# Quantum Gravity

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## Contents



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## Lecture 1

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**Exam.** Presentation on topic not seen during class, even a follow up.
Lecture notes by Tom Hartman.

**Program.** First week is general difficulties with quantum gravity, overview and approaches, some problems associated with quantum gravity. The following three weeks are black hole thermodynamics, black hole information paradox. The next three weeks are introduction to CFT. The following week is an instruction to AdS. The last three weeks are an introduction to AdS/CFT.

# 1 Introduction

**Classical gravity.** The action of general relativity is the Einstein–Hilbert action

$$S = \frac{1}{16\pi G_N}\int \mathrm{d}^D x\,\sqrt{-g}[R-2\Lambda] + S_{\text{matter}}$$

The equations of motion are Einstein's field equations

$$G_{\mu\nu} = 8\pi G_N T^{\text{matter}}_{\mu\nu}$$

Classical gravity is understood well. Solutions to these equations include black holes.

**Quantum field theory.** Quantum dynamics without gravity is quantum field theory. It describes the matter sector of the EH Lagrangian. It describes the three other fundamental forces: electromagnetic, weak and strong forces. They are unified in the Standard Model which has worked very well. Though, quantum chromodynamics is a strongly couple theory and perturbation theory is limited.

**Quantum gravity.** Quantum gravity is supposed to unify general relativity with the Standard Model by quantizing the gravitational field in a consistent manner.

However, there are some difficulties. The naive approach is taking GR action, take a metric, pick a background and expand it around such background

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{16\pi G_N}h_{\mu\nu}\,, \quad \Lambda = 0$$

As such one gets

$$S = \int \mathrm{d}^D x\,\sqrt{-g}\left[\frac{1}{2}[\partial_\mu h^{\mu\nu}\,\partial_\nu h - \partial_\mu h^{\rho\sigma}\,\partial_\rho h^\mu{}_\sigma + \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h^{\rho\sigma})(\partial_\nu h_{\rho\sigma}) - \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h)(\partial_\nu h) + \sum_{n\geq 3} G_N^{\frac{n}{2}} P_n(\partial_\mu, h_{\mu\nu})\right]$$

where $h = \operatorname{Tr} h_{\mu\nu}$. The first three terms is the kinetic term of the graviton, a spin-2 particle. There are also an infinite number of terms (given by the sum) which can be thought of as interactions. [r] Now one can think about this theory as a QFT for the field $h$. The theory has one coupling constant $G_N$ with length dimensions $D-2$. Every time one adds the coupling, one has to add momentum to have a dimensionless quantity $G_N p^{D-2}$ so at higher order the theory diverges more severely. This theory is not renormalizable, more precisely it is not power-counting renormalizable.

This means that GR should be seen as an effective field theory (EFT), because this problems involves higher energies.

The mass scale relevant is the Planck's mass

$$S = \frac{1}{16\pi G_N}\int \mathrm{d}^4x\,\sqrt{-g}R = \frac{1}{16\pi\hbar}\int \mathrm{d}^4x\,\sqrt{-g}M_P^2 R\,, \quad M_P = \sqrt{\frac{\hbar}{G_N}} \approx 1.22\times 10^{19}\,\mathrm{GeV}$$

So the action looks like the following

$$S = \frac{1}{16\pi\hbar}\int \mathrm{d}^4x\,\sqrt{-g}[M_P^2 R + R^2 + M_p^{-2}R^3 + \cdots)$$

The last terms give contributions in the UV energy spectrum, while in IR then the terms are just tiny corrections. The Planck scale is also the scale at which the Schwartzschild radius is the Compton wavelength.

**Non-renormalizability.** To solve the problem of renormalizability:

- the naive power-counting fails and there are infinite cancellations in the Feynman diagrams (for GR it has been computed at two-loops and the cancellations do not happen, same happens in maximal supergravity but higher orders);
- there is a fixed point in the RG flow but cannot be understood through power-counting: the perturbative QFT is naive and there is a non-perturbative fixed point;
- pure quantum GR is not a consistent theory, so one must add other degrees of freedom besides the graviton (string theory works in this way).

The above is also dependent on the background of the expansion, but even for the simplest, Minkowski, there are problems.

One would like to understand the path integral of quantum gravity. The partition function is

$$Z[\mathrm{b.c.}] = \int_{\partial g = \mathrm{b.c.}} [\mathcal{D}g]\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}S_{\mathrm{EH}}}$$

where b.c. stands for boundary condition. [r] one fixes the metric infinitely far away. For AdS the integral is understood non perturbatively.

**Different approaches to quantum gravity.** Different approaches to quantum gravity are

- string theory; originally string theory was developed for strong interaction;
- holography and AdS/CFT; it originated from string theory and will be developed from here on;
- loop quantum gravity; space-time is discrete and there are critical length scales, the degrees of freedom are loops like in gauge theory, one formulates a [r];
- asymptotic safety; there is a fixed point of GR in the UV, but the power-counting argument fails bc it is perturbative, but the fixed point is not perturbative;
- causal dynamical triangulation; it is a lattice approach to QG but it needs to rely on asymptotic safety.

## 1.1 Black hole entropy

The best hint towards quantum gravity is without doubt the Bekenstein–Hawking formula for black hole entropy

$$S_{\mathrm{BH}} = \frac{1}{4}\frac{k_{\mathrm{B}}c^3}{\hbar G_N}A_{\mathrm{H}}$$

Natural units are used in the following $c = \hbar = k_B = G_N = 1$. A quarter of the area of a black hole's event horizon $A_{\mathrm{H}}$ is the entropy. In this formula, there appears Boltzmann constant for statistical mechanics, the speed of light for special relativity, the universal gravitation constant for gravity and Planck's constant for quantum physics.

The formula first came from a thought experiment of Bekenstein. Entropy always increases thanks to the second law of thermodynamics. Black holes must have entropy because the information of falling objects must be kept. Hawking proved that the area of a black hole always increases

$$\Delta A \geq 0$$

So the entropy of the black hole should be proportional to the area. Hawking found the proportionality constant to be $\frac{1}{4}$.

In thermodynamics, entropy is extensive so it scales with volume, but in this case it scales with the area. The property of being extensive is related to degrees of freedoms of local systems so it is an important property. The concept of holography is that the degrees of freedom of a local quantum gravity are located on the boundary of the black hole, not its bulk.

Black holes have a lot of entropy. In statistical mechanics, entropy counts the fundamental different degrees of freedoms

$$S = \log \rho_{\text{micro}}$$

For example, Sagittarius A* the entropy is

$$S_{\text{BH}} = 2.69 \times 10^{67}\,\text{J}\,\text{K}^{-1}$$

which is about $10^{20}$ the entropy of the sun.

A success of string theory is that for supersymmetric five-dimensional black holes, a count of individual degrees of freedom coincides with the formula above.

## 1.2 Open problems

**The hierarchy problem.** The mass of the Higgs boson $m_{\text{H}} \approx 125\,\text{GeV}$, but the Standard Model predicts infinity (for the bare value of the mass). The mass is computed from the propagator and there are loop diagrams like the self-interaction that give

$$\int \mathrm{d}^4k\, \frac{1}{k^2 - m^2} \sim k_{\max}^2$$

The mass correction to the mass is $\delta m_{\text{H}}^2 \sim \# k_{\max}^2$. However $\delta m_{\text{H}} \gg m_{\text{H}}$. The cancellation of the bare mass and true mass happens to many digits. This problem is related to fine tuning and naturalness problem.

This is related to quantum gravity because there may be not new physics up to the quantum gravity energy scale and QG has to solve such hierarchy problem. If $k_{\max} \sim M_p$ then one has a fine-tuning of $10^{17}$ digits.

There are many resolutions within quantum field theory like supersymmetry.

**The cosmological constant problem.** In quantum field theory the energy of the vacuum is not very much important, but in general relativity energy bends space-time. The energy density of the vacuum is

$$\begin{aligned}\rho_{\text{vac}} = \langle 0|\,\rho\,|0\rangle = \langle 0|\,T_{00}\,|0\rangle &= \frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\sqrt{k^2+m^2} \\ &= \frac{1}{4\pi^2}\int^{k_{\max}} \mathrm{d}k\, k^2\sqrt{k^2+m^2} = \frac{k_{\max}^4}{16\pi^2}\left[1 + \frac{m^2}{k_{\max}} + \cdots\right]\end{aligned}$$

The observed energy density is

$$\Lambda_{\text{eff}} = \Lambda_{\text{bare}} + 8\pi G_N \rho_{\text{vac}}\,, \quad \rho_{\text{obs}} = \frac{\Lambda_{\text{eff}}}{8\pi G_N} \approx 10^{-47}\,\text{GeV}^4$$

At the Planck's scale one has

$$\rho_{\text{vac}} \sim 10^{71}\,\text{GeV}^4$$

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This argument is missing a part. The pressure density is

$$\langle 0|\, p_{\text{vac}}\, |0\rangle = \langle 0|\, T_{0i}\, |0\rangle \implies \frac{\rho_{\text{vac}}}{p_{\text{vac}}} = \frac{1}{3} \neq -1$$

The energy density is not $-1$ so it cannot be interpreted as the cosmological constant. The UV cutoff regularization breaks Lorentz invariance.

**Proper argument.** The energy density of the vacuum is proportional to the vacuum bubble diagram. To preserve Lorentz invariance one may use dimensional regularization:

$$\rho_{\text{vac}} = \frac{m^4}{64\pi^2} \ln \frac{m^2}{4\pi\mu^2}$$

The quartic divergence appears in the mass, while the divergence in the momentum $\mu$ is only logarithmic. In quantum field theory one does not care about the vacuum energy because all disconnected diagrams are cancelled. Computing in this way the pressure, the ratio is $-1$. So observed energy density is

$$\rho_{\text{obs}} = \rho_{\text{bare}} + \rho_{\text{vac}}$$

for the tau lepton mass one has $\rho_{\text{obs}} \sim 10^{-47}\,\text{GeV}^4$ and $\rho_{\text{vac}} \sim 10^8\,\text{GeV}^4$; so there are still 40 orders of magnitude.

**Resolution in AdS/CFT.** Starting from the Einstein–Hilbert action

$$S = \frac{1}{16\pi G_N} \int \mathrm{d}^D x\, \sqrt{-g}(R - 2\Lambda) = \frac{1}{16\pi G_N} \int \mathrm{d}^D x\, \sqrt{-g} \left[ R + \frac{(D-1)(D-2)}{l_{\text{AdS}}^2} \right]$$

where $l_{\text{AdS}}$ is a length scale of the cosmological constant. There are two length scales

$$l_p \sim G_N^{\frac{1}{D-2}}, \quad l_{\text{AdS}}$$

The effective field theory suggests that the Planck scale is the same as the AdS. [r] However gravity is semi-classical if the curvature of AdS is very little and as such the length scale is very large: this happens for $l_{\text{AdS}} \gg l_p$. This is the cosmological constant problem, the separation of the scale is not natural from the point of view of effective field theory.

AdS/CFT is a dictionary and one of its entries is

$$\frac{l_{\text{AdS}}^{D-2}}{G_N} = C_T$$

where $C_T$ is the central charge. In conformal field theory [r]

$$\langle T_{\mu\nu}(x_1) T_{\rho\sigma}(x_2) \rangle = C_T I_{\mu\nu\rho\sigma}(x_1, x_2)$$

The second factor is fixed by CFT symmetry up to one number. The central charge should be $C_T \gg 1$ to get semi-classical gravity. In CFT, the central charge counts the number of degrees of freedom. Therefore, to solve the cosmological problem one has to consider CFT with many degrees of freedom.

## 2 Black hole thermodynamics

Black holes carry an entropy proportional to the event horizon

$$S_{\text{BH}} = \frac{A_{\text{H}}}{4G_N}$$

Black holes also have temperature. The numerical factor was found by Hawking by calculating the temperature of a black hole.

There are two approaches to study the thermodynamics of black holes:

- euclidean methods, which involve thermal physics;
- QFT on a curved background.

## 2.1 Euclidean quantum field theory

The partition function is

$$Z(\beta) = \mathrm{Tr}\,\mathrm{e}^{-\beta H} \qquad \beta = \frac{1}{T}$$

In thermal physics everything can be obtained from it. The entropy is

$$S = -\partial_t (T \log Z)$$

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One would like to find the connection with path integrals. In ordinary quantum mechanics, a Schrödinger state evolves through the Lorentzian time evolution operator

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht} |\psi(0)\rangle$$

This operator is unitary. The Euclidean time $\tau$ is given by $t = -\mathrm{i}\tau$. The state is then

$$|\psi(t)\rangle = \mathrm{e}^{-\tau H} |\psi(0)\rangle$$

The Euclidean time evolution operator is not unitary. A Euclidean path integral is a transition amplitude

$$\langle \phi_2 | \mathrm{e}^{-\tau H} |\phi_1\rangle = \int_{\phi_1}^{\phi_2} [\mathcal{D}\phi]\, \mathrm{e}^{-S_{\mathrm{E}}}\,, \quad \phi_1 = \phi(\tau = 0)\,, \quad \phi_2 = \phi(\tau)$$

One integrates all field configurations with some boundary conditions weighed by some the Euclidean action.

One can also use path integrals to prepare states. Consider a quantum state

$$|\psi\rangle = \mathrm{e}^{-\beta H} |\phi_1\rangle$$

It has a wave-functional

$$\Psi(\phi) = \langle \phi | \mathrm{e}^{-\beta H} |\phi_1\rangle$$

[r] The state $\phi$ is not fixed, but depends on the specific boundary conditions. In terms of the path integral one has

$$\int_{\phi_1}^{\phi(\beta)} [\mathcal{D}\phi]\, \mathrm{e}^{-S_{\mathrm{E}}}$$

**Ground state.** Consideri a generic state

$$|\psi\rangle \sum_n c_n |n\rangle\,, \quad H|n\rangle = E_n |n\rangle$$

The evolved state is

$$\mathrm{e}^{-\beta H} |\psi\rangle = \sum_n c_n \mathrm{e}^{-\beta E_n} |n\rangle$$

Taking $\beta \to \infty$, one can extract the vacuum (which means $T \to 0$ where $T$ is the temperature). The vacuum is given by the Euclidean path integral with open boundary conditions at $\tau = 0$ and path integrating from $-\infty$; this is half of $\mathbb{R}^d$. A bra can be prepared from $\infty$ to zero. The vacuum normalization is just a path integral form $-\infty$ to $\infty$. The boundary conditions get integrated, it is similar to inserting a complete set of states

$$\langle 0|0\rangle = \sum_{\phi_1} \langle 0|\phi_1\rangle \langle \phi_1|0\rangle$$

This is the starting point to generate everything else.

**Correlation functions.** A two-point correlation function in the vacuum is

$$\langle 0| \phi(x_1)\phi(x_2) |0\rangle = \int_{\mathbb{R}^d} [\mathcal{D}\phi]\, \mathrm{e}^{-S_{\mathrm{E}}} \phi(x_1)\phi(x_2)$$

One prepares the vacuum and inserts two operators at two specific points. In Euclidean space there is no Hilbert space because one uses fields and computes functions in Euclidean coordinates.

**Thermal physics.** One would like to compute the partition function

$$Z = \operatorname{Tr} \mathrm{e}^{-\beta H} = \sum_{\phi_1} \langle\phi_1| \mathrm{e}^{-\beta H} |\phi_1\rangle$$

The sum over same boundary conditions glues them together, so one obtains a cylinder with circumference $\beta$: periodic boundary conditions. Thermal physics corresponds to periodicity in Euclidean time. The period is $\beta = \frac{1}{T}$ where $T$ is the temperature.

[r] The Green function is

$$G(\tau, x) = \langle O(\tau, x)O(0, 0)|_\beta = \operatorname{Tr}\left[\mathrm{e}^{-\beta H} O(\tau, x)O(0, 0)\right] = \operatorname{Tr}\left[O(0, 0)\mathrm{e}^{-\beta H} O(\tau, x)\right]$$

Knowing that

$$O(\beta, 0) = \mathrm{e}^{-\beta H} O(0, 0)\mathrm{e}^{\beta H}$$

one has

$$G(\tau, x) = \operatorname{Tr}\left[\mathrm{e}^{-\beta H} O(\beta, 0)O(\tau, x)\right] = G(\tau - \beta, x)$$

This is periodicity.

The path integral with open boundary conditions on one side and closed on the other is a state. If both are open then it is a density matrix of the thermal state

$$\rho = \mathrm{e}^{-\beta H}$$

The amplitude is given by definite boundary conditions

$$\langle\phi_2| \rho |\phi_1\rangle$$

**Density matrix.** An entangled state

$$|EPR\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) \neq |\psi\rangle_1 \otimes |\psi\rangle_2$$

cannot be written as two separate qubit states. [r] Tracing over one state one has

$$\langle\uparrow|_2 |EPR\rangle\langle EPR| |\uparrow\rangle_2 + \langle\downarrow|_2 |EPR\rangle\langle EPR| |\downarrow\rangle_2 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

It is a density matrix.

## 2.2 Rindler space and polar coordinates

The metric of two-dimensional Rindler space

$$\mathrm{d}s^2 = \mathrm{d}R^2 - R^2\,\mathrm{d}\eta^2\,, \quad R > 0\,, \quad \eta \in \mathbb{R}$$

The horizon of the space is $R = 0$ where the metric is singular. Setting $\eta = \mathrm{i}\theta$, the metric becomes

$$\mathrm{d}s^2 = \mathrm{d}R^2 + R^2\,\mathrm{d}\theta^2$$

that is polar coordinates. Rindler space is the Lorentzian equivalent of polar coordinates. Using the change of coordinates

$$x = R\cosh\eta\,, \quad t = R\sinh\eta$$

one obtains flat space

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2$$

One has

$$x^2 - t^2 = R^2 > 0$$

so on the Penrose diagram one occupies only the right Rindler patch. In higher dimension, the Rindler space is

$$\mathrm{d}s^2 = \mathrm{d}R^2 - R^2\,\mathrm{d}\eta^2 + \sum_{i=2}^{d}\left|\mathrm{d}x^i\right|^2$$

The trajectories of fixed $R$ are not geodesics, but are trajectories of constant acceleration.

In Minkowski space, the Hamiltonian generates time translation

$$H = \int_{\mathbb{R}} \mathrm{d}x \, T_{tt}$$

In Rindler space, the $\eta$-time translation is a boost

$$H_R = \int_{\mathbb{R}} \mathrm{d}x \, x^2 T_{tt}$$

**Euclidean path integral.** The overlap of the vacuum with itself is the path integral from $-\infty$ to $\infty$ generated by $\mathrm{e}^{-\beta H}$ for $\beta \to \infty$. Instead of opening the path integral and inserting a configuration, one opens the path integral as a pizza slice and uses a time evolution operator that gives rotations not translations

$$\mathrm{e}^{-2\pi \, \partial_\phi} \,, \quad \eta = \mathrm{i}\phi \,, \quad H_{\mathrm{Rindler}} = \partial_\phi$$

A rotation is a Euclidean-Rindler time translation.

One particular matrix element of the Rindler Hamiltonian is

$$\langle \phi_2 | \, \mathrm{e}^{-2\pi H_{\mathrm{Rindler}}} \, | \phi_1 \rangle$$

To compute the path integral over all of space, one has to glue together the two states so sum over the states

$$\langle 0 | 0 \rangle = \sum_{\phi_1} \langle \phi_1 | \, \mathrm{e}^{-2\pi H_{\mathrm{Rindler}}} \, | \phi_1 \rangle = \operatorname{Tr} \mathrm{e}^{-2\pi H_{\mathrm{Rindler}}} = Z_{\mathrm{Rindler}}$$

This is the partition function with temperature $\beta = 2\pi$. The vacuum state can be interpreted as a thermal state of Rindler space of temperature $T = \frac{1}{2\pi}$. The Rindler space is half of Minkowski space, the other half is not know and it is a temperature mass [r]. The Rindler horizon is not very different from the black hole's.

## Lecture 3

## 2.3 Hawking temperature

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The entropy of a black hole is related to its area. The Schwarzschild metric is given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2MG}{r}} + r^2 \, \mathrm{d}\Omega^2$$

Going to Euclidean time, one has

$$\mathrm{d}s_{\mathrm{E}}^2 = \left(1 - \frac{2GM}{r}\right) \mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{1 - \frac{2MG}{r}} + r^2 \, \mathrm{d}\Omega_2^2$$

One expands around the event horizon to get the near horizon expansion

$$r_{\mathrm{H}} = 2MG \,, \quad r = 2MG(1 + \varepsilon^2) \,, \quad \varepsilon \ll 1$$

The line element is

$$\begin{aligned} \mathrm{d}s_{\mathrm{E}}^2 &= \left(1 - \frac{1}{1 + \varepsilon^2}\right) \mathrm{d}\tau^2 + \frac{(2MG)^2 4\varepsilon^2 \, \mathrm{d}\varepsilon^2}{1 - \frac{1}{1 + \varepsilon^2}} + (2MG)^2 (1 + \varepsilon^2)^2 \, \mathrm{d}\Omega_2^2 \\ &= \varepsilon^2 \, \mathrm{d}\tau^2 + 16M^2G^2 \, \varepsilon^2 + 4M^2G^2 \, \mathrm{d}\Omega_2^2 \end{aligned}$$

The first two addenda are the Rindler Euclidean metric, while the last is the two-sphere. The full solutions expands the Einstein's field equations, Rindler is flat space, but the sphere has positive metric. In the vacuum, Einstein's equation give $R = 0$, but here $R \neq 0$. There is no horizon in the Euclidean Rindler metric. [r]

Employing a change of coordinates

$$\varepsilon = \frac{R}{4MG}\,, \quad \tau = \frac{\theta}{4MG}$$

one gets

$$\mathrm{d}s^2 = R^2\,\mathrm{d}\theta^2 + \mathrm{d}R^2 + 4M^2G^2\,\mathrm{d}\Omega_2^2$$

This metric is called cigar metric. The coordinate $\theta$ is $2\pi$-periodic, being smaller or greater implies a conical singularity. Going back to Euclidean time $\tau$ of an observer at infinity, one has

$$\tau \sim \tau + 8\pi MG$$

The Euclidean time has the period above, so the temperature is

$$\beta = 8\pi MG \implies T_\mathrm{H} = \frac{1}{8\pi MG}$$

For a black hole of the mass of the sun, the temperature is $T_\mathrm{H} \approx 6 \times 10^{-8}\,\mathrm{K}$. The cosmic microwave background as a temperature of $T_\mathrm{CMB} \approx 3\,\mathrm{K}$.

To compute the entropy, one uses the following thermodynamic identity

$$\mathrm{d}E = T\,\mathrm{d}S$$

Remembering that energy is mass, one has

$$\mathrm{d}M = T\,\mathrm{d}S = \frac{1}{8\pi MG}\,\mathrm{d}S$$

The entropy is proportional to the area of the black holed

$$S = \alpha A_\mathrm{H} = \alpha 4\pi r_\mathrm{H}^2 = \alpha 16\pi M^2G^2$$

Therefore

$$\mathrm{d}S = \alpha 32\pi G^2 M\,\mathrm{d}M \implies \mathrm{d}M = \frac{1}{8\pi MG}\alpha 32\pi MG^2\,\mathrm{d}M \implies \alpha = \frac{1}{4G}$$

The entropy of a black hole is

$$S_\mathrm{BH} = \frac{A_\mathrm{H}}{4G}$$

This answer was found with the anstaz of entropy been linear in the area: only this ansatz works because masses cancels.

This derivation is semi-classical: one has gone into Euclidean space and has applied classical thermodynamical equations to obtain a valid answer. Why is happens is yet unknown.

## 2.4 Quantum field theory in curved space-time

One picks a frozen background metric $g^\mathrm{b}_{\mu\nu}$ which solves Einstein's field equations and applies quantum field theory on such background without Einstein's equations. One consider a state of the QFT, it may have a non trivial stress-tensor, but this is not input into Einstein's equation to solved for a new metric: the metric is frozen.

This method is easier than quantum gravity, however it is not trivial.

**Revision of quantum field theory.** Consider a free scalar field $\phi$ of mass $m$. The equation of motion are Klein–Gordon's

$$(\Box + m^2)\phi = 0 \implies \phi(x^\mu) = \phi_0 \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\mathbf{k}\cdot\mathbf{x}}\,, \quad \omega^2 = |\mathbf{k}|^2 + m^2$$

The above is just one solution. The most general one is a superposition of plane waves. One needs to find an orthonormal set of motions onto which expand the solutions. The Klein–Gordon inner product

$$(\phi_1, \phi_2) = -\mathrm{i} \int_{\Sigma_t} (\phi_1\, \partial_t \phi_2^* - \phi_2^*\, \partial_t \phi_1)\, \mathrm{d}^{d-1}x\,, \quad \Sigma_t\,, \quad t = 0$$

where $\Sigma$ is a slice [r] a solution is given by

$$f_{\mathbf{k}}(x^\mu) = \frac{\mathrm{e}^{\mathrm{i}kx}}{\sqrt{(2\pi)^{d-1} 2\omega}}$$

The dispersion relation does not fix the sign of the energy.

The positive-frequency modes have

$$\partial_t f_{\mathbf{k}} = -\mathrm{i}\omega f_{\mathbf{k}}\,, \quad \omega > 0$$

The negative-frequency modes have

$$\partial_t f_{\mathbf{k}}^* = \mathrm{i}\omega f_{\mathbf{k}}^*\,, \quad \omega < 0$$

One has the inner products

$$(f_{\mathbf{k}_1}, f_{\mathbf{k}_2}^*) = 0\,, \quad (f_{\mathbf{k}_1}^*, f_{\mathbf{k}_2}^*) = -\delta^{(d-1)}(\mathbf{k}_1 - \mathbf{k}_2)$$

The most general solution is given by

$$\phi(t, \mathbf{x}) = \int \mathrm{d}^{d-1}k\, [a_{\mathbf{k}} f_{\mathbf{k}}(t, \mathbf{x}) + a_{\mathbf{k}}^* f_{\mathbf{k}}^*(t, x)]$$

One promotes the field to operator which implies the Fourier coefficient to be operators as well. The positive frequency modes are associated with the annihilation operators $\hat{a}_{\mathbf{k}}$. The vacuum is the state which is annihilated by all annihilation operators

$$\hat{a}_k \left|0\right\rangle = 0$$

A Fork space is built upon the vacuum

$$\hat{a}_{\mathbf{k}_1}^\dagger \cdots \hat{a}_{\mathbf{k}_n}^\dagger \left|0\right\rangle$$

The number operator counts the number of particle in a state

$$N_{\mathbf{k}} = \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}$$

Performing a boost, the momenta $k^\mu$ change, but the vacuum is Lorentz-invariant. The number of particles is conserved. In curved space-time this conservation no longer holds.

**Curved space-time.** The Lagrangian is made covariant

$$\mathcal{L} = -\frac{1}{2}\, \partial_\mu \phi\, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \rightsquigarrow \mathcal{L} = \sqrt{-g}\left[-\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi\, \nabla_\nu \phi - \frac{1}{2} m^2 \phi^2 - \xi R \phi^2\right]$$

where $R$ is the Ricci scalar; with $\xi = 0$ is the minimal coupling, while the conformal coupling is

$$\xi = \frac{d-2}{4(d-1)}$$

This last coupling means that the Lagrangian is invariant under conformal transformations (scaling and translating). The equations of motion are

$$(\nabla_\mu \nabla^\mu - m^2)\phi = 0$$

The covariant Klein–Gordon inner product is

$$(\phi_1, \phi_2) = -\mathrm{i} \int_\Sigma (\phi_1\, \nabla_\mu \phi_2^* - \phi_2^*\, \nabla_\mu \phi_1)\, n^\mu \sqrt{\gamma}\, \mathrm{d}^{d-1}x$$

where $n^\mu$ is a unit vector pointing outwards from the slice $\Sigma$ and $\gamma$ is the induced metric on the slice $\Sigma$. One needs to find an orthonormal set of solution with respect to the inner product above.

In general, the solution is not separable, as opposed to flat space-time $f_t(t)f_{\mathbf{k}}(\mathbf{x})$. One needs to pick some functions $f(t,\mathbf{x})$ such that

$$(f_i, f_j) = \delta_{ij}\,, \quad (f_i^*, f_j^*) = -\delta_{ij}$$

The scalar field is then the superposition

$$\phi = \sum_i a_i f_i + a_i^* f_i^*$$

Similar to flat space-time, one promotes the field to operators and defines the vacuum as the state annihilated by all destruction operators

$$a_i \left|0\right\rangle_f = 0\,, \quad \forall i$$

The number operator is

$$N_{fi} = a_i^\dagger a_i$$

One could have also done a different expansion

$$\phi = \sum_j (b_j g_j + b_j^\dagger g_j^*)$$

with different vacuum and different number operator

$$b\left|0\right\rangle_g = 0\,, \quad N_{gj} = b_j^\dagger b_j$$

Though no basis is the preferred one. The important property is that, in flat space-time the vacuum is unique, while in this case the vacuum is not unique

$$\left|0\right\rangle_f \neq \left|0\right\rangle_g$$

Even if there is no preferred basis, two observers cannot agree on the number of particles.

[r] To get from one basis to another, one employs Bogoliubov transformation

$$g_i = \sum_j (\alpha_{ij} f_j + \beta_{ij} f_j^*)\,, \quad f_i = \sum_j (\alpha_{ji}^* g_j - \beta_{ji} g_j^*)$$

[r] which is a change of basis. The coefficients are defined as the overlap of the wave functions

$$\alpha_{ij} = (g_i, f_j)\,, \quad \beta_{ij} = -(g_i, f_j^*)$$

They are normalized as

$$\sum_k (\alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^*) = \delta_{ij}\,, \quad \sum_k (\alpha_{ik}^*\beta_{jk} - \beta_{ik}^*\alpha_{jk}) = 0$$

The transformation can be also used to change the destruction and creation operators

$$a_i = \sum_j (\alpha_{ij} b_j + \beta_{ji}^* b_j^\dagger)\,, \quad b_i = \sum_j (\alpha_{ij}^* a_j - \beta_{ij}^* a_j^\dagger)$$

[r] The different choices do not agree. The expectation value of the number of the $g_i$ mode is

$$\begin{aligned}
{}_f\!\left\langle 0\right| n_{gi} \left|0\right\rangle_f &= {}_j\!\left\langle 0\right| \sum_{jk} (\alpha_{ij}^* a_j^\dagger - \beta_{ij} a_j)(\alpha_{ik} a_k - \beta_{ik}^* a_k^\dagger) \left|0\right\rangle_f \\
&= {}_f\!\left\langle 0\right| \sum_{jk} \beta_{ij} a_j \beta_{ik}^* a_k^\dagger \left|0\right\rangle_f = {}_f\!\left\langle 0\right| \sum_{jk} \beta_{ij}\beta_{ik}^* (\delta_{jk} + a_k^\dagger a_j) \left|0\right\rangle_f \\
&= {}_f\!\left\langle 0\right| \sum_j \left|\beta_{ij}\right|^2 \left|0\right\rangle = \sum_j \left|\beta_{ij}\right|^2 > 0
\end{aligned}$$

Different observers do not agree on the particle count. The vacuum for one gives a net positive number of particle for another observer. In the context of black hole, a geodesic measures no particles, but at constant radius one feels a flux of particle coming from Hawking radiation.

## Lecture 4

gio 14 mar 2024 14:30

This is similar to Rindler space: the Minkowski vacuum is a thermal state. The particle number is not conserved because one does not perform Lorentz boosts. By the equivalence principle, Rindler space is similar to a curved background.

**Rindler space.** Consider a trajectory

$$t(\tau) = \frac{1}{\alpha}\sinh\alpha\tau\,, \quad x(\tau) = \frac{1}{\alpha}\cos\alpha\tau$$

The acceleration is

$$a^\mu = \mathrm{d}_\tau^2 x^\mu\,, \quad a^t = \alpha\sinh\alpha\tau\,, \quad a^x = \alpha\cosh\alpha\tau\,, \quad \sqrt{a^\mu a_\mu} = \alpha$$

The above trajectory is a hyperbola in the right quadrant of Minkowski space $xt$

$$x^2(\tau) = t^2(\tau) + \frac{1}{\alpha^2}$$

The point at $t = 0$ has $x = \alpha^{-1}$. The radius is constant and given by $R = \alpha^{-1}$. Rindler time is the time perceived when constantly accelerating in Minkowski space.

**Wave equation in Rindler space.** The wave equation for a massless scalar field is given by

$$\Box\,\phi = 0$$

The line element is

$$\mathrm{d}s^2 = -R^2\,\mathrm{d}\eta^2 - \mathrm{d}R^2$$

The solutions are

$$g_k^{\mathrm{R}} = \frac{1}{\sqrt{4\pi\omega}}\mathrm{e}^{-\mathrm{i}\omega\eta}R^{\mathrm{i}k}\,, \quad \omega = |\mathbf{k}|$$

The positive frequency modes are

$$\partial_\eta g^R = -\mathrm{i}\omega g^R\,, \quad \omega > 0$$

[r] There is another change of coordinate

$$t = -R\sinh\eta\,, \quad x = -R\cosh\eta$$

It describes the left quadrant of Minkowski space. In this coordinate system, the solution to the wave equation is

$$g_k^{\mathrm{L}} = \frac{1}{\sqrt{4\pi\omega}\mathrm{e}^{\mathrm{i}\omega\eta}R^{\mathrm{i}k}}$$

The positive frequency modes are

$$\partial_{-\eta} g^{\mathrm{L}} = -\mathrm{i}\omega g^{\mathrm{L}}\,, \quad \omega > 0$$

The two solutions obtain are true for different coordinate systems [r]. The Rindler expansion of the field is

$$\varphi = \int \mathrm{d}k\,[g_k^{\mathrm{R}} b_k^{\mathrm{R}} + (g_k^R)^*(b_k^R)^\dagger + g_k^L b_k^L + (g_k^L)^*(b_k^L)^\dagger]$$

where one has

$$[b_{k_1}^L, b_{k_2}^R] = [b_{k_1}^L, (b_{k_2}^R)^\dagger] = 0$$

The Minkowski expansion of the field is

$$\varphi = \int \mathrm{d}k\,[a_k f_k + a_k^\dagger f_k^*]$$

Since there are two sets of modes, there are two inequivalent vacua, one for Rindler and one for Minkowski. The expectation values of the hamiltonian are

$$\langle 0| H |0\rangle_{\text{Rindler}} = \infty$$

This vacuum is not part of a Hilbert space because it is singular. The Bogoliubov coefficients are

$$\alpha_{ij} = (g_i, f_j)\,, \quad b_i = \sum_j \alpha_{ij} a_j + \cdots$$

from which one can compute

$$\langle 0| b_i^\dagger b_i |0\rangle_M$$

However this is lengthy and one can use a trick to see that the vacuum is thermal. One patches together the Rindler modes to obtain something related to Minkowski modes. In fact

$$\sqrt{4\pi\omega} g_k^R = (-t+x)^{\mathrm{i}\omega}\,, \quad x > |t|\,, \quad k > 0$$

for $x < 0$ one has $g_k^R = 0$. The right modes look like waves that disappear in the left patch of Minkowski space. The left modes are

$$\sqrt{4\pi\omega} g_k^L = (-t-x)^{-\mathrm{i}\omega}$$

However this does not work because the way propagates in the opposite direction to the right modes. Though, one may notice

$$\sqrt{4\pi\omega} (g_{-k}^L)^* = \mathrm{e}^{\pi\omega} (-t+x)^{\mathrm{i}\omega}$$

The factor in front can be absorbed

$$\sqrt{4\pi\omega} [g_k^L + \mathrm{e}^{-\pi\omega} (g_{-k}^R)^*] = (-t+x)^{\mathrm{i}\omega}$$

In this way the modes are not normalized. So to normalize them one defines

$$\begin{aligned} h_k^{(1)} &= \frac{1}{[\mathrm{e}^{\frac{\pi\omega}{2}}} g_k^R + \mathrm{e}^{-\frac{\pi\omega}{2}} (g_{-k}^L)^*] \sim (-t+x)^{\mathrm{i}\omega} \\ h_k^{(2)} &= \frac{1}{[\mathrm{e}^{\frac{\pi\omega}{2}}} g_k^L + \mathrm{e}^{-\frac{\pi\omega}{2}} (g_{-k}^R)^*] \sim (t+x)^{\mathrm{i}\omega} \end{aligned}$$

These are not the Minkowski modes because there an exponential between. Therefore, one has

$$\varphi = \int \mathrm{d}k\, [c_k^{(1)} h_k^{(1)} + c_k^{(1)} (h_k^{(1)})^* + c_k^{(2)} h_k^{(2)} + (c_k^{(2)})^\dagger (h_k^{(2)})^*]$$

One may show that

$$c_k^{(1)} |0\rangle_M = 0 = c_k^{(2)} |0\rangle_M$$

The coefficients $c_k$ are a superposition of only destruction operators $a_k$.

One would like to compute the expectation value of the number operator on the Minkowski vacuum

$$\langle 0| n_k^R |0\rangle_M = \langle 0| (b_k^R)^\dagger b_k^R |0\rangle_M$$

Inverting the expressions, one has

$$b_k^R = \frac{1}{\sqrt{2\sinh \pi\omega}} [\mathrm{e}^{\frac{\pi\omega}{2}} c_k^{(1)} + \mathrm{e}^{-\frac{\pi\omega}{2}} (c_k^{(2)})^\dagger]$$

Inserting this expression in the expectation value above, one obtains

$$\langle 0| n_k^R |0\rangle_M = \frac{1}{2\sinh \pi\omega} \mathrm{e}^{-\pi\omega} \langle 0| c_k^{(2)} (c_k^{(2)})^\dagger |0\rangle = \frac{1}{\mathrm{e}^{2\pi\omega} - 1} \delta(0)$$

The term $\delta(0)$ would not appear if one computes the expectation value for a wave packet instead of a single mode. In the second equality one has applied the commutation relation

$$[c_{k_1}^{(2)}, (c_{k_2}^{(2)})^\dagger] = \delta(k_1 - k_2)$$

The above is the Boltzmann–Maxwell distribution with temperature

$$T = \frac{1}{2\pi}$$

This temperature is valid for an observer at infinity. For a finite radius one would have $T = \frac{a}{2\pi}$. This is in agreement with the Euclidean Rindler time. The presence of such temperature is called Unruh effect.

The Euclidean method is faster and simpler, therefore more powerful.

## 2.5 Quantum entanglement

One has seen that the vacuum of Minkowski space is the same as a density matrix given by $\mathrm{e}^{-2\pi H_{\mathrm{Rindler}}} = \rho_{\mathrm{R}}$ of the right patch. [r] The state that has support in both left and right patches is called thermofield-double state

$$|\mathrm{TFD}(\beta)\rangle = \sum_n \mathrm{e}^{-\frac{1}{2}\beta E_n} |n\rangle_R \otimes |n\rangle_L^*$$

where $|n\rangle$ are Rindler energy eigenstates. The factors in the exponent are half Boltzmann factors. The star denotes CPT conjugation. This form already appeared

$$g_k^R + (g_{-k}^L)^*$$

The minus sign is about parity, the star is about charge conjugation, and the flow of time (up for right patch, down for left patch) is time reversal.

The Minkowski vacuum is the thermofield-double of the Rindler modes

$$|0\rangle_M = \int \mathrm{d}E_n \, \mathrm{e}^{-\pi E_n} |n\rangle_R \otimes |n\rangle_L^*$$

The TFD state is an entangled pure state. The boost operator kills the Minkowski vacuum and it is

$$K = H_R - H_L \,, \quad K |0\rangle_M = 0$$

The trace of the TFD state on the left wedge recovers a thermal state. The entropy of thermodynamics is the entropy between the left and right patch.

**Entanglement.** In quantum mechanics, a Hilbert space can be often decomposed as product of Hilbert spaces

$$\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$$

A state $|\psi\rangle$ of the Hilbert space $\mathcal{H}$ is entangled if

$$|\psi\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle$$

Entanglement implies correlation. The two point function of two operators is

$$\langle\psi| O_A O_B |\psi\rangle$$

If the state can be factorized, then the correlation function of operators with support only on one Hilbert space is factorized

$$\langle\psi| O_A O_B |\psi\rangle = \langle\psi_A| O_A |\psi_A\rangle \langle\psi_B| O_B |\psi_B\rangle$$

The entanglement can be quantified. The density matrix is

$$\rho = |\psi\rangle\langle\psi|$$

[r] the reduced density matrix is

$$\rho_A = \mathrm{Tr}_B \, \rho$$

One can count the logarithm of entangled q-bits using [r]

$$S_{\mathrm{EE}} = -\,\mathrm{Tr}\, \rho_A \log \rho_A$$

**Example.** Consider the following EPR pair

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle)$$

The density matrix is

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\uparrow\uparrow\rangle\langle\downarrow\downarrow| + |\downarrow\downarrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|) = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Its trace is $\operatorname{Tr}\rho = 1$. If $\rho$ is pure, in the eigebasis it looks like

$$\rho = \operatorname{diag}(1, 0, 0, 0)$$

Using the trace trick, one has

$$\rho_A = \operatorname{Tr}_B \rho = \langle\uparrow|_B\, \rho\, |\uparrow\rangle_B + \langle\downarrow|_B\, \rho\, |\downarrow\rangle_B = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|) = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{\dim \mathcal{H}_A} I$$

This is the maximally mixed stated. The [r] entropy is

$$S_{\text{EE}} = -\operatorname{Tr}\rho_A \log \rho_A = -2\left[\frac{1}{2}\log\frac{1}{2}\right] = \log 2$$

This corresponds to one EPR pair.

The state of the example is also highly correlated, exactly because it is entangled.

**Proposition.** A few properties. If a state $|\psi\rangle$ is pure, then $S_A = S_B$. If the state $|\psi\rangle$ is a density matrix $\rho$ then $S_A \neq S_B$ and $S_{\text{EE}}$ is not the number of EPR pairs.

Multi-partite quantum entanglement is very hard.

**Quantum field theory.** In quantum field theory, one splits space-time in two, but one finds $S_{\text{EE}} = \infty$. This is because the oscillators are highly entangled between the barrier of the two Hilbert spaces. This is because one cannot split the Hilbert space. A regularization can be done on the lattice.

**Rindler space.** Going back to Rindler space, one has

$$|\text{TFD}(\beta)\rangle = \sum_n \mathrm{e}^{-\frac{1}{2}\beta E_n} |n\rangle_L \otimes |n\rangle_R^*$$

From which

$$|\text{TFD}\rangle\langle\text{TFD}| = \sum_{nm} \mathrm{e}^{-\frac{1}{2}\beta(E_n+E_m)} |n\rangle_L |n\rangle_R \langle m|_L \langle m|_R$$

Taking the trace, one has

$$\operatorname{Tr}_{H_L} |\text{TFD}\rangle\langle\text{TFD}| = \sum_k \langle k|_L (\cdots) |k\rangle_L = \sum_k \mathrm{e}^{-\beta E_k} |k\rangle_R \langle k|_R (\cdots) |k\rangle_L = \mathrm{e}^{-\beta H_R}$$

[r] The thermal entropy is

$$S_{\text{th}} = -\operatorname{Tr}\rho_{\text{th}} \log \rho_{\text{th}} = S_{\text{EE}}(R)$$

The thermal entropy can be interpreted as entanglement entropy that purifies the state [r]. Applying this to Rindler space, one has

$$|0\rangle_M = |\text{TFD}\rangle_R = \int \mathrm{d}n\, \mathrm{e}^{-\frac{1}{2}\beta E_n} |n\rangle_L \otimes |n\rangle_R^*$$

[r] So

$$S_{\text{th}}(\text{Rindler}) = S_{\text{EE}}(\text{half space-time}) = \infty$$

This infinite entanglement is important in QFT because one can compute correlation functions for space-like separation

$$\langle\phi_L(x_1)\phi_R(x_2)\rangle = \frac{1}{(x_1-x_2)^2}$$

This is only possible because the vacuum is a very entangled state. In an unentangled scenario, one would have

$$\langle\phi\phi\rangle = \langle\phi\rangle\langle\phi\rangle = 0$$

This state has infinite energy.

There is a relation between thermal physics and entanglement. In Minkowski space, the left region is a heat bath for the right patch. [r]

## Lecture 5

### 2.6 Schwarzschild black hole

mer 20 mar 2024 14:30

The radial acceleration needed to remain at a fixed radius in the Schwarzschild metric is

$$a_r = \frac{GM}{r^2\sqrt{1-\frac{2GM}{r}}}$$

One assume that the state is like the vacuum. In free falling one sees the vacuum everywhere. The temperature at the horizon is

$$T_H = \lim_{r\to r_H}\frac{a_r}{2\pi}$$

but at the horizon the acceleration diverges to infinity. One does not measure the local temperature at the horizon, but one is at infinity and measures the temperature there. However, due to the gravitational potential, the temperature at infinity is shifted and related to the temperature at the horizon. The temperature at infinity is

$$T_\infty = \frac{V_H}{V_\infty}T_H = V_H T_H = \frac{V_H a_H}{2\pi}$$

The numerator is called surface gravity. A surface gravity can be defined generally: a Killing vector which is null on a hypersurface $\Sigma$ [r]. Simply one can think of it as the acceleration needed to remain at a certain radius. The temperature at infinity is

$$T_\infty = \frac{1}{8\pi GM}$$

which is in accordance with the Euclidean trick. In this setup it was assumed that near the horizon one is locally in the vacuum. This is a particular choice of state, but there are other choices.

**Hadamard states.** Looking at a two-point function on a scale smaller than anything else, one should recover the Minkowski's UV structure from any curved space. The correlation function is a function of the geodesic distance between two points

$$\langle\psi|\,\phi(x)\phi(x_2)\,|\psi\rangle = f(d_{\text{geodesic}}(x_1,x_2)) = f(g^{\mu\nu}x_{12\mu}x_{12\nu})$$

One defines a universal prescription

$$\sigma_\varepsilon(x_1,x_2) = g^{\mu\nu}x_{12\mu}x_{12\nu} + 2\mathrm{i}\varepsilon(t_1-t_2)+\varepsilon^2$$

The Hadamard state is

$$\langle\psi|\,\phi(x_1)\phi(x_2)\,|\psi\rangle = \frac{U(x_1,x_2)}{4\pi^2\sigma_\varepsilon} + V(x_1,x_2)\ln\sigma_\varepsilon + W(x_1,x_2)$$

where the three functions $U$, $V$ and $W$ are all smooth with $U(x_1,x_2)=1$. [r] Consequence is that the stress-energy tensor is regular on Hadamard state.

The Hartle–Hawking state is a unique state that is smooth everywhere and invariant under $\partial_t$. [r] There is a flux of particle from infinity that equilibrate the particles emitted from the black hole.

**Boulmare state.** The flux is null at future and past infinity. This state is singular on the past and future horizons. This is similar to the Rindler vacuum. The stress-energy tensor diverges.

**Unruh state.** One demands no flux at $F^-$ and is singular on $H^-$, but there is flux at $F^+$ and is regular on $H^+$. This geometry evaporates.

The evaporation cannot yet be seen from the previous calculation [r] because the mass is fixed. For a dynamical system, the emission of particles diminishes the mass of the black hole. One needs to do backreaction and update the metric to get a lower mass of the black hole.

**Greybody factors.** The Schwarzschild metric is Rindler times a two-sphere. One may use the tortoise coordinates

$$\mathrm{d}r^* = \left[1 - \frac{2GM}{r}\right]^{-1} \mathrm{d}r$$

from which

$$\mathrm{d}s^2 = \cdots$$

[r] the potential is a complicated term [r] due to spherical symmetry one can decompose the solution in spherical harmonics. One has a wave equation with some potential

$$[\partial_t^2 - \partial_{r^*}^2 + V(r)]\phi_l(t,r) = 0$$

Near the horizon, a particle still needs to tunnel through the potential to propagate to infinity. The peak is the greybody factor. When one measures the distribution of a particle number operator at infinity, the Bose–Einstein distribution gets corrected by the scattering potential

$$n_\omega = \frac{\Gamma_l(\omega)}{\mathrm{e}^{-\frac{\omega}{T_H}} - 1}$$

Closed forms of this distribution are not known, but the limits are

$$\Gamma_l(\omega) \leq 1$$

high frequency

$$\Gamma_l(\omega) \approx 1\,, \quad \omega \gg GM$$

low frequency

$$\Gamma_l(\omega) \approx \frac{A_H}{4\pi}\omega^2\,, \quad \omega \ll GM$$

So black holes are grey bodies.

# 3 Black hole thermodynamics

Black holes have a temperature associated to them and an entropy

$$T = \frac{\kappa}{2\pi}\,, \quad S = \frac{A_H}{4G}$$

where $\kappa$ is the surface gravity of the black hole. Black holes are thermodynamical objects.

| Law | Thermodynamics | BH thermodynamics |
|---|---|---|
| 0-th | Equilibrium means homogeneous T | $\kappa$ is homogeneous on horizon for stationary BH |
| 1st | Conservation of energy $\mathrm{d}E = T\,\mathrm{d}S$ | $\mathrm{d}M = T\frac{\mathrm{d}A}{8\pi G}$ |
| 2nd | $\mathrm{d}_t S \geq 0$ | $\mathrm{d}_t A_H \geq 0$, $\frac{\Delta A}{4G} + \Delta S_{\text{ext}} \geq 0$ |
| 3rd | $S$ constant as $T \to 0$ | $\kappa = 0$ cannot be attained ? |

A stationary black hole has a time-like Killing vector, a static metric has a time-like Killing vector orthogonal to spatial direction [r].
The second law for BH is called generalized second law; the third law for BH one cannot collapse all matter into an extremal $J = M$ black hole in finite time to achieve zero temperature black holes (this may be wrong, see arXiv). The relations for BH were originally formulated classically for general relativity, but can be interpreted in thermodynamics. The duality in CFT is sharper.

## 3.1 Quantum gravity path integral

The partition function is

$$Z[\text{b.c.}] = \int_{\partial g = \text{b.c.}} [\mathcal{D}g] \mathrm{e}^{-S[g]}$$

One would like to understand the above in the semi-classical limit $\hbar \to 0$.

**Saddle-point method.** One may use the saddle-point method (also known as steepest ascent or Laplace method; the name depends on the presence of an imaginary unit i). Consider an integral of the type

$$I = \int \mathrm{d}x \, \mathrm{e}^{\frac{1}{a} f(x)} \,, \quad a \ll 1$$

An approximation of such integral by finding the saddle-point. The dominant contribution comes from the maximum of $f(x)$

$$x_0 \mid f'(x_0) = 0 \,, \quad f''(x_0) < 0$$

The approximation is

$$I \approx \int \mathrm{d}x \exp\left[\frac{1}{a} f(x_0) + \frac{1}{2a}(x - x_0)^2 f''(x_0) + \cdots\right] \approx \sum_{x_0} \mathrm{e}^{\frac{1}{a} f(x_0)} \sum_{n=0}^{\infty} f_n$$

where the first sum is over all the maxima while the second is the perturbations around the maxima. The second derivative is just a Gaussian integral. In QFT, since the first derivative is zero, means that the variation of the action is null. Therefore, the first sum above is the classical equations of motion, while the second are the perturbative $\hbar$ expansion.

In the semi-classical limit, the path integral becomes

$$Z[\text{b.c.}] \approx \sum_{g_0} \mathrm{e}^{-S[g_0]} \sum_{i=0}^{\infty} (g_0)_n^i$$

where the sum is over the solutions of the equations of motion. In general, when one varies the boundary conditions, the EOM solutions can exchange dominance.

## 3.2 Semi-classical limit

The Einstein–Hilbert action is

$$S = -\frac{1}{16\pi G} \int_M \mathrm{d}^4 x \, R$$

where $M$ is a manifold. To have a well-defined variational principle, it is not sufficient that the fields go to zero, because there are derivatives of the metric, so one has to add another boundary term called Gibbons–Hawking–York to have

$$S = -\frac{1}{16\pi G} \int_M \mathrm{d}^4 x \, R - \frac{1}{8\pi G} \int_{\partial M} \mathrm{d}^3 x \sqrt{h} K$$

where $h$ is the induced metric and $K$ is the trace of extrinsic curvature, it is the second fundamental form

$$K_{\mu\nu} = \nabla_{(\mu} n_{\nu)} \,, \quad K = h^{\mu\nu} K_{\mu\nu}$$

where $n$ is a unit normal to $\partial M$ vector pointing inwards. This is needed because the EH action for Schwarzschild solution is null since $R = 0$.

The metric is

$$\mathrm{d}s^2 = \left[1 - \frac{2GM}{r}\right] \mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{1 - \frac{2GM}{r}} + r^2 \, \mathrm{d}\Omega_2^2$$

The boundary $\partial M$ is at $r = r_{\max}$. Therefore

$$h_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu = \left[1 - \frac{2GM}{r_{\max}}\right] \mathrm{d}\tau^2 + r_{\max}^2 \, \mathrm{d}\Omega_2^2$$

The normal vector is

$$n_\mu = -\left[1 - \frac{2GM}{r}\right]^{-\frac{1}{2}} \partial_r$$

The minus sign is needed to be inwards, the derivative is needed to make it orthogonal. The second fundamental form is

$$K = \frac{2r - 3GM}{\sqrt{1 - \frac{2GM}{r_{\text{max}}} r_{\text{max}}^2}}$$

The square root determinant is

$$\sqrt{h} = r_{\text{max}}^2 \left[1 - \frac{2GM}{r_{\text{max}}}\right]^{\frac{1}{2}} \sin\theta$$

The action is

$$\begin{aligned} S &= \frac{1}{8\pi G}\int_{r_{\text{max}}} (3GM - 2r_{\text{max}}) = \frac{1}{8\pi G}\int \mathrm{d}\Omega_2^2 \int_0^\beta \mathrm{d}\tau\,(3GM - 2r_{\text{max}}) \\ &= \frac{1}{8\pi G} 4\pi\beta(3GM - 2r_{\text{max}}) \end{aligned}$$

This is the solution at a fix $r_{\text{max}}$. It diverges when taking all of space. To deal with this one uses a counter term

$$S_{\text{tot}} = S_{\text{EH}} + S_{\text{GHY}} + S_{\text{ct}}$$

The counter term is not know how to be found from first principles, but one uses the action on Minkowski

$$S_{\text{ct}} = -S_{\text{EH}}^{\text{M}} - S_{\text{CHY}}^{\text{M}}$$

but this does not always work like in AdS, where one should use holographic normalization instead [r]. The EH action on Minkowski is zero [r] and the other can be calculated from the metric

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 + r^2\,\mathrm{d}\Omega_2^2$$

One remembers that Euclidean time is cyclic $\tau \sim \tau + \beta$, but in [r] one has

$$\tau \sim \tau + \beta\sqrt{1 - \frac{2GM}{r_{\text{max}}}}$$

[r] One may finally get

$$S_{\text{tot}} = \frac{3}{2}\beta M - \frac{\beta r_{\text{max}}}{G} + \frac{\beta r_{\text{max}}}{G}\sqrt{1 - \frac{2GM}{r_{\text{max}}}} = \frac{3}{2}\beta M - \beta M = \frac{1}{2}\beta M + o(r_{\text{max}}^{-1})$$

The partition function is then

$$Z[\beta] = \mathrm{e}^{-\frac{1}{2}\beta M} = \exp\left[-\frac{\beta^2}{16\pi G}\right]$$

The free energy is then

$$F = -\frac{1}{\beta}\log Z = \frac{\beta}{16\pi G} = \frac{1}{16\pi G T}$$

The entropy is then

$$S = -\partial_T F = (1 - \beta\,\partial_\beta)\log Z = -\frac{\beta^2}{16\pi G} + \frac{\beta^2}{8\pi G} = \frac{\beta^2}{16\pi G} = \frac{A_H}{4G}$$

Once one obtains the partition function, one can obtain everything. This procedure can also be done in AdS [r].

## Lecture 6

gio 21 mar 2024 14:30

For more complicated stationary black holes, the partition function also has chemical potential and [r].

One may also compute the specific heat

$$c = -T\,\partial_T^2 F = -\frac{1}{8\pi G T^2} < 0$$

The specific heat is negative but it is not desired. This is thermodynamical instability. If the mass $M$ decreases, then temperature $T$ increases; the mass decreases more rapidly and so the temperature increases. This is a first intuition of the evaporation of black holes in flat space. In AdS the case is different: the potential attractive and it is like a box.

### 3.3 Black hole information paradox

Gravity is not renormalizable. The paradox appear in infrared limit.

Consider a classical argument. Consider a black hole made from an initial state of matter. By the no-hair theorem, a black hole is uniquely specified by $M$, $Q$ and $J$. [r] The initial information seems to be lost in the black hole. This is not really a paradox since if one knew the inside the black hole, one could time reverse the system to obtain the information of the initial state.

The quantum argument is properly a paradox. It seems to violate one of the most fundamental property of quantum mechanics: unitarity. General relativity and quantum mechanics are not compatible due to unitary: probability is not conserved. One starts from a star at low-density in a pure state $|\psi\rangle$. One lets the star collapse to form the black hole. The quantum fields of this geometry are in the Unruh state, the state that is smooth in the future. Slowly, Hawking radiation depletes the black hole, so it evaporates; the particles emitted are thermal particle. Eventually the black hole disappears [r] diagr. In the end there are thermal quanta, a mixed state

$$\rho_{\text{thermal}} = \mathrm{e}^{-\beta H}$$

This is incompatible with unitarity. Time evolution implies

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht}\,|\psi(0)\rangle\ ,\quad \rho(t) = \mathrm{e}^{-\mathrm{i}Ht}\,|\psi(0)\rangle\langle\psi(0)|\,\mathrm{e}^{\mathrm{i}Ht}$$

This is the density matrix of a pure state

$$-\operatorname{Tr}\rho(t)\log\rho(t) = 0$$

However, the entropy collected from the black hole is

$$S(t) = S_{\text{th}} \neq 0 \iff \rho(t\to\infty) = \rho_{\text{thermal}} \neq \rho_{\text{pure}}$$

**Resolution.** A first option is stating that quantum mechanics is wrong. This is logically possible, but there is not other alternative.

The second option is remnants: the black hole evaporates and at some point it gets to the Planck's scale, here though one may not claim to control the calculation. This is again an ultraviolet problem. One stops at the Planck's scale: one is left with a very compact object of Planck size that has an unbounded number of quantum states (since the original black hole can be arbitrarily large). This option is a drastic departure from quantum field theory where the number of quantum states, at fixed volume and energy, is always bounded. A loop diagram [r] is the ratio of the number of degrees of freedom to the coupling constant. However, this option present a problem before the Planck scale.

The third option is a problem with Hawking's calculations. Namely, the final state is pure

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht}\,|\psi(0)\rangle$$

but the calculation is an approximation. There are corrections ($\mathrm{e}^{-S_{\text{EH}}}$) that add up to restore unitarity. This may be the right solution.

**The Page curve.** The aforementioned problem of the second option is related to the Page curve. Consider a black hole formed from a collapse of a pure state $|\psi\rangle$. At some time after, some radiation has been emitted. At any time $t_0$, the Hilbert space can be decompose as

$$\mathcal{H} = \mathcal{H}_{\mathrm{R}} \otimes \mathcal{H}_{\mathrm{BH}}$$

The Hilbert space of the black hole can be thought of as the radiation at $t > t_0$. One may study how this decomposition evolves in time. The entanglement entropy for the radiation $S^{\mathrm{R}}_{\mathrm{EE}}(t)$ increases forever according to Hawking, while the entropy of the black hole decreases. The curve compatible with unitarity is the Page curve [r] diagr. If the state is pure, then entropy cannot be bigger than the dimension of the Hilbert space.

Page's argument goes as follows. If [r] a Hilbert space can be decomposed

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \,, \quad |\mathcal{H}_A| \gg |\mathcal{H}_B|$$

The typical state is

$$S_A = S_B = \log \dim \mathcal{H}_B$$

A black hole state is typical.

At an early time $|\mathcal{H}_{\mathrm{R}}| \ll |\mathcal{H}_{\mathrm{BH}}|$ so

$$S_{\mathrm{R}} \sim tT$$

For the Schwarzschild black hole, one has

$$J = 0 \implies l = m = 0$$

[r] One may forget about the two-sphere. The entropies are

$$S_{\mathrm{BH}} = A \sim M^2 \,, \quad T \sim \frac{1}{M}$$

For $t \ll M^3$, one has $tT$; at $t \sim M^3$ is the turn-over point which has

$$|\mathcal{H}_{\mathrm{BH}}| < |\mathcal{H}_K|$$

then one has

$$\frac{A_H(t)}{4G} = S_0 \left[1 - \frac{t}{M^3}\right]^{\frac{2}{3}}$$

Before the critical time, the photons emitted are entangled with photons inside the black hole; after the critical time the photons emitted are entangled with all photons already outside. This is what must happen to have a unitarity theory; but one needs to calculate if this is actually the case.

## 3.4 Exponential corrections and thermalization

Thermalization does not mean to take a pure state, evolve it and it becomes a thermal state. Thermalization is taking a state not in equilibrium, evolve it and take an expectation value of a simple operator (coarse-grained quantity) since they are well approximated by their expectation value

$$\langle\psi(t)|\, \mathcal{E} \,|\psi(t)\rangle = \operatorname{Tr} \mathrm{e}^{-\beta H} \mathcal{E} + e^{-S}$$

The exponential correction is important for the Page curve.

In AdS/CFT, eternal black holes are thermal states: a collapsing star is a pure state, it thermalizes, it becomes a state on which the expectation value of an operator is close to the expectation value of a thermal state. Hawking is missing the exponential corretions.

# Lecture 7

## 3.5 The firewall proposal

mer 03 apr 2024 14:30

In the 70s, Hawking, Bekenstein and Gibbons discover black hole evaporation, etc. Then in '93 came about Page. In 2009, Mattur observers the Page's problems. In 2012, AMPS rephrased and formalized the problems introduced by Page through quantum entanglement in the firewall proposal. In 2021 the unitarity in the Page curve was calculated.

**Postulates.** There are four postulates made by Susskind in the 90s of Black Hole Complementarity:

1. Formation and evaporation of a black hole is unitary.
2. Outside the horizon of a black hole, effective field theory is valid, in particular, QFT in curved space and semi-classical gravity is a good approximation.
3. To a distant observer $J^+$, a black hole is an ordinary quantum system with dimension of its Hilbert space of
$$|\mathcal{H}| = \mathrm{e}^{S_{\mathrm{BH}}}$$
4. Nothing happens to an in-falling observer.

AMPS concluded that these postulates are not compatible. From the fourth postulate, the horizon is locally the vacuum state of the quantum fields [r].

AMPS proposed to give up the last postulate. For nothing to happen to the in-falling observers, one is in the Unruh vacuum [r] there is a stress-energy tensor singularity: there is infinite energy, it is a firewall. The particles are not thermal forever and there is another structure [r]. Even if the Page curve has been solve, the firewall is not really understood.

If there is a firewall, there is no space-time after the horizon.

**Proposal.** The Hilbert space is split into the black hole Hilbert space and the early radiation Hilbert space. At a time $t > t_{\mathrm{Page}}$, one may split the Hilbert space

$$\mathcal{H} = \mathcal{H}_{\mathrm{HB}} \otimes \mathcal{H}_{R,\mathrm{early}} \to \mathcal{H}_{\mathrm{in}} \otimes \mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{R,\mathrm{early}}$$

By the fourth postulate, the first two spaces have to be maximally entangled. But the first postulates implies that the out space and the radiation early space must be entangled. However, a single system cannot be maximally entangled with two systems: a photon in $\mathcal{H}_{\mathrm{out}}$ would be entangled with two systems. Consider an EPR pair and a third qubit, to preserve the maximum entanglement between the first two, one may not entangle with the third qubit [r].

**Properties of entanglement entropy.** Consider a state mixed $\rho$ or pure $|\psi\rangle\langle\psi|$ on a Hilbert space
$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$
[r] the partial trace over the system B is

$$\rho_A = \mathrm{Tr}_B\, \rho$$

The entanglement entropy (EE) is

$$S_A = -\,\mathrm{Tr}\, \rho_A \log \rho_A$$

A few properties are

- If $\rho_A$ is a positive semi-definite matrix and $\mathrm{Tr}\, \rho_A = 1$ then
$$S_A \geq 0$$
- One may define the mutual information as
$$I(A:B) \equiv S_A + S_B - S_{AB}\,, \quad S_{AB} = -\,\mathrm{Tr}\, \rho \log \rho$$
The property of sub-additivity is that $I(A:B) \geq 0$.

  For example consider an EPR pair
$$\rho = |\mathrm{EPR}\rangle\langle\mathrm{EPR}| \implies S_A = S_B = \log 2\,, \quad S_{AB} = 0$$
Therefore
$$I(A:B) = 2 \log 2 > 0$$

Consider another example

$$\rho = \frac{1}{4}\mathrm{id}$$

[r] One has

$$\rho_A = \frac{1}{4}\,\mathrm{Tr}(|{\uparrow\uparrow}\rangle\langle{\uparrow\uparrow}| + |{\uparrow\downarrow}\rangle\langle{\uparrow\downarrow}| + |{\downarrow\uparrow}\rangle\langle{\downarrow\uparrow}| + |{\downarrow\downarrow}\rangle\langle{\downarrow\downarrow}|) = \frac{1}{2}(|{\uparrow}\rangle\langle{\uparrow}| + |{\downarrow}\rangle\langle{\downarrow}|) = \frac{1}{2}\mathrm{id}$$

Therefore

$$S_A = S_B = \log 2\,, \quad S_{AB} = \log 4 \implies I(A:B) = 0$$

The two system are classically correlated because the mutual information is zero, there is no quantum superposition.

- The third property is strong sub-additivity (SSA). Consider a Hilbert space partitioned into three

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$$

Then it holds

$$S_{AB} + S_{BC} \geq S_{ABC} + S_B$$

This implies that one cannot be maximally entangled with two system.

**Multi-partite information.** For three partites there are two types of entanglement: the W type and GHZ. These are named after the homonym states

$$|W\rangle = \frac{1}{\sqrt{3}}(|{\uparrow\downarrow\downarrow}\rangle + |{\downarrow\uparrow\downarrow}\rangle + |{\downarrow\downarrow\uparrow}\rangle)\,, \quad |\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|{\uparrow\uparrow\uparrow}\rangle + |{\downarrow\downarrow\downarrow}\rangle)$$

For the latter, every qubit is maximally entangled with other 2, but pairs are not entangled. This is only tri-partite entanglement. For the former, every pair is entangled, this is bi-partite entanglement. Both of the satisfy strong sub-additivity.

Consider the W state. The density matrix is [r]

$$\rho = \frac{1}{3}(|{\uparrow\downarrow\downarrow}\rangle\langle{\uparrow\downarrow\downarrow}| + |{\uparrow\downarrow\downarrow}\rangle\langle{\downarrow\uparrow\downarrow}| + \cdots$$

Tracing out the third qubit, one obtains

$$\rho_{AB} = \mathrm{Tr}_C\,\rho = \frac{1}{3}(|{\uparrow\downarrow}\rangle\langle{\uparrow\downarrow}| + |{\uparrow\downarrow}\rangle\langle{\downarrow\uparrow}| + |{\downarrow\uparrow}\rangle\langle{\uparrow\downarrow}| + |{\downarrow\uparrow}\rangle\langle{\downarrow\uparrow}| + |{\downarrow\downarrow}\rangle\langle{\downarrow\downarrow}|) = \frac{1}{3}\begin{bmatrix} 0 & & & \\ & 1 & 1 & \\ & 1 & 1 & \\ & & & 1 \end{bmatrix}$$

In the eigenbasis, one has

$$\widetilde{\rho}_{AB} = \frac{1}{3}\,\mathrm{diag}(2,1,0,0)$$

Therefore, the entropy is

$$S_{AB} = \log 2 - \frac{2}{3}\log 3$$

Performing another partial trace, one has

$$\rho_A = \frac{1}{3}\,\mathrm{diag}(1,2)$$

whose entropy is

$$S_A = -\frac{1}{3}\log\frac{1}{2} - \frac{2}{3}\log\frac{2}{3} = \log 2 - \frac{2}{3}\log 3$$

The state reduced to 1 qubit is not a maximally mixed state since it is equal to the entanglement $S_{AB}$.

Consider the GHZ state. The density matrix is

$$\rho_{AB} = \frac{1}{2}(|{\uparrow\uparrow}\rangle\langle{\uparrow\uparrow}| + |{\downarrow\downarrow}\rangle\langle{\downarrow\downarrow}|)\,, \quad \rho_A = \frac{1}{2}(|{\uparrow}\rangle\langle{\uparrow}| + |{\downarrow}\rangle\langle{\downarrow}|)\,, \quad S_A = S_{AB} = \log 2$$

Every qubit is maximally entangled and there is no bi-partite entanglement because [r].

**Proposal.** Consider again the proposal. Naming the three Hilbert space $A$, $B$, and $C$, consider the inequality

$$S_{AB} + S_{BC} \geq S_{ABC} + S_B$$

From the fourth postulate [r] $S_{AB} = 0$. Since there is no entanglement between $AB$ and the rest then $S_{ABC} = S_C$. From first postulate, after the page time the entropy is decreasing

$$S_{BC} < S_C \implies S_C > S_B + S_C \implies S_B < 0$$

which contradicts the positivity of entanglement. Therefore strong sub-additivity is violated.

Therefore, the fourth postulate is not compatible with quantum mechanics and unitarity, in particular it contradicts strong sub-additivity of entanglement entropy.

AMPS proposed to abandon this postulate and the horizon becomes a firewall: there is no inside to a black hole (See Harlow). This conclusion comes from a wrong assumption

$$\mathcal{H} = \mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{out}} \otimes \mathcal{H}_{R,\text{early}}$$

The Hilbert space of quantum gravity cannot have this structure, one may not separate it into three parts. The reason this equation is wrong is dealt at a later time.

### Lecture 8

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**Separability.** A state is entangled if it is not separable. A pure state is separable if

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

and is not the sum of direct products. A mixed state $\rho$ is separable if

$$\rho = \sum_i P_i \rho_1^i \otimes \rho_2^i \,, \quad \sum_i P_i = 1 \,, \quad \rho_j = |\psi_j\rangle\langle\psi_j|$$

For the GHZ state, one has

$$\rho_{AB} = \operatorname{Tr}_C \rho = \frac{1}{2}(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|)$$

It is separable [r]. The W state is

$$\rho_{AB} = \operatorname{Tr}_C \rho = \frac{1}{3}(|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\uparrow\downarrow\rangle\langle\downarrow\uparrow| + |\downarrow\uparrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|)$$

This state is not separable due to the first four terms. Every pair in this state has entanglement.

# 4 AdS/CFT correspondence

Introduction to AdS/CFT, geometry of AdS/CFT, calculations.

The AdS/CFT correspondence was discovered in '97. See Magoo review for results (but not to learn). The AdS/CFT correspondence is a duality: a physical system is described by two non-trivially-connected theories.

**Example.** Some examples of dualities are:

- electromagnetism, both electric and magnetic field may describe the same system depending on the reference frame; the duality between the two fields is generated by Lorentz transformations;
- in string theory, there are many; AdS/CFT comes from there in the era of dualities;
  - ◇ .[r] Consider a periodic coordinate $x' \sim x' + R$, the spectrum of the string is

    $$E = n^2 R^2 + \frac{m^2}{R^2}$$

    The first term is the winding modes, the second term is the momentum modes. The spectrum is identical by $R \leftrightarrow \frac{1}{R}$. This is T duality.

- ◇ There is also S duality. A Yang–Mills theory may be invariant under $g \leftrightarrow \frac{1}{g}$, this is the strong-weak duality.
- ◇ Mirror symmetry; in ten dimension, exchanging CY with CY' in

$$\mathbb{R}^{1,3} \otimes CY \leftrightarrow \mathbb{R}^{1,3} \otimes CY'$$

AdS/CFT was born during the second string revolution. This period saw the introduction of dualities, M-theory (the eleven-dimensional mother theory uniting the four? string theories) and D-branes (objects where strings end). The AdS/CFT correspondence was found by stacking some D-branes a taking various limits. [r] It was derived as the correspondence between type IIB supergravity on $\mathrm{AdS}_3 \times S^5$ and $\mathrm{CFT}_4$ $N = 4$ supersymmetry. It is a duality between a theory of quantum gravity (in five dimensions?) and ordinary QFT in four dimensions. The two sides have different dimensions therefore the duality is holographic: the conformal field theory lives on the boundary of the anti-de Sitter space. It is the best understanding of holography.

**Theorem** (Weinberg–Witten)**.** The graviton (spin 2) cannot be obtained as a composite of two lower spin fields.

However, in CFT the graviton is composite, but the dimension is different.

**History.** The D-branes in '95 were followed by the AdS/CFT correspondence in '97. Up to the 00s, there was a large amount of papers: the correspondence was extended to AdS 3, 6 and 7. For AdS 4 it happened in 2007, while AdS 2 took ten more years.

**Application.** The duality can be used in two ways. The first way is study the CFT in $d$ dimensions and obtain results for quantum gravity in AdS in $d + 1$ dimensions. One may use standard quantum mechanics to learn about quantum gravity (this is one of the reason why unitarity should hold). The second way is use gravity in $\mathrm{AdS}_5$ to learn about CFT: use general relativity (which can be put on a computer) to learn about (particular) strongly coupled systems.

The AdS/CFT correspondence touches many areas of physics: quantum gravity, condensed matter theory (AdS/CMT, in '10), AdS/QCD for quark-gluon plasma, integrability (to get the spectrum of operators exactly in terms of the coupling without supersymmetry), formal QFT, quantum information theory, number theory, etc.

**Statement.** The statement of AdS/CFT is that

$$\mathcal{H}^{\mathrm{QG}} = \mathcal{H}^{\mathrm{CFT}}, \quad Z^{\mathrm{QG}}[\text{b.c.}] = Z^{\mathrm{CFT}}[J] = \int [\mathcal{D}\varphi]\, \mathrm{e}^{-S_{\mathrm{CFT}} - \int \mathrm{d}^4x\, J\varphi}$$

where $J$ is a source. The quantum integral path integral is

$$Z^{\mathrm{QG}}[\text{b.c.}] = \int_{\substack{\phi|_{\partial M} = J \\ g|_{\partial M} \sim \mathrm{AdS}}} [\mathcal{D}g]\, [\mathcal{D}\phi]\, \mathrm{e}^{-[S_{\mathrm{EM}} + S_{\mathrm{matter}}(\phi)]}$$

One only fixes the boundary conditions far away: inside there can be anything. This is a non-perturbative, background independent definition of quantum gravity (and string theory). The duality gives

$$\mathrm{AdS}_3 \times S^3 \iff N = 4 \text{ Super Yang–Mills}$$

[r]

**Roads to AdS/CFT.** There are two ways to learn AdS/CFT:

- the traditional way, old school way is top-down, from the UV to the IR. Starting from a brane system in string theory, one takes various decoupling limits and derive the duality; in this way one is guaranteed to have UV-completion. This requires string theory (perturbative, spectrum, D-branes, etc), supersymmetry (of QFT and string theory). In this way AdS/CFT becomes an advanced topic in string theory.

- the new school way is bottom-up approach, from the IR to the UV. Starting from gravitational EFT in IR, and not worrying too much about UV completion (which can be seen from the traditional way), one may see many results of AdS/CFT. The rules of UV completion are well defined and discussed later.

The second way is the one followed. Starting from the bulk EFT, one obtains the boundary CFT and through UV completion obtains the true CFT (see lecture notes by J. Kaplan, J. Penedones).

CFT has its own structure compared to QFT.

**Terminology.** The bulk is the inside of something, it is the AdS part. At the boundary lives the CFT.

## Lecture 9

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### 4.1 Conformal field theory

CFT was developed concurrently by many areas like condensed matter and statistical physics (transitions in quantum systems in low dimensions, $d = 1+1$, $d = 1+2$), standard quantum field theory (a system may have a fixed point of the RG flow, $d = 1+3$, $d = 1+2$), string theory (the theory on the world sheet of the string is a CFT, $d = 1+1$).

The mature way to understand QFT is as relevant deformations of CFT. By understanding the space of CFT, [r] one may deform it to some relevant operator [r]

$$S_{\mathrm{CFT}} \to S_{\mathrm{CFT}} + \lambda \int \mathrm{d}^{d_x} O_\Delta(x) \,, \quad \Delta < d$$

where $\Delta$ is the scaling dimension. This triggers an RG flow. [r] For massive fields, the correlation functions decay exponentially [r]. This makes no reference to Lagrangians.

Once one has reached a fixed point of the RG when the beta function vanish

$$\beta(\lambda_i) = 0$$

The system becomes scale invariant: the system behaves the same at all scales. There is no known example of a theory with Lorentz invariance, scaling invariance but not conformal invariance. In $d = 2$ there is a proof that Lorentz and scale invariance implies conformal invariance (idea for exam topic). In $d = 4$, with some reasonable assumptions, this is also true.

There are three kinds of transformations: scale, conformal and Weyl.

**Scale transformations.** Consider the set of transformations

$$x' = \lambda x \,, \quad O'_\Delta(x') = O'_\Delta(\lambda x) = \lambda^{-\Delta} O_\Delta(x)$$

where $O$ are operators. The dilation operator is

$$D = -\mathrm{i} x^\mu \, \partial_\mu$$

One may see that $O_\Delta(x)|_{x=0}$ diagonalize the dilation operator. In fact

$$\delta O = O'_\Delta(x) - O_\Delta(x) = (1+\delta\lambda)^{-\Delta} O_\Delta\left(\frac{x}{1+\delta\lambda}\right) - O_\Delta(x) \sim -\delta\lambda(\Delta + x^\mu \, \partial_\mu) O_\Delta(x)$$

By definition, one has

$$\delta O = \delta\lambda[\mathrm{i}\Delta, O_\Delta(x)]$$

[r] Therefore

$$[D, O_\Delta(0)] = \mathrm{i}\Delta O(0)$$

A QFT is scale invariant if the action is invariant under

$$\boxed{S[O_\Delta, x] = S[O'_\Delta, x']}$$

If there are any dimensional parameters in the Lagrangian, this is not true. In fact, consider the presence of

$$g\int \mathrm{d}^4x\, O_\Delta(x) \to g\lambda^{d-\Delta}\int \mathrm{d}^4x\, O_\Delta(x)$$

If $\Delta < d$ then $g$ is dimensional, while $D = d$ then $g$ is dimensionless. Since mass is dimensionful then a massive theory cannot be scale invariant.

Letting an operator be

$$O = \varphi^{n_1}\prod_{i=1}^s \partial_{\mu_i}\varphi^{n_s}$$

The dimension is

$$\Delta_0 = n\Delta\varphi + s\,, \quad n = n_s - n_1$$

[r]

There is another way to think about the transformation. One may take the QFT and couple it to the background metric (this is the same done when doing QFT in curved space time). One may undo the scale transformation by a diffeomorphism (a change of coordinates). When doing a change of coordinates, the metric also transforms

$$g_{\mu\nu} \to \lambda^2 g_{\mu\nu}$$

Therefore, a QFT is scale invariant if

$$S[O_\Delta, x, g_{\mu\nu}] = S[\lambda^{-\Delta}O_\Delta, \lambda x, g_{\mu\nu}] = S[\lambda^{-\Delta}O_\Delta, x, \lambda^2 g_{\mu\nu}]$$

The field still transforms in the other way, but the metric transforms: a theory like this is also scale invariant.

Infinitesimally, this condition is

$$0 = \frac{\delta S}{\delta g_{\mu\nu}}\,\delta g_{\mu\nu} + \frac{\delta S}{\delta\phi}\,\delta\phi$$

there is a small variation of the operator and the metric. On-shell, the second addendum is zero. The variation is

$$\lambda = 1 + \delta\lambda\,, \quad \delta g_{\mu\nu} = 2\delta\lambda\, g_{\mu\nu}\,, \quad \delta\phi = -\Delta\phi\,\delta\phi$$

Using these variations in the variation of the action, one has

$$0 = \frac{\delta S}{\delta g_{\mu\nu}}\,\delta g_{\mu\nu} = \int \mathrm{d}^4x\, T^{\mu\nu}g_{\mu\nu}2\delta\lambda = 0$$

where $T^{\mu\nu}$ is the stress-energy tensor. Since the trace appears, the integral has to be zero

$$\int \mathrm{d}^4x\, T^\mu{}_\mu = 0$$

In a quantum context, the expectation value of the stress-energy tensor is zero.

One must not that this is not a local condition, the integral over all space time is null. One would like a generalization of scale invariance that promotes the above condition to a local condition. This leads to conformal invariance.

**Weyl transformations.** Before studying conformal invariance, one shall look at Weyl invariance. The condition on the tensor is not local because Noether's trick is global. By rescaling the metric by a function depending on the space time points, one obtains Weyl invariance. A QFT is Weyl-invariant if

$$S[O_\Delta, x, g_{\mu\nu}] = S[\lambda^{-\Delta}(x)O_\Delta, x, \lambda^2(x)g_{\mu\nu}]$$

The matter part of the Lagrangian does not change on-shell, while the variation of the metric is

$$\delta g_{\mu\nu} = 2g_{\mu\nu}\,\delta\lambda(x)$$

The variation of the action is

$$0 = \frac{\delta S}{\delta g_{\mu\nu}}\,\delta g_{\mu\nu} = \int \mathrm{d}^4x\, T^{\mu\nu} g_{\mu\nu}\delta\lambda(x) \implies T_\mu{}^\mu = 0$$

Not all Weyl transformation can be undone by a change of coordinates. Any scalar quantity cannot be changed by a change of coordinates. Certain Weyl transformations can change the Ricci tensor.

One would like to know what is the set of diffeomorphism such that [r]. This leads to conformal transformations.

**Conformal transformations.** Consider an infinitesimal change of coordinate in flat space (with Euclidean signature)

$$x^\mu \to x^\mu + \varepsilon^\mu\,, \quad \delta_{\mu\nu} \to \delta_{\mu\nu} + \partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu$$

Conformal transformations are given by

$$\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu = 2\,\delta\lambda(x)\delta_{\mu\nu}$$

Intuitively, the group contains scaling and translations [r]. One may find that

$$\partial_\mu\varepsilon^\mu = \delta\lambda(x)\, d \implies \frac{2}{d}\partial_\alpha\varepsilon^\alpha\,\delta_{\mu\nu} = \partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu \tag{$\star$}$$

By applying $\partial_\beta\partial^\mu$ one obtains

$$\partial_\beta\partial_\mu\partial^\mu\varepsilon_\nu + \partial_\beta\partial_\nu\partial^\mu\varepsilon_\mu - \frac{2}{d}\partial_\beta\partial_\nu\partial_\mu\varepsilon^\mu = 0$$

Symmetrizing the expression in $\beta,\nu$ (and multiplying by $d$), one gets

$$(d-2)\partial_\beta\partial_\nu\partial_\mu\varepsilon^u + \frac{d}{2}\partial_\mu\partial^\mu(\partial_\beta\varepsilon_\nu + \partial_\nu\varepsilon_\beta) = 0$$

Taking the trace gives

$$\boxed{(d-1)\partial_\beta\partial^\beta\partial_\mu\varepsilon^\mu = 0}$$

This equation is trivial for $d=1$. In $d=1$ everything is a scale factor

$$\mathrm{d}s^2 = \mathrm{d}x^2 \to [f'(x)]^2\mathrm{d}f^2$$

Inserting ($\star$) into the equation above the boxed one, one gets

$$(d-2)\partial_\beta\partial_\nu\partial_\mu\varepsilon^u + \frac{d}{2}\partial_\mu\partial^\mu\left[\frac{2}{d}\partial_\alpha\varepsilon^\alpha\,\delta_{\mu\nu}\right] = 0 \implies \boxed{(d-2)\partial_\beta\partial_\nu\partial_\mu\varepsilon^u = 0}$$

the second addendum is the boxed equation. This equation is trivially solved in $d=2$. Therefore, $\mathrm{CFT}_2$ are very different from $\mathrm{CFT}_d$ in higher dimensions. The set of conformal transformations are much larger in two dimensions that do not exist in higher dimensions. These transformations imply that correlation functions have only one free parameter ? [r].

By applying $\partial_\alpha\partial_\beta$ on ($\star$) and use the above equation, one obtains

$$\partial_\alpha\partial_\beta\partial_\gamma\varepsilon_\delta + \partial_\alpha\partial_\beta\partial_\delta\varepsilon_\gamma = 0$$

Letting

$$T_{\alpha\beta\gamma\delta} = \partial_\alpha\partial_\beta\partial_\gamma\varepsilon_\delta$$

then this is anti-symmetric in the $3,4$ components and symmetric in the $1,2,3$ components. In fact

$$T_{\alpha\beta\gamma\delta} = T_{\alpha\gamma\beta\delta} = -T_{\alpha\gamma\delta\beta} = -T_{\alpha\delta\gamma\beta} = T_{\alpha\delta\beta\gamma} = T_{\alpha\beta\delta\gamma}$$

so it is symmetric in the last two indices, therefore it is zero. [r] Therefore

$$\partial_\alpha\partial_\beta\partial_\gamma\varepsilon_\delta = 0$$

This implies that $\varepsilon_\mu$ is at most quadratic in $x_\mu$. Translations are

$$\varepsilon^\mu = a^\mu$$

Rotations are

$$\varepsilon^\mu = \omega_\nu{}^\mu x^\nu$$

Dilations (scale) are

$$\varepsilon^\mu = \lambda x^\mu$$

Finally, special conformal transformations are

$$\varepsilon^\mu = b^\mu x^\nu x_\nu - 2x^\mu b_\nu x^\nu$$

Conformal transformations preserve angles.

Each transformation has an associated transformation:

$$P_\mu = -\mathrm{i}\,\partial_\mu\,, \quad M_{\mu\nu} = \mathrm{i}(x_\mu\partial_\nu - x_\nu\,\partial_\mu)\,, \quad D = -\mathrm{i}x^\mu\,\partial_\mu\,, \quad K_\mu = \mathrm{i}(x^2\,\partial_\mu - 2x_\mu x^\nu\,\partial_\nu)$$

These are the generators of $\mathrm{SO}(1,d+1)$. The vectors $\varepsilon^\mu$ are the conformal Killing vectors since they satisfy the conformal Killing equation. The Lie algebra generated by the vectors is given by

$$[D,K_\mu] = -\mathrm{i}K_\mu\,, \quad [D,P_\mu] = \mathrm{i}P_\mu\,, \quad [K_\mu,P_\nu] = 2\mathrm{i}(\delta_{\mu\nu}D - M_{\mu\nu})\,, \quad ???$$

The other commutators are zero. The Poincaré algebra can be extended to the conformal algebra [r].

Conformal invariance implies Weyl invariance. A change of the metric produces a change of the action

$$\delta S \sim \frac{2}{d}\int \mathrm{d}^d x\,T^{\mu\nu}(\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu) = \frac{2}{d}\int \mathrm{d}^d x\,T_{\mu\nu}\delta_{\mu\nu}\,\partial_\rho\varepsilon^\rho$$

There is no arbitrary variation from which one can state that the integral of the stress-energy tensor is zero. One must follow a more general argument. The variation of the action must be of the form

$$\delta S \sim \int \mathrm{d}^d x\,\frac{2}{d}L^{\alpha\beta}\,\partial_\alpha\partial_\beta\partial_\rho\varepsilon^\rho$$

for some $L^{\alpha\beta}$. This means that the theory is conformal invariance. The trace of the stress-energy tensor is not necessarily zero, but by integrating twice by parts one has

$$T_\mu{}^\mu = \partial_\alpha\partial_\beta L^{\alpha\beta}$$

One may write an improved stress-energy tensor

$$\begin{aligned} T^T_{\mu\nu} = T_{\mu\nu} &+ \frac{1}{d-2}(\partial_\mu\partial_\rho L^\rho{}_\nu + \partial_\nu\partial_\rho L^\rho{}_\mu - \partial_\rho\partial^\rho L_{\mu\nu} - \delta_{\mu\nu}\,\partial_\rho\partial_\sigma L^{\rho\sigma}) \\ &+ \frac{1}{(d-1)(d-2)}(\partial_\mu\partial_\nu L^\sigma{}_\sigma - \delta_{\mu\nu}\,\partial_\sigma\partial^\sigma L_\rho{}^\rho) \end{aligned}$$

The stress-tensor is symmetric, conserved, generates the same currents, but has zero trace.

Conformal invariance implies $T = 0$ and Weyl invariance is equivalent to $T = 0$. [r] Scale does not imply Weyl. Conformal implies Weyl only for $T = 0$ [r].

## Lecture 10

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[r] The special conformal transformation has a finite form

$$x^\mu \to \frac{x^\mu - b^\mu x^2}{1 - 2b^\mu x_\mu + b^2x^2}$$

The inversion map is

$$I : x^\mu \to -\frac{x^\mu}{x^2}$$

Special conformal transformations can be obtained through the inversion

$$IP_\mu I^{-1}\,,\quad I = I^{-1}$$

One may generate the whole conformal group through [r] and inversion. The inversion cannot be found infinitesimally because the inversion does not have an infinitesimal: it is not continuously connected to the identity.

The special conformal transformation is indeed found as

$$x^\mu \to \frac{x^\mu + b^\mu x^2}{x^2}$$

and invert it

$$\frac{x^\mu + b^\mu x^2}{x^2}\left[\frac{x^\mu + b^\mu x^2}{x^2}\frac{x^\mu + b^\mu x^2}{x^2}\right]^{-1} = \cdots = \frac{x^\mu + b^\mu x^2}{1 + 2x^\mu b_\mu + b^2 x^2}$$

## 4.2 Generic-point function

[r]

**One-point function.** The one-point functions can always be set to zero. From $\langle O\rangle$ one may define

$$O - \langle O\rangle I$$

**Two-point function.** A two-point function is

$$\langle O_{\Delta_1}(x_1)O_{\Delta_2}(x_2)\rangle = f(x_1, x_2) = f(x_1 - x_2) = f(x_{12}^2) = \frac{a}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}} = \frac{a}{(x_{12})^{2\Delta_1}}\delta_{\Delta_1,\Delta_2}$$

The second equality is consequence of translational invariance, the third is a consequence of rotational invariance, the fourth is a consequence of scale invariance, the last is the consequence of special conformal invariance. Without loss of generality one may set $a = 1$

$$\langle O_{\Delta_1}(x_1)O_{\Delta_2}(x_2)\rangle = \frac{\delta_{\Delta_1,\Delta_2}}{(x_{12})^{2\Delta_1}}$$

This holds for scalar operators.

For spin operators, one may build new objects [r]

$$\delta_{\mu_1\mu_2}\,,\quad \frac{x_{\mu_1}x_{\mu_2}}{x^2}$$

It is still true that $\Delta_1 = \Delta_2$ and the same spin [r]. Therefore

$$\left\langle O_{\mu_1\cdots\mu_s}O_{\mu_1\cdots\mu_{s'}}\right\rangle \propto \delta_{\Delta_1,\Delta_2}, \delta_{s_1,s_2}$$

[r] For the stress-energy tensor one has

$$\langle T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2)\rangle = \frac{c_\mathrm{T}}{x_{12}^{2d}}\left[\frac{1}{2}(I_{\mu\sigma}(x_{12})I_{\nu\rho}(x_{12}) + I_{\mu\rho}(x_{12})I_{\nu\sigma}(x_{12}) - \frac{1}{d}\delta_{\mu\nu}\delta_{\rho\sigma}\right]$$

where

$$I_{\mu\nu}(x^\mu) = \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}$$

The normalization is given by the coefficient $c_\mathrm{T}$ and cannot be set to one due to the definition of the stress-energy tensor as the variation of the matter Lagrangian. [r]

**Three-point function.** The three-point functions are completely fixed by symmetry. Conformal transformations are the set of transformations that map three points to any three other points.

One may always map three points to 0, 1 and $\infty$. Consider three points $x_1$, $x_2$, $x_3$. One may being with a translation by $-x_1$ to have $x_1 = 0$. One applies a special conformal transformation so that $b$ gives $x_3 \to \infty$. The last point can be rotated on the $x$-axis and scaled to 1.

The correlation function is a number expressed in terms of the above three fixed points, not a function. One may obtain the general form by undoing the transformations. Therefore

$$\langle O_i(x_1)O_j(x_2)O_k(x_3)\rangle = \frac{C_{ijk}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{13}^{\Delta_1-\Delta_2+\Delta_3}x_{23}^{-\Delta_1+\Delta_2+\Delta_3}}$$

where the OPE coefficient is

$$C_{ijk} = \langle O_i(0)O_j(1)O_k(\infty)\rangle$$

This holds for scalar operators: one may fixed up to one number. For spin operators there are a finite number of parameters.

CFTs are completely specified by the list of OPEs and scaling dimensions of the operators. One may think about scaling dimensions as [r] fields and the OPEs as couplings. This analogy is not perfect.

## 4.3 Radial quantization and state-operator correspondence

One shall study the representations of the conformal group.

**Representation theory.** [r] Representation are labelled by their spin [r]. Consider scalar operators

$$[D, M_{\mu\nu}] = 0\,, \quad [K_\mu, P_\nu] = 2\mathrm{i}D\delta_{\mu\nu}\,, \quad [-\mathrm{i}D, K_\mu] = -K_\mu\,, \quad [-\mathrm{i}D, P_\mu] = P_\mu$$

This is the lowering and raising algebra. There are $d$ independent copies of the raising and lowering algebra. From the lowest possible state, one builds the representation. If $\Delta$ is bounded from below, one goes to the minimum $\Delta_{\text{min}}$. Then

$$[K_\mu, O_{\Delta_{\text{min}}}(?)] = 0$$

An operator satisfying the above is a primary operator. The rest of the representation is built using $P_\mu$. The existence of the lower bound is guaranteed by unitarity. [r]

$$[D, O_\Delta(0)] = -\mathrm{i}\Delta O_\Delta(0)$$

One may build the descendent operator

$$O_{\Delta,n} \equiv P_{\mu_1}\cdots P_{\mu_n}O_\Delta(0)\,, \quad \Delta_{O_{\Delta,n}} = \Delta + n$$

[r] All the dynamics sits in the primary operators.

For an operator not at the origin, one has

$$O_\Delta(x) \to \mathrm{e}^{\mathrm{i}Px} \to \text{linear combination of all descendent operators}$$

**Time evolution.** Usually, time evolution is done with a Hamiltonian

$$H = \int \mathrm{d}^{d-1}x\, T_{tt}$$

The spectrum of the Hamiltonian on a flat space time is continuous. [r] A clever trick is evolve by dilatation

$$U(\tau) = \mathrm{e}^{\mathrm{i}D\tau}$$

The eigenbase of the dilation operator is made of scaling dimension operators [r] at the origin. One has

$$D\,|0\rangle = 0\,, \quad DO_\Delta(0)\,|0\rangle = [D, O_\Delta(0)]\,|0\rangle = -\mathrm{i}\Delta O_\Delta(0)\,|0\rangle$$

One defines

$$\boxed{|\Delta\rangle \equiv O_\Delta(0)\,|0\rangle}$$

Doing the same thing for operators, one has

$$K_\mu O_\Delta(0)\,|0\rangle = [K_\mu, O_\Delta(0)]\,|0\rangle$$

If $K_\mu\,|\Delta\rangle = 0$ then $|\Delta\rangle$ is a primary state.

Time evolution is easy on the states $|\Delta\rangle$ because

$$U(\tau)\,|\Delta\rangle = \mathrm{e}^{\mathrm{i}D\tau}\,|\Delta\rangle = \mathrm{e}^{-\Delta\tau}\,|\Delta\rangle$$

This looks like a Euclidean time evolution with energy $D$. Notice that one has a singularity at $\tau = -\infty$. This is because there is an operator at $\tau = -\infty$ to make the state in the first place.

One may interpret $\tau$ as a conformal transformation. One may do two change of coordinates: radial and exponential

$$x^\mu = rn^\mu = \mathrm{e}^\tau n^\mu$$

The line element is

$$\mathrm{d}s^2 = \mathrm{d}x_\mu\,\mathrm{d}x^\mu = \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2_{d-1} = \mathrm{e}^{2\tau}(\mathrm{d}\tau^2 + \mathrm{d}\Omega^2_{d-1})$$

The coefficient can be removed with a conformal transformation

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 + \mathrm{d}\Omega^2_{d-1}$$

Using this transformation one obtains

$$O'_\Delta(\tau) = \mathrm{e}^{\Delta\tau} O_\Delta(r)$$

The ation of the dilation operator on this is

$$\begin{aligned}
[-\mathrm{i}D, O'_\Delta(\tau)]\mathrm{e}^{-\Delta\tau} &= [-\mathrm{i}D, O_\Delta(r)] = 8r\,\partial_r + \Delta)O_\Delta(r) = (\partial_\tau + \Delta)[O'_\Delta(\tau)\mathrm{e}^{-\Delta\tau}] \\
&= [\partial_\tau O'_\Delta(\tau)]\mathrm{e}^{-\Delta\tau}
\end{aligned}$$

The dilation operator time translates [r]. Therefore

$$\boxed{O'_\Delta(\tau) = \partial_\tau O'_\Delta(\tau)}$$

There is a geometric intuition of this. For the left-hand side, consider concentric circles in the plane. The dilation operator maps one circle into another. For the right-hand side, in two dimensions, the metric is a cylinder: the axis is the time coordinate and circles are on the surface. [r] One may go from the plane to the cylinder through the exponential map.

For primary states one acted with $O_\Delta(0)\,|0\rangle$. [r] The divergence is coming form the insertion of an operator in the infinite past.

Until now, one has constructed ket states. To construct the bra one has to define complex conjugation

$$[O_\Delta(x)]^\dagger = IO_\Delta(x)I^{-1} = O'_\Delta(x) = |x|^{-2\Delta}O_\Delta(-{}^{x^\mu}\!/_{x^2})$$

Therefore

$$O_\Delta(0)^\dagger = \lim_{x\to 0} O_\Delta(x)^\dagger = \lim_{x'\to\infty} |x'|^{2\Delta} O_\Delta(x')$$

So

$$|\Delta\rangle \to \langle\Delta| = \langle 0|\,O_\Delta(0)^\dagger = \lim_{x\to\infty}|x|^{2\Delta}\,\langle 0|\,O(x)$$

One may obtain

$$\langle\Delta|\Delta\rangle = \langle 0|\,O_\Delta(0)^\dagger O_\Delta(x)\,|0\rangle = \lim_{x\to\infty}|x|^{2\Delta}\langle O(x)O(0)\rangle = \lim_{x\to\infty}|x|^{2\Delta}\frac{1}{|x|^{2\Delta}} = 1$$

**State-operator correspondence.** Every local operator located at the origin can be associated to a state [r] The opposite is also true. Consider a linear combination of operators [r]. The opposite is [r] The nature of the singularity describes what kind of operators have to be inserted. There is a bijective map between states and operators. The Hamiltonian has discrete spectrum and produces the local operators.

Consider a set of operators strictly inside the unit disk. This produces a normalizable state. This is the same as a linear combination of operators. What happens outside the disk is the same but for the bra state. Any correlation function can be reduced as the correct combination of two-point functions. These functions are constrained by symmetry. Therefore, one only needs to find the [r] coefficients. This is the OPE, the operator product extension.

## Lecture 11

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A generic $n$-point function can be written as bra states (outside unit disk) and ket states (inside the unit disk) and may be rewritten as a linear combination of operators inserted at the origin

$$\sum_i C_i \widetilde{C}_i$$

## 4.4 Unitarity bounds

The unitarity bounds are related to the existence of a minimum scaling dimension $\Delta$. [r] One has

$$P_\mu^\dagger = I P_\mu I^{-1} = K_\mu$$

The norm on some state is

$$\begin{aligned}\|P_\mu |\Delta\rangle\|^2 &= \langle\Delta| K_\mu P_\mu |\Delta\rangle = \langle\Delta| [K_\mu, P_\mu] + K_\mu, P_\mu |\Delta\rangle = \langle\Delta| [K_\mu, P_\mu] |\Delta\rangle \\ &= \langle\Delta| 2\mathrm{i}D |\Delta\rangle = 2\Delta \langle\Delta|\Delta\rangle = 2\Delta \geq 0\end{aligned}$$

recalling

$$K_\mu |\Delta\rangle = 0$$

Unitarity implies that norms are positive. Therefore, the scaling dimension has a lower bound. It must also be a real number.

The bound may be refined for scalars

$$\Delta \geq \frac{d-2}{2}$$

which can be seen from taking the norm of $P_{\mu_1} P_{\mu_2} |\Delta\rangle$. The bound is the scaling dimension of a free scalar field. For spin-ful fields one has

$$\Delta(s) \geq d - 2 + s$$

where $s$ is the spin. One may recall the scaling dimensions

$$\dim \psi = \frac{3}{2}\,, \quad \dim F_{\mu\nu} = 2\,, \quad \dim T_{\mu\nu} = d$$

The bound is saturated by conserved currents. For example, in CFT, one may prove that

$$\partial_\mu J^\mu = 0 \iff \dim J = d - 1$$

## 4.5 Operator product expansion

The correspondence between states and operator computed at the origin can be formalized. Consider two states in the unit disk and one state outside. [r] Schematically, this state overlaps with the other two.

$$O_3 \sim \langle\Delta_3|\Delta_1 \times \Delta_2\rangle \neq 0$$

Therefore

$$O_1 \times O_2 \subset O_3$$

[r] reverse sign
All operators are kind of included in the product of two operators. In CFT, it holds

$$O_i(x_1)O_j(x_2) = \sum_k O_k(x_1)\frac{C_{ijk}}{|x_{12}|^{\Delta_i-\Delta_j-\Delta_k}}$$

One may Taylor expand [r] and obtain all the operators of the CFT. This is a convergent expression (as opposed to the one divergent in QFT). In fact, there cannot be a finite radius of convergence because it sets a scale. [r]

If one may compute any correlation function, the QFT is solved. The above enables one to solve the CFT. Consider an $n$-point function

$$\begin{aligned}\langle O_1O_2\cdots O_n\rangle &= \sum_i A_i\langle O_iO_3\cdots O_n\rangle = \sum_{ij} B_{ij}\langle O_jO_4\cdots O_n\rangle = \cdots \\ &= \sum_{i_1,\cdots,i_{n-2}} Z\big\langle O_{i_{n-3}}O_n\big\rangle = \sum \Omega\end{aligned}$$

One may apply the OPE to two operators, then again, then again. Any correlation function can be reduced to numbers. One needs to know all the scaling dimensions $\Delta$, the coefficients $C_{ijk}$ and the positions $x^i$. There is no more information in the $n$-point functions than the two- and three-point functions.

If one knows the scaling dimensions $\Delta_i$ and spin $s_i$ of every operator, the OPE coefficients $C_{ijk}$ then the CFT is solved. Due to conformal invariance, the dynamical data can be repacked in the above three. This gives a classification of CFT. From this the CFT can be deformed by a relevant operator from which one obtains a QFT.

## 4.6 Conformal bootstrap

Consider the four-point function. Conformal symmetry can be used to map three points to 0, 1 and $\infty$. Through conformal transformations the four points can be put on a bidimensional plane. The four-point function

$$\langle O(x_1)O(x_2)O(x_3)O(x_4)\rangle = \frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}}f(u,v)$$

where $u$ and $v$ are conformal ratios

$$u = \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}\,, \quad v = \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}$$

The four-point function is fixed up to a function [r]. One may use the OPE to obtain

$$\langle OOOO\rangle = \frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}}\sum_i |C_{OOOi}|^2\left[\frac{x_{12}x_{34}}{x_{13}x_{24}}\right]^{\Delta_i}$$

When doing the OPE, one has to sum over all the operators of the CFT. This means to sum over primaries and descendants. The correlation function of these are fixed by the primaries. It would be nice to repackage the descendants in [r]

$$\langle OOOO\rangle = \frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}}\sum_{\text{primary}} |C_{OOOi}|^2 g_{O_i}(u,v)$$

where $g(u,v)$ is the conformal block and it is a complicated function: it resums all the descendants. The conformal blocks are fixed by conformal symmetry. They are known functions (and in closed form in even dimensions).

[r] The OPE may applied to 14 and 23

$$\langle OOOO\rangle = \frac{1}{|x_{14}|^{2\Delta}|x_{23}|^{2\Delta}} = \sum_i |C_{OOOi}|^2 g_{O_i}(v,u)$$

Confronting this with the previous, one gets

$$\sum |C_{OOOi}|^2 g_{O_i}(u,v) = \left[\frac{u}{v}\right]^{2\Delta} \sum_i |C_{OOOi}|^2 g_{O_i}(v,u)$$

This condition needs to be satisfied. This is called crossing symmetry (similar to the scattering amplitude).

The conformal bootstrap tries to solve CFT solely based on this. The scaling dimensions need to be finely tuned for the equation to hold [r]. This was done successfully for two-dimensional CFT where the expression has an algebraic formula. The above formula can be implemented numerically since there are squares. One may get bounds for the OPE coefficients. This is the numerical conformal bootstrap. The most famous result to date is the 3D Ising model which is equivalent to $\lambda\varphi^4$ in $d = 2 + 1$. The CFT living at the Wilson–Fisher fixed point of the RG flow is the 3D Ising model [r].

**Implications.** One may study the OPE limit $u \to 0$ (equivalent to $x_{12} \to 0$ and $x_{34} \to 0$. One has

$$\frac{1}{u^{2\Delta}} \sum |C|^2 g(u,v) = \sum |C|^2 g_O(v,u) \implies \frac{1}{u^{2\Delta}} \sim \sum_i |C_{OOOi}|^2 \log u$$

where $g(u,v) \to 1$ and $g_O(v,u) \sim \log u$. The left-hand side diverges. On the right-hand side, it is not clear how 1 and 2 are coming together. The RHS diverges logarithmically which is slower than a power law. Therefore, the sum must be infinite. This implies that every CFT has an infinite number of primary operators.

# 5 Two-dimensional CFT

Two important CFT beyond two dimensions are the 3D Ising model and the $d = 4$ Super-Yang–Mills [r].

The conformal Killing equation is

$$\frac{2}{d}\,\partial_\alpha \varepsilon^\alpha \delta_{\mu\nu} = \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu$$

Differentiating by $\partial^\nu$, one has

$$\begin{aligned}
\frac{2}{d}\,\partial_\mu\, \partial_\alpha \varepsilon^\alpha \delta_{\mu\nu} &= \partial_\mu \partial_\nu \varepsilon_\nu + \partial_\nu \partial_\nu \varepsilon_\mu \\
\left(\frac{2}{d} - 1\right) \partial_\mu \partial_\alpha \varepsilon^\alpha &= \partial_\nu \partial^\nu \varepsilon_\mu \\
\partial_\nu \partial^\nu \varepsilon_\mu &= 0
\end{aligned}$$

This is Laplace equation (since this is two dimensional Euclidean space). The solutions are holomorphic functions of $z$ and $\bar{z}$. One obtains the Cauchy–Riemann equations

$$\partial_x \varepsilon^x = \partial_y \varepsilon^y\,, \quad \partial_x \varepsilon^y = -\partial_y \varepsilon^x$$

These may be rewritten as

$$\partial_{\bar{z}} \varepsilon = 0\,, \quad \partial_z \bar{\varepsilon} = 0$$

where $\varepsilon = \varepsilon_x + \mathrm{i}\varepsilon_y$. Under the Weyl transformation

$$z \to f(z)\,, \quad \bar{\bar{f}}(\bar{z})$$

the metric becomes

$$\mathrm{d}z\,\mathrm{d}\bar{z} \to |\partial_z f|^2\, \mathrm{d}z\,\mathrm{d}\bar{z}$$

The absolute value is a conformal factor [r].

This does not imply that Weyl transformations are conformal transformations. Doing an infinitesimal transformation $z \to z + \varepsilon(z)$, holomorphic functions can be expanded in Laurent series

$$\varepsilon(z) = -\sum_n a_n z^{n-1}$$

and the generators are

$$l_n = -z^{n+1}\,\partial_z$$

that satisfy the Witt algebra

$$[l_m, l_n] = (m-n) l_{m+n}$$

Not all generators are globally defined.

## Lecture 12

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The generators with $n < -1$ diverge at the origin. The ones with $n > 1$ diverge at infinity. Let

$$z = \frac{1}{w}\,, \quad \partial_z w = -\frac{1}{z^2} = -w^2$$

The generators become

$$l_n \to -w^{-(n+1)}\,\partial_z w\,\partial_w = w^{-(n-1)}\,\partial_w$$

For $n > 1$ there is a pole at $w = 0$.

The non problematic generators form a closed subalgebra of the Witt algebra

$$[l_1, l_{-1}] = 2l_0\,, \quad [l_0, l_{-1}] = l_{-1}\,, \quad [l_0, l_1] = -l_1$$

This is the algebra of the group $\mathrm{SL}(2,\mathbb{C})$ which is the global conformal group. It acts as

$$z \to \frac{az+b}{cz+d}\,, \quad ad - bc = 1$$

[r] contains a subgroup

$$\mathrm{SO}(3,1) \subset \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$$

[r]

To extend this algebra one shall use the Jacobi identity. There is one single thing one may add

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m,-n}$$

The second addendum is called central extension. This is the most general expression that satisfies the Jacobi identity. The parameter $C$ is the central charge. For $n = \pm 1, 0$ the extended part does not modify the global part [r].

## 5.1 Representation of the global conformal group

A primary operator is an operator that transforms as

$$O(z,\bar{z}) \to (\partial_z f)^h (\partial_{\bar{z}} \bar{f})^{\bar{h}} O(f(z), \bar{f}(\bar{z}))$$

[r] This already includes spin. One may consider

$$z \to \lambda z\,, \quad \bar{z} \to \lambda \bar{z}\,, \quad O \to \lambda^{h+\bar{h}} O(\lambda z, \lambda \bar{z})$$

The scaling dimension of the operator is $\Delta_O = h + \bar{h}$. These transformations are generated by the dilation operator

$$O = -z\,\partial_z - \bar{z}\,\partial_{\bar{z}} = l_0 + \bar{l}_0$$

On the plane, rotations are given by

$$R = \mathrm{i}(y\,\partial_x - x\,\partial_y) = -z\,\partial_z + \bar{z}\,\partial_{\bar{z}} = l_0 - \bar{l}_0$$

The quantum number of the spin is $s = h - \bar{h}$. In complex coordinates, a rotation is

$$z \to \lambda z\,, \quad \bar{z} \to \lambda^{-1}\bar{z}$$

This leaves the norm invariant. Under this transformation, one has

$$O(z, \bar{z}) \to \lambda^{h-\bar{h}} O(\lambda z, \lambda^{-1}\bar{z})$$

so one indeed has spin.

The structure discussed for any dimension applies also in two dimensions. A two-point function is

$$\langle O_1(z_1, \bar{z}_1) O_2(z_2, \bar{z}_2)\rangle = \frac{1}{z_{12}^{2h}\bar{z}_{12}^{2\bar{h}}}\delta_{h_1 h_2}\delta_{\bar{h}_1, \bar{h}_2}$$

The exponent $2h$ is called holomorphic weight and $2\bar{h}$ is the anti-holomorphic weight.

For the quantum states one may use radial quantization [r]

$$O_{h,\bar{h}}(0,0)\,|0\rangle$$

If $O$ is a primary state, then $|\psi\rangle$ is primary. By acting with derivative and its complex conjugate, one obtains the descendants.

There is an important distinction with the higher dimension case, in particular regarding the symmetry genertors $l_n$ with $|n| > 1$. In two dimensions, there are two notions of primaries:

- quasi-primaries, they have the same conditions as the ones in higher dimension,
- primaries (also called Virasoro primaries), they are subject to a stronger condition.

All primaries are quasi-primaries but not the other way around.

## 5.2 Conserved currents in radial quantization

In two dimensions, an operator with $s = \Delta$ impies $h = 0$ or $\bar{h} = 0$. The operator depends only on one of the complex coordinates either $z$ or $\bar{z}$. The equation of a conserved current is

$$\partial_\mu j^\mu = 0 \implies \partial_\mu \bar{j}^\mu + \bar{\partial}_\mu j^\mu = 0$$

Let

$$h = \frac{\Delta + s}{2}$$

Since the current is a vector, then $s = 1$. The weightsa re

$$h(j) = \frac{\Delta+1}{2}\,, \quad \bar{h}(j) = \frac{\Delta-1}{2}\,, \quad h(\bar{j}) = \frac{\Delta-1}{2}\,, \quad \bar{h}(\bar{j}) = \frac{\Delta+1}{2}$$

The two-point functions are

$$\langle j(z,\bar{z})j(0,0)| = \frac{1}{z^2}\frac{1}{|z|^{2(\Delta-1)}}\,, \quad \langle \bar{j}(z,\bar{z})\bar{j}(0,0)\rangle = \frac{1}{\bar{z}^2}\frac{1}{|z|^{2(\Delta-1)}}\,, \quad \langle j(z,\bar{z})\bar{j}(0,0)\rangle = 0$$

The crossed two-point function vanishes because the weights are different. From the correlation equation one may extrapolate the scaling dimension

$$0 = \big\langle(\partial\bar{j} + \bar{\partial}j)(z,\bar{z})j(0,0)\big\rangle = \bar{\partial}\langle j(z,\bar{z})j(0,0)\rangle = -\frac{\Delta-1}{2}\langle j(z,\bar{z})j(0,0)\rangle$$

This means that

$$\Delta = 1 = s$$

A direct consequence of this is

$$\bar{\partial}\langle j(z,\bar{z})j(0,0)\rangle = 0 \implies \bar{\partial}j(z,\bar{z}) = 0 \implies j(z,\bar{z}) \to j(z)$$

Similarly

$$\partial\bar{j} = 0$$

The current $j$ is only a function of $z$. The conservation equation implies the above. Not only the sum of currents is zero, but also each current. This is true for any spin $s$. The conservation equation always splits into two equations.

One may use again the cylinder through the exponential map

$$z = \mathrm{e}^{\tau + \mathrm{i}\varepsilon}\,, \quad \mathbb{C} \to \mathbb{R} \times S^1$$

Having a compact space means discrete spectrum and no IR divergence. One a cylinder, the conserved charge is

$$Q = \frac{1}{2\pi} \int \mathrm{d}\varphi\, j^\tau = \frac{1}{2\pi} \int \varepsilon_{\mu\nu}\, \mathrm{d}x^\mu\, j^\nu$$

In plane coordinates, one has

$$Q = \frac{1}{2\pi\mathrm{i}} \left[ \oint \mathrm{d}z\, j(z) - \oint \mathrm{d}\bar{z}\, \bar{j}(\bar{z}) \right] = Q + \bar{Q}$$

where one has contour integral. A circle on the cylinder is circle on the plane. The integral can be calculated with the residue theorem. Knowing that there are two separate conservation equation, there are two conserved charges

$$Q = \frac{1}{2\pi\mathrm{i}} \oint \mathrm{d}z\, j(z)\,, \quad \bar{Q} = -\frac{1}{2\pi\mathrm{i}} \oint \mathrm{d}\bar{z}\, \bar{j}(\bar{z})$$

Since $\bar{\partial} j = 0$, then considering another function

$$j(z) \to \xi(z) j(z)$$

it is conserved too. For any conserved current, one may define a set of charges

$$Q_\xi = \frac{1}{2\pi\mathrm{i}} \oint \mathrm{d}z\, j(z)\xi(z)$$

and similarly for the anti-holomorphic current. The conserved charge of a state can be calculated with

$$Q_\xi \left| h, \bar{h} \right\rangle = \frac{1}{2\pi\mathrm{i}} \oint \mathrm{d}z\, \xi(z) j(z) O_{h,\bar{h}}(z, \bar{z}) \left| 0 \right\rangle$$

then applying the OPE. One shall take the correct pole in the OPE expansion. A natura basis is to take

$$\xi_n = z^{h+n-1}$$

One may define the modes of the conserved current

$$Q_n = \frac{1}{2\pi\mathrm{i}} \oint \mathrm{d}z\, j(z) z^{h+n-1}$$

The scaling dimension is $h_{Q_n} = -n$. Any operator on the plane can be Laurent-expanded

$$j(z) = \sum_n \frac{j_n}{z^{n+h}}$$

[r] inserting this in the above, one obtains

$$\begin{aligned} Q_n &= \frac{1}{2\pi\mathrm{i}} \oint \mathrm{d}z\, j_h(z) z^{n+h-1} = \frac{1}{2\pi\mathrm{i}} \oint \mathrm{d}z\, \sum_k \frac{j_k}{z^{k+h}} z^{n+k-1} \\ &= \frac{1}{2\pi\mathrm{i}} \oint \mathrm{d}z\, \sum_k \frac{j_k}{z^{1+k-n}} = \sum_k j_k \delta_{k,n} = j_n \end{aligned}$$

The coefficients of Laurent expansion of the current are the conserved charges.

On the cylinder one has a nice interpretation

$$O_{\mathrm{cyl}}(\tau + \mathrm{i}\varphi) = (\partial_{\tau+\mathrm{i}\varphi})^h O(z) = z^h O(z) = \sum_n \frac{O_n}{z^n} = \sum_n O_n \mathrm{e}^{-n(\tau+\mathrm{i}\varphi)}$$

This is the Fourier decomposition on the cylinder. There is only $\tau + \mathrm{i}\varphi$ which correspond to $z$ and not $\bar{z}$. These are like right-moving modes in Minkowski space [r]. Fourier decomposition on the cylinder maps to Laurent expansion on the plane. This is true for any spin $h$.

## 5.3 Stress-energy tensor

The stress-energy tensor is symmetric and traceless

$$T_{z\bar{z}} = T_{\bar{z}z} = 0$$

There are only two components left

$$T(z) \equiv T_{zz}\,, \quad \bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}$$

The conservation equation gives

$$\bar{\partial} T = 0\,, \quad \partial \bar{T} = 0$$

There is also the set of infinite conserved quantities

$$L_\varepsilon = \frac{1}{2\pi\mathrm{i}} \oint \mathrm{d}z\, T(z)\varepsilon(z)$$

which generate the infinitesimal conformal transformations

$$z \to z + \varepsilon(z)$$

A natural basis for the stress-energy tensor is the modes

$$L_n = \frac{1}{2\pi\mathrm{i}} \oint \mathrm{d}z\, T(z) z^{n+1}$$

Dilation correspond to

$$z \to z + \varepsilon z\,, \quad \bar{z} \to \bar{z} + \varepsilon \bar{z}$$

This is generated to the mode $n = 0$. Therefore

$$D = L_0 + \bar{L}_0$$

The rotations are generated by

$$J = L_0 - \bar{L}_0$$

One may find $K_\mu$ and $P_\mu$ through the linear combinations $L_{\pm 1}$ and $\bar{L}_{\pm 1}$. The set

$$\{L_0, L_{-1}, L_1\} + \text{c.c.}$$

form the global conformal group. The local conformal transformations are generated by

$$L_n\,, \quad |n| > 1$$

The Hamiltonian on the cylinder is the dilation operator

$$H_{\text{cyl}} \sim D = L_0 + \bar{L}_0$$

There would be an equality if the stress-energy tensor is a primary operator, but in fact is a quasi-primary operator. The true equation is

$$H_{\text{cyl}} = L_0 + \bar{L}_0 - \frac{C}{12}$$

The shift can be interpreted as a Casimir energy: the vacuum state has non-zero energy.

[r] One consequence of this is that the correlation functions of the stress-energy tensor are fixed.

## Lecture 13

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### 5.4 Commutators, contour integrals and OPE

One would like to understand commutators of operators [r]. The commutators are equal-time commutators. To pick the time ordering, one uses the i$\varepsilon$ formalism: one computes results with $\varepsilon$ written explicitly and then orderly taking the limit $\varepsilon \to 0$. In this case, the radius [r] is expanded

$$[A(z), B(z)] = \lim_{\delta\to 0^+} [A(z\mathrm{e}^{\delta})B(z) - A(z\mathrm{e}^{-\delta})B(z)]$$

In Euclidean there is no operator ordering: the operators are fields [r]. Consider the following commutator

$$\begin{aligned}\left[\oint \mathrm{d}z\, A(z), B(w)\right] &= \lim_{\delta\to 0} \oint_{C+\delta} \mathrm{d}z\, A(z)B(w) - \oint_{C-\delta} \mathrm{d}z\, A(z)B(w) \\ &= \oint_w \mathrm{d}z\, A(z)B(w)\end{aligned}$$

One finds the difference between two integrals on slightly different contours. The negative integral is equivalent to a positive integral with contour in the opposite direction. In the limit, the contours cancel each other apart from a little box around the operator. The last integral can be computed through the residue theorem [r].

**Conserved current.** Consider an operator with some charge. Under a phase rotation, it transforms as

$$O_q(z) \to \mathrm{e}^{\mathrm{i}q\alpha} O_q(z)$$

Infinitesimally, this is

$$\delta_\alpha O_q(z) = \mathrm{i}\alpha[Q, O_q(z)] = \mathrm{i}\alpha q O_q(z)$$

This can also be computed with a contour integral

$$qO_q(w) = \frac{1}{2\pi\mathrm{i}} \oint_w \mathrm{d}z\, j(?)O(w) = \text{simple pole of } [j(z)O(w)]$$

Therefore

$$q\frac{O_q(w)}{z-w} \subset j(z)O_q(w)$$

This is just one term. One may use the tower of infinite charges to involve other operators

$$O'_q(z) = \mathrm{e}^{\mathrm{i}q\alpha(z)} O_q(z)$$

By doing an expansion, one has

$$\delta\alpha_n = \mathrm{i}\alpha_n[Q_n, O_q(z)] = \mathrm{i}\alpha_n z^n q O_q(z)$$

One inserts the definition of $Q_n$ as a contour integral. Rewriting

$$z^n = [w + (z-w)]^n = \sum_{i=0}^n \binom{n}{i} w^{n-i}(z-w)^i$$

the commutator is

$$\begin{aligned}[Q_n, O_q(w)] &= \frac{1}{2\pi\mathrm{i}} \oint_w \mathrm{d}z\, z^n j(z)O_q(w) = \frac{1}{2\pi\mathrm{i}} \oint \mathrm{d}z\, \sum_{k=0}^n \binom{n}{k} w^{n-k}(z-w)^k j(z)O_q(w) \\ &= \sum_{k=0}^n \binom{n}{k} w^{n-k} \mathrm{Pole}_{k+1}[j(z)O_q(w)]\end{aligned}$$

The only operator than can appear is $O_q$ and by conformal symmetry, only the simple pole $z-w$ can appear

$$j(z)O_q(w) \sim \frac{qO_q(w)}{z-w}$$

Consider the OPE $j(z)j(w)$. In free field theory, the conserved current measures charge but does not carry any charge. Therefore

$$j(z)j(w) \sim 0$$

but this means that the correlation function is zero

$$\langle j(z)j(w)\rangle = 0$$

Due to local state correspondence, one $j$ can be sent to zero, the other to infinity and their product must be non-zero. There is a state with zero norm and this is unfavorable. [r] One may write a term with the right scaling dimension

$$j(z)j(w) \sim \frac{k}{(z-w)^2}$$

The charge is

$$[Q, j(w)] = \int \mathrm{d}z\, j(z)j(w) = 0$$

since the OPE is a quadratic pole. This still gives finite norm to states but does not mess with [r]. The two-point function is then

$$\langle j(z)j(w)\rangle = \frac{k}{(z-w)^2}$$

The $k$ in the numerator is needed since one has normalized $j$ rather than the two-point function.

The commutators of currents should be zero since the current $j$ generates the abelian U(1) group. However, one has

$$\begin{aligned}
[j_m, j_n] &= -\frac{1}{4\pi^2}\left[\oint \mathrm{d}z\, j(z)z^m, \oint \mathrm{d}w\, j(w)w^n\right] \\
&= -\frac{1}{4\pi^2}\oint_0 \mathrm{d}w\, w^n \oint_w \mathrm{d}z\, z^m j(z)j(w)z\, z^m j(z)j(w) \\
&= -\frac{1}{4\pi^2}\oint_0 \mathrm{d}w\, w^n \oint_w \mathrm{d}z\, z^m \frac{k}{(z-w)^2} \\
&= -\frac{1}{4\pi^2}\oint_0 \mathrm{d}w\, w^n \oint_w \mathrm{d}z\, \sum_{i=0}^{m}\binom{m}{i} w^{m-i}(z-w)^i \frac{k}{(z-w)^2} \\
&= \frac{1}{2\pi\mathrm{i}}\oint_0 \mathrm{d}w\, w^n w^{m-1} mk = mk\,\delta_{m+n,0}\,, \quad k>0
\end{aligned}$$

This infinite extension of U(1) is called central extension; the only non-trivial result happens for $m=-n$. The coefficient $k$ measures how much the algebra is extended. In this quantum theory the operators do not commute [r]. This is a conformal anomaly.

For non-abelian groups, one has

$$J^a(z)J^b(w) \sim \mathrm{i}f^{abc}\frac{J^c}{z-w} + \frac{k\delta^{a,b}}{(z-w)^2}$$

**Stress-energy tensor.** Consider a conformal transformation

$$O_h(z) \to (\partial_z w)^h O_h(w(z))\,, \quad z \to w = z + \varepsilon(z)$$

The variation is

$$\delta_\varepsilon O_h(z) = h\,\partial\varepsilon\, O_h(z) + \varepsilon(z)\,\partial O_h(z)$$

The OPE is

$$T(z)O(w) \sim h\frac{O(w)}{(z-w)^2} + \frac{\partial O(w)}{z-w}$$

where the 2 comes from the scaling dimension of the stress-energy tensor [r]. Similarly

$$T(z)T(w) \sim 2\frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

The two-point function is

$$\langle T(z)T(w)\rangle = 0$$

because there is a one-point function that gives zero [r]. One needs to put another term

$$T(z)T(w) \sim \frac{1}{2}\frac{c}{(z-w)^4} + 2\frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

From this, the two-point function is

$$\langle T(z)T(w)\rangle = \frac{1}{2}\frac{c}{(z-w)^4} \implies c > 0$$

This is the central extension. Therefore the Virasoro algebra is

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

This algebra is the central extension of the Witt algebra generated by

$$l_m = -z^m\,\partial_z$$

The parameter $c$ is called central charge.

The stress-energy tensor changes after inserting the central charge because the tensor is not a primary operator. It transforms as

$$T(z) = (\partial_z w)^2 T(w) + \frac{c}{12}S(w,z)\,, \quad S(w,z) = \frac{\partial_z w\,\partial_z^3 w - \frac{3}{2}(\partial_z^2 w)^2}{(\partial_z w)^2}$$

The last term is the Schwarzian derivative. If $w$ is a conformal transformation

$$w(z) = \frac{az+b}{cz+d}$$

then the Schwarzian is zero $S(w,z) = 0$. Under global conformal transformations the stress-energy tensor transforms as expected, but for local transformations, it has an added piece. Operators that transform like this are called quasi-primary.

Considering the exponential map

$$w = \mathrm{e}^z$$

from the cylinder to the plane, one has

$$T_{\text{cyl}} = z^2 T(z) - \frac{c}{24}\,, \quad S = -\frac{1}{2}$$

The Hamiltonian on the cylinder is

$$H_{\text{cyl}} = L_0 + \bar{L}_0 - \frac{c}{12}$$

one accounts for the shift due to the movement from the cylinder to the plane. The ground state has a non-zero energy but

$$E(|0\rangle) = -\frac{c}{12}$$

This is some Casimir energy that due to quantum effects has a negative vacuum energy.

## 5.5 Virasoro representation

Recall

$$O_h(0)\,|0\rangle = |h\rangle$$

Using the Virasoro algebra, one has

$$L_0 L_n = (L_n L_0 - nL_n)\,|h\rangle = (h-n)L_n\,|h\rangle$$

One concludes that $L_n$ acts as a lowering operator. There is an infinite number of lowering operators.

The vacuum state is the state annihilated by all the symmetry generators

$$L_n\,|0\rangle = 0\,, \quad \forall n$$

However, there is no such state due to the Virasoro algebra. In fact

$$[L_n, L_{-n}]\,|0\rangle = \frac{c}{12}(n^3-n)\,|0\rangle$$

This is zero for $n = \pm 1, 0$. The demand is relaxed to impose that only the lowering operators annihilate the vacuum

$$L_n\,|0\rangle = 0\,, \quad \forall n \geq -1$$

One may define a primary state as a state

$$L_n\,|0\rangle = 0\,, \quad \forall n \geq 1$$

The most general state on the Hilbert space is

$$\cdots (L_{-N})^{n_N} \cdots (L_{-2})^{n_2}(L_{-1})^{n_1}\,|h\rangle$$

The primary states have $n_i = 0$ for all $i$, while descendants have $n_i \neq 0$ for at least one $i$. [r] Note that if $|h\rangle = 0$ then $n_i = 0$.

There is a state that corresponds to the stress-energy tensor

$$|T\rangle = T(0)\,|0\rangle = \lim_{z\to 0} T(z)\,|0\rangle = \lim_{z\to 0} \sum_{n\leq -2} \frac{L_n}{z^{n+2}}\,|0\rangle = L_{-2}\,|0\rangle$$

The tensor admits a Laurent expansion in terms of the Virasoro modes [r]. From this one sees that the stress-energy tensor is a descendant. For $n \leq -3$ one has derivatives of the stress-energy tensor.

By conformal symmetry and by crossing, a CFT has an infinite number of quasi-primaries. Many CFTs have a finite number of vectors $|h\rangle$ in their Hilbert spaces. These CFTs can be solved algebraically.

**Number of ? operators.** Let $h_n = n + h$. One would like to know how many states are present at $h_n$. This does not depend on $h$ since the towers are built [r]. Let the number be $P(n)$. At each level, one counts the numbers of partitions

$$P(0) = 1\,, \quad P(1) = 1\,, \quad P(2) = 2\,, \quad P(3) = 3\,, \quad P(4) = 5$$

The generating function of this sequence is

$$F(q) = \sum_{n=0}^{\infty} P(n) q^n = \left[\prod_{i=1}^{\infty}(1-q^i)\right]^{-1}$$

This is related to Dedekinds' $\eta$ function.

Let $q = \mathrm{e}^{-\beta}$, the thermal partition function of the CFT is

$$\boxed{Z(\beta) = q^{-\frac{c}{24}}\left[F(q)(1-q) + \sum_{h>0} q^h F(q)\right]}$$

The generating function $F(q)$ is also called characters (as in abstract algebra) or conformal block.

## Lecture 14

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### 5.6 Example: free boson theories in two dimensions

The free boson action is

$$S = \frac{g}{2} \int \mathrm{d}^2 z \, \partial X \, \bar{\partial} X$$

The equations of motion are

$$DX = 0 \iff \partial \bar{\partial} X = 0$$

The field solution to the equations can be split

$$X(z, \bar{z}) = X(z) + \bar{X}(\bar{z})$$

These are the left- and right-moving waves in two dimension. This splitting implies a conserved current. A shifted field is a solution again

$$X(z) \to X(z) + f(z)$$

There is a conserved current that enforces this symmetry

$$j(z) = \mathrm{i} \, \partial X(z)$$

and the anti-holomorphic counter part. In two dimensions, the scaling dimension of a conserved current is $\Delta = d - 1$. Since $h_j = 1$ then $X$ has dimension zero. The correlation function is logarithmic

$$\langle X(z, \bar{z}) X(0, 0) \rangle = -\frac{1}{4\pi g} \ln|z|^2$$

This means that $X$ is not a good operator in the theory, but the current $\partial X$ is. [r] Knowing that the OPE of the conserved current is

$$j(z) j(0) \sim \frac{k}{z^2}$$

Then, comparing this with the above, one has

$$k = \frac{1}{4\pi g}$$

Typically, the convention is $k = 1$. This theory has also a stress-energy tensor

$$T(z) = -2\pi g \colon \partial X \, \partial X \colon$$

The normal ordering operator is the same as in QFT [r]. The two-point correlation function is

$$\langle T(z) T(0) \rangle = \frac{1}{2} \frac{1}{z^4} \implies c = 1$$

The central charge is just unity.

**Primary operators.** The current is a primary operator

$$j(z) \,, \quad h_j = 1$$

while $X(z, \bar{z})$ is not. The theory has an infinite number of Virasoro primaries called vertex operators (from string theory)

$$V(z, \bar{z}) = \colon \mathrm{e}^{\mathrm{i}\alpha X(z, \bar{z})} \colon , \quad \alpha \in \mathbb{R}$$

There is a continuous spectrum of primary operators. The quantum numbers of the operators are

$$h_\alpha = \frac{\alpha^2}{8\pi g} \,, \quad q_\alpha = \frac{\alpha}{4\pi g}$$

Knowing the OPE and the Wick contractions, one may computing anything. With respect to higher dimensions, in this free field theory, the current appears but the boson field $X$ cannot be used.

**String theory.** In string theory, on the world sheet of a propagating string lives a two-dimensional CFT. The propagation of the string in space-time is described by boson fields. The coordinates $z$ and $\bar{z}$ of the field $X(z,\bar{z})$ are coordinates of the world sheet. The coordinates of space-time (called target space) are the boson fields $X^I(z,\bar{z})$.

**Discrete boson theories.** The presence of a continuous spectrum of vertex operator is because one has taken an infinite string that propagates in space-time. To get a finite spectrum, one has to periodically identify space-time. This means identify the fields

$$X(z,\bar{z}) \sim X(z,\bar{z}) + 2\pi R m\,, \quad m \in \mathbb{Z}$$

This theory is called compact free boson theory. It has a discrete spectrum. [r]

The vertex operators are

$$V_{n,m}(z,\bar{z}) = :\exp\left[\mathrm{i}\left(\frac{n}{R}+\frac{mR}{2}\right)X(z)\right]\exp\left[\mathrm{i}\left(\frac{n}{R}-\frac{mR}{2}\right)\bar{X}(\bar{z})\right]:\,, \quad n,m \in \mathbb{Z}$$

with quantum numbers of

$$h = \frac{1}{2}q^2\,, \quad \bar{h} = \frac{1}{2}\bar{q}^2\,, \quad q = \frac{n}{r}+\frac{mR}{2}\,, \quad \bar{q} = \frac{n}{r}-\frac{mR}{2}$$

[r] When compactifying space on a circle, the momentum is quantized in units of the reciprocal of the radius. The same above: the number $n$ is related to the momentum mode units. The number $m$ is quantized in terms of the radius, it is a winding mode, how many times one wraps around the circle. The rules for computing correlations functions is similar to before: the operators get restricted to the vertex operators. This CFT satisfy many properties and it is completely solved.

If one chooses $R$ in a particular way, then there can be a symmetry $n \leftrightarrow m$: this is $T$ duality. Also, one may get $\bar{h} = 0$ or $h = 0$.

An example of computation is the following. The two-point correlation function is

$$\langle V_{n,m}(z_1,\bar{z}_1)V_{n,m}(z_2,\bar{z}_2)\rangle = \frac{1}{z_{12}^{h_{n,m}}\bar{z}_{12}^{\bar{h}_{n,m}}}$$

The three-point correlation function is

$$\langle V_{n_1,m_1}(z_1,\bar{z}_1)V_{n_2,m_2}(z_2,\bar{z}_2)V_{n_3,m_3}(z_3,\bar{z}_3)\rangle = \frac{C(n_j,m_j)}{z_{12}^{h_{n_1,m_1}+h_{n_2,m_2}-h_{n_3,m_3}}\cdots}$$

Since these operators have charges, then the sum of the operators must be zero $q_1+q_2+q_3=0$: this is charge conservation.

## 5.7 Example: Virasoro minimal models

There exists a complete classification of two-dimensional CFTs that are unitary and with central charge $0<c<1$. These are the Virasoro minimal models. To find them, it is enough to know representation theory and unitary. From the expression of the general state in the Hilbert space

$$|\psi\rangle = (L_{-n})^{n_n}\cdots(L_{-1})^{n_1}|n\rangle$$

A state that is annihilated by a combination of $L_{-i}$ is called short representation: it is killed by the generators of the characters. An example is the vacuum $|h\rangle = |0\rangle$:

$$L_{-1}|0\rangle = 0$$

**Level two.** At level two, the state present are

$$L_{-2}|h\rangle\,, \quad (L_{-1})^2|h\rangle$$

It is possible that

$$[aL_2 + b(L_{-1})^2]|h\rangle = 0$$

To check this, one looks at the matrix of inner products

$$M_2(c,h)=\begin{bmatrix}\langle h|\,L_2L_{-2}\,|h\rangle & \langle h|\,L_1^2L_{-2}\,|h\rangle\\ \langle h|\,L_2(L_{-1})^2\,|h\rangle & \langle h|\,L_1^2L_{-1}^2\,|h\rangle\end{bmatrix}=\begin{bmatrix}4h+\frac{c}{2} & 6h\\ 6h & 4h+8h^2\end{bmatrix}$$

and searches for a null eigenvalue. This guarantees the presence of null linear combination. [r] One may use the Viraroso algebra to compute the elements.

The determinant can be computed as

$$\det M_2(c,h)=32[h-h_{1,1}(c)][h-h_{1,2}(c)][h-h_{3,1}(c)]\,,\quad h_{p,q}=\frac{[(m+1)p-mq]^2-1}{4m(m+1)}\,,\quad m=\frac{1}{2}\pm\sqrt{\frac{25-c}{1-c}}$$

In this case, one has

$$h_{1,1}=0\,,\quad h_{1,2},h_{2,1}=\frac{5-c}{16}\pm\frac{1}{16}\sqrt{(1-c)(25-c)}$$

[r] Since $h_{1,1}$ is null, then there is null state if $h=0$ as can be seen from the first parenthesis in the determinant: this is the vacuum. If

$$h_{1,2}<h<h_{2,1}$$

then the determinant is negative and unitary is violated. Therefore, this range is not allowed.

There are two special values, $c=1$ and $c=25$. There are three regimes $[0,1]$, $[1,25)$ and $[25,\infty)$. In the second case, $h_{1,2}$ and $h_{2,1}$ are imaginary and there are no constraints. In the third case, $h_{1,2}$ and $h_{2,1}$ are both negative, but the determinant is positive: there are no constraints. In the first case, there is an interesting constraint. In the $(c,h)$ plane one has a sideways parabola pointing left with the minimum on $c=1$. Everything inside the parabola is excluded.

**Arbitrary level.** At level $N$, there is Kac's determinant

$$\det M_N(c,h)=\alpha_N\prod_{\substack{pq\leq N\\ p,q>0}}(h-h_{p,q})^{P(N-pq)}$$

where $\alpha_N$ are known positive coefficients. The determinant is a polynomial [r]. In the level $N$, the determinant presents all the terms of the previous level plus others. In the $(c,h)$ plane, one obtains several parabolas stacked one on the other. If the central charge $c$ is too small, then there are no unitary representations left. The only possible place of unitary representations are the intersection of the boundaries [r] see notes. Uniting all the constraints, there are no problems if

$$c=1-\frac{6}{m(m+1)}\,,\quad m\in\mathbb{N}\,,\quad m\geq 3$$

The representations are given by $h_{p,q}$ with $1\leq q\leq p\leq m-1$. All of this is a consequence of unitarity and is very constraining.

There are two more steps:

- in a given theory, one has to understand which representations actually appear and with what degeneracy; to this end one has to study modular invariance;
- one has only discussed the holomorphic part $h$; in general a representation is $|h\rangle\otimes|\bar{h}\rangle$ and one has to get a consistent theory from this tensor product; sometimes there is more than one way to combine $h$ and $\bar{h}$; one has to use modular invariance for this as well.

**Ising model.** Many of these CFTs can be obtained from the continuum limit of critical lattice models. In particular, the first Viraroso minimal model corresponds to the Ising model. It has three representations:

$$(h,\bar{h})=\begin{cases}(0,0)\\ \left(\frac{1}{16},\frac{1}{16}\right)\\ \left(\frac{1}{2},\frac{1}{2}\right)\end{cases}$$

The first is the vacuum corresponding to the identity operator, the second is the $\sigma$ operator with critical exponent $\frac{1}{16}$, the last is the energy operator $\varepsilon$.

There is a finite number of Virasoro primary operators which makes computing correlation functions easy to compute. Recall that in any consistent CFT, there must be an infinite number of global (or quasi-) primary operators. In this case, one is treating Virasoro primary operators: the global operators reorganize in three Virasoro primaries.

[r] The theory has a $\mathbb{Z}_2$ symmetry. The operator $\varepsilon$ is even, while $\sigma$ is odd. Therefore the OPE coefficients are

$$C_{\varepsilon\varepsilon\sigma} = C_{\sigma\sigma\sigma} = C_{11\varepsilon} = C_{11\sigma} = C_{\varepsilon\sigma 1} = 0$$

The third and the fourth are zero due to one-point correlation functions being zero. The last is zero by conformal symmetry. Similarly

$$C_{\varepsilon\varepsilon 1} = C_{\sigma\sigma 1} = 1$$

One also has

$$C_{\varepsilon\sigma\sigma} = \frac{1}{2}\,, \quad C_{\varepsilon\varepsilon\varepsilon} = 0$$

The second is a consequence of a non-invertible symmetry, an exotic symmetry in quantum systems. From these, one can compute any correlation function: the theory is solved.

[r] with degeneracy 1.

## Lecture 15

### 5.8 Two-dimensional CFT on the torus

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One is interested in thermal physics and the thermal partition function

$$Z(\beta) = \mathrm{Tr}_{\mathcal{H}}\, \mathrm{e}^{-\beta H} = \sum_E \rho(E)\mathrm{e}^{-\beta E}$$

The energy eigenstates on the circle corresponds to local operators on the plane. [r] A more general partition function is

$$Z(\beta, \Omega) = \mathrm{Tr}_{\mathcal{H}}\, \mathrm{e}^{-\beta H + \mathrm{i}\Omega J}$$

This is the grand-canonical partition function where $J$ is the angular momentum and $\Omega$ is a chemical potential for angular momentum. The chemical potential fixes the expectation values of the associated operator. The Hamiltonian is

$$H = L_0 + \bar{L}_0 - \frac{c}{12}\,, \quad J = L_0 + \bar{L}_0$$

One introduces the following notation

$$\beta = 2\pi\tau_2\,, \quad \Omega = 2\pi\tau_1$$

so that

$$\begin{aligned} Z(\tau_1, \tau_2) &= \mathrm{Tr}_{\mathcal{H}}\, \mathrm{e}^{-2\pi\tau_2(L_0+\bar{L}_0-\frac{c}{12})}\mathrm{e}^{2\pi\tau_1(L_0-\bar{L}_0)} = \mathrm{Tr}\, \mathrm{e}^{2\pi\mathrm{i}(\tau_1+\mathrm{i}\tau_2)(L_0-\frac{c}{24})}\mathrm{e}^{-2\pi\mathrm{i}(\tau_1+\mathrm{i}\tau_2)(\bar{L}_0-\frac{c}{24})} \\ Z(\tau, \bar{\tau}) &= \mathrm{Tr}\, q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}} \end{aligned}$$

where

$$q = \mathrm{e}^{2\pi\mathrm{i}\tau}\,, \quad \tau = \tau_1 + \mathrm{i}\tau_2$$

The partition function can be rewritten using characters where one sums over only Virasoro primary states

$$Z(\tau, \bar{\tau}) = (q\bar{q})^{-\frac{c}{24}} \sum_{h,\bar{h}} \chi_h(q)\chi_{\bar{h}}(\bar{q})\,, \quad \chi_h(q) = q^h \prod_{n\geq 1} \frac{1}{1-q^n}\,, \quad h > 0\,, \quad c > 1$$

The character $h = 0$ of the vacuum is special

$$\chi_0(q) = q^0 \prod_{n\geq 2} \frac{1}{1-q^n}$$

The character computes how many ways one can build [r]

One may write the thermal partition function as a path integral with compactified (read repeating) Euclidean time. One starts from a cylinder and glues the two ends together: one gets a torus. A two-dimensional CFT of finite size (on a spatial circle) at finite temperature lives on a torus. This was done for the typical partition function. In the case of a grand-canonical partition function, the exponential of $J$ is a rotation: one does some about of Euclidean evolution with $\beta$, then rotate by inserting the exponential and then glues the ends by taking the traces. This Configuration is a twisted cylinder. In general

$$\operatorname{Tr} e^{-\beta H + i\Omega J} \iff \text{twisted torus}$$

**Torus.** A torus can be thought of as a lattice on $\mathbb{R}^2$. Consider the generators of the lattice being $\mathbf{v}_1$ and $\mathbf{v}_2$. One may periodically identify the lattice cells. Not all choices of basis vectors lead to different tori. For example

$$\mathbf{v}_1' = \mathbf{v}_1 + \mathbf{v}_2 \,, \quad \mathbf{v}_2' = \mathbf{v}_2$$

Their lattice is the same as before. Only shift by integers give the same tori. Instead of two vectors, one may use two complex numbers, $\alpha_1$ and $\alpha_2$. If one hadn't known about the lattice identification, one would have said that the space of tori is $\mathbb{C}^2$. If one considers a complex vector $\alpha$, one may consider a matrix to describe change of basis. For example, the two matrices

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \,, \quad U = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

give the same lattice $T\alpha$ and $U\alpha$. Letting

$$S = UT^{-1}U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Then $S$ and $T$ generate the modular group $\mathrm{SL}(2, \mathbb{Z})$ [r]. Its elements are square matrices with whole integer entries and unit determinant. The unit determinant preserves the area of the torus [r]. Two complex numbers include too much information. The only information needed is just

$$\tau = \frac{\alpha_1}{\alpha_2}$$

Its transformation is

$$\tau \to \frac{\alpha_1'}{\alpha_2'} = \frac{a\alpha_1 + b\alpha_2}{c\alpha_1 + d\alpha_2} = \frac{a\tau + b}{c\tau + d}$$

The matrices transform as

$$T : \tau \to \tau + 1 \,, \quad S : \tau \to -\frac{1}{\tau}$$

CFT must satisfy modular invariance

$$\boxed{Z(\tau, \bar{\tau}) = Z(\tau', \bar{\tau}')}$$

where $\tau'$ is any modular image of $\tau$. This is similar to the crossing equation. One can use this equation in the bootstrap to constrain the spectrum of CFTs. The parameter $\tau$ is called modular parameter of the torus.

The partition function can be written as a sum of characters

$$Z(\tau, \bar{\tau}) = (q\bar{q})^{-\frac{c}{24}} \sum_{h, \bar{h}} \chi_h(q) \chi_{\bar{h}}(\bar{q})$$

The characters do not transform trivially under the modular group $\mathrm{SL}(2, \mathbb{Z})$.

Consider the transformation given by the matrix $T$ under the modular group

$$T : \tau \to \tau + 1$$

In the character, the $q^n$ term presents no problem, but $q^h$ requires attention:

$$T : \chi\chi \to \mathrm{e}^{2\pi\mathrm{i}(h-\bar{h})}\chi_h(q)\chi_{\bar{h}}(\bar{q})$$

To maintain the partition function invariant, one must set

$$h - \bar{h} = s \in \mathbb{Z}$$

This is spin quantization.

The transformation given by the $S$ matrix is more complicated

$$\chi_{h_0}(q_S) = \sum_h S_{hh_0}\chi_h(q)$$

The coefficients need to be tune in a particular way. In the minimal models, the characters are more complicated since there are null states [r]. This is very constraining.

The invariance under the modular group [r] enables one to study a region of the complex plane. Consider the $\tau$ complex plane in the region $\operatorname{Im}\tau > 0$ so that the exponential decays. For the transformation $T$, one can restrict oneself to a unit band $[-\frac{1}{2}, \frac{1}{2}]$. For the transformation $S$, one can stay outside the unit circle. This particular region is called fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$. All the other regions can be reached by applying $S$ and $T$. The particular region chosen is useful since it is centered around $\operatorname{Re}\tau = 0$ and it avoids low $\tau$ which corresponds to high temperatures (high energies) knowing that

$$2\pi J_n \tau = \beta$$

This is used in string theory to have processes ultraviolet-finite.

**Cardy formula.** Consider the $S$ transformation

$$S : \tau \to -\frac{1}{\tau}$$

The argument of the partition function transforms as

$$Z(\tau,\bar{\tau}) = Z(-{}^1\!/\!_\tau, -{}^1\!/\!_{\bar{\tau}}) \iff Z(\beta) = Z({}^{4\pi^2}\!/\!_\beta)$$

This means that high-temperature physics is equivalent to low-temperature physics which is well understood.

Consider the partition function in the limit $\tau, \bar{\tau} \to 0$ after the $S$ transformation

$$Z(\tau,\bar{\tau}) = Z(-{}^1\!/\!_\tau, -{}^1\!/\!_{\bar{\tau}}) = \operatorname{Tr}_{\mathcal{H}} \mathrm{e}^{-\frac{2\pi\mathrm{i}}{\tau}(L_0 - \frac{c}{24})}\mathrm{e}^{\frac{2\pi\mathrm{i}}{\bar{\tau}}(\bar{L}_0 - \frac{c}{24})} \approx \mathrm{e}^{\frac{2\pi\mathrm{i}c}{24\tau}}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}c}{24\bar{\tau}}}$$

Letting $\beta = 2\pi\operatorname{Im}\tau$, one has

$$Z(\beta) \sim \mathrm{e}^{\frac{c}{12}\frac{4\pi^2}{\beta}}, \quad \beta \to 0$$

Knowing that

$$S = (1 - \beta\,\partial_\beta)\ln Z$$

one obtains Cardy's formula

$$\boxed{S = c\frac{2\pi^2}{3}T}$$

The thermal entropy of a CFT at high temperature is universal and depends only on the central charge.

This formula is valid asymptotically in all CFT as $T \to \infty$.

**Remark.** This expression has corrections. The sub-leading term is

$$\mathrm{e}^{-4\pi^2 T\Delta_1}$$

where $\Delta_1$ is the smallest non-trivial operator in the theory. Cardy's formula requires a finite gap in the spectrum and one can always find a temperature high enough that the above correction is small.

This formula is used on the counting of the microstates of black holes. In general in AdS/CFT, one may calculate the Hawking entropy and it matches the above.

One is also interested in the density of state, a microcanonical relation

$$\langle E\rangle = \left\langle L_0 - \frac{c}{24}\right\rangle = \frac{1}{2\pi \mathrm{i}}\,\partial_\tau \ln Z = \frac{c}{6}\pi^2 T^2$$

Viewing $\langle E|$ as the energy of the microcanonical ensemble [r] one can invert the relation to get

$$S(h,\bar{h}) = 2\pi\sqrt{\frac{c}{6}\left(h - \frac{c}{24}\right)} + 2\pi\sqrt{\frac{c}{6}\left(\bar{h} - \frac{c}{24}\right)}$$

This is the number of states which is also the number of local operator understood in the asymptotic limit. Modula invariance can provide much other information.

In the numerical bootstrap, one can study the difference

$$Z(\tau,\bar{\tau}) - Z(\tau^{-1},\bar{\tau}^{-1})$$

[r] In all of two-dimensional CFTs there must be a primary operator with scaling dimesion

$$\Delta < \frac{c}{8.7}$$

[r] One cannot make a gap [r] too big, otherwise modular invariance is not respected.

**Higher dimensions.** In higher dimensions there is no modular invariance. The [r] function is

$$S^1 \times S^{d-1}$$

One way to think about modular invariance is switching Euclidean time and space. The $S^{d-1}$ is a sphere and not a cycle: one cannot exchange space cycles and time cycles.

### Lecture 16

gio 09 mag 2024 14:30

## 6 Anti-de Sitter space

AdS space is a solution to Einstein's field equation with negative cosmological constant

$$S = \frac{1}{16\pi G}\int \mathrm{d}^D x\,\sqrt{|g|}(R - 2\Lambda)\,, \quad \Lambda < 0$$

It is convenient to redefine the cosmological constant as

$$\Lambda = -\frac{(D-1)(D-2)}{2l_{\mathrm{AdS}}^2}$$

AdS can be understood as an hypersurface embedded into $\mathbb{R}^{D-1,2}$ where it has two time directions

$$\mathrm{d}s^2 = -\mathrm{d}X_{-1}^2 - \mathrm{d}X_0^2 + \sum_{i=1}^{D-1} \mathrm{d}X_i^2$$

All points on the hypersurface satisfy

$$-X_{-1}^2 - X_0^2 + \sum_{i=1}^{D-1} X_i^2 = -l_{\mathrm{AdS}}^2$$

It is frequent to set $l_{\mathrm{AdS}} = 1$. This equation can be recognized as an hyperbola. The time coordinates have spherical symmetry. The hypersurface is a hyperboloid. Moving along the timelike coordinates, one sees that there are closed loops, this leads to ill-defined CFTs. Physicists formally treat the universal cover of AdS and not AdS itself: time is unwrapped.

**Symmetries.** The symmetry group is $\mathrm{SO}(d-1,2)$. This is the Lorentzian conformal group in $d = D-1$ dimensions. This is the first hint that symmetries in $d$ dimensions are the isometries of $D-1$ dimensional AdS [r] where $d$ is the dimension of space-time while $D$ is the dimension of the AdS. Space-time has

$$\frac{d(d+1)}{2}$$

Killing vectors, so it is a maximally symmetric space. It also has constant negative curvature.

## 6.1 Coordinate systems

One has to solve the equation for the points on the hypersurface.

**Global AdS.** The coordinate patch is

$$\begin{aligned} X_{-1} &= l_{\mathrm{AdS}} \cosh\rho \sin t \\ X_0 &= l_{\mathrm{AdS}} \cosh\rho \cos t \\ X_i &= l_{\mathrm{AdS}} \sinh\rho\, \Omega_i \end{aligned}$$

where $\Omega_i$ is the $d-1$-sphere.

One can obtain the induced metric on the hypersurface

$$g_{\mu\nu} = G_{\alpha\beta} \frac{\mathrm{d}^\alpha X}{\mathrm{d}x^\mu} \frac{\mathrm{d}X^\beta}{\mathrm{d}x^\nu}$$

where $G$ and $X$ are the $\mathbb{R}^{d-1,2}$ metric and coordinates, while $x$ are the AdS coordinates. One finds

$$\frac{\mathrm{d}s^2}{l_{\mathrm{AdS}}^2} = -\cosh^2\rho\,\mathrm{d}t^2 - \mathrm{d}\rho^2 + \sinh^2\rho\,\mathrm{d}\Omega_{d-1}^2$$

One may also write the metric by changing coordinates $r = l_{\mathrm{AdS}} \sinh\rho$ to have

$$\mathrm{d}s^2 = -\left[\frac{r^2}{l_{\mathrm{AdS}}^2} + 1\right] \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\frac{r^2}{l_{\mathrm{AdS}}^2} + 1} r^2\,\mathrm{d}\Omega_{d-1}^2$$

**Causal structure and Penrose diagram.** One finds a compact coordinate system [r] and ignores the conformal factor. Let

$$\tan\theta = \sinh\rho \implies 0 \le \theta \le \frac{\pi}{2}$$

One has

$$\frac{\mathrm{d}s^2}{l_{\mathrm{AdS}}^2} = \frac{1}{\cos^2\theta} \left[-\mathrm{d}t^2 + \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\Omega_{d-1}^2\right]$$

The metric in the bracket is a time coordinate and a hemisphere

$$\mathbb{R}_t \times \frac{1}{2} S^{d-1}$$

One flattens the hemisphere into a circle and evolves it with time: a cylinder. In the Penrose diagram of Minkowski space, the diamond's boundary are null, so they have no timelike direction. In this case, the boundary has a time coordinate [r]. This cylinder is reminiscent of CFT.

**Poincaré AdS.** One may also obtain flat space [r]. The embedding coordinates are

$$\begin{aligned} X_{-1} &= l_{\mathrm{AdS}} \frac{t}{z} \\ X_0 &= \frac{z}{2}\left[1 + \frac{1}{z^2}(l_{\mathrm{AdS}}^2 - t^2 + \sum_{i=0}^{d-1} x_i^2)\right] \\ X_i &= l_{\mathrm{AdS}} \frac{x^i}{z}, \quad i = 1, \cdots, d-1 \\ X_d &= \frac{z}{2}\left[1 - \frac{1}{z^2}(l_{\mathrm{AdS}}^2 + t^2 - \sum_{i=1}^{d-1} x_i^2)\right] \end{aligned}$$

where $z \in [0, \infty)$. The metric is

$$\frac{\mathrm{d}s^2}{l_{\mathrm{AdS}}^2} = \frac{1}{z^2}\left[-\mathrm{d}t^2 + \mathrm{d}z^2 + \sum_{i=1}^{d-1} \mathrm{d}x_i^2\right]$$

This coordinate system covers a triangular cut of the cylinder that covers half of it. For $z = 0$ one has the boundary, while $z \to \infty$ is the Poincaré horizon [r]. The boundary metric is

$$\mathrm{d}s_{\mathrm{boundary}}^2 = -\mathrm{d}t^2 + \sum_{i=1}^{d-1} \mathrm{d}x_i^2$$

This is Minkowski metric. This is because the Penrose diagram of Minkowski is a diamond and the surface of the cylinder covered by the coordinates is a diamond glued to the cylinder. The above metric is not the only one possible, one can multiply by a conformal factor $\lambda^2$ and it would be also described by the cylinder. Therefore, the boundary is defined up to a conformal transformation.

**Rindler AdS.** In this space, constant time and radius are hyperboloids. For the previous two coordinate systems, they are spheres and planes. The metric is

$$\mathrm{d}s^2 = -\left[\frac{r^2}{l_{\mathrm{AdS}}^2} - 1\right]\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\frac{r^2}{l_{\mathrm{AdS}}^2} - 1} + r^2\,\mathrm{d}H_{d-1}^2\,, \quad \mathrm{d}H_{d-1}^2 = \mathrm{d}\iota^2 + \sinh^2\iota\,\mathrm{d}\Omega_{d-2}^2$$

This metric is similar to a black hole. There is a Rindler's horizon. This patch covers a quarter of the cylinder, similar flat space and Rindler space.

**Euclidean AdS.** One may rotate the time coordinate

$$X_0 \to \mathrm{i}X_0$$

which is equivalent to $t \to \mathrm{i}t_{\mathrm{E}}$. The symmetry group is $\mathrm{SO}(d+1,1)$. When one rotates to Euclidean, one recovers all the three above patches: they cover all Euclidean space up to a point or a line.

## 6.2 Fields on AdS

The action of a scalar field is

$$S = -\frac{1}{2}\int \mathrm{d}^{d-1}x\,\sqrt{|g|}(\partial_\mu\varphi\,\partial^\mu\varphi + m^2\varphi^2)\,, \quad (\nabla^2 - m^2)\varphi = 0$$

In the Poincaré metric, the action becomes

$$S = -\frac{1}{2}l_{\mathrm{AdS}}^{d-1}\int\prod_i \mathrm{d}x_i\,\mathrm{d}t\,\mathrm{d}z\,\frac{1}{z^{d-1}}\left[\eta^{\mu\nu}\,\partial_\mu\varphi\,\partial_\nu\varphi + \partial_z\varphi\,\partial_z\varphi + \frac{l_{\mathrm{AdS}}^2 m^2}{z^2}\varphi^2\right]$$

This is not really to a flat space QFT. One may redefine the field to have

$$\varphi_i = \phi z^{\frac{d}{2}}\,, \quad -\ln z = y$$

One has

$$\partial_z\varphi\,\partial_z\varphi = z^d\left[\left(\frac{d}{2}\right)^2\frac{\phi^2}{z^2} + \partial_z\phi\,\partial_z\phi + d\frac{1}{z}\phi\,\partial_z\phi\right]$$

The last term can be expresses as a four-derivative and becomes a boundary term in the action which can be ignored. The action is then

$$S = -\frac{1}{2}l_{\mathrm{AdS}}^{d-1}\int\prod_i \mathrm{d}x^i\,\mathrm{d}t\,\mathrm{d}y\,\left[\partial_y\phi\,\partial_y\phi + \mathrm{e}^{-2y}\eta^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi + \left(m^2 l_{\mathrm{AdS}}^2 + \frac{d^2}{4}\right)\phi^2\right]$$

[r] The mass is shifted. In flat space, if the potential is not bounded from below, the theory is ill-defined. In AdS the mass can be negative, but the term in the parenthesis has to be positive. The theory is stable if

$$m^2 l_{\mathrm{AdS}}^2 \geq -\frac{d^2}{4}$$

This is the Breitentohner–Friedmann (BF) bound.

Consider the wave equation

$$(\nabla^2 - m^2)\varphi = 0$$

[r] the field can be Fourier expanded in the $z$ direction

$$\varphi(z, t, x^i) = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x} - \mathrm{i}\omega t}\varphi_{k,\omega}(z)$$

From the wave equation, one then gets

$$z^{d+1}\,\partial_z\left[\frac{1}{z^{d-1}}\,\partial_z\varphi_{k,\omega}(z)\right] - \left[m^2 l_{\mathrm{AdS}}^2 + z^2(k^2 - \omega^2)\right]\varphi_{k,\omega}(z) = 0$$

This is Bessel's equation which defines Bessel's functions. These functions are well defined above the BF bound.

**Near-boundary solution.** Near the boundary of AdS, $z \to 0$, the solution is a power law [r]

$$\varphi(z) = z^{\Delta}$$

where $\Delta$ is not the scaling dimension of CFT. Substituting this into the equation gives

$$z^{d+1}\,\partial_z\left[\frac{\Delta}{z_{d-1}} z^{\Delta-1}\right] = z^{d+1}\Delta(\Delta - d) z^{\Delta-d-1} = m^2 l_{\mathrm{AdS}}^2 z^{\Delta}$$

This implies

$$\Delta(\Delta - d) = m^2 l_{\mathrm{AdS}}^2$$

The exponent of the power law is related to the mass. This looks like a local operator that scales [r]. The relation can be inverted to obtain

$$\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 l_{\mathrm{AdS}}^2}$$

Near the boundary, the solution to the wave equation is

$$\varphi(x^\mu, z) \sim \varphi_1(x_9^\mu z^{\Delta_+ + \varphi_0(x) z^{\Delta_-}}$$

These are the different modes that exist. The modes need to be normalized with respect to the Klein–Gordon inner product

$$\int_0^\infty \mathrm{d}z\,\sqrt{\left|g^{(d)}\right|}\varphi\,\partial_z\varphi \approx \int_0^\infty \mathrm{d}z\,\frac{1}{z^d} z^{\Delta}\Delta z^{\Delta-1}$$

remembering that the integral is over a time slice. The integral converges for

$$2\Delta - 1 - d > -1 \implies 2\Delta > d \implies \Delta > \frac{d}{2}$$

This defines normalizable modes. The normalizable mode is $\varphi_1(x^\mu)$, while the non-normalizable mode is $\varphi_0(x^\mu)$. [r] The first is related to the VEV of the operator $O_\Delta(x^\mu)$ of dimension $\Delta_+$ dual to the field $\varphi$. The second is related to the source of the operator

$$S_{\mathrm{CFT}} + \int \mathrm{d}^d x\,\varphi_0(x^\mu) O_\Delta(x^\mu)$$

[r]

**Wave equation in global AdS.** The wave equation can be solved in global AdS. Near the boundary there is a solution

$$\varphi(r,t,\Omega^i) \sim \frac{A}{r^{\Delta_+}} + \frac{B}{r^{\Delta_-}}$$

One sets $B = 0$. The solution to the full equation is a hypergeometric function. One expands the $A$-solution near $r = 0$ to have a regular and a singular term. This last is proportional to

$$\Gamma(-\omega + \Delta)^{-1}$$

To eliminate the singular term, one chooses a negative integer argument so that the Gamma function diverges. One then has

$$\omega = \Delta + n$$

[r] Frequency is quantized: the first mode is the dimension of the primary operator while $n$ is the descendants.

## Lecture 17

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In global AdS, for example, in $\mathrm{AdS}_3$ one has

$$\phi(r,t,\varphi) = R_{\omega l}(r)\mathrm{e}^{il\varphi}\mathrm{e}^{-\mathrm{i}\omega t}$$

One finds two solutions of the Klein–Gordon equation

$$R_{\omega,l}(r) = c_1 f_1(r,t,\Omega^i) + c_2 f_2(r,t,\Omega^i)$$

Expanding this mode near $r = 0$ one has

$$R \sim c_1 r^{-l} + c_2 r^l\,, \quad l > 0$$

Therefore one sets $c_1 = 0$. Expanding at $r \to \infty$ one has

$$R_{\omega,l}(r) = c_2 f_2(r) \sim r^{-\Delta_+}\#{+}r^{-\Delta_-}\frac{\Gamma(l+1)\Gamma(\sqrt{1+m^2 l_{\mathrm{AdS}}^2}}{\Gamma(\frac{1}{2}(1+l+\sqrt{1+m^2 l_{\mathrm{AdS}}^2}-\omega))\Gamma(\frac{1}{2}(1+l+\sqrt{1+m^2 l_{\mathrm{AdS}}^2})+\omega))}$$

[r] If the argument is a non-positive integer, the Gamma function diverges. One imposes renormalizability by setting

$$\frac{1}{2}(1+l+\sqrt{1+m^2 l_{\mathrm{AdS}}^2}-\omega) = -k\,, \quad k \in \mathbb{N}$$

From which one finds the quantization of the frequency $\omega$

$$\boxed{\omega = 1+\sqrt{1+m^2 l_{\mathrm{AdS}}^2}+l+2k}\,, \quad k,l \in \mathbb{N}$$

AdS puts gravity in a box. The AdS has a potential well that quantizes the frequency like a box in ordinary quantum mechanics. Therefore

$$\omega = \Delta + l + 2k$$

If $l = k = 0$ then the frequency corresponds to primary operator, while the descendants are

$$(\partial_\mu\partial^\mu)^k(\partial_\mu)^l O_\Delta$$

The frequency of the fields (which are Fourier conjugate of energy) are [r].

In free field theory, one may create any combination of particles.

## 6.3 Asymptotically AdS and FG gauge

[r] One would like to understand fluctuating spaces but with boundaries that approach AdS space. One allows for perturbations in the bulk of the space.

**Definition 6.1.** An asymptotic AdS space

- is solution to the Einstein's field equations with $\Lambda < 0$,
- preserves the conformal structure of the boundary.

The metric near the boundary is

$$g_{\mu\nu} \sim \frac{r^2}{l_{\text{AdS}}^2} g_{\mu\nu}^{(0)} + \frac{r}{l_{\text{AdS}}} g_{\mu\nu}^{(1)} + \cdots \sim \frac{l_{\text{AdS}}^2}{z^2} g_{\mu\nu}^{(0)} + \frac{l_{\text{AdS}}}{z} g_{\mu\nu}^{(1)} + \cdots$$

One may define an $\Omega$ that has a single zero at the boundary. The limit

$$\lim_{\text{boundary}} \Omega^2 g_{\mu\nu}$$

exists.

**Example.** In global AdS, one has

$$-\left(\frac{r^2}{l_{\text{AdS}}^2} + 1\right) \mathrm{d}t^2 - \frac{\mathrm{d}r^2}{\frac{r^2}{l_{\text{AdS}}^2} - 1} + r^2 \, \mathrm{d}\Omega_{d-1}^2$$

Then one has

$$\Omega = \frac{l_{\text{AdS}}^2}{r^2} \,, \quad \lim_{r\to\infty} \Omega^2 g_{\mu\nu} = -\mathrm{d}t^2 - l_{\text{AdS}}^2 \, \mathrm{d}\Omega_{d-1}^2$$

**Example.** In Poincaré AdS one has

$$\mathrm{d}s^2 = \frac{l_{\text{AdS}}^2}{z^2} (-\mathrm{d}t^2 + \mathrm{d}z^2 + \sum_{i=1}^{d-1} (\mathrm{d}x_i)^2)$$

Then one has

$$\Omega = \frac{z}{l_{\text{AdS}}} \,, \quad \lim_{z\to 0} \Omega^2 g_{\mu\nu} = -\mathrm{d}t^2 + \sum_{i=1}^{d-1} (\mathrm{d}x^i)^2$$

**Example.** A counter example is

$$g_{\mu\nu} \sim g_{\mu\nu}^{(0)} r^2 + g_{\mu\nu}^{(1)} r^4 \,, \quad r \to \infty$$

This metric does not satisfy the boundary conditions of asymptotic AdS.

**Example.** An example of non-empty asymptotic AdS is Schwarzschild AdS

$$\mathrm{d}s^2 = -\left(\frac{r^2}{l_{\text{AdS}}^2} + 1 - \frac{2M}{r}\right) \mathrm{d}t^2 + \left(\frac{r^2}{l_{\text{AdS}}^2} + 1 - \frac{2M}{r}\right) \mathrm{d}r^2 + r^2 \, \mathrm{d}\Omega_2^2$$

Near the boundary, the metric is

$$\mathrm{d}s^2 \sim -\frac{r^2}{l_{\text{AdS}}^2} \mathrm{d}t^2 + r^2 \, \mathrm{d}\Omega_{d-1}^2 - \left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 + \left(-\frac{1}{r^4} + \frac{2M}{r^5} + \cdots\right) \mathrm{d}r^2$$

The time-time component of the metric is

$$g_{tt} \sim \frac{r^2}{l_{\text{AdS}}^2} \left(1 + \frac{1}{r^2} - \frac{2M l_{\text{AdS}}^2}{r^3}\right) \mathrm{d}t^2$$

The first addendum goes like $r^0$, while the third goes like $r^{-3}$. The first is the source, the second is the vacuum expectation value of the energy-stress tensor $T_{tt}$. The third term is a perturbation to the metric.

[r] One may take

$$\Omega^1 = z f^1(x^\mu)\,, \quad \Omega_2 = z f^2(x^\mu)$$

Then the metric at the boundary is

$$g_{\text{boundary}} = \lim_{z\to 0} (\Omega^j)^2 g_{\mu\nu} = g_{\mu\nu}^{(0)} f^j(x^\mu)$$

The two transformations are the same up to a conforma factor

$$g^1_{\text{bdry}} = g^2_{\text{bdry}} \frac{f^1(x^\mu)}{f^2(x^\mu)}$$

There is a way to go from one metric to another through a change of coordinates. [r] The boundary metric is fixed up to a conformal factor.

## 6.4 Gauge transformation

In gravity one has to treat diffeomorphism invariance

$$x'^\mu = x^\mu + \varepsilon^\mu(x)$$

Two metrics related by such invariance are the same metric but in different coordinates systems. In QFT one fixes the gauge. [r] The same is possible in asymptotic AdS using the Fefferman–Graham (FG) gauge

$$\mathrm{d}s^2_{\text{AAdS}} = l^2_{\text{AdS}} \left[\frac{\mathrm{d}\rho^2}{4\rho^2} + \frac{1}{\rho} g_{\mu\nu}(\rho, x^\mu)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\right]$$

where $\mu, \nu = 1, \cdots, d$ are boundary coordinates [r] in $d+1$ space. There are no cross-terms

$$G_{\rho\mu} = 0$$

and this gives $d$ conditions since $\rho$ is fixed. The last condition is

$$G_{\rho\rho} = \frac{1}{4\rho^2}$$

In Poincaré AdS, one may just set $\rho = z^2 l^{-2}_{\text{AdS}}$.

In general, one may Taylor expand the boundary metric

$$g_{\mu\nu}(\rho, x^\mu) = g^{(0)}_{\mu\nu}(x^\mu) + \rho g^{(2)}_{\mu\nu}(x^\mu) + \cdots + \rho^{\frac{d}{2}} g^{(d)}(x^\mu) + \cdots + h^{(d)}_{\mu\nu}(x^\mu)\rho^{\frac{d}{2}} \ln\rho$$

where the last term is present only if $d$ is even. Inserting this expression into Einstein's field equations, one finds that $g^{(2k)}_{\mu\nu}$ and $h_{\mu\nu}$ are all determined by $g^{(0)}_{\mu\nu}$ except $g^{(d)}_{\mu\nu}$. Only

$$(g_\mu{}^\nu)^{(0)}\,, \quad \nabla_\mu g^{\mu\nu(0)}$$

are fixed.

[r] To fix a single metric, one obtains a regular metric on the inside. [r]

**Example.** An example is

$$g^{(2)}_{\mu\nu} a = \frac{1}{d-2}\left[R^{(0)}_{\mu\nu} - \frac{R^{(0)}}{2(d-1)} g^{(0)}_{\mu\nu}\right]\,, \quad g^{(4)}_{\mu\nu} = \frac{1}{4}(g^{(0)}_{\mu\nu})^2$$

Shifting the metric at the boundary

$$g^{(0)}_{\mu\nu} \to \mathrm{e}^{2\sigma(x^2)} g^{(0)}_{\mu\nu}$$

Consider the following change of oordinates

$$\rho = \rho' \mathrm{e}^{-2\sigma(x^\mu)} + \sum_{k=2} a_k(x^\mu)(\rho')^k\,, \quad x^\mu = x'^\mu + \sum_{k=1} a^\mu_k (\rho')^k$$

The metric is then

$$\mathrm{d}s^2 = l_{\mathrm{AdS}}^2 \left[\frac{\mathrm{d}\rho^2}{4\rho^2} + \frac{1}{\rho} g_{\mu\nu}(\rho, x')\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\right]$$

[r] By imposing that the FG gauge is preserved, then all coefficients $a_k$ and $a_k^\mu$ are fixed. For example

$$\begin{aligned}
a_2 &= -\frac{1}{2}\left[\partial_\mu\sigma\,\partial^\mu\sigma\right]\mathrm{e}^{-4\sigma} \\
a_3 &= \frac{1}{4}\mathrm{e}^{-6\sigma}\left[\frac{3}{4}(\partial_\mu\sigma)^2 + \partial^\mu\sigma\,\partial^\nu\sigma\,g_{\mu\nu}^{(0)}\right] \\
a_1^\mu &= \frac{1}{2}\,\partial^\mu\sigma\,\mathrm{e}^{-2\sigma} \\
a_2^\mu &= -\frac{1}{4}\mathrm{e}^{-4\sigma}\left[\partial_k\sigma\,g^{\mu k(2)} + \frac{1}{2}\,\partial^\mu\sigma\,(\partial_\mu\sigma)^2 + \frac{1}{2}\Gamma^{\mu(2)}_{kl}\partial^k\sigma\,\partial^l\sigma\right]
\end{aligned}$$

[r] From this, one concludes that conformal transformations on the boundary of AdS can be implemented by diffeomorphisms in AdS. This is more general than the fact that the isometry group of an empty AdS is the conformal group of the boundary. The equivalence is only in one direction, the conformal transformations imply the diffeomorphisms, but the vice versa is not always true.

There are two types of diffeomorphisms:

- large diffeomorphisms, $\rho = \rho'\mathrm{e}^{-2\sigma(x^\mu)} + \cdots$, these do not decay at the boundary and implement Weyl rescalings on the boundary,
- small diffeomorphisms, $\rho = \rho' + (\rho')^2 f(x^\mu) + \cdots$, these have no effect on the boundary.

The second diffeomorphisms are the actual gauge transformations.

**Example.** In Euclidean, one has [r]

The metric is then [r]

## Lecture 18

gio 16 mag 2024 14:30

**References.** Skenderis hep-th/0010138, FG/AAdS. Penedones TASI Lectures notes on AdS/CFT.

# 7 AdS/CFT

## 7.1 Introduction

The main statement of AdS/CFT is that

$$Z_{\mathrm{AdS}}[J] = Z_{\mathrm{CFT}}[J]$$

The partition function of gravity [r] is the same as the partition functions [r]. The right-hand side is understood

$$Z_{\mathrm{CFT}} = \int [\mathcal{D}\phi]\exp\left[-S_{\mathrm{CFT}} + \int \mathrm{d}^4x\,J(x)O(x)\right]$$

The left-hand side is

$$\int_{g\in\mathrm{AAdS}} [\mathcal{D}g\,\mathcal{D}\varphi]\mathrm{e}^{-S_{\mathrm{grav}}}$$

where

$$\varphi \sim z^\Delta\varphi_i(x) + z^{d-\Delta}\varphi_0(x)$$

and the boundary conditions $\partial\varphi$ is such that $\varphi_0(x) = J(x)$. From gravity, one takes functional derivatives with respect to $\varphi_0$ to obtain correlation functions.

This statement can be found by doing QFT on AdS [r]. One may find the propagator, the two-point function. One can take the two points to the boundary

$$O(x) = \lim_{z\to 0} \frac{1}{z^\Delta} \varphi(x,z)$$

The $n$-point correlation function becomes a CFT correlation function. This is the theory for the bulk

$$S_{\text{bulk}} = -\frac{1}{2} \int \mathrm{d}^{d-1}x \sqrt{|g|} [(\partial_\mu \varphi)^2 + m^2 \varphi^2]$$

The correlation function one may be able to calculate have the same properties as a CFT.

Any QFT on AdS produces correlation functions at the boundary that are the same of a CFT. However, there is no stress-energy tensor yet. The real CFT is obtained by adding gravity and letting it fluctuate. For QFT in AdS the [r] crossing symmetry gives zero. Once one adds a stress-energy tensor, the relations get modified.

**Dictionary.** The AdS/CFT is a dictionary between the bulk and the boundary

| Bulk (field) | Boundary (operator) |
|---|---|
| massive field $\varphi_m$ | $O_\Delta$ |
| metric $h_{\mu\nu}$ | stress tensor $T_{\mu\nu}$ |
| gauge fields $A_\mu$ | current $J^\mu$ |

| Bulk (gravitat. coupling) | Boundary (central charge) |
|---|---|
| $\frac{3l_{\text{AdS}}}{2G_{\text{N}}}$ | $c$ |

| Bulk (AAdS geometry) | Boundary (state) |
|---|---|
| AdS | $\|0\rangle$ |
| Black hole | thermal $\mathrm{e}^{-\beta H}$ |

| Bulk | Boundary |
|---|---|
| Observables: area of a string lading on $C$ | $\mathrm{Tr}_R\, \mathrm{e}^{\mathrm{i} \oint_C A_\mu \,\mathrm{d}x^\mu}$ |
| $\frac{A_{\min}(x)}{4G_{\text{N}}}$ | $SEE(A)$ |

The right-hand side is a Wilson loop are observables in gauge theory. [r] The entanglement entropy is infinite and this is reflected in the bulk since it is infinite.

There are also other correspondences.

## 7.2 Two-point correlation function

In CFT, the first correlation functions are given by

$$\langle O(x)\rangle = 0\,, \quad \langle O(x_1)O(x_2)\rangle = \frac{1}{|x_{12}|^{2\Delta}}\,, \quad \langle O_i O_j O_k\rangle = \frac{c_{ijk}}{\pi x_{ij}^{\Delta_{ijk}}} \cdot \text{coeff}$$

Last one can be a topic for the exam.

Consider Poincaré coordinates

$$S = -\frac{1}{2} \int \mathrm{d}^{d-1}x \sqrt{g} (\partial_M \varphi\, \partial^M \varphi + m^2 \varphi^2)\,, \quad (\nabla^2 - m^2)\varphi = 0$$

The Klein–Gordon equation is

$$z^{d+1}\, \partial_z (z^{-d+1}\, \partial_z \varphi) + \partial_\mu \partial^\mu \varphi - m^2 \varphi = 0$$

which has solution

$$\varphi(z,x) \sim \phi_1(x) z^{\Delta} + \phi_0(x) z^{d-\Delta}\,, \quad z \to 0$$

The propagator solves the equation [r]. One may look at the bulk-to-boundary propagator

$$K_\Delta(z,x,y) = C_\Delta \left[\frac{z}{z^2 + (x^\mu - y^\mu)^2}\right]^\Delta$$

[r] therefore

$$\varphi(z,x) = \int_{\text{bdry}} \mathrm{d}^d y\, K_\Delta(z,x,y) \phi_0(y)$$

One solves the equation at some point $x$ in the bulk by using a propagator from a point on the boundary $y$ to $x$. Near $z \to 0$, one has

$$\varphi(z,x) \sim z^\Delta C_\Delta \int \mathrm{d}^d y \frac{\phi_0(y)}{|x^\mu - y^\mu|^{2\Delta}} = z^\Delta \phi_1(x)$$

The other term is

$$\phi_0(x) = \lim_{z\to 0} z^{\Delta - d} - \varphi(z,x)$$

In fact, inserting this expression into $K_\Delta$ in the integral of $\varphi$, then one has $z^{2\Delta-d}$. Due to renormalizability [r] blows up. In the limit $z \to 0$, the integral gives a delta function. For arbitrary $z$, the integral is a regularized delta function.

Consider the action

$$\begin{aligned}
S &= -\frac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}z\, z^{-d-1}[z^2\, \partial_z\varphi\, \partial_z\varphi + \partial_\mu\varphi\, \partial^\mu\varphi + m^2\varphi^2] \\
&= -\frac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}z\, z^{-d-1}[z^2\, \partial_z\varphi\, \partial_z\varphi - \varphi\, \partial_\mu\partial^\mu\varphi + m^2\varphi^2] \\
&= -\frac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}z\, [-\partial_z(z^{-d-1}\varphi\, \partial_z\varphi) - \varphi\, \partial_\mu\partial^\mu\varphi + m^2\varphi^2] - \frac{1}{2}\int_{\text{bdry}} \mathrm{d}^d x\, z^{-d-1}\varphi\, \partial_z\varphi \\
&= -\frac{1}{2}\int_{\text{bdry}} \mathrm{d}^d x\, z^{-d-1}\varphi\, \partial_z\varphi \\
&= -\frac{1}{2}\int_{\text{bdry}} \mathrm{d}^d x\, [d\phi_0(x)\phi_1(x) + (d-\Delta)\phi_0^2(x) z^{d-2\Delta} + \cdots]
\end{aligned}$$

At the second line one has integrated by parts in $x$. At the third line one has integrated by parts in $z$. The bracket is Klein–Gordon equation

$$\varphi(\nabla^2 - m^2)\varphi = -\partial_z(z^{-d-1}\varphi\, \partial_z\varphi) - \varphi\, \partial_\mu\partial^\mu\varphi + m^2\varphi^2 = 0$$

At the fifth line, one has inserted

$$\varphi(z,x) \sim \phi_0(x) z^{\Delta-d} + \phi_1(x) z^\Delta$$

In the limit $z \to 0$, the action diverges

$$\lim_{z\to 0} S[\phi_0] = \infty$$

This is analogue to the fact that in QFT one finds infinities when dealing with the bare action.

**Holographic renormalization.** The renormalization is done through counter terms

$$S = S_{\text{bulk}} + S_{\text{bdry}} + S_{\text{ct}}$$

the counter term action is made of local counter terms of the boundary data. For example

$$\int_{\text{bdry}} \mathrm{d}^d x\, \sqrt{\gamma}\phi^2(x)\,, \quad \int_{\text{bdry}} \mathrm{d}^d x\, \sqrt{\gamma}(\alpha_1 + \alpha_2 R\gamma + \alpha_3 R\gamma^2 + \cdots)$$

The bare? action is

$$S_0 = -\frac{1}{2}\int_{\text{bdry}} \mathrm{d}^d x\, z^{-d-1} \varphi\, \partial_z \varphi$$

The counter term action

$$S_{\text{ct}} = \frac{1}{2}(d-\Delta) z^{-d} \int_{\text{bdry}} \mathrm{d}^d x\, \phi^2(z,x)$$

This counter term has a finite component that shifts around [r]. Therefore,

$$\boxed{S_0 + S_{\text{ct}}|_{z=0} = \frac{d-2\Delta}{2}\int_{\text{bdry}} \mathrm{d}^d x\, \phi_0(x)\phi_1(x)} = -\ln Z[J=\phi_0]$$

Since $\phi_1$ is an integral expression in $\phi_0$, then the above is a quadratic integral in $\phi_0$.

Taking functional derivatives with respect to $\phi_0$ gives the correlators. The one-point correlator is

$$\langle O(x)\rangle = \left.\frac{\delta \ln Z}{\delta \phi_0(x)}\right|_{\phi_0=0} = 0$$

since [r]. The two-point correlator is

$$\begin{aligned}\langle O(x_1)O(x_2)\rangle &= \left.\frac{\delta^2 \ln Z}{\delta\phi_0(x_1)\delta\phi_0(x_2)}\right|_{\phi_0=0} = \frac{\delta^2 \ln Z}{\delta\phi_0(x_1)\delta\phi_0(x_2)} \frac{d-2\Delta}{2}\int_{\text{bdry}} \mathrm{d}y\, \mathrm{d}x\, \frac{\phi_0(x)\phi_0(y)}{|x^\mu - y^\mu|^{2\Delta}} \\ &= (d-2\Delta)\frac{1}{|x_{12}|^{2\Delta}}\end{aligned}$$

The factor in front is just a normalization of the field. The three-point correlator is zero for the free theory.

There is a similar prescription for all higher-point correlation functions. These are called Witten diagrams. For example [r] diagr

$$= \int \mathrm{d}x_A\, \mathrm{d}x_B\, K(x_1,x_A)K(x_2,x_A)K(x_3,x_B)K(x_4,x_B)K(x_A,x_B)$$

[r]

**Causality.** On AdS one may send a light message on the boundary, but also through the bulk. Writing some EFTs, the bulk is faster, so the theory is not causal. One may derive bounds of EFTs' couplings that need to be satisfied for the theory to be causal.

## 7.3 Black hole coupling

Consider the partition function with no sources

$$Z_{\text{CFT}}[J=0] = Z_{\text{bulk}}[J=0]$$

The partition function on a sphere is an uninteresting number. One may instead take a periodic Euclidean time manifold

$$\mathcal{M} = S^{d-1}\times S^1$$

Then the partition function is

$$Z_{\text{CFT}}[J=0] = Z(\beta) = \operatorname{Tr} \mathrm{e}^{-\beta H}$$

while in the semi-classical limit, one has

$$Z_{\text{bulk}}[J=0] = \int [\mathcal{D}g]\, \mathrm{e}^{-S_{\text{on-shell}} - S_{\text{GHY}} - S_{\text{ct}}}$$

the boundary conditions $\partial g$ on $S^{d-1}\times S^1_\beta$ [r] are such that

$$Z_{\text{bulk}}[J=0] \approx \sum_{g_0^i} \mathrm{e}^{-S_{\text{tot}}[g_0^i]}$$

In $d = 2$, the manifold is a torus

$$S^1 \times S^1 = T^2$$

There are two solutions:

- thermal AdS, one fills the cylindrical boundary

$$\mathrm{d}s^2 = (r^2 + 1)\,\mathrm{d}t_{\mathrm{E}}^2 + \frac{\mathrm{d}r^2}{r^2 + 1} + r^2\,\mathrm{d}\varphi^2\,, \quad t_{\mathrm{E}} \sim t_{\mathrm{E}} + \beta$$

- BTZ black hole

$$\mathrm{d}s^2 = (r^2 - M^2)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2 - M^2} + r^2\,\mathrm{d}\varphi^2$$

  The period of $\beta$ sets the mass $M$. At the boundary is also a torus, but the solution is different.

In thermal AdS, the time coordinate is not contractible, while here time is contractible, but not the angular coordinate.

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The partition function is

$$Z_{\mathrm{CFT}}(\beta) = \mathrm{Tr}\,\mathrm{e}^{-\beta H_{\mathrm{CFT}}} = \int [\mathcal{D}\psi_{\mathrm{CFT}}]\,\mathrm{e}^{-S_{\mathrm{CFT}}}$$

where the path integral is on $S^1 \times S^{d-1}$, the shape is a cylinder with compactified time: a torus. The metric of the manifold is

$$\mathrm{d}s^2_{\mathrm{CFT}} = \mathrm{d}t_{\mathrm{E}}^2 + \mathrm{d}\Omega_{d-1}^2\,, \quad t_{\mathrm{E}} \sim t_{\mathrm{E}} + \beta$$

Using the GPKW [r] dictionary, one uses the bulk to compute

$$Z_{\mathrm{CFT}}[J] = Z_{\mathrm{AdS}}[J] = \int [\mathcal{D}g\,\mathcal{D}\varphi]\;\exp[??]$$

[r] The sources can be set to zero since one is not computing any correlation function.

In two dimensions the metric is

$$\mathrm{d}s^2 = \mathrm{d}t_{\mathrm{E}}^2 + \mathrm{d}\phi^2\,, \quad t_{\mathrm{E}} \sim t_{\mathrm{E}} + \beta\,, \quad \phi \sim \phi + 2\pi$$

The gravity path integral is

$$Z_{\mathrm{grav}} = \int [\mathcal{D}g]\,\mathrm{e}^{-S_{\mathrm{grav}}} \approx \sum_{g_i} \mathrm{e}^{-S_{\mathrm{grav}}^{\text{on-shell}}}$$

where the integral is evaluated on the boundary $\partial g = S^1 \times S^1$. The gravity action is evaluated for a metric that solves Einstein's field equations. The gravity action is

$$S_{\mathrm{grav}} = -\frac{1}{16\pi G}\int_M \sqrt{g}\,\mathrm{d}^3x\,(R - 2\Lambda) - \frac{1}{8\pi G}\int_{\partial M} \sqrt{n}K + S_{\mathrm{ct}}$$

The counter term is needed to prevent infinities. It is the most general function of local objects that depend only on the boundary data

$$S_{\mathrm{ct}} = \alpha_1\,\mathrm{d}^2x\,\sqrt{n} + \alpha_2 \int \mathrm{d}^2x\,\sqrt{n}R + \alpha_3 \int \mathrm{d}^2x\,\sqrt{n}R^2 + \cdots$$

The minimal subtraction scheme involves only the first term. The G??–Hawking term is not a counter term since $K$ is not a local operator.

There are two important solutions. The first one is thermal AdS which has the following metric

$$\mathrm{d}s^2 = (r^2 + l_{\mathrm{AdS}}^2)\,\mathrm{d}t_{\mathrm{E}}^2 + \frac{\mathrm{d}r^2}{\frac{r^2}{l_{\mathrm{AdS}}} + 1} + r^2\,\mathrm{d}\phi^2\,, \quad t_{\mathrm{E}} \sim t_{\mathrm{E}} + \beta$$

However the Euclidean time is compactified [r]. This is a solid torus. This is a solution to Einstein's field equations. The second solution is the BTZ black hole

$$\mathrm{d}s^2 = (r^2 - r_{\mathrm{H}}^2)\,\mathrm{d}t_{\mathrm{E}}^2 + \frac{\mathrm{d}r^2}{\frac{r^2 - r_{\mathrm{H}}^2}{l_{\mathrm{AdS}}^2}} + r^2\,\mathrm{d}\varphi^2$$

In Lorentzian space $r = r_{\mathrm{H}}$ is the horizon. The radius is in

$$r \in [r_{\mathrm{H}}, r_{\mathrm{max}}]$$

where $r_{\mathrm{max}}$ is a just a maximum range which eventually is sent to infinity. In terms of the mass, one has

$$8GM = \frac{r_{\mathrm{H}}^2}{l_{\mathrm{AdS}}^2}$$

At fixed Euclidean time, the metric is a circular corona in the radius range above. The metric does not exist inside the horizon. If one periodically identifies [r] one obtains a torus. Each time slice is a horizontal slice ofthe torus. The time coordinate can be smoothly shrunk [r]. The circle in $\varphi$ are finite.

The two solutions [r], but for the first one, the $\phi$ cycle can be contracted, but not the time; viceversa for the second solution. Going form one solutoin to the other is like exchanging time and space.

**BTZ black hole.** For the BTZ black hole, the Ricci scalar, the cosmological constant and the unit vector are

$$R = -\frac{6}{l_{\mathrm{AdS}}^2}\,, \quad \Lambda = -\frac{1}{l_{\mathrm{AdS}}^2}\,, \quad n_\mu = -\frac{\partial r}{\sqrt{\frac{r^2 - r_{\mathrm{H}}^2}{l_{\mathrm{AdS}}^2}}}$$

The Killing vector? is

$$K_{\mu\nu} = \nabla_{(\mu} n_{\nu)}\,, \quad K = \frac{r_{\mathrm{H}}^2 - 2r^2}{l_{\mathrm{AdS}} r}\sqrt{\frac{1}{r^2 - r_{\mathrm{H}}^2}}$$

The metric [r] is

$$\sqrt{h} = r\sqrt{r^2 - r_{\mathrm{H}}^2}$$

Therefore, the action is

$$\begin{aligned}
S_{\mathrm{E}} &= -\frac{1}{16\pi G}\int_0^\beta \mathrm{d}t_{\mathrm{E}} \int_0^{2\pi} \mathrm{d}\phi \int_{r_{\mathrm{H}}}^{r_{\mathrm{max}}} \mathrm{d}r\, r l_{\mathrm{AdS}}^2 \left[-\frac{4}{l_{\mathrm{AdS}}^2}\right] - \frac{1}{8\pi G}\int_0^\beta \mathrm{d}t_{\mathrm{E}} \int_0^{2\pi} \mathrm{d}\phi\, \sqrt{h} K + S_{\mathrm{ct}} \\
&= \frac{\beta 2\pi}{8\pi G l_{\mathrm{AdS}}}(r_{\mathrm{max}}^2 - r_{\mathrm{H}}^2) + \frac{2\pi\beta}{8\pi G}\frac{2r_{\mathrm{max}}^2 - r_{\mathrm{H}}^2}{l_{\mathrm{AdS}} r_{\mathrm{max}}}\frac{1}{\sqrt{r_{\mathrm{max}}^2 - r_{\mathrm{H}}^2}} r_{\mathrm{max}}\sqrt{r_{\mathrm{max}}^2 - l_{\mathrm{AdS}}^2} \\
&= \frac{\beta}{4G l_{\mathrm{AdS}}}(r_{\mathrm{max}}^2 - r_{\mathrm{H}}^2) + \frac{\beta}{4G l_{\mathrm{AdS}}}(2r_{\mathrm{max}}^2 - r_{\mathrm{H}}^2)\sqrt{\frac{r_{\mathrm{max}}^2 - l_{\mathrm{AdS}}^2}{r_{\mathrm{max}}^2 - r_{\mathrm{H}}^2}}
\end{aligned}$$

One may Taylor expand this expression at $r_{\mathrm{max}} \to \infty$. Therefore

$$S_{\mathrm{E}} = \frac{3\beta}{4G l_{\mathrm{AdS}}} r_{\mathrm{max}}^2 - \frac{r_{\mathrm{H}}^2}{2G l_{\mathrm{AdS}}} + O(r_{\mathrm{max}}^{-1})$$

The counter term is needed since the first term diverges. The counter term is

$$S_{\mathrm{ct}} = \frac{\alpha}{8\pi G l_{\mathrm{AdS}}}\int_{r_{\mathrm{max}}} \sqrt{h}\,\mathrm{d}^2x = \frac{\alpha}{8\pi G l_{\mathrm{AdS}}}\int_0^\beta \mathrm{d}t_{\mathrm{E}} \int_0^{2\pi} \mathrm{d}\phi\, r_{\mathrm{max}}\sqrt{r_{\mathrm{max}}^2 - r_{\mathrm{H}}^2} = \frac{\alpha\beta 2\pi}{8\pi G l_{\mathrm{AdS}}}\left[r_{\mathrm{max}}^2 - \frac{1}{2}r_{\mathrm{H}}^2 + \cdots\right]$$

Comparing this with the divergent term, one has to set

$$\alpha = -3$$

Therefore, the total action is

$$S = -\frac{r_{\rm H}^2\beta}{2Gl_{\rm AdS}} + \frac{3r_{\rm H}^2\beta}{8Gl_{\rm AdS}} = -\frac{r_{\rm H}^2\beta}{8Gl_{\rm AdS}}$$

The partition function is then

$$Z(\beta) = {\rm e}^{-S} = \exp\left[\frac{r_{\rm H}^2\beta}{8Gl_{\rm AdS}}\right]$$

This expression contains the horizon radius, but it cannot appear in a CFT. It needs to be expressed as [r]. Recall that one has

$$\beta = \frac{4\pi l_{\rm AdS}}{f'(r_{\rm H})} = \frac{4\pi l_{\rm AdS}}{2r_{\rm H}} = \frac{2\pi l_{\rm AdS}}{r_{\rm H}} \implies r_{\rm H} = \frac{2\pi l_{\rm AdS}}{\beta}$$

Therefore

$$Z(\beta) = \exp\left[\frac{16\pi^2 l_{\rm AdS}}{8G\beta}\right]$$

There are still gravity parameters that are not CFT parameters. Recalling from the dictionary that the central charge is [r]

$$c = ??$$

one finds

$$Z_{\rm grav}(\beta) = \exp\left[\frac{c}{12}\frac{4\pi^2}{\beta}\right]$$

This is the result for the BTZ black hole

**Thermal AdS.** For the thermal AdS solution, one has

$$Z_{\rm grav}(\beta) = \exp\left[\frac{c}{12}\beta\right]$$

**Modular invariance.** Recall that in two dimensions, a CFT on a torus must obey modular invariance

$$Z_{\rm CFT}(\beta) = Z_{\rm CFT}({}^{4\pi^2}\!/\beta) \iff Z(\tau) = Z(-\tau^{-1})\,, \quad \beta = 2\pi\,{\rm Im}\,\tau$$

Total partition function of gravity, is

$$Z_{\rm grav}(\beta) = \sum_{g_i} {\rm e}^{-S_{\text{on-shell}}} = \exp\left[\frac{c}{12}\frac{4\pi^2}{\beta}\right] + \exp\left[\frac{c}{12}\beta\right]$$

The gravity partition function is invariant under modular transformation. The presence of two solutions is important since the CFT partition function must be modular invariant.

The solutions exchange Euclidean time and space [r] this is what modular invariance does. One may generate new geometries by summing over images of the symmetry [r].

Notice that there are also other solutions, they are called ${\rm SL}(2,\mathbb{Z})$ black holes [r]. The two solutions above are invariant under $S$ but not under $T$. The other black holes are like the BTZ black holes but with some angular momentum (possible topic for exam).

**Free energy of BTZ.** The free energy of the BTZ geometry is

$$F_{\rm BTZ} = -\frac{1}{\beta}\ln Z_{\rm BTZ} = -\frac{c}{12}\frac{4\pi^2}{\beta}\,, \quad \beta < 2\pi$$

while for AdS one has

$$F_{\rm AdS} = -\frac{1}{\beta}\ln Z_{\rm AdS} = -\frac{c}{12}\,, \quad \beta > 2\pi$$

In thermodynamics one has to minimize the free energy. The phase diagram is [r]. At $\beta = 2\pi$, the derivatives of the free energy jump, this is the Hawking–Page phase transition. The dominant geometry depends on the temperature. At low temperatures AdS dominates with a constant, when increasing the energy there is a first order phase transition to the BTZ geometry.

There are many parameters that jump in the transition. An important one is entropy

$$S = (1 - \beta\,\partial_\beta)\ln Z$$

In thermal AdS it is

$$S = 0$$

while in BTZ it is

$$S = \frac{c}{6}\frac{4\pi^2}{\beta}$$

Therefore

$$S = \begin{cases} 0\,, & \beta > 2\pi \\ \frac{c}{6}\frac{4\pi^2}{\beta}\,, & \beta < 2\pi \end{cases}$$

This is discontinuous jump and since it is the first derivative of the free energy, then the phase transition is of first order.

The Beckestein–Hawking entropy is

$$S = \frac{A}{4G} = \frac{2\pi r_{\mathrm{H}}}{4G} = \frac{4\pi^2}{\beta}\frac{l_{\mathrm{AdS}}}{4G} = \frac{4\pi^2}{\beta}\frac{1}{6}\frac{3l_{\mathrm{AdS}}}{2G} = \frac{4\pi^2}{\beta}\frac{c}{6}$$

which matches the BTZ black hole entropy. The BH entropy is the right formula for the entropy only after the phase transition. In the dictionary, black holes are dual to thermal states only at high enough temperatures.

In the boundary CFT this is interpreted as follows. In general, a CFT is a gauge theory. In gauge theories (e.g. QCD) at high temperatures the theory is free, while low temperatures are hadrons; this transition is confinement. The Hawking–Page phase transition corresponds to confinement/de-confinement (thermal AdS/BTZ). The problem of confinement is easily geometrized in AdS/CFT.

The entropy can be useful to be written in terms of the energy

$$S = \frac{4\pi}{\beta}\frac{c}{6}\,, \quad \langle E\rangle = -\partial_\beta \ln Z = \frac{c}{12}\frac{4\pi^2}{\beta^2} \implies \beta = 2\pi\sqrt{\frac{c}{12}\frac{1}{E}}$$

Therefore

$$S = 2\pi\frac{c}{6}\sqrt{E\frac{12}{c}} = 2\pi\sqrt{\frac{c}{3}E} = S(E)$$

This is Cardy's formula, the entropy (i.e. density of states) of a two-dimensional CFT. [r] Thus Cardy's formula counts the micro-states of the BTZ black hole in $\mathrm{AdS}_3$.