Mathematical Methods for Physics

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*`https://github.com/M-a-s-o/notes`

Lecture 1

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1 Introduction

The course is about group theory and explaining where it appears in theoretical physics. Group theory is motivated by the fact that the current understanding of Nature is based on symmetries. Symmetries are properties that are invariant under some transformations. Utilizing Noether's theorem, one can link the conservation of charges to transformations and invariants. Symmetries play an important role in physics and are described by groups. There are several classes of symmetries.

Continuous symmetries. Symmetries can be parameterized by continuous variables. Some examples of continuous symmetries are:

- (space) translation, equivalent to the assumption that space is homogeneous: one can obtain the conservation of momentum;
- time translation, equivalent to the assumption that time is homogeneous: conservation of energy;
- space rotation, equivalent to the assumption of the isotropy of space: conservation of angular momentum;
- Lorentz and Poincaré transformations: conservation of the energy–momentum tensor.
- Local gauge symmetry: it is a symmetry of an internal degree of freedom (i.e. not related to temporal nor spacial degrees of freedom) and the transformation depends on the space-time coordinates
$$V_{ab}(\mathbf{x}, t)\phi(\mathbf{x}, t)$$
These kinds of symmetries describe the dynamics of quantum field theory and the properties of fundamental interactions.
- Internal, global symmetry, for example isospin: swapping the up and down quarks does not change the physics.

Discrete symmetries. A few examples of discrete symmetries are:

- parity $\mathbf{x} \to -\mathbf{x}$: parity is conserved in quantum electrodynamics and in quantum chromodynamics, but not in electroweak theory.
- Time reversal $t \to -t$;
- charge conjugation;
- discrete lattice: no more continuous translation symmetries, because there is only a subset of them. Lattices are used for the study of QCD in the non-perturbative regime.

This course will be dedicated to the classification and study of groups and their representations.

Definition 1.1. A group G is a set endowed with a binary operation $\circ$ (called group multiplication) that has the following properties:

- closure: $\forall g_1, g_2 \in G$, $\exists! g_3 \in G$ such that $g_3 = g_1 \circ g_2$.
- associativity: $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$, $\forall g_1, g_2, g_3 \in G$.
- identity element: $\exists e \in G$ such that $g \circ e = e \circ g = g$, $\forall g \in G$.
- inverse element: $\forall g \in G$, $\exists g^{-1} \in G$ such that $g \circ g^{-1} = g^{-1} \circ g = e$.

Example 1.2. A few examples:

- The set $\mathbb{R}$ of the real numbers endowed with summation is a group $(\mathbb{R}, +)$ with identity element 0 and inverse $-g$.
- The whole numbers with summation $(\mathbb{Z}, +)$ is a group.
- The positive integers with summation $(\mathbb{N}, +)$ are not a group: there does not exist the inverse.
- The real numbers with multiplication $(\mathbb{R}, \cdot)$ has identity element 1, but is not a group because 0 does not have an inverse. To obtain a group, one can consider $\mathbb{R} \setminus \{0\}$.
- Consider the set of square matrices of size n with either real entries $\mathrm{Mat}(n, \mathbb{R})$ or complex entries $\mathrm{Mat}(n, \mathbb{C})$. Taking the operation to be matrix multiplication, the identity to be $I = \mathrm{diag}(1, 1, 1, 1, \ldots)$, the set is not a group because the inverse element, inverse matrices, does not exist for matrices with null determinant.

Definition 1.3. The general linear group is defined as $\mathrm{GL}(n, \mathbb{R}) \equiv \{M \in \mathrm{Mat}(n, \mathbb{R}) \mid \det M \neq 0\}$ and is the largest and most general group of matrices. The definition is equivalent for matrices with complex entries.

Definition 1.4. The cyclic group $\mathbb{Z}_n$ is discrete and defined by

$$\mathbb{Z}_n \equiv \{e, a, a^2, a^3, \ldots, a^{n-1}\}\,, \quad a^n \equiv e$$

with generic operation $\circ$ where the notation implies $a^2 \equiv a \circ a$.

Example 1.5. Some examples are:

- $\mathbb{Z}_2 = \{0, 1\}$ with summation mod 2,
- the group $G = \{1, -1\}$ with multiplication,
- endowing the integers $\mathbb{Z}$ with summation mod n, one can split the set into equivalence classes that are the elements of the cyclic group. For example, if $n = 2$ then the sets

  $$\{0\} = \{0, 2, 4, \ldots\}\,, \quad \{1\} = \{1, 3, 5, \ldots\}$$

  are the elements of the cyclic group $\mathbb{Z}_2$. This group is important in particle physics.

Definition 1.6. The order of a group G, denoted by $|G|$, is given by the number of elements of the group.

Definition 1.7. A group G is finite if its order is finite. Otherwise it is infinite.

Definition 1.8. A group G is discrete if it is a countable set.

Definition 1.9. A group G is abelian if its operation is commutative $g_1 \circ g_2 = g_2 \circ g_1$, $\forall g_1, g_2$.

Example 1.10. A few examples of abelian and non-abelian groups:

- Some abelian groups are $(\mathbb{R}, +)$, $(\mathbb{Z}, +)$ which are also infinite. Though, the first is not countable, while the second is and as such it is a discrete group.
- All cyclic groups $\mathbb{Z}_n$ are abelian: $a^2 \circ a^3 = a^3 \circ a^2$.
- The general linear group $\mathrm{GL}(n, \mathbb{R})$ is not abelian: the order of matrix multiplication matters.

Definition 1.11. A ring R is an abelian group with composition $+$, endowed with a second operation $\cdot : R \times R \to R$ with the properties:

- associativity $(r_1 \cdot r_2) \cdot r_3 = r_1 \cdot (r_2 \cdot r_3)$, $\forall r_1, r_2, r_3 \in R$;
- distributivity with respect to the first composition: $r_1 \cdot (r_2 + r_3) = r_1 \cdot r_2 + r_1 \cdot r_3$ and equivalently for right multiplication.

Remark 1.12. The identity element of the relation $+$ is 0, while the inverse is $-r$. The second relation need not an identity nor an inverse.

Definition 1.13. A field $\mathbb{F}$ is a ring such that $\mathbb{F} \setminus \{0\}$ is an abelian group with respect to the ring's second operation.

Example 1.14. Some examples and counterexamples of fields are the following.

- Consider the integers with summation $(\mathbb{Z}, +)$: it is an abelian group. It is also a ring with the multiplication $n_1 \cdot n_2 \in \mathbb{Z}$. However, $(\mathbb{Z}, \cdot)$ is not a group because $n^{-1} \notin \mathbb{Z}$, therefore the set of integers is not a field.
- The real numbers with summation and multiplication is a field.
- The complex numbers are also a field.
- The cyclic group $\mathbb{Z}_n$ with summation is an abelian group. Adding the multiplication $\{n_1\} \cdot \{n_2\} \equiv \{n_1 \cdot n_2\}$, the cyclic group is a field only for n prime. See the following theorem.

The fact that real numbers and complex numbers are fields implies one can define sets of matrices with entries from those fields.

Theorem 1.15. If n is prime, then the cyclic group $\mathbb{Z}_n$ is a field (and vice versa).

Example 1.16. Some more examples:

- Consider the cyclic group $\mathbb{Z}_5$. It is an abelian group with respect to summation. One can study if $(\mathbb{Z}_5 \setminus \{0\}, \cdot)$ is an abelian group:
$$\{2\} \cdot \{3\} = \{6\} = \{1\}$$
This means that $\{3\}$ is the inverse of $\{2\}$. Continuing, one gets
$$\{4\} \cdot \{4\} = \{1 + 5 \cdot 3\} = \{1\}$$
So $\{4\}$ is its own inverse. Since the remaining elements are the additive identity $\{0\}$ and multiplicative identity $\{1\}$, the set $\mathbb{Z}_5$ is a field.
- The group $(\mathbb{Z}_4 \setminus \{0\}, \cdot)$ is not abelian because
$$\{2\} \cdot \{2\} = \{4\} = \{0\}$$
the set is not closed under multiplication. Therefore $\mathbb{Z}_4$ is not a field.

Definition 1.17. A module M with respect to a ring R is an abelian group endowed with an action $\cdot : R \times M \to M$ such that the following properties hold for every $r \in R$ and $m \in M$:

- associativity $(r_1 \cdot r_2) \cdot m = r_1 \cdot (r_2 \cdot m)$
- distributivity $r_1 \cdot (m_1 + m_2) = r \cdot m_1 + r \cdot m_2$ and $(r_1 \cdot r_2) \cdot m = r_1 \cdot m + r_2 \cdot m$

Pay attention to whom the operations apply: operations in different sets have the same notation.

Definition 1.18. A vector space V with respect to a field $\mathbb{F}$ is a module over $\mathbb{F}$ such that the multiplicative identity of $\mathbb{F}$ acts as the identity on V: $1 \cdot v = v \cdot 1 = v$, $\forall v \in V$.

Definition 1.19. An associative algebra A over a ring R is a ring and also a module with respect to R such that the action of R on A is bilinear
$$r \cdot (a_1 \cdot a_2) = (r \cdot a_1) \cdot a_2 = a_1 \cdot (r \cdot a_2)$$
Again, pay attention to whom the operations apply.

[r] image of recap

Example 1.20. The set of square matrices of size n, $\mathrm{Mat}(n, \mathbb{F})$, with summation, matrix multiplication and scalar product forms an algebra over the field $\mathbb{F}$.

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Definition 1.21. The set of vectors $\{\,|e_i\rangle\}$ is a basis of a vector space V if and only if $|v\rangle = v^i\,|e_i\rangle$, $v^i \in \mathbb{F}$ for all $|v\rangle \in V$.

Definition 1.22. The action of an inner product on a basis defines a metric $g^i{}_j = \langle e^i|e_j\rangle$. The metric g is hermitian and if $g^i{}_j = \delta^i{}_j$ then the basis is orthonormal.

Lemma 1.23. Linear operators over V form an algebra called $\mathrm{End}(V)$:

$$A:\,|v\rangle \mapsto |Av\rangle \in V\,,\quad A:\,|w\rangle = \alpha\,|v_1\rangle + \beta\,|v_2\rangle \mapsto |Aw\rangle = \alpha\,|Av_1\rangle + \beta\,|Av_2\rangle$$

Proposition 1.24. Linear operators follow these properties:

- composition: $(A \circ B)\,|v\rangle = A\,|Bv\rangle = |ABv\rangle$;
- given a basis $\{\,|e_i\rangle\}$, the operator A is a matrix $n \times n$ over a field $\mathbb{F}$;
- a basis rotation S is a linear operator which is invertible ($\det S \neq 0$):

$$|e'_i\rangle = S^j{}_i\,|e_j\rangle \implies \langle e'^i|e'_j\rangle = (S^\dagger)^i{}_k\,\langle e^k|e_l\rangle\,S^l{}_j \implies g' = S^\dagger g S$$

Definition 1.25. A linear operator A is idempotent if $A^2 = A$ (and $A \neq I$). Given a basis $\{\,|e_i\rangle\}$, an example of idempotent linear operator the projector $A = |e_i\rangle\langle e^i|$:

$$A\,|v\rangle = |e_i\rangle\langle e^i|\,v^k\,|e_k\rangle = v^i\,|e_i\rangle$$

Definition 1.26. A linear operator A is nilpotent if $\exists n \in \mathbb{N}$ such that $A^n = 0$ where 0 is the null operator. For example

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

One wants to categorize type of transformations to be able to study symmetries.

1.1 Classical groups of matrices

Linear group. The general linear group $\mathrm{GL}(n,\mathbb{C})$ is the largest group in the endomorphism algebra $\mathrm{End}(V)$. The group is parametrized by $2n^2$ real elements. The dimension of the group is given by the number of free parameters.

The special linear group is defined by

$$\mathrm{SL}(n,\mathbb{C}) \equiv \{M \in \mathrm{GL}(n,\mathbb{C}) \mid \det M = 1\} \subset \mathrm{GL}(n,\mathbb{C})$$

This set is also a group, not just a subset of matrices. The number of free parameters is $2n^2 - 1$ because of the condition on the determinant. The definition is analogous for $\mathrm{SL}(n,\mathbb{R})$. The set $\mathrm{SL}(n,\mathbb{Z})$ is also a group.

Consider a change of coordinates from x^i to $\overline{x}^j = S^j{}_i x^i$ with the intention to perform an integral over space-time. The measures of the two coordinate systems are related by the Jacobian

$$\mathrm{d}^n\overline{x} = |\partial_x \overline{x}|\,\mathrm{d}^n x = \det S\,\mathrm{d}^n x$$

Matrices in the special linear group preserve the volume.

Unitary group. Consider the groups that preserve the metric of V over $\mathbb{C}$:

$$|\overline{e}_i\rangle = U^j{}_i\,|e_j\rangle \implies \delta^i{}_j = \langle \overline{e}^i|\overline{e}_j\rangle = \left\langle e^k\right|(U^\dagger)^i{}_k U^l{}_j\,|e_l\rangle = (U^\dagger)^i{}_k U^l{}_j\,\delta^k{}_l \implies U^\dagger U = I$$

Unitary matrices belong to the unitary group $\mathrm{U}(n,\mathbb{C})$ which is defined as

$$\mathrm{U}(n,\mathbb{C}) \equiv \{M \in \mathrm{GL}(n,\mathbb{C}) \mid M^\dagger M = MM^\dagger = I\}$$

The number of free parameters is $2n^2 - n - n(n-1) = n^2$:

- the unitary condition $U^\dagger U$ imposes n conditions corresponding to the elements on the diagonal, $[U^\dagger U]^i{}_i = 1$ (no sum);
- the upper and lower triangular parts of the matrix provide $\frac{n(n-1)}{2}$ conditions each.

The unitary group U(1) describes quantum electrodynamics. The determinant of a unitary matrix is
$$\det\left(U^\dagger U\right) = |\det U|^2 = 1 \implies \det U = e^{i\varphi}\,, \quad \varphi \in \mathbb{R}$$
One can define the special unitary group as
$$\mathrm{SU}(n, \mathbb{C}) \equiv \{M \in \mathrm{U}(n, \mathbb{C}) \mid \det M = 1\}$$
The number of free parameters is $n^2 - 1$. The strong and weak interactions are described by special unitary groups.

Orthogonal group. Consider the groups that preserve the metric of V over $\mathbb{R}$. The orthogonal group is
$$\mathrm{O}(n, \mathbb{R}) \equiv \{M \in \mathrm{GL}(n, \mathbb{R}) \mid M^\top M = M M^\top = I\}$$
The number of free parameters is $n^2 - n - \frac{n(n-1)}{2} = \frac{n(n-1)}{2}$. The determinant of the orthogonal matrices is
$$\det\left(O^\top O\right) = (\det O)^2 = 1 \implies \det O = \pm 1$$
The special orthogonal group is
$$\mathrm{SO}(n, \mathbb{R}) \equiv \{M \in \mathrm{O}(n, \mathbb{R}) \mid \det M = 1\}$$
The number of free parameters is the same as above. For the unitary group and orthogonal group, the field is not usually specified, but rather understood to be $\mathbb{C}$ and $\mathbb{R}$ respectively.

Indefinite orthogonal group. Consider the groups that preserve the metrics of V over $\mathbb{R}$ with signature (p, q). Consider the metric to be
$$g^i{}_j = \eta^i{}_j = \mathrm{diag}(\underbrace{1, \dots, 1}_{p}, \underbrace{-1, \dots, -1}_{q})$$
Then
$$|\overline{e}'_i\rangle = R^j{}_i\,|e_j\rangle \implies \overline{\eta} = R^\top \eta R$$
The indefinite orthogonal group is
$$\mathrm{O}(p, q, \mathbb{R}) \equiv \{R \in \mathrm{GL}(n, \mathbb{R}) \mid \eta = R^\top \eta R\}$$
The indefinite special orthogonal group is
$$\mathrm{SO}(p, q, \mathbb{R}) \equiv \{R \in \mathrm{O}(p, q, \mathbb{R}) \mid \det R = 1\}$$
The number of free parameters is the same as the orthogonal group. The Lorentz group is $\mathrm{SO}(1, 3)$. The group $\mathrm{SO}(p, q)$ contains both ordinary rotations $\mathrm{SO}(p)$ and $\mathrm{SO}(q)$ subgroups.

1.1.1 Symplectic forms

Definition 1.27. A symplectic vector space V is a vector space over a field $\mathbb{F}$ equipped with a bilinear form $\Omega : V \times V \to \mathbb{F}$. The bilinear form is:

- anti-symmetric: $\Omega(\mathbf{v}, \mathbf{w}) = -\Omega(\mathbf{w}, \mathbf{v})$;
- non-singular: $\Omega(\mathbf{v}, \mathbf{w}) = 0$, $\forall \mathbf{w} \implies \mathbf{v} = \mathbf{0}$,

Theorem 1.28. Given a square matrix A of size n and a constant c, then
$$\det(cA) = c^n \det A$$

Proof. From the hypotheses, it follows

$$\det(cA) = \det(cI)\,\det A = c^n \det A$$

$\square$

Given a vector space V over $\mathbb{R}$, one of its bases $\{|e_i\rangle\}$ and a bilinear form $\Omega_{ij} = \Omega(|e_i\rangle, |e_j\rangle)$, the aim is to represent the form Ω as a linear operator acting on the basis of V. Using anti-symmetry and the theorem above, it follows

$$\det\Omega = \det\left(-\Omega^\top\right) = (-1)^n \det\Omega$$

Because of non-singularity, the dimension n can only be even. Using anti-symmetry and setting $\mathbf{w} = \mathbf{v}$ then

$$\Omega(\mathbf{v},\mathbf{v}) = -\Omega(\mathbf{v},\mathbf{v}) \implies \Omega_{ii} = 0\,, \quad \Omega_{ij} = -\Omega_{ji}$$

Fixing i, if $\Omega_{ij} = 0$ for all j then $|e_i\rangle = 0$. Negating this statement means if all basis' vectors are non-zero (which must hold for a vector to be part of a basis) then, for at least one j, it must be true that $\Omega_{ij} \neq 0$. One can choose $\Omega_{ij} = 0$ for all $j \neq i+1$ and $\Omega_{i,i+1} = \lambda_i$. With this choice, one can obtain the canonical form

$$\Omega = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & -1 & 0 & 1 \end{pmatrix}$$

Another canonical form obtained by change of basis from the previous is

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

given by the conditions

$$\Omega_{i,\frac{n}{2}+i} = 1\,, \quad \Omega_{ij} = 0\,, \quad \forall j \neq \frac{n}{2} + i\,, \quad i < \frac{n}{2}$$

These two forms amount to a decomposition of the vector space $V = V_1 \oplus V_1^*$ — where V_1^* is the dual space of V_1 — such that a basis of V can be written as

$$\{|e_1\rangle, \dots, |e_m\rangle, |f_1\rangle, \dots, |f_m\rangle\}\,, \quad |e_i\rangle \leftrightarrow V_1\,, \quad |f_i\rangle \leftrightarrow V_1^*$$

As such the dimension is always even

$$n = \dim V = \dim V_1 + \dim V_1^* = 2\dim V_1 = 2m$$

A generic transformation of a symplectic form

$$\Omega' = A\Omega B$$

needs to preserve the form's properties:

$$\begin{aligned}(\Omega')^\top &= B^\top \Omega^\top A^\top = -B^\top \Omega A^\top \\ &= -\Omega' = -A\Omega B \implies A = B^\top\end{aligned}$$

Symplectic group. The symplectic group is

$$\mathrm{Sp}(2n, \mathbb{F}) \equiv \{M \in \mathrm{Mat}(2n, \mathbb{F}) \mid M^\top \Omega M = \Omega\}\,, \quad (\det M)^2 = 1$$

This group also preserves the pfaffian

$$[\mathrm{Pf}(N)]^2 = \det N$$

1.2 Morphisms

Definition 1.29. A morphism ϕ is a map from a group G to another group G', $\phi : G \to G'$, $\phi(g) = g'$ with $g \in G$ and $g' \in G'$.

Definition 1.30. A homomorphism is a morphism that respects the composition law of G and G': $\forall g_1, g_2 \in G$, $\phi(g_1 \cdot g_2) = \phi(g_1) \cdot \phi(g_2)$. The first composition is in G, the second is in G'.

Definition 1.31. An isomorphism is an invertible homomorphism. It is a bijective map: $\forall g \in G$, $\phi(g) = g' \in G'$ and $\exists \phi^{-1} : G' \to G$ such that $\phi^{-1}(g') = \phi^{-1}(\phi(g)) = g$.

Definition 1.32. An automorphism is an isomorphism with $G' = G$.

Lecture 3

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Definition 1.33. A representation ρ is a homomorphism where G' is a group of matrices:

$$\rho : G \to \mathrm{GL}(n, \mathbb{C})$$

The number n is the dimension of the representation.

Example 1.34. A few examples:

- the trivial homomorphism exists: it maps all elements into the identity, $\phi(g) = e'$, $\forall g \in G$, $e' \in G'$; this is used in the context of representations because, if it exists, there is always a group element g mapped to the identity of the general linear group.
- Take a function between two cyclic groups $\phi : \mathbb{Z}_4 \to \mathbb{Z}_2$. To create a homomorphism one can take $\phi(e) = \phi(a^2) = e'$ and $\phi(a) = \phi(a^3) = a'$. One then has

$$\phi(ea) = \phi(a) = a' \,, \quad \phi(ea) = \phi(e)\phi(a) = e'a' = a'$$

- It's possible to construct an isomorphism between the definitions of $\mathbb{Z}_n$: the set $\{e, a, \dots, a^{n-1}\}$ with multiplication and the set $\{\{0\}, \{1\}, \dots, \{n-1\}\}$ with addition.
- One can also construct an isomorphism between $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$. The following should hold $\phi(g_1 + g_2) = \phi(g_1)\phi(g_2)$. Such an isomorphism is the exponential function $\mathrm{e}^{g_1+g_2} = \mathrm{e}^{g_1}\mathrm{e}^{g_2}$. Its inverse is the natural logarithm.
- An example of non trivial automorphism considers $\mathbb{Z}_3$. One can define

$$\phi(e) = e' \,, \quad \phi(a) = a'^2 \,, \quad \phi(a^2) = a'$$

  and one should check that the morphism respects the composition law

$$\phi(a)\phi(a^2) = \phi(a^3) = e' \,, \quad \phi(a)\phi(a^2) = a'^2 a = e'$$

  However, this is not true for $\mathbb{Z}_4$. Consider the mapping

$$\phi(e) = e' \,, \quad \phi(a) = a'^2 \,, \quad \phi(a^2) = a'$$

  The fourth element is given by

$$\phi(a)\phi(a^3) = b^3$$

  but

$$\phi(a)\phi(a^3) = \phi(a^4) = \phi(e) = e' \,, \quad \phi(a)\phi(a^3) = a'^2 e' = a'^2$$

Definition 1.35. Two elements h_1 and h_2 of a group G are conjugate of one another if there exists a third element g of the group such that $h_2 \equiv g^{-1} h_1 g$.

One can then build equivalence classes.

Definition 1.36. An internal automorphism is the mapping given by conjugating every element of a group while fixing g:

$$\phi_g : G \to G\,, \quad \phi_g : h \mapsto g^{-1}hg$$

The inverse mapping is

$$\phi_g^{-1} : h \mapsto ghg^{-1}$$

Lemma 1.37. The set of automorphisms of a group is itself a group $\mathrm{Aut}(G)$ with the operation of composition of functions.

Definition 1.38. Given a set $S = \{s_1, \dots, s_k\}$ of k elements, the set of all words S' created by the concatenation of the elements of the set S, together with an equivalence relation, forms a free group. A word can be of any length, with as many repeated elements as one wants. The composition is given by concatenating words. The identity is the null word.

Definition 1.39. Given a subset of words $R \subset S'$, a presentation is a group whose elements are given by the equivalence classes in S of the relation $r = e$ for all $r \in R$. The set R is the set of relations.

Remark 1.40. In terms of groups, the set S' is a group and one wants to identify the elements of S that give all the elements of the group utilizing equivalence classes. The set S is the set of generators: the minimal set that gives all the group elements.

Definition 1.41. The number of generators is called rank.

Example 1.42. The set of generators of the cyclic group $\mathbb{Z}_n$ has rank one because it only contains the element a with relation $a^n = e$: all other elements can be generated from a.

Theorem 1.43. Every finite group G admits an homomorphism of the form $\phi : G \to G'$ where G' is a free group. Therefore, a presentation always exists.

Example 1.44. A presentation for $\mathbb{Z}_n$ is $\{a, a^n = e\}$.

2 Dihedral groups

Definition 2.1. A dihedral group D_k is a group of symmetries of a k-gon.

Remark 2.2. The course considers the Euclidean plane $\mathbb{R}^2$ and its isometries, transformations that keep invariant the distance between points. Such isometries are translation, rotation and reflection.

Definition 2.3. A figure is a subset of $\mathbb{R}^2$ delimited by a closed curve.

Definition 2.4. In $\mathbb{R}^2$, a k-gon is a regular polygon figure with k side.

Three points. A 3-gon is a regular triangle. One wants to find all transformations that keep the triangle in the same situation as initially. Such transformations are the identity, rotations by $\mathrm{e}^{\mathrm{i}\frac{2}{3}\pi}$ or $\mathrm{e}^{\mathrm{i}\frac{4}{3}\pi}$, and reflections R_i by each side's axis of the triangle. The number of elements of the dihedral group is 6. [r] image

One wants to find the minimal set of transformations as to find the generators. Let $r = \mathrm{e}^{\mathrm{i}\frac{2}{3}\pi}$ and $\sigma = R_1$. Combining some transformations one gets

$$\sigma r = R_3\,, \quad r\sigma = R_2\,, \quad r^2 = \mathrm{e}^{\mathrm{i}\frac{4}{3}\pi}\,, \quad r^2\sigma = R_3$$

Note that the order of operation is from right to left. The presentation of the dihedral group is

$$\mathrm{D}_3 = \{\{r, \sigma\}, r^3 = e, \sigma^2 = e, (\sigma r)^2 = e\}$$

The rank is then 2.

Two points. Since two sides are not enough for a planar figure, one can consider two points connected by two overlapping segments. The transformations include rotations by $r = e^{i\pi}$ and reflections along the x and y axes. Let $\sigma = R_x$, then $\sigma r = R_y$. The presentation is

$$\mathrm{D}_2 = \{\{r, \sigma\}, \sigma^2 = e, r^2 = e, (\sigma r)^2 = e\}$$

This group is abelian.

One point. There is only one transformation to be made and that is a reflection about an arbitrary axis. Given $\sigma = R$ then the presentation is

$$\mathrm{D}_1 = \{\{\sigma\}, \sigma^2 = e\}$$

This group is abelian and it's isomorphic to the cyclic group: $\mathrm{D}_1 \simeq \mathbb{Z}_2$.

Four points. The transformations are rotations every $e^{i\frac{\pi}{2}}$ and four reflections, two along the diagonals, two splitting the sides. The presentation is

$$\mathrm{D}_4 = \{\{\sigma, r\}, \sigma^2 = e, r^4 = e, (\sigma r)^2 = e\}$$

Arbitrary points. The n dihedral group has $n+n$ elements given by rotations and reflections. The rank is 2 for $n > 1$. The group is not abelian for $n > 2$. The presentation is

$$\mathrm{D}_n = \{\{\sigma, r\}, r^n = e, \sigma^2 = e, (\sigma r)^2 = e\}$$

Subgroups. The rotations are a subgroup $\mathbb{Z}_n = \{r, r^n = e\}$ and so is each reflection $\mathbb{Z}_2 = \{\sigma, \sigma^2 = e\}$ along some axes: there are n copies of $\mathbb{Z}_2$.

Definition 2.5. A subgroup is a subset H of a group G which is a group itself under the same composition of G. It suffices to check

$$\forall h_1, h_2 \in H, h_1 \cdot h_2 \in H\,, \quad \forall h \in H, \exists h^{-1} \in H \mid hh^{-1} = h^{-1}h = e$$

Example 2.6. Some trivial examples are $H = G$ and $H = \{e\}$.

Lecture 4

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3 Finite groups of small order

Groups can be represented through a multiplication table, like the following

	e	g_1	g_2
e	g_1	g_2	g_2
g_1	g_1	g_1^2	$g_1 g_2$

Order one. A group of order one contains only the identity.

Order two. For a group of order two, $G = \{e, a\}$, the table is

	e	a
e	e	a
a	a	a^2

To be a group, then a^2 must be an element of G. If $a^2 = a$ then

$$a^{-1}a^2 = a\,, \quad a^{-1}a^2 = a^{-1}a = e$$

which is a contradiction. So if $a^2 = e$ then one has $\mathbb{Z}_2 = \{e, a\}$ with $a^2 = e$. It has rank one, it is abelian, it has no subgroups and it is the only group of order two.

Order three. For a group of order three, the table is

	e	a	b
e	e	a	b
a	a	a^2	ab
b	b	ba	b^2

One can follow a branching logic as before. Setting $a^2 = a$ leads to $a = e$ which means that the group is no longer of order three.

So if $a^2 = e$ then $a^{-1} = a$. The product ab must be an element of the group. Therefore

- if $ab = e$ then $b = a$;
- if $ab = b$ then $a = e$;
- if $ab = a$ then $b = e$.

These are not acceptable conditions since the order would no longer be three. So one must modify the previous condition to $a^2 = b$:

- if $ab = b$ then $a = e$, or if $ab = a$ then $b = e$, which are not acceptable;
- if $ab = e$ then b^2 must be an element of the group also:
  - ◇ if $b^2 = b$ then $b = e$, not acceptable;
  - ◇ if $b^2 = e$ then $b = a$, not acceptable;
  - ◇ if $b^2 = a$ then it is a valid condition.

The only such group is $\mathbb{Z}_3$:

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

When the multiplication table is symmetric, the group is abelian. From the multiplication table one can identify subgroups by looking at smaller multiplication tables already known (not necessarily with entries one next to each other). In this case, $\mathbb{Z}_3$ has rank one, it is the only group of order three and has no subgroups.

Order four. For a group of order four, the table is

	e	a	b	c
e	e	a	b	c
a	a	a^2	ab	ac
b	b	ba	b^2	bc
c	c	ca	cb	c^2

Repeating the same branching procedure gives two ways of filling the multiplication table. The first one leads to $\mathbb{Z}_4$. It is abelian and has rank 1 with $a^4 = e$:

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

Looking at the table one can see the sub-table of $\mathbb{Z}_2 = \{e, a^2\} = \{e, b\}$ in the upper-left corner.

The second way of filling the multiplication table gives the dihedral group D_2. It is abelian and has rank 2:

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

One can identify three subgroups $\mathbb{Z}_2$, two the corners and one at the center. There's also a fourth hidden $\mathbb{Z}_2$ subgroup that is found by combining the corners in 2×2 blocks.

These two groups constructed with the branching procedure are not isomorphic: one can look at the subgroups, in this case how many there are. Also by looking at their subgroups, it holds

$$\mathrm{D}_2 = \mathbb{Z}_2 \otimes \mathbb{Z}_2$$

where $\otimes$ is the Cartesian product later called direct product.

Higher orders. At order five there is only $\mathbb{Z}_5$. At order six there are $\mathbb{Z}_6$ and the group of permutations S_3. At order seven there is $\mathbb{Z}_7$. One may notice that for prime-number orders, there are only cyclic groups.

3.1 Groups of permutations

Definition 3.1. The set of $n!$ permutations of n objects forms a group S_n (also called the symmetric group).

Remark 3.2. A permutation is an automorphism over the set of n objects. The group of permutations S_n is a group of automorphisms.

Notation. The permutation of n elements is represented as

$$p = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ p_1 & p_2 & p_3 & \cdots & p_n \end{pmatrix}$$

In the product PQ the permutation that acts first is Q followed by P.

Representation in matrix notation. Given a permutation $p \in \mathrm{S}_n$, its $n \times n$ matrix representation is $[P]_{ij} = \delta_{jp_i}$. [r]

A representation is a homomorphism so it must preserve the group structure. In fact the matrix representation of a product of permutations must represent the product of the permutations:

$$[PQ]_{ij} = \sum_k P_{ik} Q_{kj} = \sum_k \delta_{ip_k} \delta_{kq_j} = \delta_{i,(pq)_j}$$

[r]

Permutations are used for rotations on the lattice.

Cycle notation. The cycle notation is compact way to write permutations. It has the properties

- the sum of all cycle lengths is equal to n,
- the order of the cycles is irrelevant,
- the cyclic order within a cycle is irrelevant.

For example

$$(134)(2)$$

means that 2 stays put, while 1 and 3 interchange then 3 and 4 interchange. The last property means $(134) = (341) = (413)$. It is possible rewrite a cycle of length n in terms of two cycles that pivot around the same element: $(134) = (14)(13)$.

Lemma 3.3. Every cycle can be written as the product of two cycles with repeated elements. For convention, one always chooses 1 to be the pivotal, repeated number.

Proposition 3.4. It holds

$$(1\,2\,\cdots\,m) = (1\,m)(1\,m-1)\cdots(13)(12)$$

It follows that

$$(ij) = (1i)(1j)(1i)$$

From the cycle notation, one can represent the entire group. By combining two-cycles one can get arbitrary permutations. So, the symmetric group S_n has the following generators

$$\{(12), (13), \ldots, (1n)\}\,, \quad \operatorname{rank}\mathrm{S}_n = n-1$$

The identity is given by $(12)^2(13)^2(14)^2$ but is not part of the generators.

Definition 3.5. The parity of an element $g \in \mathrm{S}_n$ is the number of two-cycles in any two-cycle decomposition.

Definition 3.6. Since the identity e is even under parity, the set of even permutations forms a subgroup called alternating group A_n. Its order is $2^{-1}n!$.

Example 3.7. Consider $\mathrm{S}_3 = \{e, (12), (13), (23), (123), (132)\}$. Let $g_1 = (12)$ and $g_2 = (23)$. It is not abelian

$$abc \overset{g_1}{\to} bac \overset{g_2}{\to} bca\,, \quad abc \overset{g_2}{\to} acb \overset{g_1}{\to} cab$$

The identity is given by $e = (12)^2(13)^2$. There are also three copies of $\mathbb{Z}_2$:

$$\{e, (12)\}\,, \quad \{e, (13)\}\,, \quad \{e, (23)\}$$

The alternating group is given by the even permutations

$$\mathrm{A}_3 = \{e, (123), (132)\}$$

At order three, there is only one group, so $\mathrm{A}_3 \simeq \mathbb{Z}_3$. At order six, one has $\mathrm{D}_3 \simeq \mathrm{S}_3$. For general groups, $\mathrm{S}_n \not\simeq \mathrm{D}_n$ because $|\mathrm{D}_n| = 2n$ and $|\mathrm{S}_n| = n!$. The dihedral group is a subset of the symmetric group: there are permutations that cannot be achieved from rotations and reflections.

Lemma 3.8 (Rearrangement)**.** If g, a, b are elements of a group G and $g \circ a = g \circ b$, then $a = b$.

Proof. Multiplying $g \circ a = g \circ b$ by g^{-1} on the left one finds the thesis. □

Corollary 3.9. Consider the group G of n elements. Multiplying every element by $h \in G$ gives a new group, mapping the group G to itself. It is an automorphism.

Definition 3.10. A morphism $\varphi_h : G_n \to \mathrm{S}_{|G|}$, $h \mapsto p_h$ such the above gives the permutations of the set G. It is a homomorphism: $\varphi_{h_1}\varphi_{h_2} = \varphi_{h_1 h_2}$. The range of φ_h is a subset of $\mathrm{S}_{|G|}$ and it is a subgroup P_h. [r]

Theorem 3.11 (Cayley's)**.** Every group G of order n is isomorphic to a subgroup of S_n called Cayley's group.

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Corollary 3.12. Two consequences are

- given a permutation P_h in Cayley's group, it cannot contain 1-cycles except for the identity $h = e$;
- a permutation P_h in Cayley's group can be decomposed in cycles of equal length. If n is prime, the corresponding permutations in Cayley's group can only contain non-factorized n-cycles. This means that one 1-cycle is the identity as it is one n-cycle: the entire Cayley's group is built from them and as such has rank one and order n. It is $\mathbb{Z}_n$.

Remark 3.13. Applying n times a cycle of length n, one gets the identity.

Theorem 3.14. If the order n of a group G is prime, then it must be isomorphic to $\mathbb{Z}_n$.

4 Structural properties of groups

Definition 4.1. Two elements are said to be conjugate $g_1 \sim g_2$ if and only if $\exists h \in G$ such that $hg_1h^{-1} = g_2$. The relation induces a partitioning of the group into equivalence classes.

Definition 4.2. If H is a subgroup of G and $a \in G$, then its conjugate group is

$$H' = \{aha^{-1}, \forall h \in H\} \equiv aHa^{-1}$$

which is also a subgroup.

Lemma 4.3. If H and H' are conjugate subgroups, then either $H \cap H' = \{e\}$ or $H = H'$.

Definition 4.4. An invariant subgroup H of G is identical to all its conjugate subgroups. It is also called normal. The set $H \subseteq G$ is invariant if and only if $gHg^{-1} = H$ for all $g \in G$.

Example 4.5. A few examples:

- Consider the group of permutations

  $$\mathrm{S}_3 = \{e, (12), (13), (23), (123), (132)\}$$

  It has three copies of $\mathbb{Z}_2$ given by the 2-cycles

  $$\{e, (12)\}\,, \quad \{e, (13)\}\,, \quad \{e, (23)\}$$

  The conjugate subgroup of the first $\mathbb{Z}_2$ is

  $$(23)^{-1}e(23) = e\,, \quad (23)^{-1}(12)(23) = (23)(12)(23) = (13)$$

  but this does not remain within the group. Remember that two-cycles are their own inverse. So $H' \neq H$ but only share the identity. Since the two sets do not coincide, by the lemma $\mathbb{Z}_2$ is not invariant. Consider instead the subgroup

  $$H_2 = \{e, (123), (132)\} \simeq \mathbb{Z}_3$$

  Its conjugate group is

  $$(12)H_2(12) = \{e, (132), (123)\} = H_2$$

  Therefore $\mathbb{Z}_3$ is invariant.

- Consider

  $$\mathbb{Z}_4 = \{e, a, a^2, a^3\}$$

  It has one subgroup

  $$\mathbb{Z}_2 = \{e, a^2\}$$

  Its conjugate is

  $$a^{-1}\{e, a^2\}a = \{e, a^2\}\,, \quad (a^3)^{-1}\{e, a^2\}a^3 = \{e, a^2\}$$

  Remember that $\mathbb{Z}_2$ is abelian. It is also abelian.

- Consider the dihedral group D_n. The relations used $r^n = e$ and $\sigma^2 = e$ corresponding to a $\mathbb{Z}_n$ and $\mathbb{Z}_2$. In general, $\mathbb{Z}_n \subset \mathrm{D}_n$ is normal, while $\mathbb{Z}_2 \subset \mathrm{D}_n$ is not invariant. On the other hand, in general $\mathbb{Z}_n \subset \mathrm{S}_n$ is not invariant.

Lemma 4.6. All subgroups of abelian groups are invariant subgroups.

Definition 4.7. A group is simple if it does not contain any non-trivial invariant subgroup.

Definition 4.8. A group is semi-simple if it does not contain any abelian invariant subgroup.

Example 4.9. Some examples:

- The group $\mathbb{Z}_4$ has a $\mathbb{Z}_2$ subgroup which is invariant and abelian: the group $\mathbb{Z}_4$ is not simple nor semi-simple.
- The group S_3 has a $\mathbb{Z}_3$ subgroup which is invariant and abelian.
- The cyclic group $\mathbb{Z}_n$ with n prime is a simple group (because it does not admit subgroups).

Remark 4.10. The simple property describes whether a group is fundamental or has substructures.

Definition 4.11. Consider a set H subgroup of G, an element $g \in G$ such that $g \notin H$. The set of elements

$$gH = \{gh_1, gh_2, \dots\}$$

is a left coset. Similarly Hg is a right coset.

Remark 4.12. Cosets have the same order of their parent group.

Lemma 4.13. Cosets are not subgroups.

Proof. Cosets do not have the identity element and g can't be the identity if it is not part of the subgroup H. □

Lemma 4.14. Two cosets either coincide completely or have no common elements.

Proof. Consider two cosets pH and qH. If $ph_i = qh_j$ then $q^{-1}p = h_j h_i^{-1} \in H$. If? it holds for all elements, then

$$q^{-1}pH = H \implies pH = qH$$

[r] □

Theorem 4.15 (Lagrange)**.** The order of a finite group must be an integer multiple of the order of any of its subgroups.

Example 4.16. Consider the symmetric group S_3 and one of its subgroup

$$H_1 = \mathrm{A}_3 = \{e, (123), (132)\}$$

The only coset that can be built is

$$(12)H_1 = (13)H_1 = (23)H_1 = \{(12), (13), (23)\}$$

Cosets induce a partitioning of the group. Consider one $\mathbb{Z}_2$ subgroup

$$\mathbb{Z}_2 \simeq H_2 = \{e, (12)\}$$

It has two cosets

$$(23)H_2 = (132)H_2 = \{(23), (132)\}\,, \quad (13)H_2 = (123)H_2 = \{(13), (123)\}$$

Theorem 4.17. If H is an invariant subgroup of G, then its left and right cosets coincide. The set of cosets endowed with the composition

$$pH \cdot qH = (p \cdot q)H$$

forms a group called quotient group of G denoted by ${}^{G}/_{H}$, [r] with order ${}^{|G|}/_{|H|}$.

Proof. It holds

$$gH = gH(g^{-1}g) = (gHg^{-1})g = Hg$$

Also

$$ph_iqh_j = p(qq^{-1})h_iqh_j = (pq)h_k$$

where $q^{-1}h_iq \in H$ and $h_j \in H$. □

Example 4.18. Two examples are:

- Consider the group $\mathbb{Z}_4$. It has an invariant subgroup

$$\mathbb{Z}_2 \simeq H = \{e, a^2\}$$

  which has only one coset

$$M = aH = \{a, a^3\} = a\mathbb{Z}_2 = a^3\mathbb{Z}_2$$

  In the sense of sets, that is multiplying all the elements together, one has

$$HH = H\,, \quad HM = MH = M\,, \quad MM = H$$

  Therefore ${}^{\mathbb{Z}_4}/_{\mathbb{Z}_2} \simeq \mathbb{Z}_2$ the partitioning sets behave as elements of $\mathbb{Z}_2$.

- The set S_3 has three elements as it does A_3. The quotient has only two elements. Since the only group of order two is $\mathbb{Z}_2$ then

$${}^{\mathrm{S}_3}/_{\mathrm{A}_3} \simeq \mathbb{Z}_2$$

4.1 Multiplication of groups

Definition 4.19. Given two groups H_1 and H_2, their direct product is the set of pairs

$$\{(h_1, h_2) \mid h_1 \in H_1, h_2 \in H_2\}$$

with the composition law

$$(h_1, h_2) \cdot (h_1', h_2') = (h_1h_1', h_2h_2') \in H_1 \otimes H_2$$

The order of the direct product is $|H_1 \otimes H_2| = |H_1||H_2|$. The sets H_1 and H_2 are invariant subgroups of $H_1 \otimes H_2$.

Remark 4.20. Consider

$$(h_1', h_2')^{-1}(e_1, h_2)(h_1', h_2') = (h_1'^{-1}, h_2'^{-1})(e_1, h_2)(h_1', h_2') = (h_1'^{-1}h_1', h_2'^{-1}h_2h_2') = (e_1, h_2'^{-1}h_2h_2')$$

[r]

Definition 4.21. Given a group $G = H_1 \otimes H_2$, it is the direct product group of the subgroups H_1 and H_2 if

- $h_1h_2 = h_2h_1$ for all $h_1 \in H_1$ and $h_2 \in H_2$;
- the intersection is $H_1 \cap H_2 = \{e\}$;
- for all $g \in G$, $\exists h_1 \in H_1, h_2 \in H_2$ such that $g = h_1h_2$.

Remark 4.22. If $G = H_1 \otimes H_2$ with H_1 and H_2 invariant subgroups then

$${}^{G}/_{H_1} \simeq H_2\,, \quad {}^{G}/_{H_2} \simeq H_1$$

[r]

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Example 4.23. Consider

$$\mathbb{Z}_6 = \{e, a, a^2, a^3, a^4, a^5\}$$

Two subgroups are

$$H_1 = \{e, a^3\} \simeq \mathbb{Z}_2\,, \quad H_2 = \{e, a^2, a^4\} \simeq \mathbb{Z}_3$$

The subgroups are abelian because the group is. They are also invariant $ghg^{-1} = hgg^{-1} = h$. Therefore one can write

$$\mathbb{Z}_6 = \mathbb{Z}_2 \otimes \mathbb{Z}_3 \implies {}^{\mathbb{Z}_6}/_{\mathbb{Z}_3} \simeq \mathbb{Z}_2$$

One may think that one can factor a cyclic group in terms of smaller cyclic groups by using factorization.

Theorem 4.24 (Chinese remainder)**.** Let p and q be coprime integers. Then

$$\mathbb{Z}_{pq} \simeq \mathbb{Z}_p \otimes \mathbb{Z}_q$$

Theorem 4.25. Let $\phi : G \to G'$ be a homomorphism. The kernel of ϕ

$$\operatorname{Ker}\phi = \{g \in G \mid \phi(g) = e'\} \equiv K$$

is an invariant subgroup of G. The quotient ${}^{G}/_{\operatorname{Ker}\phi}$ is isomorphic to G'.

Example 4.26. Consider again $\mathbb{Z}_6$ and assume not knowing its structure. Then

$$H_1 = \{e, a^3\}\,, \quad aH_1 = \{a, a^4\}\,, \quad a^2H_1 = \{a^2, a^5\}$$

There is a natural mapping ϕ using this structure. One can map every coset to the elements that multiply H_1:

$$\{e', b, b^2\}$$

The set $H_1 \simeq \mathbb{Z}_2$ is the kernel of ϕ. Therefore

$${}^{G}/_{H_1} \simeq G' = \mathbb{Z}_3$$

The direct product is restrictive because it requires subgroups to be invariant. One may want a more loose definition of product.

Definition 4.27. Given two groups G and K with different operations $g_i \cdot g_j$, $k_i \circ k_j$, assuming G acts as a group of transformations over K

$$\forall g \in G\,, \quad g : K \to K\,, \quad k_i \mapsto g(k_i) = k_j$$

the semi-direct product, $G \ltimes K$, is the group given by the direct product endowed with the composition law

$$(g_1, k_1)(g_2, k_2) = (g_1 \cdot g_2, k_1 \circ g_1(k_2))$$

Proposition 4.28. A few properties of this definition:

- the definition is the composition law of Poincaré transformations,
- the inverse element of (g, k) is

$$(e, e) = (g, k)^{-1}(g, k) \implies (g, k)^{-1} = (g^{-1}, [g^{-1}(k)]^{-1})$$

- the order is

$$|G \ltimes K| = |G||K|$$

- the set K is an invariant subgroup of $G \ltimes K$ but not of G

$$(e, k) \in K \subset G \ltimes K \implies (g, h)^{-1}(e, k)(g, h) = (g^{-1}eg, \cdots) \in K$$

- if G acts trivially on K
$$g(k) = k, \forall g \in G, \forall k \in K$$
then
$$G \ltimes K = G \otimes K$$

Proposition 4.29. Given the subgroups K and H of G, then $G = K \ltimes H$ if

- H is invariant,
- the intersection is $K \cap H = \{e\}$,
- given $g \in G$, there exists $k \in K$ and $h \in H$ such that
$$g = (k, e)(e, h)$$

If only one subgroup is invariant, then one uses a semi-direct product. If both subgroups are invariant, one uses the direct product.

Example 4.30. In the dihedral groups, one can identify two subgroups: $\mathbb{Z}_k$ and $\mathbb{Z}_2$. Only the first is invariant. Therefore
$$\mathrm{D}_k/\mathbb{Z}_k \simeq \mathbb{Z}_2 \implies \mathrm{D}_k \simeq \mathbb{Z}_2 \ltimes \mathbb{Z}_k$$

Example 4.31. Consider
$$\mathbb{Z}_4/\mathbb{Z}_2 \simeq \mathbb{Z}_2\,, \quad \mathbb{Z}_4 \not\simeq \mathbb{Z}_2 \ltimes \mathbb{Z}_2$$
Because one cannot identify a second $\mathbb{Z}_2$ subgroup inside $\mathbb{Z}_4$.

5 Classification of finite simple groups

From simple groups one can build any other group.

There are 18 infinite series of finite simple groups:

- $\mathbb{Z}_p$ with p prime,
- the alternating subgroup A_n subgroup with the permutation group with $n \geq 5$,
- classical groups of matrices over finite fields.

There are also 26 exceptional groups (simple and finite):

- the smallest such group is Mathieu group M_{11} of order $7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$.
- the biggest such group is the monster group F_1, it contains 20 out of the 26 exceptional groups. Its order is about 8.08×10^{53}. This group is associated with Galois Theory.

6 Groups of space-time symmetries

Definition 6.1. The Euclidean group $\mathrm{ISO}(n)$ (inhomogeneous special orthogonal group?) consists of all continuous linear transformations of the n-dimensional Euclidean space $\mathbb{R}^n$ which leave the norm of all vectors invariant. The norm is defined as
$$\|x\|^2 = \sum_{i=0}^{n-1} (x^i)^2$$

A generic element of the Euclidean group $\mathrm{ISO}(n)$ maps
$$x^i \mapsto x'^i = R^i{}_j x^j + b^i$$
where R is a rotation and b is a translation.

6.1 Two dimensions

Rotations. An active transformation rotates a point P and keeps the axes fixed, while a passive rotation is the opposite, the axes are rotated. The latter will be used. The rotation is then

$$\begin{bmatrix} x'_p \\ y'_p \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_p \\ y_p \end{bmatrix} = R(\theta) \begin{bmatrix} x_p \\ y_p \end{bmatrix}$$

All rotation matrices $R(\theta)$ form a group. In fact

$$R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)\,, \quad RR^\top = I$$

The matrices are orthogonal and have determinant equal to $\det R = 1$. In two dimensions, rotations form the special orthogonal group SO(2). It is abelian, its identity is $R(\theta = 0) = I$, the inverse is $R^{-1}(\theta) = R(-\theta)$.

Translations. A translation is a transformation such that

$$T(\mathbf{b}) : x^i \mapsto x'^i = T(\mathbf{b}_1 + \mathbf{b}_2)$$

It holds

$$T(\mathbf{b}_1)T(\mathbf{b}_2) = T(\mathbf{b}_1 + \mathbf{b}_2)\,, \quad T(\mathbf{0}) = e\,, \quad T^{-1}(\mathbf{b}) = T(-\mathbf{b})$$

The translations form an abelian group.

Euclidean group. An element of the Euclidean group ISO(2) is

$$g(\theta, \mathbf{b}) : \begin{bmatrix} x' \\ y' \end{bmatrix} = R \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix}$$

One may want to understand if elements g with both compositions form a group. It holds

$$g(\theta_1, \mathbf{b}_1)g(\theta_2, \mathbf{b}_2) = g(\theta_1 + \theta_2, \mathbf{b}_1 + R(\theta_1) \cdot \mathbf{b}_2)$$

In fact, applying one element g after the other, one gets

$$R_1(R_2\mathbf{x} + \mathbf{b}_2) + \mathbf{b}_1$$

The inverse is

$$g^{-1}(\theta, \mathbf{b}) = g(-\theta, -[R(-\theta) \cdot \mathbf{b}])$$

The identity is

$$g(0, \mathbf{0}) = e$$

The Euclidean group ISO(2) is a group and it is a semi-direct product

$$\mathrm{ISO}(2) = \mathrm{SO}(2) \ltimes \mathrm{T}_2$$

Exercise. Prove that T_2 is invariant and SO(2) is not.

Reflections. Consider the reflection with respect to the x axis

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

One may want a reflection about an arbitrary line. If the line has an angle θ from the x axis, then the reflected point P'' forms an angle with the origin and P' of 2θ. So a reflection about an arbitrary axis is a reflection about the x axis and then a reflection:

$$I(\theta) = R(2\theta)I(0)$$

For a passive transformation one has

$$I(\theta) = I(0)R(-2\theta)$$

Since one should get the same result for both active and passive transformations then

$$R(2\theta)I(0) = I(0)R(-2\theta) \implies I^{-1}(0)R(2\theta)I(0) = R(-2\theta)$$

Since θ is arbitrary, so is $2\theta = \varphi$:

$$I^{-1}(0)R(\varphi)I(0) = R(-\varphi) = R^{-1}(\varphi)$$

Substituting this relation, one gets

$$I(\theta) = I(0)R(-2\theta) = I(0)[I^{-1}(0)R(\theta)I(0)]R(-\theta) = R(\theta)I(0)R(-\theta)$$

[r] A reflection through the origin is

$$\begin{bmatrix} x''' \\ y''' \end{bmatrix} = -I \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

Noting that $-I = R(\pi)$. A reflection about the y axis is

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = R(\pi)I(0)$$

Therefore

- all reflections are built from a single reflection $I(0)$ and reflections;
- it holds
$$\det[I(\theta)] = \det R \, \det I(0) = \det I(0) = -1$$
- it holds
$$I^2(0) = I \,, \quad I^{-1}(0) = I(0)$$
The element $I(0)$ represents a generator of $\mathbb{Z}_2$.
- the metric transforms as
$$g' = I(0)gI(0)$$
If $g = \delta$ then
$$g' = I^2(0) = I$$
The matrix $I(0)$ preservers the 2×2 metric and as such it is an element of the orthogonal group $\mathrm{O}(2)$.
- Elements M in the orthogonal group have $\det M = \pm 1$; this group has two disjoint sets
$$\mathrm{SO}(2) = \{M \in \mathrm{O}(2) \mid \det M = 1\} \,, \quad I(0)\mathrm{SO}(2) = \{M \in \mathrm{O}(2) \mid \det M = -1\}$$
Therefore
$$\mathrm{O}(2) \simeq \mathbb{Z}_2 \ltimes \mathrm{SO}(2)$$
Because only $\mathrm{SO}(2)$ is invariant.

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6.2 Arbitrary dimensions

Definition 6.2. The set of all transformations, including reflections, of n-dimensional Euclidean space which preserve the norm of a vector is called the Euclidean group E_n. Therefore

$$\mathrm{T}_n \subset \mathrm{ISO}(n) \subset E_n \,, \quad \mathrm{SO}(n) \subset \mathrm{ISO}(n) \,, \quad \mathrm{SO}(n) \subset \mathrm{O}(n) \subset \mathrm{E}_n \,, \quad \mathrm{D}_n \subset \mathrm{O}(n)$$

The relation above can be generalized to any dimension

$$\mathrm{O}(n) \simeq \mathbb{Z}_2 \ltimes \mathrm{SO}(n)$$

The special orthogonal group is always an invariant group

$$\mathrm{O}(n)/\mathrm{SO}(n) \simeq \mathbb{Z}_2$$

[r] By adding translations?

$$\mathrm{E}_n \simeq \mathrm{O}(n) \ltimes \mathrm{T}_n \,, \quad \mathrm{E}_n/\mathrm{T}_n \simeq \mathrm{O}(n)$$

7 Representations of groups

A presentation ρ of G is a homomorphism $\rho : G \to \mathrm{GL}(V)$, $g \mapsto \rho(g)$. One defines

- the dimension of the vector space V over a field $\mathbb{F}$ is the dimension n of the representation;
- a representation if faithful if the homomorphism is also an isomorphism, otherwise the representation is degenerate;
- given a degenerate representation, the set ${}^{G}/_{\mathrm{Ker}\,\rho}$ determines a faithful representation;
- it holds
$$\rho(e) = I\,, \quad \rho(g^{-1}) = \rho^{-1}(g)\,, \quad \rho(g_1)\rho(g_2) = \rho(g_1 g_2)$$

Example 7.1. A few examples:

- every group G admits a trivial representation
$$\rho(g) = I\,, \quad \forall g \in G\,, \quad V = \mathbb{C}$$
- All groups of matrices admit a one dimensional representation
$$V = \mathbb{C}\,, \quad \forall M \in G \subset \mathrm{GL}(n, \mathbb{C})\,, \quad \rho(M) = \det M$$
The representation is degenerate
$$\rho(M_1)\rho(M_2) = \rho(M_1 M_2) = \det(M_1 M_2) = \det M_1 \, \det M_2$$
- The cyclic group $\mathbb{Z}_2 = \{e, a\}$, $a^2 = e$ admits two representations with dimension two with $V = \mathbb{R}^2$:
  - ⋄ first: $\rho(e) = I$, $\rho(a) = -I$, this can be generalized to arbitrary dimensions;
  - ⋄ second: $\rho(e) = I$, $\rho(a) = \mathrm{diag}(1, -1)$.

  There does not exist a change of basis which maps one into the other.
- The dihedral group $\mathrm{D}_2 = \{e, R_x, R_y, r\}$ has two representations in two dimensions:
$$e = I\,, \quad \rho(R_x) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\,, \quad \rho(R_y) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\,, \quad \rho(r) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$
From the representation one can also see that $\mathbb{Z}_2 \subset \mathrm{D}_2$. The group structure is preserved.
- A one dimensional representation of
$$\mathbb{Z}_n = \{e, a, \dots, a^{n-1}\}\,, \quad a^n = e\,, \quad \rho(a^n) = \rho^n(a) = \rho(e) = 1$$
and then
$$\rho(a) = \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}\,, \quad \rho(a^k) = \mathrm{e}^{\frac{2\pi \mathrm{i}}{n} k}\,, \quad k = 0, 1, \dots, n-1$$
- A representation gives a way to multiply the elements $g \in G$ by a vector $\mathbf{v} \in V$. One can see the representation ρ as a map
$$\rho : G \times V \to V\,, \quad (g, \mathbf{v}) \mapsto \rho(g)\mathbf{v}$$
Therefore ρ is a module.

Definition 7.2. The representation of a group of matrices coinciding with the group itself is a fundamental representation.

Example 7.3. Two examples:

- For the special linear group SO(2), one can use
$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$
The matrix R is two dimensional acting on a vector space over $\mathbb{R}$.
- Using orthogonal matrices to represent $\mathrm{O}(n)$ defines the orthogonal group's fundamental representation.

Theorem 7.4. If a group G admits an invariant subgroup H, then every representation of ${}^{G}\!/_{H}$ induces a degenerate representation of G.

Example 7.5. Consider the alternating group
$$\mathrm{A}_3 = \{e, (123), (321)\}$$
it is an invariant subgroup of S_3. Therefore
$$^{\mathrm{S}_3}\!/_{\mathrm{A}_3} \simeq \mathbb{Z}_2\,, \quad \rho(\mathbb{Z}_2) = \{1, -1\}$$
The representation ρ induces a one dimensional representation of S_3: the parity of the elements. For every element, there is an associated cycle which has an even or odd number of two-cycles.

Definition 7.6. A similarity transformation between two matrices is defined by an invertible matrix S such that
$$M_1 = S^{-1} M_2 S$$
Notice that it is one matrix for all group elements.

Definition 7.7. Two representations ρ_1 and ρ_2 of G are equivalent if and only if
$$\rho_1(G) \sim \rho_2(G) \iff \exists s \mid \forall g \in G\,, \rho_1(g) = S^{-1}\rho_2(g) S$$

Definition 7.8. The character $\chi_\rho(g)$ for an element $g \in G$ of a representation ρ is defined as
$$\chi_\rho(g) = \mathrm{Tr}[\rho(g)]$$
The character is a function of the equivalence classes, not of the representation
$$\mathrm{Tr}[\rho_1(g)] = \mathrm{Tr}\big[S^{-1}\rho_2(g)S\big] = \mathrm{Tr}\big[\rho_2(g)SS^{-1}\big] = \mathrm{Tr}[\rho_2(g)]$$

7.1 Irreducible representations

Definition 7.9. Let ρ be a representation of G on a vector space V. Let V_1 be a subspace of V such that
$$\rho(g)\left|v_1\right\rangle \in V_1\,, \quad \forall \left|v_1\right\rangle \in V_1$$
The set V_1 is an invariant subspace of V with respect to $\rho(G)$. If V_1 does not contain any non-trivial invariant subspace, then V_1 is said to be minimal or proper.

Definition 7.10. A representation ρ of G on V is

- irreducible (irrep) if there is no invariant subspace of V,
- reducible if there is at least one invariant subspace,
- fully reducible or decomposable if the orthogonal complement
$$V_1^\perp = \{\left|v\right\rangle \in V \mid \left\langle v_1 | v\right\rangle = 0\,, \forall \left|v_1\right\rangle \in V_1\}$$
is also invariant.

Remark 7.11. One may note:

- the representation ρ is reducible if there exists a basis for all $g \in G$ such that
$$\rho(g) = \begin{bmatrix} \rho_1(g) & \rho_2(g) \\ 0 & \rho_3(g) \end{bmatrix}$$
the representation is block diagonal, but the subspaces are mixed.
- the representation ρ is fully reducible if there exists a basis such that
$$\rho(g) = \begin{bmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{bmatrix}$$
the representation is block diagonal.

Definition 7.12. The direct sum of two representations is
$$\rho = \rho_1 \oplus \rho_2 = \begin{bmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{bmatrix}$$
acting on the vector space $V = V_1 \oplus V_2$ with $\dim V = \dim V_1 + \dim V_2$.

Definition 7.13. The direct product of two representations $\rho = \rho_1 \otimes \rho_2$ is defined on the vector space $V = V_1 \otimes V_2$ with $\dim V = \dim V_1 \, \dim V_2$ and vectors
$$|v\rangle = v^{ij} \, |e_i^{(1)}\rangle \otimes |e_j^{(2)}\rangle \in V$$
for a basis $\{ |e_i^{(j)}\rangle \}$ of V_j; such that
$$\rho(g) \, |v\rangle = v^{ij} (\rho_1(g) \, |e_i^{(1)}\rangle) \otimes (\rho_2(g) \, |e_j^{(2)}\rangle) = v^{ij} \rho_1(g)]^k{}_i [\rho_2(g)]^m{}_j \, |e_k^{(1)}\rangle \otimes |e_m^{(2)}\rangle = |v'\rangle$$
In matrix notation one has
$$v' = \rho_1(g) v \rho_2(g)^\top$$

Remark 7.14. The central problem in representation theory and physics is classification. One wants to write any representation as sums or products of irreps ρ:
$$D = \bigotimes_{\nu=1}^{m} [\rho_\nu(G)]^{b_\nu} = \bigoplus_{\mu=1}^{n} a_\mu \rho_\mu(G)$$

Example 7.15. Some examples:

- Consider the cyclic group $\mathbb{Z}_2$. The trivial representation and another representation both one dimensional are
$$\rho_0(\mathbb{Z}_2) = \{1, 1\} \, , \quad \rho_1(\mathbb{Z}_2) = \{1, -1\}$$
One can build a two dimensional representation
$$\rho_2(\mathbb{Z}_2) = \rho_0(\mathbb{Z}_2) \oplus \rho_1(\mathbb{Z}_2) = \begin{bmatrix} \rho_0(g) & 0 \\ 0 & \rho_1(g) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
Consider the representation
$$\widetilde{\rho}_2(\mathbb{Z}_2) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$
Letting $V = \mathbb{R}^2$, one basis is
$$\{|v_1\rangle \, , |v_2\rangle\} \, , \quad |v_i\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix}$$
and one gets
$$\widetilde{\rho}_2(a) \, |v_1\rangle = |v_1\rangle \ , \quad \widetilde{\rho}_2(a) \, |v_2\rangle = - |v_2\rangle$$
The subspaces generated by $\widetilde{\rho}_2 \, |v_i\rangle$ are separate. One can introduce a similarity matrix
$$S = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \implies S^{-1} \widetilde{\rho}_2(a) S = \rho_2(a)$$
So the representations $\widetilde{\rho}_2$ and ρ_2 are equivalent. The property of equivalence can be stated by other means.

Lecture 8

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- The matrix
$$\begin{bmatrix} 1 & -a \\ a & 1 \end{bmatrix}, \quad a \neq 0$$
does not admit an invariant subspace. In fact, imposing
$$\begin{bmatrix} 1 & -a \\ a & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \equiv \begin{bmatrix} x \\ y \end{bmatrix} \implies \begin{cases} x - ay = x \\ ax + y = y \end{cases} \implies x = y = 0$$
[r] The group S_3 admit a two dimensional representation with
$$\rho_2((123)) = -\frac{1}{2} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

- The fundamental representation of SO(2)
$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$
is an irreducible representation over $V = \mathbb{R}^2$. However, consider now $V = \mathbb{C}^2$ and
$$|v_\pm\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm\mathrm{i} \end{bmatrix}$$
Applying the rotation matrix, one gets
$$R(\theta)\,|v_\pm\rangle = \frac{\mathrm{e}^{\pm\mathrm{i}\theta}}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm\mathrm{i} \end{bmatrix} = \mathrm{e}^{\pm\mathrm{i}\theta}\,|v_\pm\rangle$$
So the vectors $|v_\pm\rangle$ define an invariant subspace. Defining a unitary transformation to be
$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{bmatrix}, \quad S^\dagger = S^{-1}$$
one gets a similarity transformation to diagonalize the rotation matrix
$$S^{-1}R(\theta)S = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\theta} \end{bmatrix}$$
Thus, the representation above is not an irreducible representation over $V = \mathbb{C}^2$ and is fully reducible written as a direct sum
$$\rho_2(\theta) = \rho_1(\theta) \oplus \rho_{-1}(\theta)$$
The single representations are also representations of U(1). The fundamental representation of SO(2) over $\mathbb{C}^2$ can be written as the sum of U(1) representations.

7.2 Unitary representations

Definition 7.16. A vector space is hermitian if it admits an inner product which is sesquilinear $\varphi : V \otimes V \to \mathbb{F}$, typically $\mathbb{F} = \mathbb{C}$:

- Linear in the arguments
$$\varphi(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}) = \varphi(\mathbf{v}_1, \mathbf{w}) + \varphi(\mathbf{v}_2, \mathbf{w}), \quad \varphi(\mathbf{v}, \mathbf{w}_1 + \mathbf{w}_2) = \cdots$$
- It holds
$$\varphi(\mathbf{v}, \lambda\mathbf{w}) = \lambda\varphi(\mathbf{v}, \mathbf{w})$$
- then [r]
$$\varphi(\mathbf{v}, \mathbf{w}) = [\varphi(\mathbf{w}, \mathbf{v})]^*$$

- it holds

$$\varphi(\mathbf{v},\mathbf{v}) \geq 0\,, \quad \varphi(\mathbf{v},\mathbf{v}) = 0 \iff \mathbf{v} = \mathbf{0}$$

Definition 7.17. If the operators $\rho(g)$ of a representation of a group G over a hermitian vector space are unitary

$$[\rho(g)]^{-1} = \rho(g^{-1}) = [\rho(g)]^\dagger\,, \quad \forall g \in G$$

then the representation is called unitary. That is, the representation ρ is unitary if $\rho(g)$ is a unitary matrix.

Theorem 7.18. Any representation of a finite group on a hermitian vector space V is equivalent to a unitary representation.

Proof. Since the vector space is hermitian one can build the conjugate representation $\rho^\dagger(g)$and construct

$$M = \sum_{g\in G} \rho^\dagger(g)\rho(g)$$

but ρ is not unitary. Therefore

$$\rho^\dagger(g)M\rho(g) = \rho^\dagger(g)\sum_{g'}\rho^\dagger(g')\rho(g')\rho(g) = \sum_{g'}\rho^\dagger(g'g)\rho(g'g) = \sum_{g''}\rho^\dagger(g'')\rho(g'') = M$$

Since $M = M^\dagger$ is hermitian then it is diagonalizable

$$M = U^\dagger \Lambda U\,, \quad \Lambda = \mathrm{diag}(\Lambda_1, \dots, \Lambda_n)$$

The matrix M is positive definite:

$$\langle v^\dagger M|v\rangle = \langle v^\dagger \sum_g \rho^\dagger(g)\rho(g)|v\rangle = \sum_g \|\rho(g)\,|v\rangle\|^2 \geq \|\rho(e)\,|v\rangle\|^2 = \||v\rangle\|^2 > 0$$

Therefore its eigenvalues are real and positive. Taking the squaring root of the eigenvalues then multiply by $U^\dagger U$ one gets the square root of the matrix

$$\sqrt{M} = U^\dagger\sqrt{\Lambda}U\,, \quad \sqrt{M}\sqrt{M} = M\,, \quad \sqrt{M} = (\sqrt{M})^\dagger$$

Starting from the beginning expression of M one multiplies by $M^{-\frac{1}{2}}$:

$$M = \rho^\dagger(g)M\rho(g) = [M^{-\frac{1}{2}}\rho^\dagger(g)M^{\frac{1}{2}}][M^{\frac{1}{2}}\rho(g)M^{-\frac{1}{2}}] = I$$

Therefore, the matrix

$$M^{\frac{1}{2}}\rho(g)M^{-\frac{1}{2}}$$

is unitary. □

Theorem 7.19. If the representation ρ is reducible and unitary, then it is fully reducible.

Proof. Consider an invariant subspace V_1 and its orthogonal complement $V_1^\perp$. One needs to prove that the orthogonal complement is an invariant subspace:

$$\rho(g)\,|v\rangle \in V_1\,, \quad \rho(g^{-1})\,|v\rangle \in V_1\,, \quad |v\rangle \in V_1$$

Then, using $w \in V_1^\perp$, one gets

$$\langle w|\,\rho^\dagger(g)\,|v\rangle = \langle w|\,\rho^{-1}(g)\,|v_1\rangle = \langle w|\,\rho(g^{-1})\,|v_1\rangle = 0 \implies \rho(g)\,|w\rangle \in V_1^\perp$$

So the orthogonal complement is invariant. The representation ρ is thus fully reducible. □

Corollary 7.20 (both theorems)**.** All reducible representations of finite groups are fully reducible.

Lemma 7.21 (Schur's I)**.** Let ρ be an irreducible representation of a group G on a vector space V, and A an operator on V. If A commutes with all operators

$$A\rho(g) = \rho(g)A\,, \quad \forall g \in G$$

then $A = \lambda I$.

Proof. With no loss of generality, consider A to be a hermitian operator since one can always have

$$A_\pm = \frac{A \pm A^\dagger}{2}$$

Consider an orthonormal basis formed by the eigenvectors of A

$$A\left|u_{i,\alpha}\right\rangle = \lambda_i \left|u_{i,\alpha}\right\rangle$$

where α is the index of degeneracy of an eigenvalue. Applying the hypothesis:

$$A\rho(g)\left|u_{i,\alpha}\right\rangle = \rho(g)A\left|u_{i,\alpha}\right\rangle = \rho(g)\lambda_i\left|u_{i,\alpha}\right\rangle$$

so the vector $\rho(g)\left|u_{i,\alpha}\right\rangle$ is also an eigenvector of A, so at most ρ changes α. Defining the subspace

$$V_i = \{\left|u_{i,\alpha}\right\rangle\}$$

then

$$\rho(g)\left|u_{i,\alpha}\right\rangle \in V_i$$

so the vector space V_i is an invariant subspace. However, if ρ is an irreducible representation, then it does not admit an invariant (proper) subspace: there is only one subspace, the trivial one $V_i = V$. This means that A has only one eigenvalue

$$A = \lambda I$$

□

Lemma 7.22 (Schur's II)**.** Let ρ and ρ' be two irreducible representation of a group G on the vector space V and V'. Let $A : V \to V'$ be a linear operator satisfying

$$A\rho'(g) = \rho(g)A\,, \quad g \in G$$

It follows either $A = 0$, or $V = V'$ (so A is an isomorphism) and $\rho \sim \rho'$.

Proof. The image (or range) of A in V' is

$$\operatorname{im} A = \{\left|x'\right\rangle \in V' \mid \left|x'\right\rangle = A\left|x\right\rangle\,, \quad \left|x\right\rangle \in V\}$$

It holds

$$\forall g \in G\,, \rho'(g)\left|x'\right\rangle = \rho'(g)A\left|x'\right\rangle = A\rho(g)\left|x\right\rangle \in \operatorname{im} A$$

So the image is an invariant subspace of V' with respect to $\rho'(g)$. But ρ' is an irreducible representation, therefore either $\operatorname{im} A = 0$ implies $A = 0$, or $\operatorname{im} A = V$ so A maps on the entire V'.

One can apply the same reasoning on the kernel

$$\operatorname{Ker} A = \{\left|x\right\rangle \in V \mid A\left|x\right\rangle = \left|0\right\rangle \in V'\}$$

Considering $\left|x\right\rangle \in \operatorname{Ker} A$, then

$$A\rho(g)\left|x\right\rangle = \rho'(g)A\left|x\right\rangle = \rho'(g)\left|0\right\rangle = \left|0\right\rangle \implies \rho(g)\left|x\right\rangle \in \operatorname{Ker} A$$

Therefore, the kernel is an invariant subspace of V with respect to ρ. But ρ is an irrep so ether $\operatorname{Ker} A = V$ implies $A = 0$, or $\operatorname{Ker} A = 0$ implies

$$A\left|x'\right\rangle = A\left|y'\right\rangle \implies \left|x'\right\rangle = \left|y'\right\rangle$$

This last case, combined with the last above, the map A is a bijective map. Therefore it is invertible and is an isomorphism

$$V = V'\,, \quad \dim V = \dim V'$$

Since A is invertible, then the two representation are equivalent

$$A\rho' = \rho A \implies \rho' = A^{-1}\rho A \implies \rho' \sim \rho$$

□

Notation. The labels of irreps are μ, ν. The dimension of irrep μ is d_μ. The character χ_i^μ has the index i that labels equivalence classes. The number of equivalence classes is n_c. The number of elements in an equivalence class is n_i.

Theorem 7.23 (A)**.** Irreducible representations are orthonormal to each other, valid only for finite groups:

$$\frac{d_\mu}{|G|}\sum_{g\in G}[\rho_\mu^\dagger(g)]^i{}_j[\rho_\nu(g)]^k{}_l = \delta_{\mu\nu}\delta_l^i\delta_j^k$$

The reason it describes orthonormality can be seen by the following

$$\langle g|u,i,j\rangle = \sqrt{\frac{d_\mu}{|G|}}[\rho_\mu(g)]^i{}_j \implies \sum_g \langle u,i,j|g\rangle\,\langle g|\nu,k,l\rangle = \delta_{\mu\nu}\delta_l^i\delta_j^k$$

Proof. Consider two irreps ρ_μ and ρ_ν with dimensions d_μ and d_ν. Let X be a $d_\mu\times d_\nu$ matrix

$$M_x = \sum_g \rho_\mu^\dagger(g)X\rho_\nu(g)$$

From the rearrangement lemma

$$\rho_\mu^{-1}(g)M_x\rho_\nu(g) = M_x\,, \quad \forall g\in G$$

Considering

$$M_x\rho_\nu(g) = \rho_\mu(g)M_x\,, \quad \forall g\in G$$

from the second Schur's lemma one has either $\mu\neq\nu$ and $M_x = 0$, or $\mu=\nu$ and $M_x = \lambda I$ (from the first Schur's lemma). This is the first step towards $\delta_{\mu\nu}$.

Consider the following matrices X_l^k indexed by k, l

$$[X_l^k]^i{}_j = \delta^k{}_j\delta^i{}_l$$

Therefore

$$[M_l^k]^m{}_n = \sum_g[\rho_\mu^\dagger(g)]^m{}_i[X_l^k]^i{}_j[\rho_\mu(g)]^j{}_n = \sum_g[\rho_\mu^\dagger(g)]^m{}_l[\rho_\mu(g)]^k{}_n = [\lambda_l^k]I_{d_\mu}$$

Taking the trace one gets

$$\mathrm{Tr}\big[M_l^k\big] = \sum_g[\rho_\mu^\dagger(g)]^m{}_l[\rho_\mu(g)]^k{}_m = \sum_g[\rho_\mu(g)\rho_{mu}^\dagger]^k{}_l = \sum_g I^k{}_l = |G|\delta_l^k \equiv \lambda_l^k d_\mu \implies \lambda_l^k = \frac{|G|}{d_\mu}\delta_l^k$$

from which the thesis follows. ☐

Example 7.24. Consider the cyclic group $\mathbb{Z}_2 = \{e,a\}$. One wants to construct all possible irreps using the theorem above. Starting from an irrep, one can build perpendicular irreps so independent. Starting from the trivial irrep

$$\rho(e) = \rho(a) = 1$$

One can write it as a vector $(1,1)$ which has only one orthogonal vector $(1,-1)$. So the cyclic group $\mathbb{Z}_2$ admits only two irreps.

Remark 7.25. Abelian groups have all irreps with dimension $d=1$. The theorem above is then

$$\frac{1}{|G|}\sum_g\rho_\mu^\dagger(g)\rho_\mu(g) = \delta_{\mu\nu}$$

Corollary 7.26. The number of inequivalent irreps of a finite group is

$$\sum_\mu d_\mu^2 \leq |G|$$

where the sum over μ is the sum over the irreps.

Proof. Consider the vector notation $\langle g|u,i,j\rangle$ which means the g-th component of the vector $|\mu,i,j\rangle$. This vector has two indices, so it is a matrix and as such has d_μ^2 components. The number of linearly independent vectors lass than $\dim V$ is $|G|$. ☐

Lecture 9

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Theorem 7.27 (B)**.** Irreducible representations are complete. It holds

$$\sum_\mu d_\mu^2 = |G| \, , \quad \frac{1}{|G|} \sum_\mu d_\mu \sum_{i,j=1}^{d_\mu} [\rho_\mu(g_1)]^i{}_j [\rho_\mu^\dagger(g_2)]^j{}_i = \delta_{g_1 g_2}$$

The argument of the second relation is the trace of $\rho\rho^\dagger$. The proof of the first relation can be found in the proof of Theorem 7.37.

Proposition 7.28 (Properties of characters)**.** A few properties of characters:

- the character χ labels equivalence classes of irreps; in fact, two irreps are equivalent $\rho \sim \rho'$ if it exists a matrix S such that $\rho' = S^{-1}\rho S$, therefore

$$\chi_{\rho'}(g) = \mathrm{Tr}[\rho'(g)] = \mathrm{Tr}\big[S^{-1}\rho(g)S\big] = \chi_\rho(g)$$

- The character χ also depends on conjugacy classes; two elements are conjugate if

$$g \sim g' \iff hgh^{-1} = g' \, , \quad \forall h$$

  then

$$\chi_\rho(g') = \mathrm{Tr}[\rho(g')] = \mathrm{Tr}\big[\rho(h)\rho(g)\rho(h^{-1})\big] = \mathrm{Tr}\big[\rho(h)\rho(g)\rho(h)^{-1}\big] = \chi_\rho(g)$$

- One defines

$$\chi_\mu(g) = \chi_\mu^i = \mathrm{Tr}[\rho_\mu(g)] \, , \quad g \in \zeta_i$$

  where ζ_i is a conjugacy class.

- The character of the identity is

$$\chi_\rho(e) = \mathrm{Tr}[\rho(e)] = \dim\rho = \dim V$$

  For an irrep one has $\chi_\mu(e) = d_\mu$.

Theorem 7.29 (C.1)**.** The characters of irreducible representations are orthonormal

$$\frac{1}{|G|} \sum_{i=1}^{n_c} (\chi_\mu^i)^* \chi_\nu^i n_i = \delta_{\mu\nu}$$

Proof. Consider Theorem 7.23

$$\frac{d_\mu}{|G|} \sum_g [\rho_\mu^\dagger(g)]^k{}_i [\rho_\nu(g)]^j{}_l = \delta_{\mu\nu}\delta_l^k\delta_i^j$$

Setting $i = k$ and $j = l$ then

$$\frac{1}{|G|} \sum_g \chi_\mu^*(g)\chi_\nu(G) = \delta_{\mu\nu}$$

The sum can be decomposed in a sum over classes

$$\sum_g = \sum_i \sum_{h \in \zeta_i}$$

where ζ_i is a class. However, the character χ is invariant of classes so

$$\sum_g = \sum_{i=1}^{n_c} \sum_{h \in \zeta_i} \implies \sum_g = \sum_{i=1}^{n_c} n_i$$

therefore the thesis. □

Lemma 7.30. It holds

$$\sum_{h\in\zeta_i}\rho_\mu(h)=\frac{n_i}{d_\mu}\chi^i_\mu I_{d_\mu}$$

Proof. Defining

$$A\equiv\sum_{h\in\zeta_i}\rho_\mu(h)$$

it follows

$$\rho_\mu(g)A\rho_\mu^{-1}(g)=\sum_{h\in\zeta_i}\rho_\mu(ghg^{-1})=A$$

From the first Schur's lemma one has $A=\lambda I$ and its trace is

$$\operatorname{Tr}A=\sum_{h\in\zeta_i}\operatorname{Tr}[\rho_\mu(h)]=\sum_h\chi^i_\mu=n_i\chi^i_\mu=\lambda\operatorname{Tr}I=\lambda d_\mu\implies\lambda=\frac{n_i}{d_\mu}\chi^i_\mu$$

from which follows the thesis. □

Theorem 7.31 (C.2)**.** The characters obey a completeness relation

$$\frac{n_i}{|G|}\sum_\mu\chi^i_\mu(\chi^j_\mu)^*=\delta^{ij}$$

Proof. Consider the second thesis of Theorem 7.27

$$\delta_{g_1g_2}=\frac{1}{|G|}\sum_\mu d_\mu\sum_{kl}[\rho(g_1)]^k{}_l[\rho^\dagger_\mu(g_2)]^l{}_k$$

Summing over all elements g_1, g_2 in two classes, one gets

$$\begin{aligned}n_i\delta^{ij}&=\frac{1}{|G|}\sum_\mu d_\mu\sum_{kl}\left[\sum_{g_1\in\zeta_i}[\rho(g_1)]^k{}_l\right]\left[\sum_{g_2\in\zeta_j}[\rho(g_2)]^l{}_k\right]\\&=\frac{1}{|G|}\sum_\mu d_\mu\sum_{kl}\left(\frac{n_i}{d_\mu}\chi^i_\mu\delta^k_l\right)\left(\frac{n_j}{d_\mu}(\chi^j_\mu)^*\delta^l_k\right)\\&=\frac{1}{|G|}\sum_\mu\sum_{kl}(n_i\chi^i_\mu\delta^k_l)\left(\frac{n_j}{d_\mu}(\chi^j_\mu)^*\delta^l_k\right)\\&=\frac{1}{|G|}\sum_\mu\frac{n_in_j}{d_\mu}(\chi^i_\mu)(\chi^j_\mu)^*d_\mu\end{aligned}$$

from which the thesis

$$\frac{1}{|G|}=\sum_\mu n_j(\chi^i_\mu)(\chi^j_\mu)^*=\delta^{ij}$$

□

Corollary 7.32. The number of inequivalent irreducible representations for any finite group is equal to the number n_c of conjugacy classes. The characters χ^i_μ can be put in an $n_c\times n_c$ matrix called character table. Sometimes they are normalized as

$$\chi^i_\mu\to\sqrt{\frac{n_i}{|G|}}\chi^i_\mu$$

Theorem 7.33. One can reduce a generic representation into irreducible representations

$$\rho(G)=\bigoplus_{\mu=1}^{n_c}a_\mu\rho_\mu(G)$$

then the number a_μ of times an irreducible representation ρ_μ occurs is

$$a_\mu=\sum_i\frac{(\chi^i_\mu)^*\chi^in_i}{|G|}$$

Proof. A generic representation for a finite group is similar to a unitary representation which is fully reducible, therefore it can be written as a direct sum. Since the number of irreducible representations is the number of conjugacy class, the sum runs from 1 to n_c.

Defining

$$\chi_i = \mathrm{Tr}[\rho(G)] = \sum_\mu a_\mu \chi^i_\mu$$

and using Theorem 7.29

$$\frac{1}{|G|} \sum_i n_i (\chi^i_\mu)^* \chi^i = \frac{1}{|G|} \sum_{i\nu} n_i (\chi^i_\mu)^* a_\nu \chi^i_\nu = \sum_\nu a_\nu \delta_{\mu\nu} = a_\nu$$

□

Theorem 7.34 (Irreducibility condition)**.** A necessary and sufficient condition for a representation ρ with characters χ_i to be irreducible is

$$\sum_{i=1}^{n_c} n_i |\chi_i|^2 = |G|$$

Proof. Using Theorem 7.29

$$\sum_i \frac{n_i}{|G|} (\chi^i)^* \chi^i = \sum_i \frac{n_i}{|G|} \sum_{\mu\nu} a_\mu (\chi^i_\mu)^* a_\nu \chi^i_\nu = \sum_\mu |a_\mu|^2$$

If ρ is an irreducible representation, then for one μ one has $a_\mu = 1$, while for all others $a_\mu = 0$. Conversely, if

$$\sum_i \frac{n_i}{|G|} |\chi_i|^2 = 1$$

then

$$\sum_\mu |a_\mu|^2 = 1$$

Since $a_\mu \in \mathbb{N}_0$, this is possible only if $a_\mu = 1$ for one μ and $a_\mu = 0$ for all others. □

7.3 Regular representations

Theorem 7.35. Every finite group admits a representation with $|G| \times |G|$ matrices called regular representation. For all $g_i, g_j \in G$ it holds

$$g_i g_j = g_k = g_m [\rho^{\mathrm{R}}(g_i)]^m{}_j$$

where

$$[\rho^{\mathrm{R}}(g_i)]^m{}_j = \begin{cases} 1\,, & m = k \\ 0\,, & m \neq k \end{cases}\,, \quad i, j, m = 1, 2, \ldots, |G|$$

Remark 7.36. One may notice:

- The matrices ρ^{R} have one 1 and all zeros in each row. Consider the cyclic group $\mathbb{Z}_3 = \{e, a, a^2\}$. The relations between the elements can be written as a linear system of equations

$$\begin{cases} ea = a \\ ea^2 = a^2 \\ ee = e \end{cases} \implies X \begin{bmatrix} e \\ a \\ a^2 \end{bmatrix} = \begin{bmatrix} e \\ a \\ a^2 \end{bmatrix} \implies X = \rho^{\mathrm{R}}(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  Consider now the relations with a:

$$X \begin{bmatrix} e \\ a \\ a^2 \end{bmatrix} = \begin{bmatrix} a \\ a^2 \\ e \end{bmatrix} \implies X = \rho^{\mathrm{R}}(a) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

  Note that $\rho^{\mathrm{R}}(a)$ permutes the elements.

- The matrices ρ^{R} are representations of $\mathrm{S}_{|G|}$ group elements. This is again Cayley's theorem.
- Since the matrices are permutations then

$$\mathrm{Tr}\big[\rho^{\mathrm{R}}(e)\big] = |G|\,, \quad \mathrm{Tr}\big[\rho^{\mathrm{R}}(g)\big] = 0\,, \quad g \neq e$$

Theorem 7.37. The regular representation contains every inequivalent irreducible representation d_μ times

$$\rho^{\mathrm{R}}(G) = \bigoplus_{\mu=1}^{n_c} d_\mu \rho_\mu(G)\,, \quad \sum_\mu d_\mu^2 = |G|$$

Proof. Using Theorem 7.33, since any representation can be written as sum of irreducible representations it holds

$$\rho^{\mathrm{R}}(G) = \bigoplus_{\mu=1}^{n_c} a_\mu \rho_\mu(G)\,, \quad a_\mu = \sum_i \frac{(\chi_\mu^i)^* \chi^i n_i}{|G|}\,, \quad \chi_i = \mathrm{Tr}\big[\rho^{\mathrm{R}}(G)\big]$$

Using the last property above one gets

$$\chi^i = \mathrm{Tr}\big[\rho^{\mathrm{R}}(G)\big] = \mathrm{Tr}\big[\rho^{\mathrm{R}}(e)\big] = \chi^0 = |G|$$

therefore

$$a_\mu = \frac{(\chi_\mu^0)^* \chi^0 n_0}{|G|} = (\chi_\mu^0)^* n_0 = (\chi_\mu^0)^* = \mathrm{Tr}[\rho_\mu(e)] = \mathrm{Tr}\big[I_{d_\mu}\big] = d_\mu$$

in the third equality one considers that the conjugacy class of the identity has only one element and that is the identity

$$geg^{-1} = e\,, \quad \forall g$$

so $n_0 = 1$. So every irreducible representation appears always d_μ times.

The second statement follows from

$$\mathrm{Tr}\big[\rho^{\mathrm{R}}(e)\big] = \sum_\mu d_\mu \,\mathrm{Tr}[\rho_\mu(e)] = \sum_\mu d_\mu^2 = |G|$$

the first equality is a consequence of the first thesis of the theorem. □

Lecture 10

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Example 7.38. One would like to construct all irreducible representations. The symmetric group S_3 has order $3! = 6$. One can count the irreducible representations. The trivial one always exists, $d = 1$ and $\rho_0 : G \to 1$. Knowing that abelian groups have only representations of dimension one, since S_3 is not abelian, it must have a representation with dimension greater than one.

The number of irreducible representations is equal to the number of conjugacy classes χ_μ^i. The symmetric group has a subgroup $\mathrm{A}_3 = \{e, (123), (132)\} \simeq \mathbb{Z}_3$ which is invariant: the effect of conjugation keeps all elements in the group [r]. The equivalence classes are

$$\{e\}\,, \quad \{(12), (13), (23)\}\,, \quad \{(123), (321)\}$$

One looks for the next irreducible representations ρ_1. Knowing that the quotient ${}^{\mathrm{S}_3}/_{\mathrm{A}_3} \simeq \mathbb{Z}_2$ induces a representation of the entire group, one gets the parity $\sigma(p) = \pm 1$, $p \in \mathrm{S}_3$.

The last irreducible representation can be built from orthogonality. Looking at the character table

	1	2	3
0	1	1	1
1	1	-1	1
2	2	0	-1

where the last row has been constructed considering the entries as vectors and finding the vector perpendicular to both first and second vector. In fact, one has seen that

$$\rho_2(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \,, \quad \rho_2(r) = \frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} \,, \quad \rho_2(\sigma) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

for r and σ in D_3, but $\mathrm{S}_3 \simeq \mathrm{D}_3$.

One can build the regular representation of the symmetric group S_3:

$$\rho^{\mathrm{R}}(\mathrm{S}_3) = \rho_0(\mathrm{S}_3) \oplus \rho_1(\mathrm{S}_3) \oplus 2\rho_2(\mathrm{S}_3)$$

Example 7.39. Consider the dihedral group D_2 with order $|\mathrm{D}_2| = 4$. Knowing the trivial representation to exist and that the group is abelian, it must have four irreducible representations of dimension one. The strategy is to find invariant subgroups, consider the quotient and induce the representations.

The subgroup $\{e, r, \sigma_1, \sigma_2\}$ is an invariant and has a subgroup $\mathbb{Z}_2 \simeq \{e, r\}$. The quotient is isomorphic to another $\mathbb{Z}_2$

$$\rho_1(e) = 1 \,, \quad \rho_1(r) = 1 \,, \quad \rho_1(\sigma_1) = -1 \,, \quad \rho_1(\sigma_2) = -1$$

The group has another subgroup $\{e, \sigma_1\} \simeq \mathbb{Z}_2$ whose quotient is another $\mathbb{Z}_2$. One can then build

$$\rho_2 \begin{pmatrix} e \\ r \\ \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

The character table is then

	e	r	σ_1	σ_2
0	1	1	1	1
1	1	1	-1	-1
2	1	-1	1	-1
3	1	-1	-1	1

[r] One has four conjugacy classes $\{e\}$, $\{r\}$, $\{\sigma_1\}$ and $\{\sigma_2\}$. In fact, one can note that $ghg^{-1} = h$ therefore every element is its own conjugacy class.

Example 7.40. Consider the cyclic group $\mathbb{Z}_n$. It is abelian, so one expects n one-dimensional irreps. Previously, a representation has been constructed as

$$\rho_1(a^k) = \mathrm{e}^{\frac{2\pi \mathrm{i}}{n} k} \implies \rho_\mu(a^k) = \mathrm{e}^{\frac{2\pi \mathrm{i}}{n} \mu k} \,, \quad \mu, k = 0, \dots, n-1$$

7.4 Representations of the symmetric group

The cycle decomposition is invariant under conjugation. This means that the classes are given by cycle structures.

Suppose to have ν_n n-cycles for $n \in \mathbb{N}$, then for each class of S_n, one has

$$n = \sum_{k=1}^{n} k\nu_k$$

[r]

Definition 7.41. A Young tableau of order k is a way of stacking k squares in columns ordered such that the $(i+1)$-th column is not longer than the i-th. [r] for S_4 remember that $2, 1, 1$ exists.

Theorem 7.42. The number of Young tableau is equal to the number of classes, which itself is equal to the number of irreps.

Example 7.43. The cyclic group appears in physics as a parity symmetry:

$$\mathcal{L} = \partial_\mu \phi \, \partial^\mu \phi + m\phi^2 + \lambda\phi^4 \,, \quad \phi \to -\phi$$

For a vector field, one has instead

$$m\boldsymbol{\phi}^\top \phi \implies \boldsymbol{\phi} \to \rho(g)\boldsymbol{\phi} \implies \rho(g) \in \mathrm{O}(n)$$

[r] In quantum field theory, the tame divergences, one discretizes space-time into a lattice: the continuous symmetries become finite group symmetries.

8 Continuous groups

Definition 8.1. A topological space is a set X where a topology is defined, which is a set U of subsets $U_i \subseteq X$ such that

- the empty set and X belong both to U,
- every union of U_i is a subset of U,
- every intersection of U_i is a subset of U.

Definition 8.2. The subsets U_i are called open neighbourhoods.

Remark 8.3. A topological space is the most general structure where continuity is defined.

Definition 8.4. A topological space has a Hausdorff topology if all pairs of elements $(x, y) \in X$ admit a disjoint neighbourhood. [r] disjoint?

Definition 8.5. A homeomorphism ϕ between two topological spaces A and B with Hausdorff topology is a continuous bijective (therefore invertible) map, with the inverse ϕ^{-1} also continuous. The spaces A and B are said to be homeomorphic.

Definition 8.6. A manifold M of dimension $N = \dim M$ is a space locally homeomorphic to $\mathbb{R}^n$: for every point $p \in M$ there exists an open neighbourhood U_p homeomorphic to an open neighbourhood of $\mathbb{R}^n$.

Definition 8.7. A chart (U, φ) is an open set $U \subseteq M$ with $\varphi : U \to V \subseteq \mathbb{R}^n$, $x \mapsto \{x^i\}$ an homeomorphism, where x^i are the coordinates on the chart.

Definition 8.8. An atlas is a set of charts $\{(U_\alpha, \varphi_\alpha\}$ such that

- the union of all charts is $\bigcup_\alpha U_\alpha = M$,
- the transition functions $f_{\beta\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1}$ are differentiable C^k, then the atlas is C^k.

Remark 8.9. The transition functions are functions $f_{\beta\alpha} : \mathbb{R}^n \to \mathbb{R}^n$ so the notion of derivative is well-defined. In an intersection $U_\alpha \cap U_\beta$ there are two charts, so two coordinate systems x and y. A change of coordinates is given by

$$y^i(p) = \varphi_\beta \circ \varphi_\alpha^{-1}(x^i(p)) = \varphi_\beta(\varphi_\alpha^{-1}(x^i(p))) = f_{\beta\alpha}(x^i(p))$$

Definition 8.10. A smooth manifold is a manifold M with an atlas C^p (typically C^∞).

Example 8.11. Consider the two-sphere S^2 embedded in $\mathbb{R}^3$: $x^2 + y^2 + z^2 = 1$. One can have two charts

$$U_\mathrm{S} = S^2 - (0, 0, 1) \,, \quad U_\mathrm{N} = S^2 - (0, 0, -1)$$

which are the sphere without one of the poles, with the homeomorphism $\varphi_{\mathrm{S,N}}$ a stereographic projection. The stereographic projection is singular at the poles, so one must remove one of them. The dimension of the manifold is 2 because

$$\varphi_\mathrm{S} : U_\mathrm{S} \to \mathbb{R}^2 \,, \quad \varphi_\mathrm{S} = \begin{cases} \frac{x}{1-z} \\ \frac{y}{1-z} \end{cases}$$

so a point $p \in S^2$ is given by the embedding (x, y, z) in $\mathbb{R}^3$. There is only one transition function $f_\mathrm{SN} = \varphi_\mathrm{S} \circ \varphi_\mathrm{N}^{-1}$ and it is C^∞, so one has a smooth manifold.

Definition 8.12. A scalar function $g : M \to \mathbb{R}$ is smooth, C^∞, if given a chart (U, φ) then $g \circ \varphi^{-1}$ is also smooth C^∞. This definition considers just one point.

Remark 8.13. The homeomorphism is $\varphi^{-1} : \mathbb{R}^n \to M$ and the function is $g : M \to \mathbb{R}$ therefore one has a notion of derivative through $f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}$.

[r]

Definition 8.14. Consider two smooth manifolds M_1 and M_2. A map $g : M_1 \to M_2$ smooth C^∞ at $p \in M_1$ if for a chart $(U, \varphi) \in M_1$ nd $(V, \psi) \in M_2$ then $\psi \circ g \circ \varphi^{-1}$ is smooth C^∞. If g is bijective (therefore invertible) and g, g^{-1} are smooth C^∞ everywhere, then g is a diffeomorphism.

Definition 8.15. A Lie group G is a smooth manifold C^∞ with composition $\varphi : G \times G \to G$ of class C^∞ and smooth inverse such that it holds

- closure $\forall x, y \in G$, $\exists! z \in G \mid z = \varphi(x, y)$,
- associativity $\forall x, y, z \in G$, $\varphi(x, \varphi(y, z)) = \varphi(\varphi(x, y), z)$,
- identity $\exists e \in G$, $\forall x \in G$ such that $\varphi(e, x) = \varphi(x, e) = x$,
- inverse $\forall x \in G$, $\exists x^{-1} \in G$ such that $\varphi(x, x^{-1}) = \varphi(x^{-1}, x) = e$.

Example 8.16. Examples of Lie groups already known are

$$\begin{aligned}
&\mathrm{GL}(n, \mathbb{R})\,, \quad \dim[\mathrm{GL}(n, \mathbb{R})] = n^2 \\
&\mathrm{SL}(n, \mathbb{R})\,, \quad \dim[\mathrm{SL}(n, \mathbb{R})] = n^2 - 1 \\
&\mathrm{O}(n)\,, \quad \dim \mathrm{O}(n) = \frac{n(n+1)}{2} \\
&\mathrm{SO}(n)\,, \quad \dim \mathrm{SO}(n) = \frac{n(n-1)}{2}
\end{aligned}$$

Lecture 11

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Definition 8.17. The center of a group is a subgroup of all commuting elements

$$Z(G) = \{a \in G \mid g^{-1} a g = a\,, \forall g \in G\}$$

It is an invariant abelian subgroup of G, so it admins a quotient ${}^{G}/_{Z(G)} \simeq H$.

Remark 8.18. Matrices that commute with unitary matrices are of the form cI.

Example 8.19. One can study the center of the unitary group $\mathrm{U}(n)$ for which $M^\dagger = M^{-1}$. Let $M = cI$, then

$$M^\dagger M = I \implies \det\big(M^\dagger M\big) = c^* c = |c|^2 = 1 \implies c = \mathrm{e}^{i\theta}$$

therefore, the center and the quotient group are

$$Z(\mathrm{U}(n)) = \mathrm{U}(1)\,, \quad {}^{\mathrm{U}(n)}/_{\mathrm{U}(1)} \simeq \mathrm{SU}(n)$$

Example 8.20. Consider $\mathrm{SU}(n)$ and take $M = cI$ since it is a subgroup of the above. Therefore

$$1 = \det M = c^n\,, \quad |c|^2 = 1 \implies c = \mathrm{e}^{\frac{2\pi \mathrm{i} k}{n}}$$

The center and the quotient are

$$Z(\mathrm{SU}(n)) = \mathbb{Z}_n\,, \quad {}^{\mathrm{SU}(n)}/_{\mathbb{Z}_n} \otimes \mathrm{U}(1) \simeq \mathrm{U}(n)$$

Definition 8.21. A curve γ passing through a point p is a smooth function

$$\gamma : I \to M\,, \quad t \mapsto \gamma(t)$$

where $\gamma(0) = p$, $I = [0, 1] \subset \mathbb{R}$ and M is manifold.

Remark 8.22. Considering a scalar function $g : M \to \mathbb{R}$, one can define the derivative of g along a curve γ. One evaluates g over γ

$$g_\gamma : I \subset \mathbb{R} \to \mathbb{R}\,, \quad t \mapsto g \circ \gamma(t) = g(\gamma(t))$$

Taking a chart $(U_\alpha, \varphi_\alpha) \in M$, one assigns the coordinates such that

$$x^\mu_\alpha(t) = \varphi_\alpha(\gamma(t))$$

and one has

$$g \circ \varphi_\alpha^{-1} \circ \varphi_\alpha \circ \gamma(t) = g(\gamma(t))$$

Definition 8.23. Given a curve γ passing through a point p, and a scalar function g, the tangent vecotr $t_p^{(\alpha)}$ in the chart $(U_\alpha, \varphi_\alpha)$ is

$$\mathrm{d}_t g_\gamma(t)\big|_{t=0} = \partial_{x^\mu_\alpha} g\, \mathrm{d}_t x^\mu_\alpha\big|_{t=0} = \partial_{x^\mu_\alpha} g\, b^\mu_\alpha \equiv t_p^{(\alpha)} g$$

Definition 8.24. Consider the space of curves γ passing through a point p

$$\Gamma_p = \{\gamma : \mathbb{R} \to M \mid \gamma(0) = p\}$$

Two curves are equivalent if the tangent vectors coincide for a given chart $(U_\alpha, \varphi_\alpha)$ to which p belongs:

$$\mathrm{d}_t(\varphi_\alpha \circ \gamma_1)\big|_{t=0} = \mathrm{d}_t(\varphi_\alpha \circ \gamma_2)\big|_{t=0} \implies \gamma_1 \sim \gamma_2$$

The equivalence class

$$T_p M = \{\gamma_1, \gamma_2 \in \Gamma_p \mid \gamma_1 \sim \gamma_2\}$$

is called tangent space. It is a vector space with dimension $\dim M = n$. [r]

Remark 8.25. The tangent vector $t_p^{(\alpha)}$ is a map from the algebra of smooth functions at p into itself

$$t_p^{(\alpha)} : C^\infty(U^\alpha) \to C^\infty(U^\alpha)\,, \quad g_\gamma \mapsto t_p^{(\alpha)} g_\gamma$$

Definition 8.26. A vector field v over a manifold M is a linear operator from the space of smooth functions $C^\infty(M)$ into itself, such that on each chart $(U_\alpha, \varphi_\alpha)$ is a directional derivative

$$v = b^\mu_\alpha =\, \partial_{x^\mu_\alpha}$$

Remark 8.27. A change of coordinates, that is a change of charts

$$(U_\alpha, \varphi_\alpha) \to (U_\beta, \varphi_\beta)\,, \quad p \in U_\alpha \cap U_\beta$$

is given by

$$\mathrm{d}_t g_\gamma(0) = \partial_{x^\mu_\alpha} g\, \mathrm{d}_t x^\mu_\alpha(0) = \partial_{x^\mu_\beta} g\, \mathrm{d}_t x^\mu_\beta(0)$$

and one can rewrite

$$b^\mu_\alpha = \mathrm{d}_t x^\mu_\alpha(0) = \partial_{x^\nu_\beta} x^\mu_\alpha\, \mathrm{d}_t x^\nu_\beta(0) = \partial_{x^\nu_\beta} x^\mu_\alpha\, b^\nu_\beta$$

A vector field transforms with the Jacobian.

Definition 8.28. The Lie bracket of two vector fields v and w is the commutator

$$[v, w](f) = v(w(f)) - w(v(f))$$

where one has

$$[v, w] = v^\mu \partial_\mu(w^\nu \partial_\nu) - w^\mu \partial_\mu(v^\nu \partial_\nu)$$

Remark 8.29. The Lie bracket is invariant under change of coordinates

$$u = [v, w]\,, \quad (U_\alpha, \varphi_\alpha) \to (U_\beta, \varphi_\beta)\,, \quad p \in U_\alpha \cap U_\beta$$

Given

$$u = u^\nu \partial_\nu = v^\mu \partial_\mu(w^\nu \partial_\nu) - w^\mu \partial_\mu(v^\nu \partial_\nu)\,, \quad u^\nu_\alpha = u^\sigma_\beta\, \partial_{x^\sigma_\beta} x^\nu_\alpha$$

one has

$$\begin{aligned} v^\mu_\alpha \partial_\mu w^\nu_\alpha &= v^\mu_\alpha \partial_\mu(w^\sigma_\beta\, \partial_{x^\sigma_\beta} x^\nu_\alpha) = v^\delta_\beta\, \partial_{x^\delta_\beta} x^\mu_\alpha\, \partial_{x^\mu_\alpha}(w^\sigma_\beta\, \partial_{x^\sigma_\beta} x^\nu_\alpha) \\ &= v^\delta_\beta\, \partial_{x^\delta_\beta}(w^\sigma_\beta)\, \partial_{x^\sigma_\beta} x^\nu_\alpha + v^\sigma_\beta w^\delta_\beta\, \partial_{x^\sigma_\beta x^\delta_\beta} x^\nu_\alpha \end{aligned}$$

doing the same for the second addendum, one finds that the double derivatives cancel.

Example 8.30. The orthogonal group $\mathrm{O}(n)$ is a Lie group so a manifold. To build the tangent space one needs a curve

$$O : \mathbb{R} \to \mathrm{O}(n)\,, \quad t \mapsto O(t)\,, \quad O(0) = I$$

Starting from the group defining relation

$$O(t)O^\top(t) = I$$

taking the derivative in the parameter and then evaluating at $t = 0$ one gets

$$0 = (\partial_t O)O^\top + O\,\partial_t O^\top = \partial_t O + \partial_t O^\top = A + A^\top \implies A = -A^\top$$

so the matrix A is anti-symmetric over $\mathbb{R}$. Therefore, the tangent space at the identity is

$$\mathrm{o}(n) \equiv T_e\mathrm{O}(n) = \{A \in \mathrm{M}(n, \mathbb{R}) \mid A + A^\top = 0\}$$

Example 8.31. The unitary group $\mathrm{U}(n)$ is a Lie group. Taking the curve $U(t)$ such that $U(0) = I$. From the group relation one has

$$U^\dagger U = I \implies 0 = \partial_t U^\dagger\, U + U^\dagger\, \partial_t U = \partial_t U^\dagger + \partial_t U$$

[r] The tangent space is composed of anti-hermitian matrices

$$\mathrm{u}(n) = T_e U(n) = \{A \in \mathrm{M}(n, \mathbb{C}) \mid A + A^\dagger = 0\}$$

Example 8.32. Consider $\mathrm{SU}(n)$ and take a curve $U(t)$ such that $U(0) = I$. Since this is a subgroup of $\mathrm{U}(n)$ then the tangent space is made of anti-hermitian matrices. At an infinitesimal time t one can write the curve as

$$U(t) = I + tA + o(t)$$

From the determinant one has

$$\det U(t) = 1\,, \quad \det U(t) = 1 + t\,\mathrm{Tr}(A) + o(t)$$

Using the derivative, one has

$$\partial_t \det U(0) = 0 \implies \mathrm{Tr}\, A = 0$$

Therefore the tangent space is

$$\mathrm{su}(n) = T_e\mathrm{SU}(n) = \{A \in \mathrm{u}(n) \mid \mathrm{Tr}(A) = 0\}$$

Example 8.33. The same can be done for $\mathrm{SO}(n)$ which has traceless anti-symmetric matrices.

Example 8.34. Considering $\mathrm{GL}(n, \mathbb{F})$ and curve $M(t)$ such that $M(0) = I$, one can expand

$$M(t) = I + tA + o(t)\,, \quad A \in \mathrm{M}(n, \mathbb{F})$$

and then the tangent space is

$$\mathrm{gl}(n, \mathbb{F}) = T_e\mathrm{GL}(n, \mathbb{F}) = \mathrm{M}(n, \mathbb{F})$$

Similarly for $\mathrm{SL}(n, \mathbb{F})$, one has

$$\mathrm{sl}(n, \mathbb{F}) = T_e\mathrm{SL}(n, \mathbb{F}) = \{A \in \mathrm{M}(n, \mathbb{F}) \mid \mathrm{Tr}\, A = 0\}$$

Remark 8.35. In all the cases above, the elements of the tangent space are not elements of the manifold.

Definition 8.36. A Lie algebra is a vector space $\mathcal{G}$ over a field $\mathbb{F}$ (typically $\mathbb{R}$ or $\mathbb{C}$) with a bracket (called Lie product) $[\cdot, \cdot] : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ such that

- linearity

$$[x, \alpha y + \beta z] = \alpha[x, y] + \beta[x, z]\,, \quad [\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z]\,, \quad \alpha, \beta \in \mathbb{F}\,, \quad x, y, z \in \mathcal{G}$$

- anti-symmetry

$$[x, y] = -[y, x]\,, \quad \forall x, y \in \mathcal{G}$$

- Jacobi identity

$$[x, [y, z] + [y, [z, x]] + [z, [x, y]] = 0\,, \quad \forall x, y, z \in \mathcal{G}$$

Example 8.37. A vector space $M(n, \mathbb{F})$ is a Lie algebra with

$$[M_1, M_2] = M_1 M_2 - M_2 M_1$$

Remark 8.38. Any associative algebra can be turned into a Lie algebra by imposing

$$[a_1, a_2] \equiv a_1 a_2 - a_2 a_1$$

Definition 8.39 (Structure constants)**.** Let $\mathcal{G}$ be a Lie algebra and $\{t_i\}$ a basis, that is $x = x^i t_i$ with $x^i \in \mathbb{F}$ for all $x \in \mathcal{G}$. The bracket of two elements is closed in $\mathcal{G}$

$$[t_i, t_j] = {c_{ij}}^k t_k$$

where ${c_{ij}}^k$ are called structure constants. Their properties are

- anti-symmetry ${c_{ij}}^k = -{c_{ji}}^k$,
- Jacobi identity

$${c_{ij}}^m {c_{km}}^l + {c_{ki}}^m {c_{jm}}^l + {c_{jk}}^m {c_{im}}^l = 0$$

- change of basis $t'_i = {S_i}^j t_j$

$${(c')_{ij}}^k = {S_i}^m {S_j}^n {c_{mn}}^l {(S^{-1})_l}^k$$

The first two follow from the properties of the Lie bracket.

Lecture 12

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[r] change notation of Lie algebra to Fraktur.

Definition 8.40. A morphism of Lie algebras is a mapping $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ such that

$$\phi([x, y]_{\mathcal{G}_1}) = [\phi(x), \phi(y)]_{\mathcal{G}_2}\,, \quad \forall x, y \in \mathcal{G}_1$$

Definition 8.41. An isomorphism of Lie algebras is an invertible morphism.

Definition 8.42. A representation of a Lie algebra $\mathcal{G}$ is a morphism $\rho : \mathcal{G} \to \mathrm{gl}(n, \mathbb{F})$. An injective representation ρ is called faithful.

Definition 8.43. The adjoint representation, ad, of a Lie algebra $\mathcal{G}$ is defined by

$$\mathrm{ad}_x(y) \equiv [x, y]\,, \quad \forall x, y \in \mathcal{G}$$

This representation has dimension $\dim \mathcal{G}$. The Jacobi identity guarantees that this is a representation.

Remark 8.44. Fixing a basis $x = x^i t_i$, from the regular representation one has

$$[t_i, t_j] = {c_{ij}}^k t_k$$

One can see that the structure constants give the adjoint representation

$$t_i \to {[T_i]_j}^k = -{c_{ij}}^k$$

Example 8.45. An example of a Lie algebra is $\mathbb{R}^3$ treated as a vector space with the outer product. Its properties are

- $\mathbf{v}_1 \wedge \mathbf{v}_2 = -\mathbf{v}_2 \wedge \mathbf{v}_1$
- $\mathbf{v}_1 \wedge (\mathbf{v}_2 \wedge \mathbf{v}_3) + \mathbf{v}_2 \wedge (\mathbf{v}_3 \wedge \mathbf{v}_1) + \mathbf{v}_3 \wedge (\mathbf{v}_1 \wedge \mathbf{v}_2) = 0$
- the basis satisfies $\mathbf{e}_i \wedge \mathbf{e}_j = {\varepsilon_{ij}}^k \mathbf{e}_k$, where ε is the Levi–Civita tensor, it is the structure constants of this Lie algebra.

Example 8.46. Consider the Pauli matrices σ_i. Defining

$$t_i = -\frac{\mathrm{i}}{2}\sigma_i \,, \quad \sigma_i\sigma_j = \delta_{ij} + \mathrm{i}{\varepsilon_{ij}}^k\sigma_k \,, \quad \sigma_i\sigma_j - \sigma_j\sigma_i = 2\mathrm{i}{\varepsilon_{ij}}^k\sigma_k$$

Then

$$[t_i, t_j] = {\varepsilon_{ij}}^k t_k$$

form an algebra. Note that the properties

$$\operatorname{Tr}\sigma_i = 0 \,, \quad \sigma_i + \sigma_i^\dagger = 0 \,, \quad d = 2$$

suggest that the Pauli matrices are a basis for su(2). Since ${(t_i)_j}^k$ for $i = 1, 2, 3$ and $j, k = 1, 2$ then the elements t_i are not the adjoint representation because it should have dimension 3.

Example 8.47. Given the Levi–Civita tensor, the elements L_i such that

$${[L_i]_j}^k = -{\varepsilon_{ij}}^k$$

form an algebra. In fact one has

$$[L_i, L_j] = {\varepsilon_{ij}}^k L_k \,, \quad i, j, k = 1, 2, 3$$

A generic vector $A = a^i L_i$ is such that

$$A^\top = a_i L_i^\top \implies {[L_i^\top]_j}^k = {[L_i]_k}^j = -{[L_i]_j}^k$$

So they are anti-symmetric matrices. Moreover

$${\varepsilon_{ij}}^k = 0 \implies \operatorname{Tr} A = 0$$

This is so(3) algebra and L_i is the adjoint representation because ${\varepsilon_{ij}}^k$ are structure constants.

Remark 8.48. The algebra su(2) has dimension $n^2 - 1 = 3$ and the algebra has dimension $\frac{1}{2}n(n-1) = 3$. There is an isomorphism between the two algebras. The two have the same structure constants.

8.1 Lie groups as groups of transformations

Consider the circle S^1. It is a manifold. Take two charts, the first

$$(U, \varphi) \,, \quad \varphi : S^1 \to \mathbb{R}^2 \,, \quad \rho \mapsto x^2 + y^2 = 1$$

the second

$$(U', \varphi') \,, \quad \varphi' : S^1 \to \mathbb{C} \,, \quad \rho \mapsto \mathrm{e}^{\mathrm{i}\theta}$$

A generic rotation $R(\theta)$ is a translation along the circle, so

$$R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$$

so rotations constitute a group. An infinitesimal shift is

$$R(\theta + \delta\theta) = R(\theta) + \delta\theta\, \partial_\theta R(0) + o(\delta\theta) \,, \quad J \equiv \partial_\theta R(0)$$

Setting $\theta = 0$ one gets

$$R(\delta\theta) = 1 + \delta\theta\, J + o(\delta\theta)$$

The expression above is also

$$R(\theta + \delta\theta) = R(\theta)R(\delta\theta) = R(\theta)[1 + \delta\theta\, J + o(\delta\theta)]$$

From these two relations one has

$$\mathrm{d}_\theta R(\theta)\big|_{\delta\theta=0} = JR(\theta)\,, \quad R(0) = 1 \implies R(\theta) = \mathrm{e}^{J\theta}$$

One obtains an exponential map from group properties together with smoothness conditions.

For the chart (U', φ'), a generic element of the circle is $\mathrm{e}^{\mathrm{i}\theta}$. A rotation along the circle is $\mathrm{e}^{\mathrm{i}\theta'}$: therefore $J = \mathrm{i}$ and the rotation group is U(1). For the chart (U, φ), a generic rotation

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

So, letting $\theta = 0$, one gets

$$J = \partial_\theta R(0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

To prove that $R(\theta) = \mathrm{e}^{J\theta}$ one can see that $J(-J) = I$ from which

$$R(\theta) = \mathrm{e}^{J\theta} = \sum_{n=0}^{\infty} \frac{(J\theta)^n}{n!} = I\cos\theta + J\sin\theta$$

note that reordering the terms in an infinite series is allowed only if the series absolutely converges (see Riemann series theorem).

Remark 8.49. One can construct SO(2) and U(1) from the same exponential map and J. Both groups are manifolds and are S^1 so they are isomorphic. Notice that these are groups, the example before involved Lie algebras.

Definition 8.50. The left and right translations over a group

$$Lg : h \mapsto Lg(h) = gh\,, \quad Rg : h \mapsto Rg(h) = hg$$

maps the group into itself. Consider some charts $(U_\alpha, \varphi_\alpha)$ and assign some coordinates α^μ to h, one has

$$\varphi_\alpha : G \to \mathbb{R}^d\,, \quad \alpha^\mu = [\varphi_\alpha(g)]^\mu\,, \quad d = \dim G$$

Consider a second set of charts (U_β, φ_β) and assign β^μ to g. The left translation is

$$Lg : \alpha^\mu \mapsto \varphi^\mu(\beta, \alpha)$$

Since the group is closed, then there is a set of charts $(U_\gamma, \varphi_\gamma)$ where

$$\gamma^\rho = \varphi^\rho(\beta, \alpha)$$

This is the group product gh but with coordinates.

Definition 8.51 (Generators of a Lie group)**.** Consider $g = e$ and consider a char (U, φ) such that $\beta^\mu = 0$. Performing a translation of h with g infinitesimally close to e, one has

$$\alpha^\mu \xrightarrow{Lg} \varphi^\mu(\delta\beta, \alpha) = \varphi^\mu(0, \alpha) + \partial_{\beta^\nu}\varphi^\mu\big|_{\beta=0}\delta\beta^\nu + o(\delta\beta^\nu) = \alpha^\mu + \partial_{\beta^\nu}\varphi^\mu\big|_{\beta=0}\delta\beta^\nu + o(\delta\beta^\nu)$$

Noting that

$$\partial_{\alpha^\nu}\alpha^\mu = \delta_\nu{}^\mu\,, \quad \partial_{\beta^\nu}\varphi^\mu = \partial_{\beta^\nu}\varphi^\rho\, \delta_\rho{}^\mu$$

one gets

$$\varphi^\mu(\delta\beta, \alpha) = \alpha^\mu + [\delta\beta^\nu\, \partial_{\beta^\nu}\varphi^\rho\, \partial_{\alpha^\rho}]\alpha^\mu + o(\delta\beta^\nu) = [1 + \delta\beta^\nu\, J_\nu(\alpha)]\alpha^\mu + o(\delta\beta^\nu)$$

Therefore

$$J_\nu(\alpha) = J^\rho{}_\nu\, \partial_{\alpha^\rho}\,, \quad J^\rho{}_\nu(\alpha) = \partial_{\beta^\nu}\varphi^\rho(\beta, \alpha)\big|_{\beta=0}$$

The generators J are vector fields over the group manifold, but are not elements of the group.

Example 8.52. Consider the general linear group $\mathrm{GL}(n, \mathbb{R})$ with

$$\varphi(\alpha, \lambda) \leftrightarrow N_i{}^k M_k{}^j$$

The generators are

$$J_{ij} = \partial_{M_{ij}}(NM)\big|_{M=0}$$

Setting $N = I$ one gets

$$[J_{kl}]_{ij} = \partial_{M_{ij}} M_{kl} = \delta_i{}^k \delta_j{}^l$$

One can see that $\det J = 0$ so $J \notin \mathrm{GL}(n, \mathbb{R})$, but $J \in \mathrm{gl}(n, \mathbb{R})$.

Definition 8.53. Let Lg be a left translation and v a vector field

$$v = v^\mu \, \partial_{\alpha^\mu}$$

The left translation of v is given by

$$\mathrm{d}Lg : v^\mu(\alpha) \mapsto w^\mu(\varphi(\beta, \alpha)) = \partial_{\alpha^\nu} \varphi^\mu(\beta, \alpha) \, v^\nu(\alpha)$$

Definition 8.54. The left-invariant vector fields are given by the relation

$$v^\mu(\varphi(\beta, \alpha)) = \partial_{\alpha^\nu} \varphi^\mu(\beta, \alpha) \, v^\nu(\alpha)$$

meaning $v^\mu = w^\mu$.

Theorem 8.55 (First Lie's)**.** The generators of a Lie group G are left and right invariant.

Proof. Starting from α^μ on the manifold, one uses β to translate to γ^μ. One considers an infinitesimal shift by ε along γ^μ to get to $\gamma^\mu + \delta\gamma^\mu$. This point can be reached from a different group element $\beta + \delta\beta$. So

$$\gamma^\mu = \varphi^\mu(\beta, \alpha)$$

Shifting both $\gamma \to \gamma + \delta\gamma$ and $\beta \to \beta + \delta\beta$ one gets

$$\varphi^\mu(\beta + \delta\beta, \alpha) = \gamma^\mu + \delta\gamma^\mu = \varphi^\mu(\varepsilon, \gamma) \,, \quad \beta^\mu + \delta\beta^\mu = \varphi^\mu(\varepsilon, \beta)$$

When ε is infinitesimally close to the identity, then $g = e$ from which

$$\begin{aligned}
\beta^\mu + \delta\beta^\mu &= \varphi^\mu(0, \beta) + \varepsilon^\nu \, \partial_{\varepsilon^\nu} \varphi^\mu(0, \beta) + o(\varepsilon) \geq \delta\beta^\mu = \varepsilon^\nu \, \partial_{\varepsilon^\nu} \varphi^\mu(0, \beta) + o(\varepsilon) \\
\gamma^\mu + \delta\gamma^\mu &= \varphi^\mu(0, \gamma) + \varepsilon^\nu \, \partial_{\varepsilon^\nu} \varphi^\mu(0, \gamma) + o(\varepsilon) \implies \delta\gamma^\mu = \varepsilon^\nu \, \partial_{\varepsilon^\nu} \varphi^\mu(0, \gamma) + o(\varepsilon)
\end{aligned}$$

These become

$$\delta\beta^\mu = J^\mu{}_\nu(\beta)\varepsilon^\nu \,, \quad \delta\gamma^\mu = J^\mu{}_\nu(\gamma)\varepsilon^\nu$$

From which (not rigorously follows)

$$\delta\gamma = \delta\beta J^{-1} J \implies J^\sigma{}_\nu(\beta) \, \partial_{\beta^\sigma} \gamma^\mu = J^\mu{}_\nu(\gamma)$$

and thus the generators are left invariant. In this way one can calculate the generators along the manifold. □

Lecture 13

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Theorem 8.56 (Second Lie's)**.** Given a Lie group G, the generators form a Lie algebra (which is a sub-algebra of the algebra of vector fields over M)

$$[J_\mu, J_\nu](\gamma) = c_{\mu\nu}{}^\rho J_\rho(\gamma)$$

Proof. Consider the definition

$$\psi = J^{-1}$$

and take

$$\partial_{\beta^\sigma} \gamma^\mu = J^\mu{}_\nu(\gamma) \psi^\nu{}_\sigma(\beta)$$

from the first Lie's theorem. Its second derivative is

$$\partial^2_{\beta^\rho \beta^\sigma} \gamma^\mu = \partial_{\beta^\rho}[J^\mu{}_\nu(\gamma)\psi^\nu{}_\sigma(\beta)] = \partial_{\beta^\sigma}[J^\mu{}_\nu(\gamma)\psi^\nu{}_\rho(\beta)]$$

The second equality comes from the symmetry of the second derivative. Therefore

$$\begin{aligned}
J^\mu{}_\nu(\gamma)[\partial_{\beta^\rho}\psi^\nu{}_\sigma(\beta) - \partial_{\beta^\sigma}\psi^\nu{}_\rho(\beta)] &= -[\partial_{\beta^\rho} J^\mu{}_\nu(\gamma)]\psi^\nu{}_\sigma(\beta) + [\partial_{\beta^\rho} J^\mu{}_\nu(\gamma)]\psi^\nu{}_\rho(\beta) \\
&= -[\partial_{\beta^\rho}\gamma^\lambda\, \partial_{\gamma^\lambda} J^\mu{}_\nu(\gamma)]\psi^\nu{}_\sigma(\beta) + [\partial_{\beta^\sigma}\gamma^\lambda\, \partial_{\gamma^\lambda} J^\mu{}_\nu(\gamma)]\psi^\nu{}_\rho(\beta) \\
&= -[J^\lambda{}_\alpha(\gamma)\psi^\alpha{}_\rho(\beta)\, \partial_{\gamma^\lambda} J^\mu{}_\nu(\gamma)]\psi^\nu{}_\sigma(\beta) \\
&\quad + [J^\lambda{}_\alpha(\gamma)\psi^\alpha{}_\sigma(\beta)\, \partial_{\gamma^\lambda} J^\mu{}_\nu(\gamma)]\psi^\nu{}_\rho(\beta) \\
&= \psi^\nu{}_\sigma(\beta)\psi^\alpha{}_\rho(\beta)[-(J^\lambda{}_\alpha\, \partial_{\gamma^\lambda} J^\mu{}_\nu) + (J^\lambda{}_\nu\, \partial_{\gamma^\lambda} J^\mu{}_\alpha)](\gamma) \\
&= \psi^\nu{}_\sigma(\beta)\psi^\alpha{}_\rho(\beta)[J_\alpha, J_\nu]
\end{aligned}$$

Multiply from the left by JJ to cancel $\psi\psi$ of β and multiplying by ψ to cancel $J(\gamma)$ one gets

$$\psi(\gamma)[J\,\partial J - J\,\partial J](\gamma) = J(\beta)J(\beta)[\partial\psi - \partial\psi](\beta)$$

There is a complete separation of coordinates. The only way this is satisfied is that the two expressions are the same constant. This means that the left-hand side is

$$c_{\mu\nu}{}^\rho = [J^{-1}]^\rho{}_\lambda[J^\sigma{}_\mu\, \partial_{\gamma^\sigma} J^\lambda{}_\nu - (\mu \leftrightarrow \nu)]$$

□

Remark 8.57. The generators J form a basis among left and right invariant vector fields. The theorem states that the structure constants do not depend on the point $p \in G$, but are global. The generators are defined everywhere on a group. The vector field from J is non singular because the Jacobian must exist. [r]

Definition 8.58 (Lie algebra of a Lie group)**.** The Lie algebra $\mathrm{Lie}(G)$ of a Lie group G is the tangent space T_eG considered as a vector space.

Proposition 8.59. Taking a left-invariant field to translate the tangent space T_eG to another point

$$v^\mu(\beta) = \partial_{\alpha^\nu}\varphi^\mu(\beta, 0)v^\nu(0) \equiv \bar{J}^\mu{}_\nu(\beta)v^\nu(0)$$

The generators $\bar{J}$ are the base for left-invariant fields, while J for right-invariant fields. The new vector space T_pG generated from the vector $v^\mu(\beta)$ with β^μ coordinates of p is defined everywhere because the Jacobian must exist and be non-singular. The Lie bracket

$$[v_1(\beta), v_2(\beta)]$$

is the same as the Lie bracket

$$[v_1(0), v_2(0)] \in T_eG$$

The vector spaces are different, but the Lie brackets are the same.

Example 8.60. The special orthogonal group SO(2) has elements

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

the function that couples two group elements is

$$\varphi(\alpha, \beta) = R(\alpha)R(\beta) = R(\alpha + \beta)$$

The generator is

$$J = \partial_\alpha \varphi(0, \beta)$$

Starting from the identity and T_eG one imposes $\beta = 0$ and gets

$$J = \partial_\alpha R(0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The algebra is one-dimensional because there is only one generator. The commutator is null $[J, J] = 0$ because the group is abelian.

For this group, one also has

$$J^\top + J = 0$$

Taking $O \in \mathrm{O}(2)$ then

$$(I + \delta\theta\, J)^\top (I + \delta\theta\, J) = I \iff I + \delta\theta(J^\top J) + o(\delta\theta) = I \implies 0 + o(\delta\theta) = 0$$

So the generator J is also the generator of O(2). [r]

Example 8.61. Consideri the Lie group $(\mathbb{R}, +)$. It is an abelian group, so the left and right translations are the same. A translation of $x \in G$ by $a \in G$ is

$$x \mapsto x + a = \varphi(\alpha, \beta)\,, \quad \alpha = x\,, \quad \beta = a$$

One can infer the generators from left-invariant vector fields. Consider a generic vector field

$$v = v^0(x)\,\partial_x$$

Imposing the left-invariant condition, one gets

$$v^0(\varphi(\beta, \alpha)) = \partial_{\alpha^0}\varphi^0(\beta, \alpha)v^0(\alpha) \implies v^0(x + a) = 1v^0(x) \implies v^0(x) = k = \text{const.}$$

So the vector field is

$$v = k\,\partial_x$$

Translations in one dimension admit a one-dimensional Lie algebra given by one generator: the momentum operator above.

Example 8.62. Consider $G = (R^+, \cdot)$ and $x, a \in G$. The translation is

$$x \mapsto ax = \varphi(a, x)$$

To build a left-invariant vector field one has

$$v^0(x + a) = \partial_x\varphi(a, x)v^0(x) = av^0(x) \implies v^0(x) = kx$$

the solution is a first-degree polynomial. Therefore the generator is

$$v = kx\,\partial_x$$

This group is a dilation over $\mathbb{R}^+$.

8.2 Exponential map

One has seen that generator $J_\mu \in \mathcal{G}$ is an element of the tangent space, but not of the group $J_\mu \notin G$. Close at the identity one has

$$(e + \varepsilon\alpha^\mu J_\mu) \in G$$

Taking

$$g = e + \frac{\varepsilon}{k}\alpha^\mu J_\mu$$

then $gg \in G$ and $ggg \in G$ etc. Also

$$\lim_{k\to\infty} g^k = \lim_{k\to\infty}\left(1 + \frac{\varepsilon}{k}\alpha^\mu J_\mu\right)^k = \mathrm{e}^{\varepsilon\alpha^\mu J_\mu} \in G$$

The mapping between the algebra and the group is valid for any matrix group because the exponential map admits a representation in terms of a Taylor expansion

$$M = \exp(m) = \sum_{n=0}^{\infty} \frac{m^n}{n!}$$

Proposition 8.63 (Properties)**.** A few properties

- the exponential map is smooth C^∞ and invertible,
- it holds
$$\mathrm{e}^{n+m} = \mathrm{e}^m \mathrm{e}^n \iff [m, n] = 0$$
and in general one uses the Baker–Campbell–Hausdorff formula
$$\mathrm{e}^m \mathrm{e}^n = \exp\left[m + n + \frac{1}{2}[m, n] + \frac{1}{12}([m, [m, n]] + [[m, n], n]) + \cdots\right]$$
The importance of m, n being in a Lie algebra, they are generators, then the commutators are given by the structure constants: the same structure repeats itself.
- For a change of basis S in a vector space then
$$\mathrm{e}^{S^{-1} m S} = S^{-1} \mathrm{e}^m S$$
- It holds
$$\det \mathrm{e}^m = \mathrm{e}^{\operatorname{Tr} m}$$
This implies that a group with elements that have determinants one has traceless generators in its algebra.

Theorem 8.64 (Taylor)**.** Given a Lie group G and its Lie algebra $\mathcal{G}$ of generators J_μ, and given a parametrization such that $\alpha^\mu(e) = 0$, then the elements $\alpha^\mu J_\mu \in \mathcal{G}$ define one-dimensional abelian subgroups
$$g = \mathrm{e}^{\lambda \alpha^\mu J_\mu}\,, \quad \lambda \in \mathbb{R}$$

Theorem 8.65. Every element of the group G connected with the identity e through a continuous curve belong to a single abelian subgroup as defined above.

Remark 8.66. From the generators and the exponential map one can infer if the group is compact or not.

Lecture 14

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Example 8.67. One can build an entire group through the exponential map. Consider $U \in \mathrm{SU}(2)$, $U^\dagger U = I$, $\det U = I$ and $u \in \mathrm{su}(2)$. Then
$$U = \mathrm{e}^u$$
It follows
$$I = \det U = \det \mathrm{e}^u = \mathrm{e}^{\operatorname{Tr} u} \implies \operatorname{Tr} u = 0$$
The unitary condition on U implies that u is anti-hermitian. A generic parametrization of 2×2 complex matrices is
$$u(\alpha_i) = \frac{\mathrm{i}}{2}(\alpha_0 I + \alpha_i \sigma^i)\,, \quad i = 1, 2, 3$$
where σ^i are the Pauli matrices. Taking the trace, one gets
$$\operatorname{Tr}[u(\alpha_i)] = \mathrm{i}\alpha_0$$
Imposing the trace, one gets $\alpha_i = 0$. One has a larger and a smaller algebra [r]. So
$$I(\alpha_i) = \exp\left[\frac{\mathrm{i}}{2}\alpha_0 I + \frac{\mathrm{i}}{2}\alpha_i \sigma^i\right] = \mathrm{e}^{\frac{\mathrm{i}}{2}\alpha_i} \mathrm{e}^{\frac{\mathrm{i}}{2}\alpha_i \sigma^i}$$
The first factor is a phase, so an element of U(1), while the second is an element of SU(2). [r] One can expand the second term to get
$$\mathrm{e}^{\frac{\mathrm{i}}{2}\alpha_i \sigma^i} = \left[I + \frac{\mathrm{i}}{2}\alpha_i \sigma^i - \frac{1}{8}\alpha_i \sigma^i \alpha_j \sigma^j + \cdots\right]$$

using the property $\sigma^i\sigma^j = \delta^{ij} + \mathrm{i}\varepsilon^{ij}{}_k\sigma^k$ one gets

$$\mathrm{e}^{\frac{\mathrm{i}}{2}\alpha_i\sigma^i} = \left[I + \frac{\mathrm{i}}{2}\alpha_i\sigma^i - \frac{1}{8}|\boldsymbol{\alpha}| - \frac{1}{8}\alpha_i\alpha_j\varepsilon^{ij}{}_k\sigma^k + \cdots\right]$$

Whenever one encounters at least two Pauli matrices, one can use the identity above to get either the identity or one matrix fewer. Therefore

$$\mathrm{e}^{\frac{\mathrm{i}}{2}\alpha_i\sigma^i} = \left[I\cos\frac{|\boldsymbol{\alpha}|}{2} + \mathrm{i}\hat{\alpha}_i\sigma^i\sin\frac{|\boldsymbol{\alpha}|}{2}\right] = U$$

This matrix can be easily calculated by a computer. This is the fundamental representation of SU(2). One can study the underlying manifold. From the unit vector $\hat{\alpha}_i$ one gets

$$\hat{\alpha}_1^2 + \hat{\alpha}_2^2 + \hat{\alpha}_3^2 = 1$$

that is the two-sphere S^2, which has dimension 2. The group SU(2) has dimension three, so one has to study $|\boldsymbol{\alpha}|$ also. When $|\boldsymbol{\alpha}| = 0$ then $U = I$. Similarly, when $|\boldsymbol{\alpha}| = 2\pi$ one has $U = -1$. When $|\boldsymbol{\alpha}| = 4\pi$ one has $U = I$. The parameter $|\boldsymbol{\alpha}|$ is an angle so the whole manifold is S^3. One can also see this by reparametrizing the parameter to see that it is another colatitude.

One can do the construction of the elements of SO(3). The algebra has anti-symmtric matrices. Consider

$$M \in \mathrm{SO}(3)\,,\quad M = \mathrm{e}^{\alpha_i L^i} = \exp\begin{bmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{bmatrix}$$

Using the property

$$[\alpha_i L^i]^3 = -|\boldsymbol{\alpha}|\hat{\alpha}_i L^i$$

one has

$$\begin{aligned} M &= I + |\boldsymbol{\alpha}|\hat{\alpha}_i L^i + \frac{1}{2}|\boldsymbol{\alpha}|^2(\hat{\alpha}_i L^i)^2 - \frac{1}{6}|\boldsymbol{\alpha}|^3\hat{\alpha}_i L^i + \cdots \\ &= I + \sin|\boldsymbol{\alpha}|\,\hat{\alpha}_i L^I + (1-\cos|\boldsymbol{\alpha}|)(\hat{\alpha}_i L^i)^2 \end{aligned}$$

The odd powers are rearranged into the sine, while the even powers are arranged into L^i, not the identity. The parameters $\hat{\alpha}_i$ define again a two-sphere S^2. When $|\boldsymbol{\alpha}| = 0$, then $M = I$. When $|\boldsymbol{\alpha}| = 2\pi$, then $M = I$. One has

$$M(\hat{\alpha}, |\boldsymbol{\alpha}| = \pi) = I + 2(\alpha^i L_i)^2 = M(-\hat{\alpha}, |\boldsymbol{\alpha}| = \pi)$$

One can identify anti-podal points. At $|\boldsymbol{\alpha}| = \pi$ one has ${S^2}/{\mathbb{Z}_2}$ and not S^2. The global manifold is ${S^3}/{\mathbb{Z}_2}$.

Therefore

$$\mathrm{SU}(2) \simeq S^3\,,\quad \mathrm{SO}(3) \simeq {S^3}/{\mathbb{Z}_2}\,,\quad \mathrm{SU}(3) \simeq {\mathrm{SU}(2)}/{\mathbb{Z}_2}$$

The algebra of both is built at the identity and it is the same. The exponential map gives different groups.

Example 8.68. Consider SU(3), the symmetry group of the strong force. Its generators are proportional to the Gell-Mann matrices λ_a

$$t_a = -\frac{\mathrm{i}}{2}\lambda_a\,,\quad \lambda_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}\,,\quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\,,\quad \lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}$$

also

$$\lambda_6 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_1 \end{pmatrix}\,,\quad \lambda_7 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_3 \end{pmatrix}\,,\quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Conventions for the generators t_a may vary. The generators are eight as it is the dimension of the group.

Example 8.69. One can build the generators for the pseudo-orthogonal group $\mathrm{O}(p,q)$, $\mathrm{SO}(p,q)$. The metric is

$$\eta_{\mu\nu} = \mathrm{diag}(-1,\dots,-1,1,\dots,1)$$

each p and q times. For $\Lambda \in \mathrm{O}(p,q)$ one has

$$\Lambda^\top \eta \Lambda = \eta$$

To find the generators one has to look at the Lie algebra which is the tangent space at the identity. So

$$(I+\lambda^\top)\eta(1+\lambda) = \eta\,, \quad \Lambda = \mathrm{e}^\lambda$$

Using $\eta = \eta^{-1}$ one has

$$\lambda^\top\eta + \eta\lambda = 0 \implies \eta^{-1}\lambda^\top = -\lambda\eta^{-1}$$

so $\lambda\eta^{-1}$ is anti-symmetric. To find the generators one can use this most generic anti-symmetric matrix

$$(J_{\rho\sigma})^\mu{}_\nu = (\delta_\rho{}^\mu\delta_\sigma{}^\lambda - \delta_\rho{}^\lambda\delta_\sigma{}^\mu)\eta_{\lambda\nu}$$

The indices $\rho\sigma$ parametrize the tangent space of the generators. The commutator of two generators defines the structure constants giving the algebra

$$[J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\rho}J_{\mu\sigma} - \eta_{\nu\sigma}J_{\mu\rho}$$

This is the algebra of the Lorentz group.

Example 8.70. The group $G = (\mathbb{R},+)$ of translations over $\mathbb{R}$ has generators $J = \partial_x$ which is an infinitesimal translation. The exponential map leads to a finite translation

$$\exp(aJ)f(x) = \exp(a\,\partial_x)f(x) = \left[1 + a\,\partial_x + \frac{1}{2}a^2\,\partial_x^2 + \cdots\right]f(x) = f(x+a)$$

In this way one can see how J generates momentum.

8.3 Global properties of Lie groups

Definition 8.71 (Connectedness)**.** A topological space is arc-wise connected if any two points on a manifold M can be connected by a continuous curve.

Definition 8.72. A Lie group G is simply connected if all loops drawn on the manifold G are contractible.

Definition 8.73 (Compactness)**.** If a manifold M can be embedded in a higher-dimensional flat space (like $\mathbb{R}^d$), then M is compact if there exists a ball of finite radius containing such manifold.

For groups of matrices, one can infer compactness if the range of parameters if finite.

Example 8.74. Consider $\mathrm{U}(n)$, $U^\dagger U = I$, then

$$(U_j{}^i)^* U^j{}_k = \delta^i{}_k \implies \delta^i{}_i = n = \sum_{ij} \left|U_j{}^i\right|^2$$

The group is compact because the sum is finite.

Example 8.75. The group $\mathrm{SU}(2) \simeq S^3$ is compact and simply connected. The group $\mathrm{SO}(3) \simeq {}^{S^3}\!/_{\mathbb{Z}_2}$ is compact but not simply connected; the algebras are isomorphic $\mathrm{su}(2) \simeq \mathrm{so}(3)$, they see the local structure, not the global properties.

Definition 8.76. Two groups G_1 and G_2 are locally isomorphic if there exists an isomorphic mapping $w : \mathcal{U}_{e_1} \to \mathcal{U}_{e_2}$ among open neighbourhoods of the identity.

Theorem 8.77 (Third Lie's)**.** All groups G_i whose algebras $\mathcal{G}_i$ are isomorphic, are locally isomorphic. Among all the groups, there is only one simply connected group G_0 called universal cover. Given an isomorphism $w_i : \mathcal{U}_{e_0} \to \mathcal{U}_{e_i}$, then the kernel

$$\ker w_i = D_i$$

is a discrete group and $G_i = {}^{G_0}\!/_{D_i}$.

Proof. Proof of the second statement. If both groups G_1 and G_2 are simply connected and locally isomorphic, then they are globally isomorphic. ☐

Example 8.78. Consider $\mathrm{SO}(3) \simeq {}^{\mathrm{SO}(2)}/_{\mathbb{Z}_2}$, then $\mathrm{SO}(3) = G_i$, $\mathrm{SO}(2)$ is the simply connected universal cover G_0, and $\mathbb{Z}_2 = D_i$.

Example 8.79. The group of translations $(\mathbb{R}, +)$ is simply connected because loops are just points. The group $(\mathbb{Z}, +)$ is an invariant subgroup also called $\mathbb{Z}$:

$$ {}^{(\mathbb{R},+)}/_{\mathbb{Z}} \simeq S^1 \simeq \mathrm{U}(1) $$

So a circle which is not simply connected.

Example 8.80. Consider the group $\mathrm{GL}(n, \mathbb{R})$. The set of determinants of these matrices is discontinuous because $\det M \neq 0$. The set is not connected. There are two disjoint sets. The identity is the identity matrix and determinant 1. Exponentiating the algebra, one only gets $\mathrm{SL}(n, \mathbb{R})$ (that is the set with $\det M > 0$), because it is connected with the identity. To recover the other half, one considers

$$ Mg\,, \quad g \in \mathrm{SL}(n, \mathbb{R})\,, \quad \det M < 0\,, \quad M^2 = I $$

Therefore the relation one obtains is

$$ {}^{\mathrm{GL}(n,\mathbb{R})}/_{\mathrm{SL}(n,\mathbb{R})} \simeq \mathbb{Z}_2 $$

not related to the previous theorem. This is similar to

$$ {}^{\mathrm{O}(n)}/_{\mathrm{SO}(n)} \simeq \mathbb{Z}_2 $$

Lemma 8.81. If $G_1 \simeq G_2$ then also the Lie algebras $\mathcal{G}_1 \simeq \mathcal{G}_2$ are isomorphic (but not viceversa).

Lecture 15

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Definition 8.82 (Haar measure)**.** The Haar measure is a left, or right, invariant integration measure

$$ \int_G \mathrm{d}\mu_R(g)\, f(gh) = \int_G \mathrm{d}\mu_R(gh)\, f(gh) = \int_G \mathrm{d}\mu_R(g)\, f(g) $$

Remark 8.83. If a group is abelian then left and right measure are the same $\mathrm{d}\mu_L = \mathrm{d}\mu_R$.

If a group is compact, then the volume is

$$ \mathrm{vol}(G) = \int_G \mathrm{d}\mu_R(g) < \infty\,, \quad \mathrm{d}\mu_R = \mathrm{d}\mu_L = \mathrm{d}\mu_H $$

The measure can be defined from a parametrization

$$ \mathrm{d}\mu_L(g) = \mu_L(g)\, \mathrm{d}^d\alpha = \mu_L(\varphi(\beta, \alpha))\, \mathrm{d}^d\varphi = \mu_L(\varphi)\, \det(\partial_\alpha \varphi)\, \mathrm{d}^d\alpha $$

Taking $g = e$ one defines

$$ \mathrm{d}\mu_L = [\det \partial_\alpha \varphi(\beta, 0)]^{-1}\, \mathrm{d}^d\alpha $$

One may notice that $\mu_L(\alpha = 0)$ are constants.

Example 8.84. Consider $\mathrm{SO}(2)$ which is parametrized by a single angle

$$ \varphi(\theta_1, \theta_2) = \theta_1 + \theta_2 $$

The jacobian and the measure are

$$ \partial_{\theta_1}\varphi = 1\,, \quad \mathrm{d}\mu_R = \mathrm{d}\mu_L(\theta) = \frac{\mathrm{d}\theta}{2\pi} $$

Example 8.85. Consider $\mathrm{GL}(n, \mathbb{R})$. One has

$$\varphi(M_1, M_2) = M_1 M_2\,, \quad \det \partial_M \varphi(M, N) = \det\bigl(\partial_{M_{ij}} M^{nl} N_{lm}\bigr) = \det(IN) = \det N$$

the measure is

$$\mathrm{d}\mu_L(M) = \frac{1}{\det M} \prod_{ij} \mathrm{d}M_{ij}$$

Example 8.86. Consider $\mathrm{SL}(n, \mathbb{R})$, $\det M = 1$, then

$$\mathrm{d}\mu_L(M) = \frac{1}{\det M} \prod_{ij} \mathrm{d}g_{ij}\, \delta(1 - \det M)$$

Theorem 8.87 (Orthogonality and completeness)**.** The following relations are called Peter–Weyl's. It holds

$$\int_G \mathrm{d}\mu_H(g)\, [\rho_\mu(g)]^i{}_j [\rho_\nu(g^{-1})]^k{}_l = \frac{1}{\mathrm{d}_\mu} \delta_{\mu\nu} \delta^k{}_l$$

Also

$$\int_G \mathrm{d}\mu_H(g)\, \chi_\mu(g) \chi_\nu^*(g) = \delta_{\mu\nu}$$

Similarly, given

$$\int_G \mathrm{d}\mu_H(g)\, f(g)\, \delta(g, h) = f(h)$$

then

$$\sum_\mu \mathrm{d}_\mu \operatorname{Tr}\bigl[\rho_\mu(g_1) \rho_\mu(g_2^{-1})\bigr] = \delta(g_1, g_2)$$

Likewise, given

$$\hat{\delta}(g_1, g_2) = \delta(h g_1 h^{-1}, h g_2 h^{-1})$$

one has

$$\sum_\mu \chi_\mu(g_1) \chi_\mu^*(g_2) = \hat{\delta}(g_1, g_2)$$

8.4 Structure of Lie algebras

Definition 8.88. A Lie subalgebra $\mathcal{H} \subseteq \mathcal{G}$ is a subspace which is closed under the Lie bracket, namely

$$[h_1, h_2] \in \mathcal{H}\,, \quad \forall h_1, h_2 \in \mathcal{H}$$

Definition 8.89. A Lie subspace $\mathcal{H} \subseteq \mathcal{G}$ is an invariant (also said ideal) if

$$[\mathcal{H}, \mathcal{G}] = \mathcal{H}$$

Definition 8.90. A Lie subseteq is the direct sum $\mathcal{G}_1 \oplus \mathcal{G}_2$ if it is a direct sum as a vector space $V_1 \oplus V_2$ and $[\mathcal{G}_1, \mathcal{G}_2] = 0$ (so $\mathcal{G}_1$ and $\mathcal{G}_2$ are ideals).

Example 8.91. Consider euclidean four-space's algebra so(4) (or equivalently Minkowski space after a Wick's rotation, that is with the imaginary time formalism). The generators obey

$$[L_{\mu\nu}, L_{\rho\sigma}] = \delta_{\mu\sigma} L_{\nu\rho}$$

[r] remembering that the two indices are motivated by $\Lambda = w^{\mu\nu} \lambda_{\mu\nu}$ where w is the infinitesimal parameter and λ is a generator. Inside there must be a so(3) subalgebra

$$M_i = \frac{1}{2} \varepsilon_{ijk} L^{jk}\,, \quad [M_i, M_j] = \varepsilon_{ij}{}^k M_k\,, \quad i, j, k = 1, 2, 3$$

The algebra has 6 dimensions

$$\dim \mathrm{so}(4) = \frac{n(n-1)}{2}$$

The boosts are

$$N_i = L_{i4}\,, \quad [N_i, N_j] = \varepsilon_{ijk} M^k$$

meaning that they do not form a subalgebra. Considering the change of basis

$$J_i = \frac{M_i + N_i}{2}\,, \quad K_i = \frac{M_i - N_i}{2}$$

one has

$$[J_i, J_j] = \varepsilon_{ijk} J^k\,, \quad [K_i, K_j] = \varepsilon_{ijk} K^k\,, \quad [J_i, K_j] = 0$$

In this way, one has found two ideals, both being so(3). Therefore

$$\mathrm{so}(4) \simeq \mathrm{so}(3) \oplus \mathrm{so}(3)$$

Theorem 8.92. Subalgebras exponentiate into subgroups. Ideals exponentiate into invariant subgroups.

Remark 8.93. Consider the algebra $\mathrm{u}(n)$. A matrix in this algebra is anti-hermitian $A^\dagger = -A$. Consider

$$B = aA\,, \quad a \in \mathbb{C}$$

one has

$$B^\dagger = a^* A^\dagger = -a^* A \neq -B$$

So the algebra $\mathrm{u}(n)$ is a real Lie algebra:

$$a \in \mathbb{R} \implies aA \in \mathrm{u}(n)$$

Definition 8.94. Given two vector spaces V_1 and V_2, their tensor product $V_1 \otimes V_2$ is defined from the Cartesian product $V_1 \times V_2$ up to an equivalence relation

$$(\lambda v_1) \otimes v_2 = v_1 \otimes (\lambda v_2) = \lambda(v_1 \otimes v_2)\,, \quad \forall v_1, v_2$$

Definition 8.95. Given a real vector space V_1, its complexification is $V \otimes \mathbb{C}$, that is

$$v \in V\,, \quad \lambda \in \mathbb{C} \implies \lambda v \in V \otimes \mathbb{C}$$

In this way one can complexify real Lie algebras.

Example 8.96. Consider $\mathrm{SL}(2, \mathbb{R})$. Its algebra $\mathrm{sl}(2, \mathbb{R})$ is made up of 2×2 traceless matrices over $\mathbb{R}$. The parametrization is

$$m(\alpha) = \begin{pmatrix} \alpha_0 & \alpha_1 \\ \alpha_2 & -\alpha_0 \end{pmatrix}$$

From this parametrization one can construct the generators as

$$L_i = \partial_{\alpha_i} m(\alpha)\,, \quad L_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \quad L_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,, \quad L_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The group can be recovered with the exponential map

$$M(\alpha) = \mathrm{e}^{\alpha_i L_i} \in \mathrm{SL}(2, \mathbb{R})\,, \quad \alpha_i \in \mathbb{R}$$

To complexify, one can introduce three new generators for a total of six

$$J_i = L_i\,, \quad J_k = \mathrm{i} L_{k-3}\,, \quad k = 4, 5, 6$$

One can either consider $\alpha_i J_i$ with $\alpha_i \in \mathbb{R}$ or $\alpha_i L_i$ with $\alpha_i \in \mathbb{C}$. Both ways give the Lie algebra $\mathrm{sl}(2, \mathbb{C}) \simeq \mathrm{sl}(2, \mathbb{R})_{\mathbb{C}}$.

One may note that

$$t_3 = \mathrm{i} L_0\,, \quad t_1 = \frac{\mathrm{i}}{2}(L_1 + L_2)\,, \quad t_2 = -\frac{\mathrm{i}}{2}(L_1 - L_2)$$

corresponding to the generators $t_i = \frac{\mathrm{i}}{2}\sigma_i$ of su(2). Therefore, one has found

$$\mathrm{sl}(2, \mathbb{R}) \simeq \mathrm{su}(2)\,, \quad \mathrm{sl}(2, \mathbb{R})_{\mathbb{C}} \simeq \mathrm{sl}(2, \mathbb{C}) \simeq \mathrm{su}(2)_{\mathbb{C}}$$

Definition 8.97. A simple algebra does not admit ideal. The simple algebra exponentiate into a simple group connected to the identity.

Definition 8.98. A semi-simple algebra does not admit abelian ideals (since the ideal are $\{0\}$ and $\mathcal{G}$ itself).

Definition 8.99. An abelian algebra has $[x, y] = 0$ for all $x, y \in \mathcal{G}$.

Definition 8.100. The derived algebra $\mathcal{G}^1$ or $D\mathcal{G}$ is a subalgebra of $\mathcal{G}$ whose elements are all commutators $[x, y]$ for all $x, y \in \mathcal{G}$

$$D\mathcal{G} = [\mathcal{G}, \mathcal{G}] = \{z \in \mathcal{G} \mid \exists x, y \in \mathcal{G}\,, z = [x, y]\}$$

The derived algebra is an ideal of $\mathcal{G}$ and if $\mathcal{G}$ is abelian, then $D\mathcal{G} = \mathbf{0}$.

Example 8.101. One has $D[\mathrm{su}(2)] = \mathrm{su}(2)$.
Consider the group ISO(2). A group element is

$$g(\theta, \mathbf{b}) = \begin{bmatrix} \cos\theta & \sin\theta & b_1 \\ -\sin\theta & \cos\theta & b_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

The generators are

$$x_R = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \quad x_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \quad x_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The commutators are

$$[x_1, x_2] = 0\,, \quad [x_R, x_1] = x_2\,, \quad [x_R, x_2] = -x_1$$

The subalgebra of translations is

$$t_2 = \{x_1, x_2\}$$

it is abelian and an ideal. Also

$$D[\mathrm{iso}(2)] = t_2$$

Its derived algebra contains only zero.

Definition 8.102. A Lie algebra is solvable if the k-th derived algebra $G^k = \{0\}$ for some $k \in \mathbb{N}$.

Definition 8.103 (Semi-direct sum)**.** Let $\mathcal{H}$ and $\mathcal{J}$ be two Lie algebras such that $\mathcal{J}$ admits a representation on $\mathcal{H}$, that is a homomorphism from $\mathcal{J}$ to the space of linear operators on $\mathcal{H}$

$$\forall s \in \mathcal{J}, \exists \hat{s} : \mathcal{H} \to \mathcal{H}\,, \quad h \mapsto \hat{\Delta}(h)$$

The semi-direct sum algebra $\mathcal{G} = \mathcal{J} +_s \mathcal{H}$ is $\mathcal{J} \oplus \mathcal{H}$ as a vector space and it is equipped with a Lie bracket:

$$\forall g \in \mathcal{G}\,, \quad \exists s \in \mathcal{J}, h \in \mathcal{H} \mid g = s + h$$

and

$$\forall g_1, g_2\,, \quad [g_1, g_2]_{\mathcal{G}} = [s_1 + h_1, s_2 + h_2] = [s_2, s_2]_{\mathcal{J}} + [h_1, h_2]_{\mathcal{H}} + \hat{s}_1(h_2) - \hat{s}_2(h_1)$$

The condition $g = s + h$ is important. The vector s and h are matrices that obey the previous vector linear combination, but also the Lie bracket must fulfilled: in particular the last two terms are the action of the algebra $\mathcal{J}$ on elements of the algebra $\mathcal{H}$.

Lecture 16

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Theorem 8.104 (Levi's decomposition)**.** Every complex Lie algebra $\mathcal{G}$ can be written as a vector space direct sum $\mathcal{G} = \mathcal{G}_{\text{ss}} \oplus \mathcal{G}_{\text{solv}}$ (or semi-direct sum $\mathcal{G} = \mathcal{G}_{\text{ss}} +_s \mathcal{G}_{\text{solv}}$) of a semi-simple part and a solvable part called radical. This is not a direct sum of algebras.

Corollary 8.105. If $\mathcal{G}$ is semi-simple, then it is a direct sum algebra

$$\mathcal{G} = \bigoplus_i^n \mathcal{G}_i$$

of simple $\mathcal{G}_i$.

Example 8.106. Consider

$$\text{iso}(2) = \text{so}(2) +_s t_2 \,, \quad \text{ISO}(2) = \text{SO}(2) \ltimes T_2$$

The algebra t_2 is an ideal because

$$[p_\mu, p_\nu] = 0 \,, \quad [p, J] \propto p$$

This implies that the algebra so(2) is not an ideal and is not solvable. So

$$[\text{iso}(2), t_2] = t_2 \,, \quad D(\text{iso}(2)) = t_2 \,, \quad D(t_2) = 0 \,, \quad D(D(\text{iso}(2))) = 0$$

The first commutation is between the generators which gives either zero or another generator of t_2. The algebras iso(2) and t_2 are both solvable.

Definition 8.107. A Cartan–Killing form of a real Lie algebra is defined as

$$K(x, y) = \text{Tr}(\text{ad}_x, \text{ad}_y)$$

Choosing a basis $[e_i, e_j] = {c_{ij}}^k e_k$, one may write a Killing form as

$$K_{ij} = {c_{il}}^m {c_{jm}}^l$$

Theorem 8.108 (Cartan)**.** An algebra $\mathcal{G}$ is semi-simple if $\det K \neq 0$.

Theorem 8.109. A Lie algebra is compact if it has negative semi-definite Killing form (i.e. non-positive eigenvalues).

Theorem 8.110. If a Killing form $K(x, y) = 0$, for all $x \in \mathcal{G}$ and for all $y \in D\mathcal{G}$, then $\mathcal{G}$ is solvable (as a matrix, it does not have maximal rank).

Example 8.111. Consider su(2), then

$$K_{ij} = {\varepsilon_{il}}^m {\varepsilon_{jm}}^l = -2\delta_{ij}$$

It is compact and semi-simple $\det K \neq 0$.

Example 8.112. Consider $\text{sl}(2, \mathbb{R})$, then

$$K = 2 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Its eigenvalues are 1 and -1 so it is not compact, but it is semi-simple $\det K \neq 0$.

Definition 8.113. For a semi-simple algebra $\mathcal{G}$ ($\det K \neq 0$), a Casimir operator is

$$c_2 \equiv K_{ij} t^i t^j$$

where t_i are the generators. A Casimir operator commutes with all generators.

8.5 Representations

Definition 8.114. A representation ρ of a Lie algebra is hermitian if the representation ρ is hermitian.

Theorem 8.115. Every finite-dimensional representation ρ of a compact Lie group is equivalent to a unitary representation.

Corollary 8.116. A unitary irreducible representation of a compact Lie group is finite-dimensional.

Theorem 8.117. A unitary representation of a non-compact group is infinite-dimensional.

Theorem 8.118. The Lie algebra $\mathcal{G}$ of a compact group G is also compact.

Example 8.119. One can study finite-dimensional irreducible representations of so(3)

$$[L_i, L_j] = \varepsilon_{ij}{}^k L_k$$

One can change the basis to

$$J_3 = \mathrm{i}L_3 \,, \quad J_\pm = \mathrm{i}(L_1 \pm \mathrm{i}L_2)$$

for which

$$[J_3, J_\pm] = \pm J_\pm \,, \quad [J_+, J_-] = 2J_3$$

One introduces a vector space and a vector $|m\rangle$ such that

$$J_3 \,|m\rangle = m\,|m\rangle \;, \quad m \in \mathbb{R}$$

It holds

$$J_3 J_+ \,|m\rangle = [J_3, J_+]\,|m\rangle + J_+ J_3 \,|m\rangle = (m+1) J_+ \,|m\rangle$$

So $J_+ \,|m\rangle$ is an eigenvector with eigenvalue $m+1$. Similarly for $J_- \,|m\rangle$ with eigenvalue $m-1$. One defines

$$|m \pm 1\rangle = J_\pm \,|m\rangle \;, \quad \langle m \pm 1 | m \pm 1\rangle = 1$$

and looks for finite-dimensional irreducible representations, so one has to stop after some lowering or raising. Taking

$$J_3 \,|j\rangle = j\,|j\rangle \;, \quad J_+ \,|j\rangle = 0$$

one uses the Killing form $K_{ij} = -2\delta_{ij}$ to compute the Casimir operator

$$J^2 = -\delta_{ij} L^i L^j = -L_1^2 - L_2^2 - L_3^2$$

for which

$$[J^2, L_i] = 0 \,, \quad [J^2, J_\pm] = 0$$

Using

$$J^2 = J_3^2 \pm J_3 + J_\mp J_\pm$$

to have

$$J^2 \,|j\rangle = j(j+1)\,|j\rangle$$

so $|j\rangle$ is also an eigenstate of J^2. Using the lowering operator, one has

$$J_-^{n+1} \,|j\rangle = 0$$

but the lowering operator also has to stop on a state

$$J_- \,|l\rangle = 0 \implies l \leq m \leq j$$

So one can write

$$J_3 = \sum_{m=l}^{j} m \,|m\rangle\langle m|$$

Taking the trace, one has

$$0 = \mathrm{Tr}(J_3) = l + (l+1) + \cdots + (j-1) + j$$

which is solved for $l = -j$. Therefore

$$0 = J_- \left|l\right\rangle = J_-(J_-^n \left|j\right\rangle) \implies J_-^n \left|j\right\rangle = \left|-j\right\rangle \implies n = 2j$$

which implies $j \in \frac{1}{2}\mathbb{N}_0$ and it is the angular momentum.

The irreducible representations of $\mathrm{so}(3) \simeq \mathrm{su}(2)$, $\mathrm{su}(2)_{\mathbb{C}} \simeq \mathrm{sl}(2,\mathbb{R})_{\mathbb{C}} \simeq \mathrm{sl}(2,\mathbb{C})$ are classified by the vectors in representation space

$$J^2 \left|jm\right\rangle = \left|jm\right\rangle j(j+1)\,, \quad J_3 \left|jm\right\rangle = \left|jm\right\rangle m\,, \quad J_\pm \left|jm\right\rangle = \left|jm\right\rangle [j(j+1) - m(m \pm 1)]^{\frac{1}{2}}$$

For $j = \frac{1}{2}$, $m = \pm\frac{1}{2}$ with dimension $2j+1 = 2$, one has

$$J_3 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\,, \quad J_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \implies J_i = \frac{\mathrm{i}}{2}\sigma_i$$

For $j = 1$, $m = -1, 0, 1$ with dimension 3, one has

$$J_3 = b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

These are the [r] L_i found for so(3).

In both cases, the representations are hermitian irreducible representations.

Definition 8.120. A set of fields $\phi^m(\mathbf{x})$ with $m = -j, \dots, j$ is an irreducible field of spin j if ϕ^m transform as

$$\varphi \xrightarrow{R} \varphi'\,, \quad \varphi'^m(\mathbf{x}) = [\rho_j(R)]^m{}_n \phi^n(R^{-1}\mathbf{x})$$

with ρ_j irreducible representations with index, spin j as square matrices of size $(2j+1)$.

Example 8.121. An example of spin 1 field is the electromagnetic field, while a spin $\frac{1}{2}$ field is the Pauli spinors field.

Definition 8.122. Given two representations ρ_1 on V_1 and ρ_2 on V_2, the tensor representations on $V_1 \otimes V_2$ is defined as

$$(\rho_1 \otimes \rho_2)(g)(v_1 \otimes v_2) \equiv [\rho_1(g)v_1] \otimes [\rho_2(g)v_2]$$

Example 8.123. Consider $\mathrm{SO}(n)$ and $O^i{}_j \in \mathrm{SO}(n)$ such that $O = \rho(g)$ on V acts as

$$v^i = O^i{}_j v^j\,, \quad v \in V$$

and considering $M \in V \otimes V$ such that $M^{ij} = v^i v^j$, then one has

$$M \xrightarrow{O} O^i{}_k v^k v^l O^j{}_l = OMO^\top\,, \quad (\rho \otimes \rho)(g)(v^i \otimes v^i) = OMO^\top$$

One notes the invariant subspace

$$(\rho \otimes \rho)(g)(I) = OIO^\top = I$$

this means that the representation is reducible.

One would like to find matrices left invariant. Taking the traceless matrix

$$T = M - \frac{\mathrm{Tr}\, M}{N} I\,, \quad \dim V = N$$

then

$$(\rho \otimes \rho)(g)(T) = OMO^\top - \frac{\mathrm{Tr}\, M}{N} I = OMO^\top - \frac{\mathrm{Tr}\big(OMO^\top\big)}{N} I$$

where one notes that

$$\operatorname{Tr}\bigl(OMO^\top\bigr) = \operatorname{Tr} M$$

This is another traceless matrix, the traceless matrices form an invariant subspace.

Remembering that one can decompose a matrix into its symmetric and anti-symmetric parts

$$T = \frac{T + T^\top}{2} + \frac{T - T^\top}{2} = T_+ + T_-$$

then $(\rho \otimes \rho)(T_+)$ is again a symmetric traceless matrix

$$(\rho \otimes \rho)(T_\pm) = \frac{OTO^\top \pm OT^\top O^\top}{2} = \frac{(OTO^\top) \pm (OTO^\top)^\top}{2}$$

Therefore $V \otimes V$ splits into I, traceless symmetric matrices and traceless anti-symmetric matrices.

Remark 8.124. The algebra $\mathrm{so}(n)$ is made of traceless anti-symmetric matrices which are the adjoint representation of $\mathrm{SO}(n)$

$$n \otimes n = 1 \oplus \frac{n(n-1)}{2} \oplus \frac{n(n+1)}{2} - 1$$

For $n = 3$ then

$$3 \otimes 3 = 1 \oplus 3 \oplus 5$$

Using the spin notation $2j + 1$ one has

$$(j = 1) \otimes (j = 1) = (j = 0) \oplus (j = 1) \oplus (j = 2)$$

Which is the usual rule in Quantum Mechanics to add momentum.

Lecture 17

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[r] clarification

Example 8.125. Consider $\mathrm{su}(n)$ in the fundamental representation ρ acting on V with $U = \rho(g) \in \mathrm{SU}(n)$ for which $U^i{}_j\, v^j$, $v^j \in \mathbb{C}$, $v \in \mathbb{C}^n$. On can define the anti-fundamental representation

$$\bar{U} = \bar{\rho}(g) = U^* \,, \quad (U^*)^i{}_j \bar{v}^j \,, \quad \bar{v} \in \bar{V}$$

where one has

$$U_1^* U_2^* = (U_1 U_2)^*$$

This is a representation

$$(\rho \otimes \bar{\rho})(g)(v^i \bar{v}^j) = U^i{}_k\, v^k v^l (U^*)^j{}_l = UM(U^*)^\top = UMU^\dagger$$

The identity is an invariant subspace. It holds $\operatorname{Tr}\bigl(UMU^\dagger\bigr) = \operatorname{Tr} M$ so one can decompose the matrices into traceless matrices. Therefore, one has

$$n \otimes \bar{n} = 1 \oplus (n^2 - 1)$$

The second addendum are matrices in the adjoint representation. Coupling the fundamental representation with itself, one finds

$$n \otimes n = \frac{1}{2} n(n-1) \oplus \frac{1}{2} n(n+1)$$

where the first addendum in the second member is in the anti-fundamental representation.

Example 8.126. A physical example is the following. Taking the lightest three quarks, the up, down and strange, one builds a lagrangian invariant under $\mathrm{SU}(3)$ flavor rotations. Taking the fundamental (quarks) and anti-fundamental (anti-quarks) representation one has

$$3 \otimes \bar{3} = 1 \oplus 8$$

And fundamental with itself

$$3 \otimes 3 = 6 \oplus \bar{3}$$

The singlet and octet have been observed experimentally.

With only two quarks, one has SU(2) and there is no different between fundamental and anti-fundamental. In fact one has

$$2 \otimes 2 = 1 \oplus 3$$

The triplet corresponds to the pions $\pi^{\pm}$ and π^0. In spin notation one has

$$\frac{1}{2} \oplus \frac{1}{2} = 0 \oplus 1$$

This is the isospin.

9 Minkowski groups

9.1 Lorentz group

The Minkowski space-time as a time-like metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. A group preserving the metric is $\mathrm{O}(3, 1)$ given by the matrices Λ which satisfy

$$\Lambda^\top \eta \Lambda = \eta \,, \quad \det \Lambda = \pm 1$$

The group is disconnected and split into $\pm \det \Lambda$. The group $\mathrm{SO}(3, 1)$, $\det \Lambda = 1$, is disconnected too. In fact

$$\eta_{00} = -1 = \Lambda^0{}_i \delta^i{}_j \Lambda^j{}_0 - (\Lambda^0{}_0)^2 \implies (\Lambda^0{}_0)^2 = 1 + \sum_i \Lambda^0{}_i \Lambda^0{}_i \geq 1 \implies \left|\Lambda^0{}_0\right| \geq 1$$

The restricted proper Lorentz group is

$$\mathrm{SO}^+(3, 1) = \{\Lambda \in \mathrm{SO}(3, 1) \mid \Lambda^0{}_0 \geq 1\}$$

This group is connected.

Remark 9.1. Time reversal transformations have $\Lambda^0{}_0 \leq -1$ but it is not a fundamental symmetry of nature. Parity transformations $x^0 \to x^0$, $x^i \to -x^i$ have $\det \Lambda = -1$, but parity is also broken in nature.

Theorem 9.2. There is a $2:1$ homomorphism between $\mathrm{SL}(2, \mathbb{C})$ and $\mathrm{SO}^+(3, 1)$. The universal covering group is $\mathrm{SL}(2, \mathbb{C})$.

Proof. Taking a Minkowski vector x^μ one can build a [r] vector

$$x = x_\mu \sigma^\mu \,, \quad \sigma^\mu = (I, \sigma^i) \,, \quad -\det x = -(x^0)^2 + |\mathbf{x}|^2$$

A Lorentz transformation Λ can be mapped into a complex 2×2 matrix A such that

$$x' = A x A^\dagger \,, \quad \det x' = \det x = |\det A|^2 \det x \implies |\det A|^2 = 1$$

The set of matrices A can be divided into two disjoint sets with $\det A = \pm 1$: two copies of $\mathrm{SL}(2, \mathbb{C})$. For each $\Lambda \in \mathrm{SO}^+(3, 1)$ one has $\pm A(\Lambda) \in \mathrm{SL}(2, \mathbb{C})$.

Since one obtains $\mathrm{SL}(2, \mathbb{C})$ which is connected, one infers that the map must preserve connectedness and as such the Lorentz transformation is only of the proper Lorentz group. ☐

Proposition 9.3. The algebra

$$\mathrm{so}(3, 1) = \{\lambda \in \mathrm{M}(4, \mathbb{R}) \mid \eta\lambda = -(\lambda\eta)^\top\}$$

has six generators

$$L_i = \begin{bmatrix} 0 & 0 \\ 0 & L_i^{\mathrm{so}(3)} \end{bmatrix} \,, \quad K_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \,, \quad K_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \,, \quad K_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Since $[L_i, K_i] = \varepsilon_{ijk} K^k$, one considers the complexification.

Lemma 9.4. It holds

$$\mathrm{so}(3,1)_{\mathbb{C}} \simeq \mathrm{sl}(2,\mathbb{C}) \oplus \mathrm{sl}(2,\mathbb{C})$$

Proof. One builds the combinations

$$L_i^{\pm} = \frac{1}{2}(L_i \pm \mathrm{i}K_i)\,, \quad [L_i^{\pm}, L_j^{\pm}] = \varepsilon_{ijk} L_k^{\pm}\,, \quad [L^{\pm}, L^{\mp}] = 0$$

Remembering that

$$\mathrm{su}(2)_{\mathbb{C}} \simeq \mathrm{sl}(2,\mathbb{C}) \simeq \mathrm{so}(3)_{\mathbb{C}}$$

a finite dimensional irreducible representation of the Lorentz group can be built from

$$\mathrm{su}(2) \otimes \mathrm{su}(2)$$

so built from two indices (u, v) with $v, u \in \frac{1}{2}\mathbb{N}$. The basis in representation space is $|kl\rangle$ with

$$k = -u, \dots, u\,, \quad l = -v, \dots, v$$

or use another basis $|jm\rangle$

$$j = |u - v|, \dots, v + u\,, \quad m = -j, \dots, j$$

$\square$

Example 9.5. Consider a four-vector x^μ. It requires a 4×4 matrix, therefore

$$(2v+1)(2u+1) = 4 \implies (v, u) = \left(\frac{1}{2}, \frac{1}{2}\right), \left(0, \frac{3}{2}\right), \left(\frac{3}{2}, 0\right)$$

Under SO(3) one has seen that

$$\left(\frac{1}{2}, \frac{1}{2}\right) = 0 \oplus 1$$

A singlet (the time component) and a triplet (the space components). So only this representation is the one of four-vectors. Physical fields transforming as vectors are vector bosons: photon A_μ, weak force $W_\mu^{\pm}$, Z_μ, and gluons.

Example 9.6. The representation

$$\left(0, \frac{1}{2}\right), \quad \left(\frac{1}{2}, 0\right)$$

are Weyl spinors with right and left chirality. Dirac spinors transform in

$$\left(0, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 0\right)$$

Example 9.7. The anti-self-dual two-forms are

$$(0, 1)\,, \quad (1, 0)$$

Any anti-symmetric rank-2 tensor decomposes in

$$(0, 1) \oplus (1, 0)$$

like the electromagnetic field tensor $F^{\mu\nu}$. Any symmetric rank-2 tensor transforms as $(1, 1)$ like the energy momentum tensor $T^{\mu\nu}$. Scalar fields transform as $(0, 0)$ like the Higgs field.

Remark 9.8. The finite-dimensional irreducible representations of the Lorentz group are relevant to classify possible fields in quantum field theory.

Previous theorems state that finite-dimensional irreducible representations of $\mathrm{SO}(3,1)$ are not unitary (because the group?[r] is not compact), so they are not physical and represent fields, not states.

9.2 Poincaré group

The Poincaré group is the Lorentz group with translations

$$\mathrm{ISO}(3,1) \simeq \mathrm{SO}^+(3,1) \ltimes T^{3,1}$$

Let a generic element be $g(\Lambda, b)$ where Λ is a Lorentz transformation and b is a translation. Let the generators of translations $T^{3,1}$ be P_μ

$$x'_\mu = x_\mu + b_\mu \,, \quad T(\delta b) = I + \delta b^\mu \, P_\mu \,, \quad [P_\mu, P_\nu] = 0$$

One possible representation is with ∂_μ. The commutator is zero because the group is abelian.

The generators of the Lorentz group $\mathrm{SO}^+(3,1)$ are $J_{\mu\nu}$

$$L_i = \varepsilon_{ijk} J^{jk} \,, \quad K_i = J_{i0} \,, \quad \Lambda(\delta\omega) = I + \delta\omega^{\mu\nu} \, J_{\mu\nu}$$

The Lie brackets are

$$[J_{\mu\nu}, J_{\lambda\sigma}] = J_{\lambda\nu}\eta_{\mu\sigma} - J_{\sigma\nu}\eta_{\mu\lambda} + J_{\mu\lambda}\eta_{\nu\sigma} - J_{\mu\sigma}\eta_{\nu\lambda}$$

The Lie algebra of the Poincaré group needs one additional commutator

$$[P_\mu, J_{\lambda\sigma}] = P_\lambda \eta_{\mu\sigma} - P_\sigma \eta_{\mu\lambda}$$

Unitary irreducible representations are infinite dimensional because the group? [r] is not compact, but are relevant to physics. Hilbert spaces transform under unitary transformations so the classification of representations is the classification of physical states.

In representation space, one introduces $|p\sigma\rangle$ such that

$$P_\mu \, |p\sigma\rangle = \mathrm{i} p_\mu \, |p\sigma\rangle$$

where σ labels all other degrees of freedom. Using

$$U(\Lambda, 0) = \exp(\omega_{\mu\nu} J^{\mu\nu})$$

one can show that

$$U(\Lambda^{-1}, 0) P^\mu U(\Lambda, 0) = \Lambda^\mu{}_\nu P^\nu \,, \quad U(\Lambda, 0) \, |p\sigma\rangle = |\Lambda p, \sigma\rangle$$

So U is a representation.

Definition 9.9. Given a group G, a representation ρ on V, the little group of a vector $v \in V$ is the subgroup of elements that leave v invariant

$$\mathrm{ISO}_G(v) = \{g \in G \mid \rho(g)v = v\}$$

Theorem 9.10. For every unitary irreducible representation of the little group, one can derive the corresponding induced representation of the Poincaré group by applying Lorentz transformations.

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Proof. Consider a vector q^μ such that

$$P_\mu \, |q, \sigma\rangle = \mathrm{i} q_\mu \, |q, \sigma\rangle$$

where σ labels every other degree of freedom. One can build $\mathrm{ISO}(q)$ from Λ such that a representation $U_{\sigma'\sigma}$ of ISO is

$$U(\Lambda) \, |q\sigma'\rangle = U_{\sigma'\sigma}(\Lambda) \, |q\sigma\rangle$$

Taking a Lorentz matrix Λ^p such that $p = \Lambda^p q$. This expression is not unique

$$\Lambda^p \Lambda q = \Lambda^p q$$

so one can fix it by choosing a specific Λ^p. Noting that $\widetilde{\Lambda} \in \mathrm{SO}^+(3,1)$ gives

$$\Lambda^{-1}_{\widetilde{\Lambda}^p}\widetilde{\Lambda}\Lambda^p q = \Lambda^{-1}_{\widetilde{\Lambda}^p}\widetilde{\Lambda}p = q \implies \Lambda^{-1}_{\widetilde{\Lambda}^p}\widetilde{\Lambda}\Lambda^p \in \mathrm{ISO}(q)$$

The action of the generic element $\widetilde{\Lambda}$ is

$$\begin{aligned} U(\widetilde{\Lambda})\,|p\sigma\rangle &= U(\widetilde{\Lambda})U(\Lambda^p)\,|q\sigma\rangle = U(\Lambda_{\widetilde{\Lambda}^p})U(\Lambda^{-1}_{\widetilde{\Lambda}^p})U(\widetilde{\Lambda})U(\Lambda^p)\,|q\sigma\rangle \\ &= U(\Lambda_{\widetilde{\Lambda}^p})U(\Lambda^{-1}_{\widetilde{\Lambda}^p}\widetilde{\Lambda}\Lambda^p)\,|q\sigma\rangle = U(\Lambda_{\widetilde{\Lambda}^p})U_{\sigma'\sigma}(\Lambda^{-1}_{\widetilde{\Lambda}^p}\widetilde{\Lambda}\Lambda^p)\,|q\sigma'\rangle \\ &= U_{\sigma\sigma'}(\Lambda^{-1}_{\widetilde{\Lambda}^p}\widetilde{\Lambda}\Lambda^p)\,|\widetilde{\Lambda}_p\sigma'\rangle \end{aligned}$$

One already knows how to represent translations from p_μ in $|p\sigma\rangle$ and how to represent a generic $\widetilde{\Lambda}$ from $U_{\sigma\sigma'}$, for example from the representation of the little group ISO.

Timelike. Using a spacelike metric, a timelike vector has

$$q^2 = q^\mu \eta_{\mu\nu} q^\nu = -(q^0)^2 + |\mathbf{q}|^2 < 0$$

One possible choice is the rest frame $q^\mu = (m, \mathbf{0})$ of a single particle of mass m. This vector is invariant under spatial rotations SO(3), so this group is the little group. The irreducible representations of SO(3) can be labeled by the spin j. A basis of irreducible representations for the Poincaré group is $|qj\rangle$. One then builds the one-particle irreducible states.

Separating the components of q^μ and writing explicit SO(3) internal dimension one has

$$|qj\rangle = |m, s, \mathbf{q}, \lambda\rangle$$

where m is the mass and s is the spin j. If $q^\mu = (m, \mathbf{0})$ then $|m, s, \mathbf{0}, \lambda\rangle$. Therefore

$$\begin{aligned} P_\mu P^\mu\,|m, s, \mathbf{0}, \lambda\rangle &= q^2\,|m, s, \mathbf{0}, \lambda\rangle = m^2\,|m, s, \mathbf{0}, \lambda\rangle \\ J^2\,|m, s, \mathbf{0}, \lambda\rangle &= s(s+1)\,|m, s, \mathbf{0}, \lambda\rangle \\ J_3\,|m, s, \mathbf{0}, \lambda\rangle &= \lambda\,|m, s, \mathbf{0}, \lambda\rangle \end{aligned}$$

where λ can only label internal SO(3) representation space. To generate all other vectors, one acts with a Lorentz matrix $\widetilde{\Lambda}$. One may note that $U_{\sigma\sigma'}(\Lambda^{-1}_{\widetilde{\Lambda}_p}\widetilde{\Lambda}\Lambda^p)$ depends on $\widetilde{\Lambda}$ and on p: for a fixed q one picked Λ^p such that $p = \Lambda^p q$. Therefore

$$U(\widetilde{\Lambda})\,|p\sigma\rangle = |\widetilde{\Lambda}p\sigma\rangle = U_{\sigma\sigma'}(\Lambda, p)\,|\widetilde{\Lambda}p\sigma'\rangle$$

Defining $p' = \widetilde{\Lambda}p$ and use a new notation

$$U(\widetilde{\Lambda})\,|m, s, \mathbf{p}, \lambda\rangle = U_{\lambda\lambda'}(\widetilde{\Lambda}, p)\,|m, s, \mathbf{p}', \lambda'\rangle$$

This specifies completely the action of $\widetilde{\Lambda} \in \mathrm{SO}^+(3,1)$ on the representation vector. The representation $U_{\lambda\lambda'}$ rotates the internal spin components. In general the state $|m, s, \mathbf{p}, \lambda\rangle$ is not an eigenstate of J_3. The generalized notion of intrinsic spin of moving particles $\mathbf{p} \neq 0$ is the helicity. The irreducible representation is labeled by (m, s) and additionally $\mathbf{p}$ and λ. The mass m is the eigenvalue of P^2 which is a Casimir operator $C_1 = -P_\mu P^\mu$ and p^i is the eigenvalue of P^i. There is another Casimir operator for which s and λ are eigenvalues

$$C_2 = W_\lambda W^\lambda$$

This is the Pauli–Lubanski vector (see the following definition and theorem). One can study the action of this vector upon the state considered in the rest frame $q^\mu = (m, \mathbf{0})$

$$W^\mu\,|m, s, \mathbf{0}, \lambda\rangle = \frac{1}{2}\varepsilon^{\mu\nu\lambda 0} J_{\nu\lambda} m\,|m, s, \mathbf{0}, \lambda\rangle$$

Also

$$W^0 = 0\,, \quad W^i \implies \frac{1}{2}\varepsilon^{ijk} J_{jk} = L_i$$

The second is the generator of so(3). Moreover

$$J^2 = L_i L^i = W_i W^i = W_\mu W^\mu$$

and s is its eigenvalue (see the following theorem on Poincaré group).

Lightlike. Consider a massless particle $q^2 = 0$

$$q_\mu = (\omega, 0, 0, \omega)$$

The Pauli–Lubanski is

$$W^\mu |0, s, \mathbf{q}, \lambda\rangle = \frac{1}{2}(\varepsilon^{\mu\nu\lambda 0} J_{\nu\lambda} + \varepsilon^{\mu\nu\lambda 3} J_{\nu\lambda})\omega |0, s, \mathbf{q}, \lambda\rangle$$

The first component is

$$W^0 |0, s, \mathbf{q}, \lambda\rangle = \omega J_{12} |0, s, \mathbf{q}, \lambda\rangle = \omega L_3 |0, s, \mathbf{q}, \lambda\rangle$$

similarly for the fourth, the second is

$$W^1 \to \frac{1}{2}(\varepsilon^{1230} J_{23} + \varepsilon^{1320} J_{32} - \varepsilon^{1023} J_{02} - \varepsilon^{1203} J_{20})\omega = (J_{23} + J_{20})\omega = \omega(L_1 + K_2)$$

the third is

$$W_2 \to \omega(J_{32} - J_{10}) = \omega(L_2 - K_1)$$

The commutators are

$$[W^1, W^2] = 0\,, \quad [W^1, L_3] = -\mathrm{i}W^2\,, \quad [W^2, L_3] = \mathrm{i}W_1$$

One has two translations and one rotation: this is the algebra of $\mathrm{ISO}(2) \simeq \mathrm{SO}(2) \ltimes \mathrm{T}^2$. Similar to the Poincaré group, the Casimir operator of T^2 is

$$P^2 = (W^1)^2 + (W^2)^2$$

with eigenvalues w.

The irreducible representations of SO(2) are unitary. Using $\mathrm{SO}(2) \simeq \mathrm{U}(1)$, then $\mathrm{e}^{\mathrm{i}\theta}$ is a one-dimensional irreducible representation of SO(2) and U(1). The group is abelian, so all the irreducible representations are one-dimensional. The representation $\mathrm{e}^{2\mathrm{i}\theta}$ is also an irreducible representation but has a different destination space. It wraps twice arounds the unitary circle, so two group elements have the same representation. Therefore only $\mathrm{e}^{\pm\mathrm{i}\theta}$ is faithful. The irreducible representations of SO(2) are of the form

$$\mathrm{e}^{\mathrm{i}m\theta}\,, \quad m \in \mathbb{Z}$$

The irreducible representations of ISO(2) are labeled by $|\omega\lambda\rangle$

$$[(W^1)^2 + (W^2)^2] |\omega\lambda\rangle = \omega |\omega\lambda\rangle\,, \quad W^3 |\omega\lambda\rangle = \lambda |\omega\lambda\rangle$$

In nature one observes photons with polarization $\lambda = \pm 1$ and no other Q-number, so $\omega = 0$. The most generic lightlike irreducible representation is

$$|0, s, \mathbf{q}, \omega, \lambda\rangle$$

The infinite continuous spin representation, with $\omega \neq 0$, is not observed, so only

$$|m = 0, s, \mathbf{q}, \lambda\rangle$$

is used in physics. The action of W^μ is

$$W^{0,3} |0, s, \mathbf{q}, \lambda\rangle = \lambda |0, s, \mathbf{q}, \lambda\rangle\,, \quad W^{1,2} |0, s, \mathbf{q}, \lambda\rangle = 0$$

□

Definition 9.11. The Pauli–Lubanski vector

$$W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma} J^{\nu\lambda} P^\sigma$$

It fulfills

$$W^\mu P_\mu = 0\,, \quad [W^\mu, P^\nu] = 0$$

Theorem 9.12. The independent components of the Pauli–Lubanski vector form a Lie algebra of the little group.

Proof. Consider

$$W^\mu P^\nu |p\sigma\rangle = p^\nu W^\mu |p\sigma\rangle = P^\nu W^\mu |p\sigma\rangle$$

Therefore $W^\mu |p\sigma\rangle$ is an eigenvector of P^μ with eigenvalue p^ν. Therefore

$$W^\mu |p\sigma\rangle \implies U_{\sigma\sigma'} |p\sigma'\rangle$$

and W belongs to the little group. One can also demonstrate that

$$[W^\mu, W^\nu] = \mathrm{i}\varepsilon^{\mu\nu\lambda\sigma} W_\lambda P_\sigma$$

For a fixed p^μ the operator P_σ is substituted by p_σ and the above becomes a closed algebra. □

Theorem 9.13. The irreducible unitary representations of the Poincaré group are characterized by the eigenvalues of the Casimir operators

$$-P^2\,, \quad -\frac{1}{p^2}W^2$$

9.2.1 Spinors

One can study why there are two spin indices.

Definition 9.14. The Clifford algebra is defined by the Dirac matrices γ_μ such that

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$$

These are represented by $n \times n$ matrices with $n \geq 4$.

Noting that

$$[\gamma_\mu, \gamma_\nu] \equiv 2\gamma_{\mu\nu}$$

is a representation of the Lorentz group

$$\rho(J_{\mu\nu}) = -\frac{1}{2}\gamma_{\mu\nu}$$

called spin representation. This representation ρ acts on the vector space V of spinor generic vectors ψ.

The representation of the group

$$\rho_s(\exp(\omega_{\mu\nu} J^{\mu\nu})) = \exp\left[-\frac{1}{2}\omega_{\mu\nu}\gamma^{\mu\nu}\right]$$

defines the spin group

$$\mathrm{Spin}(3,1) = \left\{\exp\left[-\frac{1}{2}\omega_{\mu\nu}\gamma^{\mu\nu}\right], \quad \omega_{\mu\nu} = -\omega_{\nu\mu}\right\}$$

The representation ρ is not irreducible: there exists an invariant subspace. Taking

$$\gamma_5 \equiv \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3\,, \quad \{\gamma_5, \gamma_\mu\} = 0\,, \quad \gamma_5^2 = I$$

one defines

$$P_{\mathrm{L,R}} = \frac{1 \mp \gamma_5}{2}\,, \quad \psi = P_\mathrm{L}\psi + P_\mathrm{R}\psi$$

Noting that

$$\gamma_5 \exp\left[-\frac{1}{2}\omega_{\mu\nu}\gamma^{\mu\nu}\right] = \exp\left[-\frac{1}{2}\omega_{\mu\nu}\gamma^{\mu\nu}\right]\gamma_5$$

it follows

$$P_{\mathrm{L}} \exp\left[-\frac{1}{2}\omega_{\mu\nu}\gamma^{\mu\nu}\right]\psi = \exp\left[-\frac{1}{2}\omega_{\mu\nu}\gamma^{\mu\nu}\right]P_{\mathrm{L}}\psi$$

The projectors $P_{\mathrm{L,R}}$ define two invariant subspaces called chiralities. One can block-diagonalize the representation

$$\rho_5(\cdots) = \begin{bmatrix} \mathrm{LL} & 0 \\ 0 & \mathrm{RR} \end{bmatrix}$$

One can choose an explicit representation of γ_μ such that

$$\gamma_5 = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$$

from which

$$\frac{1}{2}\omega_{\mu\nu}\gamma^{\mu\nu} = \begin{bmatrix} \omega_i^+\sigma_i & 0 \\ 0 & \omega_i^-\sigma_i \end{bmatrix}$$

where σ^i are the Pauli matrices and $\omega_i^\pm \in \mathbb{C}$. One has two copies of $\mathrm{su}(2)_{\mathbb{C}} \simeq \mathrm{sl}(2,\mathbb{C})$. The two copies' irreducible representations are labeled by two indices u, v: left chirality $(0, \frac{1}{2})$ and right chirality $(\frac{1}{2}, 0)$. These are Weyl spinors.

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10 Classification of semi-simple Lie algebras

A few motivations are the following.

- The Yang–Mills action is

  $$S = -\int \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$$

  where $F^{\mu\nu}$ is the field-strength tensor. The vector potential follows a local gauge U(1) symmetry. One can postulate a bigger symmetry like $\mathrm{SU}(n)$ for which

  $$F_{\mu\nu} = F^a_{\mu\nu}T_a$$

  where T_a are the generators. So the action becomes

  $$S = -\int \mathrm{Tr}\big(T^aT^b\big)F^a_{\mu\nu}F^{\mu\nu b} = -\int K^{ab}F^a_{\mu\nu}F^{\mu\nu b}$$

  where K is a negative definite Killing form with $\det K \neq 0$. This means that all degrees of freedom have kinetic terms. The condition on the determinant implies that the algebra is semi-simple.

- In general [r] $\det K \neq 0$ implies the existence of the inverse $K^{ab}K_{bc} = \delta^a{}_c$.

- The structure constants are defined from a commutator

  $$[T_a, T_b] = c_{ab}{}^cT_c$$

  They are anti-symmetry under exchange of ab. [r] One can then define

  $$c_{abc} = K_{ad}c_{bc}{}^d$$

  The left-hand side is fully anti-symmetric. In fact, for su(2) one has the Levi–Civita tensor ε_{ijk}.

- If the algebra is compact and semi-simple, one can start from a general Killing form and find a basis in which it is diagonal.

Definition 10.1. A sub-algebra $\mathcal{H} \subset \mathcal{G}$ is a Cartan sub-algebra if given the set of generators H_a of $\mathcal{H}$, with $a = 1, \dots, r$ and r is the rank of $\mathcal{G}$, is abelian

$$[H_a, H_b] = 0$$

diagonalizable and maximal (no larger abelian sub-algebra).

The dimension of the algebra is $d = \dim \mathcal{G}$ and it holds $r \leq d$.

Definition 10.2. Consider a linear combination of the Cartan generators $H = \lambda^a H_a$, with $a = 1, \dots, r$, and a generic element linear combination of linear generators $x = x^i T_i$, $i = 1, \dots, d$. From the commutator one defines the root ρ as

$$[H, x] = \lambda^a x^i [H_a, T_i] = \lambda^a x^i {c_{ai}}^j T_j \equiv \rho x^j T_j = \rho x \implies (\lambda^a {c_{ai}}^j - \rho {\delta_i}^j) = 0$$

The root is the eigenvalue of the adjoint representation

$$\mathrm{ad}_H(T_i) = [H, T_i] = \lambda^a {c_{ai}}^j T_j$$

The secular equation is

$$\det(\mathrm{ad}_H - \rho I) = 0$$

Theorem 10.3. There are $d = \dim \mathcal{G}$ roots. The root $\rho = 0$ has multiplicity r. There are $d - r$ non-zero roots.

Proof. Let $H = \lambda^a H_a$ and E_α be the solution to the secular equation. Then

$$[H, E_\alpha] = \rho_\alpha E_\alpha$$

The indices are $a = 1, \dots, r$, $\alpha = r + 1, \dots, d$. Also

$$[H, H_b] = 0$$

One can see that

$$\begin{aligned}
[H, [H_b, E_\alpha]] &= [H, H_b E_\alpha] - [H, E_\alpha H_b] = H H_b E_\alpha - H_b E_\alpha H - H E_\alpha H_b + H H_b E_\alpha \\
&= H_b H E_\alpha - H_b E_\alpha H - H E_\alpha H_b + H_b H E_\alpha \\
&= H_b [H, E_\alpha] - [H, E_\alpha] H_b = H_b \rho_\alpha E_\alpha - \rho_\alpha E_\alpha H_b = \rho_\alpha [H_b, E_\alpha]
\end{aligned}$$

This last commutator has the same eigenvalue of E_α. From the algebra, one knows that

$$[H_b, E_\alpha] = {c_{b\alpha}}^i T_i$$

If $i = 1, \dots, r$ then the commutator is zero. Therefore, one only has

$$[H_b, E_\alpha] = {c_{b\alpha}}^\beta T_\beta = \rho_{\alpha b} E_\alpha \,, \quad \beta = r + 1, \dots, d$$

Therefore, there are $d - r$ non-zero roots and they are vectors in $\mathbb{C}^r$. Also one has

$$\rho_{\alpha b} \delta_\alpha^\beta = {c_{b\alpha}}^\beta$$

□

Lemma 10.4. If ρ_α and ρ_β are roots then

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}$$

Proof. Using the Jacobi identity

$$\begin{aligned}
0 &= [H, [E_\alpha, E_\beta]] + [E_\beta, [H, E_\alpha]] + [E_\alpha, [E_\beta, H]] \\
&= [H, [E_\alpha, E_\beta]] + \rho_\alpha [E_\beta, E_\alpha] - \rho_\beta [E_\alpha, E_\beta] \\
[H, [E_\alpha, E_\beta]] &= (\rho_\alpha + \rho_\beta) [E_\alpha, E_\beta]
\end{aligned}$$

where the second commutator is $E_{\alpha+\beta} = [E_\alpha, E_\beta]$. □

Lemma 10.5. If $\rho_\alpha + \rho_\beta \neq 0$, then E_α and E_β are orthogonal with respect to the Killing form.

Proof. Knowing that

$$K([x,y],z) = K(x,[y,z])$$

it follows

$$K([E_\alpha, H], E_\beta) = K(E_\alpha, [H, E_\beta] \implies -\rho_\alpha K(E_\alpha, E_\beta) = \rho_\beta K(E_\alpha, E_\beta)$$

from which the thesis. □

Lemma 10.6. If ρ_α is a root, then $-\rho_\alpha$ is a root and $[E_\alpha, E_{-\alpha}] \in \mathcal{H}$ is an element of the Cartan sub-algebra.

Proof. If $-\rho_\alpha$ were not a root, the Killing from would be degenerate, but one is considering only semi-simple algebras, so this cannot be. Therefore

$$[H, E_{-\alpha}] = -\rho_\alpha E_{-\alpha}$$

Considering the Jacobi identity

$$\begin{aligned} [H,[E_\alpha,E_{-\alpha}]] &= -[E_{-\alpha},[H,E_\alpha]] - [E_\alpha,[E_{-\alpha},H]] \\ &= -\rho_\alpha[E_{-\alpha},E_\alpha] - (-1)(-\rho_{alhpa})[E_\alpha,E_{-\alpha}] \\ &= 0 \end{aligned}$$

Therefore

$$[E_\alpha, E_{-\alpha}] = {c_{\alpha,-\alpha}}^a H_a$$

This means that

$${c_{\alpha,-\alpha}}^\beta = 0\,, \quad \beta = r+1,\dots,d$$

□

10.1 Reduction of the Killing form

The Killing form is a matrix

$$K_{ij} = \begin{bmatrix} ab & a\beta \\ \alpha b & \alpha\beta \end{bmatrix}\,, \quad a,b = 1,\dots,r\,, \quad \alpha,\beta = r+1,\dots,d$$

One would like to study the form of the matrix knowing only that the algebra is semi-simple. Only the upper left block depends on the algebra, while the rest is fixed. [r]

$$[H_a, H_b] = 0 = {c_{\alpha\beta}}^i T_i\,, \quad {c_{ab}}^i = 0\,, \quad \forall i$$

also

$${c_{b\alpha}}^\beta = \rho_{\alpha b}\delta_\alpha^\beta$$

where ${c_{b\alpha}}^i$ is null for $i = a$ and non-zero for $i = \beta$. Examining the upper left corner of the Killing form, one sees that

$$K_{ab} = {c_{ai}}^j {c_{bj}}^i = {c_{a\beta}}^j {c_{bj}}^\beta = \rho_{\beta a}{\delta_\beta}^\alpha \rho_{\alpha b}{\delta_\alpha}^\beta = \sum_\alpha \rho_{\alpha a}\rho_{\alpha b}$$

Studying the lower right corner one has

$$K_{\alpha\beta} = {c_{\alpha i}}^j {c_{\beta j}}^i = {c_{\alpha a}}^j {c_{\beta j}}^a + {c_{\alpha\beta'}}^j {c_{\beta j}}^{\beta'}$$

Knowing that

$$[E_\alpha, E_\beta] = {c_{\alpha\beta}}^i T_i = N_{\alpha\beta}E_{\alpha+\beta}$$

then, if $i = \alpha + \beta$ one has

$${c_{\alpha\beta}}^i \neq 0$$

Also

$$[H_b, E_\alpha] = c_{b\alpha}{}^i T_i = \rho_{\alpha b} E_\alpha \implies c_{b\alpha}{}^i \neq 0 \iff i = \alpha$$

Likewise

$$[E_\alpha, E_{-\alpha}] = c_{\alpha,-\alpha}{}^a H_a$$

Therefore, the first addendum is

$$c_{\alpha a}{}^j c_{\beta j}{}^a = c_{\alpha a}{}^\alpha c_{\beta\alpha}{}^a \neq 0 \impliedby \beta = -\alpha$$

The second addendum is

$$c_{\alpha\beta'}{}^j c_{\beta j}{}^{\beta'} = c_{\alpha\beta'}{}^{\alpha+\beta'} c_{\beta,\alpha+\beta'}{}^{\beta'} \to c_{\beta,\alpha+\beta'}{}^{\beta'} \neq 0 \impliedby \beta = -\alpha$$

Therefore

$$K_{\alpha\beta} \propto \delta_{\beta,-\alpha}$$

One can choose a basis where

$$K_{\alpha,-\alpha} = 1$$

The remaining block is

$$K_{a\beta} = c_{ai}{}^j c_{\beta j}{}^i = c_{ab}{}^j c_{\beta j}{}^b + c_{a\alpha}{}^j c_{\beta j}{}^\alpha = 0$$

[r] Therefore, the Killing form is

$$K_{ij} = \begin{bmatrix} K_{ab} & 0 \\ 0 & M \end{bmatrix}, \quad M = \begin{bmatrix} A & & & \\ & A & & \\ & & A & \\ & & & \ddots \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Remark 10.7. One may note that

$$\begin{aligned} c_{\alpha,-\alpha}{}^a &= K^{ai} c_{\alpha,-\alpha,i} = K^{ab} c_{\alpha,-\alpha,b} = K^{ab} c_{b,\alpha,-\alpha} = K^{ab} c_{b,\alpha}{}^\mu K_{\mu,-\alpha} \\ &= K^{ab} c_{b,\alpha}{}^\mu \delta_{\mu,-(-\alpha)} = c_{a\alpha}{}^\alpha = \rho_{\alpha a} \end{aligned}$$

at the second equality one uses $K^{a\beta} = 0$.

Recap. It holds

$$[H_a, H_b] = 0\,, \quad [H_a, E_\alpha] = \rho_{\alpha a} E_\alpha\,, \quad [E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}$$

likewise

$$[E_\alpha, E_{-\alpha}] = \rho_{\alpha a} H_a\,, \quad K_{ij} = \begin{bmatrix} K_{ab} & 0 \\ 0 & M \end{bmatrix}$$

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Example 10.8. Consider su(2). Its three generators are given by

$$(L_i)_{jk} = -\varepsilon_{ijk}$$

Taking

$$L_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solving the eigenvalue problem, one obtains

$$\det(L_3 - \rho I) = 0 \implies \rho = 0\,, \quad \rho = \pm\mathrm{i}$$

One expects a number of zero roots equal to the rank $r = 1$. [r] By the following theorem, one has

$$H = -\mathrm{i}L_3\,, \quad L_\pm = (L_1 \pm \mathrm{i}L_2)\,, \quad L_+ \leftrightarrow E_\alpha\,, \quad L_- \leftrightarrow E_{-\alpha}$$

from which

$$[H, L_\pm] = \pm L_\pm \,, \quad [L_+, L_-] = H$$

The Killing matrix is

$$K_{ij} = \begin{bmatrix} K_{HH} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \,, \quad K_{HH} = c_{H,+}{}^{+} c_{H,+}{}^{-} + c_{H,-}{}^{+} c_{H,-}{}^{-} = 2$$

Therefore $\rho = \pm 1$.

Theorem 10.9. It is possible to obtain real roots $\rho_\alpha \in \mathbb{R}^r$.

Remark 10.10. The problem is reduced to the study of the Cartan sub-algebra. [r] The roots obey a few properties that allow one to classify all semi-simple Lie algebras.

Theorem 10.11. The [r] is not singular

$$\det K_{ab} \neq 0$$

the generators H_a are a base of the Cartan sub-algebra. One can define an inner product

$$K^{ab} K_{bc} = \delta^a{}_c \,, \quad \boldsymbol{\rho}_\alpha \cdot \boldsymbol{\rho}_\beta = \rho_{\alpha a} K^{ab} \rho_{\beta b}$$

In this way one can utilize the properties of vectors.

Theorem 10.12. If $\boldsymbol{\rho}_\alpha$ is a root, then $n\boldsymbol{\rho}_\alpha$ is not a root. Along the direction of any root, the only parallel roots [r] are $(0, \boldsymbol{\rho}_\alpha, -\boldsymbol{\rho}_\alpha)$.

Proof. From

$$[E_\alpha, E_\beta] \propto E_{\alpha+\beta}$$

one has, for $n = 2$,

$$[E_\alpha, E_\alpha] = 0$$

□

Theorem 10.13. If $\boldsymbol{\rho}_\alpha$ and $\boldsymbol{\rho}_\beta$ are roots, then

$$2 \frac{\boldsymbol{\rho}_\alpha \cdot \boldsymbol{\rho}_\beta}{|\boldsymbol{\rho}_\alpha|^2} \in \mathbb{Z}$$

The scalar product of two roots is quantized.

Proof. .[r] $\mathcal{G}$. One identifies su(2) inside the algebra. One defines

$$J_3^{(\alpha)} \equiv \frac{\rho_{\alpha a} H_a}{|\boldsymbol{\rho}_\alpha|^2} \,, \quad J_\pm^{(\alpha)} \equiv \sqrt{2} \frac{E_{\pm,\alpha}}{|\boldsymbol{\rho}_\alpha|}$$

Therefore

$$[J_3^{(\alpha)}, J_\pm^{(\alpha)}] = \sqrt{2} \frac{\rho_{\alpha a}}{|\boldsymbol{\rho}_\alpha|^2} [H_a, E_{\pm,\alpha}] = \sqrt{2} \frac{\rho_{\alpha a}}{|\boldsymbol{\rho}_\alpha|^3} (\pm \rho_{\alpha a}) E_{\pm,\alpha} = \pm J_\pm^{(\alpha)}$$

also

$$[J_+^{(\alpha)}, J_-^{(\alpha)}] = \frac{2}{|\boldsymbol{\rho}_\alpha|^2} [E_\alpha, E_{-\alpha}] = 2 J_3^{(\alpha)}$$

Taking another root and its eigenvector E_β, $\beta \neq \alpha$. Then

$$[J_3^{(\alpha)}, E_\beta] = \frac{\rho_{\alpha a}}{|\boldsymbol{\rho}_\alpha|^2} [H_a, E_\beta] = \frac{\boldsymbol{\rho}_\alpha \cdot \boldsymbol{\rho}_\beta}{|\boldsymbol{\rho}_\alpha|^2} E_\beta$$

This is again an eigenvalue equation. Noting that

$$J_3 \left|jm\right\rangle = m \left|jm\right\rangle$$

then the quantity $m = -j, \dots, j$ is quantized $m \in \frac{1}{2}\mathbb{Z}$. The thesis follows.

Applying the lowering and raising operators, one has

$$[J_{\pm}^{(\alpha)}, E_\beta] = \frac{\sqrt{2}}{|\boldsymbol{\rho}_\alpha|^2} N_{\pm\alpha,\beta} E_{\beta\pm\alpha}$$

Starting from E_β and applying the operators n times, one builds a chain of operators. Since the algebra is finite the chain is finite: the final indices obey a closure condition. □

Theorem 10.14. The elements in the sequence

$$E_{\beta-m\alpha}, \cdots, E_\beta, \cdots, E_{\beta+n\alpha}$$

are all roots if $m+n+1 \leq 4$. It also holds

$$2\frac{\boldsymbol{\rho}_\alpha \cdot \boldsymbol{\rho}_\beta}{|\boldsymbol{\rho}_\alpha|^2} = m-n \implies \frac{\boldsymbol{\rho}_\alpha \cdot \boldsymbol{\rho}_\beta}{|\boldsymbol{\rho}_\alpha|^2} \leq \frac{3}{2}$$

The inner representation has maximum spin of $\frac{3}{2}$.

Theorem 10.15. If ρ_α and ρ_β are roots, then

$$\boldsymbol{\rho}_\gamma = \boldsymbol{\rho}_\beta - 2\frac{\boldsymbol{\rho}_\alpha \cdot \boldsymbol{\rho}_\beta}{|\boldsymbol{\rho}_\alpha|^2}\boldsymbol{\rho}_\alpha$$

is also a root. The above is a reflection [r].

Definition 10.16. The finite group generated by reflections $\boldsymbol{\rho}_\gamma$ is called Weyl group.

Theorem 10.17. It holds

$$\cos^2\theta_{\alpha\beta} = \frac{\boldsymbol{\rho}_\alpha \cdot \boldsymbol{\rho}_\beta}{|\boldsymbol{\rho}_\alpha|^2}\frac{\boldsymbol{\rho}_\alpha \cdot \boldsymbol{\rho}_\beta}{|\boldsymbol{\rho}_\beta|^2} = \frac{(\boldsymbol{\rho}_\alpha \cdot \boldsymbol{\rho}_\beta)^2}{|\boldsymbol{\rho}_\alpha|^2|\boldsymbol{\rho}_\beta|^2} \in \frac{1}{4}\mathbb{Z}$$

The angles are then quantized. In particular

$$\theta_{\alpha\beta} \in \left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}\right\} \mod \frac{\pi}{2}$$

Theorem 10.18. The length ratios

$$l_{\alpha\beta} = \frac{|\boldsymbol{\rho}_\alpha|}{|\boldsymbol{\rho}_\beta|}$$

are quantized by the previous theorem.

Proof. For $\theta_{\alpha\beta} = 0, \pi$ then $\cos^2\theta_{\alpha\beta} = 0$ [r]

$$l_{\alpha,-\alpha} = 1$$

For $\theta_{\alpha\beta} = \frac{\pi}{6}, \frac{5}{6}\pi$ then

$$\cos^2\theta_{\alpha\beta} = \frac{3}{4} = \frac{\boldsymbol{\rho}_\alpha \cdot \boldsymbol{\rho}_\beta}{|\boldsymbol{\rho}_\alpha|^2}\frac{\boldsymbol{\rho}_\alpha \cdot \boldsymbol{\rho}_\beta}{|\boldsymbol{\rho}_\beta|^2} = j_1 j_2$$

since each factor is quantized. One can search every product to get the cosine

$$\frac{3}{4} = \frac{1}{2}\frac{3}{2} = \frac{3}{2}\frac{1}{2}$$

Therefore

$$l_{\alpha\beta} = 3^{\mp\frac{1}{2}}$$

For $\theta_{\alpha\beta} = \frac{\pi}{4}, \frac{3}{4}\pi$ then

$$\cos^2\theta_{\alpha\beta} = \frac{1}{2} = 1\frac{1}{2} = \frac{1}{2}1 \implies l_{\alpha\beta} = 2^{\mp\frac{1}{2}}$$

For $\theta_{\alpha\beta} = \frac{\pi}{3}, \frac{2}{3}\pi$ then

$$\cos^2\theta_{\alpha\beta} = \frac{1}{4} \implies l_{\alpha\beta} = 1$$

For $\theta_{\alpha\beta} = \frac{\pi}{2}$ then

$$l_{\alpha\beta} = 0$$

□

10.2 Classification

10.2.1 Rank 1

Consider $d = \dim \mathcal{G}$ and $r = 1$. One expects $d - r$ non-zero roots. Therefore, the index a is only 1, so $\rho_{\alpha a} \in \mathbb{R}$. The roots are ρ_α and $-\rho_\alpha$ so $d - r = 2$ and $d = 3$. Rank 1 semi-simple Lie algebras have only dimension 3: they are all isomorphic

$$\mathrm{su}(2) \simeq \mathrm{so}(3) \simeq \mathrm{sp}(2)$$

Symplectic group. A symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ is defined by matrices M such that

$$M^\top \Omega M = \Omega \,, \quad \Omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

where Ω is a symplectic form. Therefore

$$\Omega = -\Omega^\top \,, \quad \Omega^2 = I \,, \quad \Omega^\top \Omega = I$$

Letting

$$M = \mathrm{e}^s = I + s + \cdots \implies (I + s)^\top \Omega (I + s) = \Omega \implies s^\top \Omega = -\Omega s$$

One has

$$\Omega s = -s^\top \Omega = s^\top \Omega^\top = (\Omega s)^\top$$

Also

$$\Omega s = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} s = \begin{bmatrix} s_{21} & s_{22} \\ -s_{11} & -s_{12} \end{bmatrix} \equiv \begin{bmatrix} s_{21}^\top & -s_{11}^\top \\ s_{22}^\top & -s_{12}^\top \end{bmatrix}$$

Therefore

$$s_{21} = s_{21}^\top \,, \quad s_{12} = s_{12}^\top$$

these are $\frac{n}{2}(n+1)$ conditions, likewise

$$s_{11} = s_{22}^\top$$

has n^2 conditions. The dimension of the algebra is

$$\dim[\mathrm{sp}(2n)] = n^2 + 2\frac{n}{2}(n+1) = (2n+1)n$$

10.2.2 Rank 2

In the plane $\mathbb{R}^2$ one can exploit the quantization of the angles. Let the quantization be $\theta_{\alpha\beta} = \frac{\pi}{2}$, then from $(0, \rho_\alpha, -\rho_\alpha)$ one can obtain another root ρ_β and $-\rho_\beta$. Therefore $d - r = 4$ from which $d = 6$. So by analyzing the dimensions of the algebras $\mathrm{su}(n)$, $\mathrm{so}(n)$ and $\mathrm{sp}(n)$, one finds that this algebra is

$$\mathrm{so}(4) \simeq \mathrm{su}(2) \oplus \mathrm{su}(2)$$

Special unitary algebra. Considering $\theta_{\alpha\beta} = \frac{2}{3}\pi$ and starting from ρ_α one obtains ρ_β from the quantization angle. Knowing that $\rho_{\alpha+\beta}$ is also a root, one has found another root. The roots are then $d - r = 6$ from which $d = 8$ and the algebra is $\mathrm{su}(3)$. Its generators are

$$\lambda_i = \begin{bmatrix} \sigma_i & 0 \\ 0 & 0 \end{bmatrix} \,, \quad \lambda_{6,7} = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_{1,2} \end{bmatrix} \,, \quad \sigma_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

A quick way to find the generators of the Cartan sub-algebra is to look at diagonal matrices

$$H_1 = \frac{\lambda_3}{2},\,, \quad H_2 = \frac{\lambda_8}{2}$$

One can identify $\mathrm{su}(2)$ inside $\mathrm{su}(3)$

$$J_\pm^{(\alpha)} = \frac{\lambda_1 \pm \mathrm{i}\lambda_2}{2} \,, \quad J_\pm^{(\beta)} = \frac{\lambda_4 \pm \mathrm{i}\lambda_5}{2} \,, \quad J_\pm^{(\gamma)} = \frac{\lambda_4 \pm \mathrm{i}\lambda_7}{2}$$

One can calculate the roots

$$[H_1, J_\pm^{(\alpha)}] = \pm J_\pm^{(\alpha)}\,, \quad [H_2, J_\pm^{(\alpha)}] = 0$$

The root is then $\rho_{\alpha a} = (1,0)^\top$. It holds $J_3^{(\alpha)} = H_1$ [r]. Similarly

$$[H_1, J_\pm^{(\alpha+\beta)}] = \frac{1}{2} J_\pm^{(\alpha+\beta)}\,, \quad [H_2, J_\pm^{(\alpha+\beta)}] = \pm\frac{\sqrt{3}}{2} J_\pm^{(\alpha+\beta)} \implies \rho_{\alpha+\beta,a} = \frac{1}{2}(1,\sqrt{3})^\top$$

Using the Weyl group, one has

$$\boldsymbol{\rho}_\beta = \boldsymbol{\rho}_{\alpha+\beta} - \frac{\boldsymbol{\rho}_\alpha \cdot \boldsymbol{\rho}_{\alpha+\beta}}{\left|\boldsymbol{\rho}_\alpha\right|^2} = \frac{1}{2}\begin{bmatrix}1\\ \sqrt{3}\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1\\ \sqrt{3}\end{bmatrix}$$

The [r] $(0, \rho_\alpha, \rho_{-\alpha})$ form a two-dimensional irreducible representation of su(2). So [r]

$$J_3^{(\gamma)} = \frac{\rho_{\gamma a} H_a}{\left|\boldsymbol{\rho}_\gamma\right|} = -\frac{1}{2}H_1 + \frac{\sqrt{3}}{2}H_2 = \frac{1}{2}\begin{bmatrix}0 & 0\\ 0 & \sigma_3\end{bmatrix}$$

This matrix, with $J_\pm^{(\gamma)}$ one has found the su(2) algebra. Finally

$$J_3^{(\beta)} = \frac{\rho_{\beta a} H_a}{|\rho_\beta|} = \frac{1}{2}H_1 + \frac{\sqrt{3}}{2}H_2 = \frac{1}{2}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}$$

So [r] three-dimensional irreducible representation of su(2). Therefore

$$8 = 3 \oplus 2 \oplus 2 \oplus 1$$

Special orthogonal algebra. Considering $\theta_{\alpha\beta} = \frac{3}{4}\pi$, one has $d - r = 8$ and $d = 10$ which is so(5) $\simeq$ sp(4).

Exceptional algebra. Considering $\theta_{\alpha\beta} = \frac{5}{6}\pi$, one has $d - r = 12$ and $d = 14$. This is an exceptional algebra g_2 because it does not belong to any chain. [r] diagrams

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One would like to draw bidimensional diagrams for any rank.

Theorem 10.19. There exists a basis of roots such that the roots have real coefficient and span $\mathbb{R}^r$. Also K_{ab} is positive-definite in this basis.

Proof. Consider

$$K_{ab} = \sum_\alpha \rho_{\alpha a}\rho_{\alpha b}$$

from which

$$\left|\boldsymbol{\rho}_\beta\right|^2 = \rho_{\beta a} K^{ab} \rho_{\beta b} = \sum_\alpha (\boldsymbol{\rho}_\alpha \cdot \boldsymbol{\rho}_\beta)^2$$

Dividing by $\left|\boldsymbol{\rho}_\beta\right|^4$ one obtains

$$\frac{1}{\left|\boldsymbol{\rho}_\beta\right|^2} = \sum_\alpha \frac{(\boldsymbol{\rho}_\alpha \cdot \boldsymbol{\rho}_\beta)^2}{\left|\boldsymbol{\rho}_\beta\right|^2}$$

Knowing that the ratio is quantized

$$\frac{\boldsymbol{\rho}_\alpha \cdot \boldsymbol{\rho}_\beta}{\left|\boldsymbol{\rho}_\beta\right|^2} \in \frac{1}{2}\mathbb{Z} \subset \mathbb{R}$$

Therefore, $\left|\boldsymbol{\rho}_\beta\right|^{-2} \in \mathbb{R}$ and $\left|\boldsymbol{\rho}_\beta\right| \in \mathbb{R}$. From these, then $\boldsymbol{\rho}_\alpha \cdot \boldsymbol{\rho}_\beta \in \mathbb{R}$. The product can be expressed as a linear combination and there exists a basis where the coefficients are real $\rho_{\alpha a} \in \mathbb{R}$.

Knowing that

$$\left|\boldsymbol{\rho}_\beta\right|^2 = \sum_\alpha (\boldsymbol{\rho}_\alpha \cdot \boldsymbol{\rho}_\beta)^2 > 0$$

remembering that ρ_α are non-zero roots. The norm is always positive and as such the Killing form block K^{ab} is positive-definite. ☐

Example 10.20. Consider su(3) with roots

$$(1,0)\,, \quad \frac{1}{2}(1,\sqrt{3})\,, \quad \frac{1}{2}(-1,\sqrt{3})$$

They are all real. The Killing form block is

$$K_{ab} = \sum \rho_{\alpha a}\rho_{\alpha b} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{bmatrix} = \frac{3}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The Killing form block is proportional to the Kronecker delta.

One would like a more fundamental object than roots. Consider the exceptional algebra g_2. On a plane, there are at most two independent vectors, so the roots are redundant. In n dimensions, one needs n fundamental roots.

Definition 10.21. A root is positive if the last non-zero component in lexicographic order is positive.

Example 10.22. Some examples of positive roots are $\rho_\alpha r > 0$, or $\rho_{\alpha a} = 0$ and $\rho_{\alpha,r-1} > 0$.

Proposition 10.23. Positivity implies ordering: if $\boldsymbol{\rho}_\alpha - \boldsymbol{\rho}_\beta > 0$ then one defines $\boldsymbol{\rho}_\alpha > \boldsymbol{\rho}_\beta$. Also, a root ρ_α is negative if $-\rho_\alpha$ is positive.

Definition 10.24. A positive root ρ_α is simple if it cannot be written as the sum of two positive roots. One denotes a simple root by s_α with $\alpha = 1, \dots, r$, such that $\boldsymbol{\rho}_\alpha$ with $\alpha = 1, \dots, d-r$ coincides $\boldsymbol{\rho}_\alpha = \mathbf{s}_\alpha$ with $\alpha = 1, \dots, r$.

Theorem 10.25 (I)**.** If $\mathbf{s}_1$ and $\mathbf{s}_2$ are simple roots, then $\mathbf{s}_1 - \mathbf{s}_2$ is not a root.

Proof. Let $\boldsymbol{\rho}_\beta = \mathbf{s}_1 - \mathbf{s}_2$. For a positive root $\boldsymbol{\rho}_\beta > 0$, then $\mathbf{s}_1 = \mathbf{s}_2 + \boldsymbol{\rho}_\beta$, but this cannot be by definition of simple root. For a negative root $\boldsymbol{\rho}_\beta < 0$, then $\mathbf{s}_2 = \mathbf{s}_1 - \boldsymbol{\rho}_\beta$ and same as before. ☐

Theorem 10.26 (II)**.** Given $\mathbf{s}_1$ and $\mathbf{s}_2$, then $\mathbf{s}_1 \cdot \mathbf{s}_2 \leq 0$.

Proof. Let $\mathbf{s}_1$ be fixed. Being a root, the chains it is part of has $m + n + 1 = 4$. The chain is then

$$\dots\,, \quad \mathbf{s}_1 - \mathbf{s}_2\,, \quad \mathbf{s}_1\,, \quad \mathbf{s}_1 + \mathbf{s}_2\,, \quad \dots$$

From before, the difference is not a root, so the chains can only be built by addition and not subtraction. [r] Only for simply roots $m = 0$. Also it holds

$$2\frac{\mathbf{s}_1 \cdot \mathbf{s}_2}{\left|\mathbf{s}_1\right|^2} = m - n = -n \leq 0$$

☐

Theorem 10.27 (III)**.** Simple roots form a basis in $\mathbb{R}^r$. This basis is not necessarily orthogonal.

Theorem 10.28 (IV)**.** For any positive root $\boldsymbol{\rho}_\alpha$ there exists a set of coefficients $\{c_a\} \subset \mathbb{N}$ with $a = 1, \dots, r$ such that

$$\boldsymbol{\rho}_\alpha = \sum_a c_a \mathbf{s}_a$$

Once one fixes a basis, the coefficients are unique.

10.3 Classification of simple roots

The simple roots are enough to classify semi-simple algebra. One would like to find rules that allow a classification of such algebras.

Definition 10.29. Given the simple roots $\mathbf{s}_i$, the Cartan matrix is

$$C_{ij} = 2\frac{\mathbf{s}_i \cdot \mathbf{s}_j}{|\mathbf{s}_i|^2}$$

From this $C_{ii} = 2$ and $C_{ij} = 0, -1, -2, -3$, with $i \neq j$.

Theorem 10.30. A semi-simple Lie algebra can be completely reconstructed from its Cartan matrix.

Proof. The Cartan matrix C_{ij} has dimension $r \times r$. Remembering that

$$\cos^2 \theta_{ij} = \frac{(\mathbf{s}_i \cdot \mathbf{s}_j)^2}{|\mathbf{s}_i|^2 |\mathbf{s}_j|^2} = \frac{1}{4} C_{ij}^2$$

The angles between the roots are given by the Cartan matrix. Similarly, for the length ratios one has

$$\frac{|\mathbf{s}_i|^2}{|\mathbf{s}_j|^2} = \frac{C_{ij}}{C_{ji}}$$

From the Cartan matrix, one can draw all the simple roots. One can build positive roots by considering chains of the form

$$\mathbf{s}_i + k\mathbf{s}_j \, , \quad k = 0, 1, 2, 3$$

Negative roots can be built from the positive ones. From this the Killing form block is

$$K_{ab} = \sum_\alpha \rho_{\alpha a} \rho_{\alpha b}$$

$\square$

Example 10.31. Consider a 2×2 matrix, it has rank 2. The Cartan matrix is

$$C_{ij} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

There are only two simple roots. Their ratio is

$$\frac{|\mathbf{s}_i|^2}{|\mathbf{s}_j|^2} = \frac{C_{ij}}{C_{ji}} = 1$$

The angle is

$$\cos^2 \theta_{12} = \frac{1}{4}(-1)(-1) = \frac{1}{4} \implies \theta_{12} = \frac{2\pi}{3}$$

so the algebra is su(3).

10.4 Dynkin diagrams

One would like to study higher ranks with diagrams instead of matrices.

Definition 10.32. Consider unit norm vectors

$$\hat{s}_i = \frac{\mathbf{s}_i}{|\mathbf{s}_i|}$$

One may draw a bullet point for every simple root. The points are connected with $n = 0, 1, 2, 3$ lines according to the quantization condition

$$\hat{s}_i \cdot \hat{s}_j = \sqrt{\cos^2 \theta_{ij}} = -\frac{\sqrt{n}}{2}$$

Filled and open points denote the condition $|\mathbf{s}_i| > |\mathbf{s}_j|$. This is relevant only when the ratio of norms is not one

$$\frac{|\mathbf{s}_i|}{|\mathbf{s}_j|} \neq 1$$

Example 10.33. For rank 2 one needs two bullet points. [r] diagrams.

- For no lines, the roots are orthogonal and the algebra is semi-simple so(4) $\simeq$ so(3) $\oplus$ so(3).
- For one line, one has $C_{ij} = -1$ and sU(3).
- For two lines, one root must be larger than the other $|\mathbf{s}_1| > |\mathbf{s}_2|$, the algebra is so(5).
- For three lines, one root is greater than the other and the algebra is the exceptional one g_2.

One would like to find rules for the Dynkin diagrams in order to classify all semi-simple Lie algebras.

Proposition 10.34 (Rule I)**.** Disconnected diagrams correspond to algebras expressed as direct sums

$$\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \cdots$$

This is the case since the roots of $\mathcal{G}_1$, $\mathcal{G}_2$, ..., are orthogonal.

Proposition 10.35 (Rule II)**.** The number N_c of pairs of points connected by lines is less than the rank $N_c < r$.

Proof. Let

$$\hat{s} = \sum_{i=i}^{r} \hat{s}_i \,, \quad \hat{s} \cdot \hat{s} > 0$$

Then

$$0 < \hat{s} \cdot \hat{s} = r + 2 \sum_{i>j} \hat{s}_i \cdot \hat{s}_j < r - N_c$$

from which the thesis. □

Example 10.36. For $r = 3$ one has three points. A triangle diagrams [r] has $N_c = 3$ and it does not obey the rules.

Proposition 10.37 (Rule III)**.** Dynkin diagrams have no loops.

Proposition 10.38 (Rule IV)**.** Only vertices connected by at most three lines are admitted (this is not the same as stating that between two points there are at most three lines).

Proposition 10.39. The rule above implies that

- there is only one diagram with three connecting lines (corresponding to g_2);
- double links are admitted if connected by single links, no two consecutive two lines;
- branch points are possible if they do not lead to loops.

Proposition 10.40 (Rule V)**.** One may replace a chain of many single links with only one bullet point. This does not mean that the algebras are the same. This is a constraint on the diagrams.

Proposition 10.41. Along with the last proposition, this rule implies that

- a chain $2 - 1 - 1 - 2$ can be contracted to $2 - 2$, but this is not allowed, so the previous is not allowed also; this implies that one can have only one double link per diagram;
- branch points $(1,1) - 1 - 1 - (1,1)$ can be contracted to $(1,1)(1,1)$ which is not allowed; therefore, there can only be one single branch point per diagram;
- a diagram $(1,1) - 1 - 1 - 2$ is contracted to $(1,1) - 2$ and it is now allowed; so one may either use a branch point or a double link;
- the simplest connected Dynkin diagrams are chains with one line between points; they were originally called A_n by Cartan and represent su$(n+1)$.

Proposition 10.42 (Rule VI)**.** All diagrams with one double link

$$1-1-\cdots-2-\cdots-1-1 \leftrightarrow \hat{s}_1\,,\,\hat{s}_2\cdots\,,\,\hat{s}_p\,,\,\hat{v}_q\,,\,\cdots\,,\,\hat{v}_1$$

[r] One defines

$$\hat{s}=\sum_{i=1}^{p} i\hat{s}_i\,,\quad \hat{v}=\sum_{i=1}^{q} i\hat{v}_i$$

One has

$$\hat{s}\cdot\hat{s}=\sum_{i=1}^{p} i^2|\hat{s}_i|^2+2\sum_{i>j} i(\hat{s}_i\cdot\hat{s}_j)j=\sum_{i=1}^{p} i^2-\sum_{i=1}^{p} i(i-1)=\sum_{i=1}^{p} i=\frac{p(p+1)}{2}$$

Since before and after, the points are connected by single lines, then the dot product is always null unless for consecutive elements

$$\hat{s}_i\cdot\hat{s}_{i+1}=-\frac{1}{2}$$

Similarly

$$\hat{v}\cdot\hat{v}=\frac{q(q+1)}{2}$$

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The dot product is

$$\hat{s}\cdot\hat{v}=\sum_{i}^{p} i\hat{s}_i\sum_{j=1}^{q} j\hat{v}_j=p\cdot q\,\hat{s}_p\cdot\hat{v}_q=-\frac{pq}{\sqrt{2}}$$

because s_q and v_q are connected by two lines. Using Schwartz inequality one has

$$\begin{aligned}(\hat{s}\cdot\hat{v})^2&<(\hat{v}\cdot\hat{v})(\hat{s}\cdot\hat{s})\\ \frac{p^2q^2}{2}&<\frac{p(p+1)}{2}\frac{q(q+1)}{2}\\ 2pq&<(p+1)(q+1)\\ 2&<\left(1+\frac{1}{p}\right)\left(1+\frac{1}{q}\right)\end{aligned}$$

Remembering that p and q are natural numbers, there are only three possible solutions. Let $p=1$, one can have any q: a double link (full-empty) followed by an arbitrary length of single links (empty dots). This algebra is called B_n by Cartan: it is $\mathrm{so}(2n+1)$.

Letting $q=1$ then p is abritrary, a chain of links (empty dots) followed by a double link (empty-full). The algebra is called C_n by Cartan: it is $\mathrm{sp}(2n)$.

For $p=2$ then $q=2$, it is $1-2-1$ (empty-empty-full-full). This is a unique algebra; like g_2, it is the exceptional algebra F_4.

Proposition 10.43 (Rule VII)**.** One may wish to classify all diagrams with one branch point. Consider a single-linked chain from $\hat{s}_1$ to $\hat{s}_{p-1}$ then another point from which the chain separates into $\hat{t}_{u-1}$-$\hat{t}_1$ and $\hat{v}_{q-1}$-$\hat{v}_1$. One defines

$$\hat{s}=\sum_{i=1}^{p-1} i\hat{s}_i\,,\quad \hat{t}=\sum_{i=1}^{u-1} i\hat{t}_i\,,\quad \hat{v}=\sum_{i=1}^{q-1} i\hat{v}_i$$

Imposing three Schwartz inequalitites one obtains

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{u}>1$$

There are five possibile solutions labeled by a tuple (p,q,u):

- A trivial solution corresponds to $(p,1,1)$ for any p: an arbitrary length single-linked chain, it corresponds to A_n.

- A solution $(p, 2, 2)$ for any p: a single-linked chain with two single-linked branches. These were named D_n by Cartan. The solution corresponds to so$(2n)$ excluding $n = 1, 2$ since so(2) and so(4) are not simple.
- There are three exceptional algebras given by special solutions
  - ◇ Solution $(3, 3, 2)$ given by $1 - 1 - 1(-1) - 1 - 1$ (ones indicate the circles, not the links anymore).
  - ◇ Solution $(4, 3, 2)$ given by $1 - 1 - 1 - 1(-1) - 1 - 1$.
  - ◇ Solution $(5, 3, 2)$ given by $1 - 1 - 1 - 1 - 1(-1) - 1 - 1$.

In this way, all simple Lie algebras have been classified: A_n, B_n, C_n, D_n plus the exceptional G_2, F_4, E_6, E_7 and E_8.

10.5 Representations of semi-simple Lie algebras

Theorem 10.44. As one has already seen, every reducible representation of a finite-dimensional semi-simple Lie algebra is completely reducible.

This guarantees one can focus only on irreducible representations.

Remark 10.45. The roots are the eigenvalues of the adjoint representation. For a generic representation ρ, one can study the eigenvalue equation

$$\rho(H_a)v_\mu = \mu_a v_\mu$$

where H_a are the Cartan generators, $a = 1, \dots, r$.

Definition 10.46. The weight $\boldsymbol{\mu}$ of a representation ρ is a vector in $\mathbb{C}^r$ which fulfills

$$\rho(H_a)v_\mu = \mu_a v_\mu$$

Remark 10.47. Given a representation ρ acting on a vector space V, one can decompose the vector space into the subspaces spanned by the weights

$$V = \bigoplus_\mu V_\mu$$

If the representation ρ is the adjoint one, then the weights correspond to the roots.

Lemma 10.48. If $\boldsymbol{\rho}_\alpha$ is a root, $\boldsymbol{\mu}$ is a weight and ρ is a representation, then

$$\rho(E_\alpha)V_\mu \subset V_{\alpha+\mu}$$

Proof. Consider

$$[H_b, E_\alpha] = \rho_{\alpha b} E_\alpha\,, \quad \rho(H_a)v_\mu = \mu_a v_\mu$$

then

$$\begin{aligned} \rho([H_b, E_\alpha])v_\mu = \rho(H_b)\rho(E_\alpha)v_\mu - \rho(E_\alpha)\rho(H_b)v_\mu &= \rho_{\alpha b}\rho(E_\alpha)v_\mu \\ \rho(H_b)\rho(E_\alpha)v_{Mu} - \mu_b\rho(E_\alpha)v_\mu &= \rho_{\alpha b}\rho(E_\alpha)v_\mu \\ \rho(H_b)[\rho(E_\alpha)v_\mu] &= [\rho_{\alpha b} + \mu_b][\rho(E_\alpha)v_\mu] \end{aligned}$$

This is an eigenvalue equation with eigenvalue $\boldsymbol{\rho}_\alpha + \boldsymbol{\mu}$. So the vector space is denoted by $V_{\alpha+\mu}$. □

Lemma 10.49. Given two representations ρ_1 and ρ_2 with weights μ_a^1 and μ_b^2, the tensor product representation $\rho_1 \otimes \rho_2$ has weights given by all possible sums $\{\mu_a^1 + \mu_b^2\}$.

Proof. For a group G and one of its elements $g \in G$, one has

$$(\rho_1 \otimes \rho_2)(g) = \rho_1(g) \otimes \rho_2(g)$$

For the algebra, one has $g \approx 1 + a$, so

$$(\rho_1 \otimes \rho_2)(1+a) = (\rho_1 \otimes \rho_2)(1) + (\rho_1 \otimes \rho_2)(a) \approx \rho_1(1) \otimes \rho_2(1) + [\rho_1(a) \otimes \rho_2(1) + \rho_1(1) \otimes \rho_2(a)]$$

therefore

$$(\rho_1 \otimes \rho_2)(H_a) v_{\mu^1} \otimes v_{\mu^2} = [\rho_1(H_a) \otimes 1 + 1 \otimes \rho_2(H_a)] v_{\mu^1} \otimes v_{\mu^2} = (\mu_a^1 + \mu_a^2) v_{\mu^1} \otimes v_{\mu^2}$$

□

Example 10.50. Consider the algebra su(3). The Cartan generators are

$$H_1 = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad H_2 = \frac{1}{2\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

This is the fundamental representation. A simultaneous eigenvector is

$$\mathbf{v}_{\mu^1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} , \quad H_1 v_{\mu^1} = \frac{1}{2} v_{\mu^1} , \quad H_2 v_{\mu^1} = \frac{1}{2\sqrt{3}} v_{\mu^1}$$

The associated weight is given by the eigenvalues

$$\mu_a^1 = \frac{1}{2} \begin{bmatrix} 1 \\ 3^{-\frac{1}{2}} \end{bmatrix}$$

The Cartan algebra is given by the maximal canonical algebra which is made up of diagonal matrices [r]. So the other weights are

$$\mu_a^2 = \frac{1}{2} \begin{bmatrix} -1 \\ 3^{-\frac{1}{2}} \end{bmatrix} , \quad \mu_a^3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

On the plot of the adjoint representation (a hexagon) one can include the weights of the fundamental representation, which form a triangle.

The anti-fundamental representation is given from conjugate matrices

$$U^* = \rho_{\text{anti}}(g)$$

Since the algebra is made from anti-hermitian matrices, then

$$\rho_{\text{anti}}(g) = U^* = \mathrm{e}^{-u^\top} , \quad u \in \mathrm{su}(3)$$

So all the weights pick up a minus sign. One can build all the irreducible representation by filling the plane using the fundamental, anti-fundamental and adjoint representation: one combines the roots.

Lemma 10.51. If $\boldsymbol{\rho}_\alpha$ is a root, $\boldsymbol{\mu}$ is a weight of a finite-dimensional representation ρ, then

$$2 \frac{\boldsymbol{\rho}_\alpha \cdot \boldsymbol{\mu}}{|\boldsymbol{\rho}_\alpha|^2} \in \mathbb{Z}$$

Definition 10.52. A weight is called maximal if it is annihilated by all positive roots:

$$\rho(E_\alpha) v_\mu = 0 , \quad \forall \alpha \text{ positive}$$

Theorem 10.53. An irreducible representation has only one maximal weight.

Theorem 10.54. A weight $\boldsymbol{\mu} \in \mathbb{C}^r$ is the maximal weight of a finite-dimensional representation ρ_μ if and only if

$$D_i = 2\frac{\mathbf{s}_i \cdot \boldsymbol{\mu}}{|\mathbf{s}_i|^2}$$

is a non-negative integer for every simple root s_i. The indices D_i are called Dynkin indices. A weight is fundamental if all $D_i = 0$ except for one.

Remark 10.55. The classification of all finite-dimensional irreducible representations corresponds to the classification of D_i.

Example 10.56. Consider again the algebra su(3). The maximal weight of the fundamental representation is

$$\mu^1_a = \frac{1}{2}\begin{bmatrix} 1 \\ 3^{-\frac{1}{2}} \end{bmatrix}$$

The positive roots are

$$\boldsymbol{\rho}_\alpha = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{\rho}_\beta = \frac{1}{2}\begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}, \quad \boldsymbol{\rho}_\gamma = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}, \quad \gamma = \alpha + \beta$$

If μ^1_a is maximal, then by applying negative roots, one expects to get all other weights $\boldsymbol{\mu}^2$ and $\boldsymbol{\mu}^3$. In fact

$$\boldsymbol{\mu}^1 - \boldsymbol{\rho}_\gamma = -\frac{1}{2\sqrt{3}}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \boldsymbol{\mu}^3 = \boldsymbol{\mu}^1 - \boldsymbol{\rho}_\alpha - \boldsymbol{\rho}_\beta$$

Also

$$\boldsymbol{\mu}^1 - \boldsymbol{\rho}_\alpha = \frac{1}{2}\begin{bmatrix} -1 \\ 3^{-\frac{1}{2}} \end{bmatrix} = \boldsymbol{\mu}^2$$

The Dynkin indices are

$$D_1 = 2\frac{\boldsymbol{\rho}_\alpha \cdot \boldsymbol{\mu}^1}{|\boldsymbol{\rho}_\alpha|^2} = 1\,, \quad D_2 = 2\frac{\boldsymbol{\rho}_\beta \cdot \boldsymbol{\mu}^1}{|\boldsymbol{\rho}_\beta|^2} = 0$$

since α and β are the only simple roots.

Considering (only) three quarks with SU(3) symmetry organized in fundamental and anti-fundamental representations: the up quark has $(D_1, D_2) = (0, 0)$, the down quark has

$$\boldsymbol{\mu}^1 - \boldsymbol{\rho}_\alpha\,, \quad D_1 = \frac{2}{|\boldsymbol{\rho}_\alpha|^2}\boldsymbol{\rho}_\alpha \cdot (\boldsymbol{\mu}^1 - \boldsymbol{\rho}_\alpha)\,, \quad (D_1, D_2) = (-1, 1)$$

the strange quark has

$$\boldsymbol{\mu}^1 - \boldsymbol{\rho}_\alpha - \boldsymbol{\rho}_\beta\,, \quad (D_1, D_2) = (0, -1)$$

The anti-strange quark has

$$(D_1, D_2) = (0, 1)$$

Example 10.57. One may build the adjoint representation of su(3) from the fundamental. The adjoint representation has maximal weight $(1, 1)$ which corresponds to $\alpha + \beta$, so

$$\boldsymbol{\rho}_\alpha \cdot \boldsymbol{\rho}_\beta = \frac{1}{2} \implies (D_1, D_2) = (1, 1)$$

[r] diagrm

One knows that $3 \otimes \bar{3} = 8 + 1$. Since the mesons are organized in octet plus singlet, one may infer that mesons are made of quark and anti-quark.