### Homotopy Theory and Characteristic Classes

#### CUI Jiaqi East China Normal University

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#### Abstract

This is the notes of a course given by Prof. Ma Langte in 25spring at Shanghai Jiaotong University.

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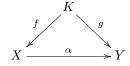
#### Part I

# Homotopy Theory

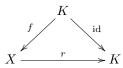
Let **TOP** be the category of topological spaces. Then we can take a quotient of **TOP** and get the homotopy category  $h-\mathbf{TOP}$ . The quotient may bring more algebraic structures. For example, Mor  $(S^1, X)$ , the homotopy classes of maps from  $S^1$  to X, is the fundamental group of X. Our goal is to study functors from hmotopy category to some algebraic categories.

Let  $\mathbf{TOP}^o$  be the pointed topological category, where the sum is wedge sum  $(X, x_0) \wedge (Y, y_0) =$  $X \sqcup Y/x_0 \sim y_0$  and the product is the smash product  $(X, x_0) \lor (Y, y_0) = X \times Y/\{x_0\} \times Y \cup X \times \{y_0\}$ . Similarly, we can take a quotient to get  $h - \mathbf{TOP}^o$ .

Let  $\mathbf{TOP}(2)$  be the category of pairs and  $h - \mathbf{TOP}(2)$  be its quotient. Fix  $K \in \mathrm{Ob}(\mathbf{TOP})$ . Let's consider  $\mathbf{TOP}^K$ , the category of spaces under K. Its objects are maps  $f: K \to X$  and morphisms are maps  $\alpha: X \to Y$  such that  $\alpha \circ f = g$ .



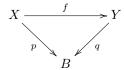
If  $K = \{*\}$  is a single point set, then  $\mathbf{TOP}^{\{*\}} = \mathbf{TOP}^o$  is the pointed topological category. Take X = K. A morphism from  $f: K \to X$  to id:  $K \to K$  is  $K \to K$  such that  $K \to K$  such that  $K \to K$  is  $K \to K$ .



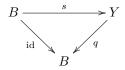
When  $K \subset X$ ,  $f = i : K \hookrightarrow X$ , we say that r is a retraction.

We have  $r: X \to K$  is a deformation retraction, if and only if  $i \circ r \simeq \mathrm{id}_X$  rel K, if and only if  $r: X \to K$  is a homotopy equivalence in  $\mathbf{TOP}^K$ .

Fix  $B \in \text{Ob}(\mathbf{TOP})$ . Let's consider  $\mathbf{TOP}_B$ , the category of spaces over B, where the objects are  $p: X \to B$  and morphisms are  $f: X \to Y$  such that  $p = q \circ f$ .



Take X = B. A morphism from id:  $B \to B$  to  $q: Y \to B$  is  $s: B \to Y$  such that  $q \circ s = \mathrm{id}_B$ .



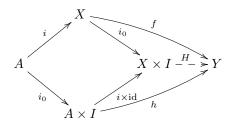
Then s is called a section of q.

Similarly, we can define  $h - \mathbf{TOP}^K$  and  $h - \mathbf{TOP}_B$ .

#### 1 Cofibrations and Fibrations

#### 1.1 Cofibrations

**Definition 1.1.** A map  $i: A \to X$  has the homotopy extension property (HEP) for a space Y if for all homotopy  $h: A \times I \to Y$  and  $f: X \to Y$  with  $f \circ i(a) = h(a, 0)$ , there exists  $H: X \times I \to Y$  satisfies



We say  $i: A \to X$  is a cofibration if it has HEP for each  $Y \in \text{Ob}(\mathbf{TOP})$ .

Recall the mapping cylinder: if  $i: A \to X$  is a map, then  $Z(i) := (A \times I) \sqcup X/(a, 1) \sim i(a)$ .

**Proposition 1.2.** Given a map  $i: A \to X$ . The followings are equivalent:

- 1.  $i: A \to X$  is a cofibration.
- 2. i has HEP for Z(i).

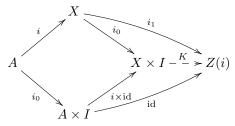
#### 3. The map

$$s: Z(i) \to X \times I$$
$$(a,t) \mapsto (f(a),t),$$
$$x \mapsto (x,1)$$

has a retraction.

*Proof.*  $(1)\Longrightarrow(2)$  is only by definition.

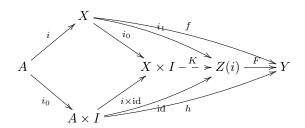
(2) $\Longrightarrow$ (1): By definition, there exists  $K: X \times I \to Z(i)$  such that the following diagram is commutative.



For any Y and homotopy  $h: A \times I \to Y$  and  $f: X \to Y$  with  $f \circ i(a) = h(a, 0)$ , we define

$$F: Z(i) \to Y$$
$$(a,t) \mapsto h(a,t)$$
$$x \mapsto f(x).$$

Then  $F \circ K$  is as desired.

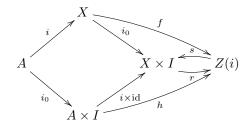


(2) $\Longrightarrow$ (3): We can easily check that the extension  $K: X \times I \to Z(i)$  in the proof of (2) $\Longrightarrow$ (1) is a retraction of s.

(3) $\Longrightarrow$ (2): Let r be a retraction of s. For any homotopy  $h: A \times I \to Z(i)$  and  $f: X \to Z(i)$  with  $f \circ i(a) = h(a,0)$ , we define

$$\sigma: Z(i) \to Z(i)$$
 
$$(a,t) \mapsto h(a,t)$$
 
$$x \mapsto f(x).$$

Then we can verify that  $H = \sigma \circ r : X \times I \to Z(i)$  extends h.



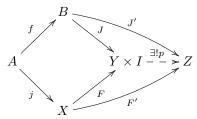
**Corollary 1.3.** When  $A \subset X$ ,  $i: A \hookrightarrow X$  is the inclusion map. Then  $i: A \to X$  is a cofibration  $\iff$   $A \times I \cup X \times \{1\}$  is a retraction of  $X \times I$ .

Therefore, we can easily check that whether  $i:A\hookrightarrow X$  is a cofibration. For example, let (X,A) be a manifold with boundary.

**Definition 1.4** (Push-Out of Cofibration). Given a commutative diagram,

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow j & & \downarrow J \\
X & \xrightarrow{F} & Y
\end{array}$$

the push-out of j along f is the initial object of this diagram, i.e.  $j: B \to Y, F: X \to Y$ , s.t.  $\forall Z$  with  $J': B \to Z, F': X \to Z$  satisfying  $J' \circ f = F' \circ j$ ,  $\exists !$  map  $p: Y \to Z$  such that the diagram is commutative.



In our setting, we can construct  $Y = X \sqcup B/f(a) \sim j(a)$  directly.

#### Part II

## Generalized Homology

#### Part III

### Characteristic Classes