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不定积分习题课

例1：求 $\int \frac{2^x 3^x}{9^x+4^x}dx$

解：原式 $= \int \frac{1}{1+\left(\frac{2}{3}\right)^{2x}}\left(\frac{2}{3}\right)^x dx \xlongequal[du=\ln\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^x dx]{u=\left(\frac{2}{3}\right)^x} \frac{1}{\ln\left(\frac{2}{3}\right)}\int \frac{du}{1+u^2}$

$= \frac{1}{\ln\left(\frac{2}{3}\right)}\arctan u + C = \frac{1}{\ln\left(\frac{2}{3}\right)}\arctan\left(\frac{2}{3}\right)^x + C$

例2：设 $y(x-y)^2 = x$，求积分 $\int \frac{dx}{x-3y}$

解：令 $x-y=t$，即 $y=x-t$.

代入 $y(x-y)^2=x$ 中，得 $(x-t)t^2=x$

解得 $x=\frac{t^3}{t^2-1}$，从而 $y=x-t=\frac{t}{t^2-1}$，$dx=\frac{t^2(t^2-3)}{(t^2-1)^2}dt$

原式 $= \int \frac{1}{\frac{t^3}{t^2-1}-3\frac{t}{t^2-1}}\cdot\frac{t^2(t^2-3)}{(t^2-1)^2}dt = \int\frac{t}{t^2-1}dt$

$= \frac{1}{2}\int\frac{d(t^2-1)}{t^2-1} = \frac{1}{2}\ln|t^2-1|+C = \frac{1}{2}\ln|(x-y)^2-1|+C$

例3：求 $\int \frac{\arctan e^x}{e^x}dx$

解：令 $u=\arctan e^x$，$v'=e^{-x}$

原式 $= -e^{-x}\cdot\arctan e^x + \int e^{-x}\frac{e^x}{1+e^{2x}}dx$

$= -e^{-x}\arctan e^x + \int\frac{1+e^{2x}-e^{2x}}{1+e^{2x}}dx$

$= -e^{-x}\arctan e^x + \int dx - \frac{1}{2}\int\frac{de^{2x}}{1+e^{2x}}$

$= -e^{-x}\arctan e^x + x - \frac{1}{2}\ln(1+e^{2x}) + C$

例4：求 $\int\frac{dx}{1+e^{\frac{x}{2}}+e^{\frac{x}{3}}+e^{\frac{x}{6}}}$

解：设 $e^{\frac{x}{6}}=t$，则 $x=6\ln t$，$dx=\frac{6}{t}dt$.

$$原式=\int\frac{6}{(1+t^3+t^2+t)t}dt=6\int\frac{1}{t(1+t)(t^2+1)}dt$$

$$\frac{1}{t(1+t)(t^2+1)}=\frac{A}{t}+\frac{B}{1+t}+\frac{Ct+D}{t^2+1}$$

$$A=\frac{1}{(1+t)(t^2+1)}\Big|_{t=0}=1$$

$$B=\frac{1}{t(t^2+1)}\Big|_{t=-1}=-\frac{1}{2}$$

$$\begin{cases}\frac{1}{4}=1-\frac{1}{4}+\frac{C+D}{2}\\ \frac{1}{30}=\frac{1}{2}-\frac{1}{6}+\frac{2C+D}{5}\end{cases}\Rightarrow\begin{cases}C=-\frac{1}{2}\\ D=-\frac{1}{2}\end{cases}$$

故 $\frac{1}{t(1+t)(t^2+1)}=\frac{1}{t}-\frac{1}{2}\frac{1}{1+t}-\frac{1}{2}\frac{t+1}{t^2+1}$

$$原式=\int\left(\frac{6}{t}-\frac{3}{1+t}-3\cdot\frac{t+1}{t^2+1}\right)dt$$

$$=6\ln|t|-3\ln|1+t|-\frac{3}{2}\ln|t^2+1|-3\arctan t+C$$

$$=x-3\ln(1+e^{\frac{x}{6}})-\frac{3}{2}\ln(1+e^{\frac{x}{3}})-3\arctan e^{\frac{x}{6}}+C.$$

例5：求 $\int\frac{1}{\sqrt[n]{(x-a)^{n+1}(x-b)^{n-1}}}dx$ （$n\in N^+$，且为奇数）

解：$原式=\int\frac{1}{(x-a)(x-b)\sqrt[n]{\frac{x-a}{x-b}}}dx$

设 $t=\sqrt[n]{\frac{x-a}{x-b}}$，即 $t^n=\frac{x-a}{x-b}$

则 $nt^{n-1}dt=\frac{a-b}{(x-b)^2}dx$.

$\Rightarrow \frac{n}{t}dt=\frac{a-b}{(x-a)(x-b)}dx$

$$原式=\int\frac{1}{t}\cdot\frac{n}{a-b}\cdot\frac{1}{t}dt=\frac{n}{a-b}\int\frac{dt}{t^2}$$

$$b-a\sqrt[n]{\frac{x-b}{x-a}}+C$$

例6：求 $\int\frac{dx}{\sin(x+a)\sin(x+b)}$ （$a-b\neq k\pi$，$k\in\mathbb{Z}$）

解：原式 $= \frac{1}{\sin(a-b)}\int \frac{\sin[(x+a)-(x+b)]}{\sin(x+a)\sin(x+b)}dx$

$= \frac{1}{\sin(a-b)}\int \frac{\sin(x+a)\cos(x+b)-\cos(x+a)\sin(x+b)}{\sin(x+a)\cdot\sin(x+b)}dx$

$= \frac{1}{\sin(a-b)}\int\left[\frac{\cos(x+b)}{\sin(x+b)} - \frac{\cos(x+a)}{\sin(x+a)}\right]dx$

$= \frac{1}{\sin(a-b)}\left[\ln|\sin(x+b)| - \ln|\sin(x+a)|\right]+C.$

例7：求 $\int \frac{dx}{(2+\cos x)\sin x}$

解：原式 $= \int \frac{\sin x\, dx}{(2+\cos x)\sin^2 x} \xlongequal{u=\cos x} -\int \frac{du}{(2+u)(1-u^2)} = \int \frac{du}{(2+u)(u^2-1)}$

$\frac{1}{(2+u)(u^2-1)} = \frac{A}{2+u} + \frac{B}{u-1} + \frac{C}{u+1}$

$A = \frac{1}{u^2-1}\Big|_{u=-2} = \frac{1}{3}$

$B = \frac{1}{(2+u)(u+1)}\Big|_{u=1} = \frac{1}{6}$

$C = \frac{1}{(2+u)(u-1)}\Big|_{u=-1} = -\frac{1}{2}$

$\Rightarrow \frac{1}{(2+u)(u^2-1)} = \frac{1}{3}\frac{1}{2+u} + \frac{1}{6}\frac{1}{u-1} - \frac{1}{2}\frac{1}{u+1}$

...

例8：求 $I_1 = \int \frac{\sin x}{a\cos x + b\sin x}dx$ 及 $I_2 = \int \frac{\cos x}{a\cos x+b\sin x}dx$

解：$\begin{cases} aI_2 + bI_1 = \int \frac{a\cos x + b\sin x}{a\cos x + b\sin x}dx = \int dx = x + C_1 \\ bI_2 - aI_1 = \int \frac{-a\sin x + b\cos x}{a\cos x + b\sin x}dx = \int \frac{d(a\cos x + b\sin x)}{a\cos x + b\sin x} \end{cases}$

$$= \ln|a\cos x + b\sin x| + C_2.$$

$$I_1 = \frac{1}{a^2+b^2}\left(bx - a\ln|a\cos x + b\sin x|\right) + C$$

$$I_2 = \frac{1}{a^2+b^2}\left(ax + b\ln|a\cos x + b\sin x|\right) + C.$$

例9：求 $\int \frac{3\sin x - 4\cos x}{\sin x + 2\cos x}\,dx$

解：$3\sin x - 4\cos x = A(\sin x + 2\cos x) + B(\sin x + 2\cos x)'$

$$= A(\sin x + 2\cos x) + B(\cos x - 2\sin x)$$

$$\Rightarrow \begin{cases} 3 = A - 2B \\ -4 = 2A + B \end{cases} \Rightarrow \begin{cases} A = -1 \\ B = -2 \end{cases}$$

故原式 $= \int \frac{-(\sin x + 2\cos x) - 2(\sin x + 2\cos x)'}{\sin x + 2\cos x}\,dx$

$$= -\int dx - 2\int \frac{d(\sin x + 2\cos x)}{\sin x + 2\cos x}$$

$$= -x - 2\ln|\sin x + 2\cos x| + C$$

例10：设 $F(x)$ 为 $f(x)$ 的原函数，且 $F(0)=1$，当 $x \geq 0$ 时，有 $f(x)\cdot F(x) = \sin^2 2x$，$F(x) \geq 0$，求 $f(x)$，$(x \geq 0)$

解：已知 $F'(x) = f(x)$，则 $F'(x)\cdot F(x) = \sin^2 2x$.

$$\Rightarrow \int F(x)\cdot F'(x)\,dx = \int \sin^2 2x\,dx$$

$$\Rightarrow \frac{1}{2}F^2(x) = \int \frac{1-\cos 4x}{2}\,dx$$

$$\Rightarrow F^2(x) = \int (1 - \cos 4x)\,dx = x - \frac{1}{4}\sin 4x + C.$$

由 $F(0) = 1$，知 $1^2 = 0 - 0 + C \Rightarrow C = 1$

又 $F(x) \geq 0$，所以 $F(x) = \sqrt{x - \frac{1}{4}\sin 4x + 1}$

从而 $f(x)=F'(x)=\dfrac{\sin^2 2x}{\sqrt{x-\frac{1}{4}\sin 4x+1}}$

定积分习题课.

例1：求 $\lim\limits_{n\to\infty}\int_0^1 \dfrac{x^n e^x}{1+e^x}dx$

解：$\lim\limits_{n\to\infty}\int_0^1 \dfrac{x^n e^x}{1+e^x}dx \overset{\checkmark}{=} \lim\limits_{n\to\infty} \dfrac{\xi^n\cdot e^\xi}{1+e^\xi}\cdot(1-0)$ $\quad \xi\in(0,1)$

$\cancel{=} 0$ $\quad$ ξ 也依赖于 n

$0<1-\frac{1}{n}<1.$ $\quad \lim\limits_{n\to\infty}(1-\frac{1}{n})^n=\frac{1}{e}\neq 0$

若题目改为 求 $\lim\limits_{n\to\infty}\int_0^a \dfrac{x^n e^x}{1+e^x}dx$ $\quad (0<a<1)$

则可用上述做法. 原因是 $\xi\in(0,a)$

而 $a^n\to 0$ (当 $n\to\infty$ 时)

此题解法：$0<\dfrac{x^n e^x}{1+e^x}<x^n.$

从而 $\underbrace{\int_0^1 0\,dx}_{=0} < \int_0^1 \dfrac{x^n e^x}{1+e^x}dx < \underbrace{\int_0^1 x^n dx}_{=\frac{1}{n+1}}$

由夹逼准则. 知 $\lim\limits_{n\to\infty}\int_0^1 \dfrac{x^n e^x}{1+e^x}dx=0.$

例2：求 $\lim\limits_{n\to\infty}\left(\dfrac{n}{n^2+1}+\dfrac{n}{n^2+2^2}+\cdots+\dfrac{n}{n^2+n^2}\right)$

解：原式 $=\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\dfrac{n}{n^2+i^2}=\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\dfrac{1}{1+(\frac{i}{n})^2}\cdot\dfrac{1}{n}.$

$=\int_0^1 \dfrac{1}{1+x^2}dx = \arctan x\Big|_0^1=\dfrac{\pi}{4}$

例3：求 $\lim\left(\dfrac{\sin\frac{\pi}{n}}{\ldots}+\dfrac{\sin\frac{2\pi}{n}}{\ldots}+\cdots+\dfrac{\sin\frac{n\pi}{n}}{\ldots}\right)$

例3：求 $\lim\limits_{n\to\infty}\left(\dfrac{\sin\frac{\pi}{n}}{n+1}+\dfrac{\sin\frac{2\pi}{n}}{n+\frac{1}{2}}+\cdots+\dfrac{\sin\frac{n\pi}{n}}{n+\frac{1}{n}}\right)$

解：原式 $=\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\dfrac{\sin\frac{i\pi}{n}}{n+\frac{1}{i}}$.

又 $\dfrac{n}{n+1}\underline{\sum\limits_{i=1}^{n}\dfrac{\sin\frac{i\pi}{n}}{n}}=\sum\limits_{i=1}^{n}\dfrac{\sin\frac{i\pi}{n}}{n+1}\le\sum\limits_{i=1}^{n}\dfrac{\sin\frac{i\pi}{n}}{n+\frac{1}{i}}\le\underline{\sum\limits_{i=1}^{n}\sin\frac{i\pi}{n}\cdot\frac{1}{n}}$.

因为 $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\dfrac{\sin\frac{i\pi}{n}}{n}=\int_0^1\sin(\pi x)\,dx=\dfrac{2}{\pi}$

所以由夹逼准则知原式 $=\dfrac{2}{\pi}$