# CHAPTER 11

# HORIZONTAL AND VERTICAL CURVES

As you know from your study of chapter 3, the center line of a road consists of series of straight lines interconnected by curves that are used to change the alignment, direction, or slope of the road. Those curves that change the alignment or direction are known as **horizontal curves,** and those that change the slope are **vertical curves.**

As an EA you may have to assist in the design of these curves. Generally, however, your main concern is to compute for the missing curve elements and parts as problems occur in the field in the actual curve layout. You will find that a thorough knowledge of the properties and behavior of horizontal and vertical curves as used in highway work will eliminate delays and unnecessary labor. Careful study of this chapter will alert you to common problems in horizontal and vertical curve layouts. To enhance your knowledge and proficiency, however, you should supplement your study of this chapter by reading other books containing this subject matter. You can usually find books such as *Construction Surveying,* FM 5-233, and *Surveying Theory and Practice,* by Davis, Foote, Anderson, and Mikhail, in the technical library of a public works or battalion engineering division.

## HORIZONTAL CURVES

When a highway changes horizontal direction, making the point where it changes direction a point of intersection between two straight lines is not feasible. The change in direction would be too abrupt for the safety of modem, high-speed vehicles. It is therefore necessary to interpose a curve between the straight lines. The straight lines of a road are called **tangents** because the lines are tangent to the curves used to change direction.

In practically all modem highways, the curves are **circular** curves; that is, curves that form circular arcs. The smaller the radius of a circular curve, the sharper the curve. For modern, high-speed highways, the curves must be flat, rather than sharp. That means they must be large-radius curves.

In highway work, the curves needed for the location or improvement of small secondary roads may be worked out in the field. Usually, however, the horizontal curves are computed after the route has been selected, the field surveys have been done, and the survey base line and necessary topographic features have been plotted. In urban work, the curves of streets are designed as an integral part of the preliminary and final layouts, which are usually done on a topographic map. In highway work, the road itself is the end result and the purpose of the design. But in urban work, the streets and their curves are of secondary importance; the best use of the building sites is of primary importance.

The principal consideration in the design of a curve is the selection of the length of the radius or the degree of curvature (explained later). This selection is based on such considerations as the design speed of the highway and the sight distance as limited by headlights or obstructions (fig. 11-1). Some typical radii you may encounter are 12,000 feet or longer on an interstate highway, 1,000 feet on a major thoroughfare in a city, 500 feet on an industrial access road, and 150 feet on a minor residential street.

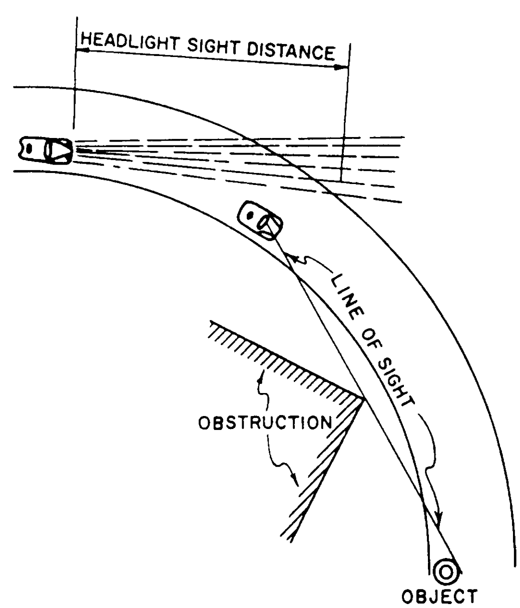


**Figure 11-1.—Lines of sight.**

## TYPES OF HORIZONTAL CURVES

There are four types of horizontal curves. They are described as follows:

1. SIMPLE. The simple curve is an arc of a circle (view A, fig. 11-2). The radius of the circle determines the sharpness or flatness of the curve.

2. COMPOUND. Frequently, the terrain will require the use of the compound curve. This curve normally consists of two simple curves joined together and curving in the same direction (view B, fig. 11-2).

3. REVERSE. A reverse curve consists of two simple curves joined together, but curving in opposite direction. For safety reasons, the use of this curve should be avoided when possible (view C, fig. 11-2).

4. SPIRAL. The spiral is a curve that has a varying radius. It is used on railroads and most modem highways. Its purpose is to provide a transition from the tangent to a simple curve or between simple curves in a compound curve (view D, fig. 11-2).

## ELEMENTS OF A HORIZONTAL CURVE

The elements of a circular curve are shown in figure 11-3. Each element is designated and explained as follows:

*PI* POINT OF INTERSECTION. The point of intersection is the point where the back and forward tangents intersect. Sometimes, the point of intersection is designated as *V (vertex).*

I INTERSECTING ANGLE. The intersecting angle is the deflection angle at the *PI.* Its value is either computed from the preliminary traverse angles or measured in the field.

Δ CENTRAL ANGLE. The central angle is the angle formed by two radii drawn from the

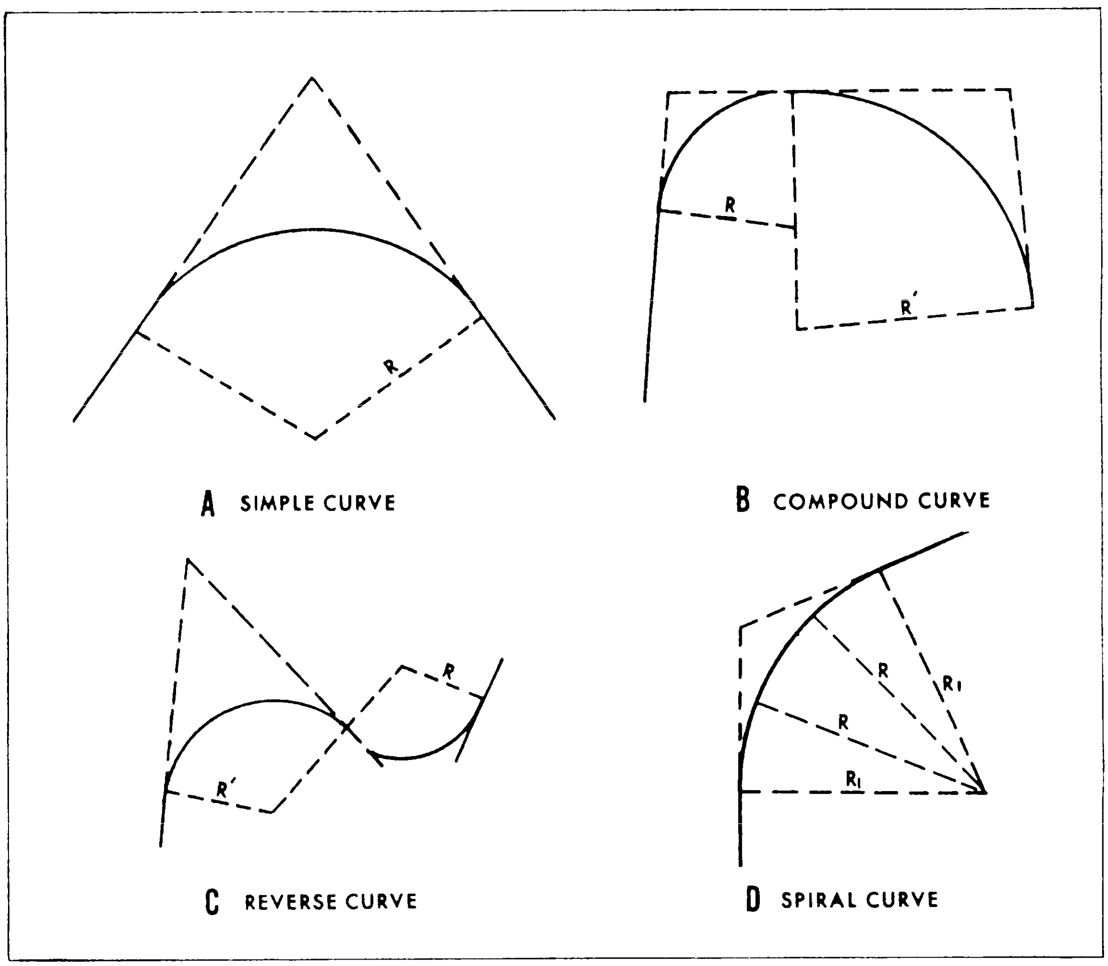


**Figure 11-2.—Horizontal curves.**

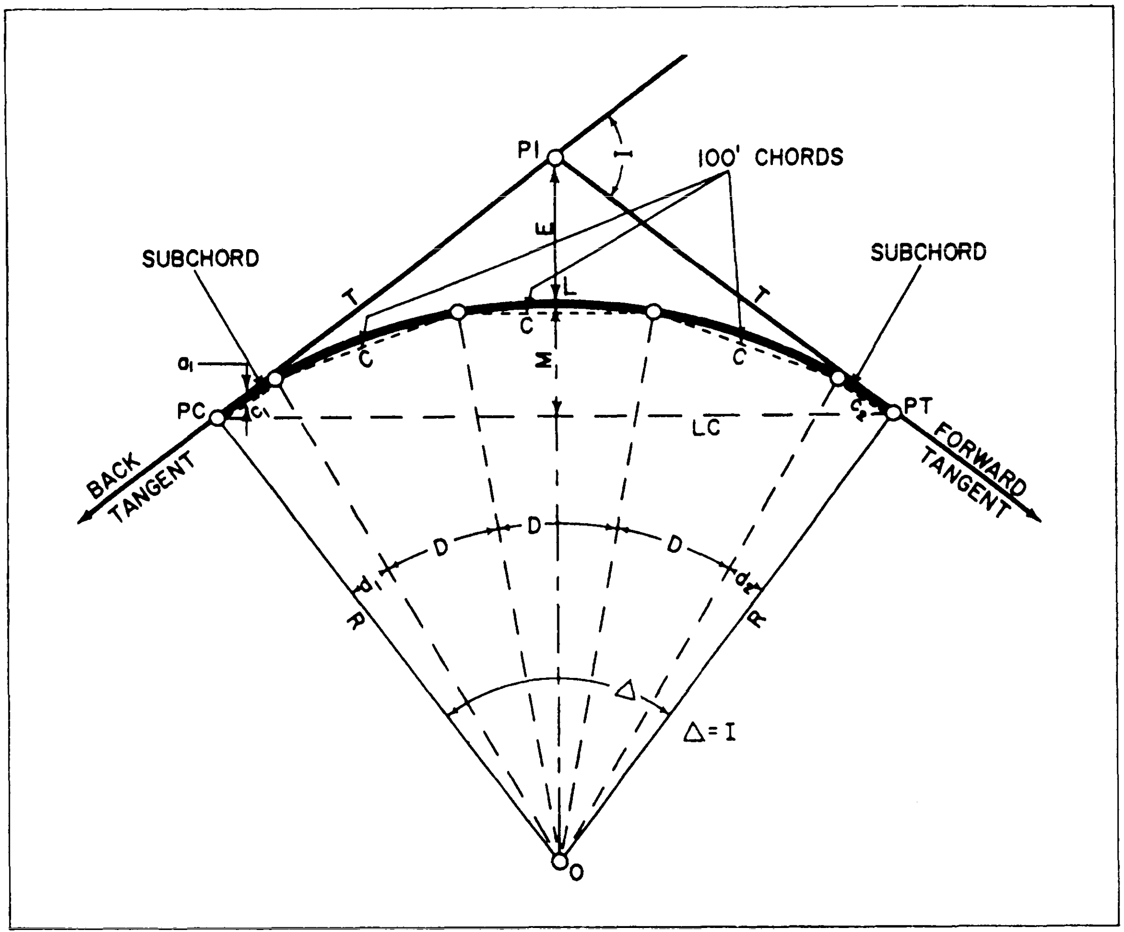


**Figure 11-3.—Elements of a horizontal curve.**

center of the circle *(O)* to the *PC* and *PT*. The value of the central angle is equal to the *I* angle. Some authorities call both the intersecting angle and central angle either *I* or Δ.

*R* RADIUS. The radius of the circle of which the curve is an arc, or segment. The radius is always perpendicular to back and forward tangents.

*PC* POINT OF CURVATURE. The point of curvature is the point on the back tangent where the circular curve begins. It is sometimes designated as *BC* (beginning of curve) or *TC* (tangent to curve).

*PT* POINT OF TANGENCY, The point of tangency is the point on the forward tangent where the curve ends. It is sometimes designated as *EC* (end of curve) or *CT* (curve to tangent).

*POC* POINT OF CURVE. The point of curve is any point along the curve.

*L* LENGTH OF CURVE. The length of curve is the distance from the *PC* to the *PT*, measured along the curve.

*T* TANGENT DISTANCE. The tangent distance is the distance along the tangents from the *PI* to the *PC* or the *PT*. These distances are equal on a simple curve.

*LC* LONG CHORD. The long chord is the straight-line distance from the *PC* to the *PT*. Other types of chords are designated as follows:

- *C* The full-chord distance between adjacent stations (full, half, quarter, or one-tenth stations) along a curve.
- $C_1$ The subchord distance between the *PC* and the first station on the curve.

$C_2$ The subchord distance between the last station on the curve and the *PT*.

*E* EXTERNAL DISTANCE. The external distance (also called the external secant) is the distance from the *PI* to the midpoint of the curve. The external distance bisects the interior angle at the *PI*.

*M* MIDDLE ORDINATE. The middle ordinate is the distance from the midpoint of the curve to the midpoint of the long chord. The extension of the middle ordinate bisects the central angle.

*D* DEGREE OF CURVE. The degree of curve defines the sharpness or flatness of the curve.

## DEGREE OF CURVATURE

The last of the elements listed above (degree of curve) deserves special attention. Curvature may be expressed by simply stating the length of the radius of the curve. That was done earlier in the chapter when typical radii for various roads were cited. Stating the radius is a common practice in land surveying and in the design of urban roads. For highway and railway work, however, curvature is expressed by the degree of curve. Two definitions are used for the degree of curve. These definitions are discussed in the following sections.

### Degree of Curve (Arc Definition)

The arc definition is most frequently used in highway design. This definition, illustrated in figure 11-4, states that the degree of curve is the central angle formed by two radii that extend from the center of a circle to the ends of an **arc** measuring 100 feet long (or 100 meters long if you are using metric units). Therefore, if you take a sharp curve, mark off a portion so that the distance along the arc is exactly 100 feet, and determine that the central angle is 12°, then you have a curve for which the degree of curvature is 12°; it is referred to as a 12° curve.

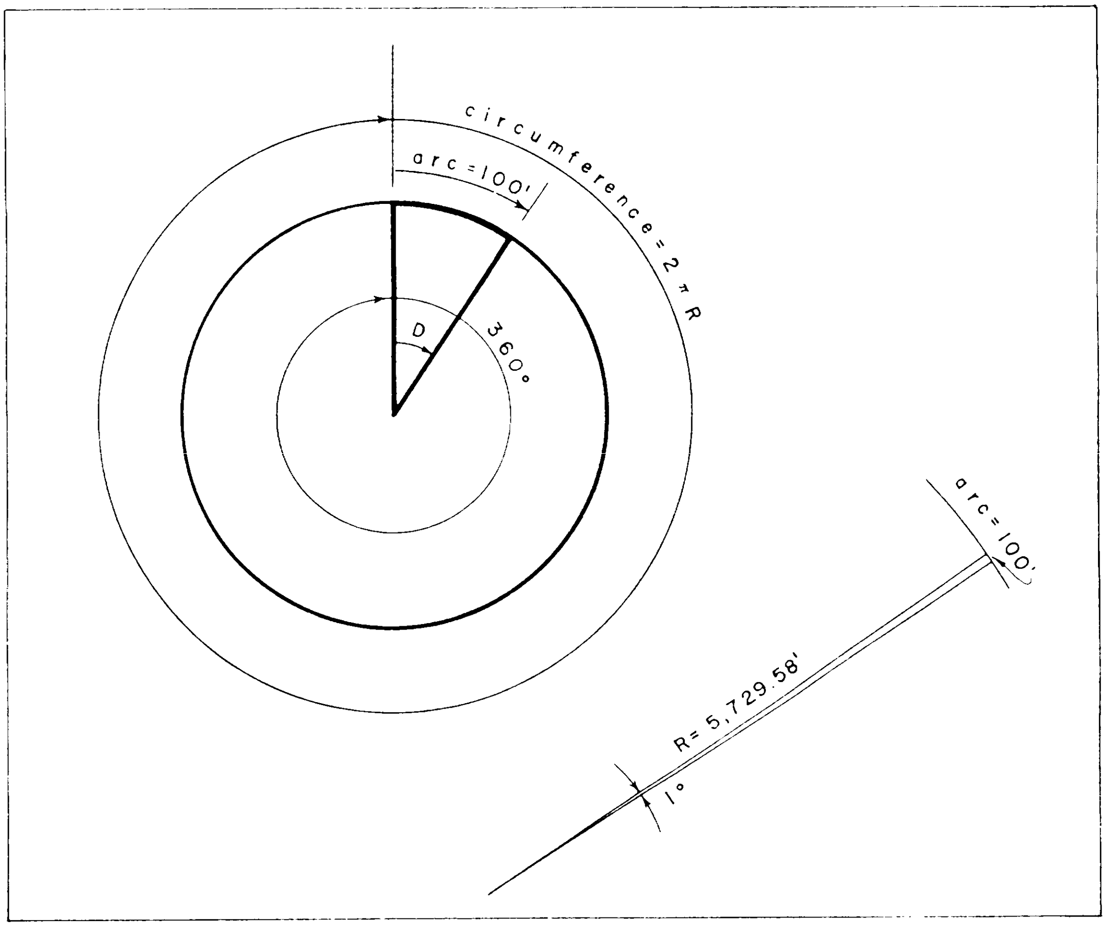


Figure 11-4.—Degree of curve (arc definition).

 

By studying figure 11-4, you can see that the ratio between the degree of curvature *(D)* and 360° is the same as the ratio between 100 feet of arc and the circumference *(C)* of a circle having the same radius. That may be expressed as follows:

$$\frac{D}{360^\circ} = \frac{100}{C}.$$

Since the circumference of a circle equals $2\pi R$, the above expression can be written as:

$$\frac{D}{360^\circ} = \frac{100}{2\pi R}.$$

Solving this expression for *R:*

$$R = \frac{5729.58}{D}$$

and also *D:*

$$D = \frac{5729.58}{R}.$$

For a 1° curve, $D = 1$; therefore $R = 5{,}729.58$ feet, or meters, depending upon the system of units you are using.

In practice the design engineer usually selects the degree of curvature on the basis of such factors as the design speed and allowable superelevation. Then the radius is calculated.

**Degree of Curve (Chord Definition)**

The chord definition (fig. 11-5) is used in railway practice and in some highway work. This definition states that the degree of curve is the central angle formed by two radii drawn from the center of the circle to the ends of a **chord** 100 feet (or 100 meters) long. If you take a flat curve, mark a 100-foot chord, and determine the central angle to be 0°30', then you have a 30-minute curve (chord definition).

From observation of figure 11-5, you can see the following trigonometric relationship:

$$\sin\left(\frac{D}{2}\right) = \frac{50}{R}.$$

Then, solving for *R:*

$$R = \frac{50}{\sin \frac{1}{2} D}.$$

For a 10 curve (chord definition), $D = 1$; therefore $R = 5{,}729.65$ feet, or meters, depending upon the system of units you are using.

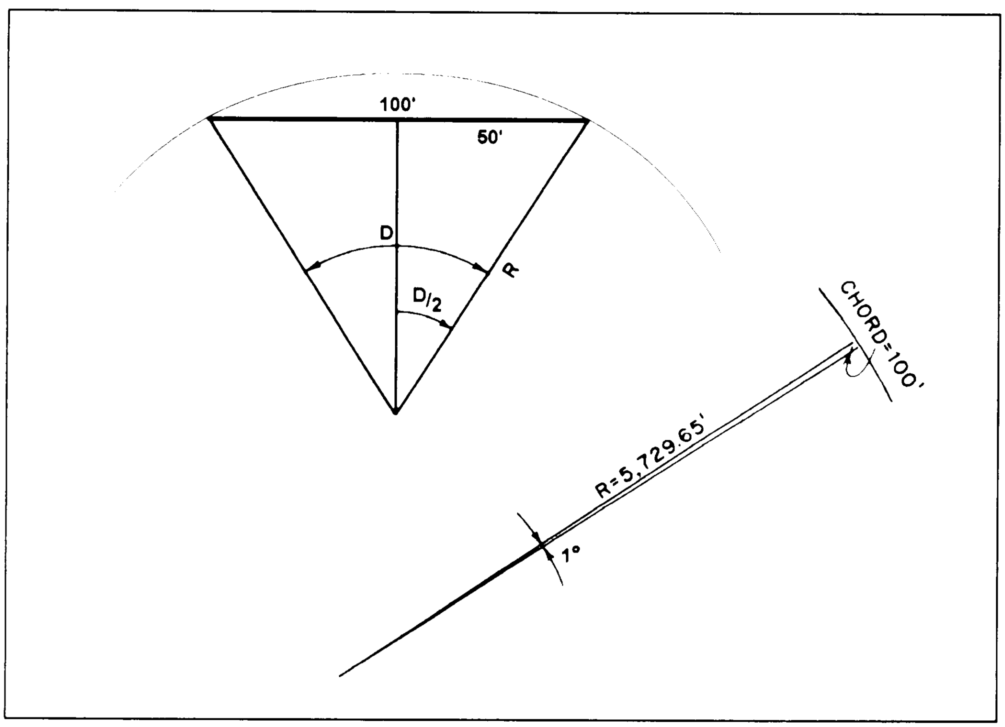


**Figure 11-5.—Degree of curve (chord definition).**

 

Notice that in both the arc definition and the chord definition, the radius of curvature is inversely proportional to the degree of curvature. In other words, the larger the degree of curve, the shorter the radius; for example, using the arc definition, the radius of a 1° curve is 5,729.58 units, and the radius of a 5° curve is 1,145.92 units. Under the chord definition, the radius of a 1° curve is 5,729.65 units, and the radius of a 5° curve is 1,146.28 units.

## CURVE FORMULAS

The relationship between the elements of a curve is expressed in a variety of formulas. The formulas for radius *(R)* and degree of curve *(D),* as they apply to both the arc and chord definitions, were given in the preceding discussion of the degree of curvature. Additional formulas you will use in the computations for a curve are discussed in the following sections.

### Tangent Distance

By studying figure 11-6, you can see that the solution for the tangent distance *(T)* is a simple right-triangle solution. In the figure, both $T$ and $R$ are sides of a right triangle, with $T$ being opposite to angle $\Delta/2$. Therefore, from your knowledge of trigonometric functions you know that

$$\tan\frac{\Delta}{2}=\frac{T}{R}$$

and solving for $T$,

$$T=R\tan\frac{\Delta}{2}.$$

### Chord Distance

By observing figure 11-7, you can see that the solution for the length of a chord, either a full chord *(C)* or the long chord *(LC),* is also a simple right-triangle solution. As shown in the figure, $C/2$ is one side of a right triangle and is opposite angle $\Delta/2$. The radius *(R)* is the hypotenuse of the same triangle. Therefore,

$$\sin\frac{\Delta}{2}=\frac{C/2}{R}$$

and solving for $C$:

$$C=2R\sin\frac{\Delta}{2}.$$

### Length of Curve

In the arc definition of the degree of curvature, length is measured along the arc, as shown in view A of figure 11-8, In this figure the relationship between $D$, $\Delta$, $L$, and a 100-foot arc length may be expressed as follows:

$$\frac{L}{100}=\frac{\Delta}{D}.$$

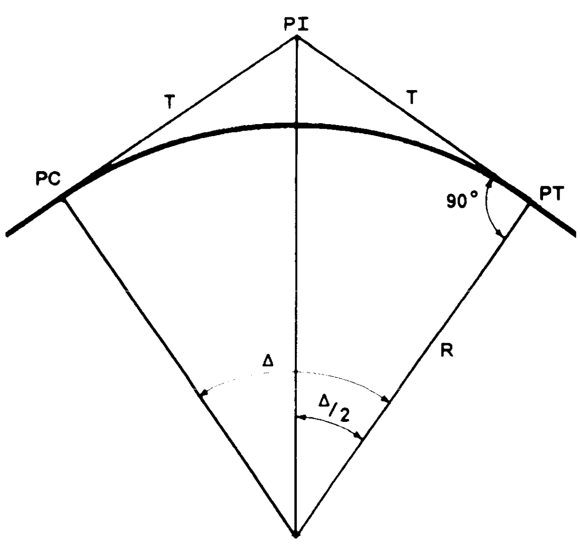


Figure 11-6.—Tangent distance.

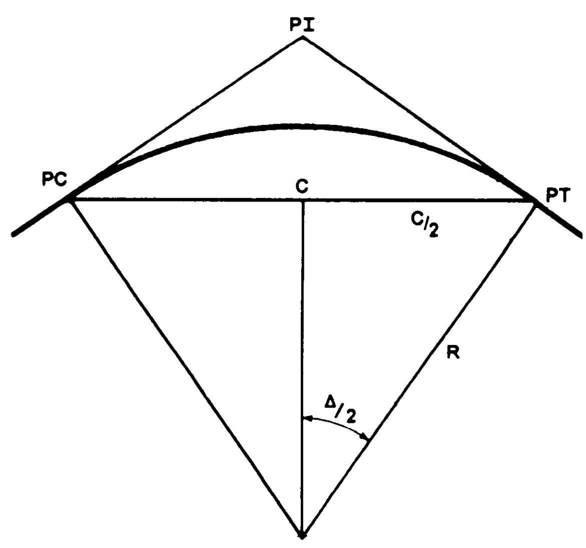


Figure 11-7.—Chord distance.

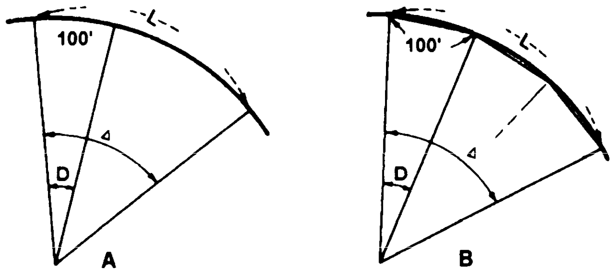


**Figure 11-8.-Length of curve.**

Then, solving for $L$,

$$L = 100\,\frac{\Delta}{D}.$$

This expression is also applicable to the chord definition. However, $L$., in this case, is not the true arc length, because under the chord definition, the length of curve is the sum of the chord lengths (each of which is usually 100 feet or 100 meters), As an example, if, as shown in view B, figure 11-8, the central angle ($\Delta$) is equal to three times the degree of curve ($D$), then there are three 100-foot chords; and the length of "curve" is 300 feet.

### Middle Ordinate and External Distance

Two commonly used formulas for the middle ordinate ($M$) and the external distance ($E$) are as follows:

$$M = R\left(1 - \cos\frac{\Delta}{2}\right)$$

$$E = T\tan\frac{\Delta}{4} = R\left(\frac{1}{\cos \Delta/2} - 1\right).$$

## DEFLECTION ANGLES AND CHORDS

From the preceding discussions, one may think that laying out a curve is simply a matter of locating the center of a circle, where two known or computed radii intersect, and then swinging the arc of the circular curve with a tape. For some applications, that can be done; for example, when you are laying out the intersection and curbs of a private road or driveway with a residential street. In this case, the length of the radii you are working with is short. However, what if you are laying out a road with a 1,000- or 12,000- or even a 40,000-foot radius? Obviously, it would be impracticable to swing such radii with a tape.

In usual practice, the stakeout of a long-radius curve involves a combination of turning **deflection angles** and measuring the length of chords ($C$, $C_1$, or $C_2$ as appropriate). A transit is set up at the PC, a sight is taken along the tangent, and each point is located by turning deflection angles and measuring the chord distance between stations. This procedure is illustrated in figure 11-9. In this figure, you see a portion of a curve that starts at the PC and runs through points (stations) A, B, and C. To establish the location of point A on this curve, you should set up your instrument at the PC, turn the required deflection angle ($d_1/2$), and then measure the required chord distance from PC to point A. Then, to establish point B, you turn deflection angle D/2 and measure the required chord distance from A to B. Point C is located similarly.

As you are aware, the actual distance along an arc is greater than the length of a corresponding chord; therefore, when using the arc definition, either a correction is applied for the difference between arc

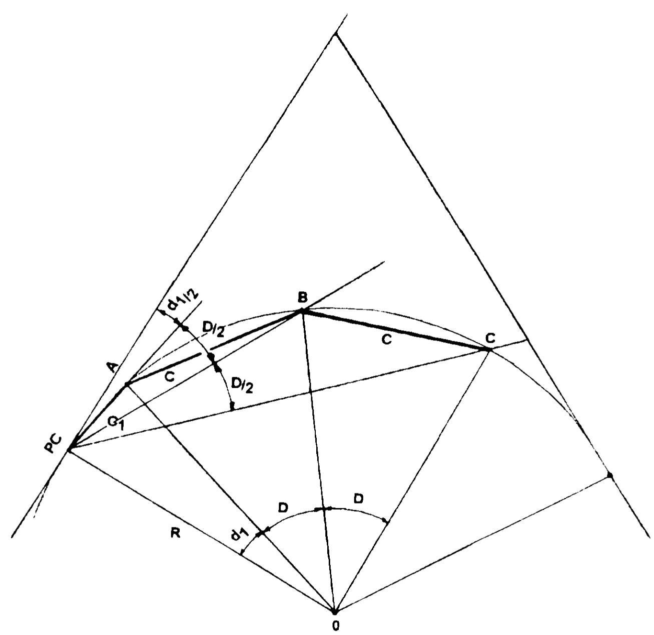


**Figure 11-9.-Deflection angles and chords.**

length and chord length, or shorter chords are used to make the error resulting from the difference negligible. In the latter case, the following chord lengths are commonly used for the degrees of curve shown:

100 feet—0 to 3 degrees of curve

50 feet—3 to 8 degrees of curve

25 feet—8 to 16 degrees of curve

10 feet-over 16 degrees of curve

The above chord lengths are the maximum distances in which the discrepancy between the arc length and chord length will fall within the allowable error for taping. The allowable error is 0.02 foot per 100 feet on most construction surveys; however, based on terrain conditions or other factors, the design or project engineer may determine that chord lengths other than those recommended above should be used for curve stakeout.

The following formulas relate to deflection angles: (To simplify the formulas and further discussions of deflection angles, the deflection angle is designated simply as $d$ rather than $d/2$.)

$$d = \left(\frac{D}{2}\right)\left(\frac{C}{100}\right)$$

Where:

$d$ = Deflection angle (expressed in degrees)

$C$ = Chord length

$D$ = Degree of curve

$$d = 0.3\ CD$$

Where:

$d$ = Deflection angle (expressed in minutes)

$C$ = Chord length

$D$ = Degree of curve

$$\sin d = \frac{C}{2R}$$

Where:

$d$ = Deflection angle (expressed in degrees)

C = Chord length

$R$ = Radius.

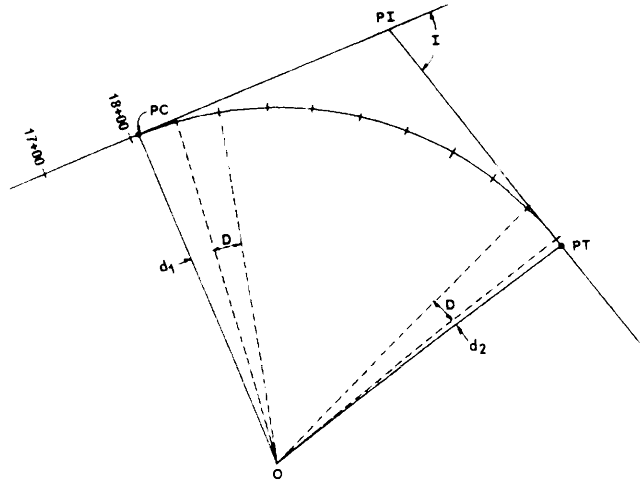


Figure 11-10.—Laying out a simple curve.

## SOLVING AND LAYING OUT A SIMPLE CURVE

Now let's solve and lay out a simple curve using the arc definition, which is the definition you will more often use as an EA. In figure 11-10, let's assume that the directions of the back and forward tangents and the location of the *PI* have previously been staked, but the tangent distances have not been measured. Let's also assume that stations have been set as far as Station 18 + 00. The specified degree of curve *(D) is* 15°, arc definition. Our job is to stake half-stations on the curve.

### Solving a Simple Curve

We will begin by first determining the distance from Station 18 + 00 to the location of the *PI*. Since these points have been staked, we can determine the distance by field measurement. Let's assume we have measured this distance and found it to be 300.89 feet. Next, we set up a transit at the *PI* and determine that deflection angle $I$ is 75°. Since $I$ always equals $\Delta$, then $\Delta$ is also 75°, Now we can compute the radius of the curve, the tangent distance, and the length of curve as follows:

$R = 5{,}729.58/D = 381.97$ ft

$T = R \tan \Delta/2 = 293.09$ ft

$L = 100\ \Delta/D = 500$ ft

From these computed values, we can determine the stations of the *PI, PC,* and *PT* as follows:

| | |
|---|---|
| Station at *PI* = (Sta. 18 + 00) + 300.89 = | 21 + 00.89 |
| Tangent distance = | (–) 2 + 93.09 |
| Station at *PC* | 18 + 07.80 |
| Length of curve = | (+) 5 + 00.00 |
| Station at *PT* | 23 + 07.80 |

By studying figure 11-10 and remembering that our task is to stake half-station intervals, you can see that the first half station after the *PC* is Station 18 + 50 and the last half station before the *PT* is 23+ 00; therefore, the distance from the *PC* to Station 18 + 00 is 42.2 feet [(18 + 50) - (18 + 07.80)]. Similarly, the distance from Station 23+ 00 to the *PT* is 7.8 feet. These distances are used to compute the deflection angles for the subchords using the formula for deflection angles *(d*= .3CD) as follows:

$$\text{Deflection angle } d_1 = .3 \times 42.2 \times 15 = 189.9' = 3°09.9'$$

$$\text{Deflection angle } d_2 = .3 \times 7.8 \times 15 = 35.1' = 0°35.1'$$

A convenient method of determining the deflection angle *(d)* for each full chord is to remember that *d* equals *1/2D* for 100-foot chords, *1/4D* for 50-foot chords, *1/8D* for 25-foot chords, and *1/20D* for 10-foot chords. In this case, since we are staking 50-foot stations, *d* = 15/4, or 3°45'.

Previously, we discussed the difference in length between arcs and chords. In that discussion, you learned that to be within allowable error, the recommended chord length for an 8- to 16-degree curve is 25 feet. Since in this example we are using 50-foot chords, the length of the chords must be adjusted. The adjusted lengths are computed using a rearrangement of the formula for the sine of deflection angles as follows:

$$C_1 = 2R \sin d_1 = 2 \times 381.97 \times \sin 3°09.9' = 42.18 \text{ ft}$$

$$C_2 = 2R \sin d_2 = 2 \times 381.97 \times \sin 0°35.1' = 7.79 \text{ ft}$$

$$C = 2R \sin d = 2 \times 381.97 \times \sin 3°45' = 49.96 \text{ ft}$$

As you can see, in this case, there is little difference between the original and adjusted chord lengths; however, if we were using 100-foot stations rather than 50-foot stations, the adjusted difference for each full chord would be substantial (over 3 inches).

Now, remembering our previous discussion of deflection angles and chords, you know that all of the deflection angles are usually turned using a transit that is set up at the *PC*. The deflection angles that we turn are found by cumulating the individual deflection angles from the *PC* to the *PT* as shown below:

| Station | Chord | Deflection angle |
|---|---|---|
| *PC* 18 + 07.80 | -------- | 0°00.0′ |
| 18 + 50 | $C_1$ 42.18 | 3°09.9′ |
| 19 + 00 | 49.96 | 6°54.9′ |
| 19 + 50 | 49.96 | 10°39.9′ |
| 20 + 00 | 49.96 | 14°24.9′ |
| 20 + 50 | 49.96 | 18°09.9′ |
| 21 + 00 | 49.96 | 21°54.9′ |
| 21 + 50 | 49.96 | 25°39.9′ |
| 22 + 00 | 49.96 | 29°24.9′ |
| 22 + 50 | 49.96 | 33°09.9′ |
| 23 + 00 | 49.96 | 36°54.9′ |
| *PT* 23 + 07.80 | $C_2$ 07.79 | 37°30′ |

Notice that the deflection angle at the *PT* is equal to one half of the *I* angle. That serves as a check of your computations. Had the deflection angle been anything different than one half of the *I* angle, then a mistake would have been made.

Since the total of the deflection angles should be one-half of the *I* angle, a problem arises when the *I* angle contains an odd number of minutes and the instrument used is a 1-minute transit. Since the *PT* is normally staked before the curve is run, the total deflection will be a check on the *PC* therefore, it should be computed to the nearest 0.5 degree. If the total deflection checks to the nearest minute in the field, it can be considered correct.

The curve that was just solved had an *I* angle of 75° and a degree of curve of 15°. When the *I* angle and degree of curve consists of both degrees and minutes, the procedure in solving the curve does not change; but you must be careful in substituting these values into the formulas for length and deflection angles; for example *I* = 42°15', *D* = 5°37'. The minutes in each angle must be changed to a decimal part of a degree. To obtain the required accuracy, you should convert them to five decimal places; but an alternate method for computing the length is to convert the *I* angle and degree of curve to minutes; thus, 42°15' = 2,535 minutes and 5°37' = 337 minutes. Substituting this information into the length formula gives the following:

$$L = 100 \times \frac{2{,}535}{337} = 752.23 \text{ feet.}$$

This method gives an exact result. By converting the minutes to a decimal part of a degree to the nearest fives places, you obtain the same result.

**Simple Curve Layout**

To lay out the simple curve (arc definition) just computed above, you should usually use the procedure that follows.

1. With the instrument placed at the *PI,* the instrumentman sights on the preceding *PI* or at a distant station and keeps the chainman on the line while the tangent distance is measured to locate the *PC.* After the *PC* has been staked out, the instrumentman then trains the instrument on the forward *PI* to locate the *PT.*

2. The instrumentman then sets up at the *PC* and measures the angle from the *PI* to the *PT.* This angle should be equal to one half of the *I* angle; if it is not, either the *PC* or the *PT* has been located in the wrong position.

3. With the first deflection angle (3°10') set on the plates, the instrumentman keeps the chainman on line as the first subchord distance (42.18 feet) is measured from the *PC.*

4. Without touching the lower motion screw, the instrumentman sets the second deflection angle (6°55') on the plates. The chainman measures the chord from the previous station while the instrumentman keeps the head chainman on line.

5. The crew stakes out the succeeding stations in the same manner. If the work is done correctly, the last deflection angle will point on the *PT.* That distance will be the subchord length (7.79 feet) from the last station before the *PT.*

When it is impossible to stake out the entire curve from the *PC,* a modified method of the procedure described above is used. Stake out the curve as far as possible from the *PC.* If a station cannot be seen from the *PC* for some reason, move the transit forward and set up over a station along the curve. Pick a station for a backsight and set the deflection angle for that station on the plates. Sight on this station with the telescope in the reverse position. Plunge the telescope and set

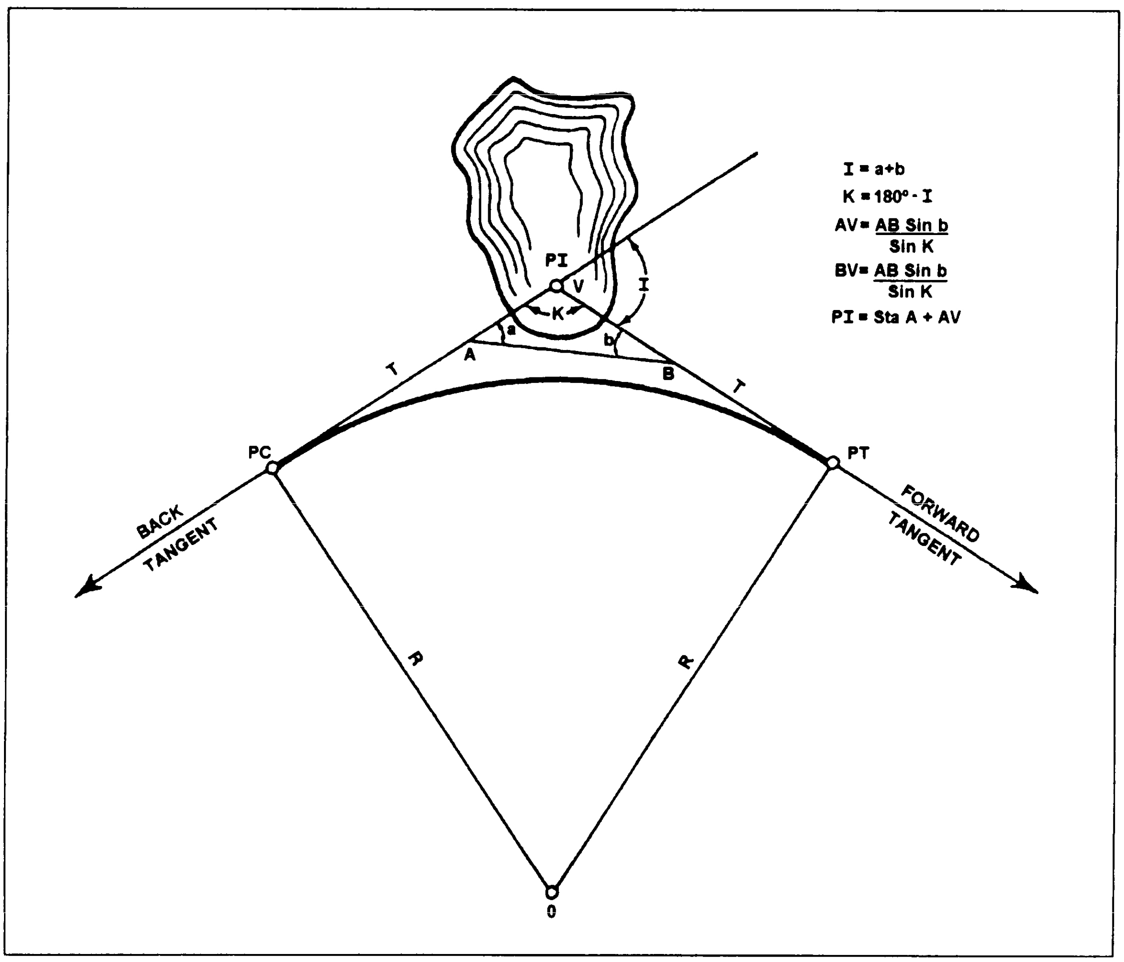


Figure 11-11.—Inaccessible *PI.*

 

the remainder of the stations in the same way as you would if the transit was set over the *PC*. If the setup in the curve has been made but the next stake cannot be set because of obstructions, the curve can be backed in. To back in a curve, occupy the *PT*. Sight on the *PI* and set one half of the *I* angle of the plates. The transit is now oriented so that, if the *PC* is observed, the plates will read zero, which is the deflection angle shown in the notes for that station. The curve stakes can then be set in the same order shown in the notes or in the reverse order. Remember to use the deflection angles and chords from the top of the column or from the bottom of the column. Although the back-in method has been set up as a way to avoid obstructions, it is also very widely used as a method for laying out curves. The method is to proceed to the approximate midpoint of the curve by laying out the deflection angles and chords from the *PC* and then laying out the remainder of the curve from the *PT*. If this method is used, any error in the curve is in the center where it is less noticeable.

So far in our discussions, we have begun staking out curves by setting up the transit at the *PI*. But what do you do if the *PI* is inaccessible? This condition is illustrated in figure 11-11. In this situation, you locate the curve elements using the following steps:

1. As shown in figure 11-11, mark two intervisible points *A* and *B* on the tangents so that line *AB* clears the obstacle.

2. Measure angles *a* and *b* by setting up at both *A* and *B*.

3. Measure the distance *AB*.

4. Compute inaccessible distance *AV* and *BV* using the formulas given in figure 11-11.

5. Determine the tangent distance from the *PI* to the *PC* on the basis of the degree of curve or other given limiting factor.

6. Locate the *PC* at a distance *T* minus *AV* from the point *A* and the *PT* at a distance *T* minus *BV* from point *B*.

## Field Notes

Figure 11-12 shows field notes for the curve we solved and staked out above. By now you should be

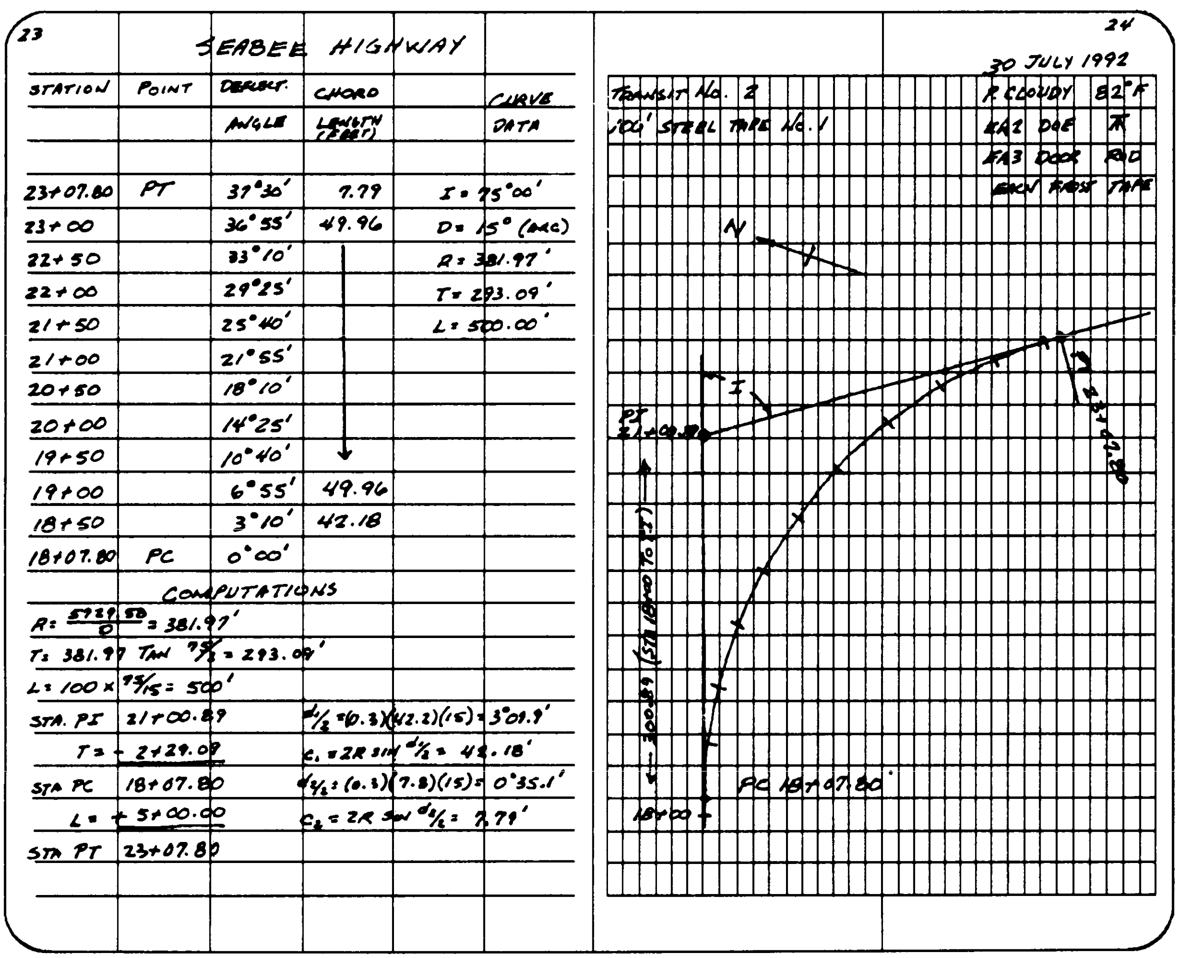


Figure 11-12.—Field notes for laying out a simple curve.

familiar enough with field notes to preclude a complete discussion of everything shown in these notes. You should notice, however, that the stations are entered in reverse order (bottom to top). In this manner the data is presented as it appears in the field when you are sighting ahead on the line. This same practice applies to the sketch shown on the right-hand page of the field notes.

For information about other situations involving inaccessible points or the uses of external and middle ordinate distance, spiral transitions, and other types of horizontal curves, study books such as those mentioned at the beginning of this chapter.

## VERTICAL CURVES

In addition to horizontal curves that go to the right or left, roads also have vertical curves that go up or down. Vertical curves at a crest or the top of a hill are called **summit curves,** or **oververticals.** Vertical curves at the bottom of a hill or dip are called **sag curves,** or **underverticals.**

### GRADES

Vertical curves are used to connect stretches of road that go up or down at a constant slope. These lines of constant slope are called **grade tangents** (fig. 11- 13). The rate of slope is called the **gradient,** or simply the **grade.** (Do not confuse this use of the term *grade* with other meanings, such as the design elevation of a finished surface at a given point or the actual elevation of the existing ground at a given point.) Grades that ascend in the direction of the stationing are designated as plus; those that descend in the direction of the stationing are designated as minus. Grades are measured in terms of percent; that is, the number of feet of rise or fall in a 100-foot horizontal stretch of the road.

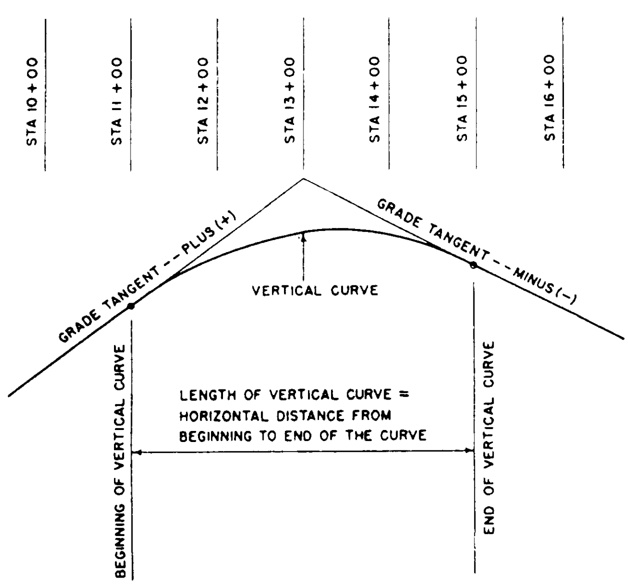


Figure 11-13.—A vertical curve.

After the location of a road has been determined and the necessary fieldwork has been obtained, the engineer designs or fixes (sets) the grades. A number of factors are considered, including the intended use and importance of the road and the existing topography. If a road is too steep, the comfort and safety of the users and fuel consumption of the vehicles will be adversely affected; therefore, the design criteria will specify **maximum grades.** Typical maximum grades are a 4-percent desired maximum and a 6-percent absolute maximum for a primary road. (The 6 percent means, as indicated before, a 6-foot rise for each 100 feet ahead on the road.) For a secondary road or a major street, the maximum grades might be a 5-percent desired and an 8-percent absolute maximum; and for a tertiary road or a secondary street, an 8-percent desired and a 10-percent (or perhaps a 12-percent) absolute maximum. Conditions may sometimes demand that grades or ramps, driveways, or short access streets go as high as 20 percent. The engineer must also consider **minimum grades.** A street with curb and gutter must have enough fall so that the storm water will drain to the inlets; 0.5 percent is a typical minimum grade for curb and gutter (that is, 1/2 foot minimum fall for each 100 feet ahead). For roads with side ditches, the desired minimum grade might be 1 percent; but since ditches may slope at a grade different from the pavement, a road may be designed with a zero-percent grade. Zero-percent grades are not unusual, particularly through plains or tidewater areas. Another factor considered in designing the finished profile of a road is the **earthwork balance;** that is, the grades should be set so that all the soil cut off of the hills may be economically hauled to fill in the low areas. In the design of urban streets, the best use of the building sites next to the street will generally be more important than seeking an earthwork balance.

### COMPUTING VERTICAL CURVES

As you have learned earlier, the horizontal curves used in highway work are generally the arcs of circles. But vertical curves are usually **parabolic.** The parabola is used primarily because its shape provides a transition and, also, lends itself to the computational methods described in the next section of this chapter. Designing a vertical curve consists principally of deciding on the proper length of the curve. As indicated in figure 11-13, the length of a vertical curve is the **horizontal distance** from the beginning to the end of the curve; the length of the curve is NOT the distance along the parabola itself. The longer a curve is, the more gradual the transition will be from one grade to the next; the shorter the curve, the more abrupt the change. The change must be gradual enough to provide the required sight distance (fig. 11- 14). The sight distance requirement will depend on the speed for which the road is designed; whether passing or nonpassing distance is required; and other assumptions, such as one's reaction time, braking time, stopping distance, height of one's eyes, and height of objects. A typical eye level used for designs is 4.5 feet or, more recently, 3.75 feet; typical object heights are 4 inches to 1.5 feet. For a sag curve, the sight distance will usually not be significant during daylight; but the nighttime sight distance must be considered when the reach of headlights may be limited by the abruptness of the curve.

## ELEMENTS OF VERTICAL CURVES

Figure 11-15 shows the elements of a vertical curve. The meaning of the symbols and the units of measurement usually assigned to them follow:

*PVC* Point of vertical curvature; the place where the curve begins.

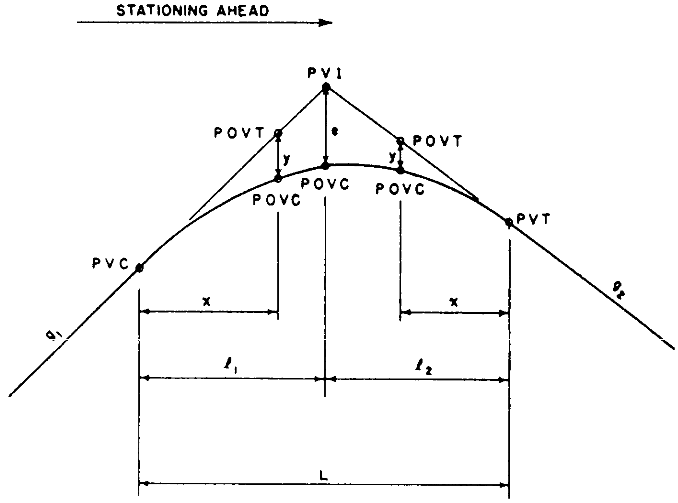


Figure 11-15.—Elements of a vertical curve.

*PVI* Point of vertical intersection; where the grade tangents intersect.

*PVT* Point of vertical tangency; where the curve ends.

*POVC* Point on vertical curve; applies to any point on the parabola.

*POVT* Point on vertical tangent; applies to any point on either tangent.

$g_1$ Grade of the tangent on which the *PVC* is located; measured in percent of slope.

$g_2$ Grade of the tangent on which the *PVT* is located; measured in percent of slope.

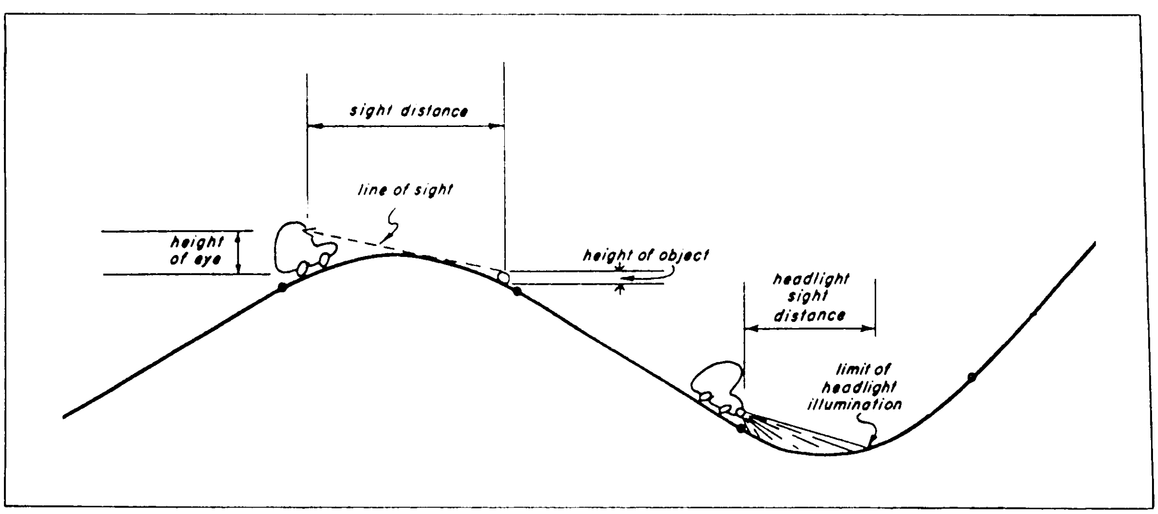


Figure 11-14.—Sight distance.

*G* The **algebraic difference** of the grades:

$$G = g_2 - g_1,$$

wherein plus values are assigned to uphill grades and minus values to downhill grades; examples of various algebraic differences are shown later in this section.

*L* Length of the curve; the **horizontal** length measured in 100-foot stations from the *PVC* to the PVT. This length may be computed using the formula ***L* = *G/r*,** where *r* is the rate of change (usually given in the design criteria). When the rate of change is not given, *L* (in stations) can be computed as follows: for a summit curve, **L = 125** x G/4; for a sag curve, **L = 100** x G/4. If *L* does not come out to a whole number of stations using these formulas, then it is usually extended to the nearest whole number. You should note that these formulas for length are for road design only, NOT railway.

$l_1$ Horizontal length of the portion of the *PVC* to the *PVI;* measured in feet.

$l_2$ Horizontal length of the portion of the curve form the *PVI* to the *PVT;* measured in feet.

*e* Vertical (external) distance from the *PVI* to the curve, measured in feet. This distance is computed using the formula e **= LG/8,** where *L* is the total length in stations and G is the algebraic difference of the grades in percent.

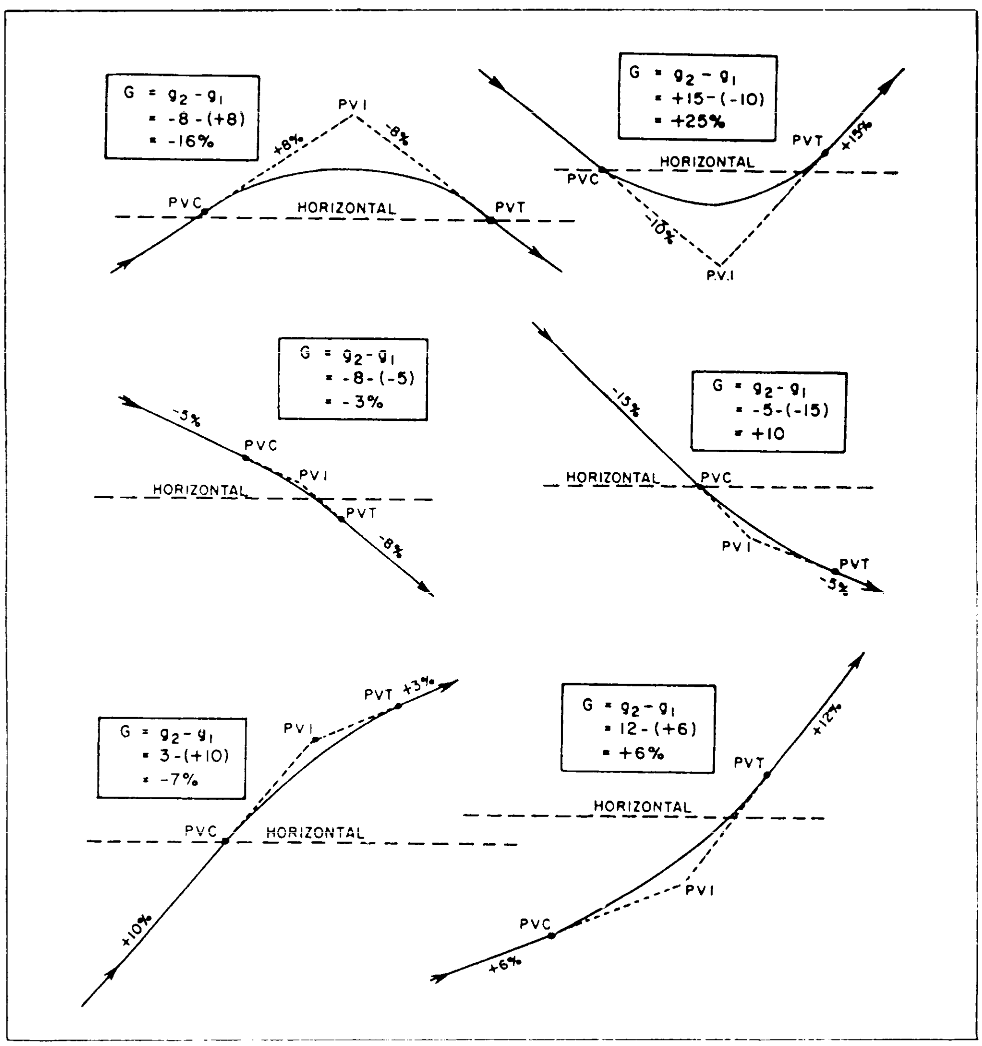


**Figure 11-16.—Algebraic differences of grades.**

$x$ Horizontal distance from the *PVC* to any *POVC* or POVTback of the *PVI*, or the distance from the PVT to any *POVC* or *POVT* ahead of the *PW*, measured in feet.

$y$ Vertical distance (offset) from any *POVT* to the corresponding *POVC*, measured in feet;

$$y = (x/l)^2(e),$$

which is the fundamental relationship of the parabola that permits convenient calculation of the vertical offsets.

The vertical curve computation takes place after the grades have been set and the curve designed. Therefore, at the beginning of the detailed computations, the following are known: $g_1$, $g_2$, $l_1$, $l_2$, $L$, and the elevation of the *PVI*. The general procedure is to compute the elevations of certain *POVTs* and then to use the foregoing formulas to compute *G*, then *e*, and then the *Ys* that correspond to the selected *POVTs*. When the $y$ is added or subtracted from the elevation of the *POVT*, the result is the elevation of the *POVC*. The *POVC* is the finished elevation on the road, which is the end result being sought. In figure 11-15, the $y$ is subtracted from the elevation of the *POVT* to get the elevation of the curve; but in the case of a sag curve, the $y$ is added to the *POVT* elevation to obtain the *POVC* elevation.

The computation of *G* requires careful attention to the signs of $g_1$ and $g_2$. Vertical curves are used at changes of grade other than at the top or bottom of a hill; for example, an uphill grade that intersects an even steeper uphill grade will be eased by a vertical curve. The six possible combinations of plus and minus grades, together with sample computations of *G*, are shown in figure 11-16. Note that the algebraic sign for *G* indicates whether to add or subtract $y$ from a *POVT*.

The selection of the points at which to compute the $y$ and the elevations of the *POVT* and *POVC* is generally based on the stationing. The horizontal alignment of a road is often staked out on 50-foot or 100-foot stations. Customarily, the elevations are computed at these same points so that both horizontal and vertical information for construction will be provided at the same point. The *PVC, PVI,* and *PVT* are usually set at full stations or half stations. In urban work, elevations are sometimes computed and staked every 25 feet on vertical curves. The same, or even closer, intervals may be used on complex ramps and interchanges. The application of the foregoing fundamentals will be presented in the next two sections under symmetrical and unsymmetrical curves.

### Symmetrical Vertical Curves

A symmetrical vertical curve is one in which the horizontal distance from the *PVI* to the *PVC* is equal to the horizontal distance from the *PW* to the *PVT*. In other words, $l_1$ equals $l_2$.

The solution of a typical problem dealing with a symmetrical vertical curve will be presented step by step. Assume that you know the following data:

$g_1$ = +97%

$g_2$ = −7%

$L$ = 400.00´, or 4 stations

The station of the *PVI* = 30 + 00

The elevation of the *PVI* = 239.12 feet

The problem is to compute the grade elevation of the curve to the nearest hundredth of a foot at each 50-foot station. Figure 11-17 shows the vertical curve to be solved.

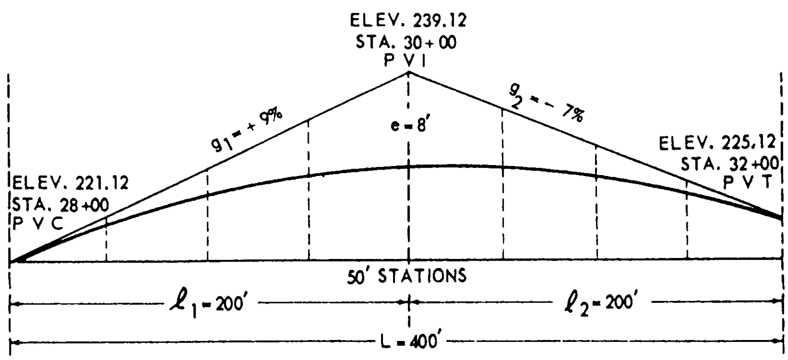


**Figure 11-17.—Symmetrical vertical curve.**

STEP 1: Prepare a table as shown in figure 11-18. In this figure, column 1 shows the stations; column 2, the elevations on tangent; column 3, the ratio of $x/l$; column 4, the ratio of $(x/l)^2$; column 5, the vertical offsets $[(x/l)^2(e)]$; column 6, the grade elevations on the curve; column 7, the first difference; and column 8, the second difference.

STEP 2: Compute the elevations and set the stations on the *PVC* and the *PVT*.

Knowing both the gradients at the *PVC* and *PVT* and the elevation and station at the *PVI,* you can compute the elevations and set the stations on the *PVC* and the *PVT*. The gradient ($g_1$) of the tangent at the *PVC is* given as +9 percent. This means a rise in elevation of 9 feet for every 100 feet of horizontal distance. Since $L$ is 400.00 feet and the curve is symmetrical, $l_1$ equals $l_2$ equals 200.00 feet; therefore, there will be a difference of 9 x 2, or 18, feet between the elevation at the *PVI* and the elevation at the *PVC.* The elevation at the *PVI* in this problem is given as 239.12 feet; therefore, the elevation at the *PVC is*

$$239.12 - 18 = 221.12 \text{ feet.}$$

Calculate the elevation at the *PVT* in a similar manner. The gradient ($g_2$) of the tangent at the *PVT* is given as –7 percent. This means a drop in elevation of 7 feet for every 100 feet of horizontal distance. Since $l_1$ equals $l_2$ equals 200 feet, there will be a difference of 7 x 2, or 14, feet between the elevation at the *PVI* and the elevation at the *PVT*. The elevation at the *PVI* therefore is

$$239.12 - 14 = 225{,}12 \text{ feet.}$$

In setting stations on a vertical curve, remember that the length of the curve *(L)* is always measured as a horizontal distance. The half-length of the curve is the horizontal distance from the *PVI* to the *PVC.* In this problem, $l_1$ equals 200 feet. That is equivalent to two 100-foot stations and may be expressed as 2 + 00. Thus the station at the *PVC* is

30 + 00 **minus** 2 + 00, or 28 + 00.

The station at the *PVT* is

30 + 00 **plus** 2 + 00, or 32 + 00.

List the stations under column 1.

STEP 3: Calculate the elevations at each 50-foot station on the tangent.

From Step 2, you know there is a 9-foot rise in elevation for every 100 feet of horizontal distance from the *PVC* to the *PVI.* Thus, for every 50 feet of horizontal distance, there will be a rise of 4.50 feet in elevation. The elevation on the tangent at station 28 + 50 is

$$221.12 + 4.50 = 225.62 \text{ feet.}$$

The elevation on the tangent at station 29 + 00 is

$$225.62 + 4.50 = 230.12 \text{ feet.}$$

| Stations | Elevations on tangent | $x/l$ | $(x/l)^2$ | Vertical offsets | Grade elevation on curve | First difference | Second difference |
|---|---|---|---|---|---|---|---|
| 28+00 (*PVC*) | 221.12 | 0 | 0 | 0 | 221.12 | | |
| | | | | | | +4.00 | |
| +50 | 225.62 | 1/4 | 1/16 | −0.50 | 225.12 | | +1.00 |
| | | | | | | +3.00 | |
| 29+00 | 230.12 | 1/2 | 1/4 | −2.00 | 228.12 | | +1.00 |
| | | | | | | +2.00 | |
| +50 | 234.62 | 3/4 | 9/16 | −4.50 | 230.12 | | +1.00 |
| | | | | | | +1.00 | |
| 30+00 (*P V I*) | 239.12 | 1 | 1 | −8.00 | 231.12 | | +1.00 |
| | | | | | | .00 | |
| +50 | 235.62 | 3/4 | 9/16 | −4.50 | 231.12 | | +1.00 |
| | | | | | | −1.00 | |
| 31+00 | 232.12 | 1/2 | 1/4 | −2.00 | 230.12 | | +1.00 |
| | | | | | | −2.00 | |
| +50 | 228.62 | 1/4 | 1/16 | − .50 | 228.12 | | +1.00 |
| | | | | | | −3.00 | |
| 32+00 (*PVT*) | 225.12 | 0 | 0 | 0 | 225.12 | | |

**Figure 11-18.—Table of computations of elevations on a symmetrical vertical curve.**

The elevation on the tangent at station 29 + 50 is

230.12 + 4.50 = 234.62 feet.

The elevation on the tangent at station 30 + 00 is

234.62 + 4.50 = 239.12 feet.

In this problem, to find the elevation on the tangent at any 50-foot station starting at the *PVC,* add 4.50 to the elevation at the preceding station until you reach the *PVI.* At this point use a slightly different method to calculate elevations because the curve slopes downward toward the *PVT.* Think of the elevations as being divided into two groups—one group running from the *PVC* to the *PVI;* the other group running from the *PVT* to the *PVI.*

Going downhill on a gradient of –7 percent from the *PVI* to the *PVT,* there will be a drop of 3.50 feet for every 50 feet of horizontal distance. To find the elevations at stations between the *PVI* to the *PVT* in this particular problem, subtract 3.50 from the elevation at the preceding station. The elevation on the tangent at station 30 + 50 is

239.12-3.50, or 235.62 feet.

The elevation on the tangent at station 31 + 50 is

235.62-3.50, or 232.12 feet.

The elevation on the tangent at station 31 + 50 is

232.12-3.50, or 228.62 feet.

The elevation on the tangent at station 32+00 (PVT) is

228.62-3.50, or 225.12 feet,

The last subtraction provides a check on the work you have finished. List the computed elevations under column 2.

STEP 4: Calculate *(e),* the middle vertical offset at the *PVI.*

First, find the *(G),* the algebraic difference of the gradients using the formula

$$G = g_2 - g_1$$
$$G = -7 - (+9)$$
$$G = -16\%$$

The middle vertical offset *(e)* is calculated as follows:

$$e = LG/8 = [(4)(-16)]/8 = -8.00 \text{ feet.}$$

The negative sign indicates e is to be subtracted from the *PVI.*

STEP 5: Compute the vertical offsets at each 50-foot station, using the formula $(x/l)^2e$. To find the vertical offset at any point on a vertical curve, first find the ratio $x/l$; then square it and multiply by *e;* for example, at station 28 + 50, the ratio of $x/l = 50/200 = 1/4$.

Therefore, the vertical offset is

$$(1/4)^2 e = (1/16)\, e.$$

The vertical offset at station 28 + 50 equals

$$(1/16)(-8) = -0.50 \text{ foot.}$$

Repeat this procedure to find the vertical offset at each of the 50-foot stations. List the results under columns 3, 4, and 5.

STEP 6: Compute the grade elevation at each of the 50-foot stations.

When the curve is on a crest, the sign of the offset will be negative; therefore, subtract the vertical offset (the figure in column 5) from the elevation on the tangent (the figure in column 2); for example, the grade elevation at station 29 + 50 is

234.62 – 4.50 = 230.12 feet.

Obtain the grade elevation at each of the stations in a similar manner. Enter the results under column 6.

**Note:** When the curve is in a dip, the sign will be positive; therefore, you will **add** the vertical offset (the figure in column 5) to the elevation on the tangent (the figure in column 2).

STEP 7: Find the turning point on the vertical curve.

When the curve is on a crest, the turning point is the highest point on the curve. When the curve is in a dip, the turning point is the lowest point on the curve. The turning point will be directly above or below the *PVI* only when both tangents have the same percent of slope (ignoring the algebraic sign); otherwise, the turning point will be on the same side of the curve as the tangent with the least percent of slope.

The horizontal location of the turning point is either measured from the *PVC* if the tangent with the lesser slope begins there or from the *PVT* if the tangent with the lesser slope ends there. The horizontal location is found by the formula:

$$x_t = \frac{gL}{G}$$

Where:

$x_t$ = distance of turning point from *PVC* or *PVT*

$g$ = lesser slope (ignoring signs)

$L$ = length of curve in stations

$G$ = algebraic difference of slopes.

 

For the curve we are calculating, the computations would be (7 x 4)/16 = 1.75 feet; therefore, the turning point is 1.75 stations, or 175 feet, from the *PVT* (station 30 + 25).

The vertical offset for the turning point is found by the formula:

$$y_t = \left(\frac{x_t}{l}\right)^2 e.$$

For this curve, then, the computation is ( $1.75/2)^2$ x 8 = 6.12 feet.

The elevation of the *POVT* at 30 + 25 would be 237.37, calculated as explained earlier. The elevation on the curve would be

$$237.37 - 6.12 = 231.25.$$

STEP 8: Check your work.

One of the characteristics of a symmetrical parabolic curve is that the second differences between successive grade elevations at full stations are constant. In computing the first and second differences (columns 7 and 8), you must consider the plus or minus signs. When you round off your grade elevation figures following the degree of precision required, you introduce an error that will cause the second difference to vary slightly from the first difference; however, the slight variation does not detract from the value of the second difference as a check on your computations. You are cautioned that the second difference will not always come out exactly even and equal. It is merely a coincidence that the second difference has come out exactly the same in this particular problem.

**Unsymmetrical Vertical Curves**

An unsymmetrical vertical curve is a curve in which the horizontal distance from the *PVI* to the *PVC* is different from the horizontal distance between the *PVI* and the *PVT*. In other words, $l_1$ does NOT equal $l_2$. Unsymmetrical curves are sometimes described as having unequal tangents and are referred to as dog legs. Figure 11-19 shows an unsymmetrical curve with a horizontal distance of 400 feet on the left and a horizontal distance of 200 feet on the right of the *PVI*. The gradient of the tangent at the *PVC* is –4 percent; the gradient of the tangent at the *PVT* is +6 percent. Note that the curve is in a dip.

As an example, let's assume you are given the following values:

Elevation at the *PVI* is 332.68

Station at the *PVI* is 42 + 00

$l_1$ is 400 feet

$l_2$ is 200 feet

$g_1$ is –4%

$g_2$ is +6%

To calculate the grade elevations on the curve to the nearest hundredth foot, use figure 11-20 as an example.

Figure 11-20 shows the computations. Set four 100-foot stations on the left side of the *PVI* (between the *PVI* and the *PVC*). Set four 50-foot stations on the right side of the *PVl* (between the *PVI* and the *PVT*). The procedure for solving an unsymmetrical curve problem is essentially the same as that used in solving a symmetrical curve. There are, however, important differences you should be cautioned about.

First, you use a different formula for the calculation of the middle vertical offset at the *PVI*. For an unsymmetrical curve, the formula is as follows:

$$e = \frac{l_1 l_2}{2(l_1 + l_2)}(g_2 - g_1).$$

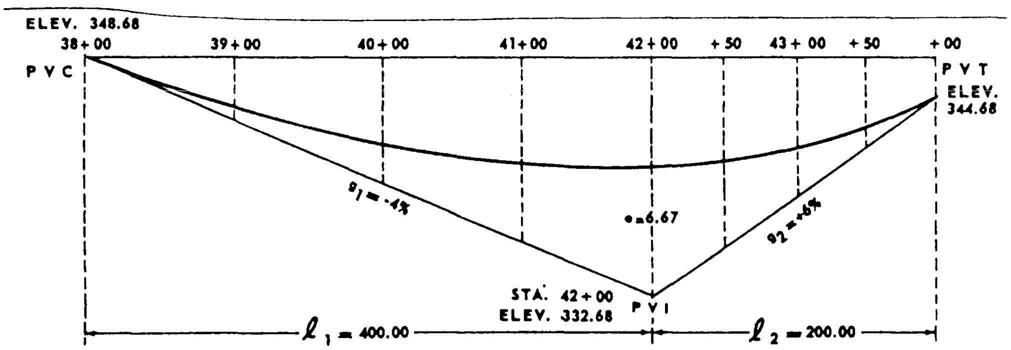


**Figure 11-19.—Unsymmetrical vertical curve.**

| Col. 1 Stations | Col. 2 Elevations on tangent | Col. 3 $x/l$ | Col. 4 $(x/l)^2$ | Col. 5 Vertical offsets | Col. 6 Grade elevation on curve |
|---|---|---|---|---|---|
| 38+00 (*PVC*) | 348.68 | 0 | 0 | 0 | 348.68 |
| 39+00 | 344.68 | 1/4 | 1/16 | + .42 | 345.10 100 foot stations. |
| 40+00 $g_1 = -4\%$ | 340.68 | 1/2 | 1/4 | + 1.67 | 342.35 |
| 41+00 | 336.68 | 3/4 | 9/16 | + 3.75 | 340.43 |
| 42+00 (*PVI*) | 332.68 | 1 | 1 | + 6.67 | 339.35 |
| 42+50 | 335.68 | 3/4 | 9/16 | + 3.75 | 339.43 |
| 43+00 $g_2 = +6\%$ | 338.68 | 1/2 | 1/4 | + 1.67 | 340.35 50 foot stations. |
| 43+50 | 341.68 | 1/4 | 1/16 | + .42 | 342.10 |
| 44+00 (*PVT*) | 344.68 | 0 | 0 | 0 | 344.68 |

**Figure 11-20.—Table of computations of elevations on an unsymmetrical vertical curve.**

In this example, then, the middle vertical offset at the *PVI* is calculated in the following manner:

$e$ = [(4 x 2)/2(4 + 2)] x [(+6) - (–4)] = 6.67 feet.

Second, you are cautioned that the check on your computations by the use of second difference does NOT work out the same way for unsymmetrical curves as for a symmetrical curve. The second difference will not check for the differences that span the *PVI.* The reason is that an unsymmetrical curve is really two parabolas, one on each side of the *PVI,* having a common *POVC* opposite the *PVI;* however, the second difference will check out back, and ahead, of the first station on each side of the *PVI.*

Third, the turning point is not necessarily above or below the tangent with the lesser slope. The horizontal location is found by the use of one of two formulas as follows:

from the *PVC*

$$x_t = \frac{(l_1)^2 g_1}{2e}$$

or from the *PVT*

$$x_t = \frac{(l_2)^2 g_2}{2e}.$$

The procedure is to estimate on which side of the *PVI* the turning point is located and then use the proper formula to find its location. If the formula indicates that the turning point is on the opposite side of the *PVI,* you must use the other formula to determine the correct location; for example, you estimate that the turning point is between the *PVC* and *PVI* for the curve in figure 11-19. Solving the formula:

$x_t = (l_1)^2(g_1)/2e$

$x_t = [(4)^2(4)]/(2 \times 6.67) = 4.80$, or Station 42 + 80.

However, Station 42 + 80 is between the *PVI* and *PVT;* therefore, use the formula:

$x_t = (l_2)^2(g_2)//2e$

$x_t = [(2)^2(6)]/(2 \times 6.67) = 1.80$, or station 42 + 20.

Station 42 + 20 is the correct location of the turning point. The elevation of the *POVT,* the amount of the offset, and the elevation on the curve is determined as previously explained.

## CHECKING THE COMPUTATION BY PLOTTING

Always check your work by plotting the grade tangents and the curve in profile on an exaggerated vertical scale; that is, with the vertical scale perhaps 10 times the horizontal scale. After the POVCs have been plotted, you should be able to draw a smooth parabolic curve through the points with the help of a ship's curve or some other type of irregular curve; if you can't, check your computations.

## USING A PROFILE WORK SHEET

After you have had some experience computing curves using a table as shown in the foregoing examples, you may wish to eliminate the table and write your computations directly on a working print of the profile. The engineer will set the grades and indicate the length of the vertical curves. You may then scale the *PVI* elevations and compute the grades if the engineer has not done so. Then, using a calculator, compute the *POVT* elevations at the selected stations. You can store the computations in some calculators. That allows you access to the grades, the stations, and the elevations stored in the calculator from one end of the profile to the other. You can then check the calculator at each previously set *PVI* elevation. Write the tangent elevation at each station on the work sheet. Then compute each vertical offset: mentally note the $x/1$ ratio; then square it and multiply by $e$ on your calculator. Write the offset on the work print opposite the tangent elevation. Next, add or subtract the offsets from the tangent elevations (either mentally or on the calculator) to get the curve elevations; then record them on the work sheet. Plot the POVC elevations and draw in the curve. Last, put the necessary information on the original tracing. The information generally shown includes grades; finished elevations; length of curve; location of *PVC, PVI, PVT,* and the *e.* Figure 11-21 shows a portion of a typical work sheet completed up to the point of drawing the curve.

## FIELD STAKEOUT OF VERTICAL CURVES

The stakeout of a vertical curve consists basically of marking the finished elevations in the field to guide the construction personnel. The method of setting a grade stake is the same whether it is on a tangent or on a curve, so a vertical curve introduces no special problem. As indicated before, stakes are sometimes set closer together on a curve than on a tangent. But that will usually have been foreseen, and the plans will show the finished grade elevations at the required stations. If, however, the field conditions do require a stake at an odd plus on a curve, you may compute the needed *POVC* elevation in the field using the data given on the plans and the computational methods explained in this chapter.

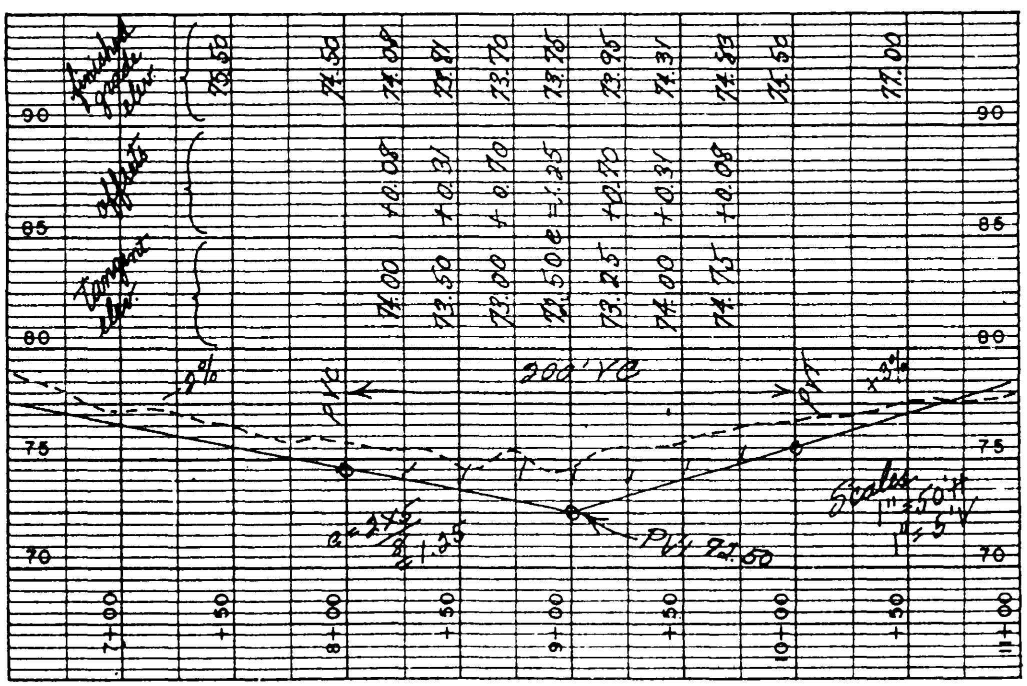


**Figure 11-21.—Profile work sheet.**

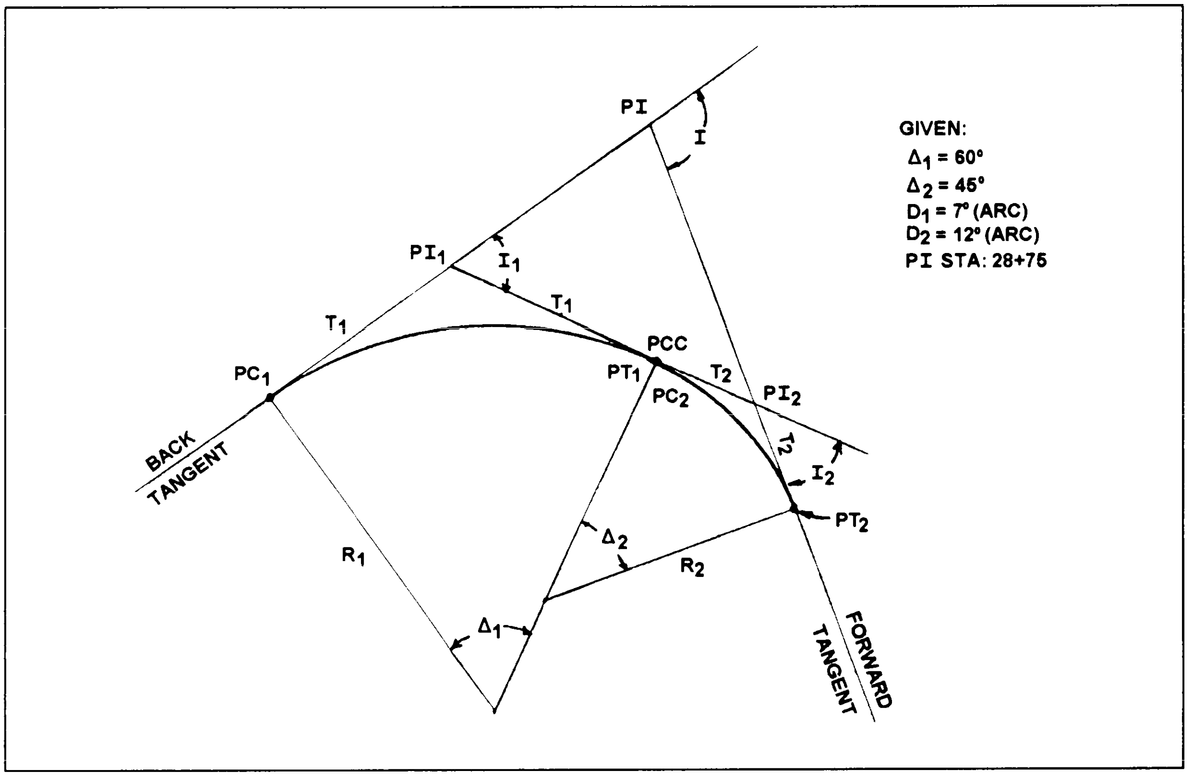


**Figure 11-22.—Compound curve.**

## QUESTIONS

*Q1. Using the data given for the compound curve shown in figure 11-22, compute the stations at $PC_1$, $PI_1$, PCC (point of compound curvature), $PI_2$, and $PT_2$.*

*Q2. Referring again to figure 11-22, assume that you are tasked to stakeout the compound curve using full stations. What deflection angles ($d_1$, $d_2$, and d) and chord lengths ($C_1$, $C_2$, and C) will you use for the 12° curve?*

*Q3. Assume that you are to set half stations for a 400-foot symmetrical vertical curve. The tangents ($g_1$ = + 3.2 percent, $g_2$ = –1.6 percent) intersect at Station 14 + 00. The tangent elevation at the PVI is 131.20 feet. Compute the following information for this curve:*

*a. Elevation at the PVC*
*b. Elevation at the PVT*
*c. Grade elevation on the curve at the PVI*
*d. Grade elevation on the curve at station 13 + 50*
*e. Station number and grade elevation on the curve at the turning point*

*Q4. Assume that you are to set half stations for a 600-foot vertical curve. The tangents ($g_1$= –3 percent, $g_2$= –8 percent) intersect at Station 15 + 00, which has an elevation of 640 feet above mean sea level. You may also assume that $l_1$= 400 feet and $l_2$ = 200 feet. Compute the following information for this curve:*

*a. Elevation at the PVC*
*b. Elevation at the PVT*
*c. Grade elevation on the curve at the PVI*
*d. Grade elevation on the curve at station 13 + 50*
*e. Station number and grade elevation on the curve at the turning point*