SC 607: Optimization **Spring 2019**

Lecture 15: March 8 2019

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15.1 Review

In the previous lecture, a necessary and sufficient condition for x^* to be a minimizer for convex optimization problem was derived. A condition on local minimizer x^*, where x^* is constrained over the tangent cone set $T(x^*, S)$, S need not be convex is found. Let us recall some of the concepts that were discussed.

Theorem 15.1. *Let S be a convex set and let f be a continuously differentiable (C^1) convex function. Consider the following convex optimization problem:*

$$\textit{minimize } f(x) \tag{15.1}$$

$$\textit{subject to } x \in S \tag{15.2}$$

Then x^* is minimizer if and only if $\nabla f(x^*)^\top (y - x^*) \geqslant 0$, for all $y \in S$.

Definintion 15.1. *For $x^* \in S$, the tangent cone $T(x^*; S)$ is defined as:*

$$T(x^*, S) := \left\{ d \;\middle|\; \exists\, x_k \subseteq S : x_k \to x^* \;\&\; t_k \downarrow 0 \text{ s.t. } d = \lim_{k \to \inf} \frac{x_k - x^*}{t_k} \right\}. \tag{15.3}$$

Tangent cone of a set captures the precise shape of the set.

Example:

In Fig 15.1a, the tangent cone is just a straight line,and in Fig 15.1b, the tangent cone at x is entire $\mathbb{R}^2$

If $x \in \overset{o}{S}$, T(x,S) = $\mathbb{R}^2$

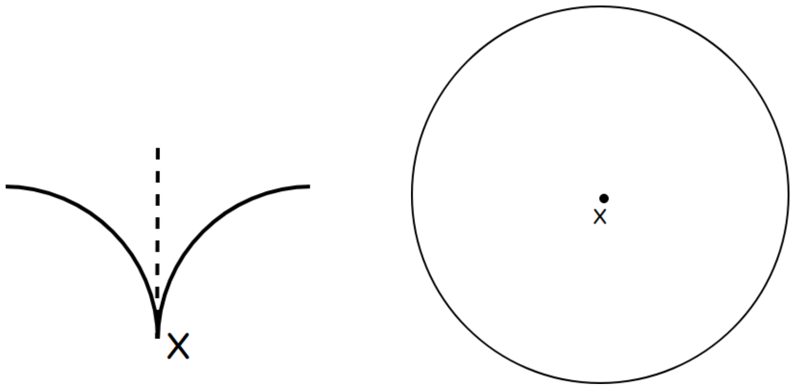
Theorem 15.2. *Consider the optimization problem:*

$$\textit{Minimize } f(x)$$
$$\textit{subject to } x \in S$$

where $f \in C^1$. If x^* is a local minimum, then $\nabla f(x^*)^\top d \geqslant 0$, $\forall d \in T(x^*, S)$.

15.2 Introduction

In this lecture, Constraint Qualifications are discussed. Farkas Lemma and Karush kuhn Tucker conditions for an optimisation problem with inequality constraints are proved



(a) (b)

Figure 15.1: Tangent cone examples

15.3 Motivation for need of Constraint Qualification(CQ)

From Theorem 15.1 local minima x^* satifies the condition $\nabla f(x^*)^\top d \geqslant 0$, $\forall d \in T(x^*, S)$. for $f \in C^1$ and $x \in S$

here d is from Tangent Cone, When the set is convex the Tangent Cone contains all the directions including limiting direction which grazes the boundary (we cannot move further in that direction) of the set.
So, for any $y \in S$ entire $y - x^*$ segment will be in the Tangent Cone

$$\nabla f(x^*)^\top (y - x^*) \in T(x^*, S) \; \forall \, y \in S \iff x^* \text{ is a global minimum}$$

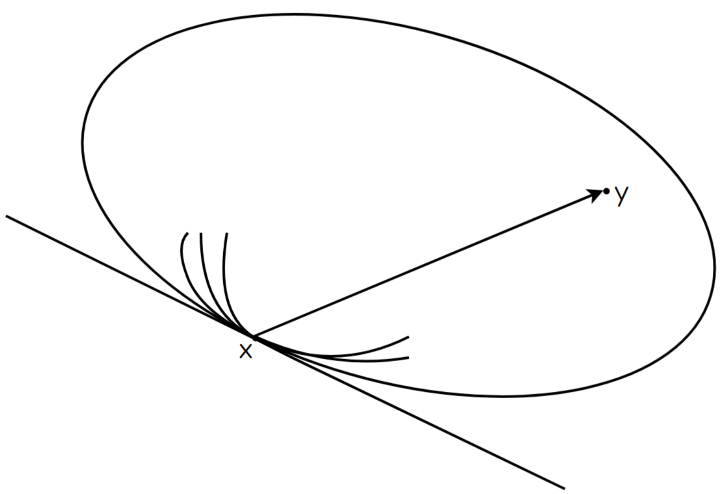


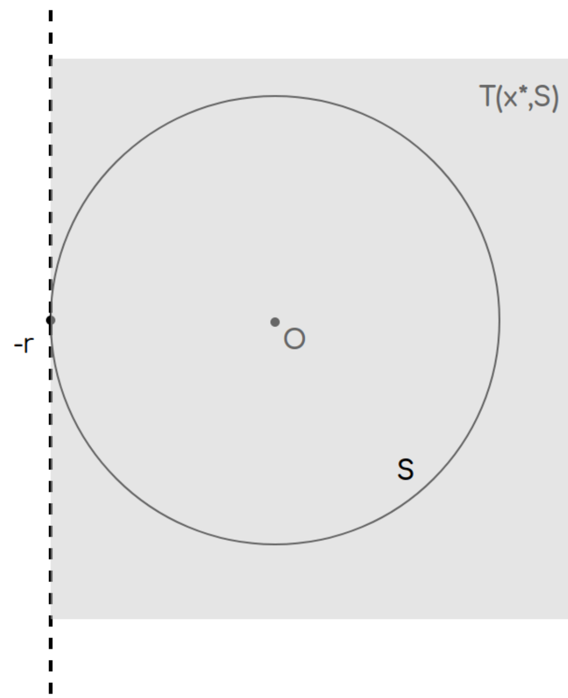
Figure 15.2: $(y - x^*) \in T(X^*, S)$ when S is convex

when the set is convex necessary condition for existance of local minima will be sufficient for finding the global minima.

15.3.0.1 Finding Tangent Cone

Consider a optimisation problem

$$\text{Minimize } f(x)$$
$$\text{subject to } g(x) = x_1^2 + x_2^2 - r^2$$
$$x \geq 0$$



Tangent cone is given by

$$\{\, d \mid \nabla g(x^*)^\intercal d \leq 0\}$$

If x^* is in interior of the set then tangent cone is entire space $\mathbb{R}^n$

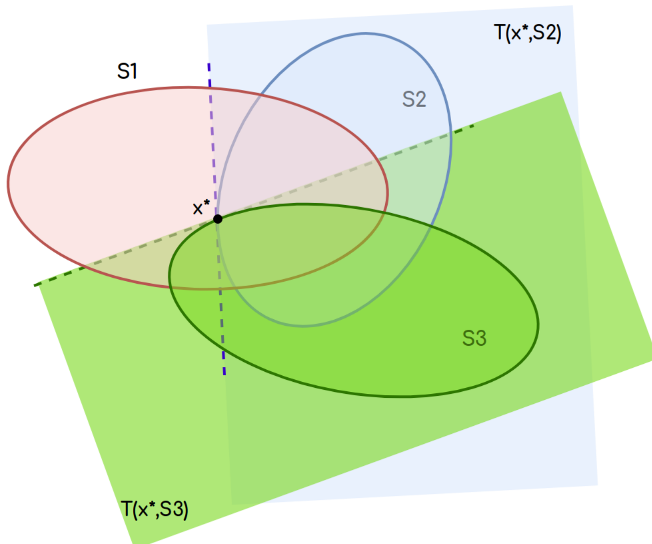
In the above case there was only one constraint so it is easy obtain the tangent cone. Consider the following optimisation problem

$$\text{minimize } f(x)$$
$$\text{subject to}$$

$$S = \{\, x \mid g_i(x) \leq 0, i = 1, 2, ...m\}$$

let

$$S = \cap_{i=1}^{m} S_i$$



where

$$S_i = \{x \mid g_i(x) \leq 0\}$$

we want $T(x^* : S)$ we have $T(x^* : S_i)$ we can say

$$T(x^* : S) \subseteq \cap_{i=1}^{m} T(x : S_i)$$

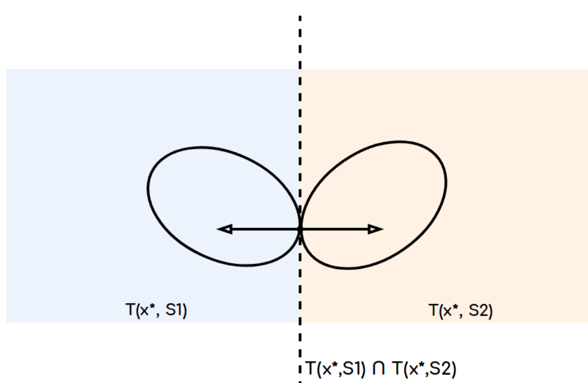
$$T(x^* : S) \not\supseteq \cap_{i=1}^{m} T(x : S_i)$$

suspose

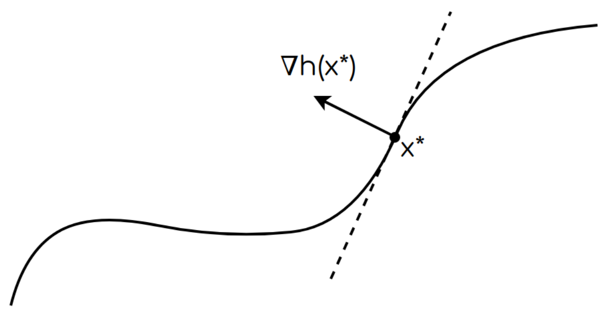
$$\text{minimize } f(x)$$
$$\text{subject to}$$

$$S = \{\, x \mid g_i(x) \leq 0, i = 1, 2, ...m\}$$

$$h_j(x) = 0, , j = 1, 2, ...p$$



same equality constraints can be represented as $\{x \mid h_j^2(x) = 0\}$ in such representation $\nabla h_j(x)$ doesnot give true normal. This might happen because of Linear Dependency of Normals. Constraints Qualification makes sure that Normals are Linearly Independent.



15.3.1 Constraint Qualification (CQ)

Let $S = \{ x \mid g_i(x) \leq 0, i = 1, 2, ...m\}, g \in C^1$ we say that a constraint qualification is satisfied at $x^* \in S$ if

$$T(x^*, S) = \{d \mid \nabla g_i(x^*)^T d \leq 0 \, \forall \, i \in I(x^*)\}$$

active constraints are such that $I(x^*) = \{i \mid g_i(x^*) = 0\}$ for other constraints $T(x^* : S) = \mathbb{R}^n$

15.4 Farkas Lemma

Lemma 15.3. *Let $A \in \mathbb{R}^{mxn}, c \in \mathbb{R}^n$ Then the following statements are equivalent*
(1) $\forall x \in \mathbb{R}^n$ such that $Ax \leq 0$, we have $c^T x \leq 0$
(2) $\exists \, \lambda \geq 0 \in \mathbb{R}^m$ such that $A^T \lambda = C$

Proof. consider

$$max c^T x$$

$$\text{subject to } Ax \leq 0$$

Dual of this can be given as

$$min 0$$

$$\text{subject to } A^T \lambda = c$$

$$\lambda \geq 0$$

(1) implies optimal value of primal is 0 which again, implies dual is feasible which implies (2)
(2) implies dual is feasible which inturn implies optimal value of dual is 0 which means primal is feasible there by optimal value of primal is 0 which implies (1)

□

15.5 Karush-kuhn-Tucker Conditions

Theorem 15.4. *Consider the problem f, $g_1, ..., g_m \in C^1$*

$$max f(x)$$

$$g_i(x) \leq 0 \, i = 1, ..., m$$

Let x^* be a local maximum and suppose that Constraint Qualification is satisfied at x^*. Then $\exists\, \lambda_1^*, \lambda_2^*, ..., \lambda_m^* \geq 0$ such that

$$\nabla f(x^*) = \sum_{n=1}^{m} \lambda_i^* \nabla g_i(x^*) \tag{15.4}$$

$$\lambda_i^* g_i(x^*) = 0 \ \forall i = 1, 2, ..., m \tag{15.5}$$

$\lambda_1^*, \lambda_2^*, ..., \lambda_m^*$ are Lagrange multipliers

Proof. x^* is a local max

$$\nabla f(x^*)^T d \leq 0 \forall \ d \in T(x^* : S)$$

but since CQ holds at x^*

$$T(x^* : S) = \{d \mid \nabla g_i(x^*)^T d \leq 0 \ \forall \ i \in I(x^*)\}$$

By Farkas Lemma $\exists \lambda_j, j \in I(x^*), \lambda_j \geq 0$ such that $\nabla f(x^*) = \sum_{j \in I(x^*)} \lambda_j^* \nabla g_i(x^*)$

$$= \sum_{j=1}^{m} \lambda_j^* \nabla g_i(x^*) \text{ and } \lambda_j^* = 0, j \notin I(x^*)$$

□

This Conditions are further generalised when equality constraints are added along with inequality constraints in next lecture.