Theory and Phenomenology of Fundamental Interactions

Maso*

July 18, 2024

Contents

*https://github.com/M-a-s-o/notes

Lecture 1

lun 04 mar 2024 12:30

The course will make more sense after taking QFT 1 and 2.

Exam. Computation of tree-level amplitude. [r] The oral part is comprised of 3 questions discussing theoretical aspects and fundamental aspects of the theory.

1 Introduction

The first part of the course is about the completion of the theoretical description of the electroweak sector of the Standard Model. Then Yukawa interaction linked to the [r] matrix: weak interactions are not diagonal with respect to flavor families. [r] Then discussion of phenomenology of the electroweak and Higgs sectors at the LHC.

The second part treats QCD, its gauge invariance; perturbative regime, the subtleties, universal divergences, properties of non-abelian gauge theories.

The last part deals with hadronic collision and its non-trivial description.

Further topics are neutrino masses, etc.

1.1 The bigger picture

The course deals with the Standard Model. One needs to understand how it must be viewed from a historical perspective and a modern perspective. To study fundamental interactions (excluding gravity) the popolar choice is quantum field theory: quantum electrodynamics, electroweak theory, quantum chromodynamics. The first is an abelian gauge theory, while the last two are non-abelian.

In these theories, the Lagrangian density is the fundamental object: it is Lorentz invariant, invariant under Poincaré transformation, and describes quantized fields. The particles are excitations of the fields. Every Lagrangian can be divided in the free term and the interaction term

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\mathrm{I}$$

The free Lagrangian contains kinetic terms, the propagators [r]. The interaction Lagrangian contains the interaction terms, which are represented as vertices in Feynman diagrams. The quantum electrodynamics Lagrangian is

$$\mathcal{L}_\mathrm{QED} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(\mathrm{i}\not{\partial} - m)\psi - eQ\bar{\psi}\gamma^\mu\psi A_\mu$$

where e is the elementary charge and Q is the quantum number of charge.

If one assumes that fields and interactions to be truly fundamental:

- then the theory has to be unitary and predictive at all energies (so predict final results and amplitudes that do not exceed 1).
- The non-trivial link to quantum field theory is that the theory must be renormalizable. The lagrangian must be constrained: the coupling constants cannot have negative mass dimension. Since fundamental interactions are well described by gauge theories, gauge invariance is a fundamental requirement of the renormalizability of the theory. [r] Once one picks a gauge group, it must produce observable phenomena and one must keep abiding its rules to keep gauge invariance.

Effective Standard Model. The Standard Model [r] but it is not a complete theory. There is no natural candidate for dark matter, gravity is not accounted for, neutrino masses are not explained. The Model has to fail at some point, at some energy scale where a phenomenon cannot be described with the field content of the Model. Therefore, the Standard Model must be an effective theory (as opposed to a fundamental theory). Therefore, it is allowed to add non-renormalizable operators: terms in the Lagrangian with negative mass dimension. If one wants to understand the physics beyond the Standard Model from a bottom-up approach, this is a middle ground: one modifies the theory enough to compute phenomena [r]. A similar story happened when going from Fermi's four-interaction theory to the intermedia vector boson theory. The interaction term for Fermi theory is $\bar{\psi}\psi\bar{\psi}\psi$ with dimension six, so the coupling constant must have mass dimension -2. [r] One may add non-renormalizable operators built from Standard Model objects that respect its symmetry group. In this paradigm one may not use renormalizability and unitarity, so the predictions are valid up to some energies.

1.2 Weyl spinors

A massless Dirac field is made of two Weyl field. A massive Dirac spinor is made of left- and right-chiral components. A term like

$$\bar{\psi}_{\mathrm{L}}\psi_{\mathrm{R}}$$

is not invariant under $\mathrm{SU}(2)_{\mathrm{L}}$.

Lorentz group. The proper Lorentz group [r] has six generators. [r] through the exponential map as

$$R(\hat{e},\theta) = \exp(-\mathrm{i}\theta\hat{e}\cdot\mathbf{J})\,, \quad B(\hat{u},\eta) = \exp(-\mathrm{i}\eta\hat{u}\cdot\mathbf{K})$$

where $\mathbf{J}$ are the generators of rotations and $\mathbf{K}$ are the generators of boosts. The explicit form of the generators can be obtained from infinitesimal transformations. For example

$$J_z = \mathrm{i}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\,, \quad K_x = \mathrm{i}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

These are a fundamental representation of the Lorentz algebra $\mathrm{SO}(1,3)$

$$[J_i, J_j] = \mathrm{i}\varepsilon_{ijk}J_k\,, \quad [K_i, K_j] = -\mathrm{i}\varepsilon_{ijk}J_k\,, \quad [J_i, K_j] = \mathrm{i}\varepsilon_{ijk}K_k$$

The above algebra can be rewritten as

$$[M_{\mu\nu}, M_{\rho\sigma}] = \mathrm{i}(\eta_{\nu\rho}M_{\mu\sigma} + \cdots)$$

where $M^{\mu\nu}$ an anti-symmetric tensor such that

$$M^{0i} = K_i\,, \quad M^{ij} = \varepsilon_{ijk}J_k$$

[r] In order to label the representations one has to use mass and spin.

The Lorentz algebra can be decomposed in two other algebras

$$\mathrm{so}(1,3) \simeq \mathrm{su}(2) \oplus \mathrm{su}(2)$$

[r] In fact, one combines

$$J_k^{\pm} = \frac{1}{2}(J_k \pm \mathrm{i}K_k) \implies [J_i^+, J_j^-] = 0\,, \quad [J_i^{\pm}, J_j^{\pm}] = \mathrm{i}\varepsilon_{ijk}J_k^{\pm}$$

In order to label the possible elementary fields associated to Lorentz group (in general the Poincaré group) one needs two non-negative half-integers (s_1, s_2). For $(0,0)$ the transformation under each su(2) is trivial, so they do not transform, they are a singlet: it is a scalar field. The next representations are $(1/2, 0)$ and $(0, 1/2)$ for right-chiral Weyl spinor and left-chiral Weyl spinor. For a vector field one has $(1/2, 1/2)$. P 164 Schwartz

For right-chiral Weyl spinors u_R, the generators are

$$J_i^+ = \frac{1}{2}\sigma_i\,, \quad J_i^- = 0 \implies J_i = \frac{1}{2}\sigma_i\,, \quad \mathrm{i}K_i = \frac{1}{2}\sigma_i$$

Therefore, for a rotation, one has

$$R = \exp(-\mathrm{i}\boldsymbol{\theta}\cdot\mathbf{J}) = \exp\left(-\frac{1}{2}\mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}\right)$$

and for a boost

$$B = \exp(-\mathrm{i}\boldsymbol{\eta}\cdot\mathbf{K}) = \exp\left(-\frac{1}{2}\boldsymbol{\eta}\cdot\boldsymbol{\sigma}\right)$$

Similarly for a left-chiral Weyl spinor. For a four-vector one has

$$J_i = J_i^+ + J_i^-$$

One may realize that there are two states that transforms as [r] The singlet component under rotation is A^0, while the triplet is A^i.

Parity. Under parity, the generators of rotations do not transform $J \to J$, while boosts do $K \to -K$. Also parity maps

$$(s_1, s_2) \to (s_2, s_1)$$

Therefore, left-chiral spinor becomes a right-chiral spinor, while a vector is still a vector.

Weyl spinors. Weyl spinors can be combined into a vector. Considering

$$\sigma_\pm^\mu = (I, \pm\boldsymbol{\sigma})$$

which is equivalent to the notation

$$\sigma^\mu = (I, \sigma^i)\,, \quad \bar{\sigma}^\mu = (I, -\sigma^i)$$

Therefore a vector is given by

$$u_\mathrm{R}^\dagger\sigma^\mu u_\mathrm{R}\,, \quad u_\mathrm{L}^\dagger\bar{\sigma}^\mu u_\mathrm{L}$$

These are bilinear objects in the spinor fields. One may use them to construct Lagrangians.

Lagrangian. One can build a Lagrangian from these fields by requiring that

$$u_\mathrm{R,L} \to \mathrm{e}^{\mathrm{i}\theta}u_\mathrm{R,L}$$

In fact, one may have

$$\mathcal{L}_\mathrm{Weyl} = \mathrm{i}u_\mathrm{R,L}^\dagger\sigma_\pm^\mu\,\partial_\mu u_\mathrm{R,L}$$

[r] The equations of motion are

$$\sigma^\mu\,\partial_\mu\psi_\mathrm{R} = 0\,, \quad \bar{\sigma}^\mu\,\partial_\mu\psi_\mathrm{L} = 0 \implies (\partial_0 \pm \boldsymbol{\sigma}\cdot\nabla)\psi_\mathrm{R,L} = 0$$

Acting on the last equation with $(\partial_0 \mp \boldsymbol{\sigma}\cdot\nabla)$ on the left side, one gets a massless Klein–Gordon equation

$$\Box\,\psi_\mathrm{R,L} = 0$$

In momentum space one has [r]

$$\psi_\mathrm{R,L} = \hat{\psi}_\mathrm{R,L}(\mathbf{k})\mathrm{e}^{-\mathrm{i}kx}\,, \quad k^0 = |\mathbf{k}|$$

where the hat indicates the Fourier transform. In momentum space, the Weyl equations are

$$[k^0 \mp (\mathbf{k}\cdot\sigma)]\hat{\psi}_\mathrm{R,L} = 0$$

So the spinor is an eigenvector of the helicity operator

$$\frac{1}{2}\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{|\mathbf{k}|}$$

[r] from this, right-chiral spinors have positive helicity and similar.

Lecture 2

mar 05 mar 2024 12:30

Weyl equations are important because electroweak theory violates parity: left- and right-chiralities transform differently.

If one relaxes the global U(1) symmetry, one can add a Lorentz scalar to the Lagrangian. Composing two left-chiral representations

$$\left(\frac{1}{2},0\right)\otimes\left(\frac{1}{2},0\right)=(0,0)\oplus(1,0)$$

One has a part that transforms as a vector under SU(2) and one as a scalar. The singlet combination can be extracted as

$$\varepsilon_{ab}u_\pm^a u_\pm^b\,,\quad \varepsilon_{ab}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

where each spinor has two complex entries. The above combination is Lorentz invariant. When checking that it is Lorentz invariant, the Levi–Civita tensor gives the determinant of some matrix, which is the exponential of the Pauli matrices, so it is 1. The combination is not zero because the spinor product is not symmetric: the spinor components are Grassmann odd fields

$$\{u_\pm^a, u_\pm^b\}=0$$

Such a term in a Lagrangian is allowed

$$\mathcal{L}_{\text{Weyl}}^\pm=\mathrm{i}u_\pm^\dagger\sigma_\pm^\mu\,\partial_\mu u_\pm-\frac{1}{2}m[\varepsilon_{ab}u_\pm^a u_\pm^b+\text{h.c.}]$$

This is a Majorana mass term, it is bilinear in the fields. This term is not invariant under U(1) of the spinors. If one would like to implement a global or local transformation such the previous cannot have a mass term like the one above. [r] For charged (under some symmetry group) chiral fermions, one cannot have a Majorana mass term. In QED a charged fermion transforms non trivially under $\mathrm{U}(1)_{\mathrm{EM}}$.

For a non charged particle of any symmetry of the Standard Model, such a term is allowed, like right-handed neutrinos.

1.3 Dirac spinors

Under parity, the right- and left-chiral representations are mapped into one another. Parity invariance means the presence of both chiralities. A Dirac spinor is a combination of Weyl spinors (in the Weyl basis)

$$\psi=\begin{bmatrix}u_-\\ u_+\end{bmatrix}=\begin{bmatrix}\psi_{\mathrm{L}}\\ \psi_{\mathrm{R}}\end{bmatrix}\,,\quad \left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$$

This is a direct sum of representations because the two chiral representations do not mix. In the Dirac basis, the Dirac spinor is

$$\psi=\frac{1}{\sqrt{2}}\begin{bmatrix}u_+ + u_-\\ u_+ - u_-\end{bmatrix}$$

The course uses Weyl basis (also called chiral basis). In this representation, the Dirac matrices are

$$\gamma^\mu=\begin{bmatrix}0 & \sigma^\mu\\ \bar{\sigma}^\mu & 0\end{bmatrix}\,,\quad \gamma^5=\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3=\begin{bmatrix}-I_2 & 0\\ 0 & I_2\end{bmatrix}$$

The projection operators are then

$$P_{\mathrm{L,R}}=\frac{1\mp\gamma^5}{2}\,,\quad P_{\mathrm{L}}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\,,\quad P_{\mathrm{R}}=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\,,\quad P_{\mathrm{L,R}}^2=P_{\mathrm{L,R}}\,,\quad P_{\mathrm{L}}P_{\mathrm{R}}=P_{\mathrm{R}}P_{\mathrm{L}}=0$$

The Dirac conjugate spinors are

$$\bar{\psi}=\psi^\dagger\gamma^0\,,\quad \bar{\psi}_{\mathrm{L,R}}=\psi_{\mathrm{L,R}}^\dagger\gamma^0=\bar{\psi}\frac{1\pm\gamma^5}{2}$$

notice how they have the opposite chirality of their non conjugate part. Some properties of the fifth gamma matrix are

$$\gamma_5 = \gamma_5^\dagger\,, \quad \gamma_5^2 = I\,, \quad \{\gamma_\mu, \gamma_5\} = 0\,, \quad \{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$$

The free Dirac lagrangian in terms of Weyl spinors is

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi = \bar{\psi}_{\mathrm{L}}\mathrm{i}\,\partial\!\!\!/\psi_{\mathrm{L}} + \bar{\psi}_{\mathrm{R}}\mathrm{i}\,\partial\!\!\!/\psi_{\mathrm{R}} - m(\bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}} + \bar{\psi}_{\mathrm{L}}\psi_{\mathrm{R}})$$

This mass term is different from Majorana's. This term couples the two chiral fields. It is invariant under $\mathrm{U}(1)_{\mathrm{EM}}$

$$\psi \to \mathrm{e}^{\mathrm{i}\alpha}\psi$$

but it is not invariant under different transformations of the chiral fields (like $\mathrm{SU}(2)_{\mathrm{L}}$). In electroweak theory, one needs to [r].

Vector theory. A vector theory does not distinguish the chiral parts of a field. As a consequence, parity is conserved. This is the reason why the fermionic current is a vector

$$\bar{\psi}\gamma^\mu\psi = \bar{\psi}_{\mathrm{L}}\gamma^\mu\psi_{\mathrm{L}} + \bar{\psi}_{\mathrm{R}}\gamma^\mu\psi_{\mathrm{R}}$$

which is invariant under

$$\psi_{\mathrm{L,R}} \to U(x)\psi_{\mathrm{L,R}}$$

Chiral theory. A chiral theory treats fields differently based on their chirality. It is parity violating. A Dirac mass term is not gauge invariant for chiral theories.

For QED and QCD a Dirac mass term is allowed; only the weak sector of the Standard Model creates problems. Under $\mathrm{SU}(2)_{\mathrm{L}}$ a left-chiral field transforms non trivially

$$\psi_{\mathrm{L}} \to U(x)\psi_{\mathrm{L}}$$

while a right-chiral field remains the same.

1.4 Conventions

A Dirac field may be written in a Fourier series as

$$\psi(t, \mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\sum_s [u_s(\mathbf{k})a_s(\mathbf{k})\mathrm{e}^{-\mathrm{i}kx} + v_s(\mathbf{k})b^\dagger(\mathbf{k})\mathrm{e}^{ikx}]$$

where u and v are wave functions? [r]. The normalization of the free spinors is

$$\sum_s u_s(\mathbf{p})\bar{u}_s(\mathbf{p}) = p\!\!\!/ + m\,, \quad \sum_s v_s(\mathbf{p})\bar{v}_s(\mathbf{p}) = p\!\!\!/ - m$$

which solve

$$(p\!\!\!/ - m)u_s = 0\,, \quad \bar{u}_s(p\!\!\!/ - m) = 0\,, \quad (p\!\!\!/ + m)v_s = 0\,, \quad \bar{v}_s(p\!\!\!/ + m) = 0$$

A fermion propagator is

$$\frac{\mathrm{i}(p\!\!\!/ + m)}{p^2 - m^2 + \mathrm{i}\varepsilon}$$

The Feynman amplitude is

$$\mathrm{i}\mathcal{M} = \sum \text{Feynman diagrams}$$

The sum over quantum numbers of external particles is

$$\sum |\mathcal{M}|^2$$

The sum and average is instead

$$\overline{\sum}|\mathcal{M}|^2$$

[r] The tree-level cross-section is obtained

$$\mathrm{d}\sigma = \mathcal{F}\overline{\sum}|\mathcal{M}|^2\mathrm{d}\phi_n$$

where $\mathcal{F}$ is the flux factor and $\mathrm{d}\phi_n$ is the phase space

$$\mathrm{d}\phi_n = (2\pi)^4\delta^{(4)}(\Delta p^\mu)\prod_{i=1}^n[\mathrm{d}k_i]$$

where the Lorentz invariant phase space measure is

$$[\mathrm{d}k_i] = \frac{\mathrm{d}^3k_i}{(2\pi)^3 2E_i}\,, \quad E_i^2 = m_i^2 + |\mathbf{k}_i|^2$$

In general, the phase space contains also symmetry factors $\frac{1}{n!}$ if the final state particles are identical bosons.

The decay width of a particle M decaying is

$$\mathrm{d}\Gamma = \frac{1}{2M}\overline{\sum}|\mathcal{M}|^2\,\mathrm{d}\phi_n$$

The total width is

$$\Gamma = \int\mathrm{d}\Gamma = \frac{1}{2M}\int\overline{\sum}|\mathcal{M}|^2\,\mathrm{d}\phi_n$$

The Dirac traces is not explicitly written but a bracket is present [r] for Bhabha scattering one has [r] wrong. For computing the cross section one needs $\mathcal{M}^*$ and therefore

$$\sum|\mathcal{M}|^2 = [\cdots]$$

Four products of momenta get shortened

$$p_1^\mu p_{2\mu} = (12) = (p_1\cdot p_2)$$

1.5 Quantum electrodynamics

One applies the gauge principle to go from a global symmetry to a local symmetry and make the lagrangian invariant. The interactions appear from the covariant derivative.

The QED lagrangian is

$$\mathcal{L}_{\mathrm{QED}} = \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - e\bar{\psi}\gamma^\mu\psi A_\mu$$

where $e = |e| > 0$. The free lagrangian

$$\mathcal{L}_0 = \bar{\psi}(\mathrm{i}\,\partial\!\!\!/ - m)\psi$$

is invariant under global U(1). Making it local and requiring invariance, the fields transform as

$$\psi'(x) = \mathrm{e}^{\mathrm{i}\alpha(x)}\psi(x)\,, \quad A'_\mu = A_\mu - \frac{1}{e}\,\partial_\mu\alpha(x)$$

The covariant derivative is then

$$D_\mu = \partial_\mu + \mathrm{i}eA_\mu$$

So that the invariant lagrangian is

$$\mathcal{L} = \bar{\psi}(\mathrm{i}D\!\!\!\!/ - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

The derivative of the field does not transform the same way as the field, however the covariant derivative does

$$(\partial_\mu\psi)' \neq \mathrm{e}^{\mathrm{i}\alpha(x)}\partial_\mu\psi\,, \quad (D_\mu\psi)' = \mathrm{e}^{\mathrm{i}\alpha(x)}D_\mu\psi$$

Therefore the term $\bar{\psi}D_\mu\psi$ is gauge invariant. The field strength tensor is also gauge invariant.

Remark. The term
$$(F_{\mu\nu}F^{\mu\nu})^2$$
is not included because its mass dimension is 8 and the theory is not renormalizable.

Remark. The term $A^\mu A_\mu$ is not gauge invariant and it corresponds to a mass term, but the photon is massless.

Lecture 3

mer 06 mar 2024 12:30

1.6 Non-abelian gauge groups

The Standard Model is based in part on $\mathrm{SU}(n)$ symmetry groups. Consider $U \in \mathrm{SU}(n)$, it can be expressed as
$$U = \exp[\mathrm{i}\theta^a t^a]\,, \quad a = 1, \cdots, n^2 - 1$$
where t^a are the generators of the group and θ^a are real parameters. Elements of such group have the properties
$$UU^\dagger = U^\dagger U = I\,, \quad \det U = 1$$
The generators, which belong to the algebra, obey
$$t^a = (t^a)^\dagger\,, \quad 1 = \det e^{t^a} = e^{\mathrm{Tr}\, t^a} \implies \mathrm{Tr}\, t^a = 0$$
So the generators are hermitian traceless matrices. The generators are normalized to give
$$\mathrm{Tr}\big(t^a t^b\big) = T_R \delta^{ab}\,, \quad T_R = \frac{1}{2}$$
The commutation relations of the Lie algebra $\mathrm{su}(n)$ are
$$[t^a, t^b] = \mathrm{i} f^{abc} t^c$$
The coefficient f^{abc} are the structure constants of the Lie algebra. For non-abelian gauge groups, the commutator is not identically zero.

Exercise. Defining the matrix
$$\tau^{ab} \equiv \mathrm{i}[t^a, t^b]$$
where a, b do not label the components. Show that
$$\mathrm{Tr}\, \tau^{ab} = 0\,, \quad (\tau^{ab})^\dagger = \tau^{ab}$$
and that f is totally anti-symmetric and real.

In general, a d-dimensional representation of an algebra is a set of $d \times d$ matrices that satisfy the commutation relation
$$[T^a, T^b] = \mathrm{i} f^{abc} T^c$$
The number of T^a is the dimension of the Lie group.

The important representations are the fundamental, anti-fundamental and the adjoint representations. The fundamental representation is an n-dimensional representation. It acts on n-dimensional objects. The adjoint representation is given by the structure constants
$$(T^a)_{bc} = \mathrm{i} f^{bac}$$
where bc are the components of the matrix T^a. It is an $n^2 - 1$ dimensional representation.

Gauge symmetry. The matrix

$$U(x) = \mathrm{e}^{\mathrm{i}\theta^a(x)t^a}$$

depends on space-time coordinates. In the fundamental representation, the matrix acts on n dimensional objects. A spinor does not transform trivially, but as

$$\psi'(x) = U(x)\psi(x)$$

The derivative transforms as

$$(\partial_\mu \psi)' = \partial_\mu[U(x)\psi(x)] = (\partial_\mu U)\psi + U\,\partial_\mu\psi$$

but this is not favorable. One uses the covariant derivative

$$(D_\mu)_{ij} = \partial_\mu \delta_{ij} + \mathrm{i}g t^a_{ij} A^a_\mu$$

where the ij indices treats gauge group components. Therefore, there are n^2-1 gauge fields A^a_μ. The covariant transforms in the same way as the field

$$(D_\mu\psi)' = D'_\mu\psi' = U(x)D_\mu\psi$$

This implies that the gauge fields transform as

$$(t^a A^a_\mu)' = t^a A'^a_\mu = U(t^a A^a_\mu)U^{-1} + \frac{\mathrm{i}}{g}(\partial_\mu U)U^{-1}$$

To define the field strength tensor one goes by analogy with quantum electrodynamics

$$\mathrm{i}q F^{\mathrm{QED}}_\mu = [D^{\mathrm{QED}}_\mu, D^{\mathrm{QED}}_\nu]\,, \quad D^{\mathrm{QED}}_\mu = \partial_\mu + \mathrm{i}qA_\mu$$

In general, one defines

$$\mathrm{i}g t^a F^a_{\mu\nu} = [D_\mu, D_\nu]$$

This gives

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$

Under gauge transformation one has

$$(t^a F^a_{\mu\nu})' = t^a F'^a_{\mu\nu} = U(t^a F^a_{\mu\nu})U^{-1}$$

The kinetic term for the gauge bosons is

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} = -\frac{1}{2}\,\mathrm{Tr}(\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu})\,, \quad \mathbf{F}_{\mu\nu} = t^a F^a_{\mu\nu}\,, \quad F^a_{\mu\nu}F^{a\mu\nu} = 2\,\mathrm{Tr}(\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu})$$

One sees that

$$\mathbf{F}'^{\mu\nu} = U\mathbf{F}^{\mu\nu}U^{-1}$$

For a non-abelian gauge group the Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\psi}_j(x)[\mathrm{i}\not{D}_{jk} - m\delta_{jk}]\psi_k(x)$$

The indices jk are the one carried by the generators t^a_{jk}.In the fundamental representation one has

$$a = 1, \ldots, n^2-1\,, \quad j,k = 1,\ldots,n$$

The covariant derivative is

$$D^\mu_{jk} = \partial^\mu\delta_{jk} + \mathrm{i}g t^a_{jk} A^\mu_a$$

There are n^2-1 gauge boson fields A^a_μ and they are massless because massive terms $m^2 A^a_\mu A^{a\mu}$ are not guage invariant.

A term $(F^a_{\mu\nu}F^{a\mu\nu})^n$ is not renormalizable for $n>1$.

The covariant derivative for the electroweak sector

$$D_\mu = \partial_\mu - \mathrm{i}g t^a A^a_\mu$$

while for the quantum chromodynamics sector is

$$D_\mu = \partial_\mu + \mathrm{i} g t^a A^a_\mu$$

[r] The interaction term

$$\bar{\psi} \not{D} \psi \rightsquigarrow -g(\bar{\psi} t^a \gamma^\mu \psi) A^a_\mu$$

generates a vertex equal to

$$-\mathrm{i} g t^a_{ji} \gamma^\mu$$

In the kinetic part FF there are three-vertices

$$FF \rightsquigarrow g f^{abc} (\partial_\mu A^a_\nu) A^b_\mu A^c_\nu$$

which is called derivative vertex. In momentum space, the derivative is a momentum [r]

$$-g f^{abc} [(p_a - p_b)^\gamma \eta^{\alpha\beta} + (p_b - p_c)^\alpha \eta^{\beta\gamma} + (p_c - p_a)^\beta \eta^{\alpha\gamma}]$$

The charged boson have to interact with the carrier of the force, which is another boson.

In the kinetic part there are also four-vertices

$$FF \rightsquigarrow g^2 f f AAAA$$

2 History of the Standard Model

2.1 Fermi theory

Fermi theory is a theory of the electroweak sector. The Lagrangian is

$$\mathcal{L} = -\frac{G_\mathrm{F}}{\sqrt{2}} J^\mu J^\dagger_\mu \,, \quad J_\mu = \bar{\psi}_l \gamma_\mu (1 - \gamma_5) \psi_{\nu_l} + \bar{\psi}_d \gamma_\mu (1 - \gamma_5) \psi_u = L_\mu + H_\mu$$

The current contains both the leptonic part and the hadronic part. Examples of weak decays are the muon beta decay and the neutron beta decay.

One can compute tree-level total decay widths Γ and differential widths $\mathrm{d}_\Omega \Gamma$ from which one obtains the Fermi constant $G_\mathrm{F} = 1.16 \times 10^{-5}\,\mathrm{GeV}^{-2}$. The Lagrangian correctly describes these two processes [r].

Fermi theory is a V-A theory because bilinears of the types

$$V^\mu = \bar{\psi}_1 \gamma^\mu \psi_2$$

transforms like a polar vector while bilinears like

$$A^\mu = \bar{\psi}_1 \gamma^\mu \gamma^5 \psi_2$$

transforms like an axial vector, or pseudo-vector. A theory of this type is in accordance with experiments. A V-A theory is maximally parity violating. Check that $V'^\mu = \Lambda^\mu{}_\nu V^\mu$ and $A'^\mu = \bar{\Lambda}^\mu{}_\nu A^\nu$ [r]. To see how it is maximally violating, one can see that from the Lagrangian one has terms like

$$A^\mu V_\mu \to -A^\mu V_\mu$$

Processes like neutrino deep-inelastic scattering. For example $\bar{\nu}_e u \to d e^+$ one has a differential cross section

$$\mathrm{d}_\Omega \sigma = \frac{G_\mathrm{F}^2}{8\pi^2} \frac{s}{4} (1 + \cos\theta)^2 \,, \quad s = (p_1 + p_2)^2$$

but this cross section implies that the scattering matrix is not unitary.

Therefore Fermi theory is not renormalizable because the coupling constant has negative mass dimensions and it is not unitary. To fix the second problem one introduces a vector gauge boson.

Lecture 4

lun 11 mar 2024 12:30

2.2 Intermediate vector boson theory

A neutrino scattering is an exchange in the t-channel of a vector boson [r] diagr. The diagram gives an amplitude

$$\mathcal{M} \sim g_W^2 J_{e\nu}^{\mu} \left[-\eta_{\mu\nu} + \frac{Q_\mu Q_\nu}{M^2} \right] \frac{1}{Q^2 - M_W^2} J_{ud}^{\nu}$$

In the limit of the momentum going to zero

$$Q^\mu = P_1^\mu - P_3^\mu \to 0$$

the amplitude is asymptotic to

$$M \sim g_W^2 J_\mu^{e\nu} J_\nu^{ud} \frac{\eta^{\mu\nu}}{Q^2 - M_W^2}$$

Confronting this result with Fermi theory, one obtains a coupling constant of

$$-\frac{G_{\mathrm{F}}}{\sqrt{2}} = \frac{1}{8} \frac{g_W^2}{Q^2 - M_W^2}$$

The intermediate vector boson is an advanced theory respect to Fermi's. Also, in the low momentum limit $Q \to 0$ (equivalent to $Q^2 \ll M_W^2$) one obtains the Fermi theory

$$\frac{G_{\mathrm{F}}}{\sqrt{2}} = \frac{g_W^2}{8M_W^2}$$

In the high energy limit, $Q^2 \gg M^2$, the differential cross section does not diverge

$$\mathrm{d}_\Omega \sigma = \frac{g_W^2}{4\pi^2} \frac{s}{(s - M_W^2)^2}\,, \quad s = (p_1 + p_2)^2$$

An explicit mass term in the Lagrangian breaks the gauge invariance

$$\mathcal{L} \sim M_W^2 W_\mu W^\mu$$

The theory presents a problem: the need of a theory with massive vector boson spoils the gauge invariance. Also a problem is that the theory is not renormalizable. The propagator of a massive vector boson is

$$G_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M^2} \left[-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{M^2} \right]$$

If one wants to understand the renormalizabilty through power counting arguments, one sees that the propagator is constant

$$G_{\mu\nu}(\mathbf{k}) \sim \frac{k_\mu k_\nu}{k^2}\,, \quad k \to \infty$$

The renormalizable nature of a theory is given by the convergence of the propagators in a Feynman diagram.

Another problem is the loss of unitarity. When scattering two vector bosons in two vector bosons, one has to build the amplitude by summing all diagrams. One can check how the amplitude scales with the energy. The behaviour of E^4 cancels and remains E^2: the amplitude grows indefinitely.

2.3 Electroweak theory

One has to associate currents to the left-chiral part of the Dirac field. A leptonic current is

$$J_\mu^{\mathrm{lept}} = \frac{1}{2} \bar{\nu} \gamma_\nu (1 - \gamma_5) e$$

The current must come from a covariant derivative. The current must be Noether current. Therefore

$$\partial_\mu - \mathrm{i}gT^a A^a_\mu$$

This term is always between two fermionic fields, so the current is

$$\bar{\psi}_i \gamma_\mu T^a_{ij} \psi_j$$

where the indices ij are associated to the transformation of the gauge group.

One defines a left-chiral leptonic doublet

$$L(x) = P_{\mathrm{L}} \begin{bmatrix} \nu_e \\ e \end{bmatrix}$$

for which the current is

$$J^{\mathrm{lept}}_\mu = \bar{L}\gamma_\mu \tau^+ L$$

[r] Confronting with the current above one has

$$\tau^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}(\tau_1 + \mathrm{i}\tau_2)$$

where τ_i are the Pauli matrices. So the gauge group is SU(2). Moreover, there is

$$(J^{\mathrm{lept}}_\mu)^\dagger = \bar{L}\gamma_\mu \tau^- L\,, \quad \tau^- = \frac{1}{2}(\tau_1 - \mathrm{i}t_2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

There is bijective correspondence between currents and generators. Since the algebra is closed, there is a third current given by

$$(J^{\mathrm{lept}}_\mu)^3 = \bar{L}\gamma_\mu [\tau^+, \tau^-] L = \bar{L}\gamma_\mu \tau^3 L$$

This current implies a bilinear combination of spinors of the type

$$e\gamma_\mu e\,, \quad \nu\gamma_\mu\nu$$

If one searches for more generators, one may compute the commutator. However, in this case the algebra is closed

$$[\tau^3, \tau^\pm] = 2\tau^\pm$$

So there are no other currents.

Leptonic sector. The leptonic electroweak sector has $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_Y$ gauge group. The Lagrangian respecting this symmetry is given by

$$\mathcal{L}_{\mathrm{f}} = \mathrm{i}\bar{L}\not{D}L + \mathrm{i}\bar{e}_{\mathrm{R}}\not{D}e_{\mathrm{R}} + \mathrm{i}\bar{\nu}_{e\mathrm{R}}\not{D}\nu_{e\mathrm{R}}$$

where

$$D_\mu = \partial_\mu - \mathrm{i}gW^j_\mu T^j - \frac{1}{2}\mathrm{i}g' Y(\psi) B_\mu$$

where $j = 1, 2, 3$ and W_μ is the field associated to the local SU(2) symmetry, while B_μ is associated to U(1). This last gauge group is not the one of electromagnetism.

The generators T^i are the generators of the Lie algebra in the representation in which the fields transform

$$T^i = \begin{cases} \frac{1}{2}\tau^i\,, & \text{left-chiral fields} \\ 0\,, & \text{right-chiral fields} \end{cases}$$

For the hypercharge symmetry, the hypercharge is an abelian group and the associated charge can depend on the field itself.

The Lagrangian $\mathcal{L}_f$ is invariant under

$$U(x) = \exp\left[\frac{1}{2}\mathrm{i}g\theta(x)\tau^i\right], \quad U(x) = \exp\left[\frac{1}{2}\mathrm{i}g'\alpha(x)Y\right]$$

The kinetic part of the Lagrangian is

$$\mathcal{L}_f^{\mathrm{kin}} = \mathrm{i}\bar{L}\,\not{\partial}L + \mathrm{i}\bar{\nu}_{e\mathrm{R}}\,\not{\partial}\nu_{e\mathrm{R}} + \mathrm{i}\bar{e}_{\mathrm{R}}\,\not{\partial}e_{\mathrm{R}}$$

One may notice that the electrons and neutrinos are both massless, but when measured they are not massless.

The charge current interaction is

$$\begin{aligned}\mathcal{L}_{\mathrm{cc}} &= \frac{1}{2}gW_\mu^1\bar{L}\gamma^\mu\tau^1L + \frac{1}{2}gW_\mu^2\bar{L}\gamma^\mu\tau^2L = \frac{g}{\sqrt{2}}[W_\mu^+\bar{L}\gamma^\mu\tau^+L + W_\mu^-\gamma^\mu\tau^-L] \\ &= \frac{g}{\sqrt{2}}[W_\mu^+\bar{\nu}_{\mathrm{L}}\gamma^\mu e_{\mathrm{L}} + W_\mu^-\bar{e}_{\mathrm{L}}\gamma^\mu\nu_{\mathrm{L}}]\end{aligned}$$

where one defines

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp \mathrm{i}W_\mu^2)$$

The Feynman rule is [r] diagr

$$\mathrm{i}\frac{g}{\sqrt{2}}\gamma^\mu\frac{1-\gamma_5}{2}$$

These are the charged currents because the carrier boson is charged.

The neutral current part is

$$\begin{aligned}\mathcal{L}_{\mathrm{nc}} &= \frac{1}{2}gW_3^\mu[\bar{\nu}_{e\mathrm{L}}\gamma_\mu\nu_{e\mathrm{L}} - \bar{e}_{\mathrm{L}}\gamma_\mu e_{\mathrm{L}}] \\ &\quad + \frac{1}{2}g'B^\mu[Y(L)(\bar{\nu}_{e\mathrm{L}}\gamma_\mu\nu_{e\mathrm{L}} + \bar{e}_{\mathrm{L}}\gamma_\mu e_{\mathrm{L}}) + Y(e_{\mathrm{R}})(\bar{e}_{\mathrm{R}}\gamma_\mu e_{\mathrm{R}}) + Y(\nu_{e\mathrm{R}})(\bar{\nu}_{e\mathrm{R}}\gamma_\mu v_{e\mathrm{R}})]\end{aligned}$$

Introducing

$$\Psi = \begin{bmatrix}\nu_{e\mathrm{L}} & e_{\mathrm{L}} & \nu_{e\mathrm{R}} & e_{\mathrm{R}}\end{bmatrix}^\top$$

the third value of the isospin is

$$T_3 = \frac{1}{2}\begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}$$

while the hypercharge is

$$Y = \mathrm{diag}[Y(L), Y(L), Y(\nu_{e\mathrm{R}}), Y(e_{\mathrm{R}})]$$

The Lagrangian becomes

$$\mathcal{L}_{\mathrm{nc}} = g(\bar{\Psi}\gamma^\mu T_3\Psi)W_\mu^3 + \frac{1}{2}g'(\bar{\Psi}\gamma^\mu Y\Psi)B_\mu$$

From these fields one would like to recognize quantum electrodynamics. One considers linear combinations of the two fields through an orthogonal rotation: the kinetic term [r]. Therefore

$$\begin{bmatrix}B_\mu \\ W_\mu^3\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}A_\mu \\ Z_\mu\end{bmatrix}$$

The angle is called Weinberg's angle. One needs to look at the kinetic terms and how the interactions change. The kinetic Lagrangian is

$$L_{\mathrm{YM,kin}} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}W_{\mu\nu}^3W_3^{\mu\nu} \to -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu}$$

where $F^{\mu\nu}$ is the field strength tensor of the electromagnetic field A^μ, and $Z^{\mu\nu}$ for the Z^μ field. There are no mixed kinetic terms $F_{\mu\nu}Z^{\mu\nu}$.

The neutral current part of the Lagrangian is

$$\mathcal{L}_{\mathrm{nc}} = \bar{\Psi}\gamma_\mu[g\sin\theta\,T_3 + \frac{1}{2}g'\cos\theta\,Y]\Psi A^\mu + \bar{\Psi}\gamma_\mu[g\cos\theta\,T_3 - \frac{1}{2}g'\sin\theta\,Y]\Psi Z^\mu$$

Noting that the interaction Lagrangian of quantum electrodynamics is

$$\mathcal{L}^{\text{int}}_{\text{QED}} = \bar{\Psi}\gamma^\mu e Q \Psi A_\mu$$

where $e > 0$ is the elementary charge and

$$Q = \operatorname{diag}(Q_\nu, Q_e, Q_\nu, Q_e) = \operatorname{diag}(0, -1, 0, -1)$$

Therefore, one has

$$g \sin\theta\, T_3 + \frac{1}{2} g' \cos\theta\, Y \equiv eQ$$

Since Y is always with g', then one can fix a value for Y and the other follow

$$Y(L) \equiv -1$$

therefore, for the left-chiral neutrino field $\nu_{e\mathrm{L}}$ one has $T_3 = \frac{1}{2}$ and the left-chiral electron field one has $T_3 = -\frac{1}{2}$. For the neutrino and electron one has

$$\frac{1}{2} g \sin\theta - \frac{1}{2} g \cos\theta = 0\,, \quad -\frac{1}{2} g \sin\theta - \frac{1}{2} g' \cos\theta = 0$$

This implies

$$g \sin\theta + g' \cos\theta = e$$

Therefore one has the Gell-Mann–Nijishima equation

$$Q = T_3 + \frac{1}{2} Y$$

For the right-chiral fields, one has $T_3 = 0$ and

$$Y(\nu_\mathrm{R}) = 0\,, \quad Y(e_\mathrm{R}) = -2$$

The right-chiral neutrino behaves like it does not exist (apart from gravity).

The Feynman rules for quantum electrodynamics $e\bar{\Psi}\gamma_\mu Q \Psi A^\mu$ and neutral currents $e\bar{\Psi}\gamma_\mu Q_Z \Psi Z^\mu$ are [r] diagr

$$\mathrm{i} e Q_f \gamma^\mu\,, \quad \mathrm{i} e \gamma_\mu (c_\mathrm{L} P_\mathrm{L} + c_\mathrm{R} P_\mathrm{R}) = \frac{\mathrm{i} e}{2 \sin\theta \cos\theta} \gamma^\mu (v_f - a_f \gamma_5)$$

where

$$Q_Z = \frac{1}{\sin\theta\cos\theta} [T_3 - Q \sin^2\theta]\,, \quad \Psi = \Psi_\mathrm{L} + \Psi_\mathrm{R}$$

and

$$c_\mathrm{L} = \frac{1}{\sin\theta\cos\theta} (T^3_f - Q_f \sin^2\theta)\,, \quad c_\mathrm{R} = -\tan\theta\, Q_f$$

likewise

$$v_f = \frac{1}{2}(c_\mathrm{L} - c_\mathrm{R}) = \frac{T^3_f - 2Q^2_f \sin^2\theta}{2 \sin\theta\cos\theta}\,, \quad a_f = \frac{1}{2}(c_\mathrm{L} + c_\mathrm{R}) = \frac{T^3_f}{2\sin\theta\cos\theta}$$

Lecture 5

mar 12 mar 2024 12:30

The pure Yang–Mills part is

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} W^i_{\mu\nu} W^{\mu\nu}_i\,, \quad i = 1, 2, 3$$

where one has

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f^{abc} W^b_\mu W^c_\nu$$

Inserting the physical fields through the relations

$$W^1_\mu = \frac{1}{\sqrt{2}} (W^+_\mu + W^-_\mu)\,, \quad W^2_\mu = \frac{\mathrm{i}}{\sqrt{2}} (W^+_\mu - W^-_\mu)$$

and

$$W^3_\mu = A_\mu \sin\theta + Z_\mu \cos\theta\,, \quad B_\mu = A_\mu \cos\theta - Z_\mu \sin\theta$$

one obtains

$$\begin{aligned}\mathcal{L}_{\mathrm{YM}} = \mathcal{L}_{\mathrm{kin}} + \mathcal{L}_{\mathrm{int}} &= -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{4} Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{2} W^+_{\mu\nu}W^{\mu\nu}_- \\ &+ \text{3-point interactions} + \text{4-point interactions}\end{aligned}$$

in the first line there are no mixed terms: the fields are mass eigenstates. The three-point vertex contains derivative so they scale with the energy of the momentum, while the four-point have no dependence on momentum.

Hadronic sector. See Ridolfi. Hadrons are not elementary particles, but are made of quarks. A neutron decays into

$$n \to pe^-\bar{\nu}_e\,, \quad |udd\rangle \to |uud\rangle + e^- + \bar{\nu}_e$$

In the low-energy limit, the decay can be explained using Fermi's four-point interaction. The interaction has to be generated from a term

$$J^{\mathrm{lept}}_\mu J^\mu_{\mathrm{had}}$$

where one has the charged currents

$$J^{\mathrm{had}}_\mu = \frac{1}{2}\bar{u}\gamma_\mu(1-\gamma_5)d$$

The up and down quarks are not the only ones. Experiments showed the presence of strange hadrons $K^\pm$, K^0, Λ^0, etc. These particles decay slowly, they have a short decay width so the interaction has a weak coupling constant: they decay weakly. One assumes that a kaon is made of another quark, the strange quark

$$\left|K^+\right\rangle = |u\bar{s}\rangle$$

One of its decay chains is

$$\left|K^+\right\rangle \to \left|\pi^0\right\rangle e^+\nu$$

It starts from a kaon and has leptons in the final state: the interaction is the weak. The strange quark is postulated to have electromagnetic charge of

$$Q_s = -\frac{1}{3}$$

The strangeness quantum number of the strange quark is $Q_s = -1$.

The natural phenomenological hypothesis is considering that the hadronic current has two parts

$$J^{\mathrm{had}}_\mu = \cos\theta\frac{1}{2}\bar{u}\gamma_\mu(1-\gamma_5)d + \sin\theta\frac{1}{2}\bar{u}\gamma_\mu(1-\gamma_5)s$$

where $\theta \approx 12^\circ$ is the Cabibbo angle.

One may extend the model supposing the current is Noether's and one may find new interactions. The hadronic current is

$$J^{\mathrm{had}}_\mu = \begin{bmatrix}\bar{u}_{\mathrm{L}} & \bar{d}_{\mathrm{L}} & \bar{s}_{\mathrm{L}}\end{bmatrix}\gamma_\mu T^+ \begin{bmatrix}u_{\mathrm{L}} \\ d_{\mathrm{L}} \\ s_{\mathrm{L}}\end{bmatrix}\,, \quad T^+ = \begin{bmatrix}0 & \cos\theta & \sin\theta \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}$$

The hermitian conjugate current is

$$(J^{\mathrm{had}}_\mu)^\dagger = \cdots\,, \quad T^- = \begin{bmatrix}0 & 0 & 0 \\ \cos\theta & 0 & 0 \\ \sin\theta & 0 & 0\end{bmatrix}$$

One imagines that the matrices $T^\pm$ are elements of a Lie group ([r] still Cabibbo angle? or which one is it?)

$$[T^+, T^-] = T^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\cos^2\theta & -\sin\theta\cos\theta \\ 0 & -\sin\theta\cos\theta & -\sin^2\theta \end{bmatrix}$$

This matrix is not diagonal. The current associated with this element is

$$\begin{aligned} J_\mu^{\mathrm{had},3} &= \begin{bmatrix} \bar{u}_\mathrm{L} & \bar{d}_\mathrm{L} & \bar{s}_\mathrm{L} \end{bmatrix} \gamma_\mu T^3 \begin{bmatrix} u_\mathrm{L} \\ d_\mathrm{L} \\ s_\mathrm{L} \end{bmatrix} \\ &= \bar{u}_\mathrm{L}\gamma_\mu u_\mathrm{L} - \cos^2\theta \bar{d}_\mathrm{L}\gamma_\mu d_\mathrm{L} - \sin^2\theta \bar{s}_\mathrm{L}\gamma_\mu s_\mathrm{L} - \sin\theta\cos\theta[\bar{d}_\mathrm{L}\gamma_\mu s_\mathrm{L} + \bar{s}_\mathrm{L}\gamma_\mu d_\mathrm{L}] \end{aligned}$$

The last parenthesis is the flavor-changing neutral current (FCNC). These currents exist in nature but are extremely suppressed. However, this theory does not predict a suppressed FCNC.

The flavor-changing charge current can be seen from

$$\left|K^+\right\rangle \to \left|\pi^0\right\rangle e^+\nu\,, \quad |u\bar{s}\rangle \to |u\bar{u}\rangle$$

mediated by

$$\sin\theta_c\, \bar{s}_\mathrm{L}\gamma^\mu u_\mathrm{L}$$

[r] Instead, for the mediating FCNC is

$$\sin\theta_c\cos\theta_c \bar{s}_\mathrm{L}\gamma^\mu d_\mathrm{L}$$

The ratio of the experimental widths is

$$\frac{\Gamma(K^+ \to \pi^+ e^+ e^-)}{\Gamma(K^+ \to \pi^0 e^+ \nu)} \approx 10^{-5}$$

while in theory one has

$$\sim \frac{(\sin\theta_c\cos\theta_c)^2}{\sin^2\theta_c} \sim 0.97$$

To resolve the issue, one may postulate the existence of a fourth quark, called charm quark. Its electric charge is

$$Q = \frac{2}{3}$$

It has to be heavy and coupled to the down and strange quarks through charged currents. The current is

$$\begin{aligned} J_\mu^{\mathrm{had}} &= \cos\theta_c\bar{u}_\mathrm{L}\gamma_\mu d_\mathrm{L} + \sin\theta_c\bar{u}_\mathrm{L}\gamma_\mu s_\mathrm{L} - \sin\theta_c\bar{c}_\mathrm{L}\gamma_\mu d_\mathrm{L} + \cos\theta_c\bar{c}_\mathrm{L}\gamma_\mu s_\mathrm{L} \\ &= \bar{u}_\mathrm{L}\gamma_\mu d'_\mathrm{L} + \bar{c}_\mathrm{L}\gamma_\mu s'_\mathrm{L} \end{aligned}$$

where one has

$$\begin{bmatrix} d' \\ s' \end{bmatrix} = \begin{bmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{bmatrix} \begin{bmatrix} d \\ s \end{bmatrix}$$

The current is then

$$J_\mu^{\mathrm{had}} = \begin{bmatrix} \bar{u}_\mathrm{L} & \bar{d}'_\mathrm{L} \end{bmatrix} \gamma_\mu \tau^+ \begin{bmatrix} u_\mathrm{L} \\ d'_\mathrm{L} \end{bmatrix} + \begin{bmatrix} \bar{c}_\mathrm{L} & \bar{s}'_\mathrm{L} \end{bmatrix} \gamma_\mu \tau^+ \begin{bmatrix} c_\mathrm{L} \\ s'_\mathrm{L} \end{bmatrix}$$

The third current is

$$\begin{aligned} J_\mu^{\mathrm{had},3} &= \begin{bmatrix} \bar{u}_\mathrm{L} & \bar{d}'_\mathrm{L} \end{bmatrix} \gamma_\mu \tau^3 \begin{bmatrix} u_\mathrm{L} \\ d'_\mathrm{L} \end{bmatrix} + \begin{bmatrix} \bar{c}_\mathrm{L} & \bar{s}'_\mathrm{L} \end{bmatrix} \gamma_\mu \tau^3 \begin{bmatrix} c_\mathrm{L} \\ s'_\mathrm{L} \end{bmatrix} \\ &= \bar{u}_\mathrm{L}\gamma_\mu u_\mathrm{L} + \bar{c}_\mathrm{L}\gamma_\mu c_\mathrm{L} - \bar{d}_\mathrm{L}\gamma_\mu d_\mathrm{L} - \bar{s}_\mathrm{L}\gamma_\mu s_\mathrm{L} \end{aligned}$$

There are no flavour-changing neutral currents.

The existence of the charm quark was discovered four years after postulating its existence. It was discovered the bound state $|c\bar{c}\rangle$ which is the J/ψ particle. The mass of the quark is $m_c \approx 1.5\,\mathrm{GeV}$.

Those who postulated its existence worked also on the GIM mechanics: there are no FCNC at tree-level, the FCNC are suppressed at 1 loop if $\Delta s \neq 0$, explains the mass difference of K_L and K_S.

At present time, the mechanism involves the mixing between three quark families: there are three angles and one complex phase; this phase implies CP violation.

Summary. The Standard Model has the following fields. The quarks are

$$\begin{bmatrix} u_{\mathrm{L}} \\ d'_{\mathrm{L}} \end{bmatrix}, \quad \begin{bmatrix} c_{\mathrm{L}} \\ s'_{\mathrm{L}} \end{bmatrix}, \quad \begin{bmatrix} t_{\mathrm{L}} \\ b'_{\mathrm{L}} \end{bmatrix}, \quad u_{\mathrm{R}} d'_{\mathrm{R}} c_{\mathrm{R}} s'_{\mathrm{R}} t_{\mathrm{R}} b'_{\mathrm{R}}$$

The leptons are

$$\begin{bmatrix} \nu_{eL} \\ e_L \end{bmatrix}, \quad \begin{bmatrix} \nu_{\mu L} \\ \mu_L \end{bmatrix}, \quad \begin{bmatrix} \nu_{\tau L} \\ \tau_L \end{bmatrix}, \quad e_R \mu_R \tau_R \nu_{eR} \nu_{\mu R} \nu_{\tau R}$$

The left-chiral leptons are organized in doublets of $\mathrm{SU}(2)_L$ with $T_3 = \pm\frac{1}{2}$, while the right-chiral are singlets with $T_j = 0$. One also has

$$Q = T_3 + \frac{1}{2} Y$$

where one has

$$Y(u_R) = \frac{4}{3}, \quad Y(d_R) = -\frac{2}{3}, \quad Y(Q_L) = \frac{1}{3}, \quad Q_u = \frac{2}{3}, \quad Q_d = -\frac{1}{3}, \quad Q_L = \begin{bmatrix} u_L \\ d'_L \end{bmatrix}, \cdots$$

The charged-current Lagrangian

$$\mathcal{L}_{cc} = \frac{g}{\sqrt{2}} \sum_f [\bar{L}_f \gamma_\mu \tau_+ L_f + \bar{Q}_f \gamma_\mu \tau_+ Q_f] W_\mu^+ + \text{h.c.}$$

where $f = 1, 2, 3$ is the index associated with the three quark and leptonic families and

$$L_f \in \{ \begin{bmatrix} \nu_{eL} \\ e_L \end{bmatrix}, \begin{bmatrix} \nu_{\mu L} \\ \mu_L \end{bmatrix}, \cdots \}, \quad Q_f = \{ \begin{bmatrix} u_L \\ d'_L \end{bmatrix}, \cdots$$

the Lagrangian is

$$\mathcal{L}_{cc} = \mathcal{L}_{\text{lept}} + \frac{g}{\sqrt{2}} \left[\sum_{f,g} (\bar{u}_L^f \gamma_\mu V^{fg} d_L^g) W_\mu^+ + \text{h.c.} \right]$$

where V_{fg} is a 3×3 non-diagonal matrix called CKM matrix. [r] one has the Feynman

$$\sim V^{12} [\gamma_\mu \frac{1}{2} (1 - \gamma_5) \frac{g}{\sqrt{2}}]$$

The neutral current Lagrangian has no FCNC but is diagonal

$$\mathcal{L}_{\text{nc}} = e(\bar{\Psi} \gamma_\mu Q \Psi) A_\mu + (\bar{\Psi} \gamma^\mu Q_Z \Psi) Z_\mu$$

2.4 Higgs mechanism

None of the fields have any masses because gauge symmetry must be respected. Masses are given through the spontaneous symmetry breaking of gauge theories: the Higgs mechanism. This gives masses to the gauge boson.

A Yukawa interaction is allowed and a Higgs field with vacuum expectation value gives masses to fermions.

One may study the number of degrees of freedom needed. In the electroweak sector, the gauge fields are massless each with two transverse polarization; though in nature three are massive, so one more degree for each is needed.

One needs a scalar field charge under the gauge group $\mathrm{SU}(2)_L \times U(1)_Y$ with non zero vacuum expectation value. The gauge group must be broken, but there must remain a gauge subgroup of $U(1)_{\mathrm{EM}}$. The simplest and minimal choice is using a doublet of scalar complex fields

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

It has four real degrees of freedom. Three of them make up the masses of three gauge boson, while the last is the Higgs field. The doublet transforms as

$$T_3\phi = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\phi \,, \quad Y(\phi) = \cdots$$

the hypercharge is not yet fixed.

To realize the spontaneous symmetry breaking one needs a potential [r], to respect gauge invariance, and the theory to be normalizable. The potential can only be

$$V(\phi) = m^2|\phi|^2 + \lambda|\phi|^4 \,, \quad |\phi|^2 = |\phi_1|^2 + |\phi_2|^2$$

The mass dimension of λ is zero, so the theory is renormalizable. A cubic term is not gauge invariant.

A spontaneous symmetry breaking implies a non zero vacuum expectation value. The minimum of the potential is

$$|\phi_{\mathrm{min}}|^2 = -\frac{m^2}{2\lambda} = \frac{1}{2}v^2 \implies m^2 < 0$$

The minimum configuration must be invariant under $U(1)$ so that this gauge group is unbroken and the photon does not gain mass. One parametrizes the minimum field as

$$\phi_{\mathrm{min}} = \frac{1}{\sqrt{2}}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \,, \quad |v_1|^2 + |v_2|^2 = v^2$$

The residual invariance implies

$$\mathrm{e}^{\mathrm{i}\alpha(x)Q}\phi_{\mathrm{min}} = \phi_{\mathrm{min}} \implies Q\phi_{\mathrm{min}} = 0$$

Lecture 6

mer 13 mar 2024 12:30

One needs to have

$$\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \frac{1}{2}\begin{bmatrix} 1+Y & 0 \\ 0 & -1+Y \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

There are two choices $v_1 = 0$ and $v_2 = |v|$ or $v_1 = |v|$ and $v_2 = 0$. For the first one has $Y(\phi) = 1$ while for the second $Y(\phi) = -1$. One picks the first choice

$$Q_{\mathrm{EM}}(\phi_1) = 1 \,, \quad Q_{\mathrm{EM}}(\phi_2) = 0$$

The doublet can rewritten as

$$\phi = \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix}$$

To obtain the content of the theory, one parametrizes the field around the vacuum expectation value. This can be done through Cartesian coordinates

$$\phi = \begin{bmatrix} \xi_1 + \mathrm{i}\xi_2 \\ \frac{1}{\sqrt{2}}(v + H(x) + \mathrm{i}\chi) \end{bmatrix}$$

but it is more useful to use complex coordinates

$$\phi = \frac{1}{\sqrt{2}}\exp\left[\frac{1}{v}\mathrm{i}\tau^a\theta^a(x)\right]\begin{bmatrix} 0 \\ v + H(x) \end{bmatrix}$$

where $H(x)$ is the Higgs field. The four fields above have zero expectation value. One imposes the unitary gauge

$$\phi(x)' = U(x)\phi(x)\,, \quad U(x) = \exp\left[-\frac{1}{v}\mathrm{i}\tau^a\theta^a(x)\right]$$

This gauge is allowed because the Lagrangian is gauge invariant. The θ^a fields are absorbed into the $W^\pm$ and Z fields as longitudinal polarization, so the associated bosons are massive. The field $H(x)$ is associated with a physical particle. The spontaneous symmetry patterns is such that three of the five degrees of freedom [r]

$$\tau_1\varphi_{\min} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{bmatrix} \neq 0\,, \quad \tau_2\phi_{\min} \neq 0$$

whereas

$$\frac{1}{2}(\tau_3 + Y)\phi_{\min} = 0$$

Physical consequences. The Lagrangian of the field is

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi)\,, \quad \phi = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ v + H(x) \end{bmatrix}$$

The potential is

$$\begin{aligned} V &= m^2|\phi|^2 + \lambda|\phi|^4 = \frac{1}{2}m^2[H^2 + 2vH] + \frac{1}{4}\lambda[H^4 + 4H^3v + 6H^2v^2 + 4Hv^3] + \text{const.} \\ &= \frac{1}{2}(2\lambda v^2)H^2 + \lambda vH^3 + \frac{1}{4}\lambda H^4 \end{aligned}$$

at the second line one uses the fact that

$$-m^2 = \lambda v^2$$

The first term is a mass term $m_{\mathrm{H}}^2 = 2\lambda v^2$. The other two terms are a three-point and four-point self-couplings of the Higgs field. The two vertices are proportional to the free parameter λ.

Within the covariant derivative there is the Higgs kinetic term and the interactions with the vector boson HVV, HHVV. The covariant derivative is

$$\begin{aligned} D_\mu\phi &= \left[\partial_\mu - \frac{1}{2}\mathrm{i}gW_\mu^i\tau^i - \frac{1}{2}\mathrm{i}g'YB_\mu\right]\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ v + H(x) \end{bmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ \partial_\mu H(x) \end{bmatrix} - \frac{\mathrm{i}}{2}\left[1 + \frac{H}{v}\right]\begin{bmatrix} gvW_\mu^+ \\ -v\sqrt{\frac{g^2+g'^2}{2}}Z_\mu \end{bmatrix} \end{aligned}$$

where $Y = 1$ for the Higgs field. Therefore, one gets

$$(D_\mu\phi)^\dagger(D^\mu\phi) = \cdots = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \left(1 + \frac{H}{v}\right)^2\left[\left(\frac{gv}{2}\right)^2 W_\mu^+W_\mu^- + \frac{v^2}{2}\frac{g^2 + g'^2}{4}Z_\mu Z^\mu\right]$$

The masses of the gauge bosons are then

$$m_W = \frac{1}{2}gv\,, \quad m_Z^2 = v^2\frac{g^2 + g'^2}{4} = \frac{m_W^2}{\cos^2\theta}$$

while the mass of the electromagnetic field is zero because a massa term does not appear. [r]

The vacuum expectation value must be

$$\frac{G_{\mathrm{F}}}{\sqrt{2}} = \frac{g^2}{8M_W^2} \implies v = (\sqrt{2}G_{\mathrm{F}})^{-\frac{1}{2}} \approx 246\,\mathrm{GeV}$$

This energy is the energy scale associated to a symmetry breaking.

Interactions. The three-point interaction is

$$\begin{aligned}\mathrm{HVV} &= \frac{2m_W^2}{v}W_\mu^+ W_-^\mu H + \frac{m_Z^2}{v}Z_\mu Z^\mu H \\ &= g m_W \mathrm{WWH} + \frac{1}{2}g\frac{m_Z}{\cos\theta}\mathrm{ZZH}\end{aligned}$$

at the second equality one uses the relations above for the masses. The Higgs boson couples in a way proportional to the mass of the vector bosons. The vertex is [r]

$$\mathrm{i}g\eta_{\mu\nu}\begin{cases} m_W\,, & \mathrm{HWW} \\ \frac{m_Z}{\cos\theta}\,, & \mathrm{HZZ}\end{cases}$$

Another interaction is the HHVV.

Remark. SSB produces a Lagrangian with massive vector boson in a gauge invariant way. This is done through a scalar field with non zero vev.

However, there are heavy fermions which have

$$\langle\bar{\psi}\psi\rangle \neq 0$$

If they were charged properly for SU(2) left [r] If such field exists, its energy is higher than the current probed energy. A model like this is called composite Higgs.

The Higgs mass is

$$m_H \approx 125\,\mathrm{GeV}$$

The mass is known from experiment and one has

$$m_H^2 = 2\lambda v^2$$

To measure directly the parameter λ one has to obtain a three-point Higgs self-coupling. The Higgs can be obtain from gluon fusion. [r] however there is a background. The dependence on λ comes from the destructive interference between the two diagrams.

Yukawa interaction. [r] One has to recover a Dirac mass. [r] The kinetic term for fermions, one can compactly write

$$\mathcal{L}_{\text{fermions}} = \sum_{f=1}^{n}\sum_{k=1}^{5}\bar{\psi}_k^{(f)}\mathrm{i}\not{D}\psi_k^{(f)}$$

where

$$k \in \{\begin{bmatrix}u_\mathrm{L}\\ d_\mathrm{L}\end{bmatrix}, u_\mathrm{R}, d_\mathrm{R}, \begin{bmatrix}\nu_{eL}\\ e_\mathrm{L}\end{bmatrix}, e_\mathrm{R}\}$$

and $f = 1, 2, 3$ so n is the number of fermionic families.

The Lagrangian is invariant under the transformation $\mathrm{U}(n)_\mathrm{F}$

$$\psi_k^{(f)} \to U^{fg}\psi_k^{(g)}$$

This is an accidental global flavour symmetry.

One has to mix different families, so the notation used is the following: the primed denotes the interaction eigenstates, the interactions with gauge bosons are diagonal; the unprimed denotes the mass eigenstates. A diagonal matrix is useful because one can read out the propagators. One would like to express everything in terms of the mass eigenstates. Let

$$Q'_\mathrm{L} = \begin{bmatrix}u'_\mathrm{L}\\ d'_\mathrm{L}\end{bmatrix}, u'_\mathrm{R}, d'_\mathrm{R}, L'_\mathrm{L} = \begin{bmatrix}\nu'_{eL}\\ e'_\mathrm{L}\end{bmatrix}, e'_\mathrm{R}$$

With the SM field content, the only term that can be added that respects Lorentz invariance, gauge invariance and renormalizability is the Yukawa interaction

$$\bar{\psi}\psi\phi$$

The Yukawa Lagrangian is

$$\mathcal{L}_{\mathrm{Y}} = -\bar{Q}'_{\mathrm{L}}\phi h'_D d'_{\mathrm{R}} - \bar{Q}'_{\mathrm{L}}\phi^c h'_U u'_{\mathrm{R}} - \bar{L}'\phi h'_E e'_{\mathrm{R}} + \text{h.c.}$$

The h' terms are $n \times n$ complex matrices in flavor space. Their entries are numbers so their mass dimension is zero. In the SM $n = 3$ so a total of 27 parameters; some are fixed by the masses and [r].

The terms are invariant under SU(2) left and hypercharge.

Also one has

$$\phi^c = \varepsilon\phi\,, \quad \varepsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \mathrm{i}\tau^2$$

The anti-fundamental representation of SU(2) is the same as the fundamental of SU(2). One has

$$\phi^c = \frac{1}{\sqrt{2}}\begin{bmatrix} v + H(x) \\ 0 \end{bmatrix}$$

In the unitary gauge, the Lagrangian is

$$\mathcal{L}_{\mathrm{Y}} = -\frac{v+H}{\sqrt{2}}[\bar{d}'_{\mathrm{L}} h'_D d'_{\mathrm{R}} + \bar{u}'_{\mathrm{L}} h'_U u'_{\mathrm{R}} + \bar{e}'_{\mathrm{L}} h'_E e'_{\mathrm{R}} + \text{h.c.}]$$

Imagining that the matrices are diagonal, one obtains the Dirac mass for each field. Though the matrices are not diagonal because there is flavour mixing. By the singular-value decomposition, for each h' with complex entries, not necessarily square, it holds

$$h' = UhV^\dagger$$

where U and V are unitary matrices, and h is a diagonal matrix with positive and real entries (i.e. eigenvalue). In this way one can write a diagonal matrix, with other two acting on the spinors, but this does not matter because spinor have to be rotated. In fact

$$\mathcal{L}_{\mathrm{Y}} = -\frac{v+H}{\sqrt{2}}[\bar{d}'_{\mathrm{L}} U_D h_D V_D^\dagger d'_{\mathrm{R}} + \bar{u}'_{\mathrm{L}} U_U h_U V_U^\dagger u'_{\mathrm{R}} + \bar{e}'_{\mathrm{L}} U_E h_E V_E^\dagger e'_{\mathrm{R}} + \text{h.c.}]$$

The matrices h are all diagonal. One defines

$$d_{\mathrm{R}} = V_D^\dagger d'_{\mathrm{R}}\,, \quad d_{\mathrm{L}} = U_D^\dagger d'_{\mathrm{L}}$$

and same for u and e. One has

$$\mathcal{L}_{\mathrm{Y}} = -\frac{v+H}{\sqrt{2}}[\bar{d}_{\mathrm{L}} h_D d_{\mathrm{R}} + \bar{u}_{\mathrm{L}} h_U u_{\mathrm{R}} + \bar{e}_{\mathrm{L}} h_E e_{\mathrm{R}} + \text{h.c.}]$$

These are the mass eigenstates. So

$$m_D = \frac{v}{\sqrt{2}} h_D = \mathrm{diag}(m_d, m_s, m_b)$$

same for m_U and m_E.

After this rotation, the kinetic terms remains diagonal [r]

Lecture 7

lun 18 mar 2024 12:30

In the quark sector there is a mixing in the charged current. There is also a source of CP violation.

One has

$$m_U = \frac{v}{\sqrt{2}} h_U = \mathrm{diag}(m_u, m_c, m_t)\,, \quad m_E = \frac{v}{\sqrt{2}} h_E = \mathrm{diag}(m_e, m_\mu, m_\tau)$$

The Yukawa Lagrangian is

$$\mathcal{L}_{\mathrm{Y}} = -\sum_f m_f(\bar{\psi}_L^{(f)}\psi_R^{(f)} + \bar{\psi}_R^{(f)}\psi_L^{(f)})\left[1 + \frac{H(x)}{v}\right]$$

In this way one has got the mass terms. There is also a three-point interaction called Yukawa coupling [r] diagr

$$-\mathrm{i}\frac{m_f}{v}$$

The coupling of the Higgs field is proportional to the mass of the particle. Mass is a consequence of the strength of the coupling.

Consequences. The kinetic term of u-type quarks is

$$\mathcal{L}_{\mathrm{kin}} \sim \bar{u}'_L \mathrm{i}\not{\partial} u'_L = \bar{u}_L \mathrm{i}\not{\partial} u_L$$

Since one has

$$d_R = V_D^\dagger d'_R \,, \quad d_L = U_D^\dagger d'_L \,, \quad u_R = V_U^\dagger u'_R \,, \quad u_L = U_U^\dagger u'_L \,, \quad e_R = V_E^\dagger e'_R \,, \quad e_R = U_E^\dagger e'_L$$

from which one has

$$d'_L = U_D d_L \implies \bar{d}'_L = \bar{d}_L U_D^\dagger$$

This also happens for d' and e', and for right-chiral fields.

The neutral current Lagrangian is diagonal in flavor space, so the same as above happens: FCNC are suppressed.

However there are flavor changing charged currents. The quark part of such currents is

$$\mathcal{L}_{\mathrm{cc}}^{\mathrm{hadron}} = \frac{g}{\sqrt{2}}[W_\mu^\dagger \bar{u}'_L \gamma^\mu d'_L + \mathrm{h.c.}]$$

In the interaction basis (the primed fields) the above is diagonal

$$(\bar{u}'_L)^{(f)} \gamma^\mu \delta^{(f)(g)} (d'_L)^{(g)}$$

[r] The physical fields are mass eigenstates?

$$\mathcal{L}_{\mathrm{cc}}^{\mathrm{hadron}} = \frac{g}{\sqrt{2}}[W_\mu^\dagger \bar{u}_L \gamma^\mu U_U^\dagger U_D d_L + \mathrm{h.c.}] \,, \quad U_U^\dagger U_D \neq I$$

The product above of unitary matrices are the Cabibbo–Kobayashi–Mashawa (CKM) matrix

$$V_{\mathrm{CKM}} = U_U^\dagger U_D$$

It is the generalization of the Cabibbo 2×2 matrix. This matrix is unitary

$$V^\dagger V = U_D^\dagger U_U U_U^\dagger U_D = I$$

The rotation of the elementary fields is needed to accomodate empirical evidences. The entry of the matrix must come from experiment. This matrix encodes flavour mixing.

The leptonic charged currents are given by

$$\mathcal{L}_{\mathrm{cc}}^{\mathrm{lepton}} = \frac{g}{\sqrt{2}}[W_\mu^\dagger \bar{\nu}'_L \gamma^\mu e'_L + \mathrm{h.c.}]$$

If neutrino ν are massless, one has the freedom to mix neutrinos with some matrix of charged leptons

$$\nu_L = U_E \nu'_L$$

This works because there is no explicit ϕ^c term in the leptonic sector (because there is no ν_R). [r] Therefore

$$\mathcal{L}_{\mathrm{cc}}^{\mathrm{lepton}} = \frac{g}{\sqrt{2}}[W_\mu^\dagger \bar{\nu}_L \gamma^\mu e_L + \mathrm{h.c.}]$$

where there is an identity matrix in leptonic families. There is no generator of mixing in the leptonic sector.

Therefore, there can be vertices of the type

$$\bar{d}u \,, \quad e^+ \bar{\nu}_e$$

but not

$$\bar{\nu}_e \mu^+$$

Neutrino masses. [r] One can include the masses of neutrinos. To construct the Lagrangian one should know if neutrinos are Dirac or Majorana particles. To the Lagrangian of the Standard Model, one may add

$$\bar{N}_R \mathrm{i}\not{\partial} N_R - \bar{L}'_L h'_N N_R \phi^c - \frac{1}{2} N_R M N_R$$

The third addendum is a Majorana mass term. Therefore, one can parametrize the mass terms in a compact way

$$\mathcal{L}_{\nu\text{mass}} \sim \begin{bmatrix} \nu^\top & N^\top \end{bmatrix} \begin{bmatrix} 0 & h_N v \\ h_N^\top v & M \end{bmatrix} \begin{bmatrix} \nu \\ N \end{bmatrix}, \quad \begin{bmatrix} \nu \\ N \end{bmatrix} = \begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \\ N_e \\ N_\mu \\ N_\tau \end{bmatrix}$$

where one consider directly the mass eigenstates. One should diagonalize the matrix and write the neutrino masses. One notices that neutrino mixing is present which has been observed.

CKM parameters. The CKM matrix is 3×3 because there are three families. Counting the parameters, one needs to add a complex number. The matrix is intrinsically complex and as such there is CP violation. The matrix is unitary

$$VV^\dagger = I_n$$

for n families. The independent parameters of the matrix are n^2. Not all these real parameters are physical. One may think the parameters as angles and phases

$$n^2 = N_{\text{angle}} + N_{\text{phase}}$$

The number of angles N_{angle} is the number of coordinate $2D$-planes of n-dimensional space

$$N_{\text{angle}} = \binom{n}{2} = \frac{1}{2} n(n-1)$$

The phases are then

$$N_{\text{phase}} = n^2 - N_{\text{angle}} = \frac{1}{2} n(n+1)$$

Not all the phases are measurable. For example, the transformation

$$u_L^{(f)} \to \mathrm{e}^{\mathrm{i}\alpha_f} u_L^{(f)}, \quad d_L^{(f)} \to \mathrm{e}^{\mathrm{i}\beta_f} d_L^{(f)}$$

leaves the Lagrangian invariant except in the charged currents $\mathcal{L}_{\text{cc}}^{\text{hadron}}$. In this way

$$V_{fg} \to V_{fg} \mathrm{e}^{\mathrm{i}(\beta_g - \alpha_f)}$$

One can re-absorb $n + (n-1)$ phases into the phases of the quark fields

$$\begin{bmatrix} u & c & t \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} d \\ s \\ b \end{bmatrix}$$

So the number of physical phases are

$$N_{\text{phys phas}} = \frac{1}{2} n(n+1) - (2n-1) = \frac{1}{2}(n-1)(n-2)$$

For $N = 1, 2$ there is no true physical phase. For $N = 3$ there is one physical phase in the CKM matrix. Therefore

$$V_{\text{CKM}} \neq V_{\text{CKM}}^*$$

This implies CP violation which is made possible only for $n \geq 3$. This CP violation is not enough to explain the violation on cosmological scale, but it is enough to explain low energy phenomenology.

The matrix is

$$V_{\mathrm{CKM}} = \begin{bmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{bmatrix}$$

The Feynman rules is [r]

$$\mathrm{i}\frac{g}{\sqrt{2}}\gamma^\mu \frac{1-\gamma_5}{2} V_{fg}$$

The matrix is unitary therefore

$$\sum_{f=1}^{3} |V_{fg}|^2 = \sum_{g=1}^{3} |V_{fg}|^2 = 1\,, \quad \sum_f V_{fg}^* V_{fh} = 0\,, \quad g \neq h$$

Geometrically, the fact that the matrix is unitary implies that the sum of the components must be a triangle in the complex plane. The entries of the matrix are measurable. The measurement give a circumference on which a vertex lies. This is the test of unitarity of the CKM matrix.

Lecture 8

mar 19 mar 2024 12:30

2.5 Charge-parity violation

See Schwartz, §29.5. QED is a vector theory, so it conserves parity and charge conjugation; CP symmetry is conserved. QCD is also a vector theory, but it does not necessarily conserve CP symmetry: this is the strong CP violation problem. According to the theory, there is room for CP violation which is not particularly suppressed, but in experiments it is indeed suppressed.

The weak sector exhibits CP violation. One may study spinor bilinear fields because they appear in interaction terms. First, one is concerned with how the spinor bilinear operators transform under CP

$$\begin{gathered} (\bar{\psi}_f \psi_g)(t, \mathbf{x}) \to \bar{\psi}_g \psi_f (t, -\mathbf{x}) \\ \bar{\psi}_f \gamma_5 \psi_g \to -\bar{\psi}_g \gamma_5 \psi_f \\ \bar{\psi}_f \not{A} \psi_g \to \bar{\psi}_g \not{A} \psi_f \\ \bar{\psi}_f \not{A} \gamma_5 \psi_g \to \bar{\psi}_g \not{A} \gamma_5 \psi_f \end{gathered}$$

[r] The charged current Lagrangian in the mass eigenbasis is

$$\mathcal{L}_{\mathrm{cc}} = \frac{g}{\sqrt{2}}[W_\mu^+ \bar{u}^f V_{fg} \gamma^\mu \frac{1}{2}(1-\gamma_5) d^g + W_\mu^- \bar{d}^f (V_{gf})^* \gamma^\mu \frac{1}{2}(1-\gamma_5) u^g] = \frac{g}{\sqrt{2}}[\mathcal{W}^+ + \mathcal{W}^-]$$

Under CP transformation, one obtains

$$\begin{aligned} \mathcal{W}^- &\to W_\mu^+ \bar{u}^g (V_{gf})^* \gamma^\mu \frac{1}{2}(1-\gamma_5) d^f \\ \mathcal{W}^+ &\to W_\mu^- \bar{d}^g V_{fg} \gamma^\mu \frac{1}{2}(1-\gamma_5) u^f \end{aligned}$$

CP symmetry is preserved if $\mathcal{W}^\mp \to \mathcal{W}^\pm$. One may check if this is the case

$$\mathrm{CP}(\mathcal{W}^+) = \bar{d}^g V_{fg} \Gamma u_f\,, \quad \Gamma = \frac{1}{2}\gamma^\mu (1-\gamma_5)$$

while

$$\mathcal{W}^- = \bar{d}^g V_{fg}^* \Gamma u^f$$

For the two to be equal, one has to have

$$V_{fg} = V_{fg}^*$$

This does not happen for three families of matter because the V matrix has a complex phase: it implies CP violation.

This violation happens in nature. In particular, in the oscillation and decay of $\bar{K}^0$-K^0 systems. The quark content of these kaons is

$$\left|K^0\right\rangle \sim |d\bar{s}\rangle \,, \quad \left|\bar{K}^0\right\rangle \sim \left|\bar{d}s\right\rangle$$

One may define the following CP eigenstates

$$\left|K^\pm\right\rangle = \frac{1}{\sqrt{2}}(\left|K^0\right\rangle \pm \left|\bar{K}^0\right\rangle)$$

with eigenvalue ± 1. The state long $|K_L\rangle$ is dominated by the eigenstate $|K^-\rangle$. If CP is conserved one would assumes that the eigenvalue is conserved. However, rarely it decays to

$$K_L \to \pi^0\pi^0$$

which has CP eigenvalue $+1$. In neutrally charged, strange mesons there is CP violation.

The parameter ε_k in the superposition of states of kaon long can be measured

$$|K_L\rangle = \left|K^-\right\rangle + \varepsilon_k \left|K^+\right\rangle$$

2.6 CKM matrix and Beyond the Standard Model tests

One of the possible ways of parametrizing the CKM matrix is

$$\begin{aligned} V_{\text{CKM}} &= \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{i\delta} \\ \cdots & \cdots & s_{23}c_{13} \\ \cdots & \cdots & c_{23}c_{13} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix} \begin{bmatrix} c_{13} & 0 & s_{13}e^{i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{bmatrix} \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

where ij are flavour indices and s and c are sine and cosine of θ_{ij}. The last matrix on the right is the Cabibbo matrix. The measurements are

$$s_{12} \approx 0.225 \,, \quad s_{23} \approx 0.04 \,, \quad s_{13} \approx 0.003 \,, \quad \delta \approx 65^\circ$$

One notices that the θ_{ij} angles are not big

$$\theta_{13} \ll \theta_{23} \ll \theta_{12}$$

and the δ angle is not small. One may check that

$$|V| \approx \begin{bmatrix} 0.97 & 0.226 & 0.003 \\ 0.226 & 0.97 & 0.04 \\ 0.008 & 0.04 & 0.99 \end{bmatrix} \approx \begin{bmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{bmatrix} + o(\lambda^3) \,, \quad \lambda \approx 0.22$$

This matrix is kind of the identity. One may write the matrix in an approximate way to understand how the physics enter in the matrix.

Wolfenstein parametrization. [r] One introduces four parameters A, λ, ρ, η to get

$$s_{12} = \lambda = \frac{|V_{us}|}{\sqrt{|V_{ud}|^2 + |V_{us}|^2}} \,, \quad s_{13} = A\lambda^2 = \lambda\left|\frac{V_{cb}}{V_{us}}\right| , \quad s_{13}e^{i\delta} = A\lambda^3(\rho + i\eta) \equiv V_{ub}^*$$

The exact CKM matrix becomes

$$V_{\text{CKM}} = \begin{bmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{bmatrix} + o(\lambda^3)$$

The pattern becomes clear and the quark mixing is

$$(12) \gg (23) \gg (13)$$

There is also a CP violating phase.

Jarlskog invariant. One would like to quantify CP violation without using an explicit parametrization

$$\mathrm{Im}(V_{ij}V_{kl}V_{il}^*V_{kj}^*) = J\sum_{mn}\varepsilon_{ikm}\varepsilon_{jln}$$

where J is the invariant. If $J \neq 0$ then there is CP violation. This parameter can be written as

$$J = c_{12}c_{23}c_{13}^2 s_{12}s_{13}s_{23}(\sin\delta) \approx \lambda^6 A^2 \eta$$

Remark. The Jarlskog invariant is not zero because all mixing angles are not zero.

Remark. The invariant is small (so CP violation is suppressed) because the mixing (angles) are small (and not because δ is small). In fact $J \sim \lambda^6$.

Unitary triangle. Since the matrix is unitary, there are six combinations of entries that must be zero. One in particular is the (db) combination

$$0 = \sum_{i=1}^3 V_{id}V_{ib}^* = V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^*$$

For this particular combination, the three addenda are of the same order λ^3. The triangle is almost equilateral. Since J does not depend on the parametrization, then the area of the triangles is the same since it is proportional to J.

Dividing by $V_{cd}V_{cb}^*$, one obtains

$$1 + \frac{V_{ub}^*V_{ud}}{V_{cd}V_{cb}^*} + \frac{V_{tb}^*V_{td}}{V_{cd}V_{cb}^*} = 0$$

[r] diagr
This is the unitary triangle. The edges have lengths of order 1. The lengths of the sides are

$$|AB| = \sqrt{(1-\rho)^2 + \eta^2}\,, \quad |OA| = \sqrt{\rho^2+\eta^2}$$

The angles are related to quantities describing how quark flavours mix. One may check if one can over-constrain the triangle (by taking more measurements than parameters) and see if everything is consistent.

3 Unitarity and optical theorem

One would like to understand why a decaying particle has a decay width, why the Higgs boson gives longitudinal degrees of freedom to vector gauge bosons and why scattering these degrees of freedom gives a unitary phenomenon.

Quantum mechanics. In ordinary quantum mechanics, a Hilbert space is complete

$$I = \sum_x \int \mathrm{d}\Pi_x \, |x\rangle\langle x| \,, \quad \mathrm{d}\Pi_x = \prod_{j\in x} \frac{\mathrm{d}^3 p_j}{(2\pi)^3 2E_j} = \prod_{j\in x}[\mathrm{d}p_j]$$

[r] where one sums over all possible states x of the theory. The phase-space $\mathrm{d}\Pi_x$ is Lorentz-invariant. The scattering matrix S may be obtained through

$$|\psi, t\rangle = \mathrm{e}^{-\mathrm{i}Ht}\,|\psi, 0\rangle \equiv S\,|\psi, 0\rangle$$

This relates the unitary of the theory to the properties of the scattering matrix. The transition density probability is

$$|\langle f|\, S\,|i\rangle|^2$$

Summing over all final states, one has

$$1 = \sum_f |\langle f|\,S\,|i\rangle|^2 = \sum_f \langle f|\,S\,|i\rangle\,(\langle f|\,S\,|i\rangle)^* = \sum_f \langle i|\,S^\dagger\,|f\rangle\,\langle f|\,S\,|i\rangle = \langle i|\,S^\dagger S\,|i\rangle \implies S^\dagger S = I$$

if one assumes that $\langle i|i\rangle = 1$.

When computing transition amplitudes, one is interested in the non-identity part of the S-matrix given by the transition matrix T

$$S = I + \mathrm{i}T\,, \quad \langle f|\,T\,|i\rangle = (2\pi)^4\delta^{(4)}(p_i - p_f)\mathcal{M}$$

In the Feynman amplitude there are only connected diagrams contributions. In momentum space, one has

$$\mathrm{i}\mathcal{M} = \sum \text{Feynman diagrams}$$

From the unitarity of the S-matrix, one obtains

$$(I + \mathrm{i}T)(I - \mathrm{i}T^\dagger) = I \implies \mathrm{i}(T^\dagger - T) = T^\dagger T$$

Calculating the expectation value between the states $\langle f|$ and $|i\rangle$, one obtains

$$\boxed{\mathcal{M}(i \to f) - \mathcal{M}^*(f \to i) = \mathrm{i}\sum_x \int \mathrm{d}\Pi_x\,(2\pi)^4\delta^{(4)}(p_i - p_x)\mathcal{M}(i \to x)\mathcal{M}^*(f \to x)}$$

This is the generalized optical theorem. There are non trivial links between amplitudes with initial and final states.

The expansion of the amplitudes in the left-hand side has powers higher of the coupling constant than the amplitudes in the right-hand side.

Lecture 9

lun 25 mar 2024 12:30

The above identity must hold order by order in perturbation theory. There is a link between tree-level and one-loop amplitudes (to compute complicated loop amplitudes, one may use concepts from tree-level amplitudes).

One may see a few applications of the theorem. Consider $|i\rangle = |f\rangle = |A\rangle$. From the optical theorem one has

$$2\mathrm{i}\,\mathrm{Im}\,\mathcal{M}(A \to A) = \mathrm{i}\sum_x \int \mathrm{d}\Pi_x\,(2\pi)^4\delta^{(4)}(p_A - p_x)|\mathcal{M}(A \to x)|^2$$

One may assume that the state $|A\rangle$ is a one-particle state. The transition width is

$$\Gamma(A \to x) = \frac{1}{2m_A}\int \mathrm{d}\Pi_x\,(2\pi)^4\delta^{(4)}(p_A - p_x)|\mathcal{M}(A \to x)|^2$$

From the above relation from the optical theorem, one obtains

$$\mathrm{Im}\,\mathcal{M}(A \to A) = m_A\Gamma_{A,\mathrm{tot}}$$

One may instead assume that the state $|A\rangle$ is a two-particle state. [r] The cross-section is

$$\begin{aligned}\sigma(A \to x) &= \frac{1}{4E_{\mathrm{CM}}|\mathbf{p}_{\mathrm{in}}|}\int \mathrm{d}\Pi_x\,(2\pi)^4\delta^{(4)}(p_A - p_x)|\mathcal{M}(A \to x)|^2 \\ &= \frac{1}{2s}\int \mathrm{d}\Pi_x\,(2\pi)^4\delta^{(4)}(p_A - p_x)|\mathcal{M}(A \to x)|^2\end{aligned}$$

At the second equality one one assumes that the two particles are massless, so the notation simplifies (see Schwartz).

From the consequence of the optical theorem, one obtains

$$\mathrm{Im}\,\mathcal{M}(A \to A) = s\sum_x \sigma(A \to x) = s\sigma_{\mathrm{tot}}(A)$$

On the left-hand side, if A is a state of two colliding particles, then the final state is a scattering without any deflection: this is called forward scattering amplitude.

Example. Consider two scalar fields

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{1}{2}M^2\Phi^2 + \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{2}\Phi\varphi\varphi$$

If $M > 2m$, then the decay $\Phi \to \varphi\varphi$ is kinematically possible. One would like to explicitly verify that

$$\operatorname{Im} \mathcal{M}(\Phi \to \Phi) = M\Gamma(\Phi \to \varphi\varphi) + o(\lambda)$$

One needs to compute the amplitude of the left-hand side [r] diagr

$$\mathrm{i}\mathcal{M} = 2\left[\frac{\mathrm{i}\lambda}{2}\right]^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\varepsilon} \frac{\mathrm{i}}{(k-p)^2 - m^2 + \mathrm{i}\varepsilon} = 2\left[\frac{\mathrm{i}\lambda}{2}\right]^2 I$$

Using Feynman parameters

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \, \frac{1}{[A + (B-A)x]^2}$$

then the above becomes

$$I \sim \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \left[\frac{1}{(k^2 - A^2 + \mathrm{i}\varepsilon)^2}\right]$$

One may apply the Pauli–Villars regularization to get

$$I \sim \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \left[\frac{1}{(k^2 - A^2 + \mathrm{i}\varepsilon)^2} - \frac{1}{(k^2 - \Lambda^2 + \mathrm{i}\varepsilon)^2}\right] = -\frac{\mathrm{i}}{16\pi^2} \log \frac{A^2}{\Lambda^2}$$

Finally

$$\mathrm{i}\mathcal{M} = -\mathrm{i}\frac{\lambda^2}{32\pi^2} \int_0^1 \mathrm{d}x \log \frac{m^2 - \mathrm{i}\varepsilon - p^2 x(1-x)}{\Lambda^2}$$

Noting that $p^2 = M^2$ since the field Φ is on-shell. So

$$\mathcal{M}(\Phi \to \Phi) = -\frac{\lambda^2}{32\pi^2} \int_0^1 \mathrm{d}x \log \frac{m^2 - M^2 x(1-x) - \mathrm{i}\varepsilon}{\Lambda^2}$$

One may notice that

$$x \in [0,1] \implies x(1-x) \leq \frac{1}{4}$$

If the numerator is negative, one has to be careful about the analytic continuation signaled by $\mathrm{i}\varepsilon$. If $M < 2m$ then the numerator is positive and the logarithm is real

$$\operatorname{Im} \mathcal{M} = 0 \implies \Gamma = 0$$

This is the case where no decay happens. If $M > 2m$ then the logarithm is of the type

$$\log(-A - \mathrm{i}\varepsilon) = \log A - \mathrm{i}\pi \,, \quad A > 0$$

In fact, one may see that

$$-A - \mathrm{i}\varepsilon = A(-1 - \mathrm{i}\varepsilon) = A\mathrm{e}^{-\mathrm{i}\pi}$$

where the branch cut is on $(-\infty, 0]$. A point infinitesimally under the branch cut is rewritten in polar coordinates. One then obtains the final result

$$\operatorname{Im} \mathcal{M}_{\mathrm{1Loop}} = \frac{\lambda^2}{32\pi} \int_0^1 \mathrm{d}x \, \theta[M^2 x(1-x) - m^2] = \frac{\lambda^2}{32\pi}\sqrt{1 - \frac{4m^2}{M^2}}$$

Since, noting Heaviside's theta function, the support of the integral is dictated by

$$x(1-x) > \frac{m^2}{M^2} \implies x_{1,2} = \frac{1}{2}\left[1 \pm \sqrt{1 - \frac{4m^2}{M^2}}\right]$$

The right-hand side is

$$\Gamma(\Phi \to \varphi\varphi) = \frac{1}{2}\frac{1}{2M}\int |\mathcal{M}|^2 \, \mathrm{d}\phi_2 \,, \quad \mathrm{d}\phi_2 = \frac{|\mathbf{p}_f|}{M}\frac{\mathrm{d}\Omega}{16\pi^2} \,, \quad M = E_1 + E_2 \,, \quad \mathcal{M}_{\text{tree}} = \frac{\lambda}{2}$$

where $\frac{1}{2}$ comes about because there are two identical bosons in the final state; also

$$E_1 = E_2 \,, \quad |\mathbf{p}_1| = |\mathbf{p}_2| \,, \quad |\mathbf{p}_f|^2 = \frac{1}{4}M^2 - m^2$$

The partial width is then

$$\Gamma = \frac{1}{2}\frac{1}{2M}\left[\frac{1}{4} - \frac{m^2}{M^2}\right]^{\frac{1}{2}} \frac{\mathrm{d}\Omega}{16\pi^2}\lambda^2 = \frac{\lambda^2}{32\pi}\frac{1}{M}\sqrt{1 - \frac{4m^2}{M^2}}$$

This is exactly the same result as before, apart from the extra factor M. Therefore

$$\operatorname{Im} M_{1\mathrm{L}} = M\Gamma$$

Width of unstable particles. One may see a first application. From this one may see why a decaying particle has a width. Consider a one-particle state and the one-particle irreducible amplitude [r] diagr

$$\cdots \equiv \mathrm{i}\Pi(p^2)$$

The full propagator is obtained through Dyson resummation [r] diagr

$$\cdots = \frac{\mathrm{i}}{p^2 - m^2} + \frac{\mathrm{i}}{p^2 - m^2}(\mathrm{i}\Pi)\frac{\mathrm{i}}{p^2 - m^2} + \cdots = \frac{\mathrm{i}}{p^2 - m^2}\left[1 - \mathrm{i}\Pi\frac{\mathrm{i}}{p^2 - m^2}\right]^{-1} = \frac{\mathrm{i}}{p^2 - m^2 + \Pi(p^2)}$$

One may assume that Π is an imaginary number

$$\cdots = \frac{\mathrm{i}}{p^2 - m^2 + \operatorname{Re}\Pi + \mathrm{i}\operatorname{Im}\Pi}$$

The quantity m is a parameter in the Lagrangian. The physical mass is not this parameter, but the pole of the real part of the propagator. The physical mass is then

$$m^2_{\text{Phys}} = m^2 - \operatorname{Re}\Pi(m^2_{\text{Ph}})$$

Therefore, the full propagator is

$$\cdots = \frac{\mathrm{i}}{p^2 - m^2_{\text{Ph}} + \mathrm{i}\operatorname{Im}\Pi(p^2)}$$

A particle is unstable if

$$\operatorname{Im}\Pi(p^2) \neq 0$$

From the result of the optical theorem for one-particle states, one has

$$\operatorname{Im}\mathcal{M}(A \to A) = m_A\Gamma_{\text{tot}}$$

The one-particle irreducible amplitude is then

$$\mathrm{i}\Pi \equiv \cdots = \mathrm{i}\mathcal{M}(A \to A) \implies \operatorname{Im}\Pi = m_A\Gamma_{\text{tot}}$$

The full propagator for an unstable particle is

$$\cdots = \frac{\mathrm{i}}{p^2 - m^2_{\text{Ph}} + \mathrm{i}m_{\text{Ph}}\Gamma_{\text{tot}}}$$

The modulus squared has a Breit–Wigner distribution. The width Γ is the same width of the distribution.

Example. One may see a second application. The numerator of the propagators is the sum over the physical spin states. For a scalar particle, one has

$$\frac{\mathrm{i}}{p^2 - m^2} \to 1$$

In fact, scalar particles have only one polarization and it is not dependent on momentum. For spin $\frac{1}{2}$ fermions, one has

$$\frac{\mathrm{i}}{p^2 - m^2}(\not{p} + m) = \frac{\mathrm{i}}{p^2 - m^2} \sum_{\text{spin}} u\bar{u}$$

For massive vector boson, one has

$$\frac{\mathrm{i}}{p^2 - m^2}\left[-\eta^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}\right]$$

The bracket can be seen as a matrix with three 1 eigenvalue and one 0 eigenvalue: of the first three, two are transverse and one is longitudinal.

The fact that the numerator is the sum of the spins can be proven from the unitarity of the theory (See Schwartz).

Unitarity bounds. One can find a third application. The optical theorem can be seen as a constraint between the amplitude and its modulus squared. Given that the modulus is

$$|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^*$$

then the amplitude cannot be arbitrarily large but has to be bounded.

One may study elastic scattering in the center-of-mass frame

$$A(p_1) + B(p_2) \to A(p_3) + B(p_4)$$

Assuming particles to be massless, one has

$$\sigma_{\text{tot}}(A + B \to A + B) = \frac{1}{2s}\frac{1}{16\pi}\int \mathrm{d}(\cos\theta)\overline{\sum}|\mathcal{M}|^2$$

One assumes that $\mathcal{M}$ is a function of the polar angle θ only due to cylindrical symmetry. In this scattering, one has

$$|\mathbf{p}_3| = |\mathbf{p}_4| = |\mathbf{p}_1| = |\mathbf{p}_2|$$

The amplitude can be decomposed through Legendre polynomials

$$\mathcal{M}(\theta) = 16\pi \sum_{j=0}^{\infty} a_j (2j+1) P_j(\cos\theta)$$

where they are normalized as

$$P_j(1) = 1 \iff \theta = 0\,; \qquad \int_{-1}^{1} P_j(\cos\theta) P_k(\cos\theta)\,\mathrm{d}(\cos\theta) = \frac{2}{2j+1}\delta_{jk}$$

Through the above decomposition, one may write

$$\sigma_{\text{tot}} = \frac{16\pi}{s}\sum_{j=0}^{\infty}(2j+1)|a_j|^2$$

Comparing this with the optical theorem, one gets constraints on a_j and the amplitude.

Lecture 10

mar 26 mar 2024 12:30

The optical theorem applied to the cross-section gives

$$\operatorname{Im}\mathcal{M}(AB \to AB, \theta = 0) = s\sum_x \sigma_{\text{tot}}(AB \to x) \geq s\sigma_{\text{tot}}(AB \to AB)$$

Substituting the above expressions, one obtains

$$16\pi\sum_{j=0}^{\infty}(2j+1)\operatorname{Im}a_j \geq 16\pi\sum_{j=0}^{\infty}(2j+1)|a_j|^2$$

This is the partial wave unitary bound. Since, in general it holds

$$|a_j| \geq \operatorname{Im}a_j$$

Then $|a_j|$ cannot be arbitrarily large since its square must be smaller than the imaginary part.

Special case. Consider

$$\sum_x \sigma_{\text{tot}}(AB \to x) \approx \sigma_{\text{tot}}(AB \to AB)$$

It follows

$$\operatorname{Im}a_j = |a_j|^2$$

Letting $a_j = x + \mathrm{i}y$, one sees that

$$y = x^2 + y^2 \implies x^2 + \left[y - \frac{1}{2}\right]^2 = \frac{1}{4}$$

which is a circumference of radius $\frac{1}{2}$ centered at $(0, \frac{1}{2})$.

In general, one can prove that

$$\forall j\,,\quad |a_j| \leq 1\,,\quad 0 \leq \operatorname{Im}a_j \leq 1\,,\quad |\operatorname{Re}a| \leq \frac{1}{2}$$

These constraints can be used to show that certain cross-sections computed through the Fermi interaction are not unitary.

Propagators for massive vector bosons and polarization states. Consider the Proca Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu\,,\quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

The equations of motion are

$$(\Box + m^2)A_\mu = 0\,,\quad \partial_\mu A^\mu = 0$$

The second equation is not the gauge-fixing Lorenz condition, but comes from the equations of motion. In fact

$$\partial_{A_\sigma}\mathcal{L} = m^2A^\sigma\,,\quad \partial_{\partial_\rho A_\sigma}\mathcal{L} = -F^{\rho\sigma}\,,\quad \partial_\rho\partial_{\partial_\rho A_\sigma}\mathcal{L} = \partial_{A_\sigma}\mathcal{L}$$

The Euler–Lagrange equations are

$$\partial_\rho F^{\rho\sigma} + m^2A^\sigma = 0$$

Expanding $F^{\rho\sigma}$ one obtains

$$(\Box + m^2)A^\sigma - \partial^\sigma(\partial_\rho A^\rho) = 0$$

Acting with a derivative, one has

$$(\Box + m^2)\partial_\sigma A^\sigma - \Box(\partial_\rho A^\rho) = 0 \implies \partial_\sigma A^\sigma = 0$$

The solution of the equations of motion in configuration space is

$$A^\mu(x) = \sum_j \frac{\mathrm{d}^3k}{(2\pi)^3}\widetilde{a}(\mathbf{k})\varepsilon^\mu_j(\mathbf{k})\mathrm{e}^{\mathrm{i}kx}\,, \quad k^0 = \sqrt{\mathbf{k}^2+m^2}$$

where j is the polarization index [r] where sum over k?. The base upon which the solution is decomposed is

$$\varepsilon^\mu_0 = (1,0,0,0)\,, \quad \varepsilon^\mu_1 = (0,1,0,0)\,, \quad \varepsilon^\mu_2 = (0,0,1,0)\,, \quad \varepsilon^\mu_3 = (0,0,0,1)$$

The constraint found from the equations of motion implies

$$\partial_\mu A^\mu = 0 \implies k_\mu\varepsilon^\mu_j(\mathbf{k}) = 0\,, \quad k^2 = m^2$$

Therefore, there are three possible solutions with the following normalization convention

$$\varepsilon^\mu_i(\varepsilon^i_\mu)^* = -1$$

Letting the wave number be

$$k^\mu = (E,0,0,k_z)$$

One has

$$\varepsilon^\mu_1 = (0,1,0,0)\,, \quad \varepsilon^\mu_1 = (0,0,1,0)\,, \quad \varepsilon^\mu_3 = \frac{1}{m}(k_z,0,0,E)\,, \quad \varepsilon^\mu_i k_\mu = 0$$

In the high-energy limit, the longitudinal polarization vector is

$$\varepsilon_3 \sim \frac{E}{m}(1,0,0,1) \sim k^\mu$$

The vector scales with the momentum itself. If there is no Higgs [r] the amplitude breaks unitarity. The Higgs boson makes unitary the high-energy limit.

From the above, one can verify that the sum of the physical polarization is equal to the numerator of the propagator

$$\sum_i \varepsilon^\mu_i\varepsilon^\nu_i = -\eta^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}$$

This does not mean that the propagator is on-shell. One may compute the above sum. One notices that

$$\sum_{j=1}^2 \varepsilon^\mu_i\varepsilon^\nu_i = \mathrm{diag}(0,1,1,0)$$

Also

$$\varepsilon^\mu_3\varepsilon^\nu_3 = \frac{1}{m^2}\,\mathrm{diag}(k_z^2,0,0,E^2) + \frac{1}{m^2}\,\mathrm{antidiag}(k_zE,0,0,k_zE)$$

Therefore

$$\begin{aligned}\sum_{i=1}^3 \varepsilon^\mu_i\varepsilon^\nu_i &= \mathrm{diag}(-1,1,1,1) + \frac{1}{m^2}\,\mathrm{diag}(k_z^2+m^2,0,0,E^2-m^2) + \frac{1}{m^2}\,\mathrm{antidiag}(k_zE,0,0,k_zE)\\ &= -\eta^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}\end{aligned}$$

[r] For massless vector bosons (like gluons) the situation requires more care.

4 Electroweak sector – phenomenology application

4.1 Kinematics

The differential cross-section of a two-scattering into n products is

$$\mathrm{d}\sigma = \mathcal{F}\overline{\sum}|\mathcal{M}|^2\,\mathrm{d}\phi_n$$

where $\mathcal{F}$ is the flux factor and the Lorentz-invariant phase-space is

$$\mathrm{d}\phi_n = (2\pi)^4 \delta^{(4)}(p_i - k_f) \prod_{i=1}^{n} [\mathrm{d}k_i] \,, \quad [\mathrm{d}k_i] = \frac{\mathrm{d}^3 k_i}{(2\pi)^3 2E_i}$$

If the particles with momenta p_1 and p_2 are massless, then

$$\mathcal{F} = \frac{1}{2s} \,, \quad s = (p_1 + p_2)^2$$

Two-body final state. If the final state has two particles $Q \to k_1 + k_2$, then the phase-space is

$$\mathrm{d}\phi_2 = \frac{1}{16\pi^2} \frac{\widetilde{k}}{Q_0} \,\mathrm{d}\cos\theta \,\mathrm{d}\varphi \,, \quad Q = p_1 + p_2$$

where

$$\widetilde{k}^2 = \frac{1}{4Q_0^2}[Q_0^2 - (m_1 + m_2)^2][Q_0^2 - (m_1 - m_2)^2] \,, \quad k_j^2 = m_j^2$$

For equal masses, one has

$$\widetilde{k}^2 = \frac{1}{4Q_0^2} Q_0^2 (Q_0^2 - 4m^2) \implies \mathrm{d}\phi_2 = \frac{1}{32\pi^2} \sqrt{1 - \frac{4m^2}{Q_0^2}} \,\mathrm{d}\cos\theta \,\mathrm{d}\varphi$$

In the center-of-mass frame one has

$$Q^\mu = (Q_0, 0, 0, 0) \,, \quad Q_0^2 = Q^2 \,, \quad \mathrm{d}\phi_2 = \frac{1}{32\pi^2} \sqrt{1 - \frac{4m^2}{Q^2}} \,\mathrm{d}\cos\theta \,\mathrm{d}\varphi$$

Instead, for massless particles, one has

$$\mathrm{d}\phi_2 = \frac{1}{32\pi^2} \,\mathrm{d}\cos\theta \,\mathrm{d}\varphi = \frac{\mathrm{d}\Omega}{32\pi^2}$$

Notice how there are two variables: there are six degrees of freedom, but four conservation of four-momentum components.

4.2 Z boson decay width

One would like to compute the decay width of a Z boson into a fermion–anti-fermion pair [r] diagr

$$\mathrm{Z} \to \mathrm{f\bar{f}} \,, \quad f = e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau, u, d, c, s, b$$

The decay into top quarks is not kinematically possible. The above vertex contributes with

$$\frac{\mathrm{i}e}{2\sin\theta\cos\theta} \gamma^\mu (v_f - a_f \gamma_5) \,, \quad v_f = T_3^f - 2Q^f \sin^2\theta \,, \quad a_f = T_3^f \,, \quad e = g\sin\theta$$

where θ is Weinberg's angle.

Particular final state. One computes the width for a given final state and the sums over all states

$$\Gamma_{ff} = \frac{1}{2M_Z} \int \mathrm{d}\phi_2 \overline{\sum} |\mathcal{M}|^2$$

[r] diagr From momentum conservation, one has

$$q = p + p' \,, \quad p^2 = p'^2 = m^2 \,, \quad q^2 = M_Z^2 = (p + p')^2 = 2pp' + 2m^2$$

The amplitude is

$$\mathrm{i}\mathcal{M} = \bar{u}(p) \frac{\mathrm{i}g}{2\cos\theta} \gamma^\mu (v - a\gamma_5) v(p') \varepsilon_\mu(q)$$

Its complex conjugate is

$$\mathcal{M}^* = \frac{g}{2\cos\theta}\bar{v}(p')\gamma^\nu(v - a\gamma_5)u(p)\varepsilon^*_\nu(p)$$

Therefore, the sum over the spin is

$$\begin{aligned}\sum|\mathcal{M}|^2 &= \frac{g^2}{4\cos^2\theta}\operatorname{Tr}[(\not{p} + m)\gamma^\mu(v - a\gamma_5)(\not{p}' - m)\gamma^\nu(v - a\gamma_5)]\sum_{\text{pol}}\varepsilon_\mu\varepsilon^*_\nu \\ &= \frac{g^2}{4\cos^2\theta}4\left[(M_Z^2 + 2m^2)v^2 + (M_Z^2 - 4m^2)a^2\right]\end{aligned}$$

where one has

$$\sum_{\text{pol}}\varepsilon_\mu\varepsilon^*_\nu = -\eta_{\mu\nu} + \frac{q_\mu q_\nu}{M_Z^2}$$

Exercise: compute the sum over spins above.

One may apply the approximation $m \approx 0$ since the heaviest particle is

$$m_{\mathrm{b}} \approx 4.5\,\mathrm{GeV} \ll M_Z$$

The second term in the polarization sum does not contribute. In fact, the amplitude is

$$\mathcal{M} \sim \bar{u}(p)\gamma^\mu(v - a\gamma_5)v(p')\varepsilon_\mu$$

when squaring it, one obtains

$$|\mathcal{M}|^2 \rightsquigarrow \bar{u}(p)\gamma^\mu(v - a\gamma_5)v(p')q_\mu = \bar{u}(p)(\not{p} + \not{p}')(v - a\gamma_5)v(p') = 0 \impliedby \bar{u}(p)\not{p} = 0$$

Therefore, one has

$$\sum|\mathcal{M}|^2 = G\operatorname{Tr}[\not{p}\gamma^\mu(v - a\gamma_5)\not{p}'\gamma^\nu(v - a\gamma_5)](-\eta_{\mu\nu})\,,\quad G = \frac{g^2}{4\cos^2\theta}$$

The trace is

$$\begin{aligned}-\eta_{\mu\nu}\operatorname{Tr} &= -\operatorname{Tr}[\not{p}\gamma^\mu(v - a\gamma_5)\not{p}'\gamma_\mu(v - a\gamma_5)] = -\operatorname{Tr}\left[\not{p}\gamma^\mu\not{p}'\gamma_\mu(v - a\gamma_5)^2\right] \\ &= -\operatorname{Tr}\left[\not{p}\gamma^\mu\not{p}'\gamma_\mu(v^2 + a^2)\right] + 2va\operatorname{Tr}[\not{p}\gamma^\mu\not{p}'\gamma_\mu\gamma_5] \\ &= 8(v^2 + a^2)pp' + 0\end{aligned}$$

At the third line one applies

$$\operatorname{Tr}[\not{p}\gamma^\mu\not{p}'\gamma_\mu] = -2\operatorname{Tr}[\not{p}\not{p}'] = -8pp'\,,\quad \gamma^\mu\gamma_\alpha\gamma_\mu = -2\gamma_\alpha$$

and also

$$\operatorname{Tr}\left[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma_\mu\gamma_5\right] \propto \varepsilon^{\alpha\mu\beta}{}_\mu = 0$$

Lecture 11

mer 27 mar 2024 10:30

The decay width is then

$$\mathrm{d}\Gamma = \frac{1}{2M_Z}\frac{1}{3}\sum|\mathcal{M}|^2\left[\frac{1}{32\pi^2}\,\mathrm{d}\cos\theta\,\mathrm{d}\varphi\right]$$

The factor $\frac{1}{3}$ comes from the averaging of the initial polarization states. One notices that

$$2pp' = (p + p')^2 = M_Z^2$$

There is no dependence of the matrix element on the polar angle. So one may integrate the phase space

$$\Gamma = \frac{1}{2M_Z}\frac{1}{3}(v_f^2 + a_f^2)8pp'\frac{g^2}{4\cos^2\theta}\frac{4\pi}{32\pi^2} = (v_f^2 + a_f^2)M_Z\frac{g^2}{48\pi\cos^2\theta}$$

The total decay width is obtained by summing over all decay channels

$$\Gamma_{\text{tot}} = \Gamma_{ee} + \Gamma_{\mu\mu} + \Gamma_{\tau\tau} + \Gamma_{\text{hadrons}} + N\Gamma_{\nu\nu}$$

In the limit of massless fermions, the first three widths are the same. The width of the hadrons is the sum over the available quark flavours. The Z boson decays in quarks but only hadrons are detected: this does not change the width, only the distribution. So

$$\Gamma_{\text{hadrons}} = \Gamma_{uu} + \Gamma_{dd} + \Gamma_{cc} + \Gamma_{ss} + \Gamma_{bb} = 2\Gamma_{uu} + 3\Gamma_{dd}$$

Quarks carry color and they live in the fundamental representation of SU(3). When one sums over the quantum numbers, the final state has a multiplicity of 3 due to the colors.

Using the parameters v_f and a_f for u, d, ν and e, and using $N_\nu = 3$, one obtains a theoretical value of

$$\Gamma_{\text{th}} \approx 2.44\,\text{GeV}$$

while the experimental value is

$$\Gamma_{\text{exp}} \approx 2.49\,\text{GeV}$$

The result has been obtain with some approximations. One has neglected the effects of the masses that give the following contribution to the phase-space $\mathrm{d}\phi_2$

$$\sqrt{1 - \frac{4m^2}{M_Z^2}}$$

One should also include the loop effects from the electroweak sector and the QCD sector.

One may measure the cross-section as a function of the center-of-mass energy $\sqrt{s}$. Since the Z boson decays, one see a Breit–Wigner centered around M_Z. From the distribution, one can measure the width.

The neutrino decays are not detected because they interact weakly, so only sees effects [r]. If one leaves the number of neutrinos as an unknown, one may fit the distribution of the Z boson to extrapolate a number. The number obtained is 3.

If this is not the case, then there may be new physics: the Z boson may couple to another field whose particles are not seen.

4.3 Forward-backward asymmetry in electron-positron annihilation

Consider the scattering

$$e^- e^+ \to \mu^+ \mu^-$$

in QED [r] diagr. The amplitude is

$$\mathrm{i}\mathcal{M}_\gamma = [\bar{u}(q)\mathrm{i}e\gamma^\mu v(\bar{q})]\frac{-\mathrm{i}\eta^{\mu\nu}}{Q^2}[\bar{v}(\bar{p})\mathrm{i}e\gamma^\nu u(p)] \implies \mathcal{M}_\gamma = \frac{e^2}{Q^2}[\bar{u}(q)\gamma^\mu v(\bar{q})][\bar{v}(\bar{p})\gamma_\mu u(p)]$$

The sum over the spins is

$$\sum \mathcal{M}^2 = \frac{8e^4}{Q^2}(u^2 + t^2)$$

where one has

$$t = (q-p)^2 = -2E_e E_\mu(1-\cos\theta)\,, \quad u = (\bar{q}-p)^2 = -2E_e E_\mu(1+\cos\theta)$$

where θ is the angle with respect to the initial collision line. The cross-section is symmetric for $\theta \to \pi - \theta$: the distribution of muons should be the same.

If one includes full electroweak effect, there is an asymmetry [r] diagr. The amplitude is

$$\begin{aligned}\mathrm{i}\mathcal{M} &= (\mathrm{i}e)^2[\bar{u}(q)\gamma_\mu v(\bar{q})]\frac{-\mathrm{i}\eta^{\mu\nu}}{Q^2}[\bar{v}(\bar{p})\gamma_\nu u(p)] \\ &\quad + \left[\frac{\mathrm{i}e}{2\sin\theta\cos\theta}\right]^2[\bar{u}(q)\gamma_\mu(v - a\gamma_5)v(\bar{q})]\mathrm{i}\frac{-\eta^{\mu\nu} + m^{-2}Q^\mu Q^\nu}{Q^2 - m^2 + \mathrm{i}m\Gamma}[\bar{v}(\bar{p})\gamma_\nu(v - a\gamma_5)u(p)]\end{aligned}$$

One may notice that $Q^\mu Q^\nu$ does not contribute because it gives

$$\bar{u}\not{Q}(v - a\gamma_5)v = \bar{u}(\not{p} + \not{\bar{p}})(v - a\gamma_5)v = 0$$

it is null due to Dirac equation. Letting

$$\bar{g} = \frac{e}{2\sin\theta\cos\theta}$$

and 1 be the electron line, 2 be the muon line; then the electron currents are

$$J_1^\mu = \bar{v}(\bar{p})\gamma^\mu u(p)\,, \quad J_1^{\mu 5} = \bar{v}(\bar{p})\gamma^\mu\gamma_5 u(p)$$

Similarly for the muon currents. The amplitude is then

$$\mathcal{M} = \frac{e^2}{s}J_1^\mu J_{2\mu} + \frac{\bar{g}^2}{s - M_Z^2 + \mathrm{i}M_Z\Gamma_Z}(v_2 J_2 - a_2 J_2^5)^\mu (v_1 J_1 - a_1 J_1^5)_\mu$$

Letting

$$X = \left(\frac{\bar{g}}{e}\right)^2 \frac{s}{s - M_Z^2 + \mathrm{i}M_Z\Gamma_Z}$$

the amplitude is

$$\begin{aligned}
\mathcal{M} &= \frac{e^2}{s}[J_1 J_2 + X(v_2 J_2 - a_2 J_2^5)^\mu (v_1 J_1 - a_1 J_1^5)_\mu] \\
&= \frac{e^2}{s}[J_1 J_2(1 + X v_1 v_2) + X(-v_1 a_2 J_1 J_2^5 - v_2 a_1 J_2 J_1^5) + X a_1 a_2 J_1^5 J_2^5]
\end{aligned}$$

The complex conjugate is

$$\mathcal{M}^* = \frac{e^2}{s}[J_1^* J_2^*(1 + X^* v_1 v_2) + X^*(-v_1 a_2 J_1^* J_2^{5*} - v_2 a_1 J_2^* J_1^{5*}) + X^* a_1 a_2 J_1^{5*} J_2^{5*}]$$

One may study the electronic line. The terms appearing are of the type

$$\sum_{\text{pol}} J_\mu J_\nu^* = \sum_{\text{tot}}[\bar{v}(\bar{p})\gamma_\mu u(p)\bar{u}(p)\gamma_\nu v(\bar{p})] = \mathrm{Tr}[\not{\bar{p}}\gamma_\mu\not{p}\gamma_\nu] = 4[\bar{p}_\mu p_\nu + p_\mu\bar{p}_\nu - p\bar{p}\eta_{\mu\nu}] = V_{\mu\nu}(p, \bar{p})$$

Also

$$\sum_{\text{pol}} J_\mu^5 J_\nu^{5*} = \mathrm{Tr}[\not{\bar{p}}\gamma_\mu\gamma_5\not{p}\gamma_\nu\gamma_5] = \mathrm{Tr}[\not{\bar{p}}\gamma_\mu\not{p}\gamma_\nu] = V_{\mu\nu}(p, \bar{p})$$

The axial terms are

$$\sum_{\text{pol}} J_\mu^5 J_\nu^* = \mathrm{Tr}[\not{\bar{p}}\gamma_\mu\gamma_5\not{p}\gamma_\nu] = \mathrm{Tr}[\not{\bar{p}}\gamma_\mu\not{p}\gamma_\nu\gamma_5] = -4\mathrm{i}\varepsilon_{\alpha\mu\beta\nu}\bar{p}^\alpha p^\beta = 4\mathrm{i}\varepsilon_{\mu\nu\alpha\beta}\bar{p}^\alpha p^\beta \equiv A_{\mu\nu}(p, \bar{p})$$

Contractions. The amplitude squared $|\mathcal{M}|^2$ gives various contractions

$$V_{\mu\nu}(p, \bar{p})V^{\mu\nu}(q, \bar{q}) = 32[(pq)(\bar{p}\bar{q}) + (p\bar{q})(\bar{p}q)] = 32[(pq)^2 + (p\bar{q})^2]$$

where one has used

$$p\bar{q} = \frac{(p - \bar{q})^2}{-2} = \frac{(\bar{p} - q)^2}{-2} = \bar{p}q$$

Likewise

$$V_{\mu\nu}A^{\mu\nu} = 0$$

because the first is symmetric while the second is anti-symmetric. Finally

$$A_{\mu\nu}(p, \bar{p})(A^{\mu\nu}(q, \bar{q}) = 16(\varepsilon_{\mu\nu\alpha\beta}\bar{p}^\alpha p^\beta)(\varepsilon^{\mu\nu\rho\sigma}\bar{q}_\rho q_\sigma) = -32[(\bar{p}\bar{q})(pq) - (p\bar{q})(\bar{p}q)]$$

where one has used

$$\varepsilon_{\mu\nu\alpha\beta} = -\varepsilon^{\mu\nu\alpha\beta}\,, \quad \varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\mu\nu\rho\sigma} = -2(\delta_\alpha^\rho\delta_\beta^\sigma - \delta_\alpha^\sigma\delta_\beta^\rho)$$

Kinematics. Letting θ be the angle of the product with the respect to the collision line. Then

$$p^\mu = (E, 0, 0, E) \,, \quad \bar{p}^\mu = (E, 0, 0, -E)$$

also

$$q^\mu = (E, 0, E\sin\theta, E\cos\theta) \,, \quad \bar{q}^\mu = (E, 0, -E\sin\theta, -E\cos\theta)$$

Some products are

$$pq = E^2(1-\cos\theta) \,, \quad p\bar{q} = E^2(1+\cos\theta) \implies s = (p+\bar{p})^2 = 4E^2$$

One obtains

$$V_{\mu\nu}V^{\mu\nu} = 4s^2(1+\cos^2\theta) \,, \quad A_{\mu\nu}A^{\mu\nu} = 8s^2\cos\theta$$

Squared amplitude. The averaged squared amplitude is

$$\begin{aligned}
\overline{\sum}|\mathcal{M}|^2 &= \frac{1}{4}\sum_{\text{in}}\sum_{\text{out}}|\mathcal{M}|^2 \\
&= \frac{1}{4}\left(\frac{e^2}{s}\right)^2 \Big[VV(|1+Xv_1v_2|^2 + |Xv_1a_2|^2 + |Xv_2a_1|^2 + |Xa_1a_2|^2) \\
&\quad + AA[(1+Xv_1v_2)(X^*a_1a_2) + 2|X|^2 v_1v_2a_1a_2(1+X^*v_1v_2)(Xa_1a_2)]\Big] \\
&= \frac{1}{4}\left(\frac{e^2}{s}\right)^2 4s^2\Big[(1+\cos^2\theta)[1+2(\operatorname{Re}X)v_1v_2 + |X|^2(v_1^2+a_1^2)(v_2^2+a_2^2)] \\
&\quad + 4\cos\theta[(\operatorname{Re}X)a_1a_2 + 2|X|^2(v_1v_2a_1a_2)]\Big]
\end{aligned}$$

Inside the first bracket, the term 1 comes purely from QED. The terms with $|X|^2$ come from the interference between Z and Z, while the terms with $\operatorname{Re}X$ come from interferences of γ and Z [r] diagr.

This matrix element is not symmetric for $\theta \to \pi - \theta$. The second line is not trivial: depending on the energy (which is contained in X), one may have more muons going forward than backwards (or viceversa).

Lecture 12

mer 03 apr 2024 12:30

Knowing that the flux and the phase space are

$$\frac{1}{2s} \,, \quad \mathrm{d}\phi_2 = \frac{\mathrm{d}\cos\theta\,\mathrm{d}\varphi}{32\pi^2}$$

the differential cross-section in the center-of-mass frame is

$$\begin{aligned}
(\mathrm{d}_\Omega\sigma)_{\text{CM}} &= \frac{1}{64\pi^2}\frac{1}{s}e^4\left[(1+\cos^2\theta)C_{VV} + 4\cos\theta C_{AA}\right] \\
&= \frac{\alpha^2}{4s}[(1+\cos^2\theta)C_{VV} + 4\cos\theta\, C_{AA}] \,, \quad \alpha = \frac{e^2}{4\pi}
\end{aligned}$$

where one has

$$\begin{aligned}
C_{VV} &= 1 + 2\operatorname{Re}X\, v_1v_2 + |X|^2(v_1^2+a_1^2)(v_2^2+a_2^2) \\
C_{AA} &= \operatorname{Re}X\, a_1a_2 + 2|X|^2 v_1v_2a_1a_2
\end{aligned}$$

The total cross-section is

$$\sigma_{\text{tot}} = \int \mathrm{d}_\Omega\sigma\,\mathrm{d}\Omega = \frac{\alpha^2}{4s}\int_0^{2\pi}\mathrm{d}\varphi\int_{-1}^{1}\mathrm{d}\cos\theta[(1+\cos^2\theta)C_{VV} + 4\cos\theta\, C_{AA}] = \frac{4\pi}{3}\frac{\alpha^2}{s}C_{VV}$$

The structure of C_{VV} implies that the cross-section, as a function of $\sqrt{s}$, has a peak corresponding to the Z boson mass m_Z which comes from the third addendum

$$X = \left(\frac{\bar{g}}{e}\right)^2 \frac{s}{s - M_Z^2 + \mathrm{i}M_Z\Gamma_Z}$$

However, for s small, then $C_{VV} \sim 1$ and the cross-section grows since it gets contributions from QED. There is a middle region that is a mix of QED and weak interactions.

The width of the Breit–Wigner has been already computed.

Asymmetry. The forward-backward asymmetry can be quantified by

$$\begin{aligned}A_s = A_{\mathrm{FB}} = \frac{\sigma_+ - \sigma_-}{\sigma_+ + \sigma_-} &= \left[\int_{\theta=\frac{\pi}{2}}^{\theta=0} \mathrm{d}_\Omega\sigma - \int_{\theta=\pi}^{\theta=\frac{\pi}{2}} \mathrm{d}_\Omega\sigma\right]\left[\int_{\theta=\frac{\pi}{2}}^{\theta=0} \mathrm{d}_\Omega\sigma + \int_{\theta=\pi}^{\theta=\frac{\pi}{2}} \mathrm{d}_\Omega\sigma\right]^{-1} \\ &= \frac{4C_{AA}[\int_0^1 \mathrm{d}\cos\theta\,\cos\theta - \int_{-1}^0 \mathrm{d}\cos\theta\,\cos\theta]}{C_{VV}\int_{-1}^1 (1+\cos^2\theta)\,\mathrm{d}\cos\theta} = \frac{3}{2}\frac{C_{AA}}{C_{VV}}\end{aligned}$$

Remark. Far from QED $s \gg 0$ and below the Z boson peak $s \ll m_Z^2$, one has

$$X = \left(\frac{\bar{g}}{e}\right)^2 \frac{s}{s - m_Z^2 + \mathrm{i}m_Z\Gamma_Z} \sim \left(\frac{\bar{g}}{e}\right)^2 \frac{-s}{m_Z^2} = \cdots = -s\frac{G_{\mathrm{F}}}{2\sqrt{2}\pi\alpha}$$

where one remembers

$$\alpha = \frac{e^2}{4\pi}\,, \quad \frac{G_{\mathrm{F}}}{\sqrt{2}} = \frac{g^2}{8m_W^2}$$

and also

$$a_1 = a_2 = T_3^f = -\frac{1}{2}\,, \quad v_1 = v_2 = T_3^f - 2Q_f\sin^2\theta_W = -\frac{1}{2} + 2\sin^2\theta_W$$

One may assume

$$X \lesssim 1 \implies \frac{sG_{\mathrm{F}}}{\alpha} \lesssim 1 \implies \sqrt{s} \lesssim \sqrt{\frac{\alpha}{G_{\mathrm{F}}}} \approx 34\,\mathrm{GeV}$$

which is exactly in the middle between pure QED and the Z boson peak. Noting that

$$C_{VV} = 1 + 2\,\mathrm{Re}\,X\,v_1v_2 + o(|X|)\,, \quad C_{AA} = \mathrm{Re}\,X\,a_1a_2 + 2|X|^2 a_1a_2v_1v_2$$

one obtains

$$C_{VV} \approx 1\,, \quad C_{AA} \approx \mathrm{Re}\,X\,a_1a_2 = \frac{1}{4}\mathrm{Re}\,X$$

Therefore, the asymmetry becomes

$$A_s = \frac{3}{2}\frac{C_{AA}}{C_{VV}} \approx \frac{3}{8}\mathrm{Re}\,X = \frac{3}{8}\frac{-s}{2\sqrt{2}}\frac{G_{\mathrm{F}}}{\pi\alpha}$$

It is small (because of X) and it is negative. There must be more backward scattering than forward.

Remark. In the region near the peak, $s \approx m_Z^2$ then X is large. Therefore

$$C_{VV} \approx |X|^2(v_1^2 + a_1^2)(v_2^2 + a_2^2)\,, \quad C_{AA} \approx 2|X|^2 a_1a_2v_1v_2$$

The asymmetry is then

$$A_s = \frac{3}{8}\frac{16v^2}{(1+4v^2)^2} \approx 12v^2 = 3(1 - 4\sin^2\theta_W)\,, \quad v = v_1 = v_2 \ll 1$$

The asymmetry is positive: there is more forward scattering.

Remark. Consider the computation as if it were done in pure QED. The differential cross-section would be proportional to a parabola in $\cos\theta$ given by $1 + \cos^2\theta$. At $\sqrt{s} \approx 30\,\mathrm{GeV}$, according to the full electroweak computation, there is more backward scattering and in fact the experimental data is to the right of the above parabola.

Computing the same observable close to the Z boson peak, the data is to the left of the parabola.

Through measurements, one can constrain the free parameters of the Standard model.

5 Higgs boson phenomenology

Higgs decay. Consider the decay of the Higgs boson into a particle, anti-particle final state $H \to A\bar{A}$ with $m_A = m_{\bar{A}} = m$

$$H(q) \to A(p_1) + \bar{A}(p_2)$$

Since the Higgs boson is a scalar particle, there is no preferred direction in its reference frame.

The decay width is

$$\mathrm{d}\Gamma = \frac{1}{2m_H} \overline{\sum} |\mathcal{M}|^2 \, \mathrm{d}\phi_2$$

The Feynman amplitude must be a function of the only scalars that can be built. Only $p_1 p_2$ is independent:

$$(p_1 q) = (p_1 + p_2) p_1 = m^2 + p_1 p_2$$

Therefore

$$|\mathcal{M}|^2 = F(p_1 p_2, m_H^2, m_A^2)$$

One finds that

$$p_1 p_2 = \frac{(p_1 + p_2)^2 - 2m_A^2}{2} = \frac{m_H^2 - 2m_A^2}{2}$$

The phase space is

$$\mathrm{d}\phi_2 = \frac{1}{8\pi} \sqrt{1 - \frac{4m^2}{m_H^2}}$$

where one has already integrated in the angles.

The decay width is then

$$\Gamma = \frac{1}{16\pi m_H} \overline{\sum} |\mathcal{M}|^2 \sqrt{1 - \frac{4m^2}{m_H^2}}$$

Higgs to fermions. There is only one Feynman diagram [r]. The amplitude is

$$\mathrm{i}\mathcal{M} = -\mathrm{i} \frac{m_f}{v} \bar{u}(p_1) v(p_2)$$

The squared amplitude is

$$\begin{aligned}
\overline{\sum} |\mathcal{M}|^2 &= N_C^{(f)} \left(\frac{m_f}{v}\right)^2 \sum_{\text{pol}} \bar{u}(p_1) v(p_2) \bar{v}(p_2) u(p_1) = N_C^{(f)} \left(\frac{m_f}{v}\right)^2 \mathrm{Tr}[(\not{p}_1 + m_f)(\not{p}_2 - m_f)] \\
&= N_C^{(f)} \left(\frac{m_f}{v}\right)^2 \mathrm{Tr}\big[\not{p}_1 \not{p}_2 - m_f^2\big] = N_C^{(f)} \left(\frac{m_f}{v}\right)^2 [4p_1 p_2 - 4m_f^2] \\
&= N_C^{(f)} \left(\frac{m_f}{v}\right)^2 2m_H^2 \left[1 - \frac{4m_f^2}{m_H^2}\right]
\end{aligned}$$

where the color summation is

$$N_C^{(\text{leptons})} = 1\,, \quad N_C^{(\text{quarks})} = 3$$

The decay width is then

$$\Gamma = \frac{1}{16\pi m_H} \left[1 - \frac{4m_f^2}{m_H^2}\right]^{\frac{3}{2}} N_C^{(f)} 2m_H^2 \left(\frac{m_f}{v}\right)^2 = N_C^{(f)} \frac{G_\mathrm{F}}{4\pi\sqrt{2}} m_H m_f^2 \left[1 - \frac{4m_f^2}{m_H^2}\right]^{\frac{3}{2}}$$

The coupling of the Higgs boson is proportional to the mass of the objects, in fact there is the term m_f^2. One expects that the partial width of the decays to heavier particles is greater than the one of lighter particles. Also, due to the color charge, quarks have greater width than similar mass fermions.

Exercise. Example of exam exercises are decays to $H \to ZZ$ and $H \to W^-W^+$ assuming $m_H > 2m_V$.

Remark. The decay width of the Higgs boson is very small with respect to the mass, contrary to the Z boson. When the Higgs boson has energy for the production of WW then the decay width becomes much greater and these decays dominate.

The reason why it grows is that the vector boson have three polarizations and the longitudinal polarization is of the type
$$\varepsilon_3^\mu = (p_z, 0, 0, E)\frac{1}{m_W}$$
The more the vector bosons are energetic, the more the longitudinal polarization grows because it depends on the energy: this is not the cause for transverse polarizations.

Remark. One also notes that the decay width to the charm quarks is smaller than the tau decay, but the former is heavier than the latter. So this is surprising.

The decay width to fermions is proportional to the number of QCD colors
$$\Gamma \sim N_C^{(f)} m_f^2$$
Since the charm is heavier than tau, then
$$\Gamma_c \sim N_c m_c^2\,, \quad N_\tau \sim m_\tau^2 \implies \left.\frac{\Gamma_c}{\Gamma_\tau}\right|_{\text{naive}} \approx 1.7$$
However, in QFT the mass of a field is a running parameter and has to be renormalized like the coupling. The mass of the charm quark has different values depending on the energy it is probed. The charm and the bottom receive large quantum correction. The running masses in the $\overline{\text{MS}}$ scheme are
$$m_b^2(\mu = m_H) \approx 0.45 m_b^2|_{\text{pole}}\,, \quad m_c^2(\mu = m_H) \approx 0.25 m_c^2|_{\text{pole}}$$
where $m^2|_{\text{pole}}$ is the kinematic mass, the one that is measured. For this reason
$$\left.\frac{\Gamma_c}{\Gamma_\tau}\right|_{\text{true}} = \frac{1}{4}\left.\frac{\Gamma_c}{\Gamma_\tau}\right|_{\text{naive}} = 0.42$$

Remark. The Higgs boson may decay to photons, but since these are massless, the decay cannot be tree-level and, in fact, it is given by a loop effect [r] diagr. The decay is the product of a destructive interference of two loop diagrams. Their sum scale like $|\mathcal{M}|^2 \propto \alpha^2$ and this is way it is suppressed. This decay channel has a distinct signature since the decay product is just photons which are easy to detect.

Remark. The Higgs may decay to four leptons through two virtual vector boson [r] diagr. This amplitude is suppressed because one boson is suppressed. The amplitude scales like
$$\frac{1}{(Q_{12}^2 - m_Z^2) + m_Z^2\Gamma_Z^2}\frac{1}{(Q_{34}^2 - m_Z^2) + m_Z^2\Gamma_Z^2}$$
If $Q_{12}^2 \approx m_Z^2$ then there is not enough energy for the other propagator. One pair of leptons is close to the Z boson peak, while the other pair is in a generic configuration. It holds always
$$Q_{1234}^2 = m_H^2$$

The Higgs boson was discovered through gluon fusion which produces the Higgs which then decays in two photons. At the LHC there is a dominant background of photons given by $q\bar{q}$. On top of the background there is a very narrow Breit–Wigner of the Higgs.

Another channel, clearer to detect, is the four-fermion decay. There are various backgrounds which produce two peaks and the Higgs produces a peak in the middle.

Lecture 13

lun 08 apr 2024 12:30

Higgs production. At LEP the Higgs boson may be produces as from an electron-positron annihilation to an energetic Z boson? that gives a Higgs boson and a Z boson. Both decay in bottom–anti-bottom quarks.

At LEP there were EW precision tests In the transition $e^+e^- \to \mu^+\mu^-$, the virtual intermediate particles give oblique radiative corrections, in particular $Z \to ZH \to Z$ [r] diagr. [r] Through the self-consistency of the data, LEP was able to find a theoretical band of the Higgs mass

There are various Higgs production channels at the LHC like the gluon fusion and the vector boson fusion [r] diagr. [r] This last process has a distinct signature. There are also the associated VH production and $t\bar{t}H$ production. These processes are useful for the vertices of the Yukawa couplings with the Higgs boson and fermions.

Challenges. A few challenges to study are: the coupling to first and second generation fermions, the Higgs self-coupling, vector boson scattering. The study of the second is related to the shape of the Mexican hat potential of the Higgs boson. The third is related to the unitarity of the Standard Model: if the Higgs boson were absent the process cross-section would violate unitarity.

5.1 Unitarity

One would like to compute the vector boson scattering to see how the Higgs boson guarantees unitarity [r] diagr. One may compute just the sub-diagram. The calculation is complicated, so one shall look at the simpler case $W^+Z \to W^+Z$ with all particles longitudinally polarized.

Assuming no Higgs boson in the Standard Model, one has three diagrams for the process [r] diagr. The longitudinal polarization is

$$\varepsilon_3^\mu(p) = \frac{1}{m}(p_z, 0, 0, E)\,, \quad \varepsilon^2 = -1\,, \quad \varepsilon^\mu p_\mu = 0$$

At high energies, one scales like $\varepsilon_3^\mu \sim E$. This is not true for transverse polarizations.

By power-counting, one may check how the above diagrams scale. The three-point vector boson vertex scales like E, while a vector boson propagators scales like E^{-2}. The first diagram scales like $E^4 \cdot E^2 \cdot E^{-2} \sim E^4$. The second is the same. The four-point vector boson vertex is E^4. Summing the three diagrams, one expects a scaling like a polynomial of order 4. However, the actual scaling is E^2. Because of this scaling, the process violates unitariy. The highest power of E is cancelled because the theory is a gauge theory.

This violation is cured by the diagram [r] for which the sum of the four diagrams is constant with the energy. See Schwartz, §29.2 for calculations.

Computation. One is interested in the high-energy limit, so the parametrization of the polarization is approximate

$$\varepsilon_3^\mu(p) \approx \frac{1}{m} p^\mu$$

but then $\varepsilon_3^\mu p_\mu \neq 0$. The clever choice is

$$\varepsilon_j^\mu = \frac{p_j^\mu}{m_j} + \frac{2m_j}{t - 2m_j^2} p_{j+2}^\mu\,, \quad \varepsilon_{j+2}^\mu = \frac{p_{j+2}^\mu}{m_j} + \frac{2m_j}{t - 2m_j^2} p_j^\mu\,, \quad t = (p_1 - p_3)^2$$

for $j = 1, 2$ where $p_{1,2}$ are the incoming particles, while $p_{3,4}$ are the outgoing particles; and $m_1 = m_W$ and $m_2 = m_Z$. In this way, one has

$$\varepsilon_i^\mu p_\mu^i = 0\,, \quad [\varepsilon_i^\mu (\varepsilon_\mu^i)^*] = -1 + o(t^{-1})$$

Calculations not in exam, only idea. See Lecture 12, p.2. Prefactor agrees with 0709.1075 eq. A7. The computation can be done with Mathematica and FeynCalc. The final result is

$$M(W_L Z_L \to W_L Z_L) = \left[e \frac{\cos\theta_W}{\sin\theta_W}\right]^2 \left[-\frac{m_Z^2}{4m_W^2}\right] [s + u + \text{const.}] = \frac{t}{v^2} + \text{const.}$$

where one uses

$$m_W = m_Z \cos\theta_W \,, \quad v = 2m_W \frac{\sin\theta_W}{e}$$

Remark. One observes that the E^4 divergent behaviour cancels between M_s, M_u and M_4 due to gauge invariance.

Remark. At high energies, one has

$$M \sim \frac{t}{v^2} \,, \quad t = (p_1 - p_3)^2 \sim -\frac{1}{2}E_{\mathrm{CM}}^2(1-\cos\theta)$$

since one ignores the mass

$$(p_1 - p_3)^2 \approx -2p_1p_2 \approx -2E_1E_2(1-\cos\theta)$$

From the optical theorem, the amplitude can be decomposed on Legendre polynomials

$$M(\theta) = 16\pi \sum_{j=0}^{\infty} a_j(2j+1)P(\cos\theta)$$

while the amplitude is

$$M(WZ \to WZ)(0) = -\frac{1}{2}E_{\mathrm{CM}}^2(1-\cos\theta)\frac{1}{v^2}$$

By comparing the two, one may extract the coefficients

$$a_0 = \frac{1}{16\pi}\left[-\frac{1}{2}E_{\mathrm{CM}}^2\right]\frac{1}{v^2} \,, \quad a_1 = \frac{1}{16\pi}\frac{1}{3}\left[-\frac{1}{2}E_{\mathrm{CM}}^2\right]\frac{1}{v^2} \,, \quad a_n = 0 \,, \quad n > 1$$

The constraint on unitarity is

$$|a_j| \leq 1 \implies E_{\mathrm{CM}} \lesssim \sqrt{32\pi}v \approx 2.5\,\mathrm{TeV}$$

At higher energy, unitarity is violated [r]. For the vector boson scattering $W_L^+W_L^+ \to W_L^+W_L^+$ one has

$$E_{\mathrm{CM}} = 800\,\mathrm{GeV}$$

To recover unitarity, there must be another mechanism (the Higgs boson) that keeps unitarity valid. At this energy there is either an elementary particle or a strongly coupled sector.

Higgs field contribution. Consider the scattering [r], the amplitude is

$$\mathcal{M}_H = \left[\frac{-e}{\sin\theta\cos\theta}\right]\varepsilon_1^\mu\varepsilon_2^\mu\varepsilon_3^\mu\varepsilon_4^\mu\eta^{\mu\alpha}\eta^{\nu\beta}\frac{m_W^2}{t-m_H^2} = -\frac{t}{v^2} + \mathrm{const.}$$

See Sc. 29.26. Therefore, the total amplitude is constant.

The amplitude of the Higgs boson contribution scales like

$$\mathcal{M}_H \sim -\frac{t^4}{t^2(t-m_H^2)} \,, \quad m_H \approx 125\,\mathrm{GeV}$$

negletcting $m_{W,Z}$ in the numerator. If the mass of the Higgs had been too large, the unitarity bound would have been violated before the contribution of the Higgs boson

$$\mathcal{M}_H \sim \frac{t^2}{m_H^2}$$

This consideration posed a bound on the mass of the Higgs

$$m_H \lesssim \sqrt{\frac{16\pi}{3}}v \approx 1\,\mathrm{TeV}$$

This is the Lee–Quigg–Thacker bound. When the boson was discovered there was no surprise that its mass is $m_H \approx 125\,\mathrm{GeV}$.

Lecture 14

mer 10 apr 2024 12:30

6 Quantum chromodynamics

The elementary fields are the quarks and the gauge fields are the gluons. These fields are called partons. They are never observed as free particles but only through indirect evidence. Historically, quarks were introduced by Gell-Mann as constituents of hadrons. Hadrons are the strongly interacting states found in nature, they are bound states of quarks.

The hadrons are divided in to families: mesons and baryons [r]. The quarks are spin $\frac{1}{2}$ Dirac fermions divided into six flavours. They have fractional charges [r]. There are three light quarks and three heavy quarks. [r]

The notation of the quark fields is $\psi_i^{(f)}(x)$ where f is the flavour index and i is the colour index. The quantum number associated to the strong dynamics (QCD) is the colour index. The gauge symmetry group of QCD is $\mathrm{SU}(3)_\mathrm{C}$, it is a non-abelian theory. Once the gauge group is defined and the matter content of the theory is defined, then the Lagrangian density is unambiguously defined.

History — quark model. Historically quarks where introduced in the quark model, also called eight-fold way, by Gell-Mann and Zweig. The observed hadrons are organized according to some pattern. A good organizing principle is to introduce an approximate flavour symmetry $\mathrm{SU}(3)_\mathrm{F}$ of three hypothetical objects: the up, down and strange quarks. [r]

One assumes the existence of three spin $\frac{1}{2}$, fractionally charged, particles belonging to the fundamental representation of $\mathrm{SU}(3)_\mathrm{F}$. A crucial point is that mesons and baryons may be organized according the irreducible representation of $\mathrm{SU}(3)_\mathrm{F}$. For mesons, one combines the fundamental and anti-fundamental representations to obtain the mesons octect $3_\mathrm{F} \otimes \bar{3}_\mathrm{F} = 8 \oplus 1$. By combining three quarks, one obtains

$$3_\mathrm{F} \otimes 3_\mathrm{F} \otimes 3_\mathrm{F} = 10 \oplus 8 \oplus 8 \oplus 1$$

made of decuplets and octets. The mesons may be arranged into an octet [r] diagr since they share similar properties, like mass and quark content. The baryons may be arranged into a decuplet [r] diagr.

The Ω^- was predicted. However, there is a problem. The Δ^{++} particle is made of three up quarks: it has $+2$ charge and $+\frac{3}{2}$ spin. It is made up of three identical quark fields and, by analyzing the decay products, the space wave function is symmetric. This means that the particle is symmetric for any permutation. The fields are fermions and Pauli exclusion principle (or Fermi–Dirac statistics) must hold. Therefore, there must be another quantum number with anti-symmetric wave function: the colour. The particle is

$$\left|\Delta^{++}\right\rangle \sim \frac{1}{\sqrt{6}} \varepsilon_{ijk} \left|u_i^\uparrow u_j^\uparrow u_k^\uparrow\right\rangle$$

One must add the assumption that quarks are colour triplets and the colour $\mathrm{SU}(3)_\mathrm{C}$ is an exacts symmetry as opposed to the flavour symmetry. Another assumption is that hadrons are colour singlets: the hadronic states are invariant under $\mathrm{SU}(3)_\mathrm{C}$ transformations.

For example, for mesons $q\bar{q}$ one has

$$3_\mathrm{C} \otimes 3_\mathrm{C} = 8_\mathrm{C} \oplus 1_\mathrm{C}$$

and the mesons are in the singlet

$$|q\bar{q}\rangle \sim \frac{1}{\sqrt{3}} \bar{q}_i q_j \delta_{ij}$$

In fact, by performing a colour transformation $\psi_i' = U_{il}\psi_l$, one has

$$\delta_{ij}\bar{\psi}\psi_j \to \delta_{ij}\bar{\psi}_i'\psi_j' = \delta_{ij}\overline{(U_{il}\psi_l)}(U_{jm}\psi_m) = \delta_{ij}\bar{\psi}_l(U^\dagger)_{li}U_{jm}\psi_m = \bar{\psi}_l\delta_{lm}\psi_m$$

For baryons, one has

$$3_\mathrm{C} \otimes 3_\mathrm{C} \otimes 3_\mathrm{C} = 10_\mathrm{C} \oplus 8_\mathrm{C} \oplus 8_\mathrm{C} \oplus 1_\mathrm{C}$$

The wave function are of the type

$$\varepsilon_{ijk}\psi_i\psi_j\psi_k \to \varepsilon_{ijk}\psi_i'\psi_j'\psi_k' = \varepsilon_{ijk}U_{ii'}U_{jj'}U_{kk'}\psi_{i'}\psi_{j'}\psi_{k'} = \varepsilon_{i'j'k'} \det U \, \psi_{i'}\psi_{j'}\psi_{k'} = \varepsilon_{i'j'k'}\psi_{i'}\psi_{j'}\psi_{k'}$$

Experimental evidence of colour. The evidence is quantified by the R-ratio

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

where

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4}{3}\pi\frac{\alpha^2}{s}$$

[r] Expresses the transition to hadrons as transition to quark–anti-quark. In the QED vertex for the quark, there is present the charge of the quarks [r]. The ratio is

$$R = \frac{N_c \sum_f Q_f^2}{Q_\mu^2} = N_c \begin{cases} \frac{2}{3}\,, & f = u, d, s \\ \frac{10}{9}\,, & f = u, d, s, c \\ \frac{11}{9}\,, & f = u, d, s, c, b \end{cases}$$

This result only includes tree-level computations.

The existence of quarks may be seen through deep-inelastic scattering $\mathrm{e}^- + p \to \mathrm{e}^- + X$ where X are hadrons. The electron interacts with the parts of the protons. One expects $N_\mathrm{C}^2 - 1$ gluons, the same as the dimension of the gauge group.

The existence of gluons may be seen through $e^+e^- \to 3$ jets [r] diagr. The gluonic fields are fundamental particles whose traces can be seen in experiments.

Colour confinement. In this model, hadrons are colour singlets even if the whole dynamic is colorful. This is due to colour confinement. One may estimate the potential of separation between two quarks in a meson

$$V_{q\bar{q}}(r) \sim C_\mathrm{F}\left[\frac{\alpha_s(r)}{r} + \sigma r + \cdots\right]$$

The potential is a bounding potential because it grows lienarly as distance increases. The separation needs ever increasing energy and at a point there creates a new quark–anti-quark pair to give two confined states.

6.1 Lagrangian

The QCD Lagrangian is based on exact colour symmetry

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\not{D} - m)\psi - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \mathcal{L}_\mathrm{GF} + \mathcal{L}_\mathrm{FP}$$

where $\psi = \psi_i$ with $i = 1, \ldots, N_c$. The covariant derivative in the fundamental representation is

$$[D_\mu]_{ij} = \partial_\mu\delta_{ij} + \mathrm{i}g_s t^a_{ij} A^a_\mu\,, \quad a = 1, \ldots, N_c^2 - 1$$

where t^a are the eight Gell-Mann matrices and A_μ is the gluon field. The field strength tensor is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f^{abc} A^b_\mu A^c_\nu\,, \quad \alpha_s = \frac{g_s^2}{4\pi}$$

where f^{abc} are the structure constants. [r] The covariant derivative in the adjoint representation is

$$[D_\mu]_{ab} = \partial_\mu\delta_{ab} + \mathrm{i}g_s(T^c)_{ab}A^c_\mu\,, \quad (T^c)_{ab} = -\mathrm{i}f^{cab} = \mathrm{i}f^{acb}$$

Propagator. From the free Lagrangian, the fermionic propagator is

$$\delta_{ij}\frac{\mathrm{i}(\not{p} + m)}{p^2 - m^2 + \mathrm{i}\varepsilon}$$

The gluon-quark-quark vertex is

$$-\mathrm{i}g_s\gamma^\alpha t^a_{ij}$$

The three-point gluonic vertex is

$$-g_s f^{abc}[\eta^{\alpha\beta}(p_a - p_b)^\gamma + \eta^{\beta\gamma}(p_b - p_c)^\alpha + \eta^{\gamma\alpha}(p_c - p_a)^\beta]$$

this comes from a term in the Lagrangian

$$\frac{g_s}{2} f^{abc} A^b_\mu A^c_\nu (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)$$

The four-point gluonic vertex is

$$\sim g_s^2 (ff)^{abcd} (\text{metric tensors})^{\alpha\beta\gamma\delta}$$

Extra terms. The extra terms in the Lagrangian are needed to quantize the theory. Although the theory is gauge invariant, there are redundant degrees of freedom. The gauge-fixing Lagrangian is present in QED.

The F.P. part of the Lagrangian is important for non-abelian gauge theories.

One may seek the origin of the problem. The QED Lagrangian is

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \implies \Box A_\mu - \partial_\mu(\partial_\nu A^\nu) = 0$$

The Green's function is the solution to

$$(\Box \eta^{\mu\nu} - \partial^\mu \partial^\nu) G_{\nu\alpha}(x - y) = -\mathrm{i}\delta^{(4)}(x - y)\delta^\mu_\alpha$$

In momentum space it is

$$(-k^2 \eta^{\mu\nu} + k^\mu k^\nu)\widetilde{G}(k) = \mathrm{i}\delta^\nu_\alpha$$

[r] This equation cannot be solved since the operator $(-k^2\eta^{\mu\nu} + k^\mu k^\nu)$ has a zero eigenvalue and cannot be naively inverted

$$(-k^2 \eta^{\mu\nu} + k^\mu k^\nu) k^\nu = 0\,, \quad k^\nu \neq 0$$

This implies that in position space one has

$$A'_\mu = A_\mu + \partial_\mu \Lambda$$

One has to add a gauge-fixing Lagrangian

$$\mathcal{L}_{\mathrm{GF}} = -\frac{1}{2\lambda}(\partial_\mu A^\mu)^2$$

[r] This is done with a Lagrangian multiplier with the constraint [r]. Therefore, the operator is

$$\left[-k^2 \eta_{\mu\nu} + \left(1 - \frac{1}{\lambda}\right) k_\mu k_\nu\right] k^\nu \neq 0 \iff k^\nu \neq 0$$

The operator can be inverted to have

$$\widetilde{G}_{\mu\nu} = \frac{\mathrm{i}}{k^2}\left[-\eta_{\mu\nu} + (1 - \lambda)\frac{k_\mu k_\nu}{k^2}\right]$$

For $\lambda = 1$ one has the Feynman gauge. For $\lambda = 0$ is the Landau gauge. The parameter λ is not arbitrary: the Green's function is not gauge invariant but the results are gauge invariant.

Lecture 15

lun 22 apr 2024 12:30

6.1.1 Faddeiv–Popov treatment

The reason why the equation cannot be solved is related to the redundancy in the gauge [r]. The gauge transformation for non-abelian gauge fields is

$$t^a (A')^a_\mu = U(t^a A^a_\mu) U^{-1} + \frac{\mathrm{i}}{g}(\partial_\mu U) U^{-1}\,, \quad U \in \mathrm{SU}(N)$$

When doing the functional quantization, one has introduce the partition function

$$Z \sim \int [\mathcal{D}A_\mu]\, \mathrm{e}^{\mathrm{i}S[A_\mu]}$$

where the measure runs over all the possible configurations of the fields. A gauge field configuration like

$$\frac{\mathrm{i}}{g}(\partial_\mu U)U^{-1}$$

is equivalent to $t^a A^a_\mu = 0$ [r]. The redundancy is problematic when quantizing a theory. The way resolve the issue uses the Faddeiv–Popov treatment. [r] One has to add a gauge-fixing term

$$\mathcal{L}_{\mathrm{GF}} = -\frac{1}{2\lambda}[f^a(A)]^2$$

and a Faddeiv–Popov Lagrangian

$$\mathcal{L}_{\mathrm{FP}} = \eta^\dagger(\delta_\theta f)\eta = (\eta^a)^\dagger[\delta_{\theta^b} f^a(A^\theta)]\eta^b$$

The η fields are ghost fields: they are N^2-1 complex scalar fields. Their spin statistics is Fermi–Dirac. These fields do not give rise to physical states.

There are two gauges: the covariant gauge and the axial gauge. Consider the covariant gauge. The gauge-fixing term is

$$\mathcal{L}_{\mathrm{GF}} = -\frac{1}{2\lambda}(\partial_\mu A^{\mu a})^2$$

the FP Lagrangian is

$$\mathcal{L}_{\mathrm{FP}} = (\partial_\mu \eta^a)^\dagger(\partial_\mu \eta^a) + (\partial_\mu \eta^a)^\dagger g f^{abc} \eta^b A^\mu_c$$

the propagators are [r] diagr. Since the ghost fields are fermions, these can appear in loops and bring a minus sign. Due to the second vertex, in the covariant gauge there is a gluon-ghost-ghost vertex. In QED, the second term is null since the gauge group is abelian $f^{abc} = 0$. This is the reason why ghosts do not appear. The ghost fields contribute to the gluon propagator

$$\widetilde{G}^{ab}_{\mu\nu} = \delta_{ab}\frac{\mathrm{i}}{k^2}\left[-\eta_{\mu\nu} + (1-\lambda)\frac{k_\mu k_\nu}{k^2}\right]$$

One may find the expression of the propagator directly from the Lagrangian. Starting from

$$\mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{FP}} \sim \frac{1}{2}A^{\mu a}\left[\Box\,\eta_{\mu\nu} - \left(1 - \frac{1}{\lambda}\right)\partial^\mu\partial^\nu\right]A^{\nu a}$$

One needs to take the Fourier transform of the bilinear term in the gauge field and consider its reciprocal.

The second gauge is the axial gauge, also called physical gauge. One picks an arbitrary fixed axis n^μ. The Lagrangians are

$$\mathcal{L}_{\mathrm{GF}} = -\frac{1}{2\lambda}(n_\mu A^{\mu a})^2$$

and

$$\mathcal{L}_{\mathrm{FP}} = (\eta^a)^\dagger[\delta^{ab} n_\mu\,\partial^\mu + g f^{abc} n_\mu A^{\mu c}]\eta^b$$

There is a three-point vertex proportional to n^μ. The gluon propagator is

$$\widetilde{G}^{\mu\nu}_{ab} = \delta_{ab}\frac{\mathrm{i}}{k^2}\left[-\eta^{\mu\nu} + \frac{n^\mu k^\nu + n^\nu k^\mu}{nk} - \frac{(n^2+\lambda k^2)k^\mu k^\nu}{(nk)^2}\right]$$

For $\lambda = 0$, then

$$n_\mu \widetilde{G}_{\mu\nu} = 0$$

This means that combining the three-point vertex with the gluon propagator gives zero. This means that ghosts decouple. The Lagrangian is complicated but the ghosts disappear. The parameters $n^2 = 0$, $\lambda = 0$ is called light-cone gauge.

Remark. The gauge-fixing Lagrangian breaks the manifest gauge invariance of the Lagrangian, but the final result is physical and does not depend on the gauge choice.

The covariant gauge has Lorentz invariance, simple propagators but has ghosts. The axis gauge do not have ghosts for $\lambda = 0$, but there is a preferred direction n^μ and spurious poles $(nk)^{-1}$ and $(nk)^{-2}$.

Propagating degrees of freedom. The propagator is schematically

$$\widetilde{G}_\mu = \frac{\mathrm{i}}{k^2} d_{\mu\nu}$$

In the covariant gauge, the numerator is not the sum of the polarizations

$$d_{\mu\nu} \neq \sum_{\text{pol}} \varepsilon_\mu(k)\varepsilon_\nu^*(k)$$

but in the axial gauge it is. For example, in the former, one has

$$\lambda = 1 \implies \widetilde{G}_{\mu\nu} = \frac{\mathrm{i}}{k^2}(-\eta_{\mu\nu})$$

while in the latter

$$d_{\mu\nu}k^\nu = 0\,, \quad d_{\mu\nu}n^\nu = 0$$

These are two constraints that remove two propagating degrees of freedom.

The reason why there are ghosts in the covariant gauge, is that there are non-physical degrees of freedom. In the axial gauge, there are no ghosts since only physical degrees of freedom propagate. An axial gauge is particularly useful when using physical intuition.

Example. Consider the light-cone gauge propagator. If the momentum of the gauge field is aligned with the z axis, then

$$k^\mu = (E, 0, 0, E)\,, \quad n^\mu = (1, 0, 0, -1)\,, \quad nk = 2E\,, \quad n^2 = 0$$

Therefore

$$d_{\mu\nu} = -\eta_{\mu\nu} + \frac{k^\mu n_\nu + k_\nu n_\mu}{nk} = \operatorname{diag}(-1, 1, 1, 1) + \operatorname{diag}(1, 0, 0, -1) = \operatorname{diag}(0, 1, 1, 0)$$

which is exactly the sum of the physical polarizations (which are the two transverse polarizations).

Sum over physical polarizations. Let the polarizations be

$$k^\mu = (E, 0, 0, E)\,, \quad \varepsilon_1^\mu = (0, 1, 0, 0)\,, \quad \varepsilon_2^\mu = (0, 0, 1, 0)$$

One may introduce an axis for which

$$\varepsilon k = 0\,, \quad \varepsilon n = 0\,, \quad nk \neq 0$$

These only leaves the two transverse polarization. Letting $n^\mu = (1, 0, 0, -1)$, the only possible polarization vectors are the transverse ones.

Since one needs an axis, then the sum of physical polarization is

$$\Sigma_{\mu\nu}(k, n) = \sum_{\text{phys pol}} \varepsilon^\mu(k)\varepsilon^\nu(k)$$

The above three conditions uniquely determine the structure

$$\Sigma_{\mu\nu} = -\eta_{\mu\nu} + \frac{n_\mu k_\nu + n_\nu k_\mu}{nk} - n^2 \frac{k_\mu k_\nu}{(nk)^2}$$

This is the numerator of the propagator of a gauge boson in the axis gauge. This can be proven by writing the most general linear combination of type 2 tensors

$$\Sigma_{\mu\nu}(k, n) = A\eta_{\mu\nu} + Bk_\mu k_\nu + cn_\mu k_\nu + Dn_\nu k_\mu + En_\mu n_\nu$$

This expression is not used in QED due to gauge invariance. In an abelian gauge theory, the above expression is equivalent to just $-\eta_{\mu\nu}$.

6.1.2 From QED to QCD

Gauge invariance in QED. Consider $q\bar{q} \to \gamma\gamma$ and let $e = g$. There are two diagrams [r] diagr. The Feynman amplitude is

$$\mathrm{i}\mathcal{M}^{\mu\nu} = \mathrm{i}M_{\mathrm{I}}^{\mu\nu} + \mathrm{i}M_{\mathrm{II}}^{\mu\nu}$$

which becomes

$$\mathcal{M}^{\mu\nu} = -g^2 \bar{v}(p') \left[\gamma^\nu \frac{\not{p} - \not{k}}{-2pk} \gamma^\mu + \gamma^\mu \frac{\not{p} - \not{k}'}{-2pk'} \gamma^\nu\right] u(p)$$

Contracting this with k_μ one has

$$\begin{aligned} k_\mu M^{\mu\nu} &= -g^2 \bar{v}(p') \left[\gamma^\nu \frac{\not{p} - \not{k}}{-2pk} \not{k} + \not{k} \frac{\not{p} - \not{k}'}{-2pk'} \gamma^\nu\right] u(p) = -g^2 \bar{v} \left[\gamma^\nu \frac{\not{p}\not{k}}{-pk} + \frac{\not{k}\not{p}'}{2pk'} \gamma^\nu\right] u \\ &= -g^2 \bar{v}(p') [-\gamma^\nu + \gamma^\nu] u(p) = 0 \end{aligned}$$

At the first line, one uses $p - k' = k - p'$ and $\not{k}^2 = 0$. At the second line, one has

$$\not{p}\not{k} = -\not{k}\not{p} + 2pk$$

and applies Dirac equation.

Therefore

$$k_\mu M^{\mu\nu} = 0\,, \quad k'_\mu M^{\mu\nu} = 0$$

since [r].

Remark. One obtains $k_\mu M^{\mu\nu} = 0$ without assuming that the second photon is physical, $\varepsilon' k' = 0$. This holds for off-shell photons.

Remark. The relation $k_\mu M^{\mu\nu} = 0$ is called Ward identity. In configuration space, this is

$$\partial_\mu J^{\mu\nu} = 0$$

So there is a conservation of charge for the first photon. The current associated the vertex is conserved: there is no charge flowing out. This is not the case for gluons and colour charge.

Remark. A part of the squared amplitude is

$$\sum_{\text{phys pol}} \varepsilon_\mu(k) \varepsilon^*_\rho(k) \mathcal{M}^{\mu\nu} (\mathcal{M}^{\rho\sigma})^*$$

The sum over physical polarizations is

$$\Sigma_{\mu\rho}(k, n) \sim -\eta_{\mu\rho} + (nk + kn)_{\mu\rho} + k_\mu k_\rho$$

[r] The second addendum contracted with the Feynman amplitude gives zero, noting that

$$n_\mu k_\rho M^*_{\rho\sigma} = 0\,, \quad k_\mu n_\rho M_{\mu\nu} = 0$$

This agrees with the fact that in QED the Σ can just be $-\eta_{\mu\nu}$ because the other terms cancel thanks to gauge invariance.

Moving to QCD. Assume that in QCD there is no three-point gluon vertex, but the theory is still not abelian. [r] diagr. The Feynman amplitude is

$$\mathcal{M}^{\mu\nu} = -g^2 \bar{v}(p') \left[\gamma^\nu \frac{\not{p} - \not{k}}{-2pk} \gamma^\mu t^b t^a + \gamma^\mu \frac{\not{p} - \not{k}'}{-2pk'} \gamma^\nu t^a t^b\right] u(p)$$

Proceeding as above, one obtains

$$k_\mu \mathcal{M}^{\mu\nu} \neq 0\,, \quad k'_\nu \mathcal{M}^{\mu\nu} \neq 0$$

One obtains instead
$$k_\mu \mathcal{M}^{\mu\nu} = -g^2 \bar{v}\gamma^\nu [t^a, t^b] u$$
The four-product is not zero since the commutator is proportional to another generator. The fact that is non-zero [r] means that there is something that tries to cancel the term above. It should have the structure $f^{abc}t^c$ and it is the three-point gluon vertex.

To the two Feynman diagrams above, one has to add the three-point gluon vertex. The digram to consider is the following [r] diagr. The Feynman amplitude is
$$\begin{aligned}\mathrm{i}\mathcal{M}^{\mu\nu} &= \bar{v}(-\mathrm{i}g\gamma^\delta t^c)u \frac{\mathrm{i}}{q^2}\left[-\eta^{\delta\gamma} + (1-\lambda)\frac{q^\delta q^\gamma}{q^2}\right] \\ &\quad \times (-g) f^{abc}[\eta^{\nu\gamma}(-k'-q)^\mu + \eta^{\gamma\mu}(q+k)^\nu + \eta^{\mu\nu}(-k+k')^\gamma]\end{aligned}$$
The $(1-\lambda)$ term gives zero. Therefore
$$k_\mu \mathcal{M}^{\mu\nu} = \frac{1}{q^2}(-g^2)\mathrm{i}f^{abc}\{\bar{v}t^c[\gamma^\nu(-q^2) + \not{k}(k')^\nu]u\} = (-g^2)\left\{\bar{v}t^c\left[\mathrm{i}f^{abc}\left(-\gamma^\nu + \frac{\not{k}(k')^\nu}{q^2}\right)\right]u\right\}$$
The full Feynman amplitude is
$$k_\mu(\mathcal{M}^{\mu\nu}_{\mathrm{I}} + \mathcal{M}^{\mu\nu}_{\mathrm{II}}) = (-g^2)\bar{v}\left\{[t^a, t^b]\left[\gamma^\nu + \left(-\gamma^\nu + \frac{\not{k}(k')^\nu}{q^2}\right)\right]\right\}u$$
The contribution from the Feynman amplitude without the three-point vertex is cancelled by such vertex, but the final result is not zero.

If ε' is a physical polarization, then
$$k_\mu \varepsilon'_\nu \mathcal{M}^{\mu\nu} = 0\,, \quad \varepsilon' k' = 0$$
If ε is physical then
$$\varepsilon_\mu k'_\nu \mathcal{M}^{\mu\nu} = 0\,, \quad k_\mu k'_\nu \mathcal{M}^{\mu\nu} = 0$$
The full gauge invariance of the amplitude is true only if the second gluon is physical. This is different from QED.

Lecture 16

mar 23 apr 2024 12:30

Sum over physical polarizations and ghost fields. Consider the following notation
$$M(v_1, v_2) = v_1^\mu v_2^\nu M_{\mu\nu}$$
One has
$$M(\varepsilon, k') = 0\,, \quad M(k, \varepsilon') = 0\,, \quad M(k, k') = 0$$
The squared amplitude is
$$|M|^2 = M_{\mu\nu}M^*_{\rho\sigma}[\varepsilon^\mu(k)(\varepsilon^*)^\rho(k)][(\varepsilon')^\nu(k')(\varepsilon'^*)^\sigma(k')]$$
Consider two auxiliary vectors given by $n^2 = 0$ and $n'^2 = 0$ in the light-like gauge. One has
$$\sum_{\text{phys pol}} \varepsilon_\mu \varepsilon^*_\rho = -\eta_{\mu\rho} + \frac{(kn + nk)_{\mu\rho}}{nk}\,, \qquad \sum_{\text{phys pol}} \varepsilon_\nu \varepsilon'^*_\sigma = -\eta_{\nu\sigma} + \frac{(k'n' + n'k')_{\nu\sigma}}{n'k'}$$
The sum over physical polarization of the squared amplitude is
$$\begin{aligned}\sum_{\text{pp}} |M|^2 &= M_{\mu\nu}M^*_{\rho\sigma}\sum(\varepsilon_\mu\varepsilon^*_\rho)\left(-\eta^{\nu\sigma} + \frac{(k'n' + n'k')^{\nu\sigma}}{n'k'}\right) \\ &= M_{\mu\nu}M^*_{\rho\sigma}\left[\left(-\eta^{\mu\rho} + \frac{(k'n' + n'k')^{\mu\rho}}{n'k'}\right)(-\eta^{\nu\sigma})\right] \\ &= MM\left[(-\eta^{\mu\rho})(-\eta^{\nu\sigma}) + \frac{(nk + kn)^{\mu\rho}}{nk}(-\eta^{\nu\sigma})\right] \\ &= M_{\mu\nu}M^*_{\rho\sigma}\left[(-\eta^{\mu\rho})(-\eta^{\nu\sigma}) + \sum_{\text{pp}}(\varepsilon'_\nu\varepsilon'^*_\sigma)\frac{(nk + kn)^{\mu\rho}}{nk} - \frac{(nk + kn)^{\mu\rho}(n'k' + k'n')^{\nu\sigma}}{(nk)(n'k')}\right]\end{aligned}$$

At the first line, the last addendum does not contribute to the sum because it gives $M(\varepsilon, k')$. At fourth line, one has applied

$$-\eta^{\nu\sigma} = \sum_{\text{pp}} \varepsilon'_\nu \varepsilon'_\sigma - \frac{(n'k' + k'n')^{\nu\sigma}}{n'k'}$$

The second addendum gives zero because

$$M(k, \varepsilon') = 0$$

The third addendum gives some terms $M(k, k') = 0$. Therefore, the sum over physical polarizations is

$$\sum_{\text{pp}} |M|^2 = M_{\mu\nu} M^*_{\rho\sigma} (-\eta_{\mu\rho})(-\eta_{\nu\sigma}) - \frac{1}{(nk)(n'k')} [M(k, n') M^*(n, k') + \text{c.c.}]$$

[r] where one has

$$M(k, n') = \left[-\mathrm{i}\frac{g^2}{q^2} f^{abc} t^c\right] (n'k')[\bar{v}\not{k}u] \,, \quad M(n, k') = \left[-\mathrm{i}\frac{g^2}{q^2} f^{abc} t^c\right] (nk)[\bar{v}\not{k}u]$$

Therefore

$$\begin{aligned}\sum_{\text{pp}} |M|^2 &= (MM)(-\eta)(-\eta) - \frac{2}{(nk)(n'k')} \left[\frac{g^2}{q^2}\right]^2 f^{cda} f^{cdb} (nk)(n'k') [\bar{v}\not{k}t^a u \bar{u}\not{k}t^b v] \\ &= (MM)(-\eta)(-\eta) - 2\left[\frac{g^2}{q^2}\right]^2 f^{cda} f^{cdb} [\bar{v}\not{k}t^a u \bar{u}\not{k}t^b v]\end{aligned}$$

The first term is the sum of physical polarization as in QED. The second term is generated by the ghost fields. [r]

For

$$\sum_{\text{pp}} (\varepsilon\varepsilon) = -\eta$$

Ghost fields have to be included to cancel non-physical polarizations [r]. Consider $q\bar{q} \to X\bar{X}$ and the diagrams [r] diagr. The first amplitude is

$$\mathrm{i}M_{G_1} = \bar{v}(-\mathrm{i}g_s t^d \gamma^\delta) u \left[-\mathrm{i}\frac{\eta^{\delta\gamma}}{q^2} \delta^{cd}\right] g_s f^{cba} k^\gamma$$

from which

$$M_{G_1} = \mathrm{i}\frac{g^2}{q^2} f^{cba} (\bar{v}\not{k}t^c u) \,, \quad M_{G_2} = M_{G_1}$$

The sum is then

$$|M_{G_1}|^2 + |M_{G_2}|^2 = 2|M_{G_1}|^2 = 2\left(\frac{g^2}{q^2}\right)^2 f^{cda} f^{cdb} (\bar{v}\not{k}t^a u \bar{u} t^b v)$$

This expression is the same as the one computed for the sum over the physical polarizations.

To get the correct result, one needs to sum ghost amplitudes with a negative sign. The ghost contribution acts as a negative probability. One may work with a covariant gauge in QCD, but has to deal with ghost fields. The other option is to sum over only physical polarizations. The ghost fields cancel the non-physical degrees of freedom.

6.2 Color algebra and color factors

The generators t^a and the structure constants f^{abc} have many components. [r] There are two representations of the algebra. The fundamental representation [r] one has

$$\mathcal{T} = t \,, \quad (\mathcal{T}^a)_{ij} = (t^a)_{ij}$$

In the adjoint representation, one has

$$\mathcal{T} = T\,, \quad (\mathcal{T}^a)_{bc} = \mathrm{i}f^{bac}$$

The commutation relations of the algebra are

$$[\mathcal{T}^a, \mathcal{T}^b] = \mathrm{i}f^{abc}\mathcal{T}^c$$

[r]

$$\mathcal{T}^2 = \sum_a \mathcal{T}^a\mathcal{T}^a$$

Proposition. It holds

$$[\mathcal{T}^2, \mathcal{T}^b] = 0\,, \quad \forall b$$

Proof. In fact

$$[T^aT^a, T^b] = T^a[T^a, T^b] + [T^a, T^b]T^a = T^a(\mathrm{i}f^{abc}T^c) + (\mathrm{i}f^{abc})T^cT^a = \mathrm{i}f^{abc}\{T^a, T^c\} = 0$$

□

Lemma (Schur's)**.** If

$$[T^2, T^b] = 0$$

then

$$T^2 = CI$$

The operator T^2 is called Casimir of the group.

In the fundamental representation, one has

$$(T^2)_{ik} = t^a_{ij}t^a_{jk} = C_\mathrm{F}I_{ik}$$

The second equality is due to Schur's lemma. The identity matrix is 3×3. In the adjoint representation, one has

$$(T^2)_{ad} = T^b_{ac}T^b_{cd} = (\mathrm{i}f^{abc})(\mathrm{i}f^{cbd}) = C_\mathrm{A}I_{ad}$$

The identity matrix is 8×8.

Remark. This Casimir operator appears in Feynman diagrams. Consider the color contribution to the following diagram

$$= t^a_{ij}t^a_{jk} = C_\mathrm{F}\delta_{ik} = C_\mathrm{F}$$

[r] diagr. Similarly

$$= (\mathrm{i}f^{abc})(\mathrm{i}f^{cbd}) = C_\mathrm{A}\delta_{ad} = C_\mathrm{A}$$

These diagrams appear when computing the amplitude squared where the color structure is

$$t^a_{ij}(t^a_{ik})^* = t^a_{ij}t^a_{ki} = t^a_{kj}$$

Remark (Normalization of fundamental representation)**.** The normalization is given by

$$\mathrm{Tr}\big(t^at^b\big) = T_R\delta^{ab}\,, \quad T_R = \frac{1}{2}$$

This is also given by

$$\mathrm{Tr}\big(t^at^b\big) = t^a_{ij}t^b_{ji}$$

There are equivalences between the diagrams.

Proposition (Fierz identity)**.** It holds

$$\delta_{ij}\delta_{lk} = \frac{1}{N}\delta_{ik}\delta_{lj} + \frac{1}{T_R}t^a_{lj}t^a_{ik}$$

Its diagrammatic representation is the following [r].

Corollary. Reorganizing the equation

$$\delta_{ij}\delta_{lk} - \frac{1}{N}\delta_{ik}\delta_{lj} = \frac{1}{T_R}t^a_{lj}t^a_{ik}$$

then one sees that [r]

Rules. The Feynman rules are

- Propagators are the identity matrix.
- Loops are traces.

From $\mathrm{Tr}\, t^a = 0$ then tadpoles are zero [r]. The vacuum bubble with fermions

$$\sum_{i=1}^{N} \delta_{ii} = N$$

The vacuum bubble with gluons is

$$\sum_{a=1}^{N^2-1} \delta_{aa} = N^2 - 1$$

Applications. One may calculate C_F. By connecting two fermionic ends [r] diagr one obtains

$$N = \frac{1}{N} + 2$$

from which

$$C_\mathrm{F} = \frac{N^2-1}{2N} = \frac{4}{3}$$

One may calculate C_A taking

$$= -\frac{1}{2N}$$

and knowing

$$[t^a, t^b] = \mathrm{i} f^{abc} t^c \,,$$

Therefore

$$= C_\mathrm{A}$$

[r] from which

$$C_\mathrm{A} = N = 3$$

Example. One has to separate color from Lorentz structure and kinematics. Consider the diagrams [r] diagr. The Feynman amplitude is

$$\mathrm{i}M^{\alpha\beta} = \bar{v}_j [S_1^{\alpha\beta} t^b_{jl} t^a_{li}] u_i + \bar{v}_j [S_2^{\alpha\beta} t^b_{jl} t^a_{li}] u_i + \bar{v}_j [S_3^{\alpha\beta} \mathrm{i} f^{abc} t^c_{ji}] u_i$$

Letting $M = M^{\alpha\beta} \varepsilon_\alpha \varepsilon'_\beta$, one ha

$$\overline{\sum} |M|^2 = \frac{1}{3}\frac{1}{3}\frac{1}{2}\frac{1}{2} \sum_{\text{pol, spin, color}} MM^*$$

When squaring the amplitude, there are nine terms. Consider only the following term

$$S_1 S_1^* (t^b_{jl} t^a_{li})(t^b_{jr} t^a_{ri})^* = S_1 S_1^* t^b_{jl} t^a_{li} t^b_{rj} t^a_{ir} = S_1 S_1^* \mathrm{Tr}\big(t^b t^a t^a t^b\big)$$

where one used $(t^a)^*_{ij} = t^a_{ji}$. [r] The diagram is $C_\mathrm{F}^2 N$. In this way one can simply the color structure by immediately computing the results.

For example, the $S_3 S_3^*$ term is $C_\mathrm{A} C_\mathrm{F} N$.

Lecture 17

mer 24 apr 2024 12:30

6.3 Running couplings

[r] The coupling constant is a function of the energy scale. One has to compute the wave function renormalizations and vertex corrections. Assembling the corrections, for QCD one obtains a beta function of

$$b_0 = \frac{11C_\mathrm{A} - 4n_\mathrm{F}T_R}{12\pi} > 0$$

where $T_R = T_\mathrm{F}$ and n_F is the number of light quarks. In QED the sign of the beta function is negative. This means that QCD exhibits asymptotic freedom and QED exhibits a Landau pole. Letting the strong coupling constant

$$\alpha_s = \frac{g_s^2}{4\pi}$$

the beta function is

$$\beta(\alpha_\mathrm{R}) \equiv \mu^2 \,\mathrm{d}_{\mu^2}\alpha_\mathrm{R} = \cdots = -b_0\alpha_\mathrm{R}^2 + o(\alpha_\mathrm{R}^2)$$

From this, the coupling constant is

$$\alpha_\mathrm{R}(\mu^2) = \frac{\alpha_\mathrm{R}(\mu_0^2)}{1 + \alpha_\mathrm{R}(\mu_0^2) b_0 \ln \frac{\mu^2}{\mu_0^2}}$$

where μ and μ_0 are arbitrary energy scales. It is convenient to express the running of the coupling by fixing the scale at which a divergence happens

$$\alpha_\mathrm{R}(\mu^2) = \frac{1}{b_0 \ln \frac{\mu^2}{\Lambda^2}}$$

From this one sees that the theory is asymptotically free for high energy, $Q \to \infty$, and has a pole for small Q [r]. The pole of QCD is at about $\Lambda \sim 200\,\mathrm{MeV}$. This point is called Λ_QCD. In the modified minimal subtraction scheme, one has

$$\Lambda_\mathrm{QCD} \approx 250\,\mathrm{MeV}\,, \quad \alpha_s(\Lambda) \to \infty$$

At this scale, perturbation theory is not applicable. This energy is of the same order of magnitude as bound states of quarks.

[r] The beta function can be expanded as

$$\beta(\alpha) = -\alpha[b_0\alpha + b_1\alpha^2 + b_2\alpha^3 + o(\alpha^3)]$$

where b_0 is one-loop, b_1 is two-loop, etc.

Dimensional regularization. In dimensional regularization, the number of space-time dimensions is extended to the reals with

$$d = 4 - 2\varepsilon$$

The trace of the metric tensor is

$$\eta^\mu{}_\mu = d$$

The Dirac algebra can be generalized and loop integrals are expressed in d dimensions. The divergences appear as poles in the regulator ε. To keep the action dimensionless, the Lagrangian density has mass dimension d and the fields have dimensions

$$\dim\psi = \frac{d-1}{2}\,, \quad \dim A = \frac{d-2}{2}$$

[r] The coupling constant has dimension

$$\dim g = \varepsilon$$

In order to keep it dimensionless, one has to add a mass scale that takes the dimension $g\mu^\varepsilon$.

Dimensional regularization is needed to cure ultraviolet and infrared divergences. Most of the current research deals with infrared divergences. These divergences are large corrections in computing some observable.

Renormalization. The bare coupling is linked to the renormalized coupling through

$$\alpha_0 = \alpha_{\mathrm{R}}\left[1 - \frac{1}{\varepsilon}\frac{\alpha_{\mathrm{R}}}{2\pi}b_0 + \cdots\right]$$

[r] the integral exposes ultraviolet poles that are cancelled by renormalizing the coupling. The above pole cancels the one that appears in the computation.

This is equivalent to counter term renormalization when fermions are massless. At one-loop, an ultraviolet divergence has a simple pole and the above relation simplifies the pole.

Euler integrals. The Euler Gamma function

$$\Gamma(z) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t}t^{z-1}\,, \quad \operatorname{Re} z > 0$$

can be analytically continued through

$$\Gamma(z+1) = z\Gamma(z)$$

A few properties are

$$\Gamma(1) = 1\,, \quad \Gamma(1/2) = \sqrt{\pi}\,, \quad \Gamma(n) = (n+1)!$$

It has simple poles for $z \in -\mathbb{N}_0$. In particular

$$-\varepsilon\Gamma(-\varepsilon) = \Gamma(1-\varepsilon) \implies \Gamma(-\varepsilon) = -\frac{1}{\varepsilon}\Gamma(1-\varepsilon) \sim -\frac{1}{\varepsilon}$$

The Euler Beta function is

$$\mathrm{B}(a,b) = \int_0^1 \mathrm{d}x\, x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

6.4 QCD in positron-electron collision

One would like to compute the total cross section of $e^+e^- \to$ hadrons. At the next lower order the computation is complicated. One shall assume that the probability of producing hadrons is the same as the one of producing strong interacting elementary particles. This can be informally motivated as follows. A positron and an electron interact and give a virtual photon that splits into $q\bar{q}$ which interact through QCD. If the photon is highly energetic beyond Λ_{QCD}, the QCD interaction is perturbative. At some point, the energy lowers and the regime is non-perturbative and the quarks are put into hadrons. One can prove that the non-perturbative regime changes only the final state, but not the cross-section

$$\sigma(e^+e^- \to \text{hadrons}) = \sigma(e^+e^- \to q\bar{q}g\cdots)\left[1 + O(\Lambda_{\mathrm{QCD}}/\sqrt{s})^6\right]$$

At high energies, the non-perturbative effects are suppressed.

Next-leading order. When adding loop corrections, many infinities appear but they cancel. The way they cancel is not trivial. [r] diagr. The first diagram is called Born term while twice the real part of the second term is called virtual correction. This last term is divergent in the infrared. One has to add all the terms that have α_s in their perturbative expansion: [r] diagr. These diagrams are infrared divergent. By a theorem, the sum of real and virtual emission diagrams is finite.

Born term. [r] diagr. The Feynman diagram is

$$\bar{u}(p)(-\mathrm{i}e_q\gamma_\alpha)v(\bar{p})\frac{-\mathrm{i}}{Q^2}\eta^{\alpha\beta}\bar{v}(2)(-\mathrm{i}e\gamma_\beta)u(1)$$

The amplitude is

$$M = (e_q e)L^\alpha H_\alpha\frac{1}{Q^2}\,, \quad L^\alpha = \bar{v}(2)\gamma^\alpha u(1)\,, \quad H^\alpha = \bar{u}(p)\gamma^\alpha v(\bar{p})$$

The squared sum is
$$\sum|M|^2 = \left(\frac{ee_q}{Q^2}\right)^2 \sum_{\text{spin, color}} L^\alpha (L^*)^{\bar\alpha} H_\alpha (H^*)_{\bar\alpha}$$
Let
$$L^{\alpha\bar\alpha} = \sum_{\text{spin}} L^\alpha (L^*)^{\bar\alpha} = \operatorname{Tr}\left(p_2 \gamma^\alpha p_1 \gamma^{\bar\alpha}\right), \quad H^{\alpha\bar\alpha} = \sum_{\text{spin}} H^\alpha (H^*)^{\bar\alpha}$$
One has
$$Q_\alpha L^{\alpha\bar\alpha} = Q_{\bar\alpha} L^{\alpha\bar\alpha} = 0$$
[r]

One is interested in non-polarized quantities [r]
$$\int \mathrm{d}\phi_n \sum_{\text{spin}} L^\alpha (L^*)^{\bar\alpha} H_\alpha (H^*)_{\bar\alpha}$$
[r] For non-polarized quantities, the integral
$$\int \mathrm{d}\phi_n \sum H_\alpha (H^*)_{\bar\alpha} = A\eta_{\alpha\bar\alpha} + BQ_\alpha Q_{\bar\alpha}$$
can only be function of Q. [r]
$$Q_\alpha H^\alpha = 0 \implies Q_\alpha (A\eta_{\alpha\bar\alpha} + BQ_\alpha Q_{\bar\alpha}) = 0 \implies A = -BQ^2$$
Therefore
$$\int \mathrm{d}\phi_n \sum_{\text{spin}} H_\alpha (H^*)_{\bar\alpha} = \int \mathrm{d}\phi_n\, H(Q^2 \eta_{\alpha\bar\alpha} - Q_\alpha Q_{\bar\alpha})$$
where H is a proportionality constant.

Similarly
$$\sum_{\text{spin}} L_\alpha L^*_{\bar\alpha} \to L(Q^2 \eta_{\alpha\bar\alpha} - Q_\alpha Q_{\bar\alpha})$$
Therefore, one has
$$\int \mathrm{d}\phi_n\, L^\alpha (L^*)^{\bar\alpha} H_\alpha (H^*)_{\bar\alpha} = \int \mathrm{d}\phi_n\, LH(Q^2 \eta_{\alpha\bar\alpha} - Q_\alpha Q_{\bar\alpha})^2 = \int \mathrm{d}\phi_n\, LH(d-1)Q^4 = \cdots$$
$$= \frac{1}{d-1} \int \mathrm{d}\phi_n\, L^\alpha L^*_\alpha H^\beta H^*_\beta$$
[r]

Consider implicit the sum over spins. One has
$$L^\mu L^*_\mu = \bar v(1)\gamma^\mu u(2) \bar u(2) \gamma_\mu u(1) = \operatorname{Tr}[\not p_1 \gamma^\mu \not p_2 \gamma_\mu] = 4(2-d) p_1 p_2$$
Also
$$H^*_\mu H^\mu = 4(2-d) pp' N$$
The cross-section is
$$\sigma_{\text{LO}} = \sigma_{\text{B}} = \frac{1}{2s} \int \mathrm{d}\phi_2 \overline{\sum} |M|^2 = \frac{1}{2s} \left[\frac{1}{2}\frac{1}{2}\right] \int \mathrm{d}\phi_2\, (LH)(L^* H^*)$$
$$= \frac{1}{2s}\frac{1}{4} \left[\frac{e_q e}{Q^2}\right]^2 \int \mathrm{d}\phi_2\, \frac{1}{d-1} (LL^*)(HH^*) = NQ_q^2 \left[\frac{1}{s}\frac{4}{d-1}\pi\alpha^2\right]$$
Notice that
$$pp' = \frac{(p+p')^2}{2} = \frac{s}{2}, \quad p_1 p_2 = \frac{s}{2}$$
The integrand does not depend on the momenta because s is fixed. To compute the phase space, see notes. [r]

6.4.1 Real emission cross-section

Preliminary considerations. The amplitude of the first diagram is

$$\mathrm{i}M_1^\alpha = \bar{u}(p)(-\mathrm{i}g_s\gamma^\mu t^a)\mathrm{i}\frac{\not{p}+\not{k}}{2pk}(\mathrm{i}e_q\gamma^\alpha)v(p')$$

The denominator may be zero

$$2pk = 2E_qE_g(1-\cos\theta_{qg})$$

In particular for, $E_g = 0$ or $\theta_{qg} = 0$. When integrating the amplitude these two divergences appear. The second diagram diverges for $\theta_{\bar{q}g} = 0$. These are infrared divergences: in particular, soft gluon and collinear divergences.

Lecture 18

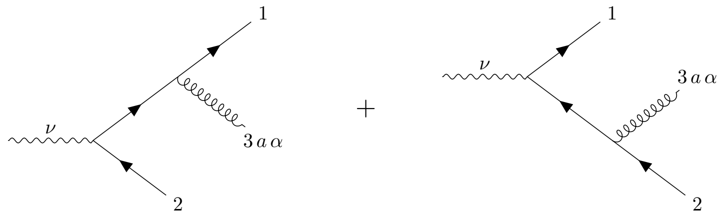
lun 29 apr 2024 10:30

Computation. One would like to compute the cross-section in dimensional regularization. For non-oriented quantities, one has found

$$\int \mathrm{d}\phi_n (LH)(LH)^* = \frac{1}{d-1}\int \mathrm{d}\phi_n\,(LL^*)(HH^*) = \frac{1}{d-1}\int \mathrm{d}\phi_n\,(-LL^*)(-HH^*)$$

One would like to compute the hadronic part given by the second parenthesis. [r] Consider the diagrams

$H^\nu \to$



$= \mathrm{I} + \mathrm{II}$

Let

$$p \leftrightarrow k_1\,,\quad p' \leftrightarrow k_2\,,\quad k \leftrightarrow k_3\,,\quad s_{ij} = (k_i + k_j)^2 = 2k_ik_j$$

Matrix element. The amplitude is

$$\begin{aligned}&-\mathrm{i}eQ_fH^\nu = \\ &= \bar{u}(1)\left[(-\mathrm{i}g_s\mu^\varepsilon\gamma^\alpha t^a)\frac{\mathrm{i}(\not{k}_1+\not{k}_3)}{s_{13}}(-\mathrm{i}eQ_f\gamma^\nu) + (-\mathrm{i}eQ_f\gamma^\nu)\frac{\mathrm{i}(-\not{k}_2-\not{k}_3)}{s_{23}}(-\mathrm{i}g_s\mu^\varepsilon\gamma^\alpha t^a)\right]\epsilon_\alpha v(2)\end{aligned}$$

From this, one gets

$$\begin{aligned}H^\nu &= g_s\mu^\varepsilon\bar{u}(1)\left[\gamma^\alpha t^a\frac{\not{k}_1+\not{k}_3}{s_{13}}\gamma^\nu + \gamma^\nu\frac{(-\not{k}_2-\not{k}_3)}{s_{23}}\gamma^\alpha t^a\right]\epsilon^\alpha v(2)\\ (H^*)^\nu &= g_s\mu^\varepsilon\bar{v}(2)\left[\gamma^\nu\frac{\not{k}_1+\not{k}_3}{s_{13}}\gamma^\beta t^a + t^a\gamma^\beta\frac{(-\not{k}_2-\not{k}_3)}{s_{23}}\gamma^\nu\right](\epsilon^*)^\beta u(1)\end{aligned}$$

Schematically, one has

$$H^\nu = g_s\mu^\varepsilon[\mathrm{I}^{\nu\alpha} + \mathrm{II}^{\nu\alpha}]\epsilon^\alpha$$

The sum of the amplitudes is

$$\sum_{s,c,p}(-H^*_\nu H^\nu) = -(g_s\mu^\varepsilon)^2\sum_{s,c}(\mathrm{I}+\mathrm{II})_{\nu\alpha}(\mathrm{I}+\mathrm{II})^*_{\nu\beta}\sum_{\mathrm{pol}}\epsilon^\alpha(\epsilon^*)^\beta$$

where s is spin, c is colour and p is polarization. Since one has to sum over physical polarizations, one may use

$$\sum_{\mathrm{pol}}\epsilon^\alpha(\epsilon^*)^\beta = -\eta^{\alpha\beta}$$

To compute the color factor one may use the previous trick to obtain

$$\mathrm{I}\cdot\mathrm{I}^* = \mathrm{II}\cdot\mathrm{II}^* = \mathrm{I}\cdot\mathrm{II}^* = C_{\mathrm{F}}N$$

Therefore, one has

$$\sum_{s,c,p}(-H_\nu H_\nu^*) = (g_s\mu^\varepsilon)^2 C_{\mathrm{F}}N[(\mathrm{I}\cdot\mathrm{I}^*) + (\mathrm{II}\cdot\mathrm{II}^*) + 2\,\mathrm{Re}(\mathrm{I}\cdot\mathrm{II}^*)]$$

One may compute each term [r] (see notes[1], p.4)

$$\mathrm{I}\cdot\mathrm{I}^* = \cdots = 2(2-d)^2\frac{s_{23}}{s_{13}} \sim \frac{E_{\bar{q}}E_g(1-\cos\theta_{23})}{E_{\bar{q}}E_g(1-\cos\theta_{13})}\,, \quad \mathrm{II}\cdot\mathrm{II}^* = 2(2-d)^2\frac{s_{13}}{s_{23}}$$

The first diagram is singular for $s_{13}\to 0$ which is the case of quark and gluon being collinear. The second diagram is singular again for collinearity. The interference term gives

$$2\,\mathrm{Re}(\mathrm{I}\cdot\mathrm{II}^*) = \frac{4(d-2)}{s_{13}s_{23}}[(d-4)s_{13}s_{23} + 2s_{12}Q^2]$$

This term is singular when the particles are collinear and when the gluon is soft.

Uniting everything gives

$$\sum_{s,c,p}(-H_\nu H_\nu^*) = (g_s\mu^\varepsilon)^2 C_{\mathrm{F}}N_{\mathrm{C}}\frac{4(d-2)}{s_{13}s_{23}}[s_{13}^2 + s_{23}^2 + 2s_{12}Q^2 - \varepsilon(s_{13}+s_{23})^2]$$

Kinematic variables. The phase space can be parametrized by choosing

$$x_i = \frac{2k_iQ}{Q^2}\,, \quad Q = k_1 + k_2 + k_3$$

In the center-of-mass frame, the above is

$$Q = (Q^0, \mathbf{0}) = (\sqrt{s}, \mathbf{0}) \implies x_i = \frac{2E_i}{Q_0}$$

where E_i is the energy of the quark (or anti-quark). The variable x is the energy fraction

$$E_i = x_i\frac{\sqrt{s}}{2}\,, \quad x_i \geq 0$$

The energy conservation law is

$$E_1 + E_2 + E_3 = \sqrt{s} \implies x_1 + x_2 + x_3 = 2$$

There is another constraint on the energy fraction

$$0 \leq (k_1+k_2)^2 = 2E_1E_2(1-\cos\theta_{12}) = (Q-k_3)^2 = Q^2 - 2Qk_3 = Q^2(1-x_3) \implies x_i \leq 1$$

From this expression, one also gets

$$s_{ij} = Q^2(1-x_k)$$

where ijk are cyclic.

The sum becomes

$$-\sum_{s,c,p}H_\nu H_\nu^* = (g_s\mu^\varepsilon)^2 C_{\mathrm{F}}N_{\mathrm{C}}4(d-2)\frac{(1-x_1)^2 + (1-x_2)^2 + 2(1-x_3)^2 - \varepsilon x_3^2}{(1-x_1)(1-x_2)}$$
$$= (g_s\mu^\varepsilon)^2 C_{\mathrm{F}}N_{\mathrm{C}}4(d-2)\frac{x_1^2 + x_2^2 - \varepsilon x_3^2}{(1-x_1)(1-x_2)}$$

[1] “e^+e^- lowest order real part”.

The sum diverges for $x_1, x_2 \to 1$ both separately and simultaneously. The constraints on the third fraction are

$$x_3 \geq 0 \implies x_1 + x_2 \leq 2$$

and

$$x_3 \leq 1 \implies 2 - x_1 - x_2 \leq 1 \implies x_2 \geq 1 - x_1$$

From these, the integration domain for the phase space can be represented as the upper right triangle of the unit square with the lower left corner on the origin. Knowing that

$$1 - x_1 = \frac{x_2 E_g}{\sqrt{s}}(1 - \cos\theta_{\bar{q}g})$$

When $x_1 \to 1$, the anti-quark and the gluon are collinear. The region where $x_1 \to 1$ is the right side of the triangle which corresponds to

$$\mathbf{k}_2 \parallel \mathbf{k}\,, \quad \bar{q} \parallel g$$

Similarly, the region of $x_2 \to 1$ is the upper side of the triangle corresponding to $\mathbf{k}_1 \parallel \mathbf{k}$. The soft gluon limit is given by

$$x_3 \to 0 \implies x_1 + x_2 = 2$$

which corresponds the upper right corner of the square. The amplitude diverges only on the edges (but not the hypotenuse). The cross-section is soft- and collinear-divergent. One expects such divergences to be logarithmic and of the form

$$\frac{\mathrm{d}x_1\,\mathrm{d}x_2}{(1 - x_1)(1 - x_2)}$$

Three-body phase space. [r] See notes, p.7. In d dimensions, the phase space is

$$\begin{aligned}
\mathrm{d}\phi_3 &= \left[\prod_{j=1}^3 \frac{\mathrm{d}^{d-1}k_j}{(2\pi)^{d-1}2E_j}\right](2\pi)^d\delta^4(Q - k_1 - k_2 - k_3) \\
&= \frac{1}{(2\pi)^{2d-3}}\frac{\mathrm{d}^{d-1}k_1}{2E_1}\frac{\mathrm{d}^{d-1}k_2}{2E_2}\delta(Q_0 - E_1 - E_2 - E_3) \\
&= \frac{1}{(2\pi)^{2d-3}}\frac{1}{4}(E_1E_2)^{d-3}\,\mathrm{d}E_1\,\mathrm{d}E_2\,\mathrm{d}\Omega_1^{(d-1)}\,\Omega_2^{(d-1)}\,\delta(Q_0 - E_1 - E_2 - E_3) \\
&= \frac{Q^2}{2(4\pi)^3}\frac{1}{\Gamma(2 - 2\varepsilon)}\left[\frac{4\pi}{Q^2}\right]^{2\varepsilon}[(1 - x_1)(1 - x_2)(1 - x_3)]^{-\varepsilon}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\delta(2 - x_1 - x_2 - x_3)
\end{aligned}$$

At the third line, one goes to spherical coordinates

$$\mathrm{d}^{d-1}k_j = E_j^{d-2}\,\mathrm{d}E_j\,\mathrm{d}\Omega_j^{(d-1)}$$

and uses the center-of-mass frame

$$E_3 = |\mathbf{k}_3| = |\mathbf{k}_1 + \mathbf{k}_2| = \sqrt{E_1^2 + E_2^2 + 2E_1E_2\cos\theta_{12}}$$

[r] At the fourth line one orients the direction of $\mathbf{k}_2$ with respect to the direction of $\mathbf{k}_1$ as to find

$$\mathrm{d}\Omega_2^{(d-1)} = \cdots = \mathrm{d}(\cos\theta_{12})\,(\sin\theta_{12})^{d-4}\,\mathrm{d}\Omega_2^{(d-2)}$$

[r] and one also inserts the energy fractions and integrates over the angles of the outgoing particles.

The integration then gives

$$\frac{\mathrm{d}x_1}{(1 - x_1)}(1 - x_1)^{-\varepsilon} \sim \frac{\mathrm{d}x_1}{(1 - x_1)^{1+\varepsilon}}$$

which produces a pole in the regulator ε.

Analysis of divergences. The matrix element is finite when integrating far from the sides of the triangle. The cross-section

$$\mathrm{d}\sigma_{\mathrm{R}}^{(d-4)} \sim \frac{\mathrm{d}x_1\,\mathrm{d}x_2}{(1-x_1)(1-x_2)}$$

is finite for

$$E_g > E_{\min}\,, \quad \theta_{qg}, \theta_{\bar{q}g} > \theta_{\min}$$

Since the divergences are logarithmic, the cross-section behaves like

$$\sigma_{\mathrm{R}}^{(d=4)} \sim \ln E_{\min}\,\ln\theta_{\min}$$

In d dimensions, the integrals are finite and can be performed. One has

$$\begin{aligned}\int\sum(-HH)\,\mathrm{d}\phi_3 = \int\Bigg[&(g_s\mu^\varepsilon)^2 C_{\mathrm{F}} N_{\mathrm{C}} 4(d-2)\frac{x_1^2+x_2^2-\varepsilon x_3^2}{(1-x_1)(1-x_2)}\Bigg]\\ &\times\frac{Q^2}{2(4\pi)^2}\frac{1}{\Gamma(2-2\varepsilon)}\left[\frac{4\pi}{Q^2}\right]^{2\varepsilon}(1-x_1)^{-\varepsilon}(1-x_2)^{-\varepsilon}(1-x_3)^{-\varepsilon}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\end{aligned}$$

with the constraint $x_1+x_2+x_3=2$. Apart from the overall factor, one has

$$\begin{aligned}I(\varepsilon) &= \int_0^1 \mathrm{d}x_1\int_{1-x_1}^1 \mathrm{d}x_2\,[x_1^2+x_2^2-\varepsilon(2-x_1-x_2)^2]\\ &\quad\times(1-x_1)^{-1-\varepsilon}(1-x_2)^{-1-\varepsilon}(x_1+x_2-1)^{-\varepsilon}\\ &= \int_0^1 \mathrm{d}x_1\int_0^1 \mathrm{d}t\,x_1[x_1^2+(1-tx_1)^2-\varepsilon(1-x_1(1-t))^2]\\ &\quad\times(1-x_1)^{-1-\varepsilon}(tx_1)^{-1-\varepsilon}(x_1)^{-\varepsilon}(1-t)^{-\varepsilon}\\ &= -2(2\varepsilon-1)[(\varepsilon-2)\varepsilon+2]\frac{\Gamma(1-\varepsilon)\Gamma^2(-\varepsilon)}{\Gamma(3-3\varepsilon)}\end{aligned}$$

At the third line, one has inserted

$$x_2 = 1-tx_1\,,\quad \mathrm{d}x_2 = x_1\,\mathrm{d}t\,,\quad t\in[0,1]$$

This expression contains a quadratic pole and a simple pole. In fact

$$\Gamma^2(-\varepsilon) = \left[\frac{\Gamma(1-\varepsilon)}{-\varepsilon}\right]^2 \sim \frac{1}{\varepsilon^2}$$

Putting everything together, the flux factor, the average over initial quantum numbers, the leptonic amplitude and the hadronic amplitude [r], one obtains

$$\sigma_{\mathrm{R}}^{(d)} = \sigma_{\mathrm{B}}^{(d)}\frac{\alpha_s}{2\pi}C_{\mathrm{F}}\left[\frac{4\pi\mu^2}{s}\right]^\varepsilon C_\Gamma\left[\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+\frac{19}{2}-\pi^2+o(\varepsilon^0)\right]\,,\quad C_\Gamma \equiv \frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}$$

where σ_{B} is the Born emission. [r] The diverges are infrared.

One has discovered that the real emission cross-section is infinite

$$\sigma_{\mathrm{R}}(e^+e^-\to q\bar{q}g) = \int_0^1 \mathrm{d}x_1\int_{1-x_1}^1 \mathrm{d}x_2\,\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

It is finite only when inserting cutoffs for E_g, θ_{qg} and $\theta_{\bar{q}g}$. Inserting cutoffs means that the gluon is not arbitrarily soft and one may distinguish the quark from the gluon [r]. The problem happens when the real emission kinematics is degenerate with respect to the Born kinematics. A final state in which the gluon is arbitrarily soft is the same as no gluon at all?. There are divergences that cancel when summing the above computation with the virtual one-loop corrections.

6.4.2 Divergences and cancellations

Theorem (Kinoshita–Lee–Nauenberg, KLN)**.** In the computation of the transition matrix T, divergences appear when there are degeneracies in the final state (for example in soft or collinear kinematics). If one sums over degenerate states, then the degeneracies cancel in the transition probabilities. Such cancellations take place order by order in perturbation theory.

The soft gluon state is degenerate with (i.e. indistinguishable from) the state without gluons, $q\bar{q}$. The collinear splitting state is degenerate with the state of the quark–anti-quark pair, $q\bar{q}$.

If one computes the virtual corrections (see notes last year) one gets

$$\sigma_{\mathrm{V}}^{(d)} = -\sigma_{\mathrm{B}}^{(d)} \frac{\alpha_s}{2\pi} C_{\mathrm{F}} \left[\frac{4\pi\mu^2}{s}\right]^{\varepsilon} C_\Gamma \left[\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + 8 - \pi^2 + o(\varepsilon^0)\right]$$

In the total cross-section, the poles cancel exactly. This is a consequences of the KLN theorem. Summing the contributions, one has

$$\sigma_{\mathrm{R}}^{(d)} + \sigma_{\mathrm{V}}^{(d)} \to \sigma_{\mathrm{B}} \frac{\alpha_s}{2\pi} C_{\mathrm{F}} \left[\frac{19}{2} - 8\right] , \quad \varepsilon \to 0$$

The total cross-section is then

$$\sigma_{\mathrm{tot}}(e^+e^- \to \text{hadrons})\big|_{\mathrm{NLO}} = \sigma_{\mathrm{B}} + \sigma_{\mathrm{R}} + \sigma_{\mathrm{V}} = \sigma_{\mathrm{B}} \left[1 + \frac{\alpha_s}{\pi}\right]$$

When summing over the kinematical configurations, the result is finite. This is seen in experimental data too (see notes[2])

Remark. The poles in ε have an infrared origin and not ultraviolet. This can be seen in three ways:

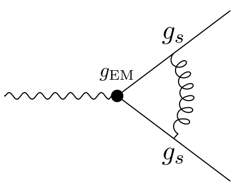
- At one-loop, an ultraviolet pole is always simple.
- In general, for massless fields, the procedure of adding counter terms means replacing the bare strong coupling α_s^0 with the renormalized version α_s^{R} with a factor

  $$\alpha_s^0 = \alpha_s^{\mathrm{R}} \left[1 - \frac{b_0}{\varepsilon} \alpha_s^{\mathrm{R}} + o(\alpha_s^2)\right]$$

  The cross-sections are observables of the order

  $$\sigma_{\mathrm{B}} \sim \alpha_s , \quad \sigma_{\mathrm{V}} \sim \alpha_s^2$$

  In the second cross-section there are ultraviolet poles. [r] When substituting the coupling constant, there is a simple pole coming from the coupling constant that cancels the ultraviolet poles. However, this procedure cannot happen in the present case since there is no coupling α_s in the Born cross-section. This means that the loop integral does not need to be renormalized [r].
- This has also a physical interpretation. The loop computation

  which is order α_s, is a correction to the QED coupling. However, the gauge group of QED, $\mathrm{U}(1)_{\mathrm{EM}}$, is disjoint from the gauge group of QCD, $\mathrm{SU}(3)_{\mathrm{C}}$: the renormalization of the electromagnetic coupling constant is not affected by the strong interactions.

[2]QCD KLN, p.5.

Remark. For the scattering $e^+e^- \to q\bar{q}(g)$, the cross-section can be computed analytically, but this is not always the case. One may have to perform some expansion in the middle of the computation. For generic observables, one may not go through all computations analytically. One may integrate only on some portions of the phase space. One has to use numerical methods.

Remark. It is possible to compute observables other than the cross-section if they are infrared safe.

Remark. In the soft and collinear limits, the observables factorize in QCD and QED.

6.4.3 Soft limit and soft factorization

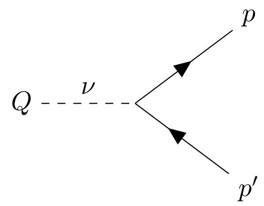
Consider an excited states emitting a quark–anti-quark pair with a soft gluon

$$\gamma^* \to q\bar{q}g$$

[r] This result can be generalized to n hard partons and one soft gluon.

Amplitude. Let Γ^ν be the amplitude describing a generic process. The Born kinematics is given by

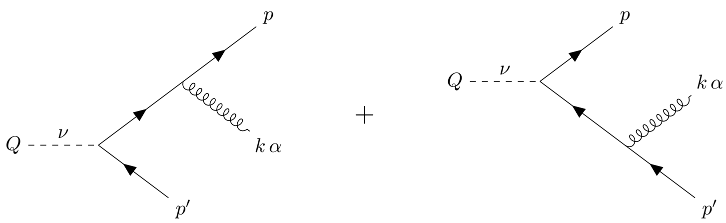
$$M_{\mathrm{B}}^\nu = \bar{u}(p)\Gamma^\nu v(p')\,,$$



where one assumes that the dashed line is colorless. The amplitude squared is

$$\sum M_{\mathrm{B}}^\nu M_{\mathrm{B}\nu}^* = [\bar{u}\Gamma^\nu v][\bar{v}\Gamma_\nu^\dagger u]N$$

The emission of a soft gluon corresponds to the sum of the two diagrams



Let the momentum k of the gluon be soft

$$k = E(1, \hat{n})\,, \quad E = k^0 \to 0$$

The real emission amplitude is

$$\begin{aligned}
M^\nu = M_{\mathrm{R}}^\nu &= \bar{u}(p)\left[(-\mathrm{i}g_s t^a\gamma^\alpha)\left(\mathrm{i}\frac{\not{p} + \not{k}}{2pk}\right)\Gamma^\nu + \Gamma^\nu\left(\mathrm{i}\frac{-\not{p}' - \not{k}}{2p'k}\right)(-\mathrm{i}g_s t^a\gamma^\alpha)\right] v(p')\epsilon_\alpha(k) \\
&\approx g_s\bar{u}\left[t^a\gamma^\alpha\frac{\not{p}}{2pk}\Gamma^\nu + \Gamma^\nu\frac{-\not{p}'}{2p'k}t^a\gamma^\alpha\right] v\epsilon_\alpha \\
&= g_s(\bar{u}_i\Gamma^\nu v_j)\left[\frac{p^\alpha}{pk} - \frac{p'^\alpha}{p'k}\right] t_{ij}^a\epsilon_\alpha \\
&= M_{\mathrm{B}}^\nu g_s\left[\frac{p^\alpha}{pk} - \frac{p'^\alpha}{p'k}\right] t^a\epsilon_\alpha
\end{aligned}$$

At the second line, since k is soft, then

$$p + k \sim p\,, \quad p' + k \sim p' \implies p \approx p'$$

At the third line, one has applied

$$\gamma^\alpha \not{p} = -\not{p}\gamma^\alpha + 2p^\alpha \,, \quad \bar{u}(p)\not{p} = 0 \,, \quad \not{p}' v(p') = 0$$

At the fourth line, in the soft limit, p and p' are the same momenta as in the Born kinematics.

Commonly, the factorization is written as

$$M_{\mathrm{R}}^\nu = M_{\mathrm{B}}^\nu g_s (J^\alpha t^a) \epsilon_\alpha$$

where the eikonal (or soft) current is

$$J^\alpha = \frac{p^\alpha}{pk} - \frac{p'^\alpha}{p'k}$$

To check gauge invariance, one shall verify that $k_\alpha J^\alpha = 0$.

Remark. For more color particles, the current is

$$J^\alpha = \sum_{i=1}^n \mathcal{T}_i \frac{p_i^\alpha}{p_i k}$$

where $\mathcal{T}$ can be the generators either in the adjoint or fundamental representations.

Remark. In the current, there is no information about the hard process. [r] The soft gluon has a vanishing energy and as such a large wavelength: it cannot resolve the quantum structure of the hard process. In classical electrodynamics there is also a concept of eikonal current (see Jackson).

Squared amplitude. The squared amplitude is given by

$$\sum_{s,c,p} M^\nu M_\nu^* = [\bar{u}\Gamma^\nu v][\bar{v}\Gamma_\nu^\dagger u] g_s^2 t_{ij}^a t_{ji}^a J_\alpha J_\beta^* \epsilon^\alpha (\epsilon^\beta)^*$$

Knowing that

$$\sum \varepsilon^\alpha (\varepsilon^\beta)^* = -\eta^{\alpha\beta} \,, \quad t_{ij}^a t_{ji}^b = C_{\mathrm{F}} N$$

one has

$$-\eta^{\alpha\beta} J_\alpha J_\beta^* = -\left[\frac{p^\alpha}{pk} - \frac{p'^\alpha}{p'k}\right]\left[\frac{p_\alpha}{pk} - \frac{p'_\alpha}{p'k}\right] = \frac{2pp'}{(pk)(p'k)}$$

Therefore, the squared amplitude factorizes too

$$\sum_{s,c,p} M^\nu M_\nu^* = \left[\sum M_{\mathrm{B}}^\nu M_{\mathrm{B}\nu}^*\right] C_{\mathrm{F}} g_s^2 \frac{2pp'}{(pk)(p'k)}$$

The leading order of the squared amplitude is in the parenthesis.

For more than two color-emitting lines, there is no exact factorization, but there is a trace.

Phase space. The phase space is given by

$$\mathrm{d}\phi_3 = \frac{\mathrm{d}^3 p}{(2\pi)^3 2p_0} \frac{\mathrm{d}^3 p'}{(2\pi)^3 2p'_0} \frac{\mathrm{d}^3 k}{(2\pi)^3 2k_0} (2\pi)^4 \delta^4 (Q - p - p' - k)$$

For a soft gluon the Dirac delta becomes

$$\delta^4(Q - p - p' - k) \to \delta^4(Q - p - p')$$

There is also the factorization of the phase space

$$\mathrm{d}\phi_3 = \mathrm{d}\phi_2 \, \frac{\mathrm{d}^3 k}{(2\pi)^3 2k_0}$$

Total real cross-section in the soft approximation. The real cross-section is

$$\begin{aligned}\mathrm{d}\sigma_{\mathrm{R}} &\approx \frac{1}{2s}(\text{leptons})\left[\sum M_{\mathrm{B}}^{\nu} M_{\mathrm{B}\nu}^{*}\right] C_{\mathrm{F}} g_s^2 \frac{2pp'}{(pk)(p'k)} \,\mathrm{d}\phi_2 \,\frac{\mathrm{d}^3 k}{(2\pi)^3 2k_0} \\ &= \mathrm{d}\sigma_{\mathrm{B}} C_{\mathrm{F}} (4\pi\alpha_s) \frac{2pp'}{(pk)(p'k)} \,\frac{\mathrm{d}^3 k}{(2\pi)^3 2k_0}\end{aligned}$$

Letting

$$p = p_0(1,0,0,1)\,, \quad p' = p_0(1,0,0,-1)$$

one has

$$pk = p_0 k_0 (1-\cos\theta)\,, \quad p'k = p_0 k_0(1+\cos\theta)\,, \quad pp' = 2p_0^2$$

where θ is the angle between the momentum p and the gluon. The one-gluon phase space is

$$\mathrm{d}^3 k = k_0^2 \,\mathrm{d}k_0 \,\mathrm{d}\cos\theta \,\mathrm{d}\varphi$$

Putting everything together, one has

$$\mathrm{d}\sigma_{\mathrm{R}} \to \mathrm{d}\sigma_{\mathrm{B}} \frac{2\alpha_s}{\pi} C_{\mathrm{F}} \frac{\mathrm{d}k_0}{k_0} \frac{\mathrm{d}\cos\theta}{(1-\cos\theta)(1+\cos\theta)}$$

There is a logarithmically divergent term of soft origin

$$\frac{\mathrm{d}k_0}{k_0}$$

and there is a term collinear divergent

$$\frac{\mathrm{d}\cos\theta}{(1-\cos\theta)(1+\cos\theta)}$$

The first term is expected since one has performed calculations in the soft limit. The presence of a factorization of the collinear divergence is an accident due to the fact that the process is simple. [r] In general, in the soft approximation there is either only a soft divergence or a soft and collinear divergence.

Lecture 19

mar 30 apr 2024 10:30

Upon integrating over the entire phase space, one has

$$\mathrm{d}\sigma_{\mathrm{R}}^{\mathrm{soft}} = \mathrm{d}\sigma_{\mathrm{B}} \,\frac{2\alpha_s}{\pi} C_{\mathrm{F}} \int_0^{\mathrm{smth}} \frac{\mathrm{d}k_0}{k_0} \int_{-1}^{1} \frac{\mathrm{d}(\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$$

Thanks to the KLN theorem, after the integration over the whole phase space at order α_s, the sum of real and virtual contributions must be finite. Therefore, one can formally express the virtual contribution as a function of the Born contribution

$$\mathrm{d}\sigma_{\mathrm{V}}^{\mathrm{soft}} = -\mathrm{d}\sigma_{\mathrm{B}} \,\frac{2\alpha_s}{\pi} C_{\mathrm{F}} \int_0^{\mathrm{smth}} \frac{\mathrm{d}k_0}{k_0} \int_{-1}^{1} \frac{\mathrm{d}(\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$$

If the gluon in the virtual loop is soft or collinear, there must be a divergence that cancels exactly the divergence of the real emission. [r] If one were to perform a full computation, then one could express the final result using factorization. One obtains a schematic result

$$\sigma_{\mathrm{R}}^{\mathrm{full}} + \sigma_{\mathrm{V}}^{\mathrm{full}} = (\sigma_{\mathrm{R}}^{\mathrm{soft}} + \mathrm{finite}_{\mathrm{R}}) + (\sigma_{\mathrm{V}}^{\mathrm{soft}} + \mathrm{finite}_{\mathrm{V}}) = \mathrm{finite}$$

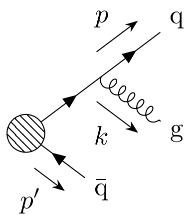
In general, the sum of real and virtual contributions depends on the kinematics [r].

6.4.4 Collinear limit and collinear factorization

From the exact formula for the real emission cross-section, one shall see that, in the collinear limit, there is factorization. Recall the real emission cross-section in four dimensions

$$\mathrm{d}\sigma_{\mathrm{R}} = \mathrm{d}\sigma_{\mathrm{B}} \frac{\alpha_s}{2\pi} C_{\mathrm{F}} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \,\mathrm{d}x_1 \,\mathrm{d}x_2$$

Consider the collinear limit in which a quark is parallel to the gluon.



One parametrizes the final state kinematics as

$$z = \frac{E_q}{E_q + E_g} = \frac{x_1}{2 - x_2} \,, \quad q^2 = (p+k)^2 = (Q - p')^2 = Q^2(1 - x_2)$$

These are the energy fraction of the final quark with respect to the initial quark and the virtuality of the propagator of the final quark (before emitting the gluon) [r]. In the collinear limit, one has $q^2 \to 0$ with $E_g \neq 0$: the internal propagator goes on-shell. The change of variables and measure are

$$x_1 = \frac{q^2 + Q^2}{Q^2} z \,, \quad x_2 = -\frac{q^2 - Q^2}{Q^2} \,, \quad \mathrm{d}q^2 \,\mathrm{d}z = \frac{Q^2}{2 - x_2} \,\mathrm{d}x_1 \,\mathrm{d}x_2$$

The collinear limit is then

$$\mathrm{d}\sigma_{\mathrm{R}} \to \mathrm{d}\sigma_{\mathrm{B}} \frac{\alpha_s}{2\pi} [P_{qq}(z) + O(q^2/Q^2)] \,\mathrm{d}z \,\frac{\mathrm{d}q^2}{q^2} \,, \quad P_{qq}(z) = C_{\mathrm{F}} \frac{1+z^2}{1-z}$$

where $P_{qq}(z)$ is the Altarelli–Parisi quark splitting function (the quark branch is z, gluon branch is $1-z$). This equation exhibits a factorization into the Born cross-section and a factor containing information only about z and q, and no information from previous processes [r]. This statement about the collinear limit is general [r].

The collinear divergence is described by the logarithmically divergent integral

$$\int_0^{\mathrm{smth}} \frac{\mathrm{d}q^2}{q^2}$$

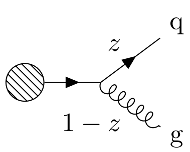
In the soft limit $E_g \to 0$, which implies $z \to 1$, one has a soft singularity logarithmically divergent [r]

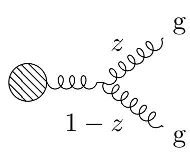
$$P(z) \sim \frac{1}{1-z} \,, \quad \int_{\mathrm{smth}}^1 \frac{\mathrm{d}z}{1-z}$$

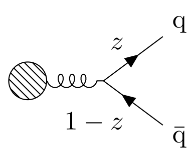
The result provided is much more general. There are various cases [r] diagr

$$P_{qq}(z) = C_{\mathrm{F}} \frac{1+z^2}{1-z} \,, \quad P_{gg}(z) = C_{\mathrm{A}} \left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right] \,, \quad P_{gq}(z) = T_{\mathrm{F}}[z^2 + (1-z)^2]$$

which correspond to the diagrams

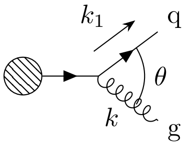






This last branch exhibits just collinear divergence? [r].

If the emitted object is massive, then there is no collinear divergence. The virtual propagator of the diagram



is of the type

$$\Gamma^{(2)} \to \frac{1}{2k^0(k_1^0 - |\mathbf{k}_1|\cos\theta)}$$

The propagator introduces a logarithmic divergence in the collinear limit since

$$|\mathbf{k}_1| \neq k_1^0\,, \quad (k_1^0)^2 = |\mathbf{k}_1|^2 + m^2$$

[r] Taking the limit and performing the integration gives a propagator of the form

$$\Gamma^{(2)} \sim \ln\left[1 + \frac{4|\mathbf{k}_1|^2}{m^2}\right]$$

There is no explicit divergence, but the result depends on the mass in such a way that if it is small, then the logarithm is large.

This discussion also holds in QED apart from color factors. In particular a charge particle emitting a photon exhibits a soft and a collinear limit.

6.5 Infrared-safe observables

One would like to understand what quantities can be computed that have finite results thanks to the cancellation between virtual and real divergences. In order to understand how observables can be safely computed, one has to study what it means to compute a differential observable.

A differential cross-section is of the type

$$\mathrm{d}\sigma^{2\to n} = \mathcal{F}\left[\overline{\sum}|\mathcal{M}|^2\right]\mathrm{d}\phi_n(k_1, \dots, k_n)$$

Consider the integration over only a subset of the n-body phase space. The cross-section above becomes

$$\mathrm{d}\sigma = \mathcal{F}\left[\overline{\sum}|\mathcal{M}|^2\right]\mathrm{d}\phi_n(k_1, \dots, k_n)\, F^{(n)}(k_1, \dots, k_n)$$

where $F^{(n)}$ is a measurement function that defines the phase space region of interest. For example

- for the total cross-section, one has $F^{(n)} = 1$.
- Consider a final state with two momenta of interest, k_1 and k_2, and the region of phase space with[3]

  $$F^{(n)}(k_1, \dots, k_n) = \theta[m^2_{\text{min}} \leq (m_1 + m_2)^2 \leq m^2_{\text{max}}]$$

  Integrating the cross-section with this constraint produces a fiducial cross-section (FXS) $\sigma|_F$.
- For a differential distribution, one uses a Dirac delta function

  $$F^{(n)}(k_1, \dots, k_n) = \delta(\bar{O} - O(k_1, \dots, k_n))$$

  to get a differential cross-section

  $$\mathrm{d}_O\sigma|_{O=\bar{O}} = \int \mathcal{F}\overline{\sum}|\mathcal{M}|^2\,\mathrm{d}\phi\,\delta(\bar{O} - O)$$

This discussion leads to various requirements that must be met by an observable in order for the results to be finite at higher orders.

[3] Where the square brackets are used in the sense of the Iverson bracket and the Heaviside step function is used for clarity.

Infrared safety. An observable with finite results must be infrared-safe. The KLN theorem works for sums over degenerate states. An observable is infrared-safe if it preserves cancellations between real and virtual contributions.

Consider the process $e^+e^- \to q\bar{q}/q\bar{q}g$. To compute the cross-section at next-leading order, one has to consider the Born, the virtual and the real contributions. In order for the cancellations to happen, the observable must be the same in the case where the gluon is soft and in the case where there is no gluon at all

$$O(\phi_3^{\text{soft}}) = O(\phi_2)$$

If this were not the case, there would be poles in the virtual cross-section in the two-body phase space ϕ_2 that would not cancel with the poles in the real cross-section when the gluon becomes soft. A similar situation happens for the collinear limit

$$O(\phi_3^{\text{coll}}) = O(\phi_2)$$

Therefore, the observable must not resolve soft and/or collinear emissions:

$$O(\phi_3^{\text{soft/coll}}) = O(\phi_2)$$

The KLN theorem ensures a cancellation if there is a sum over degenerate states. In this context, this sum involves correctly adding real and virtual contributions. If an observable does not have this property, then it cannot predict at next-leading order. This is a constraint on the theory and which quantities can be measured: one can make non-infrared-safe observables, but they cannot be compared with theory. Therefore, there is a constraint about the kind of observables that can be obtained in perturbative QCD.

A typical example of infrared-safe observable is gluon jets.

6.5.1 Sterman–Weinberg jets

A jet is a flux of collimated hadrons. For example, in the process $e^+e^- \to$ hadrons, at leading order one may see back-to-back fluxes of hadrons. One may also have a hard gluon emission which corresponds to a three-jet event. At higher orders, there can be the emission of two gluons to give four jets.

One is interested in the relative kinematics [r]. One has to define a two-jet cross-section: an event contributes to a two-jet cross-section if one can identify two cones of angle δ that contain a large fraction of the total energy, $(1-\varepsilon)E_{\text{tot}}$. This statement can also be expressed as follows: if $\varepsilon E_{\text{tot}}$ is the maximum energy outside two cones of angle δ, then one has a two-jet cross-section (δ, ε).

The historical definition of the two-jet cross-section $\sigma_{2\text{J}}^{\text{SW}}$ is infrared-safe. It can be computed in perturbative QCD.

The contributions up to order α_s are three: Born's, virtual and real.

Born contribution. For every ε and δ, one can find two cones such that σ_{B} always enters in the two-jet cross-section.

Virtual contribution. The virtual contribution is infrared-divergent, but has the same phase space as the Born contribution. The virtual contribution to the cross-section is

$$\sigma_{2\text{J}}^{(\text{V})} = -\sigma_{\text{B}} \frac{2\alpha_s}{\pi} C_{\text{F}} \int_0^{E_{\max}} \frac{\mathrm{d}k_0}{k_0} \int_{-1}^{1} \frac{\mathrm{d}(\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$$

Real contribution. Consider a real emission with a soft gluon $k_0 < \varepsilon E_{\text{tot}}$. Regardless of the angle θ between the quark and the gluon, this case is always a two-jet event since one can always find the two cones. The contribution is

$$\sigma_{2\text{J}}^{(\text{RS})} = \sigma_{\text{B}} \frac{2\alpha_s}{\pi} C_{\text{F}} \int_0^{\varepsilon E} \frac{\mathrm{d}k_0}{k_0} \int_{-1}^{1} \frac{\mathrm{d}(\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$$

Consider instead a hard gluon $k_0 > \varepsilon E$. The event is a two-jet cross-section only if the gluon is contained within one of the cones, $\theta < \delta$ or $\theta > \pi - \delta$. The contribution is

$$\sigma_{2\mathrm{J}}^{(\mathrm{RH})} = \sigma_{\mathrm{B}} \frac{2\alpha_s}{\pi} C_{\mathrm{F}} \int_{\varepsilon E}^{E} \frac{\mathrm{d}k_0}{k_0} \left[\int_{\theta=0}^{\theta=\delta} \frac{\mathrm{d}(\cos\theta)}{(1-\cos\theta)(1+\cos\theta)} + \int_{\theta=\pi-\delta}^{\theta=\pi} \frac{\mathrm{d}(\cos\theta)}{(1-\cos\theta)(1+\cos\theta)} \right]$$

Putting everything together gives

$$\sigma_{2\mathrm{J}}^{\mathrm{SW}} = \sigma_{\mathrm{B}} + \sigma_{\mathrm{V}} + \sigma^{(\mathrm{RS})} + \sigma^{(\mathrm{RH})} = \sigma_{\mathrm{B}} \left[1 - \frac{2\alpha_s}{\pi} C_{\mathrm{F}} \int_{\varepsilon E_{\mathrm{tot}}}^{E_{\mathrm{tot}}} \frac{\mathrm{d}k_0}{k_0} \int_{\theta=\delta}^{\theta=\pi-\delta} \frac{\mathrm{d}(\cos\theta)}{(1-\cos\theta)(1+\cos\theta)} \right]$$

The two integrals are finite. The first has a dependence on $\ln\varepsilon$, while the second on $\ln\delta$. This works because the observable is insensitive to soft emission [r].

The total contribution including the finite parts is

$$\sigma_{2\mathrm{J}}^{\mathrm{SW}} = \sigma_{\mathrm{B}} \left[1 - \frac{2\alpha_s}{\pi} C_{\mathrm{F}} (\ln\varepsilon \ln\delta^2 + \text{finite}) \right]$$

The logarithms have been computed in the soft limit and show cancellations of real and virtual divergences. The finite part can be obtain from a full computation, without taking any soft limit.

Remark. For any given value of the parameters ε and δ, one obtains a finite result.

Remark. If the parameters are small, then they correspond to large logarithms. The more one would like to resolve the collimated gluons from the soft gluons, the bigger the logarithms are. Schematically, one has

$$\sigma_{2\mathrm{J}}^{\mathrm{SW}} \sim \sigma_{\mathrm{B}} (1 - \alpha_s L^2)$$

If the next-leading order contribution (the second addendum) is of order 1, then the results are not reliable. If there are large logarithms in a perturbative expansion at a given order, one should resum the logarithms [r]. At the higher orders, one has

$$1 - \alpha_s L^2 + \frac{1}{2!}(\alpha_s L^2)^2 + o(\alpha_s^2) \sim \mathrm{e}^{-\alpha_s L^2}$$

This expansion misses the non-logarithm terms.

All of this makes sense since the observables are infrared-safe. The jet physics at LHC deals with this aspect of QCD.

Remark. In general, an observable $O_{n+1}(k_1, \ldots, k_{n+1})$ is infrared-safe if, in the soft limit, one has [r]

$$k_j \to 0 \implies O_{n+1}(k_1, \ldots, k_{n+1}) \to O_n(k_1, \ldots, k_{j-1}, k_{j+1}, \ldots, k_{n+1})$$

For the collinear limit, one must have

$$\mathbf{k}_i \parallel \mathbf{k}_j \implies O_{n+1}(k_1, \ldots, k_{n+1}) \to O_n(k_1, \ldots, k_i + k_j, \ldots, k_{n+1})$$

Remark. The computations at LHC are similar to the following argument. [r] Consider the cross-section

$$\begin{aligned}
\sigma|_{\mathrm{F}} &= \int \mathrm{d}\phi_n \, B_n F(\phi_n) + \int \mathrm{d}\phi_n \, V_n F(\phi_n) + \int \mathrm{d}\phi_{n+1} R_{n+1} F(\phi_{n+1}) \\
&= \int \mathrm{d}\phi_n \, B_n F(\phi_n) + \int \mathrm{d}\phi_n \left[V_n F(\phi_n) + \int \mathrm{d}\phi_{\mathrm{rad}} \, C_{n+1} F(\phi_n) \right] \\
&\quad + \int \mathrm{d}\phi_{n+1} [R_{n+1} F(\phi_{n+1}) - C_{n+1} F(\phi_n)]
\end{aligned}$$

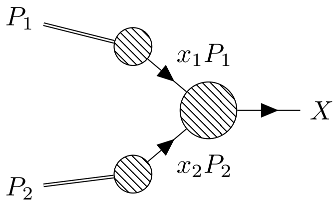
If F is a complicated function, the integral in the first equality has to be done numerically. The measures have different phase spaces and the integration would give two infinities that have to cancel. One exploits the factorization property and adds a local counter term C [r] (second equality). One has to be able to integrate the counter terms: they are simple functions (eikonal currents or Altarelli–Parisi functions) based on collinear factorization. In this way, the integration can be done analytically and one is able to cancel the quadratic and simple poles in the virtual contribution V_n.

6.6 Parton model

A scattering with hadrons in the initial state is more complicated. For hard scatterings the cross-section is

$$\sigma = \sum_{ij} \int_0^1 \mathrm{d}x_1 \, \mathrm{d}x_2 \, f_i^{H_1}(x_1) f_j^{H_2}(x_2) \hat{\sigma}_{ij}(x_1 P_1, x_2 P_2) \,, \quad Q^2 \gg \Lambda_{\mathrm{QCD}}^2$$

where $f_i^H(x)$ is the probability of having a parton i in the hadron H with a fraction x of the hadron's momentum, this is called partonic distribution function (PDF). One has to convolve the cross-section of the hard process with the probabilities f and one assumes that the transverse momentum of the parton inside the hadron is negligible. Although theoretically defined, the functions f have to be extracted from empirical data since they are involved in non-perturbative interactions [r]. They are universal functions and one can compute interesting properties about them. One is interested in the perturbative nature of the scattering process.

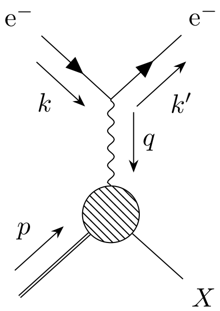


6.6.1 Deep inelastic scattering

To understand the structure of a hadron, one has to probe through deep inelastic scatterings [r]. For the proton, due to the de Broglie wavelength

$$\lambda = \frac{h}{p}$$

one needs energies of $p \sim 1\,\mathrm{GeV}$. Deep inelastic scatterings with a proton are of the type $e^- p \to e^- X$ where X is a generic final state. One is concerned only with the momentum of the electron. The electron is a charged particle and interacts through photons



Kinematics in the lab frame. One assumes the proton to be at rest, therefore

$$p^2 = m^2 \,, \quad q = k - k' \,, \quad Q^2 = -q^2 \geq 0$$

The word "deep" refers to the assumption $Q^2 \gg m^2$, while "inelastic" refers to $p_X^2 \gg m^2$ (in fact if $p_X^2 = m^2$, then the scattering is elastic).

There are two important variables

$$x = \frac{Q^2}{2pq} \,, \quad y = \frac{pq}{pk}$$

The first one is called Bjorken x. The range of both is $[0, 1]$. In the lab frame, these variables become

$$x = \frac{EE'(1 - \cos\theta)}{m(E - E')} \,, \quad y = 1 - \frac{E'}{E}$$

where one lets

$$p = (m, \mathbf{0}) \,, \quad k = (E, 0, 0, E) \,, \quad k' = (E', 0, E' \sin\theta, E' \cos\theta)$$

The two variables x and y are completely determined. The limit of the elastic scattering is $x \to 1$. In fact

$$p_X^2 = (p+q)^2 = m^2 + \frac{Q^2(1-x)}{x} \,, \quad xy = \frac{Q^2}{(s-m^2)}$$

Amplitude and cross-section. Consider the diagram above. The amplitude is

$$\mathrm{i}\mathcal{M} = \bar{u}(k')(\mathrm{i}e\gamma^\mu)u(k)\frac{-\mathrm{i}\eta_{\mu\nu}}{q^2} \langle X| eJ^\nu_{\mathrm{EM}} |p, s\rangle$$

The cross-section is

$$\mathrm{d}\sigma = \frac{1}{2s} \sum_X \overline{\sum_{\lambda\lambda' s}} |\mathcal{M}|^2 \frac{\mathrm{d}^3 k'}{(2\pi)^3 2E'} \mathrm{d}\Pi_X \, (2\pi)^4 \delta^4(p + k - k' - p_X)$$

where the phase space is

$$\mathrm{d}\Pi_X = \prod_{j\in X} \frac{\mathrm{d}^3 p_j}{(2\pi)^3 2E_j}$$

Let $L_\nu = \bar{u}\gamma_\nu v$, then

$$\mathrm{d}\sigma = \frac{1}{2s}\frac{1}{2}\frac{1}{2} \sum_{\lambda\lambda'} L^*_\mu L_\nu \sum_{X,s} \langle X| J^\mu |p, s\rangle^* \langle X| J^\nu |p, s\rangle \, \frac{e^4}{Q^4} \frac{\mathrm{d}^3 k'}{(2\pi)^3 2E'} \mathrm{d}\Pi_X \, (2\pi)^4 \delta^4(q + p - p_X)$$

The hadronic part of the formula is given by the sum over X and s, the measure $\mathrm{d}\Pi_X$, and the Dirac delta; while the leptonic part is encoded in the sum over λ and λ', and in the measure $\mathrm{d}k'$. Therefore, the leptonic tensor is

$$L_{\mu\nu} \equiv \frac{1}{2} \sum_{\lambda\lambda'} L^*_\mu L_\nu = 2[k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu}(kk')]$$

Due to gauge invariance, one has $q^\mu L_{\mu\nu} = q^\nu L_{\mu\nu} = 0$. The hadronic tensor is

$$W_{\mu\nu}(p, q) \equiv \frac{1}{2}\frac{1}{4\pi} \sum_{X,s} \langle p, s| J^\dagger_\mu |X\rangle\langle X| J_\nu |p, s\rangle \, (2\pi)^4 \delta^4(q + p - p_X) \, \mathrm{d}\Pi_X$$

This tensor is a function of only the momenta p and q since one is integrating over the final state. Since the electromagnetic current is conserved, then $q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$. The hadronic tensor parametrizes the details of the photon-proton interaction.

In this notation, the cross-section is then

$$\mathrm{d}\sigma = \frac{1}{2s}\frac{e^4}{Q^4}(4\pi) L_{\mu\nu} W^{\mu\nu} \frac{\mathrm{d}^3 k'}{(2\pi)^3 2E'} \qquad \text{(DIS HAD)}$$

Since the cross-section has dimension E^{-2} then the whole term beyond $2s$ must be dimensionless. From this, one obtains the mass dimensions

$$\dim L_{\mu\nu} = 2 \,, \quad \dim W_{\mu\nu} = 0$$

The hadronic tensor depends only on p and q, is dimensionless and is symmetric in its indices. It must be linear in $\eta^{\mu\nu}$ and $q^\mu q^\nu$, and must hold $q_\mu W^{\mu\nu} = 0$. Therefore, its most general form is

$$W^{\mu\nu}(p, q) = \left[-\eta^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right] W_1 + \frac{1}{pq}\left[p^\mu - \frac{pq}{q^2}q^\mu\right]\left[p^\nu - \frac{pq}{q^2}q^\nu\right] W_2$$

where $W_j(p, q)$ are dimensionless scalar functions.

One can express the phase space of the outgoing electron as

$$\frac{\mathrm{d}^3 k'}{(2\pi)^3 2E'} = \frac{1}{2(2\pi)^2} \frac{(s - m^2)y}{2} \,\mathrm{d}x\,\mathrm{d}y$$

In the assumption that $p^2 = m^2 \ll Q^2$, one has

$$W_{\mu\nu} L^{\mu\nu} = 2Q^2 \left[W_1(p, q) + \frac{W_2(p, q)}{2x} \frac{2(1-y)}{y^2} \right]$$

Using the above equation, one may express the cross-section in terms of x and y as

$$\mathrm{d}^2_{xy}\sigma = 4\pi\alpha^2_{\mathrm{EM}} \frac{s}{Q^4} [xy^2 W_1 + (1-y)W_2]$$

Since W_j parametrize the photon-proton interaction, they are called structure functions and may be denoted by F_j. Since these are scalar functions, they can only depend on q^2 and pq. Though, since there is unknown dynamic in the proton, the functions can also depend on a mass scale Λ^2 associated with the constituents. One expects it to be of the order

$$\Lambda^2 \sim \frac{1}{r^2}$$

where r is the size of the proton. Since the structure functions are dimensionless, they depend only on ratios

$$F_j({}^{pq}/_{Q^2}, {}^{\Lambda^2}/_{Q^2}) = F_j(x, {}^{\Lambda^2}/_{Q^2})$$

Experimental observations. In the Bjorken limit, $Q^2, pq \to \infty$ at fixed x, the structure functions are only functions of the energy fraction

$$F_j(x, {}^{\Lambda^2}/_{Q^2}) \to F_j(x)$$

There is no dependence on the momentum Q^2: this is scale invariance. See notes[4], p.7 ss.

In such limit, the Callan–Gross relation holds

$$F_2 \approx 2xF_1$$

When hitting a proton very hard, there is a scattering with the internal point-like constituents that behave as free particles. If the internal constituents were interacting, then there would be a scale dependence. A proton is a bound state and, when probed at high energies, it is like scattering on a point-like constituent as if it were free.

Lecture 20

6.6.2 Naive parton model

mer 22 mag 2024 12:30

The constituents of the proton are partons. In the parton model, one assumes that a parton carries a fraction of the momentum of the proton

$$\hat{p} = zp$$

The infinitesimal probability is

$$P(zp < \hat{p} < (z + \mathrm{d}z)p) = f(z)\,\mathrm{d}z$$

where $f(z)$ is the parton distribution function. The interaction between a proton and electron is an elastic scattering between a parton and the electron mediated by a photon exchange. This implies that the partons are charged. The hadronic cross-section is given by

$$\mathrm{d}\sigma = \sum_i \int_0^1 f_i(z)\,\mathrm{d}\hat{\sigma}(zp)\,\mathrm{d}z$$

where the sum is over the partons and $\mathrm{d}\hat{\sigma}$ is the partonic cross-section. This is a sum without interferences between amplitude. It is called incoherent sum. One should see if the structure functions F_j are predicted and if the Callan–Gross relation holds [r].

[4] QCD at ISR, leading order.

Remark. The sum is incoherent in the sense that one does not compute the amplitudes, sums them and then squares them, but one sums just their squares, there is no interferences between the amplitudes. [r] One may find a heuristic argument. The scale of the strong interaction is significantly below the electromagnetic interaction

$$m \ll Q$$

Due to the Heisenberg uncertainty principles, the interaction time-scales are

$$\tau_{\text{strong}} \gg \tau_{\text{EM}}$$

Since the electromagnetic scale is short, the other constituents of the hadron have no time to react (i.e. interact through the strong force) to the parton that has interacted through the electromagnetic force. The experimental evidence suggests that this works also for

$$m \lesssim Q$$

Remark. QCD is an asymptotically free theory: at higher energies, the coupling is weaker. This is the reason why the constituents can be considered free in a deep inelastic scattering. With a soft interaction this does not happen. (See notes p. 13)

Remark. The parton distribution function can be properly defined through QFT by using the expectation value of quark fields on hadronic states

$$\langle H| O |H\rangle \leftrightarrow f(z) \,, \quad O \sim \psi_{\text{q}}$$

This is a non-perturbative definition since hadrons are bound states.

The cross-section formula is factorized $f \,\mathrm{d}\hat{\sigma}$ and the factorization can be proven. The formula can be proven for all hadron collisions. For example the scattering proton-proton to positron-electron [r]?

$$\mathrm{d}\sigma = \sum_{ij} \int \mathrm{d}x_1 \,\mathrm{d}x_2 \, f_i(x_1)^{H_1} f_j(x_2)^{H_2} \,\mathrm{d}\hat{\sigma}(x_1 p_1 x_2 p_2)$$

For more complex final states, the proof is less formal.

Remark. The parton distribution functions are universal. [r] As long as one is interested in processes where the scale is bigger than Λ_{QCD} then the above heuristic argument applies. This implies that the distribution function does not depend on the details of the processes.

Proof of scale invariance and Callan–Gross relation. One would like to predict the property of structure functions. Consider the following scattering [r] diagr. where one has

$$\hat{p} = zp \,, \quad \hat{s} = (k + \hat{p})^2 = zs \,, \quad s = (k + p)^2$$

The parton cross-section is

$$\mathrm{d}\hat{\sigma} = \frac{1}{2\hat{s}} [\mathrm{d}k'][\mathrm{d}\hat{p}'] (2\pi)^4 \delta^{(4)}(\hat{p} + k - \hat{p}') \frac{e^4}{Q^4} L_{\mu\nu} \widetilde{W}_i^{\mu\nu}$$

where $\widetilde{W}$ is a tensor associated to the hadronic part of the process and

$$L_{\mu\nu} \propto [k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu}(kk')]$$

Assuming that partons are fermions, one has

$$\widetilde{W}_i^{\mu\nu} = \frac{1}{2} Q_i^2 \,\mathrm{Tr}\big[\hat{\not{p}} \gamma^\mu \hat{\not{p}}' \gamma^\nu\big]$$

where Q_i is the electromagnetic charge of the i-th parton. The constituents have to be spinor fields since a proton has spin $\frac{1}{2}$: it must have at least one fermionic component. The phase-space can be rewritten as

$$(2\pi)^4 [\mathrm{d}\hat{p}'] = (2\pi)^4 \frac{\mathrm{d}^4 \hat{p}'}{(2\pi)^3} \delta_+(\hat{p}'^2) = (2\pi) \mathrm{d}^4 \hat{p}' \,\frac{\delta(z - x)}{2pq}$$

where one has

$$[\mathrm{d}\hat{p}'] = \frac{\mathrm{d}^3\hat{p}'}{(2\pi)^3 2\hat{p}'_0} = \frac{\mathrm{d}^4\hat{p}'}{(2\pi)^4}\delta_+(\hat{p}^2)$$

[r] also

$$\delta_+(p^2) \leftrightarrow (p^2 = 0) + (\text{positive energy})$$

Recalling that

$$\hat{p}' = q + \hat{p} = q + zp$$

and notice that the Bjorken is

$$x = z$$

where x is measured, while z is the argument of the parton distribution function. In fact

$$0 = (\hat{p}')^2 = (q+\hat{p})^2 = q^2 + 2q\hat{p} = -Q^2 + 2\hat{p}q = -2\hat{p}qx + 2zpq \implies x = z$$

One has

$$\widetilde{W}_i^{\mu\nu} = Q_i^2 2[2z^2 p^\mu p^\nu + z(pq+qp)^{\mu\nu} - z\eta^{\mu\nu}pq]$$

also

$$\int [\mathrm{d}\hat{p}'](2\pi)^4\delta^{(4)}(\hat{p}-q-\hat{p}')\widetilde{W}^{\mu\nu} = (2\pi)z\left[\left(-\eta^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)\hat{W}_1 + \frac{1}{pq}\left(p^\mu - \frac{pq}{q^2}q^\mu\right)\left(p^\nu - \frac{pq}{q^2}q^\nu\right)\hat{W}_2\right]$$

where one has

$$\hat{W}_1 = Q_i^2\delta(z-x)\,, \quad \hat{W}_2 = Q_i^2 2z\delta(z-x)$$

Putting everything together yields

$$\begin{aligned}\mathrm{d}\sigma &= \sum_i \int_0^1 \mathrm{d}z\, f_i(z)\,\mathrm{d}\hat{\sigma}_i \\ &= \frac{2\pi}{s}\frac{e^4}{Q^4}[\mathrm{d}k']L_{\mu\nu}\left[\left(-\eta^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)\frac{1}{2}\sum_i Q_i^2 f_i(x) + \frac{1}{pq}\left(p^\mu - \frac{pq}{q^2}q^\mu\right)\left(p^\nu - \frac{pq}{q^2}q^\nu\right)\sum Q_i^2 x f_i\right]\end{aligned}$$

This cross-section must be equal to eq. DIS HAD with

$$F_1(x) = \frac{1}{2}\sum_i Q_i^2 f_i(x)\,, \quad F_2(x) = 2xF_1(x)$$

From this one predicts that F_i does not depend on the energy Q and the Callan–Gross relation also appears. The parton model enables one to predict the structure constant. If the constituents were scalar fields, then the above relation would not hold.

Remark. From the Callan–Gross relation, one has

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x\,\mathrm{d}y} \propto [xy^2F_1 + (1-y)F_2] = x[(1-y)^2+1]F_1(x)$$

By fitting one is able to extract $F_1(x)$ and therefore the sum of f_i. To find each distribution, one measures different processes to constrain many linear combinations of f_i. For example

$$|\gamma \to uu|^2 + |\gamma \to dd|^2 + \cdots \leftrightarrow F_1 \sim Q_u^2 f_u + Q_d^2 f_d + \cdots$$

Also

$$\left|W^- \to ud\right|^2 + \left|W^-\bar{d}\bar{u}\right|^2 \leftrightarrow F_1 \sim g_u f_u + g_{\bar{d}} f_{\bar{d}}$$

With enough configurations one may reconstruct the individual functions.

Sum rules. There are many relations of this distribution functions. [r] The functions also describe how the momenta of the constituents are related to the momentum of the proton

$$p = \langle p_u \rangle + \langle p_d \rangle + \langle p_s \rangle + \cdots$$

The expectation value for each is

$$\langle p_u \rangle = \int \mathrm{d}x \, x p f_u(x)$$

The total momentum is then

$$p = \sum_i \int \mathrm{d}x \, x p f_i(x) \implies \sum_i \int \mathrm{d}x \, x f_i(x) = 1$$

This is the momentum sum rule.

From deep inelastic scattering, using the naive parton model, one has

$$\sum_{\text{quarks}} \int \mathrm{d}x \, x f_i(x) \approx 0.5$$

The sum over the quarks is not enough, there must be something else: gluons.

6.6.3 Improved parton model

The naive parton model does not survive radiative corrections. Higher order corrections involve virtual and real corrections. The virtual corrections are propagator and vertex corrections. The real corrections are the real gluon emission. In naive parton model, one may not use the diagrams because the momentum entering the vertices are different [r]. If the gluon emission is hard and collinear, then the divergences do not cancel

$$\sigma_{\mathrm{R}} + \sigma_{\mathrm{V}} \sim \int_0^1 \mathrm{d}z \int_0 \frac{\mathrm{d}k_T^2}{k_T^2} \frac{1+z^2}{1-z} [\sigma_{\mathrm{B}}(z\hat{p}) - \sigma_{\mathrm{B}}(\hat{p})]$$

Renormalization absorbs the divergence into the parton distribution function. This implies that the parton distribution function acquire a dependence on the energy scale and as such run. The dependence on the scaling can be predicted.

Lecture 21

lun 27 mag 2024 12:30

Initial state radiation. One would like to compute the next-leading order of the process [r] diagr. One computes only the second diagram in the axial gauge [r]. The real correction is

$$\mathrm{R} = M(p-k) \frac{\mathrm{i}(\not{p} - \not{k})}{(p-k)^2} (-\mathrm{i} g_s t^a \gamma^\mu) u(p) \varepsilon_\mu(k) = M(p-k) \frac{\not{p} - \not{k}}{(p-k)^2} g_s t^a \gamma^\mu u(p) \varepsilon_\mu(k)$$

One would like to study the collinear case $\mathbf{p} \parallel \mathbf{k}$.

To study the kinematics one uses the Sudakov decomposition. The momentum of the gluon is decomposed on the basis

$$\{p^\mu, k_\perp^\mu, n^\mu\}$$

The four-vector n^μ is an auxiliary vector

$$n^2 = nk_\perp = pk_\perp = 0 \,, \quad np \neq 0$$

Assuming that $\mathbf{p}$ is along the z axis, then

$$p^\mu = (p^0, \mathbf{0}_\perp, p_0) \,, \quad n^\mu = (1, \mathbf{0}_\perp, -1) \,, \quad k^\mu = (k^0, \mathbf{k}_\perp, k_z) \,, \quad np = 2p^0$$

One decomposes the gluon's momentum as

$$k^\mu = (1-z) p^\mu + k_\perp^\mu + \xi n^\mu$$

where one has

$$k_\perp^\mu = (0, \mathbf{k}_\perp, 0)\,, \quad k_\perp^2 = -|\mathbf{k}_\perp|^2$$

From this, one has

$$k^\mu = ((1-z)p^0 + \xi, \mathbf{k}_\perp, (1-z)p^0 - \xi)$$

One may find the two parameters z and ξ. One has

$$0 = k^2 = [(1-z)p^\mu + k_\perp^\mu + \xi n^\mu]^2 = -|\mathbf{k}_\perp|^2 + 2(1-z)\xi np \implies \xi = \frac{|\mathbf{k}_\perp|^2}{2(1-z)np}$$

One is interested in the denominator of a propagator

$$(p-k)^2 = -2pk = -2\xi(pn) = -\frac{|\mathbf{k}_\perp|^2}{1-z}$$

In the collinear limit, $\mathbf{k}_\perp \to 0$, then $(p-k)^2 \to 0$ and the associated propagator diverges.

Phase space. The phase space is

$$[\mathrm{d}k] = \frac{\mathrm{d}^4 k}{(2\pi)^4}\delta_+(k^2)$$

where $\delta_+(k^2)$ means taking the positive energy. One has

$$\mathrm{d}k_0\,\mathrm{d}k_z = J\,\mathrm{d}z\,\mathrm{d}\xi = 2p^0\,\mathrm{d}z\,\mathrm{d}\xi = pn\,\mathrm{d}z\,\mathrm{d}\xi$$

Therefore

$$[\mathrm{d}k] = \frac{1}{(2\pi)^3}\,\mathrm{d}^2 k_\perp\,(pn)\,\mathrm{d}z\,\mathrm{d}\xi\,\delta(2pn(1-z)\xi - |\mathbf{k}_\perp|^2)$$

Integrating over $\mathrm{d}\xi$ gives

$$[\mathrm{d}k] = \frac{1}{(2\pi)^3}\frac{1}{2}\frac{\mathrm{d}z}{1-z}\,\mathrm{d}^2 k_\perp$$

Matrix element squared. One is interested only in the collinear limit, so other terms are ignored. One has

$$|M_1|^2 = g_s^2 C_F \sum_{\text{pol}} \frac{1}{(2pk)^2} M^\dagger(p-k)\,(\not{p}-\not{k})\not{\varepsilon}\not{p}\not{\varepsilon}^*(\not{p}-\not{k})M(p-k)$$

The middle momentum $\not{p}$ comes from $u\bar{u}$. The sum on polarization is done only over physical polarizations

$$\begin{aligned}\sum_{\text{pol}} \varepsilon_\mu\varepsilon_\nu^*\gamma^\mu\not{p}\gamma^\nu &= \left[-\eta_{\mu\nu} + \frac{(nk+kn)_{\mu\nu}}{nk}\right]\gamma^\mu\not{p}\gamma^\nu = -\gamma^\mu\not{p}\gamma_\mu + \frac{1}{nk}(\not{n}\not{p}\not{k} + \not{k}\not{p}\not{n}) \\ &= 2\not{p} + \frac{2}{nk}[np\not{k} - nk\not{p} + pk\not{n}] = \frac{2}{nk}[np\not{k} + pk\not{n}] \\ &= \frac{2}{1-z}[\not{k} + \xi\not{n}]\end{aligned}$$

At the first line, first equality, one substitutes the sum over physical polarization using n^μ as the auxiliary vector. At the second line, one has used

$$\not{A}\not{B}\not{C} + \not{C}\not{B}\not{A} = 2AB\not{C} + 2BC\not{A} - 2AC\not{B}$$

At the third line, one has used

$$nk = pn(1-z)\,, \quad pk = pn\xi$$

The numerator of the amplitude is then

$$\text{Num of } |M_1|^2 = (\not{p}-\not{k})[\not{k}+\xi\not{n}](\not{p}-\not{k}) = 2(pk)\not{p} + \xi(\not{p}-\not{k})\not{n}(\not{p}-\not{k})$$

One is interested only in the divergent part. Knowing that

$$p - k = zp + O(\xi, k_\perp)$$

the numerator becomes

$$\mathrm{Num} = (2pk)\not{p} + z^2\xi\not{p}\not{n}\not{p} + O(|\mathbf{k}_\perp|^3) \sim 2(pk)\not{p} + z^2\xi 2(pn)\not{p} = 2(pk)(1+z^2)\not{p}$$

At the second equality one applies?

$$\not{p}\not{n} = -\not{n}\not{p} + 2(pn)$$

[r] At the third quality one applies

$$pn = \frac{pk}{\xi}$$

Putting everything together, one has

$$\begin{aligned} |M_1|^2 &= g_s^2 \frac{C_F}{(2pk)^2} \frac{2}{1-z} [2(pk)(1+z^2)] M^\dagger(p-k) \not{p} M(p-k) \\ &= g_s^2 C_F \frac{1-z}{|\mathbf{k}_\perp|^2} 2 \left[\frac{1+z^2}{1-z}\right] M^\dagger(p-k) \not{p} M(p-k) \end{aligned}$$

At the second line one has applied.

$$2pk = \frac{|\mathbf{k}_\perp|^2}{1-z}$$

Notice that the bracket is a functional dependence on the Altarelli–Parisi splitting function.

[r] The partonic cross-section is

$$\hat{\sigma}^{(0)}(p) = \mathcal{F}(p) M^\dagger(p) \not{p} M(p)$$

The differential cross-section of the real emission of a collinear gluon is

$$\mathrm{d}\hat{\sigma}_{\mathrm{R}}^{(1)}(p) \sim \mathcal{F}(p) \left[2g_s^2 C_F \frac{1+z^2}{|\mathbf{k}_\perp|^2}\right] \left[\frac{1}{(2\pi)^3} \frac{\mathrm{d}z}{2(1-z)} \,\mathrm{d}^2 k_\perp\right] M^\dagger(p-k) \not{p} M(p-k)$$

The amplitude M are [r]. [r] The Born cross-section is

$$\begin{aligned} \hat{\sigma}^{(0)}(zp) &= \hat{\sigma}^{(0)}(p - k + O(k_\perp)) = \mathcal{F}(zp) M^\dagger(p-k)(z\not{p}) M(p-k) \\ &= \mathcal{F}(p) M^\dagger(p-k) \not{p} M(p-k) + O(k_\perp) \end{aligned}$$

Where one uses

$$zp = (p-k) + k_\perp^\mu + \xi n^\mu = p - k + O(k_\perp)\,, \quad \mathcal{F}(zp) = \frac{1}{z}\mathcal{F}(p)$$

The real emission cross-section is

$$\mathrm{d}\hat{\sigma}^{(1)}(p) \sim g_s^2 C_F \frac{1}{(2\pi)^3} \frac{1+z^2}{1-z} \frac{1}{|\mathbf{k}_\perp|^2} \,\mathrm{d}z \,\mathrm{d}^2 k_\perp \,\hat{\sigma}^{(0)}(zp)$$

Integrating in

$$\mathrm{d}^2 k_\perp = \frac{1}{2} \,\mathrm{d}(|\mathbf{k}_\perp|^2) \,\mathrm{d}\varphi$$

one finds

$$\boxed{\mathrm{d}\hat{\sigma}^{(1)}(p) \sim \frac{\alpha_s}{2\pi} C_F \frac{1+z^2}{1-z} \frac{1}{|\mathbf{k}_\perp|^2} \,\mathrm{d}z \,\mathrm{d}|\mathbf{k}_\perp|^2 \,\hat{\sigma}^{(0)}(zp)}$$

The cross-section factorizes but a term is $\hat{\sigma}^{(0)}(zp)$. It contains a collinear divergence $\mathbf{k}_\perp \to 0$

$$\frac{\mathrm{d}|\mathbf{k}_\perp|^2}{|\mathbf{k}_\perp|^2}$$

and a soft divergence (and collinear) $z \to 1$

$$\frac{\mathrm{d}z}{1-z}$$

Remark. The limit $z \to 1$ is soft and collinear. There is a double logarithmic divergences. This is the soft limit because

$$k = (1-z)p + O(k_\perp)$$

The collinear limit is $z \neq 1$ and $|\mathbf{k}| \to 0$. This is a collinear, but hard gluon.

Remark. For final state radiation, one has

$$\mathrm{d}\sigma_\mathrm{R} = \frac{\alpha_s}{2\pi} C_F \frac{1+z^2}{1-z}\,\mathrm{d}z\,\frac{\mathrm{d}p^2}{p^2}\hat{\sigma}^{(0)}(p)$$

where p is the parent emitted quark. One may show that

$$\frac{\mathrm{d}p^2}{p^2} = \frac{\mathrm{d}|\mathbf{k}_\perp|^2}{|\mathbf{k}_\perp|^2}$$

Initial and final state radiation cross-sections only differ by

$$\hat{\sigma}^{(0)}(p)\,, \quad \hat{\sigma}^{(0)}(zp)$$

Conclusion. To complete the computation of divergences, one has to include virtual divergences [r] diagr. [r] The soft divergences cancel: with a soft gluon, the initial state is the same in the virtual and real corrections, there is a degeneracy and the KLN theorem is satisfied.

For final state collinear kinematics, the real emission matrix element factorizes into $\hat{\sigma}^{(0)}(p)$. [r] The divergence cancels.

For initial state radiation, the collinear divergences do not cancel, because the initial state is not the same, not degenerate since the factorization involves

$$\mathrm{d}\sigma_\mathrm{R} \sim \hat{\sigma}^{(0)}(zp)$$

Only in the soft limit there is a degeneracy.

For all this to be consistent, the sum of real and virtual corrections in the collinear limit of the initial state radiation is

$$\boxed{\hat{\sigma}^{(1)}_{\mathrm{R+V}} \sim \frac{\alpha}{2\pi}\int \mathrm{d}z\,\left[\hat{\sigma}^{(0)}(zp) - \hat{\sigma}^{(0)}(p)\right] C_F \frac{1+z^2}{1-z}\,\frac{\mathrm{d}|\mathbf{k}_\perp|^2}{|\mathbf{k}_\perp|^2}}$$

In the limit $z \to 1$ the above (typically divergent) expression is zero. This satisfies the KLN theorem. For $z \neq 1$, then there is a logarithmic divergence

$$\int \frac{\mathrm{d}|\mathbf{k}_\perp|^2}{|\mathbf{k}_\perp|^2}$$

In dimensional regularization, the result has an exposed pole that does not cancel

$$\hat{\sigma}^{(1)}_{\mathrm{R+V}} \sim \frac{\alpha_s}{2\pi} C_F \frac{1+z^2}{1-z}\,\mathrm{d}z\,\frac{1}{\varepsilon}$$

The collinear divergence is absorbed in a different quantity.

Regularization of the splitting function. One introduces a plus-distribution [r] defined by

$$\int_0^1 \mathrm{d}x\,[g(x)]_+ f(x) \equiv \int_0^1 \mathrm{d}x\,g(x)[f(x) - f(1)]$$

For an infinitesimal δ, one has

$$\int_0^{1-\delta} \mathrm{d}x\,[g(x)]_+ f(x) \equiv \int_0^{1-\delta} \mathrm{d}x\,g(x) f(x)$$

Therefore, one may write

$$\hat{\sigma}^{(1)}_{\mathrm{R+V}} = \frac{\alpha_s}{2\pi} \int_0^1 \mathrm{d}z\, C_F \left[\frac{1+z^2}{1-z}\right]_+ \hat{\sigma}^{(0)}(zp) \int_0 \frac{\mathrm{d}|\mathbf{k}_\perp|^2}{|\mathbf{k}_\perp|^2}$$

The plus distribution is used to make the cross-section finite in the limit $z \to 1$ of the splitting function [r]. The upper bound on the momentum is of the order Q^2 of the typical hard scale of the process.

The term

$$C_F \left[\frac{1+z^2}{1-z}\right]_+ \equiv P^{(0)}_{qq}(z)$$

is the regularized Altarelli–Parisi splitting function. The superscript denotes the first order in α_s, while the subscript indicates $q \to qg$.

The knowledge of the splitting functions enables one to resum towers of log divergences.

Notice that there are configurations exhibiting collinear limits. For example $g \to q\bar{q}$ and $g \to gg$. Each one with their own splitting functions. In a physical process, all the functions contribute.

Lecture 22

mar 28 mag 2024 12:30

Improved parton model. The improved parton model is used to evaluate the integral

$$\int_0^{Q^2} \frac{\mathrm{d}|\mathbf{k}_\perp|^2}{|\mathbf{k}_\perp|^2}$$

with a cutoff μ_0^2

$$\int_{\mu_0^2}^{Q^2} \frac{\mathrm{d}|\mathbf{k}_\perp|^2}{|\mathbf{k}_\perp|^2} \sim \ln\frac{Q^2}{\mu_0^2} = L \gg 1 \iff \mu_0 \ll Q$$

Since in the cross-section, one has

$$\hat{\sigma}^{(1)}_{\mathrm{R+V}} \sim \alpha_s L$$

This means that the perturbative expansion may not be applied. So one may have to find a way to resum similar logarithms. One is able to sum towers of collinear logarithms of emission of gluons that are ever more singular

$$O \sim q_1^2 \gg q_2^2 \gg \cdots \gg q_n^2$$

The smaller the value, the higher is the virtuality and the singularity. One expects to have something like

$$(\alpha_s L)^n$$

which does not work when counting in powers of the coupling constant.

Heuristically, a cutoff is imposed on the virtuality of the intermediate quark [r] diagr. Therefore

$$(p-k)^2 \sim -|k_\perp|^2$$

An object with a small virtuality can be interpreted as a particle that has travelled a lot. This means that the gluon has been emitted way before the interaction with the off-shell photon happens. The dynamics at small values of μ_0 can be incorporated into the proton. The divergence is then reabsorbed into the partonic distribution function.

6.6.4 Renormalization of partonic distribution function

The Born cross-section is

$$\hat{\sigma}^{(0)}(\hat{p}) = \int \mathrm{d}z\, \delta(1-z)\hat{\sigma}^{(0)}(z\hat{p})$$

where $\hat{p}$ is a fraction of the momentum p of the proton. By adding real and virtual corretions gives

$$\hat{\sigma}^{(1)}_{\mathrm{R+V}} = \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu_0^2} \int \mathrm{d}z \, P^{(0)}_{qq}(z) \hat{\sigma}^{(0)}(z\hat{p})$$

in the assumption of being close to the collinear singularity. The full next-leading order result is

$$\begin{aligned}\hat{\sigma}(\hat{p}) &= \mathrm{Born} + \mathrm{Real} + \mathrm{Virtual} = \hat{\sigma}^{(0)}(\hat{p}) + \hat{\sigma}^{(1)}_{\mathrm{R+V}}(\hat{p}) \\ &= \int \mathrm{d}z \left[\delta(1-z) + \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu_0^2} P^{(0)}_{qq}(z)\right] \hat{\sigma}^{(0)}(z\hat{p})\end{aligned}$$

Let the bracket be

$$\Gamma_{qq}(z, Q^2) \equiv \delta(1-z) + \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu_0^2} P^{(0)}_{qq}(z)$$

The hadronic result is

$$\begin{aligned}\sigma(p) &= \int \mathrm{d}y \, f_q(y) \hat{\sigma}(yp) = \int \mathrm{d}y \, \mathrm{d}z \, f_q(y) \Gamma_{qq}(z, Q^2) \hat{\sigma}^{(0)}(zyp) \\ &= \int \mathrm{d}x \, \delta(x - zy) \, \mathrm{d}y \, \mathrm{d}z \, f(y) \Gamma_{qq}(z, Q^2) \hat{\sigma}^{(0)}(xp) \\ &= \int \mathrm{d}x \, \widetilde{f}_q(x, Q^2) \hat{\sigma}^{(0)}(xp)\end{aligned}$$

At the first line one has substituted $\hat{\sigma}^{(0)}(yp)$. One has

$$\widetilde{f} \equiv \int \mathrm{d}y \, \mathrm{d}z \, f(y) \Gamma_{qq}(z, Q^2) \delta(x - zy)$$

One is putting together the partonic distribution function with a divergent term. Therefore

$$\widetilde{f}(x, Q^2) = f_q(x) + \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu_0^2} \int_x^1 \frac{\mathrm{d}z}{z} \, P^{(0)}_{qq}(z) f_q(x/z)$$

One has used the facts that

$$\mathrm{d}y \, \delta(x - zy) = \frac{\delta(y - x/z)}{z}$$

which implies

$$0 < y < 1 \implies \frac{x}{z} < 1 \implies z > x$$

The function $\widetilde{f}$ contains the details of the hard process Q^2, but is not therefore universal. One introduces the factorization scale by splitting the logarithm

$$\ln \frac{Q^2}{\mu_0^2} = \ln \frac{Q^2}{\mu_F^2} + \ln \frac{\mu_F^2}{\mu_0^2}$$

The first term goes into the partonic cross-section, the second is reabsorbed into a renormalized partonic distribution function. Eventually one finds that the partonic distribution function f becomes a function also of the factorization scale.

The hadronic cross-section is

$$\boxed{\sigma(p) = \int \mathrm{d}x \, \widetilde{f}(x, \mu_F^2) \hat{\sigma}(xp, \mu_F^2)}$$

where

$$\hat{\sigma}(\hat{p}, \mu_F^2) = \hat{\sigma}^{(0)}(\hat{p}) + \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu_F^2} \int \mathrm{d}z \, P^{(0)}_{qq}(z) \hat{\sigma}^{(0)}(z\hat{p})$$

The function $\widetilde{f}(x, \mu_F^2)$ has no dependence on the hard scale, so it is universal. The cross-section $\hat{\sigma}$ has no divergences, but there is a dependence on the μ_F.

To prove this, one must check an equality between the two following expressions

$$\sigma(A) = \int \mathrm{d}x \left[f(x) + \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu_0^2} \int_0^1 \frac{\mathrm{d}z}{z} P(z) f(x/z) \right] \hat{\sigma}^{(0)}(xp)$$

$$\sigma(B) = \int \mathrm{d}x \left[f(x) + \frac{\alpha}{2\pi} \ln \frac{\mu_F^2}{\mu_0^2} \int_x^1 \frac{\mathrm{d}z}{z} P(z) f(x/z) \right] \left[\hat{\sigma}^{(0)}(xp) + \frac{\alpha}{2\pi} \ln \frac{Q^2}{\mu_F^2} \int \mathrm{d}z \, P(z) \hat{\sigma}(zxp) \right]$$

The first bracket on the second line is finite and is the renormalized partonic distribution function. The second bracket depends on the process and contains μ_F only through a logarithm.

Consider the second expression. One obtains

$$\begin{aligned}
\sigma(B) &= \int \mathrm{d}x \left[f(x) + \frac{\alpha}{2\pi} \ln \frac{\mu_F^2}{\mu_0^2} \int_x^1 \frac{\mathrm{d}z}{z} P(z) f(x/z) \hat{\sigma}(xp) \right] \\
&\quad + \int \mathrm{d}x \, f(x) \left[\frac{\alpha}{2\pi} \ln \frac{Q^2}{\mu_F^2} \int_0^1 \mathrm{d}z \, P(z) \, \hat{\sigma}(xzp) \right] + o(\alpha_s) \\
&= \cdots \\
&= \int \mathrm{d}x \left[f(x) + \frac{\alpha}{2\pi} \ln \frac{\mu_F^2}{\mu_0^2} \int_x^1 \frac{\mathrm{d}z}{z} P(z) f(x/z) \hat{\sigma}(xp) \right] \\
&\quad + \int \mathrm{d}x \int_x^1 \frac{\mathrm{d}z}{z} \frac{\alpha}{2\pi} \ln \frac{Q^2}{\mu_F^2} P(z) f(x/z) \hat{\sigma}(xp) + o(\alpha_s)
\end{aligned}$$

One introduces a delta function and matches the integration limits. This proves the equality.

The interpretation is the following. The function $f(x)$ of the naive parton model is divergent. The function

$$\boxed{\widetilde{f}_q(x, \mu_F^2) = f_q(x) + \frac{\alpha_s}{2\pi} \ln \frac{\mu_F^2}{\mu_0^2} \int_x^1 \frac{\mathrm{d}z}{z} P_{qq}^{(0)}(z) f_q(x/z)} + o(\alpha_s)$$

is finite despite its two addenda diverging. At this point, one may send $\mu_0 \to 0$ and $\widetilde{f}$ is finite. This function $\widetilde{f}$ contains all the physics of collinear divergences below μ_F. [r] The dependence on μ_F must be physical.

By taking the logarithmic derivative in μ_F^2 one obtains the DGLAP equation

$$\mu_F^2 \, \partial_{\mu_F^2} \widetilde{f}(x, \mu_F^2) = \frac{\alpha_s}{2\pi} \int_x^1 \frac{\mathrm{d}z}{z} P_{qq}^{(0)}(z) \widetilde{f}(x/z, \mu_F^2) + o(\alpha_s)$$

The initial condition of $\widetilde{f}$ must be extracted from experiments. This quantity is needed for computations with hadrons.

Remark. The function $\widetilde{f}(\mu_F)$ is not a function of the hard scattering and Q, but contain information about the structure of the hadron and the scale at which the hadron is probed.

Remark. If one measures $\widetilde{f}(x, Q^2_{\mathrm{exp}})$ at a particular experimental value, then the dependence of the function is known perturbatively.

Remark. The function $\widetilde{f}$ enters in every computation involving hadrons

$$\sigma(p) = \int \mathrm{d}x \, \widetilde{f}(x, \mu_F^2) \hat{\sigma}(xp, \mu_F^2)$$

Remark. The cross-section $\hat{\sigma}$ contains explicit logarithms in Q^2. This means that to avoid large logarithms, then one has to choose $\mu_F \sim Q$ and the large logarithms are contained within the partonic distribution function $\widetilde{f}$. One resums towers of logarithms in $\widetilde{f}(x, \mu_F = Q)$.

Remark. The partonic distribution function can be defined in quantum field theory. See Collins lectures and Schwartz §§ 32.4, 32.5.

Remark. The hadron-hadron collision has a cross-section

$$\sigma(p_1, p_2) = \sum_{ij} \widetilde{f}_i(x_1, \mu_F^2) \widetilde{f}_j(x_2, \mu_F^2) \hat{\sigma}(x_1 p_1, x_2 p_2, \mu_F^2) + O\left(\frac{\Lambda_{\mathrm{QCD}}}{Q} \right)^k$$

The result holds for energy scales higher than the non-perturbative scale.

6.6.5 Resummation of collinear logarithms

Consider

$$t = \ln \mu_F^2$$

up to a proper dimensional parameter. One has

$$\mathrm{d}f(x,t') = \frac{\alpha}{2\pi}\int_x^1 \frac{\mathrm{d}z}{z} P(z) f(x/z, t')\,\mathrm{d}t'$$

Integrating from t_0 to t, one finds

$$f(x,t) - f(x,t_0) = \frac{\alpha}{2\pi}\int_{t_0}^t \mathrm{d}t' \int_x^1 \frac{\mathrm{d}z}{z} P(z) f(x/z, t')$$

One may iterate the solution

$$f(x/z, t') = f(x/z, t_0) + \frac{\alpha}{2\pi}\int_{t_0}^{t'} \mathrm{d}t'' \int_{x/z}^1 \frac{\mathrm{d}z'}{z'} P(z') f(x/zz', t'')$$

When inserting the expression [r]. Therefore

$$f(x,t) = f(x,t_0) + \frac{\alpha}{2\pi}\ln\frac{Q^2}{Q_0^2}\int_x^1 \frac{\mathrm{d}z}{z} P(z) f(x/z, t_0) + \left(\frac{\alpha}{2\pi}\right)^2 \frac{1}{2}\ln^2\frac{Q^2}{Q_0^2}\cdots + \cdots$$

This is the solution of the DGLAP equation written iteratively. If one computes the function f at a large scale Q, then the above is a resummation of the logarithms

$$L = \ln\frac{Q^2}{Q_0^2} \implies (\alpha L)^n$$

Physically [r].

The sub-leading terms are

$$P_{qq} = P_{qq}^{(0)} + P_{qq}^{(1)} + \cdots \sim \alpha_s c_F \left(\frac{1+z^2}{1+z}\right)_+ + \alpha_s^2 + \cdots$$

One is summing two towers of logarithms

$$(\alpha L)^n\,, \quad \alpha^n L^{n-1}$$

6.6.6 DGLAP – general discussion

There are four [r] branching. For qq one has [r] diagr

$$P_{qq}^{(0)}(z) = c_F\left(\frac{1+z^2}{1-z}\right)_+$$

For gq one has

$$P_{gq}^{(0)}(z) = c_F\frac{1+(1-z)^2}{z}$$

For gg one has

$$P_{gg}^{(0)}(z) \propto c_A$$

For qg one has

$$P_{qg}^{(0)}(z) = T_F[z^2 + (1-z)^2]$$

The notation is such that P_{ij} means $j \to i$. The parton i has energy fraction z.

The first and third case are singular in the soft limit $z \to 1$. This is because one emitted object is a gluon. Therefore the singular is collinear and soft. Though, one may notice the presence of a plus-distribution which denotes the cancellation between real and virtual corrections [r].

The DGLAP equation can be written compactly as

$$\mu_F^2 \partial_{\mu_F^2} f_q(x) = \frac{\alpha}{2\pi}[P_{qq} * f_q](x)$$

where $*$ denotes the convolution. This can be interpreted as following. One integrates the energy fraction z? [r] along the emission trajectory noting that earlier times have higher momentum than later times. The contribution to a quark partaking into a hard process is made of a quark emitting a gluon and then interacting, and a gluon producing a quark–anti-quark pair.

[r] Not constant F_2, violation of scaling at $x \to 0$.

This is the basis to understand more complicated subjects.

Scaling violation. The scaling violation is predicted.

Features of PDFs. From the graphs of the PDFs one may find various features. The valence quarks peak at $\frac{1}{3}$. For the proton, the up distribution is twice as high as a down distribution. Part of this is from data and from first principles only.