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作业：4-4：2，4，12，15

有理函数的积分

有理函数

$$R(x)=\frac{P(x)}{Q(x)}=\frac{a_0x^n+a_1x^{n-1}+\cdots+a_n}{b_0x^m+b_1x^{m-1}+\cdots+b_m}$$

$m\le n$ 次，$R(x)$ 为假分式，$m>n$ 次，$R(x)$ 为真分式

有理函数 $\xlongequal{\text{相除}}$ 多项式 + 真分式

⇓ 分解

若干部分分式之和

↓ 补充：有理函数

$$\frac{P_n(x)}{q_m(x)}=\frac{P_n(x)}{\prod_{k=1}^{i}(x-a_k)^{m_k}\cdot\prod_{k=1}^{j}(x^2+2\xi_kx+\eta_k^2)^{n_k}}$$

$(\xi_k^2<\eta_k^2)$ $n<m$

可分解为简单分式之和：

$$\frac{P_n(x)}{q_n(x)}=\sum_{k=1}^{i}\sum_{r=1}^{m_k}\frac{\lambda_{kr}}{(x-a_k)^r}+\sum_{k=1}^{j}\sum_{r=1}^{n_k}\frac{\mu_{kr}x+\nu_k}{(x^2+2\xi_kx+\eta_k^2)^r}$$

例：

$$q_m(x)=(x-1)^2(x+3)^4(x^2+x+1)^2 \qquad m=10$$

$$\frac{P_n(x)}{q_m(x)}=\frac{\lambda_1}{x-1}+\frac{\lambda_2}{(x-1)^2}+\frac{\lambda_3}{x+3}+\frac{\lambda_4}{(x+3)^2}+\frac{\lambda_5}{(x+3)^3}+\frac{\lambda_6}{(x+3)^4}$$

$$+\frac{\mu_1x+\nu_1}{x^2+x+1}+\frac{\mu_2x+\nu_2}{(x^2+x+1)^2}$$

$$+\frac{\mu_1 x+\nu_1}{x^2+x+1}+\frac{\mu_2 x+\nu_2}{(x^2+x+1)^2}$$

例1：将下列真分式分解为部分分式：

① $\dfrac{1}{x(x-1)^2}$，　② $\dfrac{x+3}{x^2-5x+6}$，　③ $\dfrac{1}{(1+2x)(1+x^2)}$

解：① 拼凑法：

$$\frac{1}{x(x-1)^2}=\frac{x-(x-1)}{x(x-1)^2}=\frac{1}{(x-1)^2}-\frac{1}{x(x-1)}$$

$$=\frac{1}{(x-1)^2}-\frac{x-(x-1)}{x(x-1)}$$

$$=\frac{1}{(x-1)^2}-\frac{1}{x-1}+\frac{1}{x}$$

② 法一：赋值法：

$$\frac{x+3}{x^2-5x+6}=\frac{x+3}{(x-2)(x-3)}=\frac{\lambda_1}{x-2}+\frac{\lambda_2}{x-3}$$

$$\lambda_1=\left[(x-2)\cdot\frac{x+3}{x^2-5x+6}-(x-2)\cdot\frac{\lambda_2}{x-3}\right]\Bigg|_{x=2}=\frac{x+3}{x-3}\Bigg|_{x=2}=-5$$

$$\lambda_2=\left[(x-3)\cdot\frac{x+3}{x^2-5x+6}-(x-3)\cdot\frac{\lambda_1}{x-2}\right]\Bigg|_{x=3}=\frac{x+3}{x-2}\Bigg|_{x=3}=6$$

法二：待定系数

$$\frac{x+3}{x^2-5x+6}=\frac{\lambda_1}{x-2}+\frac{\lambda_2}{x-3}=\frac{\lambda_1(x-3)+\lambda_2(x-2)}{(x-2)(x-3)}$$

$$\Rightarrow\begin{cases}\lambda_1+\lambda_2=1\\-3\lambda_1-2\lambda_2=3\end{cases}\Rightarrow\begin{cases}\lambda_1=-5\\\lambda_2=6\end{cases}$$

③ 混合法：

$$\frac{1}{(1+2x)(1+x^2)}=\frac{\lambda_1}{1+2x}+\frac{\mu_1 x+\nu_1}{1+x^2}$$

$$\lambda_1=\left[(1+2x)\cdot\frac{1}{(1+2x)(1+x^2)}-(1+2x)\cdot\frac{\mu_1 x+\nu_1}{1+x^2}\right]\Bigg|=\frac{1}{1+x^2}\Bigg|=\frac{4}{5}$$

$$\lambda_1=\left[(1+2x)\cdot\frac{1}{(1+2x)(1+x^2)}-(1+2x)\cdot\frac{\mu_1x+v_1}{1+x^2}\right]\Bigg|_{x=-\frac{1}{2}}=\frac{1}{1+x^2}\Bigg|_{x=-\frac{1}{2}}=\frac{4}{5}$$

再分别令 $x=0,1$，代入等式两边，得

$$\begin{cases}1=\frac{4}{5}+v_1\\ \frac{1}{6}=\frac{4}{15}+\frac{\mu_1+v_1}{2}\end{cases}\Rightarrow\begin{cases}\mu_1=-\frac{2}{5}\\ v_1=\frac{1}{5}\end{cases}$$

故 $\frac{1}{(1+2x)(1+x^2)}=\frac{\frac{4}{5}}{1+2x}+\frac{-\frac{2}{5}x+\frac{1}{5}}{1+x^2}$

例2：求 $\int\frac{dx}{(1+2x)(1+x^2)}$

解：$\frac{1}{(1+2x)(1+x^2)}=\frac{1}{5}\cdot\left(\frac{4}{1+2x}+\frac{-2x+1}{1+x^2}\right)$

$$\text{原式}=\frac{1}{5}\int\left(\frac{4}{1+2x}+\frac{-2x+1}{1+x^2}\right)dx$$

$$=\frac{1}{5}\int\frac{4\,dx}{1+2x}-\frac{1}{5}\int\frac{2x\,dx}{1+x^2}+\frac{1}{5}\int\frac{dx}{1+x^2}$$

$$=\frac{2}{5}\ln|1+2x|-\frac{1}{5}\ln(1+x^2)+\frac{1}{5}\arctan x+C$$

例3：求 $\int\frac{1}{x^6(1+x^2)}dx$

解：令 $x=\frac{1}{t}$，则 $dx=-\frac{1}{t^2}dt$.

$$\text{原式}=\int\frac{-\frac{1}{t^2}dt}{\frac{1}{t^6}\left(1+\frac{1}{t^2}\right)}=-\int\frac{t^6}{1+t^2}dt$$

$$t^6=t^4(1+t^2)-t^4$$

$$-t^4=-t^2(1+t^2)+t^2$$

$$t^2=1\cdot(1+t^2)-1$$

+)

$$t^6=(t^4-t^2+1)(1+t^2)-1\Rightarrow\frac{t^6}{1+t^2}=t^4-t^2+1-\frac{1}{1+t^2}$$

$$\text{原式} = -\int \left(t^4 - t^2 + 1 - \frac{1}{1+t^2}\right) dt$$

$$= -\frac{1}{5}t^5 + \frac{1}{3}t^3 - t + \arctan t + C$$

$$= -\frac{1}{5}\frac{1}{x^5} + \frac{1}{3}\cdot\frac{1}{x^3} - \frac{1}{x} + \arctan\frac{1}{x} + C$$

☆例4：求 $\int \frac{x-2}{x^2+2x+3}dx$

解：$\text{原式} = \frac{1}{2}\int \frac{(2x+2)-6}{x^2+2x+3}dx$

$$= \frac{1}{2}\int \frac{d(x^2+2x+3)}{x^2+2x+3} - 3\int \frac{d(x+1)}{(x+1)^2+(\sqrt{2})^2}$$

$$= \frac{1}{2}\ln(x^2+2x+3) - \frac{3}{\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}} + C$$

例5：求 $\int \frac{x^2}{(x^2+2x+2)^2}dx$

解：$\text{原式} = \int \frac{x^2+2x+2-(2x+2)}{(x^2+2x+2)^2}dx$

$$= \int \frac{dx}{x^2+2x+2} - \int \frac{d(x^2+2x+2)}{(x^2+2x+2)^2}$$

$$= \int \frac{d(x+1)}{(x+1)^2+1} - \int \frac{d(x^2+2x+2)}{(x^2+2x+2)^2}$$

$$= \arctan(x+1) + \frac{1}{x^2+2x+2} + C$$

求 $\int \frac{x^2+1}{(x^2+2x+2)^2}dx = \int \frac{(x^2+2x+2)-(2x+2)+1}{(x^2+2x+2)^2}dx$

$$= \int \frac{dx}{x^2+2x+2} - \int \frac{d(x^2+2x+2)}{(x^2+2x+2)^2} + \int \frac{dx}{(x^2+2x+2)^2}$$

$$\int \frac{dx}{(x^2+2x+2)^2} = \int \frac{d(x+1)}{((x+1)^2+1)^2} = I_2 \qquad I_n = \int \frac{dx}{(x^2+a^2)^n}$$

例：求 $\int \frac{2x^3+2x^2+5x+5}{\cdots} dx$

例6：求 $\int \frac{2x^3+2x^2+5x+5}{x^4+5x^2+4}dx$

解：原式 $= \int \frac{2x^3+5x}{x^4+5x^2+4}dx + \int \frac{2x^2+5}{x^4+5x^2+4}dx$

$$= \frac{1}{2}\int \frac{d(x^4+5x^2+4)}{x^4+5x^2+4} + \int \frac{(x^2+1)+(x^2+4)}{(x^2+1)(x^2+4)}dx$$

$$= \frac{1}{2}\ln(x^4+5x^2+4) + \frac{1}{2}\arctan\frac{x}{2} + \arctan x + C$$

例7：求 $\int \frac{dx}{x^4+1}$

解：原式 $= \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^4+1}dx$

$$= \frac{1}{2}\int \frac{x^2+1}{x^4+1}dx - \frac{1}{2}\int \frac{x^2-1}{x^4+1}dx$$

$$= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$$

$$= \frac{1}{2}\int \frac{d(x-\frac{1}{x})}{(x-\frac{1}{x})^2+2} - \frac{1}{2}\int \frac{d(x+\frac{1}{x})}{(x+\frac{1}{x})^2-2}$$

$$= \frac{1}{2\sqrt{2}}\arctan\frac{x-\frac{1}{x}}{\sqrt{2}} - \frac{1}{4\sqrt{2}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + C$$

例8：求 $\int \frac{x^2+2}{(x-1)^4}dx$

解：法一：

$$\frac{x^2+2}{(x-1)^4} = \frac{(x-1)^2+2x+1}{(x-1)^4} = \frac{(x-1)^2+2(x-1)+3}{(x-1)^4}$$

$$= \frac{1}{(x-1)^2} + \frac{2}{(x-1)^3} + \frac{3}{(x-1)^4}$$

直接积分，可得结论.

法二：

设 $t=x-1$，则 $dt=dx$

原式 $=\int \frac{(t+1)^2+2}{t^4}dt$

$=\int\left(\frac{1}{t^2}+\frac{2}{t^3}+\frac{3}{t^4}\right)dt$

$=-\left(\frac{1}{t}+\frac{1}{t^2}+\frac{1}{t^3}\right)+C=\cdots$

三角函数有理式的积分

$\frac{\sin x}{3+\cos x}$ ， $\frac{\sin x\cos x}{1+\tan^3 x}$

万能变换：令 $t=\tan\frac{x}{2}$

$$\sin x=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{2t}{1+t^2}$$

$$\cos x=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}$$

$$\tan x=\frac{\sin x}{\cos x}=\frac{\frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2}}=\frac{2t}{1-t^2}$$

由 $t=\tan\frac{x}{2}$，知 $dt=\frac{1}{2}\sec^2\frac{x}{2}\,dx=\frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)dx$

即 $dt=\frac{1}{2}(1+t^2)dx$

$\Rightarrow dx=\frac{2dt}{1+t^2}$

例：求 $\int\frac{1+\sin x}{\sin x(1+\cos x)}dx$

解：设 $t=\tan\frac{x}{2}$，则 $dx=\frac{2dt}{1+t^2}$

原式 $=\int\frac{1+\frac{2t}{1+t^2}}{\frac{2t}{1+t^2}\left(1+\frac{1-t^2}{1+t^2}\right)}\cdot\frac{2dt}{1+t^2}$

$\int \frac{[illegible]}{1+t^2}\left(1+\frac{[illegible]}{1+t^2}\right)$ [illegible]

$$= \frac{1}{2}\int \left(t+2+\frac{1}{t}\right)dt$$

$$= \frac{1}{4}t^2+t+\frac{1}{2}\ln|t|+C$$

$$= \frac{1}{4}\tan^2\frac{x}{2}+\tan\frac{x}{2}+\frac{1}{2}\ln\left|\tan\frac{x}{2}\right|+C$$