Project 3 : Add Greeks to binomial tree engines

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1 - Problem

The Delta and Gamma of an option can be calculated numerically as:

$$\Delta(u) = (P(u + \delta u) - P(u))/\delta u \,;\, \Gamma(u) = (\Delta(u) - \Delta(u - \delta u))/\delta u$$

where $P(u)$ is the price of the option as a function of the underlying value and u is a small perturbation of such value; the current Delta and Gamma are $\Delta(u_0)$ and $\Gamma(u_0)$ where u_0 is the current value (that is, at $t = 0$) of the underlying.

On a binomial tree as currently implemented, there's a single point at $t = 0$ as shown in Figure 1(a). This makes it impossible to calculate quickly and accurately the Delta and Gamma of an option; we can either re-run the calculation with a perturbed value of the underlying asset (which is slow, since we need three full calculations for Delta and Gamma) or we can use points at the first time step after $t = 0$ to obtain option values for a perturbed underlying (which is not accurate, since the option value at those points would be taken at $t \neq 0$; this is what the BinomialVanillaEngine class currently does).

The problem can be solved by having three points for $t = 0$ with the center point at u_0, as shown in Figure 1(b), so that we can obtain the option values for $u + \delta u$ and $u - \delta u$ to use in the formulas for Delta and Gamma. Modify the BinomialTree class accordingly, and modify the BinomialVanillaEngine class so that it calculates Delta and Gamma correctly. Do you expect the extra points at each step to have a noticeable effect of the performance of the engine? Why? Verify your intuition by timing the code before and after your changes.

Note: the tree is built based on log u instead of u, so you'll have two different u for the points at the left and right side. Take this into account during the calculations.

2 - Solution

2 - 1 Increasing the size of the columns by 2 nodes

The *BinomialTree* is a base class for all types of trees. In order to increase the size of the tree we changed its function *size(i)*. At column *i* the number of nodes is no longer *i+1* but *i+3*, the two additional nodes correspond to the price perturbations required to compute delta and gamma at time step *i*.

2 - 2 Adapting the index argument to the extra nodes

The computation of the underlying prices at node *(i,index)* depends on the values of both *i* and *index* where *i* is the column or the time step number and *index* is the number of the nodes located at the *ith* time step. For a given *i*, in the original configuration, the *index* goes from *0 to i+1.* With the new configuration it goes now from *0 to i+2.* Hence, after increasing the size of the tree by two nodes, the real *index* needed for the underlying calculation was shifted by 1. To get back the right underlying price, as it was given by the original structure, we had to shift back *index* value by 1. Through all the derived binomial tree classes the *underlying(i,index)* function was modified to account for the aforementioned issue. We created a *new_index=index-1* which substitutes the old *index* in calculating the underlying price.

2 - 3 How it works now

Let's take the example of the *EqualProbabilitiesTree*.

2 - 3 - 1 EqualProbabilitiesBinomialTree

Let's look at the first time step:

Using the following underlying calculation formula : $u_i^{index} = x_0 e^{i.dx + new\_index.up\_}$,

we get for $i = 0$:

$$u_0 = u_0^1 = x_0 \; ; \; u_0^+ = u_0^2 = x_0 e^{2up\_} ; \; u_0^- = u_0^0 = x_0 e^{-2up\_}$$

Then,

$$delta = \frac{p(u_0^+) - p(u_0^-)}{u_0^+ - u_0^-} \quad \text{and} \quad gamma = \frac{delta^+ - delta^-}{\frac{u_0^+ - u_0^-}{2}}$$

Where
$$delta^+ = \frac{p(u_0^+) - p(u_0)}{u_0^+ - u_0}; \quad delta^- = \frac{p(u_0) - p(u_0^-)}{u_0 - u_0^-}$$

We tested for $u_0 = 36$ the new architecture on *JarrowRudd* tree, which is a derivative tree of the *EqualProbabilitiesTree* , we got a $delta$= -0.84566 which is very good approximation of the analytical $delta$=-0.85 obtained by running the pricing engine three times with three different initializations of u_0. As required, the NPV= 4.205 was not affected by the new architecture.

2 - 3 - 2 Summary of the results obtained for each of the available tree architectures

For an American Option with u_0 = 36 , strike = 40 and a date of maturity on 24 May 2021, we get:

Tree	Original NPV	Current NPV	Original Delta	Current Delta	Original Gamma	Current Gamma	Elapsed time(s)
JarrowRudd	4.205	4.205	-0.849518	-0.84566	0.0720572	0.072163	0.000599
CoxRossRubinstein	4.20472	4.20472	-0.849587	-0.84679	0.0720719	0.072156	0.000546
AdditiveEQPBinomial	4.20559	4.20559	-0.849272	-0.84642	0.0720707	0.072175	0.000518
Trigeorgis	4.20473	4.20473	-0.849586	-0.84678	0.0720719	0.072156	0.00052
Tian	4.20488	4.20488	-0.849297	-0.84683	0.0721481	0.072078	0.003956
LeisenReimer	4.19732	4.19732	-0.850354	-0.85024	0.0727937	0.071623	0.003481
Joshi4	4.19732	4.19732	-0.850355	-0.85024	0.0727937	0.071623	0.003983

While the NPV remains unchanged, we get a very good approximation of the original greeks. We notice that the execution time was reduced, because now both Gamma and Delta are being calculated at the same time step 0 by calling the ***rollback*** function only once.