2

(1)

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix} \xlongequal[r_1+r_4]{\substack{r_1+r_2 \\ r_1+r_3}} \begin{vmatrix} 10 & 10 & 10 & 10 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix} = 10 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$$

$$\xlongequal[c_2-c_1]{\substack{c_4-c_3 \\ c_3-c_2}} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & -3 \\ 3 & 1 & -3 & 1 \\ 4 & -3 & 1 & 1 \end{vmatrix}$$

将上式按r_1展开，可得

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & -3 \\ 3 & 1 & -3 & 1 \\ 4 & -3 & 1 & 1 \end{vmatrix} = 10 \begin{vmatrix} 1 & 1 & -3 \\ 1 & -3 & 1 \\ -3 & 1 & 1 \end{vmatrix} \xlongequal[r_1+r_3]{r_1+r_2} 10 \begin{vmatrix} -1 & -1 & -1 \\ 1 & -3 & 1 \\ -3 & 1 & 1 \end{vmatrix}$$

$$\xlongequal[c_2-c_3]{c_1-c_3} 10 \begin{vmatrix} 0 & 0 & -1 \\ 0 & -4 & 1 \\ -4 & 0 & 1 \end{vmatrix} = 160$$

(2)

$$\begin{vmatrix} \frac{1}{3} & -\frac{5}{2} & \frac{2}{5} & \frac{3}{2} \\ 3 & -12 & \frac{21}{5} & 15 \\ \frac{2}{3} & -\frac{9}{2} & \frac{4}{5} & \frac{5}{2} \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & \frac{3}{7} \end{vmatrix} \xlongequal{r_3-2r_1} \begin{vmatrix} \frac{1}{3} & -\frac{5}{2} & \frac{2}{5} & \frac{3}{2} \\ 3 & -12 & \frac{21}{5} & 15 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & \frac{3}{7} \end{vmatrix} \xlongequal{c_4+c_1} \begin{vmatrix} \frac{1}{3} & -\frac{5}{2} & \frac{2}{5} & -1 \\ 3 & -12 & \frac{21}{5} & 3 \\ 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & \frac{5}{7} \end{vmatrix}$$

$$= \frac{1}{2} \times (-1)^{3+2} \begin{vmatrix} \frac{1}{3} & \frac{2}{5} & -1 \\ 3 & \frac{21}{5} & 3 \\ -\frac{1}{7} & -\frac{1}{7} & \frac{5}{7} \end{vmatrix} = -\frac{1}{2} \times \frac{1}{15} \times \frac{1}{5} \times \frac{1}{7} \begin{vmatrix} 5 & 6 & -15 \\ 15 & 21 & 15 \\ -1 & -1 & 5 \end{vmatrix}$$

$$\xlongequal{r_2+r_1} -\frac{1}{2} \times \frac{1}{15} \times \frac{1}{5} \times \frac{1}{7} \begin{vmatrix} 5 & 6 & -15 \\ 20 & 27 & 0 \\ -1 & -1 & 5 \end{vmatrix} \xlongequal{r_1+3r_3} -\frac{1}{2} \times \frac{1}{15} \times \frac{1}{5} \times \frac{1}{7} \begin{vmatrix} 2 & 3 & 0 \\ 20 & 27 & 0 \\ -1 & -1 & 5 \end{vmatrix}$$

$$= -\frac{1}{2} \times \frac{1}{15} \times \frac{1}{5} \times \frac{1}{7} \times 5 \times (54-60)$$

$$= \frac{1}{35}$$

(3)

$$\begin{vmatrix} a_1 & -1 & 0 & 0 \\ a_2 & x & -1 & 0 \\ a_3 & 0 & x & -1 \\ a_4 & 0 & 0 & x \end{vmatrix} \xlongequal{r_2+xr_1} \begin{vmatrix} a_1 & -1 & 0 & 0 \\ a_1x+a_2 & 0 & -1 & 0 \\ a_3 & 0 & x & -1 \\ a_4 & 0 & 0 & x \end{vmatrix}$$

$$\xlongequal{r_3+xr_2} \begin{vmatrix} a_1 & -1 & 0 & 0 \\ a_1x+a_2 & 0 & -1 & 0 \\ a_1x^2+a_2x+a_3 & 0 & 0 & -1 \\ a_4 & 0 & 0 & x \end{vmatrix}$$

$$\xlongequal{r_4+xr_3} \begin{vmatrix} a_1 & -1 & 0 & 0 \\ a_1x+a_2 & 0 & -1 & 0 \\ a_1x^2+a_2x+a_3 & 0 & 0 & -1 \\ a_1x^3+a_2x^2+a_3x+a_4 & 0 & 0 & 0 \end{vmatrix}$$

$$=(a_1x^3+a_2x^2+a_3x+a_4)(-1)^{4+1}(-1)^3$$

$$=a_1x^3+a_2x^2+a_3x+a_4$$

(4)

$$D=\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 2^2 & 3^2 & 4^2 \\ 1 & 2^3 & 3^3 & 4^3 \\ 9 & 8 & 7 & 6 \end{vmatrix} \xlongequal{r_4+r_1} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 2^2 & 3^2 & 4^2 \\ 1 & 2^3 & 3^3 & 4^3 \\ 10 & 10 & 10 & 10 \end{vmatrix} \xlongequal{\text{提取公因子}} 10\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 2^2 & 3^2 & 4^2 \\ 1 & 2^3 & 3^3 & 4^3 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

$$\xlongequal{\text{换行}} -10\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 2^2 & 3^2 & 4^2 \\ 1 & 2^3 & 3^3 & 4^3 \end{vmatrix}$$

由范德蒙德行列式对上述4阶行列式求解，则有

$$D=(-10)\cdot(2-1)\cdot(3-1)\cdot(3-2)\cdot(4-1)\cdot(4-2)\cdot(4-3)=-120$$

(5)

$$\begin{vmatrix} k & 0 & -1 & 1 \\ 0 & k & 1 & -1 \\ -1 & 1 & k & 0 \\ 1 & -1 & 0 & k \end{vmatrix} \xlongequal{r_3+r_4} \begin{vmatrix} k & 0 & -1 & 1 \\ 0 & k & 1 & -1 \\ 0 & 0 & k & k \\ 1 & -1 & 0 & k \end{vmatrix} \xlongequal[r_4+\frac{1}{k}r_2]{r_4-\frac{1}{k}r_1} \begin{vmatrix} k & 0 & -1 & 1 \\ 0 & k & 1 & -1 \\ 0 & 0 & k & k \\ 0 & 0 & \frac{2}{k} & k-\frac{2}{k} \end{vmatrix}$$

$$\xlongequal{r_4-\frac{2}{k^2}r_3} \begin{vmatrix} k & 0 & -1 & 1 \\ 0 & k & 1 & -1 \\ 0 & 0 & k & k \\ 0 & 0 & 0 & k-\frac{4}{k} \end{vmatrix} = k\cdot k\cdot k\cdot\left(k-\frac{4}{k}\right) = k^2(k^2-4)$$