A Note on Bounded Cohomology

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Abstract

This is a note of a short lecture given by my tutor Wan Renxing about bounded cohomology.

Contents

1 Bounded Cohomology

For a group G, denote

$$C_b^n(G,\mathbb{R}) \coloneqq \{\varphi\colon G^n \to \mathbb{R} : \sup|\varphi| < \infty\}$$

where φ is just a map instead of a homomorphism. Define the boundary operator $\delta\colon C_b^n(G,\mathbb{R}) \to C_b^{n+1}(G,\mathbb{R})$ as follow: For any $\varphi \in C_b^n(G,\mathbb{R})$, let

$$\delta\varphi\left(g_0,\cdots,g_n\right) \coloneqq \varphi\left(g_1,\cdots,g_n\right) + \sum_{i=1}^{n}(-1)^i\varphi\left(g_0,\cdots,g_{i-1}g_i,\cdots,g_n\right) + (-1)^{n+1}\varphi\left(g_0,\cdots,g_{n-1}\right).$$

It's easy to chaeck that $\delta\varphi \in C_b^{n+1}(G,\mathbb{R})$ and $\delta^2 = 0$. So $(C_b^*(G,\mathbb{R}),\delta)$ is a cochain complex.

Definition 1.1. The *bounded cohomology* of G is defined by

$$H_b^*(G,\mathbb{R}) \coloneqq \frac{\operatorname{Ker}\delta^*}{\operatorname{Im}\delta^{*-1}}.$$

Fact 1.2. (1) For any group G, $H_b^1(G,\mathbb{R}) = 0$.

(2) For any solvable group G, $H_b^n(G,\mathbb{R}) = 0$, $\forall n > 0$.

(3) For any hyperbolic group G, $H_b^2(G,\mathbb{R})$ has infinite dimension.

(4) For free group F_n, $\forall n > 0$, $H_b^3\left(F_n,\mathbb{R}\right)$ has infinite dimension.

Question 1.3. What about $H_b^n\left(F_n,\mathbb{R}\right)$ for $n \geq 4$?

2 Quasimorphism

Definition 2.1. For a group G, a map $\varphi\colon G \to \mathbb{R}$ is a *quasimorphism* if $\exists D > 0$ such that

$$|\varphi(gh) - \varphi(g) - \varphi(h)| \leq D, \quad \forall g, h \in G.$$

Example 2.2. (1) The integer function $\mathbb{R} \to \mathbb{R}$, $x \mapsto \lfloor x \rfloor$ is a quasimorphism.

(2) For a manifold M with a 1-form ω, $\varphi_\omega\colon \pi_1(M) \to \mathbb{R}$, $\varphi_\omega(\alpha) \coloneqq \int_\alpha \omega$ is a quasimorphism.

Example 2.3 (Brooks Counting Quasimorphism)**.** For any free group, for example, $F_2 = \langle a, b \rangle$, and any reduced word w on it, define $C_w\colon F_2 \to \mathbb{Z}$ by

$$C_w(g) \coloneqq \text{the number of occurences of } w \text{ in } g, \quad \forall g = s_1 s_2 \cdots s_n \in F_2,\ s_i \in \{\pm a, \pm b\}.$$

Define the counting function $h_w\colon F_2 \to \mathbb{Z}$ by

$$h_w(g) \coloneqq C_w(g) - C_{w^{-1}}(g).$$

Then h_w is a quasimorphism. Especially, h_w is a homomorphism if $|w| = 1$.

Remark 2.4. Under a suitable topology on the space of all quasimorphisms of F_n, the space of all Brooks counting quasimorphisms is dense.

3 Main Theorem

Lemma 3.1. Let $\varphi\colon G \to \mathbb{R}$ be a quasimorphism, then $[\delta\varphi] \in H_b^2(G, \mathbb{R})$. Especially, if φ is unbounded, $[\delta\varphi] \neq 0$.

Proof. It follows by definition that

$$|\delta\varphi(g, h)| = |\varphi(g) + \varphi(h) - \varphi(gh)| \leq D < \infty.$$

So $[\delta\varphi] \in H_b^2(G, \mathbb{R})$. And if φ is unbounded, $\varphi \notin C_b^1(G, \mathbb{R})$. Therefore, $[\delta\varphi] \notin \operatorname{Im} \delta^1$ and then $[\delta\varphi] \neq 0$. $\square$

Theorem 3.2. For free group F_2, $H_b^2(F_2, \mathbb{R})$ has infinite dimension.

Proof. Choose two non-conjugate elements g_1, g_2 of F_2 and let $w_i = g_1^{l_i} g_2^{m_i} g_1^{n_i} g_2^{k_i}$ for $i \geq 1$ where $l_1 \ll m_1 \ll n_1 \ll k_1 \ll l_2 \ll m_2 \ll n_2 \ll k_2 \ll \infty$. We claim that

(1) For any $j > i$, $h_{w_i}(w_j) = 0$.

(2) For any $i, n \geq 1$, $h_{w_i}(w_i^n) \geq n$.

Then we prove that $\{\delta h_{w_i}\}$ is linear independent. Suppose that $\sum_{i=1}^{\infty} a_i \delta h_{w_i} = 0$, where the infinite sum is well defined by our claim (1). This means that there exists a bounded map b such that

$$\sum_{i=1}^{\infty} a_i h_{w_i} + b = 0.$$

Operating on w_1^n, we have

$$0 = a_1 h_{w_1}(w_1^n) + b(w_1^n) \geq a_i n + b(w_1^n)$$

by claim (2). Because b is bounded, let $n \to \pm\infty$, we must have $a_1 = 0$. Then doing the same things for $i = 2$, by induction, we have $a_i = 0$, $\forall i \geq 1$.

Finally, by the lemma above, linear independent $\{\delta h_{w_i}\}$ give independent classes $\{[\delta h_{w_i}]\}$ in $H_b^2(F_2, \mathbb{R})$. So we conclude that $\dim H_b^2(F_2, \mathbb{R}) = \infty$, as desired. $\square$

4 Generalization

Epstein and Fujiwara generalized Brooks counting function for any group and proved that $H_b^2(G, \mathbb{R})$ has infinite dimension for any hyperbolic group G.

Let X be a metric space and G be a group acting on X isometrically. Fix a finite directed path w in X. For any path γ in X, define

$$|\gamma|_w \coloneqq \text{the number of occurences of } w \text{ in } \gamma,$$

where "occurence" means that there is $g \in G$ such that $gw \subset \gamma$. Then for any $x, y \in X$, define

$$C_w([x,y]) \coloneqq d(x,y) - \inf_{\alpha} \left(|\alpha| - |\alpha|_w\right),$$

where $[x, y]$ denotes the geodesic connecting x, y, the infimum range over all paths in X connecting x, y and $|\alpha|$ denote the length of α in X.

They proved that $h_w = C_w - C_{w^{-1}}$ is also a quaismorphism if X is a Gromov hyperbolic space, which promises that the proof for free groups above is valid for these G, especially, for hyperbolic groups (just let hyperbolic group act on its Cayley Graph).