mathrm{d}t\,\frac{t^{n-1}}{(t+A^2)^n}=\frac{1}{(A^2)^{n-m}}\frac{\Gamma(m)\Gamma(n-m)}{\Gamma(n)}$$

A third tool is integration in spherical coordinates

$$\int\mathrm{d}^4q=\int_0^\infty\mathrm{d}q\,q^3\int_0^{2\pi}\mathrm{d}\theta_1\int_0^\pi\mathrm{d}\theta_2\,\sin\theta_2\int_0^\pi\mathrm{d}\theta_3\,\sin^2\theta_3$$

For a quantity independent of the angles, one obtains

$$\int\mathrm{d}^4q\,f(q^2)=2\pi^2\int_0^\infty\mathrm{d}q\,q^3f(q^2)=\pi^2\int_0^\infty2q\,\mathrm{d}q\,q^2f(q^2)=\pi^2\int_0^\infty\mathrm{d}q^2\,q^2f(q^2)$$

This identity can be generalized to n dimensions

$$\int\mathrm{d}^nq\,f(q^2)=\int_0^\infty\mathrm{d}q\,q^{n-1}\int_0^{2\pi}\mathrm{d}\theta_1\Big[\prod_{j=2}^{n-1}\int_0^\pi\mathrm{d}\theta_j\,(\sin\theta_j)^{j-1}\Big]f(q^2)=\frac{2\pi^{\frac{n}{2}}}{\Gamma(n/2)}\int_0^\infty\mathrm{d}q\,q^{n-1}f(q^2)$$

For dimensional regularization, one needs to analytically continue this expression to non-integer number of dimensions.