1、证明下面等式

$$\begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix} = (1+\sum_{i=1}^{n}\frac{1}{a_i})\prod_{i=1}^{n}a_i.$$

方法一:

$$\begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1+a_1 & 1 & \cdots & 1 \\ 0 & 1 & 1+a_2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 1 & \cdots & 1+a_n \end{vmatrix}$$

$$\xlongequal[i=2,3,\cdots,n+1]{r_i-r_1} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & a_1 & 0 & \cdots & 0 \\ -1 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & a_n \end{vmatrix} \xlongequal[i=2,3,\cdots,n+1]{c_1+c_i\times\frac{1}{a_{i-1}}} \begin{vmatrix} 1+\frac{1}{a_1}+\cdots+\frac{1}{a_n} & 1 & 1 & \cdots & 1 \\ 0 & a_1 & 0 & \cdots & 1 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{vmatrix}$$

$$= (1+\sum_{i=1}^{n}\frac{1}{a_i})\prod_{i=1}^{n}a_i.$$

方法二:

$$\begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix} \xlongequal[i=2,3,\cdots,n]{r_i-r_1} \begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ -a_1 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & \cdots & a_n \end{vmatrix}$$

$$\xlongequal{c_1+c_i\times\frac{a_1}{a_i}} \begin{vmatrix} 1+a_1+\sum\limits_{i=2}^{n}\frac{a_1}{a_i} & 1 & \cdots & 1 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{vmatrix} = (1+a_1+\sum_{i=2}^{n}\frac{a_1}{a_i})a_2\cdots a_n = a_1a_2\cdots a_n(1+\sum_{i=1}^{n}\frac{1}{a_i}).$$

$$\begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix} \xlongequal[i=1,2,\cdots,n]{r_i\text{提取}a_i} a_1a_2\cdots a_n \begin{vmatrix} 1+\frac{1}{a_1} & \frac{1}{a_1} & \cdots & \frac{1}{a_1} \\ \frac{1}{a_2} & 1+\frac{1}{a_2} & \cdots & \frac{1}{a_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_n} & \frac{1}{a_n} & \cdots & 1+\frac{1}{a_n} \end{vmatrix}$$

$$\xlongequal[i=2,3,\cdots,n]{r_1+r_i} a_1a_2\cdots a_n \begin{vmatrix} 1+\sum\limits_{i=1}^{n}\frac{1}{a_i} & 1+\sum\limits_{i=1}^{n}\frac{1}{a_i} & \cdots & 1+\sum\limits_{i=1}^{n}\frac{1}{a_i} \\ \frac{1}{a_2} & 1+\frac{1}{a_2} & \cdots & \frac{1}{a_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_n} & \frac{1}{a_n} & \cdots & 1+\frac{1}{a_n} \end{vmatrix}$$

$$= a_1a_2\cdots a_n(1+\sum_{i=1}^{n}\frac{1}{a_i}) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{a_2} & 1+\frac{1}{a_2} & \cdots & \frac{1}{a_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_n} & \frac{1}{a_n} & \cdots & 1+\frac{1}{a_n} \end{vmatrix} \xlongequal[i=2,3,\cdots,n]{c_i-c_1} a_1a_2\cdots a_n(1+\sum_{i=1}^{n}\frac{1}{a_i}) \begin{vmatrix} 1 & 0 & \cdots & 0 \\ \frac{1}{a_2} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_n} & \frac{1}{a_n} & \cdots & 1 \end{vmatrix}$$

$$= a_1a_2\cdots a_n(1+\sum_{i=1}^{n}\frac{1}{a_i}).$$

方法 4:

$$\begin{aligned} D_n &= \begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix} = \begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1+0 & 1+0 & \cdots & 1+a_n \end{vmatrix} \\ &= \begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix} + \begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{vmatrix} \\ &= \begin{vmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix} + a_n \begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+a_{n-1} \end{vmatrix} \\ &= a_1a_2\cdots a_{n-1} + a_nD_{n-1} = \frac{1}{a_n}\prod_{i=1}^{n}a_i + a_nD_{n-1} \\ &= \frac{1}{a_n}\prod_{i=1}^{n}a_i + a_n\left[\frac{1}{a_{n-1}}\prod_{i=1}^{n-1}a_i + a_{n-1}D_{n-2}\right] \\ &= \left(\frac{1}{a_n}+\frac{1}{a_{n-1}}\right)\prod_{i=1}^{n}a_i + a_na_{n-1}D_{n-2} \\ &= \cdots\cdots\cdots\cdots\cdots\cdots \\ &= \left(\frac{1}{a_n}+\frac{1}{a_{n-1}}+\cdots+\frac{1}{a_2}\right)\prod_{i=1}^{n}a_i + a_na_{n-1}\cdots a_2D_1 \\ &= \left(\frac{1}{a_n}+\frac{1}{a_{n-1}}+\cdots+\frac{1}{a_2}\right)\prod_{i=1}^{n}a_i + a_na_{n-1}\cdots a_2(1+a_1) \\ &= \left(1+\sum_{i=1}^{n}\frac{1}{a_i}\right)a_1a_2\cdots a_n. \end{aligned}$$

$$
\begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix} = \begin{vmatrix} 1+a_1 & 1+0 & \cdots & 1+0 \\ 1+0 & 1+a_2 & \cdots & 1+0 \\ \vdots & \vdots & \ddots & \vdots \\ 1+0 & 1+0 & \cdots & 1+a_n \end{vmatrix}
$$

$$
= \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 1 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & a_n \end{vmatrix} + \begin{vmatrix} a_1 & 1 & \cdots & \\ 0 & 1 & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & a_n \end{vmatrix} + \cdots + \begin{vmatrix} a_1 & 0 & \cdots & 1 \\ 0 & a_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} + \begin{vmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{vmatrix}
$$

$$
= a_2a_3\cdots a_n + a_1a_3\cdots a_n + \cdots + a_1a_2\cdots a_{n-1} + a_1a_2\cdots a_n
$$

$$
= \left(1+\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)a_1a_2\cdots a_n.
$$

2、计算下列余子式的值

设 $|a_{ij}|_{4\times 4} = \begin{vmatrix} 3 & 6 & 9 & 12 \\ 2 & 4 & 6 & 8 \\ 1 & 2 & 0 & 3 \\ 5 & 6 & 4 & 3 \end{vmatrix}$，试求 $A_{41}+2A_{42}+3A_{44}$，

$A_{41} + 2A_{42} + 3A_{44} = 1\cdot A_{41} + 2\cdot A_{42} + 0\cdot A_{43} + 3\cdot A_{44}$

将 1, 2, 0, 3 代替原式第4行，有

$$
原式 = \begin{vmatrix} 3 & 6 & 9 & 12 \\ 2 & 4 & 6 & 8 \\ 1 & 2 & 0 & 3 \\ 1 & 2 & 0 & 3 \end{vmatrix} = 0
$$