Solutions of *Riemannian Geometry* by do Carmo

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Abstract

This is my personal solutions of the exercises in *Riemannian Geometry* by do Carmo.
I'm sorry for some notation abuse for partial derivative $\frac{\partial}{\partial x}$ and $\frac{d}{dx}$.
I haven't finished the solutions of Exercise 0.3, 2.5, 3.5 d), 3.14, 6.12 b), 8.6 e), 8.12 b), 10.6 a).

Contents

0 Differentiable Manifolds

Exercise 0.1 (Product manifold)**.** Let M and N be differentiable manifolds and let $\{(U_\alpha, \mathbf{x}_\alpha)\}$, $\{(V_\beta, \mathbf{y}_\beta)\}$ differentiable structures on M and N, respectively. Consider the cartesian product $M \times N$ and the mappings $\mathbf{z}_{\alpha\beta}(p,q) = (\mathbf{x}_\alpha(p), \mathbf{y}_\beta(q))$, $p \in U_\alpha$, $q \in V_\beta$.

(a) Prove that $\{(U_\alpha \times V_\beta, \mathbf{z}_{\alpha\beta})\}$ is a differentiable structure on $M \times N$ in which the projections $\pi_1 \colon M \times N \to M$ and $\pi_2 \colon M \times N \to N$ are differentiable. With this differnetiable structure $M \times N$ is called the *product manifold* of M with N.

(b) Show that the product manifold $S^1 \times \cdots \times S^1$ of n circles S^1, where $S^1 \subset \mathbb{R}^2$ has the usual differentiable structure, is diffeomorphic to the n-torus $\mathbb{T}^n$ (as the quotient space $\mathbb{R}^n/G$, where G is of integral translations of $\mathbb{R}^n$, acts on $\mathbb{R}^n$).

Proof. For (a), we have

$$\mathbf{z}_{\gamma\delta}^{-1} \circ \mathbf{z}_{\alpha\beta} = (\mathbf{x}_\gamma^{-1} \circ \mathbf{x}_\alpha, \mathbf{y}_\delta^{-1} \circ \mathbf{y}_\beta),$$

so $\{(U_\alpha \times V_\beta, \mathbf{z}_{\alpha\beta})\}$ is a differentiable structure. And notice that

$$d\pi_1 = \begin{bmatrix} I & 0 \end{bmatrix}, \quad d\pi_2 = \begin{bmatrix} 0 & I \end{bmatrix},$$

So π_1, π_2 are differentiable.

For (b),

$$S^1 \times \cdots \times S^1 \cong [0,1] \times \cdots \times [0,1]/_{0\sim 1} \cong \mathbb{R} \times \cdots \times \mathbb{R}/_{x \sim x+n} \cong \mathbb{R}^n/G.$$

□

Exercise 0.2. Prove that the tangent bundle of a differnetiable manifold M is orientable (even though M may not be).

Proof. See 4.1 EXAMPLE.

$$\mathbf{y}_\beta^{-1} \circ \mathbf{y}_\alpha(q_\alpha, v_\alpha) = \mathbf{y}_\beta^{-1}(\mathbf{x}_\alpha(q_\alpha), d\mathbf{x}_\alpha(v_\alpha)) = ((\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha)(q_\alpha), d(\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha)(v_\alpha)).$$

Then

$$d(\mathbf{y}_\beta^{-1} \circ \mathbf{y}_\alpha) = ((d\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha), d\left(d(\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha)\right)) = (d(\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha), d(\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha)).$$

So

$$\det\left(\mathbf{y}_\beta^{-1} \circ \mathbf{y}_\alpha\right) = \det\left(\frac{\partial y_i}{\partial x_j}\right)_{i,j=1}^{2n} = \det\left(\frac{\partial y_i}{\partial x_j}\right)_{i,j=1}^{n} \cdot \det\left(\frac{\partial y_i}{\partial x_j}\right)_{i,j=n+1}^{2n} = \left(\det\left(\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha\right)\right)^2 > 0.$$

That means the tangent bundle is orientable. □

Exercise 0.3. Prove that:

(a) a regular surface $S \subset \mathbb{R}^3$ is an orientable manifold if and only if there exists a differentiable mapping of $N \colon S \to \mathbb{R}^3$ with $N(p) \perp T_p(S)$ and $|N(p)| = 1$, for all $p \in S$.

(b) the Möbius band is non-orientable.

Proof. I don't know how to write it down. □

Exercise 0.4. Show that the projective plane $\mathbb{R}P^2$ is non-orientable.

Proof. Firstly, we prove that if the manifold M is orientable, then any open subset of M is an orientable submanifold. Notice that the differentiable structure $\{(U_\alpha, \mathbf{x}_\alpha)\}$ of M is also a differentiable structure of any open set $U \subset M$, so if M is orientable, then U is orientable.

$\mathbb{R}P^2$ can be obtained by gluing a clown hat and a disk along their boundries. And the clown hat is diffeomorphic to a Möbius band, which is non-orientable.

If $\mathbb{R}P^2$ is orientable, then Möbius band is orientable. This is a contradiction. □

Exercise 0.5 (Embedding of $\mathbb{R}P^2$ in $\mathbb{R}^4$)**.** Let $F\colon \mathbb{R}^3 \to \mathbb{R}^4$ be given by

$$F(x,y,z) = (x^2 - y^2, xy, xz, yz), \qquad (x,y,z) = p \in \mathbb{R}^3.$$

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere with the origin $0 \in \mathbb{R}^3$. Observe that the restriction $\varphi = F|_{S^2}$ is such that $\varphi(p) = \varphi(-p)$, and consider the mapping $\widetilde{\varphi}\colon \mathbb{R}P^2 \to \mathbb{R}^4$ given by

$$\widetilde{\varphi}([p]) = \varphi(p), \ [p] = \text{equiv. class of } p = \{p, -p\}.$$

Prove that:

(a) $\widetilde{\varphi}$ is an immersion.

(b) $\widetilde{\varphi}$ is injective; together with (a) and the compactness of $\mathbb{R}P^2$, this implies that $\widetilde{\varphi}$ is an embedding.

Proof. Locally,

$$\widetilde{\varphi}([p]) = \varphi(p) = \varphi(x,y) = F|_{S^2} = (x^2 - y^2, xy, \pm x\sqrt{1-x^2-y^2}, \pm y\sqrt{1-x^2-y^2}), \qquad p = (x,y,z) \in S^2.$$

Then

$$d\widetilde{\varphi}_p = \begin{bmatrix} 2x & -2y \\ y & x \\ \pm\frac{1-2x^2-y^2}{\sqrt{1-x^2-y^2}} & \mp\frac{xy}{\sqrt{1-x^2-y^2}} \\ \mp\frac{xy}{\sqrt{1-x^2-y^2}} & \pm\frac{1-x^2-2y^2}{\sqrt{1-x^2-y^2}} \end{bmatrix},$$

where

$$\det \begin{bmatrix} 2x & -2y \\ y & x \end{bmatrix} = 2(x^2+y^2) > 0.$$

So $\operatorname{rank} \begin{bmatrix} 2x & -2y \\ y & x \end{bmatrix} = 2$. Then we have $\operatorname{rank} d\widetilde{\varphi}_p = 2$, which means $d\widetilde{\varphi}$ is injective. It follows that $\widetilde{\varphi}$ is an immersion.

If $\widetilde{\varphi}([p]) = \widetilde{\varphi}([q])$, then $\varphi(p) = \varphi(q)$. So either $p = q$ or $p = -q$. Anyway, we have $[p] = [q]$. That means $\widetilde{\varphi}$ is injective.

Because $\mathbb{R}P^2$ is compact, $\widetilde{\varphi}(\mathbb{R}P^2)$ is compact. By the finite covering theorem, $\widetilde{\varphi}(\mathbb{R}P^2)$ has the same topology as the subspace topology induced by $\mathbb{R}^4$. Finally, we have $\widetilde{\varphi}$ is an embedding. ☐

Exercise 0.6 (Embedding of the Klein bottle in $\mathbb{R}^4$)**.** Show that the mapping $G\colon \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$G(x,y) = ((r\cos y + a)\cos x, (r\cos y + a)\sin x, r\sin y\cos\frac{x}{2}, r\sin y\sin\frac{x}{2}), \qquad (x,y) \in \mathbb{R}^2$$

induces an embedding of the Klein bottle into $\mathbb{R}^4$.

Proof. Notice that $\forall n, m \in \mathbb{N}$, $(x,y) \in \mathbb{R}^2$,

$$G(x + 4n\pi, y + 4m\pi) = G(x,y).$$

So G can induces a mapping $\widetilde{G}\colon \mathbb{T}^2 \to \mathbb{R}^4$. Then for $p = (x,y)$, we have $-p = (-x, y + 2\pi)$. And

$$\widetilde{G}(-p) = \widetilde{G}(-x, y+2\pi) = \widetilde{G}(x,y) = \widetilde{G}(p).$$

So $\widetilde{G}$ can induces a mapping $K \to \mathbb{R}^4$ and we also denote it by $\widetilde{G}$. And we can restrict $x, y \in [0, 4\pi)$, where $G = \widetilde{G}$.

$$d\widetilde{G}_{(x,y)} = \begin{bmatrix} -(r\cos y + a)\sin x & -r\cos x\sin y \\ (r\cos y + a)\cos x & -r\sin x\sin y \\ -\frac{r}{2}\sin\frac{x}{2}\sin y & r\cos y\cos\frac{x}{2} \\ \frac{r}{2}\cos\frac{x}{2}\sin y & r\cos y\sin\frac{x}{2} \end{bmatrix}.$$

We also have

$$\det \begin{bmatrix} -(r\cos y + a)\sin x & -r\cos x\sin y \\ (r\cos y + a)\cos x & -r\sin x\sin y \end{bmatrix} = r^2\cos y\sin y + ar\sin y,$$

$$\det\begin{bmatrix} -\frac{r}{2}\sin\frac{x}{2}\sin y & r\cos y\cos\frac{x}{2} \\ \frac{r}{2}\cos\frac{x}{2}\sin y & r\cos y\sin\frac{x}{2} \end{bmatrix} = -\frac{r^2}{2}\sin y\cos y.$$

If

$$\begin{cases} r^2\cos y\sin y + ar\sin y = 0, \\ -\frac{r^2}{2}\sin y\cos y = 0, \end{cases}$$

we have $y = 0, \pi, 2\pi, 3\pi$. Then

$$d\widetilde{G}_{(x,y)} = \begin{bmatrix} -(\pm r + a)\sin x & 0 \\ (\pm r + a)\cos x & 0 \\ 0 & \pm r\cos\frac{x}{2} \\ 0 & \pm r\sin\frac{x}{2} \end{bmatrix}.$$

So $\operatorname{rank} d\widetilde{G}_{(x,y)} = 2$ at any point (x, y). That means $\widetilde{G}$ is an immersion.

It is easy to see that $\widetilde{G}$ is injective and the rest of proof is same as Exercise 0.5. ☐

Exercise 0.7 (Infinite Möbius band)**.** Let $C = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 = 1\}$ be a right circular cylinder, and let $A\colon C \to C$ be the symmetry with respect to the oringin $0 \in \mathbb{R}^3$, that is, $A(x,y,z) = (-x,-y,-z)$. Let M be the quotient space of C with respect to the equivalence relation $p \sim A(p)$, and let $\pi\colon C \to M$ be the projection $\pi(p) = \{p, A(p)\}$.

(a) Show that it is possible to give M a differentiable structure such that π is a local diffeomorphism.

(b) Prove that M is non-orientable.

Proof. For a small open neighborhood U of $p \in M$, $\pi^{-1}(U)$ are two symmetric open disk of C. So the differentiable structure of M can be induced by C, which makes π a local diffeomorphism. So (a).

For (b), if not, notice that M contain an open finite Möbius band, which is orientable by Exercise 0.4. That is a contradiction. ☐

Exercise 0.8. Let M_1 and M_2 be differentiable manifolds. Let $\varphi\colon M_1 \to M_2$ be a local diffeomorphism. Prove that if M_2 is orientable, then M_1 is orientable.

Proof. Locally, let $\mathbf{x}_\alpha = \varphi^{-1} \circ \mathbf{y}_\alpha$ and $U_\alpha = \varphi^{-1}(V_\alpha)$ be the differentialble structure of M_1 induced by M_2. Then

$$\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha = (\varphi^{-1} \circ \mathbf{y}_\beta)^{-1} \circ (\varphi^{-1} \circ \mathbf{y}_\alpha) = \mathbf{y}_\beta^{-1} \circ \varphi \circ \varphi^{-1} \circ \mathbf{y}_\alpha = \mathbf{y}_\beta^{-1} \circ \mathbf{y}_\alpha.$$

So we have

$$\det\left(\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha\right) = \det\left(\mathbf{y}_\beta^{-1} \circ \mathbf{y}_\alpha\right) > 0,$$

which means M_1 is orientable. ☐

Exercise 0.9. Let $G \times M \to M$ be a properly discontinuous action of a group G on a differentiable manifold M.

(a) Prove that the manifold M/G is orientable if and only if there exists an orientation of M that is preserved by all the diffeomorphisms of G.

(b) Use (a) to show that the projection plane $\mathbb{R}P^2$, the Klein bottle and the Möbius band are non-orientable.

(c) Prove that $\mathbb{R}P^n$ is orientable if and only if n is odd.

Proof. ($\Rightarrow$): We choose an orientation $\{(V_\alpha, \mathbf{y}_\alpha)\}$ of M/G. $\forall p \in M$, there exists an open neighborhood $p \in U$ such that $\pi(U)$ is a subset of a V_α. Let $\mathbf{x} = \pi^{-1} \circ \mathbf{y}_\alpha$ on U. Then we claim that $\{(U, \mathbf{x})\}$ is an orientation of M by Exercise 0.8.

Locally, we can choose U such that $U \cap gU = \varnothing$, $\forall g \in G$. And we have $\pi(U) = \pi(gU) \subset V_\alpha$. So U and gU have the same differentiable structure induced by $\{(V_\alpha, \mathbf{y}_\alpha)\}$. That means φ_g preserves the orientation of M for all g.

($\Leftarrow$): We can choose an orientation $\{(U_\alpha, \mathbf{x}_\alpha)\}$ of M, such that $U_\alpha \cap gU_\alpha = \varnothing$ for all g and α. So $\forall [p] \in M/G$, we choose $p \in U_\alpha$ and let $V_\alpha = \pi(U_\alpha)$ and $\mathbf{y}_\alpha = \pi \circ \mathbf{x}_\alpha$.

$$\mathbf{y}_\beta^{-1} \circ \mathbf{y}_\alpha = (\pi \circ \mathbf{x}_\beta)^{-1} \circ (\pi \circ \mathbf{x}_\alpha) = \mathbf{x}_\beta^{-1} \circ \pi^{-1} \circ \pi \circ \mathbf{x}_\alpha = \mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha.$$

implies $\{(U_\alpha, \mathbf{x}_\alpha)\}$ is an orientation of M/G. Whence (a).

Thus we only need to prove that the map: $p \mapsto -p$ reserves any orientation of $S^2, \mathbb{T}^2, C$. They are embedding submanifolds of $\mathbb{R}^3$ and $[p \mapsto -p]$ reverses the orientation of $\mathbb{R}^3$: if

$$\mathbf{x}_\alpha(x, y, z) = -\mathbf{x}_\beta(x', y', z'),$$

we have

$$(x', y', z') = (-x, -y, -z).$$

So

$$d((-\mathbf{x}_\beta)^{-1} \circ \mathbf{x}_\alpha) = d(-\mathbf{x}_\beta^{-1}) \cdot d\mathbf{x}_\alpha = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -I_3.$$

It follows that

$$\det\left(d((-\mathbf{x}_\beta)^{-1} \circ \mathbf{x}_\alpha)\right) = \det(-I_3) = (-1)^3 = -1 < 0.$$

Finally we have the map: $p \mapsto -p$ reserves any orientation of $S^2, \mathbb{T}^2, C$, as subspace topology.

Like (b), we know that $\mathbb{R}P^n \cong S^n/_{p \sim -p}$. For the standard differentiable structure $\{(\mathbf{x}_\alpha, U_\alpha)\}$ of $\mathbb{R}^{n+1}$,

$$\det\left(d((-\mathbf{x}_\beta)^{-1} \circ \mathbf{x}_\alpha)\right) = \det(-I_{n+1}) = (-1)^{n+1}.$$

So we know that the map: $p \mapsto -p$ preserves the orientation of $\mathbb{R}^n$ if and only if n is odd, so is any orientation of the embedding submanifold S^n, as the subspace topology. By (a), this is equivalent to $\mathbb{R}P^n$ is orientable; whence (c). ☐

Exercise 0.10. Show that the topology of the quotient differentiable manifold M/G of a properly discontinuous action of G on M is Hausdorff if and only if the following condition holds: given two non-equivalent points $p_1, p_2 \in M$, there exist neighborhoods U_1, U_2 of p_1 and p_2, respectively, such that $U_1 \cap gU_2 = \varnothing$ for all $g \in G$.

Proof. $(\Rightarrow)$: $\forall [p_1] \neq [p_2] \in M/G$, we choose disjoint open neighborhoods $[p_1] \in V_1$ and $[p_2] \in V_2$. We can also choose V_1, V_2 such that $\pi^{-1}(V_1)$ and $\pi^{-1}(V_2)$ are disjoint union of some open sets. Let $p_1 \in U_1 \subset \pi^{-1}(V_1)$ and $p_2 \in U_2 \subset \pi^{-1}(V_2)$. We claim that $U_1 \cap gU_2 = \varnothing$ for all $g \in G$. If not, we suppose $q \in U_1 \cap gU_2$. So

$$[q] \in \pi(U_1 \cap gU_2) = \pi(U_1) \cap \pi(gU_2) = \pi(U_1) \cap \pi(U_2) = V_1 \cap V_2,$$

which is a contradiction.

$(\Leftarrow)$: $\forall [p_1] \neq [p_2] \in M/G$, we choose neighborhoods U_1, U_2 of p_1 and p_2, respectively, such that $U_1 \cap gU_2 = \varnothing$ for all $g \in G$. We claim that $\pi(U_1) \cap \pi(U_2) = \varnothing$. If not, we suppose $[q] \in \pi(U_1) \cap \pi(U_2)$. That means there exists $g_1, g_2 \in G$ and $q_1 \in U_1$, $q_2 \in U_2$ such that $q = g_1 q_1 = g_2 q_2$. So $q_1 = (g_1^{-1} g_2) q_2$, which means $U_1 \cap (g_1^{-1} g_2) U_2 \neq \varnothing$. That is a contradiction. Finally we have M/G is Hausdorff. ☐

Exercise 0.11. Let us consider the two following differentiable structures on the real line $\mathbb{R}$: $(\mathbb{R}, \mathbf{x}_1)$, where $\mathbf{x}_1 \colon \mathbb{R} \to \mathbb{R}$ is given by $\mathbf{x}_1(x) = x$, $x \in \mathbb{R}$; $(\mathbb{R}, \mathbf{x}_2)$, where $\mathbf{x}_2 \colon \mathbb{R} \to \mathbb{R}$ is given by $\mathbf{x}_2(x) = x^3$, $x \in \mathbb{R}$. Show that:

(a) the identity mapping $i \colon (\mathbb{R}, \mathbf{x}_1) \to (\mathbb{R}, \mathbf{x}_2)$ is not a diffeomorphism; therefore, the maximal structures determined by $(\mathbb{R}, \mathbf{x}_1)$ and $(\mathbb{R}, \mathbf{x}_2)$ are distinct.

(b) the mapping $f \colon (\mathbb{R}, \mathbf{x}_1) \to (\mathbb{R}, \mathbf{x}_2)$ given by $f(x) = x^3$ is a diffeomorphism; that is, even though the differentiable structure $(\mathbb{R}, \mathbf{x}_1)$ and $(\mathbb{R}, \mathbf{x}_2)$ are distinct, they determine diffeomorphic differentiable manifolds.

Proof. For (a), we say

$$\mathbf{x}_2^{-1} \circ i \circ \mathbf{x}_1(x) = \mathbf{x}_2^{-1} \circ i(x) = \mathbf{x}_2^{-1}(x) = \sqrt[3]{x}.$$

So

$$d\left(\mathbf{x}_2^{-1} \circ i \circ \mathbf{x}_1\right)_x = \frac{1}{3\sqrt[3]{x^2}}.$$

which is not invertible at $x = 0$. That means i is not a diffeomorphism.

Similarly,

$$\mathbf{x}_2^{-1} \circ f \circ \mathbf{x}_1(x) = \mathbf{x}_2^{-1} \circ f(x) = \mathbf{x}_2^{-1}(x^3) = x,$$

which means it is a identity map. Obviously, f is a diffeomorphism; whence (b). ☐

Exercise 0.12 (The orientable double covering)**.** Let M^n be a non-orientable differentiable manifold. For each $p \in M$, consider the set B of bases of T_pM and say that two bases are equivalent if they are related by a matrix with positive determinant. This is an equivalent relation and separates B into two disjoint sets. Let $\mathcal{O}_p$ be the quotient space of B with respect to this equivalence relation. $O_p \in \mathcal{O}_p$ will be called an *orientation* of T_pM. Let $\overline{M}$ be the set

$$\overline{M} = \{ (p, O_p) |\, p \in M, O_p \in \mathcal{O}_p \} .$$

Let $\{(U_\alpha, \mathbf{x}_\alpha)\}$ be a maximal differentiable structure on M, and define $\overline{\mathbf{x}}_\alpha \colon U_\alpha \to \overline{M}$ by

$$\overline{\mathbf{x}}_\alpha \left(u_1^\alpha, \cdots, u_n^\alpha\right) = \left(\mathbf{x}_\alpha \left(u_1^\alpha, \cdots, u_n^\alpha\right), \left[\frac{\partial}{\partial u_1^\alpha}, \cdots, \frac{\partial}{\partial u_n^\alpha} \right] \right),$$

where $(u_1^\alpha, \cdots, u_n^\alpha) \in U_\alpha$ and $[\frac{\partial}{\partial u_1^\alpha}, \cdots, \frac{\partial}{\partial u_n^\alpha}]$ denotes the element of $\mathcal{O}_p$ determined by the basis $\{\frac{\partial}{\partial u_1^\alpha}, \cdots, \frac{\partial}{\partial u_n^\alpha}\}$. Prove that:

(a) $\{(U_\alpha, \mathbf{x}_\alpha)\}$ is a differentiable structure on $\overline{M}$ and that the manifold $\overline{M}$ so obtained is orientable.

(b) The mapping $\pi \colon \overline{M} \to M$ given by $\pi(p, O_p) = p$ is differentiable and surjective. In addition, each $p \in M$ has a neighborhood $U \subset M$ such that $\pi^{-1}(U) = V_1 \cup V_2$, where V_1 and V_2 are disjoint open sets in $\overline{M}$ and π restricted to each V_i, $i = 1, 2$, is a diffeomorphism onto U. For this reason, $\overline{M}$ is called the *orientable double cover of* M.

(c) The sphere S^2 is the orientable double cover of $\mathbb{R}P^2$ and the torus $\mathbb{T}^2$ is the orientable double cover of the Klein bottle K.

Proof. The proof is similar to Exercise 0.2 and Exercise 0.9. □

1 Riemannian Metrics

Exercise 1.1. Prove that the antipodal mapping $A\colon S^n \to S^n$ given by $A(p) = -p$ is an isometry of S^n. Use this fact to introduce a Riemannian metric on the real projective space $\mathbb{R}P^n$ such that the natural projection $\pi\colon S^n \to \mathbb{R}P^n$ is a local isometry.

Proof. Notice that as the embedding submanifold of $\mathbb{R}^{n+1}$, $\forall v \in T_pS^n n$, we have

$$dA_p(v) = -v \in T_{-p}S^n.$$

It follows that

$$\left\langle dA_p(u), dA_p(v)\right\rangle_{A(p)} = \langle -u, -v\rangle_{\mathbb{R}^{n+1}} = \langle u, v\rangle_p\,,$$

which means A is an isometry.

Locally, let $p = \pi^{-1}([p])$ be any representative points. We define

$$\langle u, v\rangle_{[p]} = \left\langle d\pi_p^{-1}u, d\pi_p^{-1}v\right\rangle_p.$$

Then we check the metric is well defined. We have $\pi^{-1}([p]) = \{p, -p\}$. So we just need to check

$$\left\langle d\pi_{A(p)}^{-1}u, d\pi_{A(p)}^{-1}v\right\rangle_{A(p)} = \left\langle -d\pi_p^{-1}u, -d\pi_p^{-1}v\right\rangle_{-p} = \left\langle d\pi_p^{-1}u, d\pi_p^{-1}v\right\rangle_p.$$

By the definition, $\forall u, v \in T_pS^n$,

$$\langle d\pi_p u, d\pi_p v\rangle_{\pi(p)} = \langle d\pi_p u, d\pi_p v\rangle_{[p]} = \left\langle d\pi_p^{-1}\circ d\pi_p u, d\pi_p^{-1}\circ d\pi_p v\right\rangle_p = \langle u, v\rangle_p\,,$$

which proves that π is a local isometry. ☐

Exercise 1.2. Introduce a Riemannian metric on the torus $\mathbb{T}^n$ in such a way that the natural projection $\pi\colon \mathbb{R}^n \to \mathbb{T}^n$ given by

$$\pi(x_1, \cdots, x_n) = (\mathrm{e}^{\mathrm{i}\,x_1}, \cdots, \mathrm{e}^{\mathrm{i}\,x_n}), \qquad (x_1, \cdots, x_n) \in \mathbb{R}^n,$$

is a local isometry. Show that with this metric $\mathbb{T}^n$ is isometric to the flat torus.

Proof. Locally, we can see $\mathbb{T}^n = S^1 \times \cdots \times S^1$ as $[0,1) \times \cdots \times [0,1) \subset \mathbb{R}^n$. So for all $v \in T_p\mathbb{T}^n$, we can find it in $\mathbb{R}^n$. Then we define the metric induced by $i\colon \mathbb{T}^n \to [0,1) \times \cdots \times [0,1) \subset \mathbb{R}^n$. For all $p = (x_1, \cdots, x_n) \in \mathbb{R}^n$,

$$d\pi_p = \operatorname{diag}\{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\,x_1}, \cdots, \mathrm{i}\,\mathrm{e}^{\mathrm{i}\,x_n}\}.$$

So we have

$$\begin{aligned}\langle d\pi_p(u), d\pi_p(v)\rangle_{\pi(p)} &= \left\langle (\mathrm{i}\,\mathrm{e}^{\mathrm{i}\,x_1} u_1, \cdots, \mathrm{i}\,\mathrm{e}^{\mathrm{i}\,x_n} u_n), (\mathrm{i}\,\mathrm{e}^{\mathrm{i}\,x_1} v_1, \cdots, \mathrm{i}\,\mathrm{e}^{\mathrm{i}\,x_n} v_n)\right\rangle_{\mathbb{C}^n} \\ &= \sum_{i=1}^n \mathrm{i}\,\mathrm{e}^{\mathrm{i}\,x_1} u_i \cdot \overline{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\,x_1} v_i} = \sum_{i=1}^n u_i v_i = \langle u, v\rangle_{\mathbb{R}^n}\,,\end{aligned}$$

which means π is a local isometry.

For $\pi|_{[0,1)\times\cdots\times[0,1)}\colon \mathbb{T}^n \to \mathbb{T}^n$, we also have

$$\langle u, v\rangle_{flat} = \sum_{i=1}^n \langle d\pi_i(u), d\pi_i(v)\rangle_{\mathrm{e}^{\mathrm{i}\,x_i}} = \sum_{i=1}^n u_i v_i = \langle u, v\rangle_{\mathbb{R}^n} = \langle d\pi_p(u), d\pi_p(v)\rangle_{\pi(p)}.$$

It follows that $\mathbb{T}^n$ with this metric is isometric to the flat torus. ☐

Exercise 1.3. Obtain an isometric immersion of the flat torus $\mathbb{T}^n$ into $\mathbb{R}^{2n}$.

Proof. For $p = (\mathrm{e}^{\mathrm{i}\,x_1}, \cdots, \mathrm{e}^{\mathrm{i}\,x_n}) \in \mathbb{T}^n = S^1 \times \cdots \times S^1$, we can see $\mathrm{e}^{\mathrm{i}\,x_i} \in \mathbb{C}$ as $(\sin x_i, \cos x_i) \in \mathbb{R}^2$. So we have a map

$$\begin{aligned}\varphi\colon \mathbb{T}^n &\to \mathbb{R}^{2n} \\ (\mathrm{e}^{\mathrm{i}\,x_1}, \cdots, \mathrm{e}^{\mathrm{i}\,x_n}) &\mapsto (\sin x_1, \cos x_1, \cdots, \sin x_n, \cos x_n).\end{aligned}$$

We denote $e^{\mathrm{i}\,x_i}$ as z, so $\sin x_i = \sin z_i$, $\cos x_i = \cos z_i$. Then

$$d\varphi_p = \begin{bmatrix} \cos z_1 & & \\ -\sin z_1 & & \\ & \ddots & \\ & & \cos z_n \\ & & -\sin z_n \end{bmatrix}.$$

So for all $u = (u_1, \cdots, u_n) \in T_p\mathbb{T}^n$,

$$d\varphi_p(u) = (u_1 \cos x_1, -u_1 \sin x_1, \cdots, u_n \cos x_n, -u_n \sin x_n).$$

Finally, by Exercise 1.2, we have

$$\begin{aligned} &\langle d\varphi_p(u), d\varphi_p(v)\rangle_{\mathbb{R}^{2n}} \\ =&\langle (u_1 \cos x_1, -u_1 \sin x_1, \cdots, u_n \cos x_n, -u_n \sin x_n), (v_1 \cos x_1, -v_1 \sin x_1, \cdots, v_n \cos x_n, -v_n \sin x_n)\rangle_{\mathbb{R}^{2n}} \\ =&\sum_{i=1}^{n}(u_i v_i \cos^2 x_i + u_i v_i \sin^2 x_i) = \sum_{i=1}^{n} u_i v_i = \langle u, v\rangle_{\mathbb{R}^n} = \langle u, v\rangle_{flat}, \end{aligned}$$

which means φ is an isometric immersion of the flat torus $\mathbb{T}^n$ into $\mathbb{R}^{2n}$. □

Exercise 1.4. A function $g\colon \mathbb{R} \to \mathbb{R}$ given by $g(t) = yt + x$, $t, x, y \in \mathbb{R}$, $y > 0$, is called a *proper affine function*. The subset of all such functions with respect to the usual composition law forms a Lie group G. As a differentiable manifold G is simply the upper half-plane $\{(x, y) \in \mathbb{R}^2 | y > 0\}$ with the differentiable structure induced from $\mathbb{R}^2$. Prove that:

(a) The left-invariant Riemannian metric of G which at the neutral element $e = (0, 1)$ coincides with the Euclidean metric ($g_{11} = g_{22} = 1$, $g_{12} = 0$) is given by $g_{11} = g_{22} = \frac{1}{y^2}$, $g_{12} = 0$.

(b) Putting $(x, y) = z = x + \mathrm{i}\,y$, $\mathrm{i} = \sqrt{-1}$, the transformation $z \mapsto z' = \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$ is an isometry of G.

Proof. We denote $g(t) = yt + x$ as $g = (x, y) \in \mathbb{H}^2$. The composition is

$$(yt + x) \circ (y't + x') = yy't + x'y + x.$$

So the product of G is given by

$$(x, y) \cdot (x', y') = (x'y + x, yy'),$$

and $g^{-1} = (-\frac{x}{y}, \frac{1}{y})$. So we have $L_{(x,y)}((a, b)) = (x, y) \cdot (a, b) = (ay + x, by)$. Then we have $dL_{(x,y)} = \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix}$. For $\partial_1 = \frac{\partial}{\partial x} = (1, 0)$ and $\partial_2 = \frac{\partial}{\partial y} = (0, 1)$,

$$g_{11} = \langle \partial_1, \partial_1 \rangle_g = \langle dL_{g^{-1}}\partial_1, dL_{g^{-1}}\partial_1 \rangle = \left\langle \begin{bmatrix} \frac{1}{y} & 0 \\ 0 & \frac{1}{y} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{y} & 0 \\ 0 & \frac{1}{y} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} \frac{1}{y} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{y} \\ 0 \end{bmatrix} \right\rangle = \frac{1}{y^2}.$$

Similarly, we have $g_{22} = \frac{1}{y^2}$, $g_{12} = 0$; whence (a).

For (b), notice that

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = -\frac{4dzd\overline{z}}{(z - \overline{z})^2}.$$

For the transformation, we have

$$dz' = \frac{dz}{(cz + d)^2}.$$

After some calculations, we have

$$ds^2 = -\frac{4dzd\overline{z}}{(z - \overline{z})^2} = -\frac{4dz'd\overline{z'}}{(z - \overline{z'})^2}.$$

That means the matric of G under the transformation is same as original G, which is we desired. □

Exercise 1.5. Prove that the isometries of $S^n \subset \mathbb{R}^{n+1}$, with the inudced metric, are the restrictions to S^n of the linear orthogonal maps of $\mathbb{R}^{n+1}$.

Proof. For all $f \in \mathrm{Isom}(S^n)$, we construct a map $F\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ as below:

$$F(x) = \begin{cases} 0, & x = 0, \\ f\left(\frac{x}{|x|}\right) \cdot |x| & x \neq 0. \end{cases}$$

For all $x, y \neq 0 \in \mathbb{R}^{n+1}$, because the metric of S^n is induced by $\mathbb{R}^{n+1}$ and f is an isometry of S^n, we have

$$\begin{aligned} \langle F(x), y \rangle_{\mathbb{R}^{n+1}} &= \left\langle f\left(\frac{x}{|x|}\right)|x|\,, y \right\rangle_{\mathbb{R}^{n+1}} = |x| \cdot |y| \left\langle f\left(\frac{x}{|x|}\right), \frac{y}{|y|} \right\rangle_{S^n} \\ &= |x| \cdot |y| \left\langle \frac{x}{|x|}, f\left(\frac{y}{|y|}\right) \right\rangle_{S^n} = \left\langle x, f\left(\frac{y}{|y|}\right)|y| \right\rangle_{\mathbb{R}^{n+1}} = \langle x, F(y) \rangle_{\mathbb{R}^{n+1}}\,. \end{aligned}$$

That means $F \in \mathrm{O}_{n+1}(\mathbb{R})$ and $f = F|_{S^n}$, as desired. ☐

Exercise 1.6. Show that the relation "M is locally isometric to N" is not a symmetric relation.

Proof. Let $M = \mathbb{R}$ and $N = \mathbb{R}^2$. It is easy to see that $i\colon \mathbb{R} \to \mathbb{R}^2$ is a local isometry. Conversely, for any map $f\colon \mathbb{R}^2 \to \mathbb{R}$, the kernel of df_p is not empty by counting the dimensions. So for any neighborhood U of p, we can choose $v \neq 0 \in \mathrm{Ker}(df_p)$ which is short enough so that $v \in U$, then we have

$$\langle df_p v, df_p v \rangle_{\mathbb{R}} = \langle 0, 0 \rangle_{\mathbb{R}} = 0 \neq \langle v, v \rangle_{\mathbb{R}^2}\,,$$

which implies that f is not a local isometry. So N is not locally isometric to M, as desired. ☐

Exercise 1.7. Let G be a compact connected Lie group ($\dim G = n$). The object of this exercise is to prove that G has a bi-invariant Riemannian metric. To do this, take the following approach:

(a) Let ω be a differential n-form on G invariant on the left, that is, $L_x^*\omega = \omega$, for all $x \in G$. Prove that ω is right invariant.

(b) Show that there exists a left invariant differential n-form ω on G.

(c) Let $\langle\ ,\ \rangle$ be a left invariant metric on G. Let ω be a positive differential n-form on G which is invariant on the left, and define a new Riemannian metric $\langle\langle\ ,\ \rangle\rangle$ on G by

$$\langle\langle u, y \rangle\rangle_y = \int_G \langle (dR_x)_y u, (dR_x)_y v \rangle_{yx}\, \omega, \qquad x, y \in G,\ u, v \in T_y(G).$$

Prove that this new Riemannian metric $\langle\langle\ ,\ \rangle\rangle$ is bi-invariant.

Proof. For (a), for any $a \in G$, $L_x^* R_a^* \omega = R_a^* L_x^* \omega = R_a^* \omega$. So $R_a^*\omega$ is left invariant. Because all n-forms span a real linear space of dimension 1, we have $R_a^*\omega = f(a)\omega$ where f is a real function on G. For any tangent vector $X_1, \cdots, X_n$,

$$\begin{aligned} R_{ab}^*\omega(X_1, \cdots, X_n) &= \omega(dR_{ab}X_1, \cdots, dR_{ab}X_n) \\ &= \omega(dR_a dR_b X_1, \cdots, dR_a dR_b X_n) = R_a^*\omega(dR_b X_1, \cdots, dR_b X_n) = R_b^* R_a^* \omega(X_1, \cdots, X_n). \end{aligned}$$

So we have $R_{ab}^*\omega = R_b^* R_a^* \omega$. Then we have

$$f(ab)\omega = R_{ab}^*\omega = R_b^* R_a^*\omega = R_b^*(R_a^*\omega) = R_b^*(f(a)\omega) = f(b)f(a)\omega = f(a)f(b)\omega.$$

So we have $f(ab) = f(a)f(b)$. Then $f\colon G \to (\mathbb{R} - \{0\})^\times$ is a homomorphism. Because G is compact and connected, so is $f(G)$. It's a fact that the only compact connected subgroup of $(\mathbb{R} - \{0\})^\times$ is $\{1\}$. It follows that $f \equiv 1$ on G, which means $R_a^*\omega = f(a)\omega = \omega$. So ω is right invariant.

For (b), we choose linear independent $X_{1e}, \cdots, X_{ne} \in T_eG$, and let $X_{ia} = dL_a X_{ie}$ such that they are linear independent too then are basis of T_aG at any $a \in G$. At any $a \in G$, we choose their dual basis $dx_1, \cdots, dx_n$ and denote $\omega = dx_1 \cdots dx_n$. We need to calculate

$$L_x^*\omega(Y_1, \cdots, Y_n) = \omega(dL_x Y_1, \cdots, dL_x Y_n)$$

for any $x \in G$ and vector fields $Y_1, \cdots, Y_n$. Because $X_1, \cdots, X_n$ is a basis, by linearity, we only need to calculate $L_x^*\omega(f_1Y_1, \cdots, f_nY_n)$ for $Y_1, \cdots, Y_n \in \{X_1, \cdots, X_n\}$ where $f_1, \cdots, f_n$ are arbitrary functions on G. As we defined, $X_1, \cdots, X_n$ are left invariant, so we have

$$\begin{aligned} L_x^*\omega(f_1Y_1, \cdots, f_nY_n) &= \omega(dL_x(f_1Y_1), \cdots, dL_x(f_nY_n)) \\ &= \omega(f_1 dL_x Y_1, \cdots, f_n dL_x Y_n) = \omega(f_1Y_1, \cdots, f_nY_n), \end{aligned}$$

which means $L_x^*\omega = \omega$ so ω is left invariant, as desired.

Finally, for any $x, y, z \in G$ and $u, v \in T_yG$, by definition, firstly we need to calculate

$$\langle\langle (dL_z)_y u, (dL_z)_y v\rangle\rangle_{zy} = \int_G \langle (dR_x)_{zy}(dL_z)_y u, (dR_x)_{zy}(dL_z)_y v\rangle_{zyx}\, \omega.$$

It is easy to see that R_x and L_z are commutable. By differentiating $R_x \circ L_z = L_z \circ R_x$ and chain rule, we have

$$(dR_x)_{zy}(dL_z)_y = (dL_z)_{yx}(dR_x)_y.$$

Because $\langle\ ,\ \rangle$ is a left invariant metric, we have

$$\begin{aligned} \langle\langle (dL_z)_y u, (dL_z)_y v\rangle\rangle_{zy} &= \int_G \langle (dR_x)_{zy}(dL_z)_y u, (dR_x)_{zy}(dL_z)_y v\rangle_{zyx}\, \omega \\ &= \int_G \langle (dL_z)_{yx}(dR_x)_y u, (dL_z)_{yx}(dR_x)_y v\rangle_{zyx}\, \omega \\ &= \int_G \langle (dR_x)_y u, (dR_x)_y v\rangle_{yx}\, \omega \\ &= \langle\langle u, y\rangle\rangle_y\,. \end{aligned}$$

So $\langle\langle\ ,\ \rangle\rangle$ is left invariant. Similarly, by definition, $R_x \circ R_y = R_{yx}$. Then we have

$$(dR_x)_{yz}(dR_z)_y = (dR_{zx})_y.$$

By (b), ω is also right invariant. Then

$$\begin{aligned} \langle\langle (dR_z)_y u, (dR_z)_y v\rangle\rangle_{yz} &= \int_G \langle (dR_x)_{yz}(dR_z)_y u, (dR_x)_{yz}(dR_z)_y v\rangle_{yzx}\, \omega \\ &= \int_G \langle (dR_{zx})_y u, (dR_{zx})_y v\rangle_{yzx}\, \omega \\ &= \int_G \langle (dR_{zx})_y u, (dR_{zx})_y v\rangle_{yzx}\, R_z^*\omega \\ &= \int_G R_z^*\left(\langle (dR_x)_y u, (dR_x)_y v\rangle_{yx}\, \omega\right) \\ &= \int_G \langle (dR_x)_y u, (dR_x)_y v\rangle_{yx}\, \omega \\ &= \langle\langle u, v\rangle\rangle_y\,, \end{aligned}$$

where the penultimate equality is the change of variables $xz \mapsto x$, which shows that $\langle\langle\ ,\ \rangle\rangle$ is right invariant. So we proved that any compact connected Lie group has a bi-invariant Riemannian metric. See reference: *https://math.stackexchange.com/questions/72333.* □

2 Affine Connections; Riemannian Connections

Exercise 2.1. Let M be a Riemannian manifold. Consider the mapping

$$P = P_{c,t_0,t} \colon T_{c(t_0)}M \to T_{c(t)}M$$

defined by: $P_{c,t_0,t}(v)$, $v \in T_{c(t_0)}M$, is the vector obtained by parallel transporting the vector v along the curve c. Show that P is an isometry and that, if M is oriented, P preserves the orientation.

Proof. For all $V \in T_{c(t_0)}M$, we denote the parallel transport along c by $V(t)$. We choose a orthonormal basis $\{V_1, \cdots, V_n\} \subset T_{c(t_0)}M \cong \mathbb{R}^n$. Then for all vector $V = \sum_{i=1}^n a_i V_i$, we consider $\left(\sum_{i=1}^n a_i V_i\right)(t)$ and $\sum_{i=1}^n a_i V_i(t)$. Notice that

$$\left(\sum_{i=1}^n a_i V_i\right)(t)\bigg|_{t=t_0} = V(t_0) = \sum_{i=1}^n a_i V_i(t)\bigg|_{t=t_0},$$

by the uniqueness of parallel transport, we must have

$$P\left(\sum_{i=1}^n a_i V_i\right) = \left(\sum_{i=1}^n a_i V_i\right)(t) = \sum_{i=1}^n a_i V_i(t) = \sum_{i=1}^n a_i P(V_i).$$

Notice that

$$P_{c,t_0,t} \circ P_{-c,a+b-t_0,a+b-t} = \mathrm{Id},$$

where $-c$ is the opposite curve $(-c)(s) = c(a+b-s)$. It follows that P is a linear isomorphism. We also have $dP = P$. Let ∇ be the affine connection which is compatible with $\langle\ ,\ \rangle$ (it exists by Levi-Civita Theorem). Then we have

$$\langle dP(V_i), dP(V_j)\rangle = \langle P(V_i), P(V_j)\rangle = \langle V_i(t), V_j(t)\rangle = \langle V_i(t_0), V_j(t_0)\rangle = \langle V_i, V_j\rangle, \quad \forall 1 \le i \le j \le n.$$

By the linearity of metirc, we conclude that P is an isometry.

For $t = t_0$, P is the identity. So the determinant of P is $1 > 0$. If for t, $P_{c,t_0,t}$ have negative determinant, by the continuity of the function det, there must be a time $t_0 < s < t$ such that $\det(P_{c,t_0,s}) = 0$, which contradicts P is a linear isomorphism for any t. □

Exercise 2.2. Let X and Y be differentiable vector fields on a Riemannian manifold M. Let $p \in M$ and let $c \colon T \to M$ be an integral curve of X through p, *i.e.* $c(t_0) = p$ and $\frac{dc}{dt} = X(c(t))$. Prove that the Riemannian connection of M is

$$(\nabla_X Y)(p) = \frac{d}{dt}\left(P^{-1}_{c,t_0,t}(Y(c(t)))\right)\bigg|_{t=t_0},$$

where $P_{c,t_0,t} \colon T_{c(t_0)}M \to T_{c(t)}M$ is the parallel transport along c, from t_0 to t (this shows how the connection can be reobtained from the concept of parallelism).

Proof. We suppose that $\{e_1, \cdots, e_n\}$ is an orthonormal basis of T_pM. We denote the parallel transport of $\{e_1, \cdots, e_n\}$ along c by $\{e_1(t), \cdots, e_n(t)\}$. Then we can write $Y(c(t)) = \sum_{i=1}^n y_i(t) e_i(t)$. By the definition of covariant derivative, we have

$$\begin{aligned}(\nabla_X Y)(p) = \left(\nabla_{c'(t_0)} Y\right)(p) = \frac{DY(c(t))}{dt}\bigg|_{t=t_0} &= \sum_{i=1}^n \frac{D\left(y_i(t)e_i(t)\right)}{dt}\bigg|_{t=t_0} \\ &= \sum_{i=1}^n \left(\frac{dy_i(t)}{dt} e_i(t) + y_i(t)\frac{De_i(t)}{dt}\right)\bigg|_{t=t_0} = \sum_{i=1}^n \frac{dy_i(t)}{dt}\bigg|_{t=t_0} e_i(t_0).\end{aligned}$$

Besides, by the linearity of $P^{-1}_{c,t_0,t}$ (by Exercise 2.1), we also have

$$\begin{aligned}\frac{d}{dt}\left(P^{-1}_{c,t_0,t}(Y(c(t)))\right)\bigg|_{t=t_0} &= \frac{d}{dt}\left(P^{-1}_{c,t_0,t}\left(\sum_{i=1}^n y_i(t)e_i(t)\right)\right)\bigg|_{t=t_0} = \sum_{i=1}^n \frac{d}{dt}\left(y_i(t)P^{-1}_{c,t_0,t}(e_i(t))\right)\bigg|_{t=t_0} \\ &= \sum_{i=1}^n \frac{d}{dt}\left(y_i(t)e_i(t_0)\right)\bigg|_{t=t_0} = \sum_{i=1}^n \frac{dy_i(t)}{dt}\bigg|_{t=t_0} e_i(t_0).\end{aligned}$$

Finally, we conclude

$$\left(\nabla_X Y\right)(p) = \sum_{i=1}^{n} \left.\frac{dy_i(t)}{dt}\right|_{t=t_0} e_i(t_0) = \left.\frac{d}{dt}\left(P^{-1}_{c,t_0,t}\left(Y(c(t))\right)\right|_{t=t_0},$$

as desired. □

Exercise 2.3. Let $f\colon M^n \to \overline{M}^{n+k}$ be an immersion of a differentiable manifold M into a Riemannian manifold $\overline{M}$. Assume that M has the Riemannian metric induced by f. Let $p \in M$ and let $U \subset M$ be a neighborhood of p such that $f(U) \subset \overline{M}$ is a submanifold of $\overline{M}$. Further, suppose that X, Y are differentiable vector fields on $f(U)$ which extend to differentiable vector fields $\overline{X}, \overline{Y}$ on an open set of $\overline{M}$. Define $(\nabla_X Y)(p) =$ tangential component of $\overline{\nabla}_{\overline{X}}\overline{Y}(p)$, where $\overline{\nabla}$ is the Riemannian connection of $\overline{M}$. Prove that ∇ is the Riemannian connection of M.

Proof. It's easy to check that ∇ is a affine connection.

For any vector field V of M, we suppose that $c\colon I \to M$ is a curve such that $\frac{dc}{dt}\big|_{t=t_0} = V_p$ and let $\overline{c} = f \circ c$. Then we have

$$\left.\frac{d\overline{c}}{dt}\right|_{t=t_0} = \left.\frac{d(f\circ c)}{dt}\right|_{t=t_0} = df_p \cdot \left.\frac{dc}{dt}\right|_{t=t_0} = df_p\left(V_p\right).$$

Because f is an immersion, we can firstly consider $f(M)$.

By the local notion of of affine connection:

$$\nabla_X Y = \sum_k \left(\sum_{ij} x_i y_j \Gamma^k_{ij} + X(y_k)\right) X_k,$$

$\nabla_X Y(p)$ depends on $x_i(p)$, $y_k(p)$ and the derivatives $X(y_k)(p)$ of y_k by X. This means $\overline{\nabla}_{\overline{X}}\overline{Y}$ at any $q \in f(M)$ depends only on $\overline{X}(q) = X(q)$ and the vectors $\overline{X}(q') = X(q')$ for $q' \in f(M)$ near q. That is to say $\overline{\nabla}_{\overline{X}}\overline{Y}$ is independent of the choice of the extensions. So we will write $\overline{\nabla}_X Y$ instead of $\overline{\nabla}_{\overline{X}}\overline{Y}$ for simplicity.

For any parallel vector field V on $f(M)$ along c, we have

$$0 = \frac{DV}{dt} = \nabla_{c'} V = \text{tangential component of } \overline{\nabla}_{c'} V = \text{tangential component of } \frac{D\overline{V}}{dt}.$$

It follows that $\frac{D\overline{V}}{dt} \perp Tf(M)$. Then for any two parallel vector fields $X, Y \in Tf(M)$,

$$\frac{d}{dt}\langle X, Y\rangle = \frac{d}{dt}\left\langle \overline{X}, \overline{Y}\right\rangle = \left\langle \frac{D\overline{X}}{dt}, \overline{Y}\right\rangle + \left\langle \overline{X}, \frac{D\overline{Y}}{dt}\right\rangle = \left\langle \frac{D\overline{X}}{dt}, Y\right\rangle + \left\langle X, \frac{D\overline{Y}}{dt}\right\rangle = 0.$$

This means $\langle X, Y\rangle =$ constant. So ∇ is compatible with the metric of $f(M) \subset \overline{M}$. There is another proof: For any $X, Y, Z \in Tf(M)$, because $\nabla_X Y$ is the tangential component of $\overline{\nabla}_{\overline{X}}\overline{Y}$, which is the projection onto $Tf(M)$. Then we have

$$\left\langle \overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z}\right\rangle = \left\langle \overline{\nabla}_{\overline{X}}\overline{Y}, Z\right\rangle = \left\langle \nabla_X Y + (\nabla_X Y)^{\perp}, Z\right\rangle = \left\langle \nabla_X Y, Z\right\rangle.$$

So we have

$$X\langle Y, Z\rangle = \overline{X}\left\langle \overline{Y}, \overline{Z}\right\rangle = \left\langle \overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z}\right\rangle + \left\langle \overline{Y}, \overline{\nabla}_{\overline{X}}\overline{Z}\right\rangle = \left\langle \nabla_X Y, Z\right\rangle + \left\langle Y, \nabla_X Z\right\rangle.$$

Then for any $X, Y \in Tf(M)$,

$$\begin{aligned}
\nabla_X Y - \nabla_Y X &= \text{tangential component of } \overline{\nabla}_{\overline{X}}\overline{Y} - \text{tangential component of } \overline{\nabla}_{\overline{Y}}\overline{X} \\
&= \text{tangential component of } \left(\overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X}\right) = \text{tangential component of } \left[\overline{X}, \overline{Y}\right] \\
&= \text{tangential component of } \left(\overline{XY} - \overline{YX}\right) = \text{tangential component of } (XY - YX) \\
&= XY - YX = [X, Y].
\end{aligned}$$

So ∇ is symmetric. Whence ∇ is the Riemannian connection of $f(M)$.

Finally, we can induce the Riemannian connection ∇ of M by the immersion f: For any $X, Y \in TM$, let

$$\nabla_X Y = df^{-1}\left(\nabla_{df(X)} df(Y)\right)$$

where df^{-1} is well defined on $Tf(M)$ and we also use the notation ∇ for the Riemannian connection of M without confusion, as desired. □

Exercise 2.4. Let $M^2 \subset \mathbb{R}^3$ be a surface in $\mathbb{R}^3$ with the induced Riemannian metric. Let $c\colon I \to M$ be a differentiable curve on M and let V be vector field tangent to M along c; V can be thought of as a smooth function $V\colon I \to \mathbb{R}^3$, with $V(t) \in T_{c(t)}M$.

a) Show that V is parallel if and only if $\frac{dV}{dt}$ is perpendicular to $T_{c(t)}M \subset \mathbb{R}^3$ where $\frac{dV}{dt}$ is the usual derivative of $V\colon I \to \mathbb{R}^3$.

b) If $S^2 \subset \mathbb{R}^3$ is the unit sphere of $\mathbb{R}^3$, show that the velocity field along great circles, parametrized by arc length, is a parallel field. A similar argument holds for $S^n \subset \mathbb{R}^{n+1}$.

Proof. We will prove for any n directly.

a) is just a special case of Exercise 2.4:

$$0 = \frac{DV}{dt} = \nabla_{c'}V = \text{tangential component of } \overline{\nabla}_{c'}V = \text{tangential component of } \frac{dV(c(t))}{dt},$$

where the last equation is induced by the fact that Riemannian connection on Euclidean space is trivial. So V is parallel if and only if $\frac{dV}{dt}$ is perpendicular to TM.

For b), we can assume the great circle is

$$\begin{aligned} c\colon I &\to S^n \\ t &\mapsto (\cos t, \sin t, 0, \cdots, 0). \end{aligned}$$

Then the velocity field is

$$c'(t) = (-\sin t, \cos t, 0, \cdots, 0).$$

Then

$$\frac{dc'}{dt} = c''(t) = (-\cos t, -\sin t, 0, \cdots, 0).$$

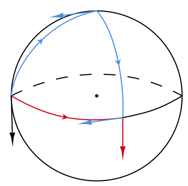
By symmetry, we only need to check $\frac{dc'}{dt} \perp T_pS^n$ at $p = (1, 0, \cdots, 0)$. Obviously, now we have $t = 0$ so $\frac{dc'}{dt} = (-1, 0, \cdots, 0)$ is perpendicular to T_pS^n which can be spanned by

$$(0, 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1).$$

Finally, by a), we have the velocity field along great circles is parallel. □

Exercise 2.5. In Euclidean space, the parallel transport of a vector between two points does not depend on the curve jointing the two points. Show, by example, that this fact may not be true on an arbitrary Riemannian manifold.

Solution.



□

Exercise 2.6. Let M be a Riemannian manifold and let p be a point of M. Consider a constant curve $f\colon I \to M$ given by $f(t) = p$, for all $t \in I$. Let V be a vector field along f (that is, V is a differentiable mapping of I into T_pM). Show that $\frac{DV}{dt} = \frac{dV}{dt}$, that is to say, the covariant derivative coincides with the usual derivative of $V\colon I \to T_pM$.

Proof. We have the local notion of covariant derivative:

$$\frac{DV}{dt} = \sum_k \left(\frac{dv^k}{dt} + \sum_{i,j} v^j \frac{dx^i}{dt} \Gamma^k_{ij} \right) X_k,$$

where we can write $X_k = \frac{\partial}{\partial x^k}$. For the constant curve f, x_i are constant so $\frac{dx^i}{dt} = 0$. By definition, then we have

$$\frac{DV}{dt} = \sum_k \left(\frac{dv^k}{dt} + \sum_{i,j} v^j \frac{dx^i}{dt} \Gamma_{ij}^k \right) \frac{\partial}{\partial x^k} = \sum_k \frac{dv^k}{dt} \frac{\partial}{\partial x^k} = \frac{dV}{dt},$$

as desired. □

Exercise 2.7. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere, c an arbitrary parallel of latitude on S^2 and V_0 a tangent vector to S^2 at a point of c. Describe geometrically the parallel transport of V_0 along c.

Solution. See the colored curves in Exercise 2.5. □

Exercise 2.8. Consider the upper half-plane

$$\mathbb{R}^2_+ = \left\{ (x, y) \in \mathbb{R}^2 \mid y > 0 \right\}$$

with the metric given by $g_{11} = g_{22} = \frac{1}{y^2}$, $g_{12} = 0$.

a) Show that the Christoffel symbols of the Riemannian connection are: $\Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{22}^1 = 0$, $\Gamma_{11}^2 = \frac{1}{y}$, $\Gamma_{12}^1 = \Gamma_{22}^2 = -\frac{1}{y}$.

b) Let $v_0 = (0, 1)$ be a tangent vector at point $(0, 1)$ of $\mathbb{R}^2_+$. Let $v(t)$ be the parallel transport of v_0 along the curve $x = t$, $y = 1$. Show that $v(t)$ makes an angle t with the direction of the y-axis, measured in the clockwise sense.

Proof. It's easy to see that

$$\begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{bmatrix}^{-1} = \begin{bmatrix} y^2 & 0 \\ 0 & y^2 \end{bmatrix}.$$

By the classical expression for the Christoffel symbols of the Riemannian connection:

$$\Gamma_{ij}^m = \frac{1}{2} \sum_k \left(\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij} \right) g^{km},$$

we can calculate, for example,

$$\begin{aligned}
\Gamma_{12}^1 &= \frac{1}{2} \sum_{k=1}^{2} \left(\frac{\partial}{\partial x_1} g_{2k} + \frac{\partial}{\partial x_2} g_{k1} - \frac{\partial}{\partial x_k} g_{12} \right) g^{k1} \\
&= \frac{1}{2} \left[\left(\frac{\partial}{\partial x_1} g_{21} + \frac{\partial}{\partial x_2} g_{11} - \frac{\partial}{\partial x_1} g_{12} \right) g^{11} + \left(\frac{\partial}{\partial x_1} g_{22} + \frac{\partial}{\partial x_2} g_{21} - \frac{\partial}{\partial x_2} g_{12} \right) g^{21} \right] \\
&= \frac{1}{2} \left[\left(\frac{\partial}{\partial x} 0 + \frac{\partial}{\partial y} \frac{1}{y^2} - \frac{\partial}{\partial x} 0 \right) y^2 + \left(\frac{\partial}{\partial x} \frac{1}{y^2} + \frac{\partial}{\partial y} 0 - \frac{\partial}{\partial y} 0 \right) 0 \right] = -\frac{1}{y}.
\end{aligned}$$

The others are similar, whence a).

For b), we suppose that $v(t) = v^1(t) \frac{\partial}{\partial x} + v^2(t) \frac{\partial}{\partial y}$ and $c = (x_1(t), x_2(t)) = (t, 1)$. We use the local notion of covariant derivative

$$\frac{DV}{dt} = \sum_k \left(\frac{dv^k}{dt} + \sum_{i,j} v^j \frac{dx^i}{dt} \Gamma_{ij}^k \right) X_k$$

to calculate

$$\begin{aligned}
0 = \frac{Dv}{dt} &= \left(\frac{dv^1}{dt} + v^2 \frac{dx^1}{dt} \left(-\frac{1}{y} \right) \right) \frac{\partial}{\partial x} + \left(\frac{dv^2}{dt} + v^1 \frac{dx^1}{dt} \frac{1}{y} + v^2 \frac{dx^2}{dt} \left(-\frac{1}{y} \right) \right) \frac{\partial}{\partial y} \\
&= \left(\frac{dv^1}{dt} - v^2 \right) \frac{\partial}{\partial x} + \left(\frac{dv^2}{dt} + v^1 \right) \frac{\partial}{\partial y},
\end{aligned}$$

where we have $y = x_2 = 1$ on c. So we have a system

$$\begin{cases} \frac{dv^1}{dt} - v^2 = 0, \\ \frac{dv^2}{dt} + v^1 = 0, \end{cases}$$

with the initial value $v(0) = v_0 = (0, 1)$. Then the solution is

$$\begin{cases} v^1(t) = -\cos t, \\ v^2(t) = \sin t, \end{cases}$$

which shows that $v(t) = -\cos t \frac{\partial}{\partial x} + \sin t \frac{\partial}{\partial y}$ is as desired. □

Exercise 2.9 (Pseudo-Riemannian metrics)**.** A *pseudo-Riemannian* metric on a smooth manifold M is a choice, at every point $p \in M$, of a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$ on T_pM which varies differentiably with p. Except for the fact that $\langle\ ,\ \rangle$ need not be positive definite, all of the definitions about Riemannian metrics make sense for a pseudo-Riemannian metric.

a) Show that the theorem of Levi-Civita extends to pseudo-Riemannian metrics: Given a pseudo-Riemannian manifold M, there exists a unique affine connection ∇ on M satisfying the conditions:

   1) ∇ is symmetric.
   2) ∇ is compatible with the pseudo-Riemannian metric.

   The connection so obtained is called the *pseudo-Riemannian connection.*

b) Introduce a pseudo-Riemannian metric on $\mathbb{R}^{n+1}$ by using the quadratic form:

$$Q(x_0, \cdots, x_n) = -x_0^2 + x_1^2 + \cdots + x_n^2, \quad (x_0, \cdots, x_n) \in \mathbb{R}^{n+1}.$$

   Show that the parallel transport corresponding to the Levi-Civita connection of this metric coincides with the usual parallel trnasport of $\mathbb{R}^{n+1}$ (this pseudo-Riemannian metric is called the *Lorentz metric*; for $n = 3$, it appears naturally in relativity).

Proof. The proof of a) is exactly the same, because it only relies on "non-degeneration".

For b), because $Q(x_0, \cdots, x_n) = -x_0^2 + x_1^2 + \cdots + x_n^2$, it's easy to see that

$$g_{ij} = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle = \begin{cases} -1, & i = j = 0, \\ 1, & 1 \le i = j \le n, \\ 0, & i \ne j \end{cases}$$

and then $g^{ij} = g_{ij}$. By the classical expression for the Christoffel symbols of the Riemannian connection:

$$\Gamma_{ij}^m = \frac{1}{2} \sum_k \left(\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij} \right) g^{km},$$

we can calculate $\Gamma_{ij}^m = 0$ for any $0 \le i, j, m \le n$. Then for any parallel transport $V(t) = \sum_{k=0}^n v^k(t) \frac{\partial}{\partial x_k}$, by the local notion of covariant derivative

$$\frac{DV}{dt} = \sum_k \left(\frac{dv^k}{dt} + \sum_{i,j} v^j \frac{dx^i}{dt} \Gamma_{ij}^k \right) X_k,$$

we can calculate

$$0 = \frac{DV}{dt} = \sum_k \left(\frac{dv^k}{dt} + \sum_{i,j} v^j \frac{dx^i}{dt} \Gamma_{ij}^k \right) \frac{\partial}{\partial x_k} = \sum_k \frac{dv^k}{dt} \frac{\partial}{\partial x_k}.$$

So we must have $\frac{dv^k}{dt} = 0$ and then $v^k(t)$ are constants c^k, which means that the parallel transport $V(t) = \sum_{k=0}^n c^k \frac{\partial}{\partial x_k}$ is the usual parallel trnasport of $\mathbb{R}^{n+1}$. □

3 Geodesic; Convex Neighborhoods

Exercise 3.1 (Geodesics of a surface of revolution)**.** Denote by (u,v) the cartesian coordinates of $\mathbb{R}^2$. Show that the function $\varphi\colon U \subset \mathbb{R}^2 \to \mathbb{R}^3$ given by $\varphi(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$,

$$U = \left\{ (u,v) \in \mathbb{R}^2 \middle| \, u_0 < u < u_1; v_0 < v < v_1 \right\},$$

where f and g are differentiable functions, with $f'(v)^2 + g'(v)^2 \neq 0$ and $f(v) \neq 0$, is an immersion. The image $\varphi(U)$ is the surface generated by the rotation of the curve $(f(v), g(v))$ around the axis $0z$ and is called a *surface of revolution* S. The image by φ of the curves $u =$ constant and $v =$ constant are called *meridians* and *parallels*, respectively, of S.

a) Show that the induced metric in the coordinates (u,v) is given by

$$g_{11} = f^2, \qquad g_{12} = 0, \qquad g_{22} = (f')^2 + (g')^2.$$

b) Show that local equations of a geodesic γ are

$$\frac{d^2u}{dt^2} + \frac{2ff'}{f^2}\frac{du}{dt}\frac{dv}{dt} = 0,$$

$$\frac{d^2v}{dt^2} - \frac{ff'}{(f')^2 + (g')^2}\left(\frac{du}{dt}\right)^2 + \frac{f'f'' + g'g''}{(f')^2 + (g')^2}\left(\frac{dv}{dt}\right)^2 = 0.$$

c) Obtain the following geometric meaning of the equations above: the second equation is, except for meridians and parallels, equivalent to the fact that the "energy" $|\gamma'(t)|^2$ of a geodesic is constant along γ; the first equation signifies that if $\beta(t)$ is the oriented angle, $\beta(t) < \pi$, of γ with a parallel P intersecting γ at $\gamma(t)$, then

$$r\cos\beta = \text{constant},$$

where r is the radius of the parallel P (the equation above is called *Clairaut's relation*).

d) Use Clairaut's relation to show that a geodesic of the paraboloid

$$\big(f(v) = v, g(v) = v^2, 0 < v < \infty, -\varepsilon < u < 2\pi + \varepsilon\big),$$

which is not a meridian, intersects itself an infinite number of times.

Proof. Firstly, we calculate

$$d\varphi_{(u,v)} = \begin{bmatrix} -f(v)\sin u & f'(v)\cos u \\ f(v)\cos u & f'(v)\sin u \\ 0 & g'(v) \end{bmatrix}.$$

Because $f', g', f \neq 0$, we can make row reduction:

$$\begin{bmatrix} -f(v)\sin u & f'(v)\cos u \\ f(v)\cos u & f'(v)\sin u \\ 0 & g'(v) \end{bmatrix} \rightsquigarrow \begin{bmatrix} -f(v)\sin u & 0 \\ f(v)\cos u & 0 \\ 0 & g'(v) \end{bmatrix}.$$

It follows from $\sin u$ and $\cos u$ can not equal to 0 simultaneously that

$$\operatorname{rank}\begin{bmatrix} -f(v)\sin u & f'(v)\cos u \\ f(v)\cos u & f'(v)\sin u \\ 0 & g'(v) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} -f(v)\sin u & 0 \\ f(v)\cos u & 0 \\ 0 & g'(v) \end{bmatrix} = 2.$$

So φ is an immersion.

We choose the basis of $T\mathbb{R}^2$ $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$. Then we have

$$d\varphi_{(u,v)}\left(\frac{\partial}{\partial u}\right) = \begin{bmatrix} -f(v)\sin u & f'(v)\cos u \\ f(v)\cos u & f'(v)\sin u \\ 0 & g'(v) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -f(v)\sin u \\ f(v)\cos u \\ 0 \end{bmatrix},$$

$$d\varphi_{(u,v)}\left(\frac{\partial}{\partial v}\right) = \begin{bmatrix} -f(v)\sin u & f'(v)\cos u \\ f(v)\cos u & f'(v)\sin u \\ 0 & g'(v) \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} f'(v)\cos u \\ f'(v)\sin u \\ g'(v) \end{bmatrix}.$$

So we have

$$g_{11}(u,v) = \left\langle d\varphi_{(u,v)}\left(\frac{\partial}{\partial u}\right), d\varphi_{(u,v)}\left(\frac{\partial}{\partial u}\right)\right\rangle = \left\langle \begin{bmatrix} -f(v)\sin u \\ f(v)\cos u \\ 0 \end{bmatrix}, \begin{bmatrix} -f(v)\sin u \\ f(v)\cos u \\ 0 \end{bmatrix}\right\rangle = f^2(v).$$

Likewise, we have

$$g_{12}(u,v) = \left\langle d\varphi_{(u,v)}\left(\frac{\partial}{\partial u}\right), d\varphi_{(u,v)}\left(\frac{\partial}{\partial v}\right)\right\rangle = \left\langle \begin{bmatrix} -f(v)\sin u \\ f(v)\cos u \\ 0 \end{bmatrix}, \begin{bmatrix} f'(v)\cos u \\ f'(v)\sin u \\ g'(v) \end{bmatrix}\right\rangle = 0$$

and

$$g_{22}(u,v) = \left\langle d\varphi_{(u,v)}\left(\frac{\partial}{\partial v}\right), d\varphi_{(u,v)}\left(\frac{\partial}{\partial v}\right)\right\rangle = \left\langle \begin{bmatrix} f'(v)\cos u \\ f'(v)\sin u \\ g'(v) \end{bmatrix}, \begin{bmatrix} f'(v)\cos u \\ f'(v)\sin u \\ g'(v) \end{bmatrix}\right\rangle = \left(f'(v)\right)^2 + \left(g'(v)\right)^2,$$

Whence a).

We also have

$$\begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}^{-1} = \begin{bmatrix} f^2 & 0 \\ 0 & (f')^2 + (g')^2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{f^2} & 0 \\ 0 & \frac{1}{(f')^2+(g')^2} \end{bmatrix}.$$

From the classical expression for the Christoffel symbols of the Riemannian connection:

$$\Gamma_{ij}^m = \frac{1}{2}\sum_k \left(\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij}\right) g^{km},$$

we can calculate, for example,

$$\Gamma_{11}^1 = \frac{1}{2}\left[\left(\frac{\partial}{\partial u} g_{11} + \frac{\partial}{\partial u} g_{11} - \frac{\partial}{\partial u} g_{11}\right) g^{11} + \left(\frac{\partial}{\partial u} g_{12} + \frac{\partial}{\partial u} g_{21} - \frac{\partial}{\partial v} g_{11}\right) g^{21}\right] = 0.$$

Likewise, we have

$$\Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{21}^2 = \Gamma_{22}^1 = 0, \qquad \Gamma_{11}^2 = -\frac{ff'}{(f')^2 + (g')^2}, \qquad \Gamma_{12}^1 = \Gamma_{21}^1 = \frac{f'}{f}, \qquad \Gamma_{22}^2 = \frac{f'f'' + g'g''}{(f')^2 + (g')^2}.$$

Then the geodesic equations

$$\frac{d^2 x_k}{dt^2} + \sum_{i,j=1}^n \Gamma_{ij}^k \frac{dx_i}{dt}\frac{dx_j}{dt} = 0$$

will be

$$\frac{d^2 u}{dt^2} + 2\frac{f'}{f}\frac{du}{dt}\frac{dv}{dt} = 0, \qquad \text{for } k = 1$$

and

$$\frac{d^2 v}{dt^2} - \frac{ff'}{(f')^2 + (g')^2}\left(\frac{du}{dt}\right)^2 + \frac{f'f'' + g'g''}{(f')^2 + (g')^2}\left(\frac{dv}{dt}\right)^2 = 0, \qquad \text{for } k = 2.$$

When b).

For c), we suppose that $\gamma(t) = (u(t), v(t)) = (f(v(t))\cos(u(t)), f(v(t))\sin(u(t)), g(v(t)))$ is a geodesic. We have

$$\gamma'(t) = (f'v'\cos u - fu'\sin u, f'v'\sin u + fu'\cos u, g'v')$$

and then

$$
\begin{aligned}
|\gamma'(t)|^2 &= (f'v'\cos u - fu'\sin u)^2 + (f'v'\sin u + fu'\cos u)^2 + (g'v')^2 \\
&= (f')^2(v')^2 + f^2(u')^2 + (g')^2(v')^2 \\
&= \left((f')^2 + (g')^2\right)(v')^2 + f^2(u')^2.
\end{aligned}
$$

So we have

$$
\frac{d}{dt}|\gamma'(t)|^2 = 2(f'f'' + g'g'')(v')^3 + 2\left((f')^2 + (g')^2\right)v'v'' + 2ff'(u')^2v' + 2f^2u'u''.
$$

By the first geodesic equation, we have

$$
u'' = -2\frac{f'}{f}u'v'.
$$

Then we have

$$
\begin{aligned}
\frac{d}{dt}|\gamma'(t)|^2 &= 2(f'f'' + g'g'')(v')^3 + 2\left((f')^2 + (g')^2\right)v'v'' + 2ff'(u')^2v' + 2f^2u'u'' \\
&= 2(f'f'' + g'g'')(v')^3 + 2\left((f')^2 + (g')^2\right)v'v'' + 2ff'(u')^2v' + 2f^2u'\left(-2\frac{f'}{f}u'v'\right) \\
&= 2(f'f'' + g'g'')(v')^3 + 2\left((f')^2 + (g')^2\right)v'v'' - 2ff'(u')^2v' \\
&= 2v'\left((f')^2 + (g')^2\right)\left(v'' - \frac{ff'}{((f')^2 + (g')^2)}(u')^2 + \frac{f'f'' + g'g''}{((f')^2 + (g')^2)}(v')^2\right).
\end{aligned}
$$

which means that $\gamma(t)$ satisfies the second geodesic equation if and only if $\frac{d}{dt}|\gamma'(t)|^2 = 0$, if and only if the energy $|\gamma'(t)|^2$ is a constant.

The parallel P intersecting γ at $\gamma(t)$ will be

$$
P_t(s) = \left(f(v(t))\cos s, f(v(t))\sin s, g(v(t))\right).
$$

Then

$$
P_t'(s) = \left.\frac{dP_t}{ds}\right|_{s=u(t)} = \left(-f\sin u, f\cos u, 0\right).
$$

Because $\beta(t) < \pi$, we have

$$
\cos\beta = \frac{\langle \gamma'(t), P_t'(s)\rangle}{|\gamma'(t)|\cdot|P_t'(s)|} = \frac{-f\sin u\,(f'v'\cos - fu'\sin u) + f\cos u\,(f'v'\sin u + fu'\cos u)}{f\,|\gamma'(t)|} = \frac{fu'}{|\gamma'(t)|}.
$$

By the calculation above, for the geodesic γ, we have $|\gamma'(t)|$ is a constant, say c. Because $r = f$, we have

$$
\frac{d}{dt}(r\cos\beta) = \frac{d}{dt}\left(f\frac{fu'}{|\gamma'(t)|}\right) = \frac{d}{dt}\left(\frac{f^2u'}{c}\right) = \frac{1}{c}\left(2ff'v'u' + f^2u''\right) = \frac{1}{cf^2}\left(2\frac{f'}{f}v'u' + u''\right) = 0,
$$

where the last "=" comes from the first geodesic equation. So we must have $r\cos\beta = $ constant.

Finally, for d), I don't know how to solve the ODEs. See references:
https://math.stackexchange.com /questions/3632077
https://mathworld.wolfram.com/ParaboloidGeodesic.html. □

Exercise 3.2. It is possible to introduce a Riemannian metric in the tangent bundle TM of a Riemannian manifold M in the following manner. Let $(p, v) \in TM$ and V, W be tangent vectors in TM at (p, v). Choose curves in TM

$$
\alpha\colon t \mapsto (p(t), v(t)), \quad \beta\colon s \mapsto (q(s), w(s)),
$$

with $p(0) = q(0) = p$, $v(0) = w(0) = v$, and $V = \alpha'(0)$, $W = \beta'(0)$. Define an inner product on TM by

$$
\langle V, W\rangle_{(p,v)} = \langle d\pi(V), d\pi(W)\rangle_p + \left\langle \frac{Dv}{dt}(0), \frac{Dw}{ds}(0)\right\rangle_p,
$$

where $d\pi$ is the differential of $\pi\colon TM \to M$, $\pi((p, v) = p$.

a) Prove that this inner product is well-defined and introduces a Riemannian metric on TM.

b) A vector at $(p,v) \in TM$ that is orthogonal (for the metric above) to the fiber $\pi^{-1}(p) \approx T_pM$ is called a *horizontal vector.* A curve

$$t \mapsto (p(t), v(t))$$

in TM is *horizontal* if its tangent vector is horizontal for all t. Prove that the curve

$$t \mapsto (p(t), v(t))$$

is horizontal if and only if the vector field $v(t)$ is parallel along $p(t)$ in M.

c) Prove that the geodesic field is a horizontal vector field (i.e., it is horizontal at every point).

d) Prove that the trajectories of the geodesic field are geodesic on TM in the metric above.

e) A vector at $(p,v) \in TM$ is call *vertical* if it is tangent to the fiber $\pi^{-1}(p) \approx T_pM$. Show that:

$$\begin{aligned}\langle W, W\rangle_{(p,v)} &= \langle d\pi(W), d\pi(W)\rangle_p\,, \quad \text{ if } W \text{ is horizontal,}\\ \langle W, W\rangle_{(p,v)} &= \langle W, W\rangle_p\,, \quad \text{ if } W \text{ is vertical,}\end{aligned}$$

where we are identifying the tangent space to the fiber with T_pM.

Proof. Firstly, we need to check that the inner product is independent of the choices of α and β. By the definition of covariant derivative:

$$\frac{Dv}{dt} = \sum_{i=1}^{n} \frac{D\left(v^i X_i\right)}{dt} = \sum_{i=1}^{n}\left(\frac{dv^k}{dt}X_i + v^i\frac{DX_i}{dt}\right),$$

where X_i is the parallel transports of a orthonormal basis $X_i(0)$ of T_pM along $p(t)$, we have

$$\frac{Dv}{dt}(0) = \sum_{i=1}^{n}\left(\left.\frac{dv^k}{dt}\right|_{t=0} X_i(0)\right) = \sum_{i=1}^{n}\left((\alpha'(0))^{n+i}\right)X_i(0) = \sum_{i=1}^{n} V^{n+i}X_i(0),$$

which proves that the inner product is well defined. The symmetry and linearity is induced by the symmetry of the Riemannina metric on M and linearity of $d\pi$ and covariant derivative. We also have $\langle V, V\rangle_{(p,v)} = \langle d\pi(V), d\pi(V)\rangle_p + \left\langle \frac{Dv}{dt}(0), \frac{Dv}{ds}(0)\right\rangle_p \geq 0$ and $\langle V, V\rangle = 0$ if and only if $\langle d\pi(V), d\pi(V)\rangle_p = 0$ and $\left\langle \frac{Dv}{dt}(0), \frac{Dv}{ds}(0)\right\rangle_p = 0$. Notice that the differential of the projection $\pi\colon (p,v) \mapsto p$ will be

$$\begin{aligned}d\pi\colon T_{(p,v)}TM &\to T_pM\\ (v^1, \cdots, v^{2n}) &\mapsto (v_1, \cdots, v^n).\end{aligned}$$

So these equivalent to

$$\langle d\pi(V), d\pi(V)\rangle_p = \left\langle (V^1, \cdots, V^n), (V^1, \cdots, V^n)\right\rangle = 0$$

and

$$\left\langle \frac{Dv}{dt}(0), \frac{Dv}{ds}(0)\right\rangle_p = \left\langle (V^{n+1}, \cdots, V^{2n}), (V^{n+1}, \cdots, V^{2n})\right\rangle = 0,$$

i.e. $V = 0$. It follows that the inner product is positive-definite which makes it a Riemannian metric on TM; whence a).

For b), we firstly need to study how the tangent space of T_pM embeds in the tangent space of TM. Notice that the inclusion $i_p\colon T_pM \to TM$ will be

$$v \mapsto (p, v).$$

So the differential of i at any $v \in T_pM$ will be

$$\begin{aligned}di_p\colon T_vT_pM &\to T_{(p,v)}TM\\ w &\mapsto (0, w).\end{aligned}$$

It follows that for a horizontal curve $\gamma(t) = (p(t), v(t))$, for any $w \in T_{v(t)}T_{p(t)}M$, we must have

$$\langle \gamma'(t), (0, w)\rangle_{(p(t),v(t))} = 0.$$

This means

$$\begin{aligned}
0 = \langle \gamma'(t), (0, w)\rangle_{(p(t),v(t))} &= \langle d\pi(\gamma'(t)), d\pi((0, w))\rangle_{p(t)} + \left\langle \frac{Dv}{dt}(t), \frac{D\overline{w}}{dt}(0)\right\rangle_{p(t)} \\
&= \langle p'(t), 0\rangle_{p(t)} + \left\langle \frac{Dv}{dt}(t), \frac{D\overline{w}}{dt}(0)\right\rangle_{p(t)} = \left\langle \frac{Dv}{dt}(t), \frac{D\overline{w}}{dt}(0)\right\rangle_{p(t)},
\end{aligned}$$

where $\overline{w}(t)$ is a curve which is the parallel transport of w along the constant curve p, such that $(0, w) = (p', \overline{w}')$. So we have $\frac{D\overline{w}}{dt}(0) = w$. Then the formula above tell us that for any $w \in T_{v(t)}T_{p(t)}M \cong \mathbb{R}^n$,

$$\left\langle \frac{Dv}{dt}(t), w\right\rangle_{p(t)} = \left\langle \frac{Dv}{dt}(t), \frac{D\overline{w}}{dt}(0)\right\rangle_{p(t)} = 0,$$

which can happens if and only if $\frac{Dv}{dt} = 0$, i.e. the vector field $v(t)$ is parallel along $p(t)$.

Then for c), we suppose that G is the geodesic field and for any $(p, v) \in TM$, $\widetilde{\gamma}(t) = (\gamma(t), \gamma'(t))$ is the trajectory such that $G(p, v) = \widetilde{\gamma}'(0) \in T_{(p,v)}TM$, where $\gamma(t)$ is a geodesic of M. By definition, we have $\frac{D\gamma'}{dt} = 0$. So the curve $\widetilde{\gamma}(t) = (\gamma(t), \gamma'(t))$ is horizontal by b). Then the tangent vector of $\widetilde{\gamma}$, i.e. $\widetilde{\gamma}'(0) = G(p, v)$ is horizontal, also by definition. So G is a horizontal vector field.

With the same notations above, for any curve $\widetilde{\alpha}(t) = (\alpha(t), v(t))$ on TM, we have

$$\begin{aligned}
\ell(\widetilde{\alpha}) &= \int_a^b |\widetilde{\alpha}'(t)| \,\mathrm{d}t = \int_a^b \sqrt{\langle \widetilde{\alpha}'(t), \widetilde{\alpha}'(t)\rangle}\,\mathrm{d}t \\
&= \int_a^b \sqrt{\langle d\pi\left(\widetilde{\alpha}'(t)\right), d\pi\left(\widetilde{\alpha}'(t)\right)\rangle + \left\langle \frac{Dv}{dt}(t), \frac{Dv}{dt}(t)\right\rangle}\,\mathrm{d}t \\
&\geq \int_a^b \sqrt{\langle d\pi\left(\widetilde{\alpha}'(t)\right), d\pi\left(\widetilde{\alpha}'(t)\right)\rangle}\,\mathrm{d}t = \int_a^b \sqrt{\langle \alpha'(t), \alpha'(t)\rangle}\,\mathrm{d}t = \int_a^b |\alpha'(t)| \,\mathrm{d}t = \ell(\alpha),
\end{aligned}$$

and the inequality is verified if and only if v is parallel along α. For any $(p, v) \in TM$, we can choose a convex neighborhoods $(p, v) \in W \subset TM$ and let $\pi(W) = V \in p$. Then if $\widetilde{\gamma}$ is not a geodesic, there are two points $Q_1 = (q_1, v_1)$ and $Q_2 = (q_2, v_2)$ in $\widetilde{\gamma} \cap W$, such that there exists a curve $\widetilde{\alpha}(t) = (\alpha(t), v(t))$ in W from Q_1 to Q_2 such that $\ell(\widetilde{\alpha}) < \ell(\widetilde{\gamma})$. It follows that

$$\ell(\alpha) \leq \ell(\widetilde{\alpha}) < \ell(\widetilde{\gamma}) = \ell(\gamma),$$

where the last equality is because $\gamma(t)$ is a geodesic of M and then $\gamma'(t)$ is parallel along $\gamma(t)$, which is a contradiction to the fact that $\gamma(t)$ is a geodesic of M who has the minimizing property.

Finally, by the proof of b), for any horizontal vector $W = \widetilde{\gamma}'(0) = (p'(0), w'(0))$ at (p, v), we must have $\frac{Dw}{dt}(0) = 0$. So we have

$$\langle W, W\rangle_{(p,v)} = \langle d\pi(W), d\pi(W)\rangle_p + \left\langle \frac{Dw}{dt}(0), \frac{Dw}{dt}(0)\right\rangle_p = \langle d\pi(W), d\pi(W)\rangle_p.$$

Let $W = \widetilde{\gamma}'(0) = (p'(0), w'(0))$ be a vertical vector at (p, v). Because the embedding of T_vT_pM into $T_{(p,v)}TM$ is $v \mapsto (0, v)$, we claim that any vector in $(T_vT_pM)^{\perp}$ will be like $(v, 0)$ where $v \in \mathbb{R}^n$ by some linear algebra. Then for any vector $V = (\alpha'(0), v'(0)) = (\alpha'(0), 0) \in (T_vT_pM)^{\perp}$, we have

$$\frac{Dv}{dt}(0) = \sum_{i=1}^n V^{n+i} X_i(0) = 0$$

by the calculation at the beginning of the proof. Then we have

$$\begin{aligned}
0 = \langle W, V\rangle_{(p,v)} &= \langle d\pi(W), d\pi(V)\rangle_p + \left\langle \frac{Dw}{dt}(0), \frac{Dv}{dt}(0)\right\rangle_p \\
&= \langle p'(0), \alpha'(0)\rangle_p + \left\langle \frac{Dw}{dt}(0), 0\right\rangle_p = \langle p'(0), \alpha'(0)\rangle_p.
\end{aligned}$$

This is equivalent to $p'(0) = 0$, i.e. $W = (0, w'(0)) \in T_pM \subset T_{(p,v)}TM$. So we have

$$\begin{aligned}
\langle W, W \rangle_{(p,v)} &= \langle d\pi(W), d\pi(W) \rangle_p + \left\langle \frac{Dw}{dt}(0), \frac{Dw}{dt}(0) \right\rangle_p \\
&= \langle 0, 0 \rangle_p + \left\langle \sum_{i=1}^n W^{n+i} X_i(0), \sum_{i=1}^n W^{n+i} X_i(0) \right\rangle_p = \langle W, W \rangle_p .
\end{aligned}$$

□

Exercise 3.3. Let G be a Lie group, $\mathcal{G}$ its Lie algebra and let $X \in \mathcal{G}$. The trajectories of X determine a mapping $\varphi\colon (-\varepsilon, \varepsilon) \to G$ with $\varphi(0) = e$, $\varphi'(t) = X(\varphi(t))$.

a) Prove that $\varphi(t)$ is defined for all $t \in \mathbb{R}$ and that $\varphi(t+s) = \varphi(t) \cdot \varphi(s)$, ($\varphi\colon \mathbb{R} \to G$ is then called a *1-parameter subgroup* of G).

b) Prove that if G has a bi-invariant metric $\langle\ ,\ \rangle$ then the geodesics of G that start from e are 1-parameter subgroups of G.

Proof. For $t_0 \in (-\varepsilon, \varepsilon)$, we denote $g = \varphi(t_0)$ and define

$$\widetilde{\varphi}(t) = g^{-1} \cdot \varphi(t + t_0) = L_{g^{-1}}(\varphi(t + t_0)), \qquad t \in (-\varepsilon - t_0, \varepsilon - t_0).$$

Then we have $\widetilde{\varphi}(0) = e$ and

$$\widetilde{\varphi}'(0) = dL_{g^{-1}}(\varphi'(t_0)) = dL_{g^{-1}}\left(X(\varphi(t_0))\right) = dL_{g^{-1}} dL_{\varphi(t_0)}(X) = dL_{g^{-1}\varphi(t_0)}(X) = dL_e(X) = X.$$

By the uniqueness of trajectory, we must have

$$\widetilde{\varphi}(t) = g^{-1} \cdot \varphi(t + t_0) = \varphi(t_0)^{-1} \cdot \varphi(t + t_0) = \varphi(t), \qquad t \in (-\varepsilon - t_0, \varepsilon - t_0).$$

So we extend the domain of definition of φ to $(-\varepsilon - t_0, \varepsilon)$. Similarly, we can define $\varphi(t)$ of all $t \in \mathbb{R}$. Meanwhile, we also have

$$\varphi(t_0 + t) = \varphi(t + t_0) = \varphi(t_0) \cdot \varphi(t).$$

So the 1-parameter subgroup of G is well defined, whence a).

For b), firstly, for any $X, Y \in \mathcal{G}$ which can be seen as left-invariant vector fields, by the equation (9) of Chapter 2: for any vector fields X, Y, Z,

$$\langle X, \nabla_Z Y \rangle = \frac{1}{2} \left(Z \langle X, Y \rangle + Y \langle X, Z \rangle - X \langle Y, Z \rangle + \langle Z, [X, Y] \rangle + \langle Y, [X, Z] \rangle - \langle X, [Y, Z] \rangle \right),$$

we have

$$\langle X, \nabla_Y Y \rangle = \frac{1}{2} \left(Y \langle X, Y \rangle + Y \langle X, Y \rangle - X \langle Y, Y \rangle + \langle Y, [X, Y] \rangle + \langle Y, [X, Y] \rangle - \langle X, [Y, Y] \rangle \right).$$

In this formula, because $\langle\ ,\ \rangle$ is bi-invariant and X, Y are left-invariant, as a function on G ,we have

$$\langle X, Y \rangle_g = \langle dL_g(X), dL_g(Y) \rangle_g = \langle X, Y \rangle_e = \text{constant}.$$

Then we have $Y[X, Y] = 0$ and $X \langle Y, Y \rangle = 0$ similarly. And the Lie bracket of two same vector fields will vanish:

$$[Y, Y] = YY - YY = 0,$$

so we have

$$\langle X, \nabla_Y Y \rangle = \frac{1}{2} \left(\langle Y, [X, Y] \rangle + \langle Y, [X, Y] \rangle \right) = \langle Y, [X, Y] \rangle .$$

from the equation above. Then by the equation (3) of Chapter 1:

$$\langle [U, X], V \rangle = - \langle U, [V, X] \rangle, \qquad \forall U, V, X \in \mathcal{G},$$

we have

$$\langle X, \nabla_Y Y \rangle = \langle Y, [X, Y] \rangle = - \langle [Y, Y], X \rangle = 0.$$

It follows that $\nabla_Y Y = 0$. Then for any geodesic γ of G that start from e, we denote $\gamma'(0) = X$ and φ the trajectory of X which we call a 1-parameter subgroup of G. So we have

$$\frac{D}{dt}\left(\frac{d\varphi}{dt}\right) = \frac{D\varphi'}{dt} = \nabla_{\varphi'} X = \nabla_X X = 0.$$

Thus the 1-parameter subgroups are geodesics. But by the uniqueness of geodesic, we must have $\gamma = \varphi$. So the geodesic $\gamma = \varphi$ is a 1-parameter subgroup of G. $\square$

Exercise 3.4. A subset A of a differentiable manifold M is *contractible* to a point $a \in A$ when the mapping Id_A and $k_a \colon x \in A \mapsto a \in A$ are homotopic (with base point a). A is *contractible* if it is contractible to one of its points.

1. Show that a convex neighborhood in a Riemannian manifold M is a contractible subset (with respect to any of its points).
2. Let M be a differentiable manifold. Show that there exists a covering $\{U_\alpha\}$ of M with the following properties:
   i) U_α is open and contractible, for each α.
   ii) If $U_{\alpha_1}, \cdots, U_{\alpha_r}$ are elemants of the covering, then $\bigcap_{i=1}^r U_{\alpha_i}$ is contractible.

Proof. For a concex neighborhood $A \subset M$, we fix $\forall a \in A$. We construct a homotopy from Id_A to k_a as follows: for any $b \in A$, because A is strongly convex, we can choose a geodesic $\gamma_b \subset A$ from a to b. And we define

$$f_s(t) = f(s,t) = \gamma_b(st).$$

It is a homotopy from $f_1 = \gamma_b$ to $f_0 = \gamma_b(0) = k_a$. By the definition of convex neighborhood, there is unique geodesic $\gamma_b \subset A$ from a to b. Then for any two different points b, c, either $\gamma_b \cap \gamma_c = \{a\}$ or one is contained in another. So we can well define a diffeomorphism for any $s \in [0,1]$:

$$\begin{aligned} F_s = F(s,\ ) \colon A &\to A \\ b &\mapsto \gamma_b(s). \end{aligned}$$

It is easy to check that $F_0 = k_a$ and $F_1 = \mathrm{Id}_A$. So F_s is a homotopy from Id_A to k_a, and then A is contractible, whence a).

For b), for any point $a \in M$ we can choose an open convex neighborhood U_a. Then $\{U_a\}_{a\in M}$ is a covering of M where U_a is open and contractible, by 1. Then for any two U_a and U_b such that $U_a \cap U_b \neq \varnothing$, for any $c, d \in U_a \cap U_b$, because U_a and U_b are strongly convex, the geodesic γ_{cd} connecting c and d must be contained in U_a and U_b, then $\gamma_{cd} \subset U_a \cap U_b$. It follows that $U_a \cap U_b$ is strongly convex. Then $\bigcap_{i=1}^r U_{a_i}$ is strongly convex by induction and then contractible by 1, as desired. $\square$

Exercise 3.5. Let M be a Riemannian manifold and X be a vector field on M. Let $p \in M$ and let $U \subset M$ be a neighborhood of p. Let $\varphi \colon (-\varepsilon, \varepsilon) \times U \to M$ be a differentiable mapping such that for any $q \in U$ the curve $t \mapsto \varphi(t, q)$ is a trajectory of X passing through q at $t = 0$ (U and φ are given by the fundamental theorem for ordinary differential equations). X is called a *Killing field* (or an *infinitesimal isometry*) if, for each $t_0 \in (-\varepsilon, \varepsilon)$, the mapping $\varphi(t_0,\ ) \colon U \subset M \to M$ is an isometry. Prove that:

a) A vector field v on $\mathbb{R}^n$ may be seen as a map $v \colon \mathbb{R}^n \to \mathbb{R}^n$; we say that the field is linear if v is a linear map. A linear field on $\mathbb{R}^n$, defined by a matrix A, is a Killing field if and only if A is anti-symmetric.

b) Let X be a Killing field on M, $p \in M$, and let U be a normal neighborhood of p on M. Assume that p is a unique point of U that satisfies $X(p) = 0$. Then, in U, X is tangent to the geodesic spheres centered at p.

c) Let X be a differentiable vector field on M and let $f \colon M \to N$ be an isometry. Let Y be a vector field on N defined by $Y(f(p)) = df_p(X(p))$, $p \in M$. Then Y is a Killing field if and only if X is also a Killing vector field.

d) X is Killing $\Leftrightarrow \langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle = 0$ for all vector fields Y, Z on M (the equation above is called the *Killing equation*).

e) Let X be a Killing field on M with $X(q) \neq 0$, $q \in M$. Then there exists a system of coordinates $(x_1, \cdots, x_n)$ in a neighborhood of q, so that the coefficients g_{ij} of the metric in this system coordinates do not depend on x_n.

Proof. For the special case a), for any $x \in \mathbb{R}^n$, we have

$$\frac{d\varphi}{dt}(t,x) = v(\varphi(t,x)) = A\varphi(t,x).$$

Then we have $\varphi(t,x) = \mathrm{e}^{tA}\,\varphi(0,x) = \mathrm{e}^{tA}\,x$ by solving the ODEs for the initial value $\varphi(0,x) = x$. Then for any $t_0 \in (-\varepsilon, \varepsilon)$, $\forall p \in U$ and $\forall y \in T_pU$, we have and denote

$$d\varphi(t_0,p)(y) = d\varphi(t_0,\ )_p(y) = e^{t_0 A} y.$$

By definition, X is Killing if and only if $d\varphi(t_0,\ )$ are isometries for all $t_0 \in (-\varepsilon, \varepsilon)$, if and only if $\forall p \in U$ and $\forall x, y \in T_pU$,

$$\begin{aligned} \langle x, y\rangle_p &= \langle d\varphi(t_0,p)(x), d\varphi(t_0,p)(y)\rangle_{\varphi(t_0,p)} = \left\langle e^{t_0A}x, e^{t_0A}y\right\rangle_{\mathbb{R}^n} \\ &= \left\langle x, \left(e^{t_0A}\right)^T e^{t_0A}y\right\rangle_{\mathbb{R}^n} = \left\langle x, \mathrm{e}^{t_0A^T}\,\mathrm{e}^{t_0A}\,y\right\rangle_{\mathbb{R}^n} = \left\langle x, \mathrm{e}^{t_0(A^T+A)}\,y\right\rangle_{\mathbb{R}^n}. \end{aligned}$$

This can happens if and only if $\mathrm{e}^{t_0(A^T+A)}$ by the positive define of Riemannian metric. This means $A^T + A = 0$, i.e. A is anti-symmetric.

For b), $\forall q \in U$, let $q \in S_\delta(p)$. We will prove $\varphi(t,q) \subset S_\delta(p)$. We denote the unique geodesic $\gamma \subset U$ from p to q, then we have $\ell(\gamma) = \delta$. For any t_0 near 0, we claim that $\varphi(t_0, \gamma(t))$ is also a geodesic: $X(p) = 0$, so $\varphi(t_0,p) \equiv p$. If not, we denote $\widetilde{\gamma} \subset U$ the geodesic from $\varphi(t_0,\gamma(0)) = \varphi(t_0,p) = p$ to $\varphi(t_0,\gamma(1)) = \varphi(t_0,q)$. Because $\varphi(t_0,\ )$ is an isometry, it is a diffeomorphism. Then there is a curve $\alpha \subset U$ from p to q such that $\varphi(t_0,\alpha(t)) = \widetilde{\gamma}(t)$. Then we have

$$\begin{aligned} \ell(\alpha) &= \int_0^1 \langle \alpha'(t), \alpha'(t)\rangle^{\frac{1}{2}}\ \mathrm{d}t = \int_0^1 \langle d\varphi(t_0,\alpha(t))\alpha'(t), d\varphi(t_0,\alpha(t))\alpha'(t)\rangle^{\frac{1}{2}}\ \mathrm{d}t = \int_0^1 \langle \widetilde{\gamma}'(t), \widetilde{\gamma}'(t)\rangle^{\frac{1}{2}}\ \mathrm{d}t \\ &= \ell(\widetilde{\gamma}) < \ell(\varphi(t_0,\gamma(t))) = \int_0^1 \langle d\varphi(t_0,\gamma(t))\gamma'(t), d\varphi(t_0,\gamma(t))\gamma'(t)\rangle^{\frac{1}{2}}\ \mathrm{d}t = \int_0^1 \langle \gamma'(t), \gamma'(t)\rangle^{\frac{1}{2}}\ \mathrm{d}t = \ell(\gamma), \end{aligned}$$

which is contradict to the fact that γ is a geodesic from p to q with the minimal property. So we prove the claim. (In fact, it's true for any local isometry. We will often use this claim later.) From the calculation above, we also know that $\ell(\varphi(t_0,\gamma(t))) = \ell(\gamma) = \delta$. So $\varphi(t_0,\gamma(t)) = \varphi(t_0,q) \in S_\delta(p)$, for all $t_0 \in (-\varepsilon,\varepsilon)$. This means the trajectory $\varphi(t,q) \subset S_\delta(p)$. It follows that, in U, $X(\varphi(t,q)) = \frac{d\varphi}{dt}(t,q) \in TS_\delta(p)$, i.e. X is tangent to the geodesic spheres centered at p.

For c), let $\varphi(t,p)$ be the trajectories of X. Because f is an isometry and then a diffeomorphism, we can denote $\widetilde{\varphi}(t, f(p)) = f(\varphi(t,p))$ be curves on N. Then we have

$$\frac{d\widetilde{\varphi}}{dt}(t,f(p)) = df_{\varphi(t,p)}\frac{d\varphi}{dt}(t,p) = df_{\varphi(t,p)}(X_{\varphi(t,p)}) = Y(f(\varphi(t,p))) = Y(\widetilde{\varphi}(t,f(p))).$$

By the uniqueness, $\widetilde{\varphi}(t,f(p))$ are trajectories of Y. Then Y is a Killing field, if and only if $\forall t_0 \in (-\varepsilon,\varepsilon)$, $\widetilde{\varphi}(t_0,\ )$ is an isometry, i.e. $\forall f(p) \in N$, $\forall v, w \in T_{f(p)}N$,

$$\langle d\widetilde{\varphi}(t_0,f(p))(v), d\widetilde{\varphi}(t_0,f(p))(w)\rangle_{\widetilde{\varphi}(t,f(p))} = \langle v, w\rangle_{f(p)}.$$

Because f is an isometry,

$$\begin{aligned} \langle v, w\rangle_p = \langle v, w\rangle_{\mathbb{R}^n} = \langle v, w\rangle_{f(p)} &= \langle d\widetilde{\varphi}(t_0,f(p))(v), d\widetilde{\varphi}(t_0,f(p))(w)\rangle_{\widetilde{\varphi}(t,f(p))} \\ &= \left\langle df_{\varphi(t_0,p)}d\varphi(t_0,p)(v), df_{\varphi(t_0,p)}d\varphi(t_0,p)(w)\right\rangle_{\widetilde{\varphi}(t,f(p))} \\ &= \langle d\varphi(t_0,p)(v), d\varphi(t_0,p)(w)\rangle_{\varphi(t,p)}. \end{aligned}$$

This happens if and only if $\varphi(t_0,\ )$ is an isometry for any $t_0 \in (-\varepsilon,\varepsilon)$, i.e. X is a Killing field.

For d), by continuity, it suffices to prove it for points $q \in U$ such that $X(q) \neq 0$. By b), we can choose a point $p \in U$, such that $X(p) = 0$, and a geodesic shpere, say $S_\delta(p)$ such that $q \in S_\delta(p)$ and $X(q) \in T_qS_\delta(p)$.

Because $S_\delta(p) \subset U$ is a submanifold with codimension 1, we can choose coordinates $(x_1, \cdots, x_{n-1})$ near $q \in S$ such that $(x_1, \cdots, x_{n-1}, t)$ are coordinates near $q \in U$, where $t \in (-\varepsilon, \varepsilon)$ is the parameter of the trajectory $\varphi(t, q)$. Then $X(\varphi(t,q)) = \frac{d\varphi}{dt}(t,q)$, i.e. $X = \frac{d}{dt}$. We denote $x_n = t$ and $X_i = \frac{d}{dx_i}$ for $i = 1, \cdots, n$. Then we have

$$[X_i, X_j] = \frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial^2}{\partial x_j \partial x_i} = 0.$$

By linearity, we only need to prove the proposition for $Y, Z \in \{X_1, \cdots, X_n\}$. By the equation (9) of Chapter 2:

$$\langle X, \nabla_Z Y \rangle = \frac{1}{2} \left(Z \langle X, Y \rangle + Y \langle X, Z \rangle - X \langle Y, Z \rangle + \langle Z, [X,Y] \rangle + \langle Y, [X,Z] \rangle - \langle X, [Y,Z] \rangle \right),$$

we have

$$\begin{aligned}
&\left\langle \nabla_{X_j} X, X_i \right\rangle + \left\langle \nabla_{X_i} X, X_j \right\rangle \\
=& \frac{1}{2} \left(X_j \langle X_i, X \rangle + X \langle X_i, X_j \rangle - X_i \langle X, X_j \rangle + \langle X_j, [X_i, X] \rangle + \langle X, [X_i, X_j] \rangle - \langle X_i, [X, X_j] \rangle \right. \\
&\left. + X_i \langle X_j, X \rangle + X \langle X_j, X_i \rangle - X_j \langle X, X_i \rangle + \langle X_i, [X_j, X] \rangle + \langle X, [X_j, X_i] \rangle - \langle X_j, [X, X_i] \rangle \right) \\
=& X \langle X_i, X_j \rangle = \frac{d}{dt} \langle X_i, X_j \rangle .
\end{aligned}$$

Notice that for any smooth function f on M, by the chain rule,

$$\begin{aligned}
d\varphi(t, \ )_q(X_i(q))(f) &= X_i(q)(f(\varphi(t, \ ))) = \left. \frac{d}{dx_i} \right|_p (f(\varphi(t, \ ))) \\
&= \sum_{j=1}^n \left. \frac{df}{d\varphi(t, \ )_j} \right|_{\varphi(t,q)} \cdot \left. \frac{d\varphi(t, \ )_j}{dx_i} \right|_q = \left. \frac{df}{dx_i} \right|_{\varphi(t,q)} = X_i(\varphi(t,q))(f).
\end{aligned}$$

Then we have

$$\langle X_i(\varphi(t,q)), X_j(\varphi(t,q)) \rangle_{\varphi(t,q)} = \langle d\varphi(t, \ )_q(X_i(q)), d\varphi(t, \ )_q(X_i(q)) \rangle_{\varphi(t,q)} .$$

So if X is a Killing field, we have $\varphi(t, \ )$ are isometries for all $t \in (-\varepsilon, \varepsilon)$. Then

$$\langle X_i(\varphi(t,q)), X_j(\varphi(t,q)) \rangle_{\varphi(t,q)} = \langle d\varphi(t, \ )_q(X_i(q)), d\varphi(t, \ )_q(X_i(q)) \rangle_{\varphi(t,q)} = \langle X_i(q), X_j(q) \rangle_q$$

is independent of t. So we conclude

$$\left\langle \nabla_{X_j} X, X_i \right\rangle + \left\langle \nabla_{X_i} X, X_j \right\rangle = \frac{d}{dt} \langle X_i(\varphi(t,q)), X_j(\varphi(t,q)) \rangle_{\varphi(t,q)} = \frac{d}{dt} \langle X_i(q), X_j(q) \rangle_q = 0.$$

For another direction, if

$$\begin{aligned}
0 &= \left\langle \nabla_{X_j} X, X_i \right\rangle + \left\langle \nabla_{X_i} X, X_j \right\rangle \\
&= \frac{d}{dt} \langle X_i(\varphi(t,q)), X_j(\varphi(t,q)) \rangle_{\varphi(t,q)} \\
&= \frac{d}{dt} \langle d\varphi(t, \ )_q(X_i(q)), d\varphi(t, \ )_q(X_i(q)) \rangle_{\varphi(t,q)} ,
\end{aligned}$$

by continuity, we must have the fact that $\varphi(t, \ )$ are isometries for all $t \in (-\varepsilon, \varepsilon)$ on $\{X_1, \cdots, X_n\}$. By linearity, $\varphi(t, \ )$ are isometries on $\mathrm{Span}\{X_1, \cdots, X_n\} = T_q M$, i.e. X is a Killing field. (See reference: https://math.stackexchange.com/questions/2504446.)

As the calculation above, we also know that if X is a Killing field, near q, for all $i, j = 1, \cdots, n$,

$$\frac{d}{dx_n} g_{ij} = \frac{d}{dt} \langle X_i, X_j \rangle = \left\langle \nabla_{X_j} X, X_i \right\rangle + \left\langle \nabla_{X_i} X, X_j \right\rangle = 0.$$

Then coefficients g_{ij} do not depend on $x_n = t$, as desired; whence e).

I think $\varphi_{t_0}(\ )$ is a better notation than $\varphi(t_0, \ )$. I will use it in the following exercises. $\square$

Exercise 3.6. Let X be a Killing field on a connected Rieamnnain manifold M. Assume that there exists a point $q \in M$ such that $X(q) = 0$ and $\nabla_Y X(q) = 0$, for all $Y(q) \in T_q M$. Prove that $X \equiv 0$.

Proof. We claim that $(d\varphi_t)_q = \mathrm{Id}$ for all t: Notice that because $X(q) = 0$, we have $\nabla_X Y(q) = 0$ for all vector field Y. Then by the symmetry of Riemannina connection,

$$[X, Y](q) = (\nabla_X Y - \nabla_Y X)(q) = 0.$$

Recall Proposition 5.4 of Chapter 0:

$$[X, Y](p) = \lim_{t \to 0} \frac{1}{t}(Y - d\varphi_t Y)(\varphi_t(p)),$$

where φ_t is the local flow of X, we have

$$0 = [X, Y](q) = \lim_{t \to 0} \frac{1}{t}(Y - (d\varphi_t)_q Y)(\varphi_t(q)) = \lim_{t \to 0} \frac{1}{t}(Y - (d\varphi_t)_q Y)(q).$$

It is easy to see that φ_0 is identity, so $(d\varphi_0)_q = \mathrm{Id}$. Then we have

$$0 = [X, Y](q) = \lim_{t \to 0} \frac{1}{t}(Y - (d\varphi_t)_q Y)(q) = \lim_{t \to 0} \frac{1}{t}((d\varphi_0)_q Y - (d\varphi_t)_q Y)(q) = -\left.\frac{d}{dt}\right|_{t=0} (d\varphi_t)_q(Y)(q)$$

for any vector field Y. This means $\left.\frac{d}{dt}\right|_{t=0} (d\varphi_t)_q = 0$. So $(d\varphi_t)_q$ doesn't depend on t near 0, is a invertible matrix, say A. By the definition of trajectory, we have $\varphi_{t+s} = \varphi_t \circ \varphi_s$, then we have $(d\varphi_{t+s})_q = (d\varphi_t)_q \cdot (d\varphi_s)_q$ by chain rule. This means $A = A^2$ so we must have $(d\varphi_t)_q = A = \mathrm{Id}$ for all t. So we proved our claim.

Then we will prove φ_t is the identity locally for all t. We choose any geodesic sphere, say $S_\delta(q)$, in a convex neighborhood of q. For any $p \in S_\delta(q)$, we can choose the unique geodesic γ from q to p. From the proof of Exercise 3.5 b), we know that $\widetilde{\gamma} = \varphi_t \circ \gamma$ is a geodesic from $\widetilde{\gamma}(0) = \varphi_t \circ \gamma(0) = q$ to $\widetilde{\gamma}(1) = \varphi_t \circ \gamma(1) = \varphi_t(p) \in S_\delta(q)$. But $(d\varphi_t)_q = \mathrm{Id}$, so

$$\widetilde{\gamma}'(0) = (d\varphi_t)_q \cdot \gamma'(0) = \gamma'(0).$$

This means γ and $\widetilde{\gamma}$ are geodesics both from q with the same initial velocity and the same length. By the uniqueness of geodesic, $\gamma = \widetilde{\gamma}$. So we must have

$$\varphi_t(p) = \widetilde{\gamma}(1) = \gamma(1) = p.$$

So φ_t is identity on a convex neighborhood of q for all t.

Finally, in such the convex neighborhood,

$$X(\varphi_t(p)) = \frac{d}{dt}\varphi_t(p) = 0.$$

Then we can repeat this progress on the convex neighborhood until the whole M. So $X \equiv 0$ on M. □

Exercise 3.7 (Geodesic frame)**.** Let M be a Riemannian manifold of dimension n and let $p \in M$. Show that there exists a neighborhood $U \in M$ of p and n vector fields $E_1, \cdots, E_n$ on U, orthonormal at each point of U, such that, at p, $\nabla_{E_i} E_j(p) = 0$. Such a family E_i, $i = 1, \cdots, n$, of vector fields is called a *(local) geodesic frame* at p.

Proof. We can choose U as a convex neighborhood of p, say $B_\varepsilon(p)$. Then we choose a orthonormal basis $\{E_1, \cdots, E_n\}$ of T_pM. For any $q \in U$, we have the unique geodesic $\gamma \subset U$ from p to q. Let $E_i(\gamma(t))$ be the parallel transports of E_i along γ. By the uniqueness of geodesic, we well defined vector fields $E_i(q), \cdots, E_n(q)$ for all $q \in U$ (See the proof of Exercise 3.4 a)).

By the compatiblity of ∇ with $\langle\ ,\ \rangle$, for any $1 \le i, j \le n$,

$$\langle E_i(q), E_j(q)\rangle_q = \text{constant} = \langle E_i(p), E_i(p)\rangle_p = \langle E_i, E_j\rangle = \delta_{ij}.$$

So $\{E_1(q), \cdots, E_n(q)\}$ is a orthonormal basis at each $q \in U$. Let γ_i be the geodesic from p such that $\gamma_i'(0) = E_i(p)$. By the uniqueness of the solution of ODEs, we can say locally, near p, γ_i is the trajectory of $E_i(p)$. Then we have

$$\nabla_{E_i}(E_j)(p) = \nabla_{\gamma_i'} E_j(p) = \frac{DE_j}{dt}(p)$$

where $\frac{DE_j}{dt}$ stands for the covariant derivative of E_j along γ_i. By our definition, it's a parallel transport and then $\frac{DE_j}{dt} = 0$. So we have

$$\nabla_{E_i}(E_j)(p) = \frac{DE_j}{dt}(p) = 0.$$

So we proved the existence of geodesic frame, as desired. □

Exercise 3.8. Let M be a Riemannian manifold. Let X be a vector field on M and f be a differentiable function on M. Define the *divergence* of X as a function $\operatorname{div} X\colon M \to \mathbb{R}$ given by $\operatorname{div} X(p) =$ the trace of the linear mapping $Y(p) \mapsto \nabla_Y X(p)$, $p \in M$, and the *gradient* of f as a vector field $\operatorname{grad} f$ on M defined by

$$\langle \operatorname{grad} f(p), v\rangle = df_p(v), \qquad p \in M,\ v \in T_pM.$$

a) Let E_i, $i = 1, \cdots, n = \dim M$, be a geodesic frame at $p \in M$. Show that

$$\operatorname{grad} f(p) = \sum_{i=1}^{n} (E_i(f))\, E_i(p),$$

$$\operatorname{div} X(p) = \sum_{i=1}^{n} E_i(f_i)(p), \qquad \text{where } X = \sum_{i=1}^{n} f_i E_i.$$

b) Suppose that $M = \mathbb{R}^n$, with coordinates $(x_1, \cdots, x_n)$ and $\frac{\partial}{\partial x_i} = (0, \cdots, 0, 1, 0, \cdots, 0) = e_i$. Show that:

$$\operatorname{grad} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} e_i,$$

$$\operatorname{div} X = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}, \qquad \text{where } X = \sum_{i=1}^{n} f_i e_i.$$

Proof. Firstly, we suppose $\operatorname{grad} f(p) = \sum_{i=1}^{n} f_i(p)E_i(p)$ where $f_i(p)$ are smooth functions of p. Because $E_1, \cdots, E_n$ is geodesic frame and then a orthonormal basis, we have

$$\langle \operatorname{grad} f(p), E_i(p)\rangle_p = \left\langle \sum_{j=1}^{n} f_i(p)E_j(p), E_i(p) \right\rangle_p = \sum_{j=1}^{n} \langle f_j(p)E_j(p), E_i(p)\rangle_p = f_i(p).$$

And by the definition, we have

$$f_i(p) = \langle \operatorname{grad} f(p), E_i(p)\rangle = df_p(E_i(p)).$$

Then we will study how any real number r acts on a smooth function $g(x)$ on $\mathbb{R}$ as $r \in \mathbb{R} \cong T_{f(p)}\mathbb{R}$. We choose a curve $\beta(t) = tr + f(p)$. Then we have $\beta(0) = f(p)$ and $\beta'(0) = r$. Then by definition, we have

$$r(g) = \left.\frac{d}{dt}\right|_{t=0} (g \circ \beta)(t) = \left.\frac{dg}{dx}\right|_{f(p)} \cdot \left.\frac{d\beta}{dt}\right|_{t=0} = r \cdot \left.\frac{dg}{dx}\right|_{f(p)},$$

by chain rule, where $\cdot$ is the product on $\mathbb{R}$. So we have

$$(E_i(p)(f))\,(g) = (E_i(p)(f)) \cdot \left.\frac{dg}{dx}\right|_{f(p)}.$$

And by the definition of the differential of a smooth map, we suppose $\alpha(t)$ is a curve on M such that $\alpha(0) = p$ and $\alpha'(0) = E_i(p)$, and have

$$df_p(E_i(p))(g) = E_i(p)(g \circ f) = \left.\frac{d}{dt}\right|_{t=0} (g \circ f \circ \alpha)(t) = \left.\frac{dg}{dx}\right|_{f(p)} \cdot \left.\frac{d}{dt}\right|_{t=0} (f \circ \alpha)(t) = (E_i(p)(f)) \cdot \left.\frac{dg}{dx}\right|_{f(p)}.$$

So we have

$$f_i(p) = df_p(E_i(p)) = E_i(p)(f)$$

and then

$$\operatorname{grad} f(p) = \sum_{i=1}^{n} (E_i(p)(f))E_i(p).$$

To compute the divergence of $X = \sum_{i=1}^{n} f_i E_i$, we need to calculate the trace of the linear map $Y(p) \mapsto \nabla_Y X(p)$. We know from linear algebra that the trace of a linear map is independent of the choice of the basis of the linear spaces, so we can still use $E_1(p), \cdots, E_n(p)$ as the basis of $T_p(M)$. Let $A = (a_{ij})_{1\le i,j\le n}$ be the matrix of the linear map $Y(p) \mapsto \nabla_Y X(p)$ under this basis and we also use it to represent the linear map. Then we have

$$a_{ij} = \left(A(E_j(p))\right)_i .$$

Then by definition, we have

$$\begin{aligned} A(E_j(p)) = \nabla_{E_j} X(p) = \nabla_{E_j}\left(\sum_{i=1}^n f_i E_i\right)(p) &= \sum_{i=1}^n \nabla_{E_j}\left(f_i E_i\right)(p) \\ = \sum_{i=1}^n \big(f_i(p)\nabla_{E_j}E_i(p) + (E_j(p)(f_i))E_i(p)\big) &= \sum_{i=1}^n (E_j(p)(f_i))E_i(p) \end{aligned}$$

where the last equation is by Exercise 3.7. Then we have

$$a_{ij} = \left(A(E_j(p))\right)_i = E_j(p)(f_i).$$

Then

$$\operatorname{div} X(p) = \operatorname{Tr}(A) = \sum_{i=1}^n a_{ii} = \sum_{i=1}^n E_i(p)(f_i).$$

So we proved a).

For the special case $M = \mathbb{R}^n$ with coordinates $(x_1, \cdots, x_n)$, we know (at page 56) that $e_i = \frac{\partial}{\partial x_i}$ is an orthonaomal basis and a geodesic frame. So we can compute directly from a):

$$\operatorname{grad} f = \sum_{i=1}^n e_i(f)e_i = \sum_{i=1}^n \frac{\partial f}{\partial x_i} e_i$$

and

$$\operatorname{div} X = \sum_{i=1}^n e_i(f_i) = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}$$

where $X = \sum_{i=1}^n f_i e_i$; whence b). $\square$

Exercise 3.9. Let M be a Riemannian manifold. Define an operator $\Delta\colon \mathcal{D}(M) \to \mathcal{D}(M)$ (the *Laplacian* of M) by

$$\Delta f = \operatorname{div} \operatorname{grad} f, \qquad f \in \mathcal{D}(M).$$

a) Let E_i be a geodesic frame at $p \in M$, $i = 1, \cdots, n = \dim M$. Prove that

$$\Delta f(p) = \sum_{i=1}^n E_i(E_i(f))(p).$$

Conclude that if $M = \mathbb{R}^n$, Δ coincides with the usual Laplacian, namely, $\Delta f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$.

b) Show that

$$\Delta(f \cdot g) = f\Delta g + g\Delta f + 2\left\langle \operatorname{grad} f, \operatorname{grad} g\right\rangle .$$

Proof. For a), by Exercise 3.8, we can calculate directly:

$$\Delta f(p) = \operatorname{div} \operatorname{grad} f(p) = \operatorname{div}\left(\sum_{i=1}^n E_i(f)E_i\right)(p) = \sum_{i=1}^n E_i(p)(E_i(f)).$$

And for the special case $M = \mathbb{R}^n$, we have

$$\Delta f = \sum_{i=1}^n e_i(e_i(f)) = \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_i}(f)\right) = \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right) = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}.$$

For b), by Exercise 3.8 and Leibniz law of tangent vectors, we have

$$
\begin{aligned}
\Delta(f\cdot g) &= \operatorname{div}\operatorname{grad}(f\cdot g) = \operatorname{div}\left(\sum_{i=1}^{n} E_i(f\cdot g)E_i\right)\\
&= \operatorname{div}\left(\sum_{i=1}^{n}\left(gE_i(f)+fE_i(g)\right)E_i\right) = \sum_{i=1}^{n} E_i\left(gE_i(f)+fE_i(g)\right)\\
&= \sum_{i=1}^{n}\left(E_i(g)\cdot E_i(f)+gE_i(E_i(f))+E_i(f)\cdot E_i(g)+fE_i(E_i(g))\right)\\
&= \sum_{i=1}^{n} fE_i(E_i(g))+\sum_{i=1}^{n} gE_i(E_i(f))+2\sum_{i=1}^{n}E_i(f)\cdot E_i(g)\\
&= f\sum_{i=1}^{n} E_i(E_i(g))+g\sum_{i=1}^{n}E_i(E_i(f))+2\left\langle \sum_{i=1}^{n}E_i(f)E_i,\sum_{i=1}^{n}E_i(g)E_i\right\rangle\\
&= f\Delta g+g\Delta f+2\left\langle \operatorname{grad} f,\operatorname{grad} g\right\rangle,
\end{aligned}
$$

where $E_1,\cdots,E_n$ is a geodesic frame then a orthonoamal basis of T_pM at any $p\in M$ (see Exercise 3.8), as desired. □

Exercise 3.10. Let $f\colon [0,1]\times[0,a]\to M$ be a parametrized surface such that for all $t_0\in[0,a]$, the curve $s\mapsto f(s,t_0)$, $s\in[0,1]$, is a geodesic parametrized by arc length, which is orthogonal to the curve $t\mapsto f(0,t)$, $t\in[0,a]$, at the point $f(0,t_0)$. Prove that, for all $(s_0,t_0)\in[0,1]\times[0,a]$, the curve $s\mapsto f(s,t_0)$, $t\mapsto f(s_0,t)$ are orthogonal.

Proof. Firstly, by the compatiblity of ∇ with $\langle\ ,\ \rangle$ (Chapter 2 Proposition 3.2), we have

$$
\frac{d}{ds}\left\langle\frac{\partial f}{\partial s},\frac{\partial f}{\partial t}\right\rangle=\left\langle\frac{D}{ds}\frac{\partial f}{\partial s},\frac{\partial f}{\partial t}\right\rangle+\left\langle\frac{\partial f}{\partial s},\frac{D}{ds}\frac{\partial f}{\partial t}\right\rangle,
$$

where $\frac{D}{ds}$ stands for the covariant derivative along the geodesic $s\mapsto f(s,t)$. Then $\frac{D}{ds}\frac{\partial f}{\partial s}=0$ and so $\left\langle\frac{D}{ds}\frac{\partial f}{\partial s},\frac{\partial f}{\partial t}\right\rangle$. For another term, use again Chapter 2 Proposition 3.2, we have

$$
\frac{\partial}{\partial t}\left\langle\frac{\partial f}{\partial s},\frac{\partial f}{\partial s}\right\rangle=\left\langle\frac{D}{dt}\frac{\partial f}{\partial s},\frac{\partial f}{\partial s}\right\rangle+\left\langle\frac{\partial f}{\partial s},\frac{D}{dt}\frac{\partial f}{\partial s}\right\rangle=2\left\langle\frac{\partial f}{\partial s},\frac{D}{dt}\frac{\partial f}{\partial s}\right\rangle.
$$

By Chapter 3 Lemma 3.4, for a symmetric connection and a parameterized surface s,

$$
\frac{D}{dv}\frac{\partial s}{\partial u}=\frac{D}{du}\frac{\partial s}{\partial v}.
$$

Then we have

$$
\left\langle\frac{\partial f}{\partial s},\frac{D}{ds}\frac{\partial f}{\partial t}\right\rangle=\left\langle\frac{\partial f}{\partial s},\frac{D}{dt}\frac{\partial f}{\partial s}\right\rangle=\frac{1}{2}\frac{\partial}{\partial t}\left\langle\frac{\partial f}{\partial s},\frac{\partial f}{\partial s}\right\rangle.
$$

Notice that $\frac{\partial f}{\partial s}$ is a parallel vector field along the geodesic $s\mapsto f(s,t)$, by the definition of compatible connection, we have $\left\langle\frac{\partial f}{\partial s},\frac{\partial f}{\partial s}\right\rangle$ is a constant. So the last term is also zero. So we have $\frac{d}{ds}\left\langle\frac{\partial f}{\partial s},\frac{\partial f}{\partial t}\right\rangle=0$, which means $\left\langle\frac{\partial f}{\partial s},\frac{\partial f}{\partial t}\right\rangle_{(s,t)}$ is independent of s. So for any $s\in[0,1]$ and $t\in[0,a]$, we have

$$
\left\langle\frac{\partial f}{\partial s},\frac{\partial f}{\partial t}\right\rangle_{(s_0,t_0)}=\left\langle\frac{\partial f}{\partial s},\frac{\partial f}{\partial t}\right\rangle_{(0,t_0)}=0
$$

where we use the condition that $s\mapsto f(s,t)$ is orthogonal to $t\mapsto f(s,t)$ at $f(0,t_0)$. This means $s\mapsto f(s,t)$ is orthogonal to $t\mapsto f(s,t)$ for any $(s_0,t_0)\in[0,1]\times[0,a]$, as desired.

In fact we have proved this proposition in the proof of Gauss Lemma at Page 70. □

Exercise 3.11. Let M be an oriented Riemannian manifold. Let ν be a differential form of degree $n = \dim M$ defined in the following way:

$$\nu(v_1, \cdots, v_n)(p) = \pm\sqrt{\det(\langle v_i, v_j\rangle)} = \text{oriented volume of } \{v_1, \cdots, v_n\}, \quad p \in M,$$

where $v_1, \cdots, v_n \in T_pM$ are linearly independent, and the oriented volume is affected by the sign $+$ or $-$ depending on whether or not the basis $\{v_1, \cdots, v_n\}$ belongs to the orientation of M; ν is called the *volume element* of M. For a vector field X on M define the *interior product* $i(X)\nu$ of X with ν as the $(n-1)$-form:

$$i(X)\nu(Y_2, \cdots, Y_n) = \nu(X, Y_2, \cdots, Y_n),$$

where $Y_2, \cdots, Y_n$ are vector fields on M. Prove that

$$d\left(i(X)\nu\right) = \operatorname{div} X\nu.$$

This result implies that the notion of the divergence of X makes sense on an oriented differentiable manifold on which a "volume element" has been chosen, that is, an n-form ν which takes positive values on positive bases.

Proof. Firstly, for any $p \in M$, we choose a geodesic frame of M, say $E_1, \cdots, E_n$. Then we choose its dual basis $e_1, \cdots, e_n$, i.e. $e_i(E_j) = \delta_{ij}$ for all $1 \le i, j \le n$. Then we claim that $\nu = e_1 \wedge \cdots \wedge e_n$ is a volume elemant of M: For any $X_i = \sum_{j=1}^n x_{ij}E_j \in T_pM$, $1 \le i \le n$, by the definition of differential form, we have

$$\nu(X_1, \cdots, X_n) = e_1 \wedge \cdots \wedge e_n(X_1, \cdots, X_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n e_{\sigma(i)}(X_i).$$

By linearity, we have

$$e_i(X_j) = e_i\left(\sum_{k=1}^n x_{jk}E_k\right) = \sum_{k=1}^n x_{jk}e_i(E_k) = x_{ji}.$$

Then we have

$$\nu(X_1, \cdots, X_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n e_{\sigma(i)}(X_i) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n x_{i\sigma(i)} = \det(x_{ij})_{1 \le i,j \le n}.$$

On the other hand, for any $1 \le i, j \le n$, because $E_1, \cdots, E_n$ is a orthonormal basis, we have

$$\langle X_i, X_j\rangle = \left\langle \sum_{k=1}^n x_{ik}E_k, \sum_{l=1}^n x_{jl}E_l \right\rangle = \sum_{k,l=1}^n \langle x_{ik}E_k, x_{jl}E_l\rangle = \sum_{k=1}^n x_{ik}x_{jk}.$$

Let $A = (x_{ij})_{1 \le i,j \le n}$ and $B = (\langle X_i, X_j\rangle)_{1 \le i,j \le n}$, we have

$$AA^T = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & \ddots & \vdots \\ x_{1n} & \cdots & x_{nn} \end{pmatrix} = \left(\sum_{k=1}^n x_{ik}x_{jk}\right)_{1 \le i,j \le n} = (\langle E_i, E_j\rangle)_{1 \le i,j \le n} = B.$$

So we have

$$\det(B) = \det(AA^T) = \det(A)\det(A^T) = \det(A)\det(A) = \det(A)^2.$$

It follows that

$$\nu(X_1, \cdots, X_n) = \det(A) = \pm\sqrt{\det(B)} = \pm\sqrt{\det(\langle v_i, v_j\rangle)}.$$

So we proved $\nu = e_1 \wedge \cdots \wedge e_n$ is a volume element of M. In fact, in the end of Chapter 1, we defined the volume form by

$$\nu = \sqrt{\det(\langle X_i, X_j\rangle)_{1 \le i,j \le n}}e_1 \cdots e_n.$$

Because $X_1, \cdots, X_n$ is a geodesic frame and then orthonormal, $\det(\langle X_i, X_j\rangle)_{1 \le i,j \le n} = \det(I_n) = 1$. So $\nu = e_1 \wedge \cdots \wedge e_n$.

Let $X=\sum_{i=1}^n f_iE_i$ be any vector field. Let $\theta_i=e_1\wedge\cdots\wedge\widehat{e_i}\wedge\cdots\wedge e_n$, $1\le i\le n$. Then for any vector fields $Y_2,\cdots,Y_n$, by the definition of anti-symmetric form, we have

$$\begin{aligned}i(X)\nu(Y_2,\cdots,Y_n)&=\nu(X,Y_2,\cdots,Y_n)=e_1\wedge\cdots\wedge e_n(X,Y_2,\cdots,Y_n)\\&=\sum_{i=1}^n(-1)^{i+1}e_i(X)\theta_i(Y_2,\cdots,Y_n)=\sum_{i=1}^n(-1)^{i+1}f_i\theta_i(Y_2,\cdots,Y_n)\end{aligned}$$

where we use $e_i(X)=e_i\left(\sum_{j=1}^n f_jE_j\right)=\sum_{j=1}^n f_je_i\left(E_j\right)=f_i$. This means

$$i(X)\nu=\sum_{i=1}^n(-1)^{i+1}f_i\theta_i.$$

It then follows that

$$d(i(X)\nu)=\sum_{i=1}^n(-1)^{i+1}df_i\wedge\theta_i+\sum_{i=1}^n(-1)^{i+1}f_id\theta_i.$$

In the last term,

$$d\theta_i=\sum_{\substack{j=1\\ j\neq i}}^n e_1\wedge\cdots\wedge de_j\wedge\cdots\wedge\widehat{e_i}\wedge\cdots\wedge e_n.$$

And by the next lemma:

Lemma 1. For any 1-form ω and any vector fields X,Y, we have

$$d\omega(X,Y)=X(\omega(Y))-Y(\omega(X))-\omega([X,Y]).$$

at p, we have

$$de_k(E_i,E_j)=E_i(e_k(E_j))-E_j(e_k(E_i))-e_k([E_i,E_j])=-e_k(\nabla_{E_i}E_j-\nabla_{E_j}E_i)=0,\qquad\forall 1\le i,j,k\le n,$$

where we use the fact that $e_k(E_i)=\delta_{ki},\ e_k(E_j)=\delta_{kj}$ are all constants and the symmetry of of Riemannian connection and our assumption that $E_1,\cdots,E_n$ is geodesic frame then $\nabla_{E_i}E_j(p)=0$ by Exercise 3.7, respectively. Then we have $de_k=0$ and by linearity, we have $d\theta_i=0$ at p. It follows that

$$\begin{aligned}d(i(X)\nu)&=\sum_{i=1}^n(-1)^{i+1}df_i\wedge\theta_i=\sum_{i=1}^n(-1)^{i+1}\sum_{j=1}^n E_j(f_i)e_j\wedge\theta_i\\&=\sum_{i=1}^n(-1)^{i+1}E_i(f_i)e_i\wedge\theta_i=\sum_{i=1}^n E_i(f_i)e_1\wedge\cdots\wedge e_n=\sum_{i=1}^n E_i(f_i)\nu,\end{aligned}$$

where the equality of the line feed is by the fact that $de_j\wedge de_j=0$, $\forall 1\le j\le n$. Then by Exercise 3.8 a), we have

$$d(i(X)\nu)(p)=\operatorname{div}X(p)\nu.$$

Since p is arbitrary, we complete the proof.

Proof of Lemma 1. By linearity, we can assume $\omega=gdf$. Then $d\omega=dg\wedge df+gd^2f$. Because 1-forms can be spanned by $dx_1,\cdots.dx_n$, we have

$$d^2f=d(df)=d\left(\sum_{i=1}^n\frac{\partial f}{\partial x_i}dx_i\right)=\sum_{i,j=1}^n\frac{\partial^2 f}{\partial x_i\partial x_j}dx_i\wedge dx_j=\sum_{i<j}\left(\frac{\partial^2 f}{\partial x_i\partial x_j}-\frac{\partial^2 f}{\partial x_j\partial x_i}\right)dx_i\wedge dx_j=0.$$

So we have $d\omega=dg\wedge df$. Then

$$d\omega(X,Y)=dg\wedge df(X,Y)=dg(X)df(Y)-df(X)dg(Y)=X(g)Y(f)-X(f)Y(g).$$

On the other hand, $\omega(X)=gdf(X)=gX(f)$. Then

$$Y(\omega(X))=Y(gX(f))=Y(g)X(f)+gY(X(f)).$$

Similarly, we have $X(\omega(Y)) = X(g)Y(f) + gX(Y(f))$. Then we have

$$\begin{aligned}
d\omega(X,Y) &= X(g)Y(f) - X(f)Y(g)\\
&= X(\omega(Y)) - gX(Y(f)) - Y(\omega(X)) + gY(X(f))\\
&= X(\omega(Y)) - Y(\omega(X)) - (gX(Y(f)) - gY(X(f)))\\
&= X(\omega(Y)) - Y(\omega(X)) - g[X,Y](f)\\
&= X(\omega(Y)) - Y(\omega(X)) - gdf([X,Y])\\
&= X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]),
\end{aligned}$$

as desired. □

□

Exercise 3.12 (Theorem of E. Hopf)**.** Let M be a compact orientable Riemannian manifold which is also connected. Let f be a differentiable function on M with $\Delta f \geq 0$. Then $f =$ constant. In particular, the harmonic functions on M, that is, those for which $\Delta f = 0$, are constant.

Proof. We denote $X = \operatorname{grad} f$ and ν is a volume form of M (See Exercise 3.11). By Exercise 3.11 and Stokes Theorem, we have

$$0 \leq \int_M \Delta f \nu = \int_M \operatorname{div}\operatorname{grad} f \nu = \int_M \operatorname{div} X \nu = \int_M d(i(X)\nu) = \int_{\partial M} i(X)\nu = 0,$$

where the last term is because M is compact and then $\partial M = \varnothing$. Then we must have $\int_M \Delta f \nu = 0$. Because ν is a volume form of a connected manifold M and $\Delta f \geq 0$, we have $\Delta f = 0$. Similarly, use Exercise 3.11 and Stokes Theorem, we also have

$$\int_M \Delta\left(\frac{f^2}{2}\right)\nu = 0.$$

On the other hand, by Exercise 3.9 b),

$$\Delta\left(\frac{f^2}{2}\right) = \frac{1}{2}\left(f\Delta f + f\Delta f + 2\left\langle \operatorname{grad} f, \operatorname{grad} f\right\rangle\right) = f\Delta f + \left\langle \operatorname{grad} f, \operatorname{grad} f\right\rangle = \left|\operatorname{grad} f\right|^2.$$

Then

$$\int_M \left|\operatorname{grad} f\right|^2 \nu = \int_M \Delta\left(\frac{f^2}{2}\right)\nu = 0.$$

Also by $\left|\operatorname{grad} f\right|^2 \geq 0$ and the connectedness of M, we must have $\operatorname{grad} f = 0$. By Exercise 3.8 a),

$$0 = \operatorname{grad} f = \sum_{i=1}^n (E_i(f))E_i$$

where $E_1, \cdots, E_n$ is a geodesic frame of M and then a orthonormal basis. This means $E_i(f) = 0$, $\forall 1 \leq i \leq n$ and then $X(f) = 0$ for any vector field X on M. So we must have f is a constant, as desired. □

Exercise 3.13. Let M be a Riemannian manifold and X be a vector field on M. Let $p \in M$ such that $X(p) \neq 0$. Choose a coordinate system $(t, x_2, \cdots, x_n)$ in a neighborhood U of p such that $\frac{\partial}{\partial t} = X$. Show that if $\nu = g dt \wedge dx_2 \wedge \cdots \wedge dx_n$ is a volume element of M, then

$$i(X)\nu = g dx_2 \wedge \cdots \wedge dx_n.$$

Conclude from this, using the result of Exercise 3.11, that

$$\operatorname{div} X = \frac{1}{g}\frac{\partial g}{\partial t}.$$

This proves that $\operatorname{div} X$ intuitively measures the degree of variation of the volume element of M along the trajectories of X.

Proof. Firstly, for any vector fields $Y_2, \cdots, Y_n$ on M, by definition,

$$\begin{aligned}
&i(X)\nu(Y_2, \cdots, Y_n) \\
=& \nu(X, Y_2, \cdots, Y_n) = g dt \wedge dx_2 \wedge \cdots \wedge dx_n(X, Y_2, \cdots, Y_n) \\
=& g\left(dt(X) dx_2 \wedge \cdots \wedge dx_n(Y_2, \cdots, Y_n) + \sum_{i=2}^{n} (-1)^{i+1} dx_i(X) dt \wedge dx_2 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n(Y_2, \cdots, Y_n) \right).
\end{aligned}$$

Because $dt, dx_2, \cdots, dx_n$ is the dual basis of $X = \frac{\partial}{\partial t}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_2}$, we have $dt(X) = 1$ and $dx_i(X) = 0$ for any $2 \leq i \leq n$. Then we have

$$i(X)\nu(Y_2, \cdots, Y_n) = g dx_2 \wedge \cdots \wedge dx_n(Y_2, \cdots, Y_n)$$

and then

$$i(X)\nu = g dx_2 \wedge \cdots \wedge dx_n.$$

Then we calculate

$$d(i(X)\nu) = d(g dx_2 \wedge \cdots \wedge dx_n) = dg \wedge dx_2 \wedge \cdots \wedge dx_n + g d(dx_2 \wedge \cdots \wedge dx_n).$$

In the last term,

$$d(dx_2 \wedge \cdots \wedge dx_n) = \sum_{i=2}^{n} (-1)^i dx_2 \wedge \cdots \wedge d^2 x_i \wedge \cdots \wedge dx_n.$$

By Lemma 1, for any $d^2 x_i$, $2 \leq i \leq n$ and any two $\frac{\partial}{\partial x_j}$, $2 \leq j \leq n$ or $\frac{\partial}{\partial t}$ (let $t = x_1$),

$$\begin{aligned}
d^2 x_i \left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right) &= d\left(dx_i\right) \left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right) \\
&= \frac{\partial}{\partial x_j} \left(dx_i \left(\frac{\partial}{\partial x_k} \right) \right) - \frac{\partial}{\partial x_k} \left(dx_i \left(\frac{\partial}{\partial x_j} \right) \right) - dx_i \left(\left[\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right] \right),
\end{aligned}$$

where $dx_i \left(\frac{\partial}{\partial x_k} \right) = \delta_{ik}$ and $dx_i \left(\frac{\partial}{\partial x_j} \right) = \delta_{ij}$ are both canstants and

$$\left[\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right] = \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} - \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} = \frac{\partial^2}{\partial x_j \partial x_k} - \frac{\partial^2}{\partial x_k \partial x_j} = 0.$$

So we have $d^2 x_i \left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right) = 0$ and then $d^2 x_i = 0$. So the last term will vanish. Then we have

$$\begin{aligned}
d(i(X)\nu) &= dg \wedge dx_2 \wedge \cdots \wedge dx_n \\
&= \left(\frac{\partial}{\partial t}(g) dt + \sum_{i=2}^{n} \frac{\partial}{\partial x_i}(g) dx_i \right) \wedge dx_2 \wedge \cdots \wedge dx_n \\
&= \frac{\partial g}{\partial t} dt \wedge dx_2 \wedge \cdots \wedge dx_n + \sum_{i=2}^{n} \frac{\partial g}{\partial x_i} dx_i \wedge dx_2 \wedge \cdots \wedge dx_n \\
&= \frac{1}{g} \frac{\partial g}{\partial t} g dt \wedge dx_2 \wedge \cdots \wedge dx_n + \sum_{i=2}^{n} (-1)^i \frac{\partial g}{\partial x_i} dx_i \wedge dx_i \wedge dx_2 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge \wedge dx_n \\
&= \frac{1}{g} \frac{\partial g}{\partial t} \nu,
\end{aligned}$$

where we use $dx_i \wedge dx_i = dx_i dx_i - dx_i dx_i = 0$. By Exercise 3.11,

$$\operatorname{div} X \nu = d(i(X)\nu) = \frac{1}{g} \frac{\partial g}{\partial t} \nu.$$

So we have

$$\operatorname{div} X = \frac{1}{g} \frac{\partial g}{\partial t},$$

as desired. So $\operatorname{div} X$ intuitively measures the degree of variation of the volume element of M along the trajectories of X, i.e. the $\frac{1}{g}$ multiply the derivative of the volume g along X. $\square$

Exercise 3.14 (Liouville's Theorem)**.** Prove that if G is the geodesic field on TM then $\operatorname{div} G = 0$. Conclude from this that the geodesic flow preserves the volume of TM.

Proof. Firstly, at any $p \in M$, we choose a normal neighborhood of p, say U. Then by definition, the exponent mapping $\exp_p$ is a isometry onto U. Then we can choose a ball $B_\varepsilon(0)$ of T_pM such that $\exp_p$ maps it into U diffeomorphically. We denote $B_\varepsilon(p) = \exp_p(B_\varepsilon(0))$ by U again. Then for any $q \in U$, there is unique tangent vector $v = \sum_{i=1}^n u_i e_i \in B_\varepsilon(0)$ such that $q = \exp_p(v)$, where $e_1, \cdots, e_n$ is a orthonormal basis of T_pM. So we have $(u_1, \cdots, u_n)$ is coordinates near p and $\exp_p$ is the transport mapping from $B_\varepsilon(0) \subset T_pM \cong \mathbb{R}^n$ to U.

Under this coordinates, by the definition of exponent mapping and the the homogeneity of geodesic, the geodesics of M from p will be

$$\gamma(t) = \gamma(t, p, v) = \gamma(1, p, tv) = \exp_p(tv) = \exp_p\left(\sum_{i=1}^n t u_i e_i\right) = (tu_1, \cdots, tu_n).$$

We denote $\gamma(t) = (x_1(t), \cdots, x_n(t))$. By the geodesic equation:

$$0 = \sum_{k=1}^n \left(\frac{d^2 x_k}{dt^2} + \sum_{i,j=1}^n \Gamma_{ij}^k \frac{dx_i}{dt}\frac{dx_j}{dt}\right)\frac{\partial}{\partial x_k},$$

we must have

$$0 = \frac{d^2 x_k}{dt^2} + \sum_{i,j=1}^n \Gamma_{ij}^k \frac{dx_i}{dt}\frac{dx_j}{dt} = \sum_{i,j=1}^n \Gamma_{ij}^k u_i u_j, \qquad \forall 1 \le k \le n.$$

Because $u_1, \cdots, u_n$ are arbitrary in $B_\varepsilon(0)$, we have $\Gamma_{ij}^k(p) = 0$, $\forall 1 \le i, j, k \le n$. Then we have

$$\nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j}(p) = \sum_{k=1}^n \Gamma_{ij}^k(p) \frac{\partial}{\partial u_k}(p) = 0, \qquad \forall 1 \le i, j \le n.$$

Then choose a geodesic frame $E_1, \cdots, E_n$ at p. Suppose that for any $1 \le i \le n$, we have $E_i = \sum_{j=1}^n a_{ij} \frac{\partial}{\partial u_j}$, i.e. $(a_{ij})_{1 \le i,j \le n}$ is the transition matrix from $\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_n}$ to $E_1, \cdots, E_n$, which is orthogonal. We denote $(a_{ij})_{1 \le i,j \le n}$ by A and the metric matrices $\left(\left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right\rangle\right)_{1 \le i,j \le n}$ and $(\langle E_i, E_j\rangle)_{1 \le i,j \le n}$ by g and g' respectively. Then we have

$$g' = AgA^T.$$

Then

$$g = A^{-1} g' (A^T)^{-1} = A^{-1} I_n A = A^{-1} A = I_n.$$

This means $\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_n}$ is an orthonormal basis of T_qM, $\forall q \in U$. So it is a geodesic frame of M at p. Then for any vector field $X = \sum_{i=1}^n x_i \frac{\partial}{\partial u_i}$ on U, by Exercise 3.8 b), we have

$$\operatorname{div} X(p) = \sum_{i=1}^n \frac{\partial}{\partial u_i}(x_i)(p) = \sum_{i=1}^n \left.\frac{\partial x_i}{\partial u_i}\right|_p.$$

For any $(q, v) \in TU$, we suppose $p = (u_1, \cdots, u_n)$ and $v = \sum_{j=1}^n v_j \frac{\partial}{\partial u_j}$. Then we let $(u_1, \cdots, u_n, v_1, \cdots, v_n)$ be the coordinates of $(q, v) \in TU$. We have proved that $\frac{\partial}{\partial u_1}, \cdots \frac{\partial}{\partial u_n}$ is a geodesic frame of M and we know that $\frac{\partial}{\partial v_1}, \cdots, \frac{\partial}{\partial v_n}$ is the geodesic frame of the Euclidean case $T_qM \cong \mathbb{R}^n$. By the metric defined in Exercise 3.2, for any $1 \le i, j \le n$,

$$\begin{aligned}
\left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial v_j}\right\rangle_{q,v} &= \left\langle d\pi\left(\frac{\partial}{\partial u_i}\right), d\pi\left(\frac{\partial}{\partial v_j}\right)\right\rangle_q + \left\langle \frac{D(t0)}{dt}(0), \frac{D(sv_j)}{ds}(0)\right\rangle_p \\
&= \left\langle \frac{\partial}{\partial u_i}, 0\right\rangle_p + \left\langle 0, \frac{D(sv_j)}{ds}(0)\right\rangle_p = 0.
\end{aligned}$$

So $\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_n}, \frac{\partial}{\partial v_1}, \cdots, \frac{\partial}{\partial v_n}$ is a orthonormal basis of $T_{(q,v)}TU$. From the classical expression for the Christoffel symbols of the Riemannian connection:

$$\Gamma_{ij}^m = \frac{1}{2} \sum_k \left(\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij} \right) g^{km},$$

the Christoffel symbols of this coordinates are all 0. Then by the definition of Christoffel symbols:

$$\nabla_{X_i} X_j = \sum_{k=1}^n \Gamma_{ij}^k X_k,$$

we know that for any $1 \leq i, j \leq n$, $\nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial v_j}$ and $\nabla_{\frac{\partial}{\partial v_j}} \frac{\partial}{\partial u_i}$ are all 0. So $\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_n}, \frac{\partial}{\partial v_1}, \cdots, \frac{\partial}{\partial v_n}$ is a geodesic frame of TU. By the beginning of the proof of Exercise 3.11, we know that

$$\nu = du_1 \wedge \cdots \wedge du_n \wedge dv_1 \wedge \cdots \wedge dv_n$$

is the volume element of TU.

In fact, I don't know about effect of the hint "Calculate the colume element of the natural metric of TM at (q, v), $q \in U$, $v \in T_q M$, and show that it is the volume element of the product metric on $U \times U$ at the point (q, q). Since the divergence of G only depends on the volume element, and G is horizontal, we can calculate $\operatorname{div} G$ in the product metric.", but I still check it here. On the other hand, let's consider the product manifold $U \times U$ with the product metric (See Example 2.7 in Page 42). Let $u_1, \cdots, u_n$ and $v_1, \cdots, v_n$ be the coordinates of the first and last U as above. Then $\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_n}$ and $\frac{\partial}{\partial v_1}, \cdots, \frac{\partial}{\partial v_n}$ are geodesic frames of U respectively. Under the product metric, at any $q \in U$ and for any $1 \leq i, j \leq n$,

$$\begin{aligned}\left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial v_j} \right\rangle_{(q,q)} &= \left\langle d\pi_1 \left(\frac{\partial}{\partial u_i} \right), d\pi_1 \left(\frac{\partial}{\partial v_j} \right) \right\rangle_q + \left\langle d\pi_2 \left(\frac{\partial}{\partial u_i} \right), d\pi_2 \left(\frac{\partial}{\partial v_j} \right) \right\rangle_q \\ &= \left\langle d\pi_1 \left(\frac{\partial}{\partial u_i} \right), 0 \right\rangle_q + \left\langle 0, d\pi_2 \left(\frac{\partial}{\partial v_j} \right) \right\rangle_q = 0.\end{aligned}$$

So $\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_n}, \frac{\partial}{\partial v_1}, \cdots, \frac{\partial}{\partial v_n}$ is a orthonormal basis of $T_{(q,q)}(U \times U)$. Same thing as above, we know that $\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_n}, \frac{\partial}{\partial v_1}, \cdots, \frac{\partial}{\partial v_n}$ is a geodesic frame of $U \times U$ and then

$$\nu = du_1 \wedge \cdots \wedge du_n \wedge dv_1 \wedge \cdots \wedge dv_n$$

is the volume element of $U \times U$. So we conclude that the volume element of the natural metric of TM at (q, v) is the same as it of the product metric of $U \times U$ at (q, q). By Exercise 3.11, we can calculate $\operatorname{div} G$ by $d(i(G)\nu)$ since the linear space of n-form of M has dimension 1. So $\operatorname{div} G$ only depends on the volume element. By Exercise 3.2 c), G is horizontal, so we can calculate $\operatorname{div} G$ in the product metric.

By the definition of geodesic field, let $x(t) = (x_1(t), \cdots, x_n(t), y_1(t), \cdots, y_n(t))$ be a trajectory of G, where $\gamma(t) = (x_1(t), \cdots, x_n(t))$ is a geodesic of M and $\gamma'(t) = (x_1'(t), \cdots, x_n'(t)) = (y_1(t), \cdots, y_n(t))$. At the beginning of the proof we know that

$$\gamma(t) = (x_1(t), \cdots, x_n(t)) = (tx_1, \cdots, tx_n).$$

Then we have

$$(y_1(t), \cdots, y_n(t)) = (x_1'(t), \cdots, x_n'(t)) = (x_1, \cdots, x_n).$$

Under the geodesic frame $\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_n}, \frac{\partial}{\partial v_1}, \cdots, \frac{\partial}{\partial v_n}$, we can calculate

$$G(x(t)) = x'(t) = \sum_{i=1}^n x_i'(t) \left. \frac{\partial}{\partial u_i} \right|_{x(t)} + \sum_{j=1}^n y_j'(t) \left. \frac{\partial}{\partial v_j} \right|_{x(t)}.$$

Then by Exercise 3.8,

$$\operatorname{div} G(x(t)) = \sum_{i=1}^n \left. \frac{\partial}{\partial u_i} \right|_{x(t)} (x_i'(t)) + \sum_{j=1}^n \left. \frac{\partial}{\partial v_j} \right|_{x(t)} (y_j'(t)) = \sum_{i=1}^n \left. \frac{\partial}{\partial u_i} \right|_{x(t)} (x_i) + \sum_{j=1}^n \left. \frac{\partial}{\partial v_j} \right|_{x(t)} (0) = 0.$$

See reference: https://math.stackexchange.com/questions/2221237.

I'm not sure how to calculate the volume of areas of TM under the geodesic flow. □

4 Curvature

Exercise 4.1. Let G be a Lie group with a bi-invariant metric $\langle\ ,\ \rangle$. Let X, Y, Z be unit left invariant vector fields on G.

a) Show that $\nabla_X Y = \frac{1}{2}[X, Y]$.

b) Conclude from a) that $R(X, Y)Z = \frac{1}{4}[[X, Y], Z]$.

c) Prove that, if X and Y are orthonormal, the sectional curvature $K(\sigma)$ of G with respect to the plane σ generated by X and Y is given by
$$K(\sigma) = \frac{1}{4}\,||[X, Y]||^2 .$$
Therefore, the sectional curvature $K(\sigma)$ of a Lie group with bi-invariant metric is non-negative and is zero if and only if σ is generated by vectors X, Y which commute, that is, such that $[X, Y] = 0$.

Proof. For a), from the proof of Exercise 3.3 b), we know that for any vector field Y on M, we have $\nabla_Y Y = 0$. Then

$$\begin{aligned}
\nabla_Y X &= \nabla_Y X - \nabla_Y Y \\
&= \nabla_Y (X - Y) \\
&= \nabla_{Y-X+X}(X - Y) \\
&= \nabla_{Y-X}(X - Y) + \nabla_X (X - Y) \\
&= -\nabla_{Y-X}(Y - X) + \nabla_X X - \nabla_X Y \\
&= -\nabla_X Y.
\end{aligned}$$

By the symmetry of the connection, then we have

$$[X, Y] = \nabla_X Y - \nabla_Y X = 2\nabla_X Y,$$

i.e.

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

Then by the definition of $R(X, Y)Z$, we have

$$\begin{aligned}
R(X, Y)Z &= \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z \\
&= \frac{1}{2}\nabla_Y [X, Z] - \frac{1}{2}\nabla_X [Y, Z] + \frac{1}{2}[[X, Y], Z] \\
&= \frac{1}{4}[Y, [X, Z]] - \frac{1}{4}[X, [Y, Z]] + \frac{1}{2}[[X, Y], Z] \\
&= \frac{1}{4}\left([Y, [X, Z]] - [X, [Y, Z]]\right) + \frac{1}{2}[[X, Y], Z].
\end{aligned}$$

By the Jacobi identity of vector fields:

$$[Y, [X, Z]] + [Z, [Y, X]] + [X, [Z, Y]] = 0,$$

we have

$$[Y, [X, Z]] - [X, [Y, Z]] = [Y, [X, Z]] + [X, [Z, Y]] = -[Z, [Y, X]] = -[[X, Y], Z].$$

Then we have

$$\begin{aligned}
R(X, Y)Z &= \frac{1}{4}\left([Y, [X, Z]] - [X, [Y, Z]]\right) + \frac{1}{2}[[X, Y], Z] \\
&= -\frac{1}{4}[[X, Y], Z] + \frac{1}{2}[[X, Y], Z] \\
&= \frac{1}{4}[[X, Y], Z];
\end{aligned}$$

whence b).

For c), for any orthonormal vector fields X, Y, for $\sigma = \text{Span}\{X, Y\}$, by b), we have

$$K(\sigma) = K(X,Y) = \frac{(X,Y,X,Y)}{|X \wedge Y|^2} = \frac{\langle R(X,Y)X, Y\rangle}{|X|^2|Y|^2 - \langle X, Y\rangle^2} = \frac{\left\langle \frac{1}{4}[[X,Y],X], Y\right\rangle}{1^2 \cdot 1^2 - 0^2} = \frac{1}{4}\langle [[X,Y],X],Y\rangle.$$

Then by the equation (3) of Chapter 1:

$$\langle [U,X],V\rangle = -\langle U,[V,X]\rangle, \quad \forall U,V,X \in \mathcal{G},$$

we have

$$K(\sigma) = \frac{1}{4}\langle [[X,Y],X],Y\rangle = -\frac{1}{4}\langle [X,Y],[Y,X]\rangle = \frac{1}{4}\langle [X,Y],[X,Y]\rangle = \frac{1}{4}||[X,Y]||^2.$$

Therefore, the sectional curvature $K(\sigma) = \frac{1}{4}||[X,Y]||^2 \geq 0$ and "=" if and only if σ is generated by vectors X, Y which commute, that is, such that $[X,Y] = 0$. □

Exercise 4.2. Let X be a Killing field on a Riemannian manifold M. Define a mapping

$$\begin{aligned} A_X \colon \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ Z &\mapsto \nabla_Z X. \end{aligned}$$

Consider the function $f(q) = \langle X, X\rangle_q$ on M. Let $p \in M$ be a critical point of f (that is, $df_p = 0$). Prove that for any vector field $Z \in \mathfrak{X}(M)$, at p,

a) $\langle A_X(Z), X\rangle_p = 0$.

b) $\langle A_X(Z), A_X(Z)\rangle_p = \frac{1}{2}Z_p(Z(\langle X, X\rangle)) + \langle R(X,Z)X, Z\rangle_p$.

Proof. For a), for any vector field Z, let $\varphi(t)$ denotes the trajectory of Z start at p, which is any critical point of f, then by the chain rule, we have

$$\left.\frac{df(\varphi(t))}{dt}\right|_{t=0} = df_p(\varphi'(0)) = 0.$$

On the other hand, by Chapter 2 Proposition 3.2, we have

$$\begin{aligned} \frac{df(\varphi(t))}{dt} &= \frac{d}{dt}\langle X(\varphi(t)), X(\varphi(t))\rangle = \left\langle \frac{DX}{dt}, X\right\rangle + \left\langle X, \frac{DX}{dt}\right\rangle \\ &= 2\left\langle \frac{DX}{dt}, X\right\rangle = 2\langle \nabla_{\varphi'}X, X\rangle = 2\langle \nabla_Z X, X\rangle. \end{aligned}$$

So we have

$$\langle A_X(Z), X\rangle_p = \langle \nabla_Z X, X\rangle_p = \frac{1}{2} \cdot \left.\frac{df(\varphi(t))}{dt}\right|_{t=0} = 0.$$

For b), let's study the right side of the equation and denote it by $S(p)$. For the first term of S, by the compatiblity of the connection (Chapter 2 Corollary 3.3), we have

$$\frac{1}{2}Z(Z\langle X, X\rangle) = \frac{1}{2}Z(\langle \nabla_Z X, X\rangle + \langle X, \nabla_Z X\rangle) = Z\langle \nabla_Z X, X\rangle.$$

By the Killing equation (Exercise 3.5 d)),

$$\frac{1}{2}Z(Z\langle X, X\rangle) = Z\langle \nabla_Z X, X\rangle = -Z\langle \nabla_X X, Z\rangle.$$

By Chapter 2 Corollary 3.3 again, we have

$$\frac{1}{2}Z(Z\langle X, X\rangle) = -Z\langle \nabla_X X, Z\rangle = -\langle \nabla_Z \nabla_X X, Z\rangle - \langle \nabla_X X, \nabla_Z Z\rangle.$$

So the right side will reduce to

$$
\begin{aligned}
S &= \frac{1}{2}Z(Z(\langle X, X\rangle)) + \langle R(X,Z)X, Z\rangle \\
&= -\langle \nabla_Z\nabla_X X, Z\rangle - \langle \nabla_X X, \nabla_Z Z\rangle + \langle \nabla_Z\nabla_X X, Z\rangle - \langle \nabla_X\nabla_Z X, Z\rangle + \left\langle \nabla_{[X,Z]}X, Z\right\rangle \\
&= -\langle \nabla_X X, \nabla_Z Z\rangle - \langle \nabla_X\nabla_Z X, Z\rangle + \left\langle \nabla_{[X,Z]}X, Z\right\rangle.
\end{aligned}
$$

For the second term, by Chapter 2 Corollary 3.3,

$$\langle \nabla_X\nabla_Z X, Z\rangle = X\langle \nabla_Z X, Z\rangle - \langle \nabla_Z X, \nabla_X Z\rangle,$$

where, by Killing equation,

$$\langle \nabla_Z X, Z\rangle = \frac{1}{2}\left(\langle \nabla_Z X, Z\rangle + \langle \nabla_Z X, Z\rangle\right) \equiv 0.$$

So the second term will be

$$\langle \nabla_X\nabla_Z X, Z\rangle = -\langle \nabla_Z X, \nabla_X Z\rangle.$$

For the last term, by the Killing equation and the symmetry of the connection,

$$\left\langle \nabla_{[X,Z]}X, Z\right\rangle = -\langle \nabla_Z X, [X,Z]\rangle = -\langle \nabla_Z X, \nabla_X Z - \nabla_Z X\rangle = -\langle \nabla_Z X, \nabla_X Z\rangle + \langle \nabla_Z X, \nabla_Z X\rangle.$$

Under the calculation above, S will reduce to

$$
\begin{aligned}
S &= -\langle \nabla_X X, \nabla_Z Z\rangle - \langle \nabla_X\nabla_Z X, Z\rangle + \left\langle \nabla_{[X,Z]}X, Z\right\rangle \\
&= -\langle \nabla_X X, \nabla_Z Z\rangle + \langle \nabla_Z X, \nabla_X Z\rangle - \langle \nabla_Z X, \nabla_X Z\rangle + \langle \nabla_Z X, \nabla_Z X\rangle \\
&= -\langle \nabla_X X, \nabla_Z Z\rangle + \langle \nabla_Z X, \nabla_Z X\rangle.
\end{aligned}
$$

For any vector field $Z \in \mathfrak{X}(M)$, by the Killing equation and a),

$$\langle \nabla_X X, Z\rangle_p = -\langle \nabla_Z X, X\rangle_p = -\langle A_X(Z), X\rangle_p = 0.$$

So we must have $\nabla_X X(p) = 0$. Then

$$S(p) = -\langle \nabla_X X, \nabla_Z Z\rangle_p + \langle \nabla_Z X, \nabla_Z X\rangle_p = \langle A_X(Z), A_X(Z)\rangle_p,$$

as desired. □

Exercise 4.3. Let M be a compact Riemannian manifold of even dimension whose sectional curvature is positive. Prove that every Killing field X on M has a singularity (i.e., there exists $p \in M$ such that $X(p) = 0$).

Proof. We still use the notation of Exercise 4.2. Let p be the minimal point of f. Then we have $df_p = 0$. Suppose that $X(p) \neq 0$. We define a linear mapping

$$
\begin{aligned}
A\colon T_pM &\to T_pM \\
y &\mapsto A_X(Y)(p) = \nabla_Y X(p),
\end{aligned}
$$

where Y is an extension of $y \in T_pM$. By the local notion of of affine connection:

$$\nabla_X Y = \sum_k \left(\sum_{ij} x_i y_j \Gamma_{ij}^k + X(y_k)\right) X_k,$$

we know that $A(y)$ doesn't depend on the choice of Y: for another extension Y', we have

$$
\begin{aligned}
\nabla_{Y'}X(p) &= \sum_{k=1}^n \left(\sum_{i,j=1}^n y_i'(p)x_j(p)\Gamma_{ij}^k(p) + \left(\sum_{l=1}^n y_l'(p)X_l(p)\right)(x_k)\right) X_k(p) \\
&= \sum_{k=1}^n \left(\sum_{i,j=1}^n y_i x_j(p)\Gamma_{ij}^k(p) + \left(\sum_{l=1}^n y_l X_l(p)\right)(x_k)\right) X_k(p) \\
&= \sum_{k=1}^n \left(\sum_{i,j=1}^n y_i(p)x_j(p)\Gamma_{ij}^k(p) + \left(\sum_{l=1}^n y_l(p)X_l(p)\right)(x_k)\right) X_k(p) = \nabla_Y X(p).
\end{aligned}
$$

And it's easy to check that A is linear. So our mapping A is well defined.

Let $E \subset T_pM$ be the linear subspace which is orthogonal to $X(p)$. Then let's study $A|_E$. Firstly, for any $y \in E$, by Exercise 4.2 a),

$$\langle A(y), X\rangle_p = \langle \nabla_Y X, X\rangle_p = \langle A_X(Y), X\rangle_p = 0,$$

which means $A(y)$ is orthogonal to $X(p)$ and then $A(y) \in E$. So we can consider the map $A\colon E \to E$. Suppose that $A(y) = 0$. If $y \neq 0$, by Exercise 4.2 b),

$$0 = \langle A(y), A(y)\rangle_p = \langle A_X(Y), A_X(Y)\rangle_p = \frac{1}{2} Y_p(Y\langle X, X\rangle) + \langle R(X,Y)X, Y\rangle_p .$$

Let's estimate the right side. Choose a coordinates $(x_1, \cdots, x_n)$ of p and suppose $y = (y_1, \cdots, y_n)$. Because we have proved that $A(y)$ is independent of the choice of extension Y, there is no harm in choosing $Y = \sum_{i=1}^n y_i \frac{\partial}{\partial x_i}$ with constants coefficients. Then we calculate

$$\begin{aligned} Y_p(Y\langle X, X\rangle) = Y_p(Y(f)) &= \left(\sum_{i=1}^n y_i \left.\frac{\partial}{\partial x_i}\right|_p\right)\left(\left(\sum_{j=1}^n y_j \frac{\partial}{\partial x_j}\right)(f)\right) = \left(\sum_{i=1}^n y_i \left.\frac{\partial}{\partial x_i}\right|_p\right)\left(\sum_{j=1}^n y_j \frac{\partial f}{\partial x_j}\right) \\ &= \sum_{i,j=1}^n y_i \left.\frac{\partial}{\partial x_i}\right|_p \left(y_j \frac{\partial f}{\partial x_j}\right) = \sum_{i,j=1}^n y_i y_j \left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_p = y \cdot \mathrm{Hess}_p(f) \cdot y^T \geq 0, \end{aligned}$$

where the last inequality is from the fact that p is the minimal point of f then the Hessian matrix $\mathrm{Hess}_p(f)$ is positive definite. Besides, M has positive sectional curvature, then

$$\begin{aligned} \langle R(X,Y)X, Y\rangle_p &= K_p(X,Y)\,|X(p) \wedge Y(p)|^2 \\ &= K_p(X,Y)\left(|X(p)|^2\,|Y(p)|^2 - \langle X(p), Y(p)\rangle^2\right) \\ &= K_p(X,Y)\,|X(p)|^2\,|Y(p)|^2 > 0. \end{aligned}$$

In a word, we have

$$0 = \frac{1}{2} Y_p(Y\langle X, X\rangle) + \langle R(X,Y)X, Y\rangle_p > 0,$$

which is a contradiction. So we must have $y = 0$ and then A is injective. By linear algebra, we know that $A\colon E \to E$ is an isomorphism. Moreover, for any $y, z \in T_pM$, by the Killing equation (Exercise 3.5 d)),

$$\langle A(y), z\rangle = \langle \nabla_Y X, Z\rangle = -\langle \nabla_Z X, Y\rangle = -\langle A(z), y\rangle .$$

This means A is anti-symmetric.

Finally, let's count the dimensions. By definition, E has codimension 1, so we have

$$\dim E = \dim T_pM - 1 = \dim M - 1.$$

On the other hand, because $A\colon E \to E$ is an anti-symmetric isometry, we have $\det A \neq 0$ and

$$\det A = \det(-A^T) = (-1)^{\dim E} \det A^T = (-1)^{\dim E} \det A.$$

So we must have $(-1)^{\dim E} = 1$, i.e. $\dim E$ is even. Then $\dim M = \dim E + 1$ is odd, which is contradict to our condition that M has even dimension. So we must have $X(p) = 0$, i.e. X has a singularity. □

Exercise 4.4. Let M be a Riemannian manifold with the following property: given any two points $p, q \in M$, the parallel transport from p to q does not depend on the curve that joint p to q. Prove that the curvature of M is identically zero, that is, for all $X, Y, Z \in \mathfrak{X}(M)$, $R(X,Y)Z = 0$.

Proof. Let's consider (define) a parametrized surface $f\colon U \to M$ where

$$U = \{\,(s,t) \in \mathbb{R}^2 \big| -\varepsilon < t < 1+\varepsilon, -\varepsilon < s < 1+\varepsilon, \varepsilon > 0\} \subset \mathbb{R}^2$$

and $f(s,0) = f(0,0)$ for all $-\varepsilon < s < 1+\varepsilon$. For any $V_0 \in T_{f(0,0)}M$, we define a vector field on f as follow: if $t = 0$, let $V(s,0) = V(f(s,0)) = V(f(0,0)) = V_0$ for any s; if $t \neq 0$, let $V(s,t) = V(f(s,t))$ be the parallel transport of V_0 along the curve $t \mapsto f(s,t)$ from $f(s,0) = f(0,0)$ to $f(s,t)$, for any s. By Chapter 4 Lemma 4.1, we have

$$\frac{D}{dt}\frac{D}{ds}V - \frac{D}{ds}\frac{D}{dt}V = R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)V.$$

Because $V(s,t)$ is constructed by the parallel transport along $t \mapsto f(s,t)$, we have $\frac{DV}{dt} = 0$. On the other hand, our property says for any s, $V(s,1)$ also can be generated by parallel transport along the curve $t \mapsto f(0,t)$ from $f(0,0) = f(s,0)$ to $f(0,1)$ and the curve $s \mapsto f(s,1)$ from $f(0,1)$ to $f(s,1)$. Then we have $\frac{DV}{ds}(s,1) = 0$, for any s. And then, $\frac{D}{dt}\frac{D}{ds}V(0,1) = 0$. So the equation above reduces to

$$R_{f(0,1)}\left(\frac{\partial f}{\partial t}(0,1), \frac{\partial f}{\partial s}(0,1)\right)V(0,1) = 0.$$

For any $p \in M$ and $X, Y, Z \in \mathfrak{X}(M)$, we can construct a parametrized surface f such that $\frac{\partial f}{\partial t}(0,1) = X(f(0,1))$, $\frac{\partial f}{\partial s}(0,1) = Y(f(0,1))$, $V(0,1) = V_0 = Z(f(0,1))$ and $f(0,1) = p$ (see Distribution Theory). So by arbitrariness, we must have that the curvature R is identically zero, as desired. $\square$

Exercise 4.5. Let $\gamma\colon [0,\ell] \to M$ be a geodesic and let $X \in \mathfrak{X}(M)$ be such that $X(\gamma(0)) = 0$. Show that

$$\nabla_{\gamma'}(R(\gamma', X)\gamma')(0) = (R(\gamma', X')\gamma')(0),$$

where $X' = \frac{DX}{dt}$.

Proof. Firstly, we need a lemma:

Lemma 2. For a tensor T of order r, if there exists $x_i = 0$, then

$$T(x_1, \cdots, x_r) = 0.$$

Denote $X(t) = X(\gamma(t))$ for all vector fields $X \in \mathfrak{X}(M)$. By Lemma 2, because $X(0) = X(\gamma(0)) = 0$ and ∇R is a tensor of order 5, by definition, at $t = 0$, for any vector field $Z \in \mathfrak{X}(M)$, we have

$$\nabla_{\gamma'}(\gamma', X, \gamma', Z)(0) = \nabla(\gamma', X, \gamma', Z, \gamma')(0) = \nabla(\gamma', 0, \gamma', Z, \gamma')(0) = 0.$$

On the other hand, by definition,

$$\begin{aligned}\nabla_{\gamma'}(\gamma', X, \gamma', Z)(0) &= \nabla(\gamma', X, \gamma', Z, \gamma')(0)\\ &= \gamma'\left(R(\gamma', X, \gamma', Z)\right)(0) - R(\nabla_{\gamma'}\gamma', X, \gamma', Z)(0) - R(\gamma', \nabla_{\gamma'}X, \gamma', Z)(0)\\ &\quad - R(\gamma', X, \nabla_{\gamma'}\gamma', Z)(0) - R(\gamma', X, \gamma', \nabla_{\gamma'}Z)(0).\end{aligned}$$

Because γ is a geodesic, we have $\nabla_{\gamma'}\gamma' = \frac{D}{dt}\gamma' = \frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0$. Then by Lemma 2, the 2nd and 4th terms will be zero. And because $X(0) = 0$, by Lemma 2 again, the 5th will also be zero. So we have

$$0 = \nabla_{\gamma'}(\gamma', X, \gamma', Z)(0) = \gamma'\left(R(\gamma', X, \gamma', Z)\right)(0) - R(\gamma', \nabla_{\gamma'}X, \gamma', Z)(0).$$

In the first term, by definition and Chapter 2 Corollary 3.3,

$$\begin{aligned}\gamma'\left(R(\gamma', X, \gamma', Z)\right)(0) &= \gamma'\left(\langle R(\gamma', X)\gamma', Z)\rangle\right)(0)\\ &= \langle \nabla_{\gamma'}R(\gamma', X)\gamma', Z\rangle(0) + \langle R(\gamma', X)\gamma', \nabla_{\gamma'}Z\rangle(0)\\ &= \langle \nabla_{\gamma'}R(\gamma', X)\gamma', Z\rangle(0) + R(\gamma', X, \gamma', \nabla_{\gamma'}Z)(0)\\ &= \langle \nabla_{\gamma'}R(\gamma', X)\gamma', Z\rangle(0),\end{aligned}$$

where the last equality is because $X(0) = 0$ and Lemma 2. And because

$$\nabla_{\gamma'}X = \frac{D}{dt}X = X',$$

by definition, the last term will be

$$R(\gamma', \nabla_{\gamma'}X, \gamma', Z)(0) = R(\gamma', X', \gamma', Z)(0) = \langle R(\gamma', X')\gamma', Z\rangle(0).$$

Above all, we have

$$0 = \langle \nabla_{\gamma'} R(\gamma', X)\gamma', Z\rangle (0) - \langle R(\gamma', X')\gamma', Z\rangle (0),$$

i.e.

$$\langle \nabla_{\gamma'} R(\gamma', X)\gamma', Z\rangle (0) = \langle R(\gamma', X')\gamma', Z\rangle (0).$$

By the arbitrariness of $Z \in \mathfrak{X}(M)$ and the completeness of Riemannian metric $\langle\ ,\ \rangle$, we must have

$$\nabla_{\gamma'} R(\gamma', X)\gamma'(0) = (R(\gamma', X')\gamma')(0),$$

as desired.

Proof of Lemma 2. Calculate directly:

$$T(x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n) = T(x_1, \cdots, x_{i-1}, 2 \cdot 0, x_{i+1}, \cdots, x_n) = 2 \cdot T(x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n).$$

So we must have

$$T(x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n) = 0$$

. □

□

Exercise 4.6 (Locally symmetric spaces)**.** Let M be a Riemannian manifold. M is a *locally symmetric space* if $\nabla R = 0$, where R is the curvature tensor of M. (The geometric significance of this condition will be given in Exercise 8.14).

a) Let M be a locally symmetric space and let $\gamma\colon [0, \ell) \to M$ be a geodesic of M. Let X, Y, Z be parallel vector fields along γ. Prove that $R(X, Y)Z$ is a parallel vector field along γ.

b) Prove that if M is a locally symmetric, connected, and has dimension two, then M has constant sectional curvature.

c) Prove that if M has constant (sectional) curvature, then M is a locally symmetric space.

Proof. For a), for any vector field $W \in \mathfrak{X}(M)$, by definition, and because X, Y, Z are parallel along γ and then $\nabla_{\gamma'} X, \nabla_{\gamma'} Y, \nabla_{\gamma'} Z = 0$, by Lemma 2, and by Chapter 2 Corollary 3.3, we have

$$\begin{aligned}
0 &= \nabla R(X, Y, Z, W, \gamma') \\
&= \gamma' (R(X, Y, Z, W)) - R(\nabla_{\gamma'} X, Y, Z, W) - R(X, \nabla_{\gamma'} Y, Z, W) \\
&\quad - R(X, Y, \nabla_{\gamma'} Z, W) - R(X, Y, Z, \nabla_{\gamma'} W) \\
&= \gamma' (R(X, Y, Z, W)) - R(X, Y, Z, \nabla_{\gamma'} W) \\
&= \gamma' (\langle R(X, Y)Z, W\rangle) - \langle R(X, Y)Z, \nabla_{\gamma'} W\rangle \\
&= \langle \nabla_{\gamma'} (R(X, Y)Z), W\rangle .
\end{aligned}$$

By the arbitrariness of $W \in \mathfrak{X}(M)$ and the completeness of Riemannian metric $\langle\ ,\ \rangle$, we must have

$$\nabla_{\gamma'} (R(X, Y)Z) = 0.$$

Then

$$\frac{D}{dt} (R(X, Y)Z) = \nabla_{\gamma'} (R(X, Y)Z) = 0,$$

i.e. $R(X, Y)Z$ is parallel along γ.

For b), because M has dimension 2, it has only one sectional curvature $K_p = K_p(T_pM)$ at any $p \in M$. For any $p \neq q \in M$, let γ be the geodesic connected p and q. Choose orthonormal vectors $x, y \in T_pM$ and extend them to X, Y by parallel transports along γ such that X, Y are geodesic frame at p (see Exercise 3.7). Then we have

$$K_p = K_p(X(p), Y(p)) = \frac{R_p(X(p), Y(p), X(p), Y(p))}{|X(p)|^2 \, |Y(p)|^2 - \langle X(p), Y(p)\rangle^2} = R_p(X(p), Y(p), X(p), Y(p)), \quad \forall p \in \gamma.$$

By a), $R(X,Y)X$ is also parallel along γ, then by the definition of compatible connection (Chapter 2 Definition 3.1), we have
$$R(X,Y,X,Y) = \langle R(X,Y)X, Y\rangle = \text{constant}$$
along γ. Then
$$K_p = R_p(X(p),Y(p),X(p),Y(p)) = \text{constant} = R_q(X(q),Y(q),X(q),Y(q)) = K_q.$$
By the arbitrariness of p and q on a connected manifold M, we know that K_p is constant of $p \in M$, i.e. M has constant sectional curvature, as desired.

Finally, if M has constant sectional curvature, by Chapter 4 Lemma 3.4, there exists $K_0 \in \mathbb{R}$ such that for any vector fields $X,Y,Z,W \in \mathfrak{X}(M)$,
$$R(X,Y,Z,W) = K_0\left(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle\right).$$
Denote
$$R'(X,Y,Z,W) = \langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle.$$
It is easy to check that R' is a tensor of order 4. By, definition, It's easy to check that the rules of covariant differential and derivative of tensors are same as the case of vector fields. Then we have
$$\nabla R = \nabla(K_0R') = K_0\nabla R'.$$
For any vector fields $X,Y,Z,W,T \in \mathfrak{X}(M)$,
$$\begin{aligned}\nabla R'(X,Y,Z,W,T) &= T\left(R'(X,Y,Z,W)\right) - R'(\nabla_T X,Y,Z,W) - R'(X,\nabla_T Y,Z,W)\\ &\quad - R'(X,Y,\nabla_T Z,W) - R'(X,Y,Z,\nabla_T W)\\ &= T(\langle X,Z\rangle\langle Y,W\rangle) - T(\langle Y,Z\rangle\langle X,W\rangle)\\ &\quad - \langle\nabla_T X,Z\rangle\langle Y,W\rangle + \langle Y,Z\rangle\langle\nabla_T X,W\rangle\\ &\quad - \langle X,Z\rangle\langle\nabla_T Y,W\rangle + \langle\nabla_T Y,Z\rangle\langle X,W\rangle\\ &\quad - \langle X,\nabla_T Z\rangle\langle Y,W\rangle + \langle Y,\nabla_T Z\rangle\langle X,W\rangle\\ &\quad - \langle X,Z\rangle\langle Y,\nabla_T W\rangle + \langle Y,Z\rangle\langle X,\nabla_T W\rangle.\end{aligned}$$
By Chapter 2 Corollary 3.3, we have
$$\begin{aligned}\nabla R'(X,Y,Z,W,T) &= T(\langle X,Z\rangle\langle Y,W\rangle) - T(\langle Y,Z\rangle\langle X,W\rangle)\\ &\quad - \langle\nabla_T X,Z\rangle\langle Y,W\rangle + \langle Y,Z\rangle\langle\nabla_T X,W\rangle\\ &\quad - \langle X,Z\rangle\langle\nabla_T Y,W\rangle + \langle\nabla_T Y,Z\rangle\langle X,W\rangle\\ &\quad - \langle X,\nabla_T Z\rangle\langle Y,W\rangle + \langle Y,\nabla_T Z\rangle\langle X,W\rangle\\ &\quad - \langle X,Z\rangle\langle Y,\nabla_T W\rangle + \langle Y,Z\rangle\langle X,\nabla_T W\rangle.\\ &= \langle Y,W\rangle T(\langle X,Z\rangle) + \langle X,Z\rangle T(\langle Y,W\rangle) - \langle X,W\rangle T(\langle Y,Z\rangle) - \langle Y,Z\rangle T(\langle X,W\rangle)\\ &\quad - \langle Y,W\rangle T(\langle X,Z\rangle) + \langle X,\nabla_T Z\rangle\langle Y,W\rangle + \langle Y,Z\rangle T(\langle X,W\rangle) - \langle Y,Z\rangle\langle X,\nabla_T W\rangle\\ &\quad - \langle X,Z\rangle T(\langle Y,W\rangle) + \langle X,Z\rangle\langle Y,\nabla_T W\rangle + \langle X,W\rangle T(\langle Y,Z\rangle) - \langle X,W\rangle\langle Y,\nabla_T Z\rangle\\ &\quad - \langle X,\nabla_T Z\rangle\langle Y,W\rangle + \langle Y,\nabla_T Z\rangle\langle X,W\rangle - \langle X,Z\rangle\langle Y,\nabla_T W\rangle + \langle Y,Z\rangle\langle X,\nabla_T W\rangle\\ &= \langle Y,W\rangle T(\langle X,Z\rangle) + \langle X,Z\rangle T(\langle Y,W\rangle) - \langle X,W\rangle T(\langle Y,Z\rangle) - \langle Y,Z\rangle T(\langle X,W\rangle)\\ &\quad - \langle Y,W\rangle T(\langle X,Z\rangle) + \langle X,\nabla_T Z\rangle\langle Y,W\rangle + \langle Y,Z\rangle T(\langle X,W\rangle) - \langle Y,Z\rangle\langle X,\nabla_T W\rangle\\ &\quad - \langle X,Z\rangle T(\langle Y,W\rangle) + \langle X,Z\rangle\langle Y,\nabla_T W\rangle + \langle X,W\rangle T(\langle Y,Z\rangle) - \langle X,W\rangle\langle Y,\nabla_T Z\rangle\\ &\quad - \langle X,\nabla_T Z\rangle\langle Y,W\rangle + \langle Y,\nabla_T Z\rangle\langle X,W\rangle - \langle X,Z\rangle\langle Y,\nabla_T W\rangle + \langle Y,Z\rangle\langle X,\nabla_T W\rangle\\ &= 0.\end{aligned}$$
Then we must have $\nabla R' = 0$. So
$$\nabla R = K_0\nabla R' = 0,$$
i.e. M is a locally symmetric space; whence c). □

Exercise 4.7. Prove the *2nd Bianchi Identity*:

$$\nabla R(X,Y,Z,W,T)+\nabla R(X,Y,W,T,Z)+\nabla R(X,Y,T,Z,W)=0$$

for all $X,Y,Z,W,T\in\mathfrak{X}(M)$.

Proof. At any $p\in M$, we can choose a geodesic frame $e_1,\cdots,e_n$ of M (see Exercise 3.7). By linearity, we only need to prove it for $X,Y,Z,W,T\in\{e_1,\cdots,e_n\}$. Because $\nabla_{e_i}e_j(p)=0$, $\forall 1\le i,j\le n$, by definition and by Lemma 2, at p, we have

$$\begin{aligned}\nabla R(e_i,e_j,e_k,e_l,e_h)(p)&=e_hR(e_i,e_j,e_k,e_l)(p)-R(\nabla_{e_h}e_i,e_j,e_k,e_l)(p)-R(e_i,\nabla_{e_h}e_j,e_k,e_l)(p)\\&\quad-R(e_i,e_j,\nabla_{e_h}e_k,e_l)(p)-R(e_i,e_j,e_k,\nabla_{e_h}e_l)(p)\\&=e_hR(e_i,e_j,e_k,e_l)(p)-R(0,e_j,e_k,e_l)-R(e_i,0,e_k,e_l)\\&\quad-R(e_i,e_j,0,e_l)-R(e_i,e_j,e_k,0)\\&=e_hR(e_i,e_j,e_k,e_l)(p).\end{aligned}$$

By Chapter 4 Proposition 2.5 (d), Chapter 2 Corollary 3.3 and definition, we have

$$\begin{aligned}e_hR(e_i,e_j,e_k,e_l)(p)&=e_hR(e_k,e_l,e_i,e_j)(p)\\&=e_h\left\langle R(e_k,e_l)e_i,e_j\right\rangle(p)\\&=\left\langle\nabla_{e_h}(R(e_k,e_l)e_i),e_j\right\rangle_p+\left\langle R(e_k,e_l)e_i,\nabla_{e_h}e_j\right\rangle_p\\&=\left\langle\nabla_{e_h}(R(e_k,e_l)e_i),e_j\right\rangle_p+\left\langle R(e_k,e_l)e_i,0\right\rangle_p\\&=\left\langle\nabla_{e_h}(R(e_k,e_l)e_i),e_j\right\rangle_p\\&=\left\langle\nabla_{e_h}\nabla_{e_l}\nabla_{e_k}e_i-\nabla_{e_h}\nabla_{e_k}\nabla_{e_l}e_i+\nabla_{e_h}\nabla_{[e_k,e_l]}e_i,e_j\right\rangle_p\\&=\left\langle\nabla_{e_h}\nabla_{e_l}\nabla_{e_k}e_i,e_j\right\rangle_p-\left\langle\nabla_{e_h}\nabla_{e_k}\nabla_{e_l}e_i,e_j\right\rangle_p+\left\langle\nabla_{e_h}\nabla_{[e_k,e_l]}e_i,e_j\right\rangle_p.\end{aligned}$$

For the last term, by the definition of curvature R, we have

$$\begin{aligned}\left\langle\nabla_{e_h}\nabla_{[e_k,e_l]}e_i,e_j\right\rangle_p&=\left\langle R([e_k,e_l],e_h)e_i+\nabla_{[e_k,e_l]}\nabla_{e_h}e_i-\nabla_{[[e_k,e_l],e_h]}e_i,e_j\right\rangle_p\\&=\left\langle R([e_k,e_l],e_h)e_i,e_j\right\rangle_p+\left\langle\nabla_{[e_k,e_l]}\nabla_{e_h}e_i,e_j\right\rangle_p-\left\langle\nabla_{[[e_k,e_l],e_h]}e_i,e_j\right\rangle_p.\end{aligned}$$

By the symmetry of Riemannian metric,

$$[e_k,e_l](p)=\nabla_{e_k}e_l(p)-\nabla_{e_l}e_k(p)=0-0=0.$$

Then by Lemma 2, the first term will be

$$\left\langle R([e_k,e_l],e_h)e_i,e_j\right\rangle_p=R([e_k,e_l],e_h,e_i,e_j)(p)=R(0,e_h,e_i,e_j)(p)=0.$$

Above all, we have

$$\begin{aligned}&\nabla R(e_i,e_j,e_k,e_l,e_h)(p)=e_hR(e_i,e_j,e_k,e_l)(p)\\&=\left\langle\nabla_{e_h}\nabla_{e_l}\nabla_{e_k}e_i,e_j\right\rangle_p-\left\langle\nabla_{e_h}\nabla_{e_k}\nabla_{e_l}e_i,e_j\right\rangle_p+\left\langle\nabla_{e_h}\nabla_{[e_k,e_l]}e_i,e_j\right\rangle_p\\&=\left\langle\nabla_{e_h}\nabla_{e_l}\nabla_{e_k}e_i,e_j\right\rangle_p-\left\langle\nabla_{e_h}\nabla_{e_k}\nabla_{e_l}e_i,e_j\right\rangle_p\\&\quad+\left\langle R([e_k,e_l],e_h)e_i,e_j\right\rangle_p+\left\langle\nabla_{[e_k,e_l]}\nabla_{e_h}e_i,e_j\right\rangle_p-\left\langle\nabla_{[[e_k,e_l],e_h]}e_i,e_j\right\rangle_p\\&=\left\langle\nabla_{e_h}\nabla_{e_l}\nabla_{e_k}e_i,e_j\right\rangle_p-\left\langle\nabla_{e_h}\nabla_{e_k}\nabla_{e_l}e_i,e_j\right\rangle_p+\left\langle\nabla_{[e_k,e_l]}\nabla_{e_h}e_i,e_j\right\rangle_p-\left\langle\nabla_{[[e_k,e_l],e_h]}e_i,e_j\right\rangle_p.\end{aligned}$$

Permute k, l, h, we have

$$
\begin{aligned}
&\nabla R(e_i, e_j, e_k, e_l, e_h)(p) + \nabla R(e_i, e_j, e_l, e_h, e_k)(p) + \nabla R(e_i, e_j, e_h, e_k, e_l)(p) \\
=& \langle \nabla_{e_h}\nabla_{e_l}\nabla_{e_k} e_i, e_j\rangle_p - \langle \nabla_{e_h}\nabla_{e_k}\nabla_{e_l} e_i, e_j\rangle_p + \left\langle \nabla_{[e_k,e_l]}\nabla_{e_h} e_i, e_j\right\rangle_p - \left\langle \nabla_{[[e_k,e_l],e_h]} e_i, e_j\right\rangle_p \\
&+ \langle \nabla_{e_k}\nabla_{e_h}\nabla_{e_l} e_i, e_j\rangle_p - \langle \nabla_{e_k}\nabla_{e_l}\nabla_{e_h} e_i, e_j\rangle_p + \left\langle \nabla_{[e_l,e_h]}\nabla_{e_k} e_i, e_j\right\rangle_p - \left\langle \nabla_{[[e_l,e_h],e_k]} e_i, e_j\right\rangle_p \\
&+ \langle \nabla_{e_l}\nabla_{e_k}\nabla_{e_h} e_i, e_j\rangle_p - \langle \nabla_{e_l}\nabla_{e_h}\nabla_{e_k} e_i, e_j\rangle_p + \left\langle \nabla_{[e_h,e_k]}\nabla_{e_l} e_i, e_j\right\rangle_p - \left\langle \nabla_{[[e_h,e_k],e_l]} e_i, e_j\right\rangle_p \\
=& \left\langle \left(\nabla_{e_h}\nabla_{e_l} - \nabla_{e_l}\nabla_{e_h} + \nabla_{[e_l,e_h]}\right)\nabla_{e_k} e_i, e_j\right\rangle_p \\
&+ \left\langle \left(\nabla_{e_k}\nabla_{e_h} - \nabla_{e_h}\nabla_{e_k} + \nabla_{[e_h,e_k]}\right)\nabla_{e_l} e_i, e_j\right\rangle_p \\
&+ \left\langle \left(\nabla_{e_l}\nabla_{e_k} - \nabla_{e_k}\nabla_{e_l} + \nabla_{[e_k,e_l]}\right)\nabla_{e_h} e_i, e_j\right\rangle_p \\
&- \left\langle \nabla_{[[e_k,e_l],e_h]+[[e_l,e_h],e_k]+[[e_h,e_k],e_l]} e_i, e_j\right\rangle_p \\
=& \langle R(e_l, e_h)\nabla_{e_k} e_i, e_j\rangle_p + \langle R(e_h, e_k)\nabla_{e_l} e_i, e_j\rangle_p + \langle R(e_k, e_l)\nabla_{e_h} e_i, e_j\rangle_p \\
&- \left\langle \nabla_{[[e_k,e_l],e_h]+[[e_l,e_h],e_k]+[[e_h,e_k],e_l]} e_i, e_j\right\rangle_p \\
=& R(e_l, e_h, \nabla_{e_k} e_i, e_j)(p) + R(e_h, e_k, \nabla_{e_l} e_i, e_j)(p) + R(e_k, e_l, \nabla_{e_h} e_i, e_j)(p) \\
&- \left\langle \nabla_{[[e_k,e_l],e_h]+[[e_l,e_h],e_k]+[[e_h,e_k],e_l]} e_i, e_j\right\rangle_p .
\end{aligned}
$$

Because $e_1, \cdots, e_n$ is a geodesic frame at p, we have $\nabla_{e_i} e_j(p) = 0$, $\forall 1 \le i, j \le n$. So the first three tensors will be zeros, by Lemma 2. And by the Jacobi identity of vector field, we have

$$[[e_k, e_l], e_h] + [[e_l, e_h], e_k] + [[e_h, e_k], e_l] = 0.$$

So the last term is also zero. Finally, we have

$$\nabla R(e_i, e_j, e_k, e_l, e_h)(p) + \nabla R(e_i, e_j, e_l, e_h, e_k)(p) + \nabla R(e_i, e_j, e_h, e_k, e_l)(p) = 0.$$

By the arbitrariness of $p \in M$ and linearity, we finish our proof. □

Exercise 4.8 (Schur's Theorem)**.** Let M^n be a connected Riemannian manifold with $n \ge 3$. Suppose that M is *isotropic*, that is, for each $p \in M$, the sectional curvature $K(p, \sigma)$ does not depend on $\sigma \subset T_pM$. Prove that M has constant sectional curvature, that is, $K(p, \sigma)$ also does not depend on p.

Proof. Firstly, we define a tensor R' of order 4: for any $X, Y, Z, W \in \mathfrak{X}(M)$, let

$$R'(X, Y, Z, W) = \langle X, Z\rangle \langle Y, W\rangle - \langle Y, Z\rangle \langle X, W\rangle .$$

At any $p \in M$, because $K(p, \sigma)$ doesn't depend on σ, denoting the curvature function by $K(p)$, by Chapter 4 Lemma 3.4, we have

$$R = KR'.$$

By, definition, It's easy to check that the rules of covariant differential and derivative of tensors are same as the case of vector fields. Then for any $U \in \mathfrak{X}(M)$,

$$\nabla_U R = \nabla_U (KR') = K\nabla_U R' + U(K)R'.$$

For the first term, by the proof of Exerxise 4.6 c), we know $\nabla R' = 0$. Then for any $X, Y, Z, W \in \mathfrak{X}(M)$,

$$\nabla_U R'(X, Y, Z, W) = \nabla R'(X, Y, Z, W, U) = 0.$$

So the first term will vanish. Using the 2nd Bianchi identity (Exercise 4.7), for any $X, Y, Z, W, U \in \mathfrak{X}(M)$, we have

$$
\begin{aligned}
0 &= \nabla R(X, Y, Z, W, U) + \nabla R(X, Y, W, U, Z) + \nabla R(X, Y, U, Z, W) \\
&= \nabla_U R(X, Y, Z, W) + \nabla_Z R(X, Y, W, U) + \nabla_W R(X, Y, U, Z) \\
&= U(K)R'(X, Y, Z, W) + U(Z)R'(X, Y, W, U) + U(W)R'(X, Y, U, Z) \\
&= U(K)\left(\langle X, Z\rangle \langle Y, W\rangle - \langle Y, Z\rangle \langle X, W\rangle\right) \\
&\quad + Z(K)\left(\langle X, W\rangle \langle Y, U\rangle - \langle Y, W\rangle \langle X, U\rangle\right) \\
&\quad + W(K)\left(\langle X, U\rangle \langle Y, Z\rangle - \langle Y, U\rangle \langle X, Z\rangle\right).
\end{aligned}
$$

At p, because $n \geq 3$, we can extend $Z(p)$ to a orthogonal basis of T_pM (for example, the geodesic frame at p, except for the condition of $|Z(p)| = 1$). Let W, Y be other two different elements of the basis and $U = Y$. Then at p, the equation above will be

$$\begin{aligned}
0 &= U(K)\left(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle\right)\\
&\quad + Z(K)\left(\langle X,W\rangle\langle Y,U\rangle - \langle Y,W\rangle\langle X,U\rangle\right)\\
&\quad + W(K)\left(\langle X,U\rangle\langle Y,Z\rangle - \langle Y,U\rangle\langle X,Z\rangle\right)\\
&= Y(K)\left(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle\right)\\
&\quad + Z(K)\left(\langle X,W\rangle\langle Y,Y\rangle - \langle Y,W\rangle\langle X,Y\rangle\right)\\
&\quad + W(K)\left(\langle X,Y\rangle\langle Y,Z\rangle - \langle Y,Y\rangle\langle X,Z\rangle\right)\\
&= Y(K)\left(\langle X,Z\rangle\cdot 0 - 0\cdot\langle X,W\rangle\right)\\
&\quad + Z(K)\left(\langle X,W\rangle\cdot 1 - 0\cdot\langle X,Y\rangle\right)\\
&\quad + W(K)\left(\langle X,Y\rangle\cdot 0 - 1\cdot\langle X,Z\rangle\right)\\
&= Z(K)\langle X,W\rangle - W(K)\langle X,Z\rangle\\
&= \langle X, Z(K)W - W(K)Z\rangle .
\end{aligned}$$

By the arbitrariness of X and the completion of Riemannian metric $\langle\ ,\ \rangle$, we must have

$$Z(K)W - W(K)Z = 0.$$

But W and Z are different elements of a basis, then they are linear independent, at p. So we must have

$$Z(K) = 0.$$

By the arbitrariness of Z, we must have

$$K = \text{constant},$$

i.e. $K(p,\sigma) = K$ also doesn't depend on p, as desired. □

Exercise 4.9. Prove that the scalar curvature $K(p)$ at $p \in M$ is given by

$$K(p) = \frac{1}{\omega_{n-1}}\int_{S^{n-1}} \operatorname{Ric}_p(x)\,\mathrm{d}S^{n-1},$$

where ω_{n-1} is the area of the sphere S^{n-1} in T_pM and dS^{n-1} is the area elements on S^{n-1}.

Proof. We use the intrinsic definiton of Ricci curvature:

$$\operatorname{Ric}_p(x) = \frac{1}{n-1}Q(x,x)$$

where

$$Q(x,y) = \text{trace of the mapping } z \mapsto R(x,z)y$$

is a quadratic form. By linear algebra, after suitable basis change, we can choose a orthonormal basis $e_1, \cdots, e_n \in T_pM$, such that for any $x = \sum_{i=1}^n x_i e_i$ and $y = \sum_{j=1}^n y_i e_i$,

$$Q(x,y) = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}\begin{pmatrix} y_1\\ y_2\\ \vdots\\ y_n \end{pmatrix} = \sum_{i=1}^n \lambda_i x_i y_i,$$

where $\lambda_1, \cdots, \lambda_n$ are eigenvalues of Q, which are all real numbers. Then for any unit vector $x = \sum_{i=1}^n x_i e_i$,

$$\operatorname{Ric}_p(x) = \frac{1}{n-1}Q(x,x) = \frac{1}{n-1}\sum_{i=1}^n \lambda_i x_i^2.$$

For $\nu = (x_1, \cdots, x_n) \in S^{n-1}$, denoting $V = (\lambda_1 x_1, \cdots, \lambda_n x_n)$, using Divergence Theorem, we have

$$\int_{S^{n-1}} \operatorname{Ric}_p(x) \, \mathrm{d}S^{n-1} = \int_{S^{n-1}} \left(\frac{1}{n-1} \sum_{i=1}^{n} \lambda_i x_i^2 \right) \mathrm{d}S^{n-1} = \frac{1}{n-1} \int_{\partial B^n} \langle V, \nu \rangle \, \mathrm{d}S^{n-1} = \frac{1}{n-1} \int_{B^n} \operatorname{div} V \, \mathrm{d}B^n.$$

Using Exercise 3.8, for $V = \sum_{i=1}^n \lambda_i x_i e_i$, where $e_1, \cdots, e_n$ is the geodesic frame generated by $e_1, \cdots, e_n \in T_pM$ (see Exercise 3.7), we have

$$\operatorname{div} V = \sum_{i=1}^{n} e_i(\lambda_i x_i) = \sum_{i=1}^{n} \lambda_i.$$

Then

$$\int_{B^n} \operatorname{div} V \, \mathrm{d}B^n = \int_{B^n} \left(\sum_{i=1}^{n} \lambda_i \right) \mathrm{d}B^n = \left(\sum_{i=1}^{n} \lambda_i \right) \int_{B^n} \mathrm{d}B^n = \operatorname{vol}(B^n) \sum_{i=1}^{n} \lambda_i.$$

Notice that

$$\operatorname{vol}(B^n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} = \frac{1}{n} \cdot \frac{n \cdot \pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} = \frac{1}{n} w_{n-1}$$

and for any $1 \le i \le n$,

$$\lambda_i = Q(e_i, e_i) = (n-1) \operatorname{Ric}_p(e_i).$$

Above all, we have

$$\begin{aligned} \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \operatorname{Ric}_p(x) \, \mathrm{d}S^{n-1} &= \frac{1}{(n-1)\omega_{n-1}} \int_{B^n} \operatorname{div} V \, \mathrm{d}B^n = \frac{\operatorname{vol}(B^n)}{(n-1)\omega_{n-1}} \sum_{i=1}^{n} \lambda_i \\ &= \frac{1}{n(n-1)} \sum_{i=1}^{n} \lambda_i = \frac{1}{n(n-1)} \sum_{i=1}^{n} ((n-1) \operatorname{Ric}_p(e_i)) \\ &= \frac{1}{n} \sum_{i=1}^{n} \operatorname{Ric}_p(e_i) = K(p), \end{aligned}$$

as desired. □

Exercise 4.10 (Einstein manifolds)**.** A Riemannian manifold M^n is called an *Einstein manifold* if, for all $X, Y \in \mathfrak{X}(M)$, $\operatorname{Ric}(X, Y) = \lambda \langle X, Y \rangle$, where $\lambda \colon M \to \mathbb{R}$ is a real function. Prove that:

a) If M^n is connected and Einstein, with $n \ge 3$, then λ is constant on M.

b) If M^3 is a connected Einstein manifold then M^3 has constant sectional curvature.

Proof. For a), let $e_1, \cdots, e_{n \ge 3}$ be a geodesic frame at any $p \in M$ (see Exercise 3.7). By the 2nd Bianchi identity (Exercise 4.7), for any $1 \le i, j, k, h, s \le n$, we have

$$\begin{aligned} 0 &= \nabla R(e_h, e_i, e_j, e_k, e_s) + \nabla R(e_h, e_i, e_k, e_s, e_j) + \nabla R(e_h, e_i, e_s, e_j, e_k) \\ &= \nabla_{e_s} R(e_h, e_i, e_j, e_k) + \nabla_{e_j} R(e_h, e_i, e_k, e_s) + \nabla_{e_k} R(e_h, e_i, e_s, e_j). \end{aligned}$$

By definition, using the fact that $\nabla_{e_i} e_j(p) = 0$ for any $1 \le i, j \le n$ (because $e_1, \cdots, e_n$ is a geodesic frame) and Lemma 2, we have

$$\begin{aligned} \nabla_{e_s} R(e_h, e_i, e_j, e_k)(p) &= e_s(R(e_h, e_i, e_j, e_k))(p) - R(\nabla_{e_s} e_h, e_i, e_j, e_k)(p) - R(e_h, \nabla_{e_s} e_i, e_j, e_k)(p) \\ &\quad - R(e_h, e_i, \nabla_{e_s} e_j, e_k)(p) - R(e_h, e_i, e_j, \nabla_{e_s} e_k)(p) \\ &= e_s(R_{hijk})(p) - R(0, e_i, e_j, e_k)(p) - R(e_h, 0, e_j, e_k)(p) \\ &\quad - R(e_h, e_i, 0, e_k)(p) - R(e_h, e_i, e_j, 0)(p) \\ &= e_s(R_{hijk})(p) \end{aligned}$$

where

$$R_{hijk} = R(e_h, e_i, e_j, e_k)$$

is the curvature tensor. Permuting j, k, s, we have

$$\begin{aligned} 0 &= \nabla_{e_s} R(e_h, e_i, e_j, e_k)(p) + \nabla_{e_j} R(e_h, e_i, e_k, e_s)(p) + \nabla_{e_k} R(e_h, e_i, e_s, e_j)(p) \\ &= e_s(R_{hijk})(p) + e_j(R_{hiks})(p) + e_k(R_{hisj})(p). \end{aligned} \tag{*}$$

Because $e_1, \cdots, e_n$ is a geodesic frame, we have

$$g_{ij} = \langle e_i, e_j \rangle = \delta_{ij}.$$

Then we also have

$$g^{ij} = \delta^{ij} = \delta_{ij} = g_{ij}.$$

By the symmetry of Riemannian metric $\langle \ , \ \rangle$, for any $1 \le i, j \le n$, $g_{ij}, g_{ji}, g^{ij}, g^{ji}$ are all equal to $\delta_{ij} = \delta_{ji}$. Multiplying (*) by $\delta_{hj}\delta_{ik}$ and summing on i, k, h, j, we have

$$\begin{aligned} 0 &= \sum_{i,k,h,j=1}^{n} \delta_{hj}\delta_{ik} \left(e_s(R_{hijk})(p) + e_j(R_{hiks})(p) + e_k(R_{hisj})(p) \right) \\ &= \sum_{i,k,h,j=1}^{n} \delta_{hj}\delta_{ik} e_s(R_{hijk})(p) + \sum_{i,k,h,j=1}^{n} \delta_{hj}\delta_{ik} e_j(R_{hiks})(p) + \sum_{i,k,h,j=1}^{n} \delta_{hj}\delta_{ik} e_k(R_{hisj})(p). \end{aligned}$$

For the first part, because δ_{hj} and δ_{ik} are all constant, using the relation between Ricci tensor and curvature tensor (see Chapter 4 Section 4 Page 98): for any $1 \le i, k \le n$,

$$\frac{1}{n-1} R_{ik} = \frac{1}{n-1} \sum_j R^j_{ijk} = \frac{1}{n-1} \sum_{sj} R_{ijks} g^{sj},$$

we have

$$\begin{aligned} \sum_{i,k,h,j=1}^{n} \delta_{hj}\delta_{ik} e_s(R_{hijk})(p) &= e_s \left(\sum_{i,k,h,j=1}^{n} \delta_{hj}\delta_{ik} R_{hijk} \right)(p) = e_s \left(\sum_{h,j=1}^{n} \delta_{hj} \left(\sum_{i,k=1}^{n} R_{hijk}\delta_{ki} \right) \right)(p) \\ &= e_s \left(\sum_{j=1}^{n} \delta_{jj} \left(\sum_{i,k=1}^{n} R_{jijk} g^{ki} \right) \right)(p) = e_s \left(\sum_{j=1}^{n} \delta_{jj} R_{jj} \right)(p) \\ &= e_s \left(\sum_{j=1}^{n} \lambda \langle e_j, e_j \rangle \right)(p) = e_s \left(\sum_{j=1}^{n} \lambda \right)(p) = e_s(n\lambda)(p) = n e_s(\lambda)(p). \end{aligned}$$

For the second part, permuting last two indices of R_{hiks} (by Chapter 4 Proposition 2.5 (c)), similarly calculating, we have

$$\begin{aligned} \sum_{i,k,h,j=1}^{n} \delta_{hj}\delta_{ik} e_j(R_{hiks})(p) &= -\sum_{i,k,h,j=1}^{n} \delta_{hj}\delta_{ik} e_j(R_{hisk})(p) = -\sum_{h,j=1}^{n} \left(\delta_{hj} e_j \left(\sum_{i,k=1}^{n} \delta_{ik} R_{hisk} \right) \right)(p) \\ &= -\sum_{j=1}^{n} \left(\delta_{jj} e_j \left(\sum_{i,k=1}^{n} R_{jisk} g^{ki} \right) \right)(p) = -\sum_{j=1}^{n} e_j \left(R_{js} \right)(p) \\ &= -\sum_{j=1}^{n} e_j \left(\lambda \langle e_j, e_s \rangle \right)(p) = -e_s(\lambda \langle e_s, e_s \rangle)(p) = -e_s(\lambda)(p). \end{aligned}$$

For the last part, permuting first two indices of R_{hisj} (also by Chapter 4 Proposition 2.5 (c)), similarly calculating, we have

$$\begin{aligned}\sum_{i,k,h,j=1}^{n}\delta_{hj}\delta_{ik}e_k(R_{hisj})(p) &= -\sum_{i,k,h,j=1}^{n}\delta_{hj}\delta_{ik}e_k(R_{ihsj})(p) = -\sum_{i,k=1}^{n}\delta_{ik}e_k\left(\sum_{h,j=1}^{n}\delta_{hj}R_{ihsj}\right)(p)\\ &= -\sum_{i=1}^{n}\delta_{ii}e_i\left(\sum_{h,j=1}^{n}R_{ihsj}g^{jh}\right)(p) = -\sum_{i=1}^{n}e_i\left(R_{is}\right)(p)\\ &= -\sum_{i=1}^{n}e_i\left(\lambda\left\langle e_i,e_s\right\rangle\right)(p) = -e_s(\lambda\left\langle e_s,e_s\right\rangle)(p) = -e_s(\lambda)(p).\end{aligned}$$

Above all, we have

$$\begin{aligned}0 &= \sum_{i,k,h,j=1}^{n}\delta_{hj}\delta_{ik}e_s(R_{hijk})(p) + \sum_{i,k,h,j=1}^{n}\delta_{hj}\delta_{ik}e_j(R_{hiks})(p) + \sum_{i,k,h,j=1}^{n}\delta_{hj}\delta_{ik}e_k(R_{hisj})(p)\\ &= ne_s(\lambda)(p) - e_s(\lambda)(p) - e_s(\lambda)(p) = (n-2)e_s(\lambda)(p).\end{aligned}$$

Because $n \geq 3$, we must have $e_s(\lambda)(p) = 0$. And by the arbitrariness of $p \in M$, we have $e_s(\lambda) = 0$. Finally, by the arbitrariness of s the connectedness of M, on M, we must have

$$\lambda = \text{constant},$$

as desired.

For the special case $n = 3$, using the same notation of a), by the relation between Ricci tensor and curvature tensor: for any $1 \leq i,k \leq n$,

$$\frac{1}{n-1}R_{ik} = \frac{1}{n-1}\sum_{j}R_{ijk}^{j} = \frac{1}{n-1}\sum_{sj}R_{ijks}g^{sj},$$

again, let $k = i$, we have

$$\lambda = \lambda\left\langle e_i,e_i\right\rangle = R_{ii} = \sum_{s,j=1}^{3}R_{ijis}g^{sj} = \sum_{j=1}^{3}R_{ijij}g^{jj} = \sum_{j=1}^{3}R_{ijij}.$$

By Chapter 4 Proposition 2.5 (b), for any $1 \leq i \leq n$, permuting the first two indices, we have

$$R_{iiii} = -R_{iiii}.$$

Then we must have $R_{iiii} = 0$. And by Chapter 4 Proposition 2.5 (b)(c) simultaneously, permuting the first two indices and the last two indices, we have

$$R_{ijij} = -R_{jiij} = R_{jiji}, \qquad \forall 1 \leq i,j \leq n.$$

Extending them for any $1 \leq i \leq 3$, we have the linear system:

$$\begin{cases}0 + R_{1212} + R_{1313} = \lambda,\\ R_{2121} + 0 + R_{2323} = \lambda,\\ R_{3131} + R_{3232} + 0 = \lambda,\\ R_{1212} = R_{2121},\\ R_{2323} = R_{3232},\\ R_{3131} = R_{1313}.\end{cases}$$

We can solve the system:

$$\begin{cases}R_{1212} = R_{2121} = \frac{\lambda}{2},\\ R_{2323} = R_{3232} = \frac{\lambda}{2},\\ R_{3131} = R_{1313} = \frac{\lambda}{2}.\end{cases}$$

These are all the sectional curvature of M^3. So we have

$$R_{1212} = R_{2121} = R_{2323} = R_{3232} = R_{3131} = R_{1313} = \frac{\lambda}{2},$$

are all constant function $\frac{\lambda}{2}$, i.e. M^3 has constant sectional curvature, whence b). □

5 Jacobi Fields

Exercise 5.1. Let M be a Riemannian manifold with sectional curvature identically zero. Show that, for every $p \in M$, the mapping $\exp_p\colon B_\varepsilon(0) \subset T_pM \to B_\varepsilon(p)$ is an isometry, where $B_\varepsilon(p)$ is a normal ball at p.

Proof. For any $v \in B_\varepsilon(0) \subset T_pM$ and $w \in T_v(B_\varepsilon(0)) \cong \mathbb{R}^n$, we can choose a curva $v(s) \subset B_\varepsilon(0)$ such that $v(0) = v$ and $v'(0) = w$. Let $f(t,s) = \exp_p(tv(s))$, then we know that

$$J(t) = \frac{\partial f}{\partial s}(t,0) = (d\exp_p)_{tv}(tw) = t(d\exp_p)_{tv}(w)$$

is a Jacobi field along the geodesic $\gamma(t) = \exp_p(tv)$. Because the sectional curvature of M is identically zero, by Chapter 5 Example 2.3, we know that

$$\frac{D^2J}{dt^2} = -KJ = 0.$$

And by Chapter 5 Proposition 2.4, we know that

$$\frac{DJ}{dt}(0) = w.$$

We choose a geodesic frame $e_1, \cdots, e_n$ at p (see Exercise 3.7). Then they are orthomormal. Let $J(t) = \sum_{i=1}^n a_i(t)e_i(t)$ and $w = \sum_{i=1}^n w_ie_i$. Because $e_1, \cdots, e_n$ are generated by parallel transport along γ, we have

$$\frac{DJ}{dt} = \sum_{i=1}^n \left(\frac{da_i}{dt}e_i + a_i\frac{De_i}{dt}\right) = \sum_{i=1}^n a_i'e_i$$

and

$$\frac{D^2J}{dt^2} = \sum_{i=1}^n \left(\frac{da_i'}{dt}e_i + a_i'\frac{De_i}{dt}\right) = \sum_{i=1}^n a_i''e_i.$$

Solving the system:

$$\begin{cases} \sum_{i=1}^n a_i''(t)e_i(t) = \frac{D^2J}{dt^2}(t) = 0; \\ \sum_{i=1}^n a_i'(0)e_i(0) = \frac{DJ}{dt}(0) = w = \sum_{i=1}^n w_ie_i; \\ \sum_{i=1}^n a_i(0)e_i(0) = J(0) = t(d\exp_p)_{tv}(w)\big|_{t=0} = 0, \end{cases}$$

we have

$$J(t) = \sum_{i=1}^n tw_ie_i(t).$$

Then we have

$$(d\exp_p)_v(w) = (d\exp_p)_{tv}(tw)\big|_{t=1} = J(1) = \sum_{i=1}^n w_ie_i(1).$$

Similarly, for any $u = \sum_{j=1}^n u_ie_i \in T_v(B_\varepsilon(0))$, we have

$$(d\exp_p)_v(u) = \sum_{j=1}^n u_je_j(1).$$

Then we must have

$$\begin{aligned}\left\langle (d\exp_p)_v(w), (d\exp_p)_v(u)\right\rangle_{\exp_p(v)} &= \left\langle \sum_{i=1}^n w_ie_i(1), \sum_{j=1}^n u_je_j(1)\right\rangle_{\exp_p(v)} \\ &= \sum_{i,j=1}^n w_iu_j\left\langle e_i(1), e_j(1)\right\rangle = \sum_{i=1}^n w_iu_i = \langle w,u\rangle_{\mathbb{R}^n} = \langle w,u\rangle_v.\end{aligned}$$

By the arbitrariness of v, w, u, we conclude that $\exp_p\colon B_\varepsilon(0) \subset T_pM \to B_\varepsilon(p)$ is an isometry. $\square$

Exercise 5.2. Let M be a Riemannian manifold, $\gamma\colon [0,1] \to M$ a geodesic, and J a Jacobi field along γ. Prove that there exists a parametrized surface $f(t,s)$, where $f(t,0) = \gamma(t)$ and the curve $t \mapsto f(t,s)$ are geodesics, such that $J(t) = \frac{\partial f}{\partial s}(t,0)$.

Proof. Firstly, we choose a curve $\lambda(s) \subset M$ such that $\lambda(0) = \gamma(0)$ and $\lambda'(0) = J(0)$. And we choose a vector field $W(s)$ along λ such that $W(0) = \gamma'(0)$ and $\frac{DW}{ds}(0) = \frac{DJ}{dt}(0)$. Let

$$f(t,s) = \exp_{\lambda(s)}(tW(s)).$$

By definition, we have

$$f(t,0) = \exp_{\lambda(0)}(tW(0)) = \exp_{\gamma(0)}(t\gamma'(0)) = \gamma(t),$$

and for any s,

$$f(t,s) = \exp_{\lambda(s)}(tW(s)) = \gamma(t, \lambda(s), W(s))$$

are geodesics.

Finally, let's study $\frac{\partial f}{\partial s}(t,0)$. Because for any s, $f(t,s)$ are geodesics, we have

$$\frac{D}{dt}\left(\frac{\partial f}{\partial t}\right) = 0.$$

Then by Chapter 4 Lemma 4.1 and Chapter 3 Lemma 3.4,

$$0 = \frac{D}{ds}\left(\frac{D}{dt}\left(\frac{\partial f}{\partial t}\right)\right) = \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial t} - R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t} = \frac{D}{dt}\frac{D}{dt}\frac{\partial f}{\partial s} - R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t}$$

as we have done in the text. So $\frac{\partial f}{\partial s}(t,0)$ satisfies the Jacobi equation and then a Jacobi field. At $t = 0$, we have

$$\frac{\partial f}{\partial s}(0,0) = \left.\frac{df(0,s)}{ds}\right|_{s=0} = \left.\frac{d\lambda}{ds}\right|_{s=0} = \lambda'(0) = J(0).$$

And by Chapter 3 Lemma 3.4 and Proposition 2.9, we also have

$$\begin{aligned}\frac{D}{dt}\frac{\partial f}{\partial s}(0,0) = \frac{D}{ds}\frac{\partial f}{\partial t}(0,0) &= \frac{D}{ds}\left(\left.(d\exp_{\lambda(s)})_{tW(s)}(W(s))\right|_{t=0}\right)(0) \\ &= \frac{D}{ds}\left((d\exp_{\lambda(s)})_0(W(s))\right)(0) = \frac{DW}{ds}(0) = \frac{DJ}{dt}(0).\end{aligned}$$

By the uniqueness of the solution of ODEs (in fact, Jacobi equations) about initial values, we must have

$$J(t) = \frac{\partial f}{\partial s}(t,0),$$

as deisred. $\square$

Exercise 5.3. Let M be a Riemannian manifold with non-positive sectional curvature. Prove that, for all p, the conjugate locus $C(p)$ is empty.

Proof. If not, by Chapter 5 Proposition 2.4, we can assume $J(t) = (d\exp_p)_{tv}(tw)$ is a non-zero Jacobi field along γ with $\gamma(0) = p$ and $\gamma'(0) = v$ such that $J(0) = J(a) = 0$, i.e. $\gamma(a) \in C(p) \neq \varnothing$. Then by Chapter 2 Proposition 3.2 and Jacobi equation

$$\frac{D^2 J}{at^2} + R(\gamma', J)\gamma' = 0,$$

we have

$$\frac{d}{dt}\left\langle \frac{DJ}{dt}, J\right\rangle = \left\langle \frac{D^2 J}{dt^2}, J\right\rangle + \left\langle \frac{DJ}{dt}, \frac{DJ}{dt}\right\rangle = -\left\langle R(\gamma', J)\gamma', J\right\rangle + \left|\frac{DJ}{dt}\right|^2 = -R(\gamma', J, \gamma', J) + \left|\frac{DJ}{dt}\right|^2.$$

Because M has non-positive sectional curvature, we know

$$\frac{d}{dt}\left\langle \frac{DJ}{dt}, J\right\rangle = -R(\gamma', J, \gamma', J) + \left|\frac{DJ}{dt}\right|^2 \geq 0.$$

This means $\left\langle \frac{DJ}{dt}, J \right\rangle(t) = \left\langle \frac{DJ}{dt}, J \right\rangle_{\gamma(t)}$ doesn't decrease about t. But we also have

$$\left\langle \frac{DJ}{dt}, J \right\rangle(0) = \left\langle \frac{DJ}{dt}(0), J(0) \right\rangle = \left\langle \frac{DJ}{dt}, 0 \right\rangle = 0 = \left\langle \frac{DJ}{dt}, 0 \right\rangle = \left\langle \frac{DJ}{dt}, J(a) \right\rangle = \left\langle \frac{DJ}{dt}, J \right\rangle(a),$$

so we must have

$$\left\langle \frac{DJ}{dt}, J \right\rangle(t) = \text{constant} = 0.$$

Then by Chapter 2 Proposition 3.2,

$$\frac{d}{dt}\langle J, J \rangle = \left\langle \frac{DJ}{dt}, J \right\rangle + \left\langle J, \frac{DJ}{dt} \right\rangle = 2\left\langle \frac{DJ}{dt}, J \right\rangle = 0.$$

This means $|J(t)|^2 = \langle J(t), J(t) \rangle = \langle J, J \rangle(t)$ is a constant. And we have

$$|J(t)|^2 = |J(0)|^2 = 0.$$

So we conclude $J(t) = 0$, which is contradict to our assumption that $J(t)$ is a non-zero Jacobi field. □

Exercise 5.4. Let $b < 0$ and let M be a manifold with constant negative sectional curvature equal to b. Let $\gamma\colon [0, \ell] \to M$ be a normalized geodesic, and let $v \in T_{\gamma(\ell)}M$ such that $\langle v, \gamma'(\ell) \rangle = 0$ and $|v| = 1$. Since M has negative curvature, $\gamma(\ell)$ is not conjugate to $\gamma(0)$ (see Exercise 5.3). Show that the Jacobi field J along γ determined by $J(0) = 0$, $J(\ell) = v$ is given by

$$J(t) = \frac{\sinh(t\sqrt{-b})}{\sinh(\ell\sqrt{-b})} w(t),$$

where $w(t)$ is the parallel transport along γ of the vector

$$w(0) = \frac{u_0}{|u_0|}, \qquad u_0 = \left(d\exp_p\right)^{-1}_{\ell\gamma'(0)}(v)$$

and where u_0 is considered as a vector $T_{\gamma(0)}M$ by the identification $T_{\gamma(0)}M \cong T_{\ell\gamma'(0)}(T_{\gamma(0)}M)$.

Proof. Firstly, by Chapter 5 Example 2.3, the Jacobi field $J_1(t)$ along γ satisfied $J_1(0) = 0$ and $\frac{DJ_1}{dt}(0) = w(0)$ is given by

$$J_1(t) = \frac{\sinh(t\sqrt{-b})}{\sqrt{-b}} w(t).$$

In the text, we need $\langle \gamma'(t), w(t) \rangle = 0$ and $|w| = 1$. But it's easy to check that the proposition is true without the conditions. On the other side, by Chapter 5 Corollary 2.5, $J_1(t)$ can be also given by

$$J_1(t) = (d\exp_{\gamma(0)})_{t\gamma'(0)}(t\frac{DJ}{dt}(0)) = (d\exp_{\gamma(0)})_{t\gamma'(0)}(tw(0)).$$

Then we have

$$\begin{aligned} J_1(\ell) = (d\exp_{\gamma(0)})_{\ell\gamma'(0)}(\ell w(0)) &= \frac{\ell}{|u_0|}(d\exp_{\gamma(0)})_{\ell\gamma'(0)}(|u_0|\, w(0)) \\ &= \frac{\ell}{|u_0|}(d\exp_{\gamma(0)})_{\ell\gamma'(0)}(u_0) = \frac{\ell}{|u_0|}v = \frac{\ell}{|u_0|}J(\ell). \end{aligned}$$

Using the uniqueness of solution of ODEs with the initial value

$$\begin{cases} J_1(0) = 0 = \frac{\ell}{|u_0|}0 = \frac{\ell}{|u_0|}J(0); \\ J_1(\ell) = \frac{\ell}{|u_0|}J(\ell), \end{cases}$$

we must have

$$J_1(t) = \frac{\ell}{|u_0|}J(t),$$

i.e.

$$J(t) = \frac{|u_0|}{\ell} J_1(t) = \frac{|u_0|}{\ell} \frac{\sinh(t\sqrt{-b})}{\sqrt{-b}} w(t).$$

In addition, since w is generated by parallel transport, by the definition of compatible metric (Chapter 2 Definition 3.1), we have

$$|w(t)|^2 = \langle w(t), w(t) \rangle = \langle w(0), w(0) \rangle = \left\langle \frac{u_0}{|u_0|}, \frac{u_0}{|u_0|} \right\rangle = 1.$$

Then

$$1 = |v| = |J(\ell)| = \left| \frac{|u_0|}{\ell} \frac{\sinh(\ell\sqrt{-b})}{\sqrt{-b}} w(\ell) \right| = \frac{|u_0|}{\ell} \frac{\sinh(\ell\sqrt{-b})}{\sqrt{-b}} |w(\ell)| = \frac{|u_0|}{\ell} \frac{\sinh(\ell\sqrt{-b})}{\sqrt{-b}},$$

i.e.

$$\frac{|u_0|}{\ell} = \frac{\sqrt{-b}}{\sinh(\ell\sqrt{-b})}.$$

So we have

$$J(t) = \frac{|u_0|}{\ell} \frac{\sinh(t\sqrt{-b})}{\sqrt{-b}} w(t) = \frac{\sqrt{-b}}{\sinh(\ell\sqrt{-b})} \frac{\sinh(t\sqrt{-b})}{\sqrt{-b}} w(t) = \frac{\sinh(t\sqrt{-b})}{\sinh(\ell\sqrt{-b})} w(t),$$

as desired. □

Exercise 5.5 (Jacobi fields and conjugate points on locally symmetric spaces)**.** Let $\gamma\colon [0, \infty) \to M$ be a geodesic in a locally symmetric space M (see Exercise 4.6) and let $v = \gamma'(0)$ be its velocity at $p = \gamma(0)$. Define a linear transformation $K_v\colon T_pM \to T_pM$ by

$$K_v(x) = R(v, x)v, \qquad x \in T_pM.$$

a) Prove that K_v is self-adjoint.

b) Choose an orthonormal basis $\{e_1, \cdots, e_n\}$ of T_pM that diagonalizes K_v, that is,

$$K_v(e_i) = \lambda_i e_i, \qquad i = 1, \cdots, n.$$

Extend the e_i to fields along γ by parallel transport. Show that, for all t,

$$K_{\gamma'(t)}(e_i(t)) = \lambda_i e_i(t),$$

where λ_i does not depend on t.

c) Let $J(t) = \sum_{i=1}^n x_i(t)e_i(t)$ be a Jacobi field along γ, Show that the Jacobi equation is equivalent to the system

$$\frac{d^2x_i}{dt^2} + \lambda_i x_i = 0, \qquad i = 1, \cdots, n.$$

d) Show that the conjugate points of p along γ are given by $\gamma(\frac{\pi k}{\sqrt{\lambda_i}})$, where k is a positive integer and λ_i is a positive eigenvalue of K_v.

Proof. Firstly, for a), for any $x, y \in T_pM$, using Chapter 4 Proposition 2.5 (d), we have

$$\langle K_v(x), y \rangle = \langle R(v, x)v, y \rangle = R(v, x, v, y) = R(v, y, v, x) = \langle R(v, y)v, x \rangle = \langle K_v(y), x \rangle = \langle x, K_v(y) \rangle.$$

This means $K_v^* = K_v$, i.e. K_v is self-adjoint.

For b), because $e_1(t), \cdots, e_n(t)$ are generated by parallel transport along γ and $\gamma'(t)$ is naturally parallel along geodesic γ, by Exerxise 4.6 a), we have

$$K_{\gamma'(t)}(e_i(t)) = R(\gamma'(t), e_i(t))\gamma'(t), \qquad \forall 1 \le i \le n$$

is parallel along γ. Then by the definition of compatible connection (Chapter 2 Definition 3.1), we have

$$\left\langle K_{\gamma'(t)}(e_i(t)), e_j(t) \right\rangle = \left\langle K_{\gamma'(0)}(e_i(0)), e_j(0) \right\rangle = \langle K_v(e_i), e_j \rangle = \langle \lambda_i e_i, e_j \rangle = \lambda_i \delta_{ij}.$$

Because $e_1(t), \cdots, e_n(t)$ is a geodesic frame (see Exercise 3.7), they are orthonormal for any t. Then we have

$$K_{\gamma'(t)}(e_i(t)) = \sum_{j=1}^{n} \lambda_i \delta_{ij} e_j(t) = \lambda_i e_i(t),$$

where λ_i, $1 \le i \le n$ are constants, do not depend on t, as desired.

Then because $e_1(t), \cdots, e_n(t)$ are parallel along γ, by the linearity of curvature tensor R, and b), the Jacobi equation will be

$$\begin{aligned}
0 &= \frac{D}{dt}\frac{D}{dt}J(t) + R(\gamma'(t), J(t))\gamma'(t) \\
&= \sum_{i=1}^{n} x_i''(t)e_i(t) + R\left(\gamma'(t), \sum_{i=1}^{n} x_i(t)e_i(t)\right)\gamma'(t) = \sum_{i=1}^{n} x_i''(t)e_i(t) + \sum_{i=1}^{n} x_i(t)R(\gamma'(t), e_i(t))\gamma'(t) \\
&= \sum_{i=1}^{n} x_i''(t)e_i(t) + \sum_{i=1}^{n} x_i(t)\lambda_i e_i(t) = \sum_{i=1}^{n} \left(x_i''(t) + \lambda_i x_i(t)\right) e_i(t).
\end{aligned}$$

Because $e_1(t), \cdots, e_n(t)$ is a orthonormal basis, the Jacobi equation will be equivalent to the system

$$x_i''(t) + \lambda_i x_i(t) = 0, \qquad \forall 1 \le i \le n;$$

whence c).

Finally, for d), with the initial values $J(0) = \sum_{i=1}^{n} x_i(0)e_i(0) = 0$, i.e.

$$x_i(0) = 0, \qquad \forall 1 \le i \le n,$$

solving the Jacobi equation above, we have for $1 \le i \le n$ such that $\lambda_i > 0$,

$$x_i(t) = A\sin\left(\sqrt{\lambda_i}t\right) \qquad \text{or} \qquad x_i(t) = 0,$$

where A is any non-zero constant; and for other is,

$$x_i(t) = 0.$$

So for any $\gamma\left(\frac{\pi k}{\sqrt{\lambda_i}}\right)$ where k is a positive integer and λ_i is a positive eigenvalue of K_v, let

$$J(t) = \sin\left(\sqrt{\lambda_i}t\right) e_i(t)$$

be a Jacobi field (i.e. for $j = i$, let $x_j(t) = \sin\left(\sqrt{\lambda_i}t\right) e_i(t)$ and for $j \ne i$, let $x_j(t) = 0$), we have

$$J\left(\frac{\pi k}{\sqrt{\lambda_i}}\right) = \sin\left(\sqrt{\lambda_i}\frac{\pi k}{\sqrt{\lambda_i}}\right) e_i\left(\frac{\pi k}{\sqrt{\lambda_i}}\right) = \sin(\pi k)\, e_i\left(\frac{\pi k}{\sqrt{\lambda_i}}\right) = 0.$$

This means $\gamma\left(\frac{\pi k}{\sqrt{\lambda_i}}\right) \in C(p)$. On the other hand, if $\gamma(t_0) \in C(p)$, ther exists a non-zero Jacobi field $J(t)$ such that $J(t_0) = 0$. Assuming $x_i(t) = A\sin\left(\sqrt{\lambda_i}t\right) \ne 0$ where $\lambda_i > 0$, we must have

$$x_i(t_0) = A\sin\left(\sqrt{\lambda_i}t_0\right) = 0.$$

So

$$\sqrt{\lambda_i}t_0 = \pi k, \qquad k \in \mathbb{Z}_+,$$

i.e.

$$t_0 = \frac{\pi k}{\sqrt{\lambda_i}}, \qquad k \in \mathbb{Z}_+.$$

So the conjugate points of p must have the form of $\gamma\left(\frac{\pi k}{\sqrt{\lambda_i}}\right)$ with $k \in \mathbb{Z}_+$ and λ_i a positive eigenvalue of K_v. Above all, the conjugate points of p along γ are given by $\gamma(\frac{\pi k}{\sqrt{\lambda_i}})$, where $k \in \mathbb{Z}_+$ and λ_i is a positive eigenvalue of K_v, as desired. $\square$

Exercise 5.6. Let M be a Riemannian manifold of dimension two (in this case we say that M is a surface). Let $B_\delta(p)$ be a normal ball around the point $p \in M$ and consider the parametrized surface

$$f(\rho, \theta) = \exp_p(\rho v(\theta)), \quad 0 < \rho < \delta, \; -\pi < \theta < \pi,$$

where $v(\theta)$ is a circle of radius 1 (I think this is a typo) in T_pM parametrized by the central angle θ.

a) Show that (ρ, θ) are coordinates in an open set $U \subset M$ formed by the open ball $B_\delta(p)$ minus the ray

$$\exp_p(-\rho v(0)), \quad 0 \leq \text{ (I think this is a typo agian) } \rho < \delta.$$

Such coordinates are called *polar coordinates* at p.

b) Show that the coefficients g_{ij} of the Riemannian metric in these coordinates are:

$$g_{12} = 0, \quad g_{11} = \left|\frac{\partial f}{\partial \rho}\right|^2 = |v(\theta)|^2 = 1, \quad g_{22} = \left|\frac{\partial f}{\partial \theta}\right|^2.$$

c) Show that, along the geodesic $f(\rho, 0)$, we have

$$(\sqrt{g_{22}})_{\rho\rho} = -K(p)\rho + R(\rho),$$

where

$$\lim_{\rho \to 0} \frac{R(\rho)}{\rho} = 0$$

and $K(p)$ is the sectional curvature of M at p.

d) Prove that

$$\lim_{\rho \to 0} \frac{(\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}}} = -K(p).$$

This last expression is the value of Gaussian curvature of M at p given a polar coordinates. This fact from the theory of surfaces, and d) shows that, in dimension two, the sectional curvature coincides with the Gaussian curvature.

Proof. Firstly, for a), because $B_\delta(p)$ is a normal ball, by definition, $\exp_p$ is a diffeomorphism on $B_\delta(0)$, and so does $f(\rho, \theta) = \exp_p(\rho v(\theta))$. Then we have

$$\begin{aligned}
U &= f(\{(\rho, \theta) | 0 \leq \rho < \delta, \; -\pi < \theta < \pi\}) \\
&= f(\{(\rho, \theta) | 0 \leq \rho < \delta, \; -\pi < \theta \leq \pi\} - \{(\rho, \pi) | 0 \leq \rho < \delta\}) \\
&= f(\{(\rho, \theta) | 0 \leq \rho < \delta, \; -\pi < \theta \leq \pi\}) - f(\{(\rho, \pi) | 0 \leq \rho < \delta\}) \\
&= f(B_\delta(0)) - f(\{(\rho, \pi) | 0 \leq \rho < \delta\}) \\
&= B_\delta(p) - f(\{(\rho, \pi) | 0 \leq \rho < \delta\}).
\end{aligned}$$

And we can solve

$$v(\theta) = (\cos\theta, \sin\theta).$$

Then

$$v(\pi) = (-1, 0) = -(1, 0) = -v(0).$$

So we have

$$f(\{(\rho, \pi) | 0 \leq \rho < \delta\}) = f(\rho, \pi) = \exp_p(\rho v(\pi)) = \exp_p(-\rho v(0)), \quad 0 \leq \rho < \delta.$$

We conclude that (ρ, θ) are polar coordinates in an open set $U \subset M$ formed by the open ball $B_\delta(p)$ minus the ray

$$\exp_p(-\rho v(0)), \quad 0 \leq \rho < \delta,$$

as desired.

Then under these coordinates, at any (ρ_0, θ_0), let

$$\gamma_1(t) = f(\rho_0 + t, \theta_0) = \exp_p((\rho_0 + t)v(\theta_0)) \quad \text{and} \quad \gamma_2(t) = f(\rho_0, \theta_0 + t) = \exp_p(\rho_0 v(\theta_0 + t))$$

be the trajectories of $\frac{\partial}{\partial \rho}\Big|_{(\rho_0,\theta_0)}$ and $\frac{\partial}{\partial \theta}\big|_{(\rho_0,\theta_0)}$. Then we have

$$\frac{\partial}{\partial \rho}\bigg|_{(\rho_0,\theta_0)} = \gamma_1'(0) = (d\exp_p)_{\rho_0 v(\theta_0)}(v(\theta_0)) \quad \text{ and } \quad \frac{\partial}{\partial \theta}\bigg|_{(\rho_0,\theta_0)} = \gamma_2'(0) = (d\exp_p)_{\rho_0 v(\theta_0)}(\rho_0 v'(\theta_0)).$$

So we can compute the coefficients: by Gauss Lemma (Chapter 3 Lemma 3.5),

$$\begin{aligned} g_{12} = g_{21} &= \left\langle \frac{\partial}{\partial \rho}, \frac{\partial}{\partial \theta} \right\rangle = \left\langle (d\exp_p)_{\rho v(\theta)}(v(\theta)), (d\exp_p)_{\rho v(\theta)}(\rho v'(\theta)) \right\rangle \\ &= \left\langle (d\exp_p)_{\rho v(\theta)}(\rho v(\theta)), (d\exp_p)_{\rho v(\theta)}(v'(\theta)) \right\rangle \\ &= \langle \rho v(\theta), v'(\theta) \rangle = \rho \langle v(\theta), v'(\theta) \rangle . \end{aligned}$$

By definition, $|v(\theta)|^2 = 1$, then we have

$$0 = \left(|v(\theta)|^2\right)' = \langle v(\theta), v(\theta)\rangle' = \langle v'(\theta), v(\theta)\rangle + \langle v(\theta), v'(\theta)\rangle = 2\langle v(\theta), v'(\theta)\rangle .$$

So we have

$$g_{12} = g_{21} = \rho \langle v(\theta), v'(\theta)\rangle = 0.$$

Similarly, we have

$$\begin{aligned} g_{11} &= \left\langle \frac{\partial}{\partial \rho}, \frac{\partial}{\partial \rho} \right\rangle = \left\langle (d\exp_p)_{\rho v(\theta)}(v(\theta)), (d\exp_p)_{\rho v(\theta)}(v(\theta)) \right\rangle \\ &= \frac{1}{\rho} \left\langle (d\exp_p)_{\rho v(\theta)}(\rho v(\theta)), (d\exp_p)_{\rho v(\theta)}(v(\theta)) \right\rangle \\ &= \frac{1}{\rho} \langle \rho v(\theta), v(\theta)\rangle = \langle v(\theta), v(\theta)\rangle = |v(\theta)|^2 = 1. \end{aligned}$$

For another calculation, we have

$$\frac{\partial}{\partial \rho}\bigg|_{(\rho_0,\theta_0)} = \gamma_1'(0) = \frac{\partial f}{\partial \rho}\bigg|_{(\rho_0,\theta_0)} \cdot \frac{d(\rho_0+t)}{dt}\bigg|_{t=0} + \frac{\partial f}{\partial \theta}\bigg|_{(\rho_0,\theta_0)} \cdot \frac{d\theta_0}{dt}\bigg|_{t=0} = \frac{\partial f}{\partial \rho}\bigg|_{(\rho_0,\theta_0)}$$

and

$$\frac{\partial}{\partial \theta}\bigg|_{(\rho_0,\theta_0)} = \gamma_1'(0) = \frac{\partial f}{\partial \theta}\bigg|_{(\rho_0,\theta_0)} \cdot \frac{d\theta_0}{dt}\bigg|_{t=0} + \frac{\partial f}{\partial \theta}\bigg|_{(\rho_0,\theta_0)} \cdot \frac{d(\theta_0+t)}{dt}\bigg|_{t=0} = \frac{\partial f}{\partial \theta}\bigg|_{(\rho_0,\theta_0)}.$$

So we also have

$$g_{11} = \left\langle \frac{\partial}{\partial \rho}, \frac{\partial}{\partial \rho} \right\rangle = \left\langle \frac{\partial f}{\partial \rho}, \frac{\partial f}{\partial \rho} \right\rangle = \left|\frac{\partial f}{\partial \rho}\right|^2 \quad \text{ and } \quad g_{22} = \left\langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \right\rangle = \left\langle \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \theta} \right\rangle = \left|\frac{\partial f}{\partial \theta}\right|^2.$$

So we proved b).

For c), by Chapter 5 Proposition 2.4, the Jacobi field along the geodesic $f(\rho,0)$ satisfies the initial values $J(0) = 0$ and $\left|\frac{DJ}{d\rho}(0)\right| = |v'(0)| = |(-\sin\theta,\cos\theta)|_{\theta=0}| = |(0,1)| = 1$ can be

$$J(\rho) = \frac{\partial f}{\partial \theta}(\rho, 0),$$

where by Chapter 3 Proposition 2.9,

$$(f(\rho,0))'(0) = \frac{\partial f}{\partial \rho}\bigg|_{(0,0)} = \frac{\partial}{\partial \rho}\bigg|_{(0,0)} = (d\exp_p)_0(v(0)) = v(0).$$

Then by b) and Chapter 5 Corollary 2.10, we have

$$\sqrt{g_{22}}(\rho,0) = \left|\frac{\partial f}{\partial \theta}(\rho,0)\right| = |J(\rho)| = \rho - \frac{1}{6}K(p,\sigma)\rho^3 + \widetilde{R}(\rho), \quad \lim_{\rho\to 0}\frac{\widetilde{R}(\rho)}{\rho^3} = 0,$$

where because M has dimension two, it has only one sectional curvature, we can denote $K(p,\sigma) = K(p)$, and using the fact that

$$\left\langle (f(\rho,0))'(0), \frac{DJ}{d\rho}(0) \right\rangle = \langle v(0), v'(0) \rangle = 0$$

again, we know that $\sigma = \operatorname{Span}\left\{ (f(\rho,0))'(0), \frac{DJ}{d\rho}(0) \right\}$ is non-degenerated. Differentiating it twice, we have

$$(\sqrt{g_{22}})_{\rho\rho} = -K(p)\rho + R(\rho),$$

where $R(\rho) = \widetilde{R}''(\rho)$ with

$$0 = \lim_{\rho \to 0} \frac{\widetilde{R}(\rho)}{\rho^3} = \lim_{\rho \to 0} \frac{\widetilde{R}'(\rho)}{3\rho^2} = \lim_{\rho \to 0} \frac{\widetilde{R}''(\rho)}{6\rho} = \lim_{\rho \to 0} \frac{R(\rho)}{6\rho}$$

by L'Hospital's rule, i.e.

$$\lim_{\rho \to 0} \frac{R(\rho)}{\rho} = 0.$$

Finally, also by L'Hospital's rule, we have

$$\lim_{\rho \to 0} \frac{(\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}}} = \lim_{\rho \to 0} \frac{-K(p)\rho + R(\rho)}{\rho - \frac{1}{6}K(p)\rho^3 + \widetilde{R}(\rho)} = \lim_{\rho \to 0} \frac{-K(p) + R'(\rho)}{1 - \frac{1}{2}K(p)\rho^2 + \widetilde{R}'(\rho)}.$$

For the remainder terms, by L'Hospital's rule,

$$0 = \lim_{\rho \to 0} \frac{R(\rho)}{\rho} = \lim_{\rho \to 0} R'(\rho) \quad \text{and} \quad 0 = \lim_{\rho \to 0} \frac{\widetilde{R}(\rho)}{\rho^3} = \lim_{\rho \to 0} \frac{\widetilde{R}'(\rho)}{3\rho^2}.$$

So we must have

$$\lim_{\rho \to 0} R(\rho) = 0 \quad \text{and} \quad \lim_{\rho \to 0} \widetilde{R}'(\rho) = 0.$$

Then

$$\lim_{\rho \to 0} \frac{(\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}}} = \lim_{\rho \to 0} \frac{-K(p) + R'(\rho)}{1 - \frac{1}{2}K(p)\rho^2 + \widetilde{R}'(\rho)} = \lim_{\rho \to 0} \frac{-K(p)}{1} = -K(p);$$

whence d). This last expression is the value of Gaussian curvature of M at p given a polar coordinates. So we conclude that, in dimension two, the sectional curvature coincides with the Gaussian curvature. $\square$

Exercise 5.7. Let M be a Riemannian manifold of dimension two. Let $p \in M$ and let $V \subset T_pM$ be a neighborhood of the origin where $\exp_p$ is a diffeomorphism. Let $S_r(0) \subset V$ be a circle of radius r centered at the origin, and let L_r be the length of the curve $\exp_p(S_r)$ in M. Prove that the sectional curvature at $p \in M$ is given by

$$K(p) = \lim_{r \to 0} \frac{3}{\pi} \frac{2\pi r - L_r}{r^3}.$$

Proof. Firstly, using the same notations as Exercise 5.6, the curve $\exp_p(S_r)$ will be

$$f(r,\theta) = \exp_p(rv(\theta)), \qquad -\pi < \theta < \pi.$$

Then we have

$$\frac{df(r,\theta)}{d\theta}(\theta) = \frac{\partial f}{\partial \theta}(r,\theta).$$

By definition and Exercise 5.6 b), we have

$$\begin{aligned} L_r &= \int_{-\pi}^{\pi} \sqrt{\left\langle \frac{df(r,\theta)}{d\theta}(\theta), \frac{df(r,\theta)}{d\theta}(\theta) \right\rangle}\, \mathrm{d}\theta \\ &= \int_{-\pi}^{\pi} \sqrt{\left\langle \frac{\partial f}{\partial \theta}(r,\theta), \frac{\partial f}{\partial \theta}(r,\theta) \right\rangle}\, \mathrm{d}\theta = \int_{-\pi}^{\pi} \left| \frac{\partial f}{\partial \theta}(r,\theta) \right| \mathrm{d}\theta \\ &= \int_{-\pi}^{\pi} \sqrt{g_{22}}(r,\theta)\, \mathrm{d}\theta. \end{aligned}$$

By the proof of Exercise 5.6 c), we know that

$$\sqrt{g_{22}}(r,0) = r - \frac{1}{6}K(p)r^3 + \widetilde{R}(r), \qquad \lim_{r\to 0}\frac{\widetilde{R}(r)}{r^3} = 0.$$

In fact, it's true for all $-\pi < \theta < \pi$:

$$\sqrt{g_{22}}(r,\theta) = r - \frac{1}{6}K(p)r^3 + \widetilde{R}(r,\theta), \qquad \forall -\pi < \theta < \pi,$$

where

$$\lim_{r\to 0}\frac{\widetilde{R}(r,\theta)}{r^3} = 0 \text{ uniformly on } \theta.$$

To see this, we can change the coordinates to $(r, \theta - \theta_0)$. Then the computation similar to the proof of Exercise 5.6 c) yeilds the same expression for $\sqrt{g_{22}}(r,0)$, which is now actually along $\exp_p(rv(\theta_0))$ (see reference: https://math.stackexchange.com/questions/1832106). So we have

$$L_r = \int_{-\pi}^{\pi} \sqrt{g_{22}}(r,\theta)\,\mathrm{d}\theta = \int_{-\pi}^{\pi}\left(r - \frac{1}{6}K(p)r^3 + \widetilde{R}(r,\theta)\right)\mathrm{d}\theta = 2\pi r - \frac{\pi r^3}{3}K(p) + \int_{-\pi}^{\pi} R(r,\theta)\,\mathrm{d}\theta,$$

i.e.

$$K(p) = \frac{3}{\pi}\left(\frac{2\pi r - L_r}{r^3} + \int_{-\pi}^{\pi}\frac{\widetilde{R}(r,\theta)}{r^3}\,\mathrm{d}\theta\right).$$

Because $\lim_{r\to 0}\frac{\widetilde{R}(r,\theta)}{r^3} = 0$ uniformly on θ, we can commute the limit process with integration. Then we have

$$\begin{aligned}
K(p) &= \frac{3}{\pi}\left(\frac{2\pi r - L_r}{r^3} + \int_{-\pi}^{\pi}\frac{\widetilde{R}(r,\theta)}{r^3}\,\mathrm{d}\theta\right) \\
&= \lim_{r\to 0}\frac{3}{\pi}\left(\frac{2\pi r - L_r}{r^3} + \int_{-\pi}^{\pi}\frac{\widetilde{R}(r,\theta)}{r^3}\,\mathrm{d}\theta\right) = \lim_{r\to 0}\frac{3}{\pi}\frac{2\pi r - L_r}{r^3} + \frac{3}{\pi}\lim_{r\to 0}\int_{-\pi}^{\pi}\frac{\widetilde{R}(r,\theta)}{r^3}\,\mathrm{d}\theta \\
&= \lim_{r\to 0}\frac{3}{\pi}\frac{2\pi r - L_r}{r^3} + \frac{3}{\pi}\int_{-\pi}^{\pi}\lim_{r\to 0}\frac{\widetilde{R}(r,\theta)}{r^3}\,\mathrm{d}\theta = \lim_{r\to 0}\frac{3}{\pi}\frac{2\pi r - L_r}{r^3} + \frac{3}{\pi}\int_{-\pi}^{\pi} 0\,\mathrm{d}\theta = \lim_{r\to 0}\frac{3}{\pi}\frac{2\pi r - L_r}{r^3},
\end{aligned}$$

as desired. ☐

Exercise 5.8. Let $\gamma\colon [0,a] \to M$ be a geodesic and let X be a Killing field on M.

a) Show that the restriction $X(\gamma(s))$ of X to $\gamma(s)$ is a Jacobi field along γ.

b) Use a) to show that if M is connected and there exists $p \in M$ with $X(p) = 0$ and $\nabla_Y X(p) = 0$, for all $Y(p) \in T_pM$, then $X = 0$ on M (i.e. Exercise 3.6).

Proof. Firstly, let $\varphi(t,q)$ be the trajectory of X. In the proof of Exercise 3.5 b), we proved a claim that $\varphi(t_0,\gamma(s))$ are also a geodesics for any t_0 near 0. Denote $f(t,s) = \varphi(t,\gamma(s))$. Then we have

$$\frac{D}{ds}\frac{\partial f}{\partial s}(t,s) = \frac{D}{ds}\frac{\partial\varphi(t,\gamma(s))}{\partial s}(t,s) = 0.$$

As we have seen in the text (where we induced the Jacobi equation), by Chapter 4 Lemma 4.1, Chapter 3 Lemma 3.4 and Chapter 4 Proposition 2.5 b), we have

$$\begin{aligned}
0 &= \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial s}(0,s) \\
&= \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial s}(0,s) + R\left(\frac{\partial f}{\partial s},\frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial s}(0,s) \\
&= \frac{D}{ds}\frac{D}{ds}\frac{\partial f}{\partial t}(0,s) + R\left(\frac{\partial f}{\partial s}(0,s),\frac{\partial f}{\partial t}(0,s)\right)\frac{\partial f}{\partial s}(0,s).
\end{aligned}$$

Noticing that $f(0,s) = \varphi(0,\gamma(s)) = \gamma(s)$, we have

$$\frac{\partial f}{\partial t}(0,s) = \left.\frac{\partial}{\partial t}\right|_{t=0} \varphi(t,\gamma(s)) = X(\varphi(0,\gamma(s))) = X(\gamma(s))$$

and

$$\frac{\partial f}{\partial s}(0,s) = \frac{\partial \varphi(0,\gamma(s))}{\partial s}(0,s) = \frac{\partial \gamma}{\partial s}(s) = \gamma'(s).$$

Then the equation above will be

$$\begin{aligned} 0 &= \frac{D}{ds}\frac{D}{ds}\frac{\partial f}{\partial t}(0,s) + R\left(\frac{\partial f}{\partial s}(0,s), \frac{\partial f}{\partial t}(0,s)\right)\frac{\partial f}{\partial s}(0,s) \\ &= \frac{D}{ds}\frac{D}{ds}X(\gamma(s)) + R\left(\gamma'(s), X(\gamma(s))\right)\gamma'(s). \end{aligned}$$

This means $X(\gamma(s))$ satisfies the Jacobi equation along the geodesic $\gamma(s)$, i.e. $X(\gamma(s))$ is a Jacobi field along γ; whence a).

Then for b), for any $q \in M$ near p, we can choose a geodesic $\gamma(s)$ from p to q and denote $\gamma(a) = q$. By a), $X(\gamma(s))$ is a Jacobi field along γ, with the initial values

$$X(0) = X(\gamma(0)) = X(p) = 0 \quad \text{and} \quad \frac{DX}{ds}(0) = \frac{DX(\gamma(s))}{ds}(\gamma(0)) = \nabla_{\gamma'}X(p) = 0.$$

By the uniqueness of the solution of ODEs about initial values, we must have

$$X(\gamma(s)) = X(s) = 0.$$

Then

$$X(q) = X(\gamma(a)) = X(a) = 0.$$

By the arbitrariness of q, we know that $X = 0$ on an open neighborhood of p. Doing the same thing for every point of the set

$$U = \{\, q \in M \,|\, X(q) = 0\} \neq \varnothing,$$

we know that U is open. On the other side, U is the preimage of a continuous (in fact, smooth) mapping X of a close set $\{0\}$, so U is also close. Then by the connectedness of M, we must have $U = M$, i.e. $X = 0$ on M, as desired. $\square$

6 Isometric Immersions

Exercise 6.1. Let M_1 and M_2 be Riemannian manifolds, and consider the product $M_1 \times M_2$, with the product metric. Let ∇^1 be the Riemannian connection of M_1 and let ∇^2 be the Riemannian connection of M_2.

a) Show that the Riemannian connection ∇ of $M_1 \times M_2$ is given by

$$\nabla_{Y_1+Y_2}(X_1+X_2) = \nabla^1_{Y_1}X_1 + \nabla^2_{Y_2}X_2, \quad \forall X_1, Y_1 \in \mathfrak{X}(M_1),\ X_2, Y_2 \in \mathfrak{X}(M_2).$$

b) For every $p \in M_1$, the set

$$(M_2)_p = \{(p,q) \in M_1 \times M_2 | q \in M_2\}$$

is a submanifold of $M_1 \times M_2$, naturally diffeomorphic to M_2. Prove that $(M_2)_p$ is a totally geodesic submanifold of $M_1 \times M_2$.

c) Let $\sigma(x,y) \subset T_{(p,q)}(M_1 \times M_2)$ be a plane such that $x \in T_pM_1$ and $y \in T_qM_2$. Show that $K(\sigma) = 0$.

Proof. Firstly, for a), by the proof of Chapter 2 Theorem 3.6, we only need to check the identity:

$$\langle Z, \nabla_Y X\rangle = \frac{1}{2}\left(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle [X,Z],Y\rangle - \langle [Y,Z],X\rangle - \langle [X,Y],Z\rangle\right)$$

for any $X = X_1+X_2, Y = Y_1+Y_2, Z = Z_1+Z_2 \in \mathfrak{X}(M_1 \times M_2) \cong \mathfrak{X}(M_1) \oplus \mathfrak{X}(M_2)$. For X_i, Y_j in different M_i, e.g. X_1, Y_2, we have

$$\nabla_{Y_2}X_1 = \nabla_{0+Y_2}(X_1+0) = \nabla^1_0 X_1 + \nabla^2_{Y_2}0 = 0.$$

Then

$$\begin{aligned}\langle Z, \nabla_Y X\rangle &= \langle Z_1, \nabla_{Y_1}X_1\rangle + \langle Z_1, \nabla_{Y_1}X_2\rangle + \langle Z_1, \nabla_{Y_2}X_1\rangle + \langle Z_1, \nabla_{Y_2}X_2\rangle \\ &\quad + \langle Z_2, \nabla_{Y_1}X_1\rangle + \langle Z_2, \nabla_{Y_1}X_2\rangle + \langle Z_2, \nabla_{Y_2}X_1\rangle + \langle Z_2, \nabla_{Y_2}X_2\rangle \\ &= \left\langle Z_1, \nabla^1_{Y_1}X_1\right\rangle + \left\langle Z_1, \nabla^2_{Y_2}X_2\right\rangle + \left\langle Z_2, \nabla^1_{Y_1}X_1\right\rangle + \left\langle Z_2, \nabla^2_{Y_2}X_2\right\rangle.\end{aligned}$$

By the product metric (Chapter 1 Example 2.7), $\nabla^2_{Y_2}X_2 \in \mathfrak{X}(M_2)$, we have

$$\left\langle Z_1, \nabla^2_{Y_2}X_2\right\rangle = \langle Z_1, 0\rangle_{M_1} + \langle 0, \nabla_{Y_2}X_2\rangle_{M_2} = 0.$$

Similarly, $\left\langle Z_2, \nabla^1_{Y_1}X_1\right\rangle = 0$. So the left side will be

$$\begin{aligned}\langle Z, \nabla_Y X\rangle &= \left\langle Z_1, \nabla^1_{Y_1}X_1\right\rangle + \left\langle Z_1, \nabla^2_{Y_2}X_2\right\rangle + \left\langle Z_2, \nabla^1_{Y_1}X_1\right\rangle + \left\langle Z_2, \nabla^2_{Y_2}X_2\right\rangle \\ &= \left\langle Z_1, \nabla^1_{Y_1}X_1\right\rangle + \left\langle Z_2, \nabla^2_{Y_2}X_2\right\rangle.\end{aligned}$$

Then for the right side, we have

$$\begin{aligned}X\langle Y,Z\rangle &= X_1\langle Y_1,Z_1\rangle + X_1\langle Y_1,Z_2\rangle + X_1\langle Y_2,Z_1\rangle + X_1\langle Y_2,Z_2\rangle \\ &\quad + X_2\langle Y_1,Z_1\rangle + X_2\langle Y_1,Z_2\rangle + X_2\langle Y_2,Z_1\rangle + X_2\langle Y_2,Z_2\rangle\end{aligned}$$

and

$$\begin{aligned}\langle [X,Z],Y\rangle &= \langle [X_1,Z_1],Y_1\rangle + \langle [X_1,Z_2],Y_1\rangle + \langle [X_2,Z_1],Y_1\rangle + \langle [X_2,Z_2],Y_1\rangle \\ &\quad + \langle [X_1,Z_1],Y_2\rangle + \langle [X_1,Z_2],Y_2\rangle + \langle [X_2,Z_1],Y_2\rangle + \langle [X_2,Z_2],Y_2\rangle.\end{aligned}$$

Similarly, for different $i,j = 1,2$, by the product metric, $\langle X_i, Y_j\rangle = 0$. And $\langle Y_j, Z_j\rangle = \langle Y_j, Z_j\rangle_{M_j} \in C(M_j)$, so $X_i\langle Y_j, Z_j\rangle = 0$. And for any $f \in C(M_1 \times M_2)$,

$$[X_i, Z_j] = X_i(Z_j(f)) - Z_j(X_i(f)),$$

where $Z_j(f) \in C(M_j)$ and $X_i(f) \in C(M_i)$, so we have $[X_i, Z_j] = X_i(Z_j(f)) - Z_j(X_i(f)) = 0$. And $[X_i, Z_i] \in \mathfrak{X}(M_i)$, so $\langle [X_i, Z_i], Y_j\rangle = 0$. by product metric. Above all, we have

$$\begin{aligned}X\langle Y,Z\rangle &= X_1\langle Y_1,Z_1\rangle + X_1\langle Y_1,Z_2\rangle + X_1\langle Y_2,Z_1\rangle + X_1\langle Y_2,Z_2\rangle \\ &\quad + X_2\langle Y_1,Z_1\rangle + X_2\langle Y_1,Z_2\rangle + X_2\langle Y_2,Z_1\rangle + X_2\langle Y_2,Z_2\rangle \\ &= X_1\langle Y_1,Z_1\rangle + X_2\langle Y_2,Z_2\rangle\end{aligned}$$

and

$$\begin{aligned}\langle [X,Z],Y\rangle &= \langle [X_1,Z_1],Y_1\rangle + \langle [X_1,Z_2],Y_1\rangle + \langle [X_2,Z_1],Y_1\rangle + \langle [X_2,Z_2],Y_1\rangle \\ &\quad + \langle [X_1,Z_1],Y_2\rangle + \langle [X_1,Z_2],Y_2\rangle + \langle [X_2,Z_1],Y_2\rangle + \langle [X_2,Z_2],Y_2\rangle \\ &= \langle [X_1,Z_1],Y_1\rangle + \langle [X_2,Z_2],Y_2\rangle .\end{aligned}$$

So the right side will be

$$\begin{aligned}&\frac{1}{2}\left(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle [X,Z],Y\rangle - \langle [Y,Z],X\rangle - \langle [X,Y],Z\rangle\right)\\ =\ &\frac{1}{2}\left(X_1\langle Y_1,Z_1\rangle + X_2\langle Y_2,Z_2\rangle + Y_1\langle Z_1,X_1\rangle + Y_2\langle Z_2,X_2\rangle - Z_1\langle X_1,Y_1\rangle - Z_2\langle X_2,Y_2\rangle\right.\\ &\quad \left.- \langle [X_1,Z_1],Y_1\rangle - \langle [X_2,Z_2],Y_2\rangle - \langle [Y_1,Z_1],X_1\rangle - \langle [Y_2,Z_2],X_2\rangle - \langle [X_1,Y_1],Z_1\rangle - \langle [X_2,Y_2],Z_2\rangle\right)\\ =\ &\frac{1}{2}\left(X_1\langle Y_1,Z_1\rangle + Y_1\langle Z_1,X_1\rangle - Z_1\langle X_1,Y_1\rangle - \langle [X_1,Z_1],Y_1\rangle - \langle [Y_1,Z_1],X_1\rangle - \langle [X_1,Y_1],Z_1\rangle\right)\\ &\quad + \frac{1}{2}\left(X_2\langle Y_2,Z_2\rangle + Y_2\langle Z_2,X_2\rangle - Z_2\langle X_2,Y_2\rangle - \langle [X_2,Z_2],Y_2\rangle - \langle [Y_2,Z_2],X_2\rangle - \langle [X_2,Y_2],Z_2\rangle\right)\\ =\ &\left\langle Z_1,\nabla^1_{Y_1}X_1\right\rangle + \left\langle Z_2,\nabla^2_{Y_2}X_2\right\rangle\\ =\ &\left\langle Z,\nabla_Y X\right\rangle,\end{aligned}$$

as desired. Then by uniqueness of Riemannian connection, we proved that ∇ is the Riemannian connection of $M_1\times M_2$, with the product metric.

Then for b), by the product metric, for any $X\in\mathfrak{X}(M_1)$ and $Y\in\mathfrak{X}(M_2)$,

$$\langle X,Y\rangle = \langle X,0\rangle_{M_1} + \langle 0,Y\rangle_{M_2} = 0.$$

So we have $(T_q((M_2)_p))^\perp = T_p(M_1)\subset T_{(p,q)}(M_1\times M_2)$, for any $q\in M_2$. Then for any $\eta\in(T_q((M_2)_p))^\perp = T_p(M_1)$ and $x,y\in T_q((M_2)_p)$, we have $B(x,y) = (\nabla_X Y(q))^N$, where X,Y are local extensions of X,Y. Because $x,y\in T_q((M_2)_p)$, we have $X,Y\in\mathfrak{X}((M_2)_p)$. Then

$$\nabla_X Y = \nabla_{0+X}(0+Y) = \nabla^1_0 0 + \nabla^2_X Y = \nabla^2_X Y\in\mathfrak{X}((M_2)_p),$$

i.e. $(\nabla_X Y)^N = 0$. So we have

$$H_\eta(x,y) = \langle B(x,y),\eta\rangle = \left\langle (\nabla_X Y)^N,\eta\right\rangle = \langle 0,\eta\rangle = 0.$$

By the arbitrariness of x,y, we have $H_\eta = 0$; and by the arbitrariness of η, we know $(M_2)_p$ is geodesic at $q\in(M_2)_p = M_2$; finally by the the arbitrariness of q, we conclude that $(M_2)_p$ is a totally geodesic submanifold of $M_1\times M_2$.

Finally, for c), by definition, we have

$$K(\sigma) = K(x,y) = \langle R(X,Y)X,Y\rangle = \left\langle \nabla_Y\nabla_X X - \nabla_X\nabla_Y X + \nabla_{[X,Y]}X, Y\right\rangle,$$

where $X,Y\in\mathfrak{X}(M_1\times M_2)$ are local extensions of $x,y\in T_{(p,q)}(M_1\times M_2)$. Because $x\in T_p(M_1)$ and $y\in T_q(M_2)$, in the proof of a), we have proved that $\nabla_Y X = 0$ and $[X,Y] = 0$. In addition, by the proof of b), we proved $\nabla_X X = \nabla^1_X X\in\mathfrak{X}(M_1)$, so we also have $\nabla_Y(\nabla_X X) = 0$. Above all, we have

$$\nabla_Y\nabla_X X - \nabla_X\nabla_Y X + \nabla_{[X,Y]}X = 0 - \nabla_X 0 + \nabla_0 X = 0.$$

Then we conclude

$$K(\sigma) = \left\langle \nabla_Y\nabla_X X - \nabla_X\nabla_Y X + \nabla_{[X,Y]}X, Y\right\rangle = \langle 0,Y\rangle = 0,$$

as desired. □

Exercise 6.2. Show that $\mathbf{x}\colon\mathbb{R}^2\to\mathbb{R}^4$ given by

$$\mathbf{x}(\theta,\varphi) = \frac{1}{\sqrt{2}}(\cos\theta,\sin\theta,\cos\varphi,\sin\varphi),\qquad \forall(\theta,\varphi)\in\mathbb{R}^2$$

is an immersion of $\mathbb{R}^2$ into the unit sphere $S^3\subset\mathbb{R}^4$, whose image $\mathbf{x}\left(\mathbb{R}^2\right)$ is a torus T^2 with the sectional curvature zero in the induced metric.

Proof. Firstly, at any $(\theta, \varphi) \in \mathbb{R}^2$,

$$(d\mathbf{x})_{(\theta,\varphi)} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \\ 0 & -\sin\varphi \\ 0 & \cos\varphi \end{pmatrix},$$

where $-\sin\theta$ and $\cos\theta$ can not vanish simultaneously, and so do $-\sin\varphi$ and $\cos\varphi$. So we have

$$\operatorname{rank}\left((d\mathbf{x})_{(\theta,\varphi)}\right) = \operatorname{rank}\left(\frac{1}{\sqrt{2}} \begin{pmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \\ 0 & -\sin\varphi \\ 0 & \cos\varphi \end{pmatrix}\right) = 2 = \dim(\mathbb{R}^2),$$

i.e. $\mathbf{x}$ is an immersion. We also have

$$|\mathbf{x}(\theta,\varphi)|^2 = \frac{1}{2}\left(\cos^2\theta + \sin^2\theta + \cos^2\varphi + \sin^2\varphi\right) = \frac{1}{2}(1+1) = 1,$$

i.e. $|\mathbf{x}(\theta,\varphi)| = 1$, then $\mathbf{x}(\theta,\varphi) \in S^3 \subset \mathbb{R}^4$. So we conclude that $\mathbf{x}$ is an immersion of $\mathbb{R}^2$ into the unit sphere $S^3 \subset \mathbb{R}^4$. Moreover, by the periodicity of sin and cos, we have

$$\mathbf{x}\left(\mathbb{R}^2\right) = \mathbf{x}\left([0,2\pi]\times[0,2\pi]\right),$$

where we also have $\mathbf{x}(\theta,0) = \mathbf{x}(\theta,2\pi)$ and $\mathbf{x}(0,\varphi) = \mathbf{x}(2\pi,\varphi)$ for all $\theta,\varphi \in [0,2\pi]$. This means $\mathbf{x}\left(\mathbb{R}^2\right) = \mathbf{x}\left([0,2\pi]\times[0,2\pi]\right)$ is given by attach the opposite sides of the square $[0,2\pi]\times[0,2\pi]$ along the same direction, which is a torus T^2. Now we calculate its sectional curvature. Because it has dimension 2, we can choose the basis $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial\varphi}$ of $T(T^2)$ to compute its sectional curvature, where we have

$$d\mathbf{x}\left(\frac{\partial}{\partial\theta}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \\ 0 & -\sin\varphi \\ 0 & \cos\varphi \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sin\theta & \cos\theta & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(-\sin\theta\frac{\partial}{\partial x_1} + \cos\theta\frac{\partial}{\partial x_2}\right)$$

and

$$d\mathbf{x}\left(\frac{\partial}{\partial\varphi}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \\ 0 & -\sin\varphi \\ 0 & \cos\varphi \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & -\sin\varphi & \cos\varphi \end{pmatrix} = \frac{1}{\sqrt{2}}\left(-\sin\varphi\frac{\partial}{\partial x_3} + \cos\varphi\frac{\partial}{\partial x_4}\right).$$

Because $\mathbb{R}^4$ is flat, we have $\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right\rangle = \delta_{ij}$. Then under the induce metric, we have

$$\begin{aligned} g_{11} = \left\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right\rangle &= \left\langle d\mathbf{x}\left(\frac{\partial}{\partial\theta}\right), d\mathbf{x}\left(\frac{\partial}{\partial\theta}\right)\right\rangle \\ &= \frac{1}{2}\left\langle -\sin\theta\frac{\partial}{\partial x_1} + \cos\theta\frac{\partial}{\partial x_2}, -\sin\theta\frac{\partial}{\partial x_1} + \cos\theta\frac{\partial}{\partial x_2}\right\rangle = \frac{1}{2}\left(\sin^2\theta + \cos^2\theta\right) = \frac{1}{2}, \end{aligned}$$

$$\begin{aligned} g_{12} = g_{21} = \left\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi}\right\rangle &= \left\langle d\mathbf{x}\left(\frac{\partial}{\partial\theta}\right), d\mathbf{x}\left(\frac{\partial}{\partial\varphi}\right)\right\rangle \\ &= \frac{1}{2}\left\langle -\sin\theta\frac{\partial}{\partial x_1} + \cos\theta\frac{\partial}{\partial x_2}, -\sin\varphi\frac{\partial}{\partial x_3} + \cos\varphi\frac{\partial}{\partial x_4}\right\rangle = 0, \end{aligned}$$

$$\begin{aligned} g_{22} = \left\langle \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\varphi}\right\rangle &= \left\langle d\mathbf{x}\left(\frac{\partial}{\partial\varphi}\right), d\mathbf{x}\left(\frac{\partial}{\partial\varphi}\right)\right\rangle \\ &= \frac{1}{2}\left\langle -\sin\varphi\frac{\partial}{\partial x_3} + \cos\varphi\frac{\partial}{\partial x_4}, -\sin\varphi\frac{\partial}{\partial x_3} + \cos\varphi\frac{\partial}{\partial x_4}\right\rangle = \frac{1}{2}\left(\sin^2\varphi + \cos^2\varphi\right) = \frac{1}{2}. \end{aligned}$$

This means the induce metric of T^2 is half of the flat metric of $\mathbb{R}^2$. It will not change the connection and so does the sectional curvature, by definition. So we must have $K(p) = 0$ at any $p \in T^2$, as deisred. (In fact, we have $T^2 \cong S^1 \times S^1 = \left\{\left(e^{i\theta}, e^{i\varphi}\right) \middle| \theta,\varphi \in \mathbb{R}\right\}$. Then $\frac{\partial}{\partial\theta}$ is a vector field on the first S^1 and $\frac{\partial}{\partial\varphi}$ on the second. By Exercise 6.1 c), we must have $K(p) = 0$ at any $p \in T^2$.) ∎

Exercise 6.3. Let M be a Riemannian manifold and let $N \subset K \subset M$ be submanifolds of M. Suppose that N is totally geodesic in K and that K is totally geodesic in M. Prove that N is totally geodesic in M.

Proof. At any $p \in N$, by Chapter 6 Proposition 2.9, any geodesic γ of N starting from p is a geodesic of K; using the proposition again, we have that it is a geodesic of M. Using the proposition inversely, we have N is geodesic in M, at p. Finally, by the arbitrariness of $p \in N$, we conclude that N is totally geodesic in M, as deisred. $\square$

Exercise 6.4. Let $N_1 \subset M_1$, $N_2 \subset M_2$ be totally geodesic submanifolds of the Riemannian manifolds M_1 and M_2, respectively. Prove that $N_1 \times N_2$ is a totally geodesic submanifold of the product $M_1 \times M_2$ with the product metric.

Proof. Firstly, let's prove a claim. For any $(p_1, p_2) \in M_1 \times M_2$, let

$$\gamma(t) = (\gamma_1(t), \gamma_2(t))$$

be a path of $M_1 \times M_2$ starting at (p_1, p_2), where $\gamma_i(t)$ is a path of M_i starting at p_i, $i = 1, 2$. We claim that $\gamma(t)$ is a geodesic if and only if $\gamma_1(t)$ and $\gamma_2(t)$ are all geodesics: Let $e_1, \cdots, e_{m_1}$ and $f_1, \cdots, f_{m_2}$ be the geodesic frame (see Exercise 3.7) of M_1 at p_1 and M_2 at p_2, respectively. Under the product metric (see Chapter 1 Example 2.7), we have

$$\langle e_i, f_j \rangle = \langle e_i + 0, 0 + f_j \rangle = \langle e_i, 0 \rangle_{N_1} + \langle 0, f_j \rangle_{N_2} = 0 + 0 = 0, \quad \forall 1 \le i \le m_1,\ \forall 1 \le j \le m_2.$$

On $T(M_1 \times M_2)$, we also have $\mathrm{Span}\{e_1, \cdots, e_{m_1}, f_1, \cdots, f_{m_2}\} = T(M_1 \times M_2)$ and $\mathrm{Span}\{e_1, \cdots, e_{m_1}\} \cap \mathrm{Span}\{f_1, \cdots, f_{m_2}\} = \varnothing$. Then

$$\gamma'(t) = (\gamma_1'(t), \gamma_2'(t)) = \gamma_1'(t) + \gamma_2'(t),$$

where $\gamma_1'(t) \in \mathrm{Span}\{e_1, \cdots, e_{m_1}\}$ and $\gamma_2'(t) \in \mathrm{Span}\{f_1, \cdots, f_{m_1}\}$. So by Exercise 6.1 a), if $\gamma(t)$ is a geodesic, we have

$$0 = \frac{D\gamma'}{dt} = \nabla_{\gamma'}\gamma' = \nabla_{\gamma_1'+\gamma_2'}(\gamma_1' + \gamma_2') = \nabla^1_{\gamma_1'}\gamma_1' + \nabla^2_{\gamma_2'}\gamma_2' = \frac{D_1\gamma_1'}{dt} + \frac{D_2\gamma_2'}{dt},$$

where $\frac{D_1\gamma_1'}{dt} \in \mathrm{Span}\{e_1, \cdots, e_{m_1}\}$ and $\frac{D_2\gamma_2'}{dt} \in \mathrm{Span}\{f_1, \cdots, f_{m_2}\}$. Because

$$\mathrm{Span}\{e_1, \cdots, e_{m_1}\} \cap \mathrm{Span}\{f_1, \cdots, f_{m_2}\} = \varnothing,$$

we must have $\frac{D_1\gamma_1'}{dt} = 0$ and $\frac{D_2\gamma_2'}{dt} = 0$, i.e. $\gamma_1(t)$ and $\gamma_2(t)$ are all geodesics. Conversely, if $\gamma_1(t)$ and $\gamma_2(t)$ are all geodesics, we have

$$\frac{D\gamma'}{dt} = \nabla_{\gamma'}\gamma' = \nabla_{\gamma_1'+\gamma_2'}(\gamma_1' + \gamma_2') = \nabla^1_{\gamma_1'}\gamma_1' + \nabla^2_{\gamma_2'}\gamma_2' = \frac{D_1\gamma_1'}{dt} + \frac{D_2\gamma_2'}{dt} = 0 + 0 = 0,$$

i.e. $\gamma(t)$ is a geodesic. Now we have proved our claim. And it's easy to see from the proof that, on $N_1 \times N_2$, our claim is also true.

At any $(p_1, p_2) \in N_1 \times N_2$, for any geodesic $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ of $N_1 \times N_2$ starting at (p_1, p_2), by the claim, $\gamma_i(t)$ is a geodesic of N_i, $i = 1, 2$. Using Chapter 6 Proposition 2.9 and the conditions, we have $\gamma_i(t)$ is also a geodesic of M_i, $i = 1, 2$. Then by the claim conversely, $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ is a geodesic of $M_1 \times M_2$. By the arbitrariness of the geodesic $\gamma(t)$, using Chapter 6 Proposition 2.9 conversely, we know that $N_1 \times N_2$ is geodesic in $M_1 \times M_2$ at (p_1, p_2). By the arbitrariness of $(p_1, p_2) \in N_1 \times N_2$, we conclude that $N_1 \times N_2$ is a totally geodesic submanifold of $M_1 \times M_2$ with the product metric, as desired. $\square$

Exercise 6.5. Prove that the sectional curvature of the Riemannian manifold $S^2 \times S^2$ with the product metric, where S^2 is the unit sphere in $\mathbb{R}^3$, is non-negative. Find a totally geodeisc, flat torus, T^2, embedded in $S^2 \times S^2$.

Proof. Let $\{(\rho_i, \theta_i) | 0 \le \rho \le \pi, -\pi < \theta_i \le \pi\}$ be the polar coordinates (see Exercise 5.6 a)) of the ith S^2, $i = 1, 2$. Under the product metric, we have $\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\rho_1}, \frac{\partial}{\partial\theta_2}, \frac{\partial}{\partial\rho_2}$ is a basis of $T(S^2 \times S^2)$ and $\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\rho_i}$ is the basis of the ith $T(S^2)$, as a submanifold. Now we can compute $K(\sigma)$ for $\sigma = \mathrm{Span}\{x, y\}$. If x, y in the same $T(S^2)$, e.g. the first one, by Exercise 6.1 a), we have

$$\nabla_X Y = \nabla_{X+0}(Y + 0) = \nabla^1_X Y + \nabla^2_0 0 = \nabla^1_X Y,$$

where X, Y are local extensions of x, y. This means the Riemannian connection of $S^2 \times S^2$ restricted to the first S^2 is same as the Riemannian connection of the first S^2, and so does the curvature. Using Chapter 6 Example 2.8, we have

$$K(\sigma) = K_{S^2}(\sigma) = 1 \geq 0.$$

And it's true for x, y in the second $T(S^2)$. If x, y in differnet $T(S^2)$, by Exercise 6.1 c), we know that

$$K(\sigma) = 0 \geq 0.$$

Because S^2 has dimension 2, we have finished all the possibility of $\sigma = \mathrm{Span}\{x, y\}$. Above all, we have $K(\sigma) \geq 0$, i.e. the sectional curvature of $S^2 \times S^2$ under the product metric is non-negative, as desired.

Let's choose the great circles $\left\{ \left(\frac{\pi}{2}, \theta_i\right) \middle| -\pi < \theta_i \leq \pi \right\} = S^1$ in the ith S^2 with the basis $\frac{\partial}{\partial \theta_i}$ of its tangent space, $i = 1, 2$. Then we have the product of the two S^1s is a torus $T^2 = S^1 \times S^1$, i.e.

$$T^2 = \left\{ \left(\frac{\pi}{2}, \theta_1, \frac{\pi}{2}, \theta_2\right) \middle| -\pi < \theta_1, \theta_2 \leq \pi \right\} \subset S^2 \times S^2.$$

And because we get the T^2 just by assigning some coordinates of $S^2 \times S^2$, it is embedded in $S^2 \times S^2$. Because the geodesic of S^1 are segments of itself and the great circles are geodesic of S^2 (see Chapter 3 Example 2.11), any geodesic of S^1 is geodesic of S^2. By Chapter 6 Proposition 2.9, S^1 is totally geodesic in S^2, for the two $S^1 \subset S^2$. Then by Exercise 6.4, our $T^2 = S^1 \times S^1$ is a totally geodesic submanifold of $S^2 \times S^2$. Finally, because $\frac{\partial}{\partial \theta_i}$ is in the ith $T(S^1)$, $i = 1, 2$, by Exercise 6.1 c), the only sectional curvature of our $T^2 = S^1 \times S^1$ will be

$$K = K\left(\sigma\left(\frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \theta_2}\right)\right) = 0.$$

This means our T^2 is flat, as desired. □

Exercise 6.6. Let G be a Lie group with a bi-variant metric. Let H be a Lie group and let $h\colon H \to G$ be an immersion that is also a homomorphism of groups (that is, H is a Lie subgroup of G). Show that h is a totally geodesic immersion.

Proof. Firstly, we prove that the bi-invariant metric of G can induce a bi-invariant metric of H. By the induce metric, we have

$$\langle X, Y \rangle_H = \langle dh(X), dh(Y) \rangle_G, \qquad \forall X, Y \in \mathfrak{X}(H).$$

For any $g \in H$,

$$\langle dL_g(X), dL_g(Y) \rangle_H = \langle dh(dL_g(X)), dh(dL_g(Y)) \rangle_G.$$

Let α, β be the trajectories of X, Y. We have

$$(h(L_g(\alpha)))' = dh(dL_g(X)).$$

On the other hand,

$$h(L_g(\alpha)) = h(g \cdot \alpha) = h(g) \cdot h(\alpha) = L_{h(g)}(h(\alpha)).$$

So we also have

$$(h(L_g(\alpha)))' = (L_{h(g)}(h(\alpha))' = dL_{h(g)}(dh(X)).$$

Then we have

$$dh(dL_g(X)) = (h(L_g(\alpha)))' = dL_{h(g)}(dh(X))$$

and

$$dh(dL_g(Y)) = (h(L_g(\beta)))' = dL_{h(g)}(dh(Y)),$$

similarly. Above all, because $\langle\ ,\ \rangle_G$ is bi-invariant, we have

$$\begin{aligned} \langle dL_g(X), dL_g(Y) \rangle_H &= \langle dh(dL_g(X)), dh(dL_g(Y)) \rangle_G \\ &= \left\langle dL_{h(g)}(dh(X)), dL_{h(g)}(dh(Y)) \right\rangle_G \\ &= \langle dh(X), dh(Y) \rangle_G = \langle X, Y \rangle_H. \end{aligned}$$

This means $\langle\ ,\ \rangle_H$ is left-invariant. Similarly, we can prove $\langle\ ,\ \rangle_H$ is also right-invariant. So we proved that the induce metric $\langle\ ,\ \rangle_H$ of $\langle\ ,\ \rangle_G$ is also bi-invariant.

Then at $e \in H$, for any geodesic γ of H starting at e, by Exercise 3.3 b), it is a 1-parameter subgroup of H. Under the homomorphism, $h(\gamma)$ is naturally a 1-parameter subgroup of G. From the proof of Exercise 3.3 b), we also know that any 1-parameter subgroup of a Lie group is a geodesic. So $h(\gamma)$ is also a geodesic of G. It follows that H is totally geodesic in G at e.

At any other $g \in H$, let's consider the map $L_{g^{-1}}$. Because $\langle\ ,\ \rangle_H$ is bi-invariant,

$$L_{g^{-1}}\colon H \to L_{g^{-1}}(H) = g^{-1}H \cong H$$

is an isometry. Then by the minimal property of geodesic (Chapter 3 Corollary 3.9), under $L_{g^{-1}}$, any geodesic γ of H starting at g is also a geodesic $L_{g^{-1}}(\gamma)$ of $L_{g^{-1}}(H)$ starting at e. Because L_g is an isometry and a homomorphism, the map

$$h \circ L_g\colon L_{g^{-1}}(H) \to G$$

is an immersion an a homomorphism. So by the proof above, we have $h \circ L_g(L_{g^{-1}}(\gamma))$ is a geodesic of G, where

$$h \circ L_g\left(L_{g^{-1}}(\gamma)\right) = h\left(L_{gg^{-1}}(\gamma)\right) = h(L_e(\gamma)) = h(\gamma),$$

i.e. $h(\gamma)$ is also a geodesic of G. So, to sum up, we proved that, under h, any geodesic γ of H starting at g is also a geodesic $h(\gamma)$ of G. This means H is totally geodesic in G at g. By the arbitrariness of $g \in H$, we conclude that H is a totally geodesic submanifold of G, i.e. h is a totally geodesic immersion, as desired. ☐

Exercise 6.7. Show that if M is a totally geodesic submanifold of $\overline{M}$, then, for any tangent fields to M, ∇ and $\overline{\nabla}$ coincide.

Proof. Firstly, because M is totally geodesic in $\overline{M}$, by definition, at any $p \in M$, for any $\eta \in (T_pM)^\perp$, $H_\eta = 0$. Then for any $x, y \in T_pM$, by definition, for any $\eta \in (T_pM)^\perp$,

$$\langle B(x,y), \eta\rangle = H_\eta(x,y) = 0.$$

Because $B(x,y) \in (T_pM)^\perp$, we must have $B(x,y) = 0$. By the arbitrariness of $p \in M$ and $x, y \in T_pM$, for any $X, Y \in \mathfrak{X}(M)$, we have $B(X,Y) = 0$. Then by definition,

$$0 = B(X,Y) = \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y,$$

where $\overline{X}, \overline{Y}$ are local extensions of X, Y. Then for any tangent fields X, Y to M, we have $\overline{X} = X$ and $\overline{Y} = Y$. It follow that

$$\nabla_X Y = \overline{\nabla}_{\overline{X}}\overline{Y} = \overline{\nabla}_X Y.$$

By the arbitrariness of tangent fields X, Y to M, we conclude that ∇ and $\overline{\nabla}$ coincide, for any tangent fields to M, as desired. ☐

Exercise 6.8 (The Clifford torus)**.** Consider the immersion $\mathbf{x}\colon \mathbb{R}^2 \to \mathbb{R}^4$ given in Exercise 6.2.

a) Show that the vectors

$$e_1 = (-\sin\theta, \cos\theta, 0, 0), \qquad e_2 = (0, 0, -\sin\varphi, \cos\varphi)$$

form an orthonormal basis of the tangent space, and that the vector

$$n_1 = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi), \qquad n_2 = \frac{1}{\sqrt{2}}(-\cos\theta, -\sin\theta, \cos\varphi, \sin\varphi)$$

form an orthonormal basis of the normal space.

b) Use the fact that

$$\langle S_{n_k}(e_i), e_j\rangle = -\left\langle \overline{\nabla}_{e_i} n_k, e_j\right\rangle = \left\langle \overline{\nabla}_{e_i} e_j, n_k\right\rangle,$$

where $\overline{\nabla}$ is the covariant derivative (that is, the usual derivative, see the end of Chapter 2) of $\mathbb{R}^4$, and $i, j, k = 1, 2$, to establish that the matrices of S_{n_1} and S_{n_2} with respect to the basis $\{e_1, e_2\}$ are

$$S_{n_1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad S_{n_2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

c) From Exercise 6.2, $\mathbf{x}$ is an immersion of the torus T^2 into S^3 (the Clifford torus). Show that $\mathbf{x}$ is a minimal immersion.

Proof. Firstly, for a), from the proof of Exercise 6.2, we know that

$$e_1 = (-\sin\theta, \cos\theta, 0, 0) = \sqrt{2}d\mathbf{x}\left(\frac{\partial}{\partial\theta}\right) \quad \text{and} \quad e_2 = (0, 0, -\sin\varphi, \cos\varphi) = \sqrt{2}d\mathbf{x}\left(\frac{\partial}{\partial\varphi}\right)$$

is a basis of the tangent space. And

$$\langle e_1, e_1\rangle = 2\left\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right\rangle = 2g_{11} = 1,$$

$$\langle e_1, e_2\rangle = 2\left\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi}\right\rangle = 2g_{12} = 0,$$

$$\langle e_2, e_2\rangle = 2\left\langle \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\varphi}\right\rangle = 2g_{22} = 1.$$

So e_1, e_2 is an orthonormal basis of the tangent space. Moreover, on the Euclidean space $\mathbb{R}^4$, calculating directly, we have

$$\langle e_1, n_1\rangle = \left\langle (-\sin\theta, \cos\theta, 0, 0), \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi)\right\rangle_{\mathbb{R}^4} = \frac{1}{\sqrt{2}}(-\sin\theta\cos\theta + \cos\theta\sin\theta) = 0,$$

$$\langle e_2, n_1\rangle = \left\langle (0, 0, -\sin\varphi, \cos\varphi), \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi)\right\rangle_{\mathbb{R}^4} = \frac{1}{\sqrt{2}}(-\sin\varphi\cos\varphi + \cos\varphi\sin\varphi) = 0.$$

So we have $n_1 \in \left(T\left(\mathbf{x}\left(\mathbb{R}^2\right)\right)\right)^\perp$, i.e. the normal space. With similar calculations, we can also check that n_2 is in the normal space. Noticing that n_1 and n_2 are linear independent and counting the dimension, we know that n_1, n_2 is a basis of the normal space, by linear algebra. Then we also have

$$\begin{aligned}\langle n_1, n_1\rangle &= \left\langle \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi), \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi)\right\rangle_{\mathbb{R}^4} \\ &= \frac{1}{2}\left(\cos^2\theta + \sin^2\theta + \cos^2\varphi + \sin^2\varphi\right) = \frac{1}{2}(1+1) = 1,\end{aligned}$$

$$\begin{aligned}\langle n_1, n_2\rangle &= \left\langle \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi), \frac{1}{\sqrt{2}}(-\cos\theta, -\sin\theta, \cos\varphi, \sin\varphi)\right\rangle_{\mathbb{R}^4} \\ &= \frac{1}{2}\left(-\cos^2\theta - \sin^2\theta + \cos^2\varphi + \sin^2\varphi\right) = \frac{1}{2}(-1+1) = 0,\end{aligned}$$

$$\begin{aligned}\langle n_2, n_2\rangle &= \left\langle \frac{1}{\sqrt{2}}(-\cos\theta, -\sin\theta, \cos\varphi, \sin\varphi), \frac{1}{\sqrt{2}}(-\cos\theta, -\sin\theta, \cos\varphi, \sin\varphi)\right\rangle_{\mathbb{R}^4} \\ &= \frac{1}{2}\left(\cos^2\theta + \sin^2\theta + \cos^2\varphi + \sin^2\varphi\right) = \frac{1}{2}(1+1) = 1.\end{aligned}$$

So n_1, n_2 is an orthonormal basis of the normal space.

For b), denoting $X_i = \frac{\partial}{\partial x_i}$, $i = 1, 2, 3, 4$ which is the basis of $T\mathbb{R}^4$, we have $\langle X_i, X_j\rangle = \delta_{ij}$ and $\nabla_{X_i}X_j = 0$, $\forall i, j = 1, 2, 3, 4$. Then we have

$$e_1 = (-\sin\theta, \cos\theta, 0, 0) = -\sin\theta X_1 + \cos\theta X_2, \qquad e_2 = (0, 0, -\sin\varphi, \cos\varphi) = -\sin\varphi X_3 + \cos\varphi X_4,$$

$$n_1 = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi) = \frac{1}{\sqrt{2}}\left(\cos\theta X_1 + \sin\theta X_2 + \cos\varphi X_3 + \sin\varphi X_4\right),$$

$$n_2 = \frac{1}{\sqrt{2}}(-\cos\theta, -\sin\theta, \cos\varphi, \sin\varphi) = \frac{1}{\sqrt{2}}\left(-\cos\theta X_1 - \sin\theta X_2 + \cos\varphi X_3 + \sin\varphi X_4\right).$$

By the fact,

$$\langle S_{n_1}(e_1), e_1\rangle = \left\langle \overline{\nabla}_{e_1} e_1, n_1 \right\rangle,$$

where

$$\begin{aligned}\overline{\nabla}_{e_1} e_1 &= \overline{\nabla}_{-\sin\theta X_1+\cos\theta X_2}\left(-\sin\theta X_1+\cos\theta X_2\right)\\ &= -\sin\theta\overline{\nabla}_{X_1}\left(-\sin\theta X_1+\cos\theta X_2\right)+\cos\theta\overline{\nabla}_{X_2}\left(-\sin\theta X_1+\cos\theta X_2\right).\end{aligned}$$

Now

$$\begin{aligned}\overline{\nabla}_{X_1}\left(-\sin\theta X_1+\cos\theta X_2\right) &= -\sin\theta\overline{\nabla}_{X_1}X_1+X_1\left(-\sin\theta\right)X_1+\cos\theta\overline{\nabla}_{X_1}X_2+X_1\left(\cos\theta\right)X_2\\ &= X_1\left(-\sin\theta\right)X_1+X_1\left(\cos\theta\right)X_2\end{aligned}$$

and

$$\begin{aligned}\overline{\nabla}_{X_2}\left(-\sin\theta X_1+\cos\theta X_2\right) &= -\sin\theta\overline{\nabla}_{X_2}X_1+X_2\left(-\sin\theta\right)X_1+\cos\theta\overline{\nabla}_{X_2}X_2+X_2\left(\cos\theta\right)X_2\\ &= X_2\left(-\sin\theta\right)X_1+X_2\left(\cos\theta\right)X_2.\end{aligned}$$

Noticing that $x_1 = \frac{1}{\sqrt{2}}\cos\theta$ and $x_2 = \frac{1}{\sqrt{2}}\sin\theta$, we have $x_1^2+x_2^2 = \frac{1}{2}$ and $\tan\theta = \frac{x_2}{x_1}$ and then $\theta = \arctan\left(\frac{x_2}{x_1}\right)$. So we have

$$\frac{\partial\theta}{\partial x_1} = \frac{-\frac{x_2}{x_1^2}}{1+\left(\frac{x_2}{x_1}\right)^2} = \frac{-x_2}{x_1^2+x_2^2} = -2x_2 = \sqrt{2}\sin\theta \quad \text{and} \quad \frac{\partial\theta}{\partial x_2} = \frac{\frac{1}{x_1}}{1+\left(\frac{x_2}{x_1}\right)^2} = \frac{x_1}{x_1^2+x_2^2} = 2x_1 = \sqrt{2}\cos\theta.$$

Then we have

$$X_1\left(-\sin\theta\right) = \frac{\partial}{\partial x_1}\left(-\sin\theta\right) = \frac{\partial\left(-\sin\theta\right)}{\partial\theta}\frac{\partial\theta}{\partial x_1} = \sqrt{2}\cos\theta\sin\theta,$$

$$X_1\left(\cos\theta\right) = \frac{\partial}{\partial x_1}\left(\cos\theta\right) = \frac{\partial\left(\cos\theta\right)}{\partial\theta}\frac{\partial\theta}{\partial x_1} = \sqrt{2}\sin^2\theta$$

and

$$X_2\left(-\sin\theta\right) = \frac{\partial}{\partial x_2}\left(-\sin\theta\right) = \frac{\partial\left(-\sin\theta\right)}{\partial\theta}\frac{\partial\theta}{\partial x_2} = -\sqrt{2}\cos^2\theta,$$

$$X_2\left(\cos\theta\right) = \frac{\partial}{\partial x_2}\left(\cos\theta\right) = \frac{\partial\left(\cos\theta\right)}{\partial\theta}\frac{\partial\theta}{\partial x_2} = -\sqrt{2}\sin\theta\cos\theta.$$

So

$$\overline{\nabla}_{X_1}\left(-\sin\theta X_1+\cos\theta X_2\right) = X_1\left(-\sin\theta\right)X_1+X_1\left(\cos\theta\right)X_2 = \sqrt{2}\left(\cos\theta\sin\theta X_1+\sin^2\theta X_2\right)$$

and

$$\overline{\nabla}_{X_2}\left(-\sin\theta X_1+\cos\theta X_2\right) = X_2\left(-\sin\theta\right)X_1+X_2\left(\cos\theta\right)X_2 = -\sqrt{2}\left(\cos^2\theta X_1-\sin\theta\cos\theta X_2\right).$$

Above all, we have

$$\begin{aligned}\overline{\nabla}_{e_1} e_1 &= -\sin\theta\overline{\nabla}_{X_1}\left(-\sin\theta X_1+\cos\theta X_2\right)+\cos\theta\overline{\nabla}_{X_2}\left(-\sin\theta X_1+\cos\theta X_2\right)\\ &= -\sin\theta\cdot\sqrt{2}\left(\cos\theta\sin\theta X_1+\sin^2\theta X_2\right)+\cos\theta\cdot\sqrt{2}\left(-\cos^2\theta X_1-\sin\theta\cos\theta X_2\right)\\ &= \sqrt{2}\left(-\sin^2\theta\cos\theta X_1-\sin^3\theta X_2-\cos^3\theta X_1-\sin\theta\cos^2\theta X_2\right)\\ &= \sqrt{2}\left(-\left(\sin^2\theta+\cos^2\theta\right)\cos\theta X_1-\left(\sin^2\theta+\cos^2\theta\right)\sin\theta X_2\right)\\ &= \sqrt{2}\left(-\cos\theta X_1-\sin\theta X_2\right).\end{aligned}$$

Finally, we conclude

$$\langle S_{n_1}(e_1), e_1\rangle = \left\langle \overline{\nabla}_{e_1} e_1, n_1\right\rangle = \left\langle \sqrt{2}\left(-\cos\theta X_1 - \sin\theta X_2\right), \frac{1}{\sqrt{2}}\left(\cos\theta X_1 + \sin\theta X_2 + \cos\varphi X_3 + \sin\varphi X_4\right)\right\rangle$$
$$= -\cos^2\theta - \sin^2\theta = -1.$$

With the same calculations for $j = 2$, we can get

$$\langle S_{n_1}(e_1), e_2\rangle = \left\langle \overline{\nabla}_{e_1} e_2, n_1\right\rangle = 0.$$

Because e_1, e_2 are orthonormal, we have

$$S_{n_1}(e_1) = -e_1 + 0e_2.$$

Doing the same calculations for $i = 2$, we have

$$S_{n_1}(e_2) = 0e_1 - e_2.$$

This means the matrix of S_{n_1} with respect to the basis $\{e_1, e_2\}$ is

$$S_{n_1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With the same whole calculations above for $k = 2$, we conclude that the matrix of S_{n_2} with respect to the basis $\{e_1, e_2\}$ is

$$S_{n_2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So we proved b).

Finally, by counting the dimension, we know that, in S^3, $\left(T\left(T^2\right)\right)^\perp$ has dimension 1. And in $\mathbb{R}^4$, for any point $\frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi)$ of the torus $\mathbf{x}\left(\mathbb{R}^2\right)$ immersed in S^3, noticing that

$$\left\langle \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi), n_2\right\rangle = \left\langle \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi), \frac{1}{\sqrt{2}}(-\cos\theta, -\sin\theta, \cos\varphi, \sin\varphi)\right\rangle_{\mathbb{R}^4}$$
$$= \frac{1}{2}\left(-\cos^2\theta - \sin^2\theta + \cos^2\varphi + \sin^2\varphi\right) = 0,$$

we conclude $n_2 \in T\left(S^3\right)$, because we all know how the tangent vectors be like of S^n in $\mathbb{R}^{n+1}$. In b), we also know that $n_2 \notin T\left(T^2\right)$. So we must have $n_2 \in \left(T\left(T^2\right)\right)^\perp$, where T^2 is an immersed submanifold of S^3. Then n_2 is a basis of $\left(T\left(T^2\right)\right)^\perp$. It follows that $\forall \eta = kn_2 \in \left(T\left(T^2\right)\right)^\perp$,

$$\langle S_\eta(e_i), e_j\rangle = \langle B(e_i, e_j), \eta\rangle = \langle B(e_i, e_j), kn_2\rangle = k\,\langle B(e_i, e_j), n_2\rangle = k\,\langle S_{n_2}(e_i), e_j\rangle\,, \quad \forall i, i = 1, 2,$$

by the definition of S_η. So we must have $S_\eta = kS_{n_2}$. Then by b),

$$\operatorname{Tr}(S_\eta) = \operatorname{Tr}(kS_{n_2}) = \operatorname{Tr}\left(k\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \operatorname{Tr}\left(\begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}\right) = -k + k = 0.$$

We conclude that, for $\mathbf{x}\left(\mathbb{R}^2\right) = T^2$ as an immersed submanifold of S^3, for any $\eta \in \left(T\left(T^2\right)\right)^\perp$, $\operatorname{Tr}(S_\eta) = 0$, i.e. $\mathbf{x}\colon \mathbb{R}^2 \to S^3 \subset \mathbb{R}^4$ is a minimal immersion; whence c). ☐

Exercise 6.9. Let $f\colon M^n \to \mathbb{R}^{m+n}$ be an immersion. Let $\eta \in (T_pM)^\perp$, $p \in M$ and $V = T_pM \oplus \mathbb{R}\eta \subset \mathbb{R}^{m+n}$, where $\mathbb{R}\eta = \{\lambda\eta | \lambda \in \mathbb{R}\}$. Let $\pi\colon \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ be the orthogonal projection onto $T_pM \oplus \mathbb{R}\eta$. Since η is transversal to M at p, $\pi|_U$ is an embedding, where U is a sufficiently small neighborhood of p in M. Let $M' = \pi(U) \subset T_pM \oplus \mathbb{R}\eta$ and let $S'_\eta\colon T_pM' = T_pM \to T_pM'$ be the operator associated to the second fundamental form of M' at p in the direction of η. Show that $S'_\eta = S_\eta$, where $S_\eta\colon T_pM \to T_pM$ is the operator associated to the second fundamental form of M' at p in the direction of η.

Proof. Let N and N' be the local extension of η normal to M and M', respectively. Then by Chapter 6 Proposition 2.3, for any $x \in T_pM = T_pM'$,

$$S_\eta(x) = -\left(\overline{\nabla}_x N\right)^T \quad \text{and} \quad S'_\eta(x) = -\left(\overline{\nabla}_x N'\right)^T.$$

We can choose U as a normal neighborhood of p. Because $\pi|_U$ is an embeding, we can see U as a embedded submanifold of $T_pM \oplus \mathbb{R}\eta$. Then the projection π will be identity on U, and then $d\pi|_{T_qM} = \mathrm{Id}$, $\forall q \in U$. Noticing that $N' = d\pi(N)$ is a local extension of η normal to M': It's easy to check that

$$N' = d\pi(N) = N \quad \text{on } U$$

and then $N'(p) = N(p) = \eta$ and $N' = N \in (TM)^\perp = (TM')^\perp$ near p. Then on U, we have

$$S'_\eta(x) = -\left(\overline{\nabla}_x N'\right)^T = -\left(\overline{\nabla}_x (d\pi(N))\right)^T = -\left(\overline{\nabla}_x N\right)^T = S_\eta(x).$$

By the arbitrariness of $x \in T_pM = T_pM'$, we conclude that $S'_\eta = S_\eta$, as desired. $\square$

Exercise 6.10. Let $f\colon M^n \to \overline{M}^{n+k}$ be an isometric immersion and let $S_\eta\colon TM \to TM$ be the operator associated to the second fundamental form of f along the normal field η. Consider S_η as a tensor of order 2 given by

$$S_\eta(X,Y) = \langle S_\eta(X), Y\rangle, \qquad \forall X, Y \in \mathfrak{X}(M).$$

Observe that saying the operator S_η is self-adjoint is equivalent to saying that the tensor S_η is symmetric, that is,

$$S_\eta(X,Y) = S_\eta(Y,X), \qquad \forall X, Y \in \mathfrak{X}(M).$$

Prove that for all $V \in \mathfrak{X}(M)$, the tensor $\nabla_V S_\eta$ is symmetric.

Proof. Because S_η is self-adjoint as our definition in the context, by the observation, S_η is a symmetric tensor. Then by definition, for any $X, Y \in \mathfrak{X}(M)$, we have

$$\begin{aligned}
\nabla_V S_\eta(X,Y) = \nabla S_\eta(X,Y,V) &= V\left(S_\eta(X,Y)\right) - S_\eta\left(\nabla_V X, Y\right) - S_\eta\left(X, \nabla_V Y\right) \\
&= V\left(S_\eta(Y,X)\right) - S_\eta\left(\nabla_V Y, X\right) - S_\eta\left(Y, \nabla_V X\right) = \nabla S_\eta(Y,X,V) = \nabla_V S_\eta(Y,X),
\end{aligned}$$

i.e. the tensor $\nabla_V S_\eta$ is symmetric, as desired. $\square$

Exercise 6.11. Let $f\colon \overline{M}^{n+1} \to \mathbb{R}$ be a differentiable function. Define the *Hessian*, $\operatorname{Hess} f$ of f at $p \in \overline{M}$ as the linear operator

$$\begin{aligned}
\operatorname{Hess} f\colon T_p\overline{M} &\to T_p\overline{M} \\
Y &\mapsto \overline{\nabla}_Y \left(\operatorname{grad} f\right),
\end{aligned}$$

where $\overline{\nabla}$ is the Riemannian connection of $\overline{M}$. Let a be a regular value of f and let $M^n \subset \overline{M}^{n+1}$ be the hypersuperface in $\overline{M}$ defined by

$$M = \left\{ p \in \overline{M} \,\middle|\, f(p) = a \right\}.$$

Prove that:

a) The Laplacian $\overline{\Delta} f$ is given by

$$\overline{\Delta} f(p) = \operatorname{Tr}(\operatorname{Hess} f).$$

b) If $X, Y \in \mathfrak{X}\left(\overline{M}\right)$, then

$$\langle (\operatorname{Hess} f)(Y), X\rangle = \langle Y, (\operatorname{Hess} f)(X)\rangle.$$

Conclude that $\operatorname{Hess} f$ is self-adjoint, hence determines a symmetric bilinear form on $T_p\overline{M}$, $p \in \overline{M}$, given by

$$(\operatorname{Hess} f)(X,Y) = \langle (\operatorname{Hess} f)(X), Y\rangle, \qquad \forall X, Y \in T_p\overline{M}.$$

c) The mean curvature H of $M \subset \overline{M}$ is given by

$$nH = -\operatorname{div}\left(\frac{\operatorname{grad} f}{|\operatorname{grad} f|}\right).$$

d) Observe that every embedded hypersurface $M^n \subset \overline{M}^{n+1}$ is locally the inverse image of a regular value. Conclude from c) that the mean curvature H of such a hypersuperface is given by

$$H = -\frac{1}{n} \operatorname{div} N,$$

where N is an appropriate local extension of the unit normal vector field on $M^n \subset \overline{M}^{n+1}$.

Proof. Firstly, for a), let $E_1, \cdots, E_{n+1}$ be a geodesic frame of $\overline{M}$ at p (see Exercise 3.7). Then by Exercise 3.8, we know that

$$\overline{\Delta} f = \sum_{i=1}^{n+1} E_i \left(E_i \left(f \right) \right).$$

On the other side, by Exercise 3.8, $\operatorname{grad} f = \sum_{j=1}^{n+1} E_j \left(f \right) E_j$. Then for any $1 \leq i \leq n+1$,

$$\begin{aligned}(\operatorname{Hess} f) \left(E_i(p) \right) &= \overline{\nabla}_{E_i} \left(\operatorname{grad} f \right)(p) = \overline{\nabla}_{E_i} \left(\sum_{j=1}^{n+1} E_j \left(f \right) E_j \right)(p) \\ &= \sum_{j=1}^{n+1} \overline{\nabla}_{E_i} \left(E_j \left(f \right) E_j \right)(p) = \sum_{j=1}^{n+1} \left(E_j \left(f \right)(p) \overline{\nabla}_{E_i} E_j(p) + E_i \left(E_j \left(f \right) \right)(p) E_j(p) \right).\end{aligned}$$

Because $E_1, \cdots, E_{n+1}$ is a geodesic frame, by Exercise 3.7, we have $\langle E_i, E_j \rangle = \delta_{ij}$ and $\overline{\nabla}_{E_i} E_j(p) = 0$, $\forall 1 \leq i, j \leq n+1$. Then

$$(\operatorname{Hess} f) \left(E_i(p) \right) = \sum_{j=1}^{n+1} \left(E_j \left(f \right) \overline{\nabla}_{E_i} E_j(p) + E_i \left(E_j \left(f \right) \right)(p) E_j(p) \right) = \sum_{j=1}^{n+1} \left(E_i \left(E_j \left(f \right) \right)(p) E_j(p) \right).$$

These mean the matrix of $\operatorname{Hess} f$ under the basis $E_1(p), \cdots, E_{n+1}(p)$ is

$$\left(E_i \left(E_j \left(f \right) \right)(p) \right)_{1 \leq i, j \leq n+1}.$$

By linear algebra, $\operatorname{Tr}(\operatorname{Hess} f)$ is independent of the choice of the basis, so we have

$$\operatorname{Tr}(\operatorname{Hess} f) = \sum_{i=1}^{n+1} \left(E_i \left(E_i \left(f \right) \right)(p) \right) = \overline{\Delta} f(p),$$

as deisred.

Then for b), by linearity, we only need to prove it for $X, Y = E_1, \cdots, E_{n+1}$. At any $p \in \overline{M}$, in the proof of a), we have seen that

$$(\operatorname{Hess} f) \left(E_i(p) \right) = \sum_{k=1}^{n+1} \left(E_i \left(E_k \left(f \right) \right)(p) E_k(p) \right).$$

Reviewing that $E_1, \cdots, E_{n+1}$ is a geodesic frame of $\overline{M}$ at p and then $\langle E_i, E_j \rangle = \delta_{ij}$ and $\overline{\nabla}_{E_i} E_j(p) = 0$, $\forall 1 \leq i, j \leq n+1$, we have

$$\begin{aligned}\left\langle (\operatorname{Hess} f) \left(E_i \right), E_j \right\rangle_p &= \left\langle (\operatorname{Hess} f) \left(E_i(p) \right), E_j(p) \right\rangle = \left\langle \sum_{k=1}^{n+1} \left(E_i \left(E_k \left(f \right) \right)(p) E_k(p) \right), E_j(p) \right\rangle \\ &= \sum_{k=1}^{n+1} \left\langle \left(E_i \left(E_k \left(f \right) \right)(p) E_k(p) \right), E_j(p) \right\rangle = E_i \left(E_j \left(f \right) \right)(p).\end{aligned}$$

Switching E_i, E_j, we have

$$\left\langle (\operatorname{Hess} f) \left(E_j \right), E_i \right\rangle_p = E_j \left(E_i \left(f \right) \right)(p).$$

Under the symmetric connection, we have

$$\begin{aligned}E_i \left(E_j \left(f \right) \right)(p) - E_j \left(E_i \left(f \right) \right)(p) &= [E_i, E_j](f)(p) \\ &= \left(\nabla_{E_i} E_j - \nabla_{E_j} E_i \right)(f)(p) = \left(\nabla_{E_i} E_j(p) - \nabla_{E_j} E_i(p) \right)(f) = 0(f) = 0,\end{aligned}$$

i.e.

$$E_i\left(E_j\left(f\right)\right)(p)=E_j\left(E_i\left(f\right)\right)(p).$$

It follows that

$$\left\langle\left(\operatorname{Hess} f\right)\left(E_i\right), E_j\right\rangle_p=E_i\left(E_j\left(f\right)\right)(p)=E_j\left(E_i\left(f\right)\right)(p)=\left\langle\left(\operatorname{Hess} f\right)\left(E_j\right), E_i\right\rangle_p.$$

By the arbitrariness of $p\in\overline{M}$, we conclude

$$\left\langle\left(\operatorname{Hess} f\right)\left(E_i\right), E_j\right\rangle=\left\langle\left(\operatorname{Hess} f\right)\left(E_j\right), E_i\right\rangle=\left\langle E_i,\left(\operatorname{Hess} f\right)\left(E_j\right)\right\rangle,$$

as desired. This means $\operatorname{Hess} f$ is self-adjoint, hence determines a symmetric bilinear form on $T_p\overline{M}$, $p\in\overline{M}$:

$$\left(\operatorname{Hess} f\right)(X,Y)=\left\langle\left(\operatorname{Hess} f\right)(X),Y\right\rangle=\left\langle\left(\operatorname{Hess} f\right)(Y),X\right\rangle=\left(\operatorname{Hess} f\right)(Y,X),\qquad\forall X,Y\in T_p\overline{M}.$$

For c), by the definition in Exercise 3.8,

$$\langle\operatorname{grad} f(p), v\rangle=df_p(v),\qquad\forall p\in\overline{M},\ v\in T_p\overline{M}.$$

But on M, we have $f|_M=a$ is a constant, then $df_p=0$, $\forall p\in M$. So we have

$$\langle\operatorname{grad} f(p), v\rangle=df_p(v)=0(v)=0,\qquad\forall p\in M,\ v\in T_pM,$$

which means $\operatorname{grad} f(p)\in\left(T_pM\right)^{\perp}$, $\forall p\in M$. As in Chapter 6 Example 2.4, we denote $\eta=\frac{\operatorname{grad} f}{|\operatorname{grad} f|}$. Moreover, by the proof of Exercise 3.7, we know that we can let $E_{n+1}(p)=\eta=\frac{\operatorname{grad} f}{|\operatorname{grad} f|}$ and extends it to a basis $E_1(p),\cdots,E_{n+1}(p)$ of $T_p\overline{M}$ and generate a geodesic frame $E_1,\cdots,E_{n+1}=\eta$ of $\overline{M}$ at p. Counting the dimension, we know that $E_1,\cdots,E_n$ is a geodesic frame of M. Then by Chapter 6 Example 2.4, the mean curvature will be

$$H(p)=\frac{1}{n}\left(\lambda_1+\cdots+\lambda_n\right)=\frac{1}{n}\operatorname{Tr}\left(S_\eta\right),$$

where $\lambda_1,\cdots,\lambda_n$ are real eigenvalues of S_η. Then because the geodesic frame $E_1,\cdots,E_n$ are orthonormal, by Chapter 6 Proposition 2.3, we have

$$nH(p)=\operatorname{Tr}\left(S_\eta\right)=\sum_{i=1}^{n}\left\langle S_\eta\left(E_i\right), E_i\right\rangle_p=-\sum_{i=1}^{n}\left\langle\overline{\nabla}_{E_i}\eta, E_i\right\rangle_p.$$

On the other side, by Exercise 3.8, on $\overline{M}$,

$$-\operatorname{div}\left(\frac{\operatorname{grad} f}{|\operatorname{grad} f|}\right)(p)=-\operatorname{div}\eta(p)=-\operatorname{Tr}\left(\left[Y(p)\to\overline{\nabla}_Y\eta(p)\right]\right)=-\sum_{i=1}^{n+1}\left\langle\overline{\nabla}_{E_i}\eta, E_i\right\rangle_p.$$

Noticing that for $i=n+1$, $E_{n+1}(p)=\eta$ as we defined. Because $E_1,\cdots,E_{n+1}=\eta$ is a geodesic frame, we have $\nabla_\eta\eta(p)=0$. Then

$$\left\langle\overline{\nabla}_{E_{n+1}}\eta, E_{n+1}\right\rangle_p=\left\langle\overline{\nabla}_\eta\eta,\eta\right\rangle_p=\left\langle\overline{\nabla}_\eta\eta(p),\eta(p)\right\rangle=\langle 0,\eta(p)\rangle=0.$$

It follows that

$$-\operatorname{div}\left(\frac{\operatorname{grad} f}{|\operatorname{grad} f|}\right)(p)=-\sum_{i=1}^{n+1}\left\langle\overline{\nabla}_{E_i}\eta, E_i\right\rangle_p=-\sum_{i=1}^{n}\left\langle\overline{\nabla}_{E_i}\eta, E_i\right\rangle_p=nH(p).$$

By the arbitrariness of $p\in M$, we conclude that

$$nH=-\operatorname{div}\left(\frac{\operatorname{grad} f}{|\operatorname{grad} f|}\right),$$

as desired.

Finally, as we have done in the proof of c), $N=E_{n+1}=\eta=\frac{\operatorname{grad} f}{|\operatorname{grad} f|}$ is a local extension of the unit normal vector field on $M^n\subset\overline{M}^{n+1}$. And then by c) again, we have

$$H=-\frac{1}{n}\operatorname{div}\left(\frac{\operatorname{grad} f}{|\operatorname{grad} f|}\right)=-\frac{1}{n}\operatorname{div} N;$$

whence d). ☐

Exercise 6.12 (Singularities of a Killing field)**.** Let X be a Killing vector field on aRiemannian manifold M. Let

$$N = \{p \in M | \, X(p) = 0\} .$$

Prove that:

a) If $p \in N$, and $V \subset M$ is a normal neighborhood of p, with $q \in N \cap V$, then the radial geodesic segment γ joint p to q is contained in N. Conclude that $\gamma \cap V \subset N$.

b) If $p \in N$, there exists a neighborhood $V \subset M$ of p such that $V \cap N$ is a submanifold of M (this implies that every connected component of N is a submanifold of M).

c) The codimension, as a submanifold of M, of a connected component N_k of N is even. Assume the following fact: if a sphere has a non-vanishing differentiable vector field on it then its dimension must be odd.

Proof. For a), let $\varphi(t, p)$ be the trajectory of X at any $p \in M$. In the proof of Exercise 3.5 b), we know that for any geodesic γ of M starting at p, $\varphi(t_0, \gamma(t))$ is also a geodesic starting at p for any $t_0 \in (-\varepsilon, \varepsilon)$, and if $\ell(\gamma(t)) = \delta$, $\ell\left(\varphi(t_0, \gamma(t))\right) = \delta$ too. Then we claim that, in normal neighborhood of p, $\varphi(t_0, \gamma) \subset B_{\ell(\gamma)}(p)$: If not, we say that in a normal neighborhood of p, there exists $\varphi(t_0, \gamma(s)) \notin B_{\ell(\gamma)}(p)$. Then we get a contradiction:

$$\ell\left(\varphi\left(t_0, \gamma|_{[0,s]}\right)\right) > \ell(\gamma) > \ell\left(\gamma|_{[0,s]}\right) .$$

We proved our claim. Using the claim, we can get the fact that $\varphi\left(t_0, B_\delta(p)\right) \subset B_\delta(p)$ for any $t_0 \in (-\varepsilon, \varepsilon)$ and normal neighborhood $B_\delta(p)$ of p.

Back to a), Denote $\gamma(a) = q$. Because $q \in N$ and then $X(q) = 0$, we have $\varphi(t_0, \gamma(a)) = \varphi(t_0, q) = q$, $\forall t_0 \in (-\varepsilon, \varepsilon)$. So we also know that $\varphi(t_0, \gamma) \subset V$ is also a geodesic from p to q. By the uniqueness of geodesic in normal neighborhood (Chapter 3 Proposition 3.6 and Corollary 3.9), we must have $\varphi(t_0, \gamma(t)) = \gamma(t)$, $\forall t \in [0, a]$, i.e. $\varphi(t_0, \ )$ fixes every point of γ. This means $X(\gamma(t)) = 0$, $\forall t \in [0, a]$, i.e $\gamma \subset N$, as desired. And we also conclude $\gamma \cap V \subset \gamma \subset N$.

For b), we can choose V as a strongly convex normal neighborhood of $p \in N$. If $V \cap N = \{p\}$, it is a 0-dimensional submanifold, as desired. If not, for any $q \in V \cap N$, by a), let $\gamma \subset N$ be the radial geodesic from p to q. By definition, we know that γ is generated by $\exp_p$ along a vector $v \in T_pM$, i.e. $\gamma(t) = \exp_p(tv)$. Denote $V = B_\delta(p)$. Then for any real number k such that $kv \in B_\delta(0)$, $\exp_p(kv) \in \gamma$. Because $\exp_p$ is a diffeomorphism on V, we have a differentiable 1-1 corresponding from the line

$$\left\{ k\frac{v}{|v|} \,\middle|\, k \in (-\delta, \delta) \right\}$$

to $\gamma \subset N \cap V$ under $\exp_p$. Doing the same thing for all $q \in V \cap N$, we know that $\exp_p^{-1}(V \cap N)$ is a open ball of a linear subspace of T_pM (Why?) and $\exp_p^{-1}(V \cap N) \subset B_\delta(0)$. So $\exp_p^{-1}(V \cap N)$ is a submanifold of $B_\delta(0) \subset T_pM$. Then under the diffeomorphism $\exp_p$, $V \cap N = \exp_p\left(\exp_p^{-1}(V \cap N)\right)$ is a submanifold of $\exp_p\left(B_\delta(0)\right) = B_\delta(p)$ which is a open set of M. So we conclude $V \cap N$ is a submanifold of M, as desired.

Finally for c), let $V \subset M$ be a normal neighborhood of p and denote

$$N_k^\perp = \exp_p\left((T_pN_k)^\perp \cap \exp_p^{-1}(V)\right).$$

We have $N_k \cap N_k^\perp = \{p\}$ and $N_k \cup N_k^\perp = M$ locally. Then $X(q) \neq 0$, $\forall q \neq p \in N_k^\perp$, i.e. p is a isolate zero of X on $N_k^\perp$. By Exercise 3.5 b), X is tangent to the geodesic spheres of $N_k^\perp$ centered at p. Moreover, by our definition of $N_k^\perp$, we also have $X \neq 0$ on $N_k^\perp$ and then on the geodesic spheres. Then by the fact, the geodesic spheres has odd dimension. So we have $\dim N_k^\perp =$ dimension of geodesic spheres $+ 1$ is even. We conclude

$$\operatorname{codim} N_k = \dim N_k^\perp$$

is even, as desired. $\square$

7 Complete Manifolds; Hopf-Rinow and Hadamard Theorems

Exercise 7.1. If M, N are Riemannian manifolds such that the inclusion $i\colon M \subset N$ is an isometric immersion, show by an example that the strict inequality $d_M > d_N$ can occur.

Solution. Consider the isometry embedding $S^2 \to \mathbb{R}^3$ as $M = S^2$ and $N = \mathbb{R}^3$. As we all known, the geodesic of S^2 are great circles. So for any $p, q \in S^2$, the minimal geodesic γ connecting p, q are short great circle which is the intersection of the plane at which $0, p, q$ are located and S^2. But in $\mathbb{R}^3$, the minimal geodesic connecting p, q is the straight line connecting p, q. It's obviously that the length of the great circle γ is strictly longer than it of the line, i.e. $d_M > d_N$, as desired. $\square$

Exercise 7.2. Let $\widetilde{M}$ be a covering space of a Riemannian manifold M. Show that it is possible to give $\widetilde{M}$ a Riemannian structure such that the covering map $\pi\colon \widetilde{M} \to M$ is a local isometry (this metric is called the *covering metric*). Show that $\widetilde{M}$ is complete in the covering metric if and only if M is complete.

Proof. Firstly, for any $p \in \widetilde{M}$, we can choose a open neighborhood U of p such that $\pi\colon U \to \pi(U)$ is a diffeomorphism to the open neighborhood $\pi(U)$ of $\pi(p)$ on M. Then on U, for any $X, Y \in T_p\widetilde{M}$, we define

$$\langle X, Y\rangle_p = \langle d\pi(X), d\pi(Y)\rangle_{\pi(p)}$$

by the induced Riemannian metric (Chapter 1 Example 2.5). In particular, we have defined the metric at any $p \in \widetilde{M}$. Because the covering map π is smooth of p, we know that this construction gives a Riemannian metric on the whole $\widetilde{M}$. By our construction, π is naturally a local isometry. Now we get a covering metric on $\widetilde{M}$.

For the second result, because π is a local isometry, by the minimal property of geodesic, we claim that for any geodesic $\widetilde{\gamma}(t)$, $\pi\left(\widetilde{\gamma}(t)\right)$ is a geodesic of M and vice versa (see the claim in the proof of Exercise 3.5 b)), it's true for all local isometry instead of $\varphi(t_0, \ )$). Then if $\widetilde{M}$ is complete, for any geodesic $\gamma(t)$ of M, by the path lifting property of covering space, we have a curve $\widetilde{\gamma}(t)$ such that $\pi\left(\widetilde{\gamma}(t)\right) = \gamma(t)$. By our claim, $\widetilde{\gamma}(t)$ is a geodesic on $\widetilde{M}$. Because $\widetilde{M}$ is complete, $\widetilde{\gamma}(t)$ is defined for all $t \in \mathbb{R}$. Then by uniqueness of geodesic, $\gamma(t) = \pi\left(\widetilde{\gamma}(t)\right)$ can be also defined for all $t \in \mathbb{R}$. Then by the arbitrariness of $\gamma(t)$, M is complete. Conversely, if M is complete, for any geodesic $\widetilde{\gamma}(t)$ of $\widetilde{M}$, by our claim again, $\gamma(t) = \pi\left(\widetilde{\gamma}(t)\right)$ is a geodesic on M and is defined for all $t \in \mathbb{R}$. By the uniqueness of path lifting, $\widetilde{\gamma}(t)$ is the lifting of $\gamma(t)$. This means for any $t \in \mathbb{R}$, $\widetilde{\gamma}(t)$ must exists as the preimage of $\gamma(t)$ under covering map π, i.e. $\widetilde{\gamma}(t)$ is also defined for all $t \in \mathbb{R}$. So by the arbitrariness of $\widetilde{\gamma}(t)$, $\widetilde{M}$ is complete. $\square$

Exercise 7.3. Let $f\colon M_1 \to M_2$ be a local diffeomorphism of a manifold M_1 onto a Riemannian manifold M_2. Introduce on M_1 a Riemannian metric such that f is a local isometry. Show by an example that if M_2 is complete, M_1 need not be complete.

Solution. Let $M_1 = \mathbb{R}^2 - \{(0,0)\}$, $M_2 = \mathbb{R}$ and f be the inclusion. Obviously, f is local diffeomorphism and we can induce a Riemannian metric on $M_1 = \mathbb{R}^2 - \{(0,0)\}$ by f (see Chapter 1 Example 2.5). Then naturally, f is a local isometry. And on $M_2 = \mathbb{R}^2$, the geodesics are straight lines and then can be defined for all $t \in \mathbb{R}$, i.e. $M_2 = \mathbb{R}^2$ is geodesically complete. But on $M_1 = \mathbb{R}^2 - \{(0,0)\}$, let $\left\{\left(0, \frac{1}{n}\right)\right\}_{n=1}^{\infty} \subset \mathbb{R}^2 - \{(0,0)\} = M_1$ be a sequence. As we all know, $\left\{\left(0, \frac{1}{n}\right)\right\}_{n=1}^{\infty}$ is a Cauchy sequence under the Euclidean metric and has limit

$$\lim_{n\to\infty}\left(0, \frac{1}{n}\right) = (0,0) \notin \mathbb{R}^2 - \{(0,0)\} = M_1.$$

So M_1 is not complete as a metric space. Then by Hopf-Rinow Theorem (Chapter 7 Theorem 2.8 c) $\Leftrightarrow$ d)), M_1 is not geodesically complete, as desired. $\square$

Exercise 7.4. Consider the universal covering

$$\pi\colon M \to \mathbb{R}^2 - \{(0,0)\}$$

of the Euclidean plane minus the origin. Introduce the covering metric on M (see Exercise 7.2). Show that M is not complete and not extendible, and that the Hopf-Rinow theorem is not true for M (this shows that the definition of non-extendibility, though natural, is not a satisfactory one).

Proof. Let $\mathbb{R}^2 - \{(0,0)\}$ be endowed with the induced metric by the conclusion $\mathbb{R}^2 - \{(0,0)\} \hookrightarrow \mathbb{R}^2$ which is Euclidean. By Exerxise 7.2, we can introduce the covering metric on M. Also by Exercise 7.2, we know that M is complete if and only if $\mathbb{R}^2 - \{(0,0)\}$ is complete. But in Exercise 7.3, we have proved that $\mathbb{R}^2 - \{(0,0)\}$ is not complete. So M is not complete.

To show that M is not extendible, we prove by contradiction. If not, say there is an isometric embedding $M \hookrightarrow M'$, we can choose a $p' \in \partial M$ and a convex neighborhood $W' \subset M'$ of p'.

We claim that $W' - \{p'\} \subset M$: Firstly, for any $p \in M$, we can choose a neighborhood $U \subset M$ of p which is diffeomorphic to an open subset of $\mathbb{R}^2 - \{(0,0)\}$, say V. Notice that for $\pi(p)$, there is unique geodesic from $\pi(p)$ that can not be defined for all $t \in \mathbb{R}$, i.e. the geodesic towards $(0,0)$, under the Euclidean metric. Then under the covering metric, in M, there also exists $\boxed{\text{unique}}$ geodesic from p that can not be extended to all $t \in \mathbb{R}$, whose image under π is the straight line from $\pi(p)$ to $(0,0)$. Then for any $x \in W' \cap M$, let $\widetilde{\gamma}$ be the geodesic in M' from x to p'. Notice that $\widetilde{\gamma}$ must have an initial segment in M, because M is open in M', the segment must stops before p', i.e. it is the unique geodesic in M from x that can not be extended to all $t \in \mathbb{R}$ and approaches p' any close. We conclude that $\widetilde{\gamma} \cap W' - \{p'\} \subset M$. To see that, if not, $\widetilde{\gamma}$ must intersect ∂M with another point, say q'. Then similarly, $\widetilde{\gamma}$ must stop at q'. So $\widetilde{\gamma}$ can not go through q' and then approach p' arbitrarily close, which is a contradiction. Finally, for any $z \in W' - \{p'\}$ and $z \notin \widetilde{\gamma}$, because W' is convex, let's consider the geodesic $\gamma \neq \widetilde{\gamma}$ connecting z and x. If γ is not in M, it must intersect ∂M with a point q' and $q' \neq p'$, because $z \notin \widetilde{\gamma}$. Then similarly, the segment of γ in M from x to q' is another geodesic from x that can not be extended to all $t \in \mathbb{R}$, which is contradict to the uniqueness. So we have $\gamma \cap W' \subset M$, hence $z \in M$ especially. So we proved our claim.

By our claim, let $\pi(W' - \{p'\}) = W$ is an open subset of $\mathbb{R}^2 - \{(0,0)\}$. We assert that W is around $(0,0)$. If not, W is simple connected, then $\pi^{-1}(W)$ is a union of some simple connected open subsets of M without intersection. So $W' - \{p'\}$ is an inner subset of M, which can not touch the boundary ∂M. Then it is away from p', which is impossible as our choice of W'. Then let's consider a loop $\alpha\colon [0,1] \to W$ around $(0,0)$. Let $\widetilde{\alpha}$ be its preimage under π in $W' - \{p'\}$. Because M is universal covering, it's simple connected. And W' is an convex neighborhood of $p' \in \partial M$, so $W' - \{p'\}$ is also a simple connected subset of M: If not, i.e. $p' \in \partial M$ is in the inner of W', the boundary ∂M must go through W', which is contradict to our claim that $W' - \{p'\} \subset M$. Then $\widetilde{\alpha}$ must homotopic to a trivial path in $W' - \{p'\}$, i.e. a single point, say q'. Let $\widetilde{F}$ be the homotopy map, i.e. $\widetilde{F}\colon [0,1] \times [0,1] \to W' - \{p'\}$ such that $\widetilde{F}(0,t) = \widetilde{\alpha}(t)$ and $\widetilde{F}(1,t) = q'$ for all $t \in [0,1]$. Then let's consider $\pi\left(\widetilde{F}\right)$. We have $\pi\left(\widetilde{F}(0,t)\right) = \pi(\widetilde{\alpha}(t)) = \alpha(t)$ and $\pi\left(\widetilde{F}(1,t)\right) = \pi(q')$ for any $t \in [0,1]$, i.e. $\pi\left(\widetilde{F}\right)$ is a homotopy map from α to a trivial path $\pi(q')$ in W. But we know that W has non-trivial fundamental group so the loop α around $(0,0)$ is in a non-trivial homotopy class, which is a contradiction. Now we proved that M is not extendible.

Finally, I don't know what the proposition "Show that the Hopf-Rinow Theorem is not true for M" means, because the theorem is proved for any Riemannian manifold. However, it doesn't matter.

(In fact, the universal covering is given by

$$\begin{aligned} \pi\colon M = \mathbb{C} &\to \mathbb{R}^2 - \{(0,0)\} = \mathbb{C}^* \\ z &\mapsto \mathrm{e}^z . \end{aligned}$$

The universal covering space M is a typical example of Riemann surface.) □

Exercise 7.5. A *divergent curve* in a non-compact Riemannian manifold M is a differentiable mapping $\alpha\colon [0,\infty) \to M$ such that for any compact set $K \subset M$ there exists $t_0 \in (0,\infty)$ with $\alpha(t) \notin K$ for all $t > t_0$ (that is, α "escapes" every compact set in M). Define the length of a divergent curve by

$$\lim_{t\to\infty} \int_0^t |\alpha'(t)| \, \mathrm{d}t.$$

Prove that M is complete if and only if the length of any divergent curve is unbounded.

Proof. If M is complete, by Hopf-Rinow Theorem (Chapter 7 Theorem 2.8 d) ⇔ d)), in M, there exists a sequence of compact subset $\{K_n\}_{n=1}^{\infty}$ with $K_n \subset \operatorname{int}(K_{n+1})$ and $\bigcup_{n=1}^{\infty} K_n = M$, such that if $q_n \notin K_n$, then $d(p,q_n) \to \infty$, where $p \in M$. Let α be any divergent curve in M. We can choose a point sequence $q_n = \alpha(t_n)$ of α such that $q_n \in K_{n+1} - K_n$. Let $p = \alpha(t_0)$. Then we have

$$\lim_{t\to\infty} \int_0^t |\alpha'(t)| \, \mathrm{d}t \geq \lim_{n\to\infty} \int_{t_0}^{t_n} |\alpha'(t)| \, \mathrm{d}t = \lim_{n\to\infty} \left| \alpha|_{[t_0,t_n]} \right| \geq \lim_{n\to\infty} d(\alpha(t_0), \alpha(t_n)) = \lim_{n\to\infty} d(p,q_n) = \infty,$$

i.e. the length of α is unbounded. By the arbitrariness of divergent curve α, we proved that the length of any divergent curve in M is unbounded.

For another direction, we will prove it by contradiction. If M is not complete, there exists a geodesic γ in M which can not defined for all $t \in \mathbb{R}$. Suppose that γ can be only defined on $[0, t_0)$. And γ can not be extended at t_0, by the theory of extension of the solution of ODEs, i.e. the right side of the definition domain of γ is open. Then we reparametrize γ such that it is defined on $[0, \infty)$. In fact, let

$$\widetilde{\gamma}(t) = \gamma\left(t_0 - \frac{t_0}{t+1}\right),$$

we have

$$\widetilde{\gamma}(0) = \gamma(0) \quad \text{ and } \quad \lim_{t\to\infty} \widetilde{\gamma}(t) = \lim_{t\to\infty} \gamma\left(t_0 - \frac{t_0}{t+1}\right) = \lim_{t\to t_0} \gamma(t),$$

as desired. Because γ is defined on a finite interval and then have finite length, we have

$$\lim_{t\to\infty} \int_0^t |\widetilde{\gamma}'(t)|\, \mathrm{d}t = \lim_{t\to\infty} \left|\widetilde{\gamma}|_{[0,t]}\right| = \lim_{t\to t_0} \left|\gamma|_{[0,t]}\right| = \left|\gamma|_{[0,t_0)}\right| < \infty.$$

So $\widetilde{\gamma}$ is not a divergent curve. By definition, there is a compact subset $K \subset M$ such that for any $t_0 \in (0, \infty)$, there exists $t > t_0$ such that $\widetilde{\gamma}(t) \in K$. So we can choose a sequence $t_n \to \infty$ such that $\widetilde{\gamma}(t_n) \in K$, $\forall n \in \mathbb{N}_+$. (For example, choose $t_n > n$ such that $\widetilde{\gamma}(t_n) \in K$.) Because K is compact, $\{\widetilde{\gamma}(t_n)\}_{n=1}^{\infty}$ has a convergent subsequence, say $\{\widetilde{\gamma}(t_{n_k})\}_{k=1}^{\infty}$ and suppose $\widetilde{\gamma}(t_{n_k}) \to p \in K \subset M$ when $t_{n_k} \to \infty$. Then because $\widetilde{\gamma}$ is continue, we have

$$\lim_{t\to t_0} \gamma(t) = \lim_{t\to\infty} \widetilde{\gamma}(t) = \lim_{k\to\infty} \widetilde{\gamma}(t_{n_k}) = p \in K \subset M,$$

i.e. γ can be extended to t_0: $\gamma(t_0) = p \in K \subset M$, which is a contradict to our choice of γ. So M is complete. $\square$

Exercise 7.6. A geodesic $\gamma \colon [0, \infty) \to M$ in a Riemannian manifold M is called a *ray starting from* $\gamma(0)$ if it minimizes the distance between $\gamma(0)$ and $\gamma(s)$, for any $s \in (0, \infty)$. Assume that M is complete, non-compact, connected and let $p \in M$. Show that M contains a ray starting from p.

Proof. In the proof, for any $v \in T_pM$, by the homogeneity of geodesics (Chapter 3 Lemma 2.6), we use $|v|$ to present the length of the geodesic $\gamma(t) = \exp_p(tv)$ from p to $q = \exp_p(v)$.

Since M is complete and non-compact, it is unbounded. Then for all $n \in \mathbb{N}_+$, there is a $q_n \in M$ such that $d(p, q_n) > n$. Let $v_n \in T_pM$ such that the geodesic $\exp_p(tv_n)$ minimizing the distance between p and q_n. And we have $|v_n| = d(p, q_n) > n$.

Let $w_n = \frac{v_n}{|v_n|} \in S_1(0) \subset T_pM$. Notice that if v minimizes the the distance of p and $\exp_p(v)$, then does tv for all $0 \le t \le 1$. Since $w_n = \frac{v_n}{|v_n|} = \frac{1}{|v_n|} v_n$ with $\frac{1}{|v_n|} \le \frac{1}{n} < 1$, $\exp_p(tw_n)$ also minimizes the the distance of p and $\exp_p(w_n)$. Because $S_1(0) \subset T_pM$ is compact, $\{w_n\}_{n=1}^{\infty}$ has a convergent subsequence, say $\{w_{n_k}\}_{k=1}^{\infty}$. Suppose $w_{n_k} \to w_0 \in S_1(0) \subset T_pM$ when $k \to \infty$.

Now we claim that $\exp_p(tw_0)$ also minimizes the distance of p and $\exp_p(w_0)$: Let $U \subset T_pM$ be the set of all $v \in T_pM$ such that the geodesic $\exp_p(tv)$ minimizing the distance of p and $\exp_p(v)$. Define a function

$$\begin{aligned} f \colon T_pM &\to \mathbb{R}_+ \\ v &\mapsto |v| - d(p, \exp_p(v)). \end{aligned}$$

It's easy to see that f is continue. Then $U = f^{-1}(0)$ is close, since it' a preimage of a close set $\{0\}$ under a continue function f. Then $S_1(0) \cap U$ is also close. So we must have $w_0 \in S_1(0) \cap U \subset U$, since it is the limit of a sequence $\{w_{n_k}\}_{k=1}^{\infty} \subset S_1(0) \cap U$, which proving our claim.

By the notice above again, for any $0 \le s \le n$, $sw_n = \frac{s}{|v_n|} v_n$ with $\frac{s}{|v_n|} < \frac{n}{n} = 1$, so $sw_n \in U$. Let $n \to \infty$, we have for any $s \in \mathbb{R}_+$, $sw_0 \in U$. This means the geodesic $\exp_p(tsw_0)$ about t minimizes the distance between p and $\exp_p(sw_0)$ for any $s \in \mathbb{R}_+$, i.e. the same geodesic $\gamma(s) = \exp_p(sw_0)$ minimizes the distance between $\gamma(0) = p$ and $\gamma(s) = \exp_p(sw_0)$ for any $s \in \mathbb{R}_+$. So $\gamma(s)$ is a ray of M starting from p, as desired. $\square$

Exercise 7.7. Let M and $\overline{M}$ be non-compact Riemannian manifolds and let $f \colon M \to \overline{M}$ be a diffeomorphism. Assume that $\overline{M}$ is complete and that there exists a constant $c > 0$ such that

$$|v| \ge c\, |df_p(v)|,$$

for all $p \in M$ and all $v \in T_pM$. Prove that M is complete.

Proof. Firstly, for any $p, q \in M$, let γ be the geodesic connecting p, q which minimizing the distance of p and q. Then $f(\gamma)$ is a curve in $\overline{M}$ connecting $f(p)$ and $f(q)$. And we have

$$d_{\overline{M}}(f(p), f(q)) \leq \ell(f(\gamma)) = \int_0^1 |f(\gamma(t))'| \, \mathrm{d}t = \int_0^1 \left|df_{\gamma(t)}(\gamma'(t))\right| \mathrm{d}t \leq \int_0^1 \frac{1}{c} |\gamma'(t)| \, \mathrm{d}t = \frac{1}{c}\ell(\gamma) = \frac{1}{c} d_M(p, q).$$

So we conclude

$$d_{\overline{M}}(f(p), f(q)) \leq \frac{1}{c} d_M(p, q), \qquad \forall p, q \in M.$$

Then for any close and bounded subset U of M, we have

$$\operatorname{diam}(f(U)) = \sup_{p,q\in U} d_{\overline{M}}(f(p), f(q)) \leq \frac{1}{c} \sup_{p,q\in U} d_M(p, q) = \frac{1}{c} \operatorname{diam}(U) < \infty,$$

i.e. $f(U)$ is bounded. Besides, because f is a diffeomorphism and so is f^{-1}, $f(U) = \left(f^{-1}\right)^{-1}(U)$ is also close. Using Hopf-Rinow Theorem (Chapter 7 Theorem 2.8 d) $\Rightarrow$ b)), we know that $f(U)$ is compact in $\overline{M}$. Also because f is a diffeomorphism, $U = f^{-1}(f(U))$ is compact in M. By the arbitrariness of U, using Hopf-Rinow Theorem again (Chapter 7 Theorem 2.8 b) $\Rightarrow$ d)), we have that M is complete, as desired. $\square$

Exercise 7.8. Let M be a connected complete Riemannian manifold, $\overline{M}$ a connected Riemannian manifold, and $f\colon M \to \overline{M}$ a differentiable mapping that is locally an isometry. Assume that any two points of $\overline{M}$ can be joined by a unique geodesic of $\overline{M}$. Prove that f is injective and surjective (and, therefore, f is a global isometry).

Proof. **Injective:** If there are $p \neq q \in M$ such that $f(p) = f(q)$, we can choose a geodesic γ in M connecting p, q. Because f is a local isometry, $f(\gamma)$ is also a geodeisc of $\overline{M}$ (see the claim in the proof of Exercise 3.5 b)). And we have $f(\gamma(0)) = f(p) = f(q) = f(\gamma(1))$, so $f(\gamma)$ is a close geodesic.

We claim that $f(\gamma)$ is not a trivial geodesic, i.e. a point. That's because f is a local isometry, there is a neighborhood U of p such that f maintain the distance in U. Then for a point $r \in U \cap \gamma$, we have $d_{\overline{M}}(f(p), f(r)) = d_M(p, r) > 0$, which means $f(r) \neq f(p)$. So $f(\gamma)$ is not a point. For the same reason, $f(\gamma)$ can not double back: Let's consider $\gamma(s)$ such that $f(\gamma(s))$ is the retracing point, and let V be the distance-maintain neighborhood of $\gamma(s)$. There are two points $\gamma(s_1), \gamma(s_2)$ on both sides such that $f(\gamma(s_1)) = f(\gamma(s)2)$. Then $0 = d_{\overline{M}}(f(\gamma(s_1)), f(\gamma(s)2)) = d_M(\gamma(s_1), \gamma(s)2) > 0$, which is a contradiction.

Now we can choose a inner point $f(\gamma(t_0))$ of $f(\gamma)$. Then let $\gamma_1(t) = f(\gamma(t_0 t))$ and $\gamma_2(t) = f(\gamma((t_0 - 1)t + 1))$. We have $\gamma_1(0) = f(\gamma(0)) = f(p)$, $\gamma_1(1) = f(\gamma(t_0))$, $\gamma_2(0) = f(\gamma(1)) = f(q) = f(p)$, $\gamma_2(1) = f(\gamma(t_0))$, i.e γ_1 and γ_2 are different geodesics of $\overline{M}$ from $f(q)$ to $f(\gamma(t_0))$, which is contradict to the uniqueness of geodesic in $\overline{M}$. So we must have $p = q$. Then f is injective.

Surjective: Because M is complete, by Chapter 7 Proposition 2.3, M is non-extendible. But if f is not surjective, $f\colon M \to \overline{M}$ is an extension of M. So we get a contradiction. Then f is surjective. (In fact, we can generalize the proof of hapter 7 Proposition 2.3: If f is not surjective, let $x \in f(M)$ and $y \in \overline{M} - f(M)$ and $\overline{\gamma}$ the unique geodesic in $\overline{M}$ connecting x, y. Suppose $\overline{\gamma}'(0) = v \in T_x\overline{M} \cong T_{f^{-1}(x)}M$. Let $\gamma(t) = \exp_{f^{-1}(x)}\left(t df_x^{-1}(v)\right)$ be a geodesic in M. Because M is complete, γ can be defined on $\mathbb{R}$. In particular, $\gamma(1) \in M$ and then $f(\gamma(1)) \in f(M)$. But we have $f(\gamma(1)) = \overline{\gamma}(1) = y \in \overline{M} - f(M)$ and then $f(\gamma(1)) \notin f(M)$, which is a contradiction. So f is surjective.) $\square$

Exercise 7.9. Consider the upper half-plane

$$\mathbb{R}^2_+ = \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, y > 0 \right\}$$

with the Riemannian metric given by

$$g_{11} = 1, \qquad g_{12} = 0, \qquad g_{22} = \frac{1}{y}.$$

Show that the length of the vertical segment

$$\{(0, y) |\, \varepsilon \leq y \leq 1\} \subset \mathbb{R}^2_+$$

extends to 2 as $\varepsilon \to 0$. Conclude from this that such a metric is not complete. (Observe, nevertheless, that when $y \to 0$ the length of vectors, in this metric, becomes arbitrarily large.)

Proof. Let $\gamma_\varepsilon(t) = (0,t)$, $t \in [\varepsilon, 1]$ be the vertical segment. Then

$$\gamma_\varepsilon'(t) = (0,1) = \left.\frac{\partial}{\partial y}\right|_t,$$

and

$$|\gamma_\varepsilon'(t)| = \sqrt{\langle \gamma_\varepsilon'(t), \gamma_\varepsilon'(t)\rangle} = \sqrt{\left\langle \left.\frac{\partial}{\partial y}\right|_t, \left.\frac{\partial}{\partial y}\right|_t \right\rangle} = \sqrt{g_{22}(t)} = \sqrt{\frac{1}{t}} = \frac{1}{\sqrt{t}}.$$

So we have

$$\ell(\gamma_\varepsilon) = \int_\varepsilon^1 |\gamma'(t)|\,\mathrm{d}t = \int_\varepsilon^1 \frac{1}{\sqrt{t}}\,\mathrm{d}t = 2\sqrt{t}\Big|_\varepsilon^1 = 2 - 2\sqrt{\varepsilon}.$$

We conclude

$$\lim_{\varepsilon\to 0} \ell(\gamma_\varepsilon) = \lim_{\varepsilon \to 0}\left(2 - 2\sqrt{\varepsilon}\right) = 2 - 0 = 2.$$

Let $p = (0,1)$, then for any sequence of compact subsets $K_n \subset M$, $K_n \subset \operatorname{int}(K_{n+1})$ and $\bigcup_{n=1} K_n = M$, we can choose $q_n \in \lim_{\varepsilon\to 0}\gamma_\varepsilon$ such that $q_n \notin K_n$. But we have

$$d(p, q_n) \leq \ell\left(\left.\lim_{\varepsilon\to 0}\gamma_\varepsilon\right|_{[p,q_n]}\right) \leq \ell\left(\lim_{\varepsilon\to 0}\gamma_\varepsilon\right) = \lim_{\varepsilon\to 0}\ell(\gamma_\varepsilon) = 2 < \infty.$$

By Hopf-Rinow Theorem (Chapter 7 Theorem 2.8 e) $\Leftrightarrow$ d)), the upper half-plane $\mathbb{R}^2_+$ is not complete under the given Riemannian metric, as desired. □

Exercise 7.10. Prove that the upper half-plane $\mathbb{R}^2_+$ with the Lobatchevski metric:

$$g_{11} = g_{22} = \frac{1}{y^2}, \qquad g_{12} = 0,$$

is complete.

Proof. By Chapter 3 Example 3.10, the geodesics of the upper half-plane $\mathbb{R}^2_+$ with the Lobatchevski metric are vertical lines and semi circles whose terminals are on x-axis. By the homogeneity of geodesics (Chapter 3 Lemma 2.6), we use can $|v|$ to present the length of the geodesic $\gamma(t) = \exp_p(tv)$ from p to $q = \exp_p(v)$. Then we $\boxed{\text{claim}}$ that the geodesic $\gamma(t) = \exp_p(tv)$ can be defined on $\mathbb{R}$, if and only if $\ell(\gamma) = \lim_{t\to\infty}|tv| = \infty$.

For the first case, let $\gamma(t) = (x(t), y(t)) = (a, t)$, $t \in (0, \infty)$ be any vertical line. We have $\gamma'(t) = (0,1) = \left.\frac{\partial}{\partial y}\right|_t$. Then we have

$$\begin{aligned}\ell(\gamma) &= \int_0^\infty |\gamma'(t)|\,\mathrm{d}t = \int_0^\infty \sqrt{\langle\gamma'(t),\gamma'(t)\rangle}\,\mathrm{d}t = \int_0^\infty \sqrt{\left\langle \left.\frac{\partial}{\partial y}\right|_t, \left.\frac{\partial}{\partial y}\right|_t\right\rangle}\,\mathrm{d}t \\ &= \int_0^\infty \sqrt{g_{22}(t)}\,\mathrm{d}t = \int_0^\infty \frac{1}{y(t)}\,\mathrm{d}t = \int_0^\infty \frac{1}{t}\,\mathrm{d}t = \ln|_0^\infty = \ln(\infty) - \ln(0) = \infty - (-\infty) = \infty.\end{aligned}$$

So γ can be defined on $\mathbb{R}$, by our claim.

For the second case, let $\gamma(t) = (x(t), y(t)) = (r\cos t + a, r\sin t)$, $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be any semi-circle. We have $\gamma'(t) = (-r\sin t, r\cos t) = -r\sin t\left.\frac{\partial}{\partial x}\right|_t + r\cos t\left.\frac{\partial}{\partial y}\right|_t$. Then we have

$$\begin{aligned}|\gamma'(t)|^2 &= \langle\gamma'(t),\gamma'(t)\rangle = \left\langle -r\sin t\left.\frac{\partial}{\partial x}\right|_t + r\cos t\left.\frac{\partial}{\partial y}\right|_t, -r\sin t\left.\frac{\partial}{\partial x}\right|_t + r\cos t\left.\frac{\partial}{\partial y}\right|_t\right\rangle \\ &= r^2\sin^2 t\cdot g_{11}(t) - 2r^2\sin t\cos t\cdot g_{12}(t) + r^2\cos^2 t\cdot g_{22}(t) = r^2\frac{1}{y^2(t)} = r^2\frac{1}{r^2\sin^2(t)} = \frac{1}{\sin^2 t}.\end{aligned}$$

It follows that

$$\begin{aligned}\ell(\gamma) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\gamma'(t)|\,\mathrm{d}t = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|\sqrt{|\gamma'(t)|^2}\right|\mathrm{d}t = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left|\frac{1}{\sin t}\right|\mathrm{d}t = 2\int_0^{\frac{\pi}{2}}\left|\frac{1}{\sin t}\right|\mathrm{d}t \\ &= 2\int_0^{\frac{\pi}{2}}\frac{1}{\sin t}\,\mathrm{d}t \overset{s=\cos t}{=} \int_0^1\left(\frac{1}{1-s} + \frac{1}{1+s}\right)\mathrm{d}s = \ln\left(\frac{1+s}{1-s}\right)\Bigg|_0^1 = \ln(\infty) - \ln(1) = \infty - 0 = \infty,\end{aligned}$$

where the fourth equality is because $\left|\frac{1}{\sin t}\right|$ is even and I omit following integral calculation. So γ can be defined on $\mathbb{R}$, by our claim.

Above all, we proved that any geodesic of the upper half-plane $\mathbb{R}^2_+$ with the Lobatchevski metric can be defined on $\mathbb{R}$. By definition, $\mathbb{R}^2_+$ with the Lobatchevski metric is complete. $\square$

Exercise 7.11. Let M be a complete Riemannian manifold, and let X be a differentiable vector field on M. Suppose that there exists a constant $c > 0$ such that $|X(p)| > c$, for all $p \in M$. Prove that the trajectories of X, that is, the curve $\varphi(t)$ in M with $\varphi'(t) = X(\varphi(t))$, are defined for all values of t.

Proof. This is WRONG. For a counterexample, let $M = \mathbb{R}$ and $X(x) = (x^2+1) \left.\frac{\partial}{\partial x}\right|_x$. As we all know $M = \mathbb{R}$ with the Euclidean metric is complete and X is differentiable. Let $c = 0.1$, for any $x \in \mathbb{R} = M$, we have

$$|X(x)| = \sqrt{\langle X(x), X(x)\rangle} = \sqrt{\left\langle (x^2+1) \left.\frac{\partial}{\partial x}\right|_x, (x^2+1) \left.\frac{\partial}{\partial x}\right|_x \right\rangle} = \sqrt{(x^2+1)^2 g_{11}} = x^2+1 > 0.1 = c.$$

So our example satisfies all the conditions. But by solving ODE, we have

$$\varphi(t) = (x(t)) = (\tan t)$$

is the trajectory of X, which can not be defined at $\frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$. $\square$

Exercise 7.12. A Riemannian manifold is said to be *homogeneous* if given $p, q \in M$ there exists an isometry of M which takes p into q. Prove that any homogeneous manifold is complete.

Proof. Let M be any homogeneous manifold. For any $p \in M$ and $v \in T_pM$ with $|v| = 1$, we will prove that the geodesic $\gamma(t) = \exp_p(tv)$ can be defined on $\mathbb{R}$. Then by the homogeneity of geodesics (Chapter 3 Lemma 2.6), we can use r to present the length of the geodesic $\gamma(t)|_{[0,r]} = \exp_p(tv)\big|_{[0,r]}$. Let $B_{2r}(p)$ be a normal ball of p, then $\gamma(r) = \exp_p(rv)$ is defined. Let φ be a isometry of M which taking p into $\gamma(r) = \exp_p(rv)$. Then because φ maintains geodesics, $\varphi(B_{2r}(p)) = B_{2r}(\varphi(p)) = B_{2r}(\gamma(r)) = B_{2r}(\exp_p(rv))$ is a normal ball (see the claim in the proof of Exercise 3.5 b)). So we can extend $\gamma(r) = \exp_p(rv)$ to $\gamma(2r) = \exp_p(2rv)$ along $\gamma(t) = \exp_p(tv)$ in $B_{2r}(\exp_p(rv))$, i.e. $\gamma(t) = \exp_p(tv)$ can be extended to $2r$. Doing the same thing again and again (Let φ_n be a isometry of M which taking $\gamma((n-1)r) = \exp_p((n-1)rv)$ into $\gamma(nr) = \exp_p(nrv)$. Then $\varphi_n(B_{2r}(\gamma((n-1)r))) = B_{2r}(\gamma(nr)) = B_{2r}(\exp_p(nrv))$ is a normal ball. So we can extend $\gamma((n-1)r) = \exp_p((n-1)rv)$ to $\gamma(nr) = \exp_p(nrv)$ along $\gamma(t) = \exp_p(tv)$ in $B_{2r}(\exp_p(nrv))$, i.e. $\gamma(t) = \exp_p(tv)$ can be extended to nr.), we know that $\gamma(t) = \exp_p(tv)$ can be extended to nr, $\forall n \in \mathbb{N}_+$, i.e. the geodesic $\gamma(t) = \exp_p(tv)$ can be defined on $\mathbb{R}$. By the arbitrariness of $p \in M$ and $v \in T_pM$, any geodesic of M can be defined on $\mathbb{R}$. By definition, the homogeneous manifold M is complete, as desired. $\square$

Exercise 7.13. Show that the point $p = (0,0,0)$ of the paraboloid

$$S = \left\{ (x,y,z) \in \mathbb{R}^3 \middle| z = x^2 + y^2 \right\}$$

is a pole of S and, nevertheless, the curvature of S is positive.

Proof. We will use Exercise 3.1. Firstly, we give S a revolution parameters: Let

$$\varphi(u,v) = \left(v\cos u, v\sin u, v^2\right), \qquad 0 < v < \infty,\ -\varepsilon < u < 2\pi + \varepsilon,$$

i.e. $f(v) = v$, $g(v) = v^2$ is as desired (see Exercise 3.1 d)). Then for any $w \in T_pS$ with $|w| = 1$, we suppose $w = (\cos u_0, \sin u_0)$. Let

$$\gamma(t) = \varphi(u(t), v(t)) = \left(v(t)\cos u(t), v(t)\sin u(t), v^2(t)\right) = \left(t\sin u_0 \cos u_0, t\sin^2 u_0, t^2 \sin^2 u_0\right),$$

i.e. $u(t) = u_0$, $v(t) = t\sin u_0$, be a meridian of S starting at $\gamma(0) = \varphi(u_0, 0) = (0,0,0) = p$ and then it's a geodesic of S from p. We have

$$\gamma'(0) = \left(\sin u_0 \cos u_0, \sin^2 u_0, 2t\sin^2 u_0\right)\Big|_{t=0} = \left(\sin u_0 \cos u_0, \sin^2 u_0, 0\right).$$

And by the proof of Exercise 3.1 a), we also have

$$d\varphi_{(u_0,0)}(w) = d\varphi_{(u_0,0)}\left(\cos u_0 \frac{\partial}{\partial u} + \sin u_0 \frac{\partial}{\partial v}\right) = \begin{bmatrix} 0 & \cos u_0 \\ 0 & \sin u_0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \cos u_0 \\ \sin u_0 \end{bmatrix} = \begin{bmatrix} \sin u_0 \cos u_0 \\ \sin^2 u_0 \\ 0 \end{bmatrix} = \gamma'(0).$$

This means $\gamma(t)$ is the geodesic of S from p along w, which can be defined for all $t \in \mathbb{R}$. Then we also have $\gamma'(t) \neq 0$, $\forall t \in \mathbb{R}$. Suppose that $\gamma(t_0)$ is a conjugate point of $p = \gamma(0)$ and $J(t)$ is a non- trivial Jacobi field along $\gamma(t)$ such that $J(t_0) = 0$. By Chapter 5 Proposition 3.6, we have

$$\langle J(t), \gamma'(t)\rangle = \langle J'(0), \gamma'(0)\rangle\, t + \langle J(0), \gamma'(0)\rangle, \qquad t \in [0, t_0].$$

Because $J(t)$ is non- trivial, $J(0)$ and $J'(0)$ are not both zero. And $\gamma'(0) = w \neq 0$. So by the positive-definition of $\langle\ ,\ \rangle$, the right side at t_0 have

$$\langle J'(0), \gamma'(0)\rangle\, t_0 + \langle J(0), \gamma'(0)\rangle > 0.$$

But on the left side, we have

$$\langle J(t_0), \gamma'(t_0)\rangle = \langle 0, \gamma'(t_0)\rangle = 0.$$

Now we conclude

$$0 = \langle J(t_0), \gamma'(t_0)\rangle = \langle J'(0), \gamma'(0)\rangle\, t_0 + \langle J(0), \gamma'(0)\rangle > 0,$$

which is a contradiction. So p has no conjugate points along γ. Then by the arbitrariness of $v \in T_pS$, any geodesic of S starting at p has no conjugate points of p. So p has no conjugate points, i.e. p is a pole of S.

Now we calculate the curvature of S. From the proof of Exercise 3.1 b), at (u, v), we have

$$g_{11} = v^2, \qquad g_{12} = g_{21} = 0, \qquad g_{22} = 1 + 4v^2,$$

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^1_{22} = 0, \qquad \Gamma^2_{11} = -\frac{v}{1+4v^2}, \qquad \Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{v}, \qquad \Gamma^2_{22} = \frac{4v}{1+4v^2}.$$

Because S has dimension 2, let $U = \frac{\partial}{\partial u}$ and $V = \frac{\partial}{\partial v}$ and $\sigma = \mathrm{Span}\{U, V\}$, we have

$$K = K(\sigma) = \frac{R(U,V,U,V)}{|U \wedge V|^2},$$

where

$$R(U,V,U,V) = \langle R(U,V)U, V\rangle = \left\langle \nabla_V \nabla_U U - \nabla_U \nabla_V U + \nabla_{[U,V]} U, V\right\rangle.$$

By the local notion of of affine connection:

$$\nabla_X Y = \sum_k \left(\sum_{ij} x_i y_j \Gamma^k_{ij} + X(y_k) \right) X_k,$$

we can calculate

$$\nabla_U U = -\frac{v}{1+4v^2} V \quad \text{ and } \quad \nabla_V U = \frac{1}{v} U \quad \text{ and } \quad \nabla_V V = \frac{4v}{1+4v^2} V.$$

Then

$$\begin{aligned}
\nabla_V \nabla_U U &= \nabla_V \left(-\frac{v}{1+4v^2} V\right) = -\frac{v}{1+4v^2} \nabla_V V + V\left(-\frac{v}{1+4v^2}\right) V \\
&= -\frac{v}{1+4v^2}\frac{4v}{1+4v^2} V + \frac{\partial}{\partial v}\left(-\frac{v}{1+4v^2}\right) V = -\frac{v}{1+4v^2}\frac{4v}{1+4v^2} V - \frac{1-4v^2}{(1+4v^2)^2} V = -\frac{1}{(1+4v^2)^2} V
\end{aligned}$$

and

$$\begin{aligned}
\nabla_U \nabla_V U &= \nabla_U \left(\frac{1}{v} U\right) = \frac{1}{v} \nabla_U U + U\left(\frac{1}{v}\right) U \\
&= \frac{1}{v}\left(-\frac{v}{1+4v^2} V\right) + \frac{\partial}{\partial u}\left(\frac{1}{v}\right) U = -\frac{1}{1+4v^2} V + 0U = -\frac{1}{1+4v^2} V.
\end{aligned}$$

And noticing that

$$[U,V]=\frac{\partial}{\partial u}\frac{\partial}{\partial v}-\frac{\partial}{\partial v}\frac{\partial}{\partial u}=0,$$

we have

$$\nabla_{[U,V]}U=\nabla_0 U=0.$$

So we have

$$\nabla_V\nabla_U U-\nabla_U\nabla_V U+\nabla_{[U,V]}U=-\frac{1}{\left(1+4v^2\right)^2}V-\left(-\frac{1}{1+4v^2}V\right)+0=\frac{4v^2}{\left(1+4v^2\right)^2}V.$$

Then We conclude

$$\begin{aligned}R(U,V,U,V)&=\left\langle\nabla_V\nabla_U U-\nabla_U\nabla_V U+\nabla_{[U,V]}U,V\right\rangle\\&=\left\langle\frac{4v^2}{\left(1+4v^2\right)^2}V,V\right\rangle=\frac{4v^2}{\left(1+4v^2\right)^2}g_{22}(u,v)=\frac{4v^2}{\left(1+4v^2\right)^2}\left(1+4v^2\right)=\frac{4v^2}{1+4v^2}.\end{aligned}$$

While

$$|U\wedge V|^2=|U|^2|V|^2-\langle U,V\rangle^2=\langle U,U\rangle\langle V,V\rangle-\langle U,V\rangle^2=g_{11}(u,v)g_{22}(u,v)-g_{12}^2(u,v)=v^2\left(1+4v^2\right).$$

It follows that

$$K=\frac{R(U,V,U,V)}{|U\wedge V|^2}=\frac{4v^2}{v^2\left(1+4v^2\right)\left(1+4v^2\right)}=\frac{4}{\left(1+4v^2\right)^2}>0,$$

i.e. S has positive curvature, as desired.

Above all, we conclude that there is a Riemannian manifold with positive curvature has a pole. $\square$

8 Spaces of Constant Curvature

Exercise 8.1. Consider, on a neighborhood in $\mathbb{R}^n$, $n > 2$ the metric

$$g_{ij} = \frac{\delta_{ij}}{F^2}$$

where $F \neq 0$ is a function of $(x_1, \cdots, x_n) \in \mathbb{R}^n$. Denote by $F_i = \frac{\partial F}{\partial x_i}$, $F_{ij} = \frac{\partial^2 F}{\partial x_i \partial x_j}$, etc.

a) Show that a necessary and sufficient condition for the metric to have constant curvature K is

$$\begin{cases} F_{ij} = 0, & \forall i \neq j, \\ F\left(F_{jj} + F_{ii}\right) = K + \sum_{i=1}^n \left(F_i\right)^2, & \forall i, j. \end{cases} \tag{*}$$

b) Use (*) to prove that the metric g_{ij} has constant curvature K if and only if

$$F = G_1\left(x_1\right) + \cdots G_n\left(x_n\right),$$

where

$$G_i\left(x_i\right) = a x_i^2 + b_i x_i + c_i$$

and

$$\sum_{i=1}^n \left(4ac_i - b_i^2\right) = K.$$

c) Put $a = \frac{K}{4}$, $b_i = 0$, $c_i = \frac{1}{n}$ and obtain the formula of Riemann

$$g_{ij} = \frac{\delta_{ij}}{\left(1 + \frac{K}{4} \sum_{i=1}^n x_i^2\right)^2} \tag{**}$$

for a metric g_{ij} of constant curvature K. If $K < 0$ the metric g_{ij} is defined in a ball of radius $\sqrt{\frac{4}{-K}}$.

d) If $K > 0$, the metric (**) is defined on all of $\mathbb{R}^n$. Show that such a metric on $\mathbb{R}^n$ is not complete.

Proof. For a), in the context (at the bottom of Page 161), we know that

$$K_{ij} = \left(-\sum_{l=1}^n f_l^2 + f_i^2 + f_j^2 + f_{ii} + f_{jj}\right) F^2,$$

where $f = \ln F$. So we have

$$f_i = \frac{F_i}{F} \quad \text{and} \quad f_{ii} = \frac{F_{ii}F - F_i^2}{F^2}.$$

Then

$$\begin{aligned} K_{ij} &= \left(-\sum_{l=1}^n f_l^2 + f_i^2 + f_j^2 + f_{ii} + f_{jj}\right) F^2 \\ &= \left(-\sum_{l=1}^n \left(\frac{F_l}{F}\right)^2 + \left(\frac{F_i}{F}\right)^2 + \left(\frac{F_j}{F}\right)^2 + \frac{F_{ii}F - F_i^2}{F^2} + \frac{F_{jj}F - F_j^2}{F^2}\right) F^2 \\ &= -\sum_{l=1}^n F_l^2 + F_{ii}F + F_{jj}F. \end{aligned}$$

So the metric have constant curvature K if and only if

$$K = K_{ij} = -\sum_{l=1}^n F_l^2 + F_{ii}F + F_{jj}F,$$

i.e.

$$F\left(F_{jj}+F_{ii}\right)=K+\sum_{l=1}^{n}F_l^2.$$

And the first equation of (*) is by the proof of Chapter 8 Theorem 5.2 (in the middle of Page 170):

$$F_{ij}=\frac{\partial^2\rho}{\partial x_i\partial x_j}=\sigma\delta_{ij}=0,\qquad \forall i\neq j,$$

if the metric has constant curvature K.

Then for b), if the metric has constant curvature K, by a), by the first equation of (*), because $F_{ij}=0$ for $i\neq j$, we know that F_i only depends on x_i. So we can suppose $F=\sum_{i=1}^n G_i\left(x_i\right)$ where G_i is a function of x_i. Differentiating the second equation about x_i, we have

$$F_i\left(F_{ii}+F_{jj}\right)+F\left(F_{iii}+F_{jji}\right)=\sum_{l=1}^{n}2F_lF_{li}.$$

By $F=\sum_{i=1}^n G_i\left(x_i\right)$, $F_{jji}=0$ and $F_{li}=0$ for $l\neq i$. So the equation above will be

$$F_i\left(F_{ii}+F_{jj}\right)+FF_{iii}=2F_iF_{ii}.$$

Noticing that FF_{iii} is about $x_1,\cdots,x_n$ while the other terms are only about i,j, we must have $FF_{iii}=0$, i.e. $F_{iii}=0$. This means G_i are polynomial of degree at most 2. Denote $G_i(x_i)=a_ix_i^2+b_ix_i+c_i$. Back to the second equation of (*), we have

$$\left(2a_i+2a_j\right)\sum_{k=1}^{n}\left(a_kx_k^2+b_kx_k+c_k\right)=K+)\sum_{k=1}^{n}\left(4a_k^2x_k^2+4a_kb_kx_k+b_k^2\right),$$

i.e.

$$\sum_{k=1}^{n}\left\{\left[4a_k^2-\left(2a_i+2a_j\right)a_k\right]x_i^2+\left[4a_kb_k-\left(2a_i+2a_j\right)b_k\right]x_i+\left[b_k^2-\left(2a_i+2a_j\right)c_k\right]\right\}+K=0.$$

All the coefficients must be zeros, so we have the system: $\forall 1\leq k\leq n$,

$$\begin{cases}4a_k^2-\left(2a_i+2a_j\right)a_k=0,\\ 4a_kb_k-\left(2a_i+2a_j\right)b_k=0,\\ \sum_{k=1}^n\left[b_k^2-\left(2a_i+2a_j\right)c_k\right]+K=0.\end{cases}$$

Solving it, we have $a_1=\cdots=a_n=a$. Then we have

$$G_i\left(x_i\right)=a_ix_i^2+b_ix_i+c_i=ax_i^2+b_ix_i+c_i,\qquad \forall 1\leq i\leq n$$

and

$$K=\sum_{k=1}^{n}\left[\left(2a_i+2a_j\right)c_k-b_k^2\right]=\sum_{k=1}^{n}\left(4ac_k-b_k^2\right),$$

as desired. The converse direction is easy to check by a) (In fact, all the inductions above are "$\Leftrightarrow$").

For c), putting $a=\frac{K}{4}$, $b_i=0$, $c_i=\frac{1}{n}$, we have

$$F=\sum_{i=1}^{n}G_i\left(x_i\right)=\sum_{i=1}^{n}\left(\frac{K}{4}x_i^2+\frac{1}{n}\right)=1+\frac{K}{4}\sum_{i=1}^{n}x_i^2.$$

Then

$$g_{ij}=\frac{\delta_{ij}}{F^2}=\frac{\delta_{ij}}{\left(1+\frac{K}{4}\sum_{i=1}^n x_i^2\right)}.$$

If $K<0$, because g_{ij} is defined on a connected subset, the denominator can not reach zero. So g_{ij} is defined on

$$\left\{\left(x_1,\cdots,x_n\right)\in\mathbb{R}^n\,\middle|\,1+\frac{K}{4}\sum_{i=1}^{n}x_i^2>0\right\}=\left\{\left(x_1,\cdots,x_n\right)\in\mathbb{R}^n\,\middle|\,\sum_{i=1}^{n}x_i^2<\frac{4}{-K}\right\}=B_{\sqrt{\frac{4}{-K}}}(0).$$

Finally for d), if $K > 0$, g_{ij} is differentiable on $\mathbb{R}^n$, so the metric (**) is defined on all of $\mathbb{R}^n$. As the beginning of Chapter 8, we can assume that the metric has constant curvature $K = 1$. If the metric is complete, by Chapter 8 Theorem 4.1, $\mathbb{R}^n$ with this metric has universal covering S^n. And $\mathbb{R}^n$ is simple connected, so $\mathbb{R}^n$ with this metric is isometric to S^n. In particular, they are homomorphic to each other. But because the function $\frac{1}{\left(1+\frac{K}{4}\sum_{i=1}^n x_i^2\right)^2}$ is bounded:

$$0 < \frac{1}{\left(1+\frac{K}{4}\sum_{i=1}^n x_i^2\right)^2} \leq 1,$$

the topology induced by the metric $g_{ij} = \frac{\delta_{ij}}{\left(1+\frac{K}{4}\sum_{i=1}^n x_i^2\right)^2}$ is same as the Euclidean topology induced by Euclidean metric $g_{ij} = \delta_{ij}$. So we know that $\mathbb{R}^n$ with this metric has trivial homology groups, while S^n does't have ones, which is contradict to the fact that $\mathbb{R}^n$ with this metric is homomorphic to S^n. So such a metric on $\mathbb{R}^n$ is not complete. $\square$

Exercise 8.2. Show that if M^k is a closed, totally geodeisc submanifold of H^n, $k \leq n$, then M^k is isometric to H^k. Determine all the totally geodesic submanifolds of H^n.

Proof. Let $\varphi \colon M \to H^n$ be the isometric immersion. Then we can consider M as $\varphi(M)$. Because M is close, any close and bounded subset of $\varphi(M)$ is also close and bounded in H^n, which is complete. Then by Hopf-Rinow Theorem (Chapter 7 Theorem 2.8 d) $\Rightarrow$ b)), that subset is compact. Using the theorem conversely (b) $\Rightarrow$ d)), $\varphi(M)$ is complete. And because M is totally geodesic, $\forall \eta \in (T_pM)^\perp$, $H_\eta = 0$. Then $B(X,Y) = \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y = 0$. So the connection of M is same as it of H^n, so is the curvature. This means M also has constant curvature -1 as H^n. Above all, we conclude that M is a complete Riemannian manifold of constant curvature -1, so M is isometric to H^k. (In fact, because M is totally geodesic, by Chapter 6 Proposition 2.9, any geodesic of M is a geodesic of H^n, which is on a 2-dimensional linear subspace of H^n. So M is an intersection of H^n and a k-dimensional linear subspace of $\mathbb{R}^n$.)

Then for any close totally geodesic submanifold of H^n with dimension k, it is isometric to H^k. For any such a non-close submanifold, we can extend it in H^n to a close one which is isometric to H^k. So it is isometric to a subset of $H^k \subset H^n$. So now we have determined all the totally geodesic submanifolds of H^n. $\square$

Exercise 8.3 (Another model of the hyperbolic space)**.** Consider on $\mathbb{R}^{n+1}$ the quadratic form

$$Q\left(x_0, x_1, \cdots, x_n\right) = -x_0^2 + \sum_{i=1}^n x_i^2, \qquad \left(x_0, \cdots, x_n\right) \in \mathbb{R}^{n+1}.$$

With the pseudo-Riemannian metric $(\ ,\ )$ induced by Q (see Exercise 2.9), $\mathbb{R}^{n+1}$ will be denoted by L^{n+1} (the Lorentzian space). Denote by H_k^n, $k = -\frac{1}{r^2}$, the connected component cooresponding to $x_0 > 0$ of the regular surface of $\mathbb{R}^{n+1}$ given by $Q(x) = -r^2$, $r > 0$. (Geometrically $Q(x) = -r^2$ is a hyperboloid of two sheets and H_k^n is the sheet contained in the half-space $x_0 > 0$.)

a) Show that for all $x \in H_k^n$, the vector $\eta = \frac{x}{r}$ is normal to the tangent space $T_x\left(H_k^n\right)$.

b) Prove that $(\eta, \eta) = -1$, and that it is possible to choose a basis $b_0, \cdots, b_n$ of L^{n+1} with $b_0 = \eta$, $(b_i, b_j) = \delta_{ij}$, $(b_i, b_0) = 0$, $\forall i, j = 1, \cdots, n$. (Use the fact that the index of a quadratic form does not depend on the basis chosen to represent it.) Conclude that the metric induced by L^{n+1} on H_k^n is Riemannian.

c) Use the pseudo-Riemannian connection $\overline{\nabla}$ of L^{n+1} (see Exercise 2.9) to show that $S_\eta = \left(-\frac{1}{r}\right)\mathrm{Id}$. Conclude that $B(X,Y) = \frac{\eta}{r}(X,Y)$, and use the Gauss formula to show that the sectional curvature of H_k^n is constant and equal $k = -\frac{1}{r^2}$.

d) Let $\mathrm{O}(1, n+1)$ be the subgroup of linear transformations of $\mathbb{R}^{n+1}$ which preserve the metric $(\ ,\ )$. Show that the elements of $\mathrm{O}(1, n+1)$ with $\det > 0$ are isometries of H_k^n and that given $X, Y \in H_k^n$ and orthonormal bases $\{v_1, \cdots, v_n\} \in T_X\left(H_k^n\right)$ and $\{w_1, \cdots, w_n\} \in T_Y\left(H_k^n\right)$, the restriction to H_k^n of the "linear" transformation which takes

$$\frac{X}{r} \to \frac{Y}{r} \quad \text{and} \quad v_i \to w_i,\ \forall i = 1, \cdots, n$$

is an isometry of H_k^n. Conclude then that H_k^n has constant curvature (which we already know from c)) and that H_k^n is complete.

e) Show that H^n_{-1} is isometric to the hyperbolic space H^n.

f) Show that the symmetries of H^n_k with respect to the plane P with passes through the origin of $\mathbb{R}^{n+1}$ and contains the x_0-axis are isometries of H^n_k. Conclude that all of the geodesic of H^n_k which pass through $(r, 0, \cdots, 0)$ are obtained as intersections $H^n_k \cap P$.

Proof. For a), denote $x = (p_0, \cdots, p_n) \in H^n_k$ where $p_0 = \sqrt{\sum_{i=1}^n p_i^2 + r^2}$. For any $v \in T_x(H^n_k)$, there is a curve $\alpha(t) = (x_0(t), \cdots, x_n(t))$ in H^n_k where $x_0(t) = \sqrt{\sum_{i=1}^n x_i^2(t) + r^2}$ such that $\alpha(0) = x$ and $\alpha'(0) = v$. Then

$$\begin{aligned} v &= \alpha'(0) = (x_0'(0), x_1'(0), \cdots, x_n'(0)) \\ &= \left(\frac{\sum_{i=1}^n x_i(0) x_i'(0)}{\sqrt{\sum_{i=1}^n x_i^2(0) + r^2}}, x_1'(0), \cdots, x_n'(0) \right) = \left(\frac{\sum_{i=1}^n p_i x_i'(0)}{\sqrt{\sum_{i=1}^n p_i^2 + r^2}}, x_1'(0), \cdots, x_n'(0) \right). \end{aligned}$$

Under the pseudo-Riemannian metric Q, we have

$$\begin{aligned} (\eta, v) &= \left(\frac{x}{r}, v \right) = \frac{1}{r} (x, v) \\ &= \frac{1}{r} \left(-\sqrt{\sum_{i=1}^n p_i^2 + r^2} \cdot \frac{\sum_{i=1}^n p_i x_i'(0)}{\sqrt{\sum_{i=1}^n p_i^2 + r^2}} + \sum_{i=1}^n p_i x_i'(0) \right) = \frac{1}{r} \left(-\sum_{i=1}^n p_i x_i'(0) + \sum_{i=1}^n p_i x_i'(0) \right) = 0, \end{aligned}$$

i.e. $\eta \perp v \in T_x(H^n_k)$. By the arbitrariness of $v \in T_x(H^n_k)$, we conclude that $\eta = \frac{x}{r}$ is normal to the tangent space $T_x(H^n_k)$.

Then for b), we have

$$\begin{aligned} (\eta, \eta) &= \frac{1}{r^2} (x, x) = \frac{1}{r^2} ((p_0, \cdots, p_n), (p_0, \cdots, p_n)) = \frac{1}{r^2} \left(-p_0^2 + \sum_{i=1}^n p_i^2 \right) \\ &= \frac{1}{r^2} \left(-\sqrt{\sum_{i=1}^n p_i^2 + r^2}^{\,2} + \sum_{i=1}^n p_i^2 \right) = \frac{1}{r^2} \left(-\sum_{i=1}^n p_i^2 - r^2 + \sum_{i=1}^n p_i^2 \right) = \frac{1}{r^2} (-r^2) = -1. \end{aligned}$$

Counting the dimension, we know that H^n_k has dimension n and $T(H^n_k)^\perp$ has dimension 1. For any $v \in T_x(H^n_k)$, $(\eta, v) = 0$. Then we can extend $b_0 = \eta$ to a basis $b_0, b_1, \cdots, b_n$ of $T(H^n_k) \subset T(L^{n+1})$ such that Q has the form

$$\begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

under the basis, i.e. $(b_i, b_j) = \delta_{ij}$ and $(b_i, b_0) = 0$, $1 \le i, j \le n$, as desired. And then the metric of H^n_k induced by Q on L^{n+1} is

$$Q_{11} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} = \mathrm{Id}_n$$

under the basis $b_1, \cdots, b_n$, which is positive-definite and then is Riemannian.

For c), by Chapter 6 Proposition 2.3, $S_\eta(v) = -\left(\overline{\nabla}_v N\right)^T$ for any $v \in T_x(H^n_k)$ where N is a local extension of η normal to H^n_k. Denote $v = (x_0, \cdots, x_n)$. By a), we must have $(\eta, v) = 0$, i.e.

$$0 = (\eta, v) = \left(\frac{x}{r}, v \right) = \frac{1}{r} (x, v) = \frac{1}{r} ((p_0, \cdots, p_n), (x_0, \cdots, x_n)) = \frac{1}{r} \left(-p_0 x_0 + \sum_{i=1}^n p_i x_i \right).$$

So we have

$$x_0 = \frac{\sum_{i=1}^n p_i x_i}{p_0} = \frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i^2 + r^2}.$$

Let $\alpha(t)=\left(\sqrt{\sum_{i=1}^{n}\left(p_i+tx_i\right)^2+r^2}, p_1+tx_1,\cdots,p_n+tx_n\right)$ be a curve on H_k^n. Then we have

$$\alpha'(t)=\left(\frac{\sum_{i=1}^{n}\left(p_i+tx_i\right)x_i}{\sqrt{\sum_{i=1}^{n}\left(p_i+tx_i\right)^2+r^2}}, x_1,\cdots,x_n\right)$$

and then

$$\alpha'(0)=\left(\frac{\sum_{i=1}^{n}p_ix_i}{\sqrt{\sum_{i=1}^{n}p_i^2+r^2}}, x_1,\cdots,x_n\right)=v.$$

By a), for all t, $\frac{\alpha(t)}{r}\in\left(T_{\alpha(t)}\left(H_k^n\right)\right)^{\perp}$, so we can choose $N(\alpha(t))=\frac{\alpha(t)}{r}$ as a local extension of η along $\alpha(t)$. Then by definition,

$$\overline{\nabla}_vN=\overline{\nabla}_{\alpha'}N(0)=\frac{DN}{dt}(0).$$

And by Exercise 2.9 b), the parallel transports of L^{n+1} are same as of $\mathbb{R}^{n+1}$, then the covariant derivative $\frac{DN}{dt}$ along $\alpha(t)$ is the usual derivative (see the end of Chapter 2). So we can calculate

$$\frac{DN}{dt}=\frac{d}{dt}\left(\frac{1}{r}\alpha(t)\right)=\frac{1}{r}\alpha'(t).$$

Then we have

$$\overline{\nabla}_vN=\frac{DN}{dt}(0)=\frac{1}{r}\alpha'(0)=\frac{1}{r}v\in T_x\left(H_k^n\right).$$

So

$$\left(\overline{\nabla}_vN\right)^T=\left(\frac{1}{r}v\right)^T=\frac{1}{r}v.$$

We conclude that

$$S_\eta(v)=-\left(\overline{\nabla}_vN\right)^T=-\frac{1}{r}v.$$

By the arbitrariness of $v\in T_x\left(H_k^n\right)$, we have

$$S_\eta=\left(-\frac{1}{r}\right)\operatorname{Id}.$$

For any $X,Y\in\mathfrak{X}\left(H_k^n\right)$, because $B(X,Y)\in\left(T_x\left(H_k^n\right)\right)=\operatorname{Span}\{\eta\}$, we can denote $B(X,Y)=k(X,Y)\eta$. Then by definition,

$$\left(S_\eta(X),Y\right)=\left(B(X,Y),\eta\right),$$

where

$$\left(S_\eta(X),Y\right)=\left(-\frac{1}{r}X,Y\right)=-\frac{1}{r}\left(X,Y\right)\qquad\text{and}\qquad\left(B(X,Y),\eta\right)=\left(k(X,Y)\eta,\eta\right)=-k(X,Y).$$

So we have

$$k(X,Y)=-\left(B(X,Y),\eta\right)=-\left(S_\eta(X),Y\right)=\frac{1}{r}\left(X,Y\right),$$

i.e.

$$B(X,Y)=k(X,Y)\eta=\frac{1}{r}\left(X,Y\right)\eta=\frac{\eta}{r}\left(X,Y\right).$$

By Gauss formula (Chapter 6 Proposition 3.1 (a)), we have

$$\left(\overline{R}(X,Y)Z,T\right)=\left(R(X,Y)Z,T\right)-\left(B(Y,T),B(X,Z)\right)+\left(B(X,T),B(Y,Z)\right),\qquad\forall X,Y,Z,T\in\mathfrak{X}\left(H_k^n\right),$$

where $\overline{R}$ is the curvature of L^{n+1}, which is zero (see the proof of Exercise 2.9 b)). Then for $X,Y,Z,T\in\{b_1,\cdots,b_n\}$, which is a orthonormal basis of $\mathfrak{X}\left(H_k^n\right)$, we have

$$\begin{aligned}R_{ijkl}=\left(R(b_i,b_j)b_k,b_l\right)&=\left(\overline{R}(b_i,b_j)b_k,b_l\right)+\left(B(b_j,b_l),B(b_i,b_k)\right)-\left(B(b_i,b_l),B(b_j,b_k)\right)\\&=\left(\frac{\eta}{r}(b_j,b_l),\frac{\eta}{r}(b_i,b_k)\right)-\left(\frac{\eta}{r}(b_i,b_l),\frac{\eta}{r}(b_j,b_k)\right)=-\frac{1}{r^2}\left(\delta_{jl}\delta_{ik}-\delta_{il}\delta_{jk}\right).\end{aligned}$$

By Chapter 4 Corollary 3.5, H_k^n has constant sectional curvature $-\frac{1}{r^2} = k$.

For d), $\forall A \in \mathrm{O}(1, n+1)$ with $\det A > 0$, notice that $\forall x \in H_k^n$,

$$Q(Ax) = (Ax, Ax) = (x, x) = Q(x) = -r^2,$$

and $\det A > 0$ guarantees that Ax is located in the upper sheet of the hyperboloid, so $Ax \in H_k^n$. This means A is a diffeomorphism from H_k^n to H_k^n. And for any $x \in H_k^n$ and $X, Y \in T_x(H_k^n)$, because A is linear, $dA_x = A$, then

$$(dA_x(X), dA_x(Y))_{Ax} = (AX, AY)_Q = (X, Y)_x\,.$$

So A is an isometry of H_k^n.

Denote $v_0 = \frac{X}{r}$ and $w_0 = \frac{Y}{r}$ and B the linear transformation

$$\frac{X}{r} \mapsto \frac{Y}{r} \qquad \text{and} \qquad v_i \mapsto w_i.$$

By b) and a), $v_0 = \frac{X}{r}$ and $w_0 = \frac{Y}{r}$ have "length" -1 and orthogonal to $v_1, \cdots, v_n$ and $w_1, \cdots, w_n$, respectively. So $\{v_0, \cdots, v_n\}$ and $\{w_0, \cdots, w_n\}$ are two orthonormal bases of L^{n+1}. Then we can calculate

$$(Bv_0, Bv_0) = (w_0, w_0) = -1 = (v_0, v_0)\,,$$

$$(Bv_i, Bv_0) = (w_i, w_0) = 0 = (v_i, v_0)\,,$$

$$(Bv_i, Bv_j) = (w_i, w_j) = \delta_{ij} = (v_i, v_j)\,, \qquad \forall 1 \le i, j \le n.$$

So B preserves the metric $(\ ,\ )$. It follows that $B \in \mathrm{O}(1, n+1)$. Our construction of B also shows that B maps H_k^n to itself (the upper sheet). Then by similar proof above, B is an isometry of H_k^n.

Using this fact, at any point $x \in H_k^n$, we can choose an isometry of H_k^n sending x to $p = (r, 0, \cdots, 0)$, which preserve the metric and then the sectional curvature. So for any section σ, $K(x, \sigma) = K(p, \sigma)$. And notice that $e_i = (0, 0, \cdots, 0, 1, 0, \cdots, 0)$, $1 \le i \le n$ is a orthonormal basis of $T_p(H_k^n)$, by the symmetries of Q and $e_1, \cdots, e_n$, $K(p, \sigma)$ is a constant about $\sigma = \mathrm{Span}\{e_i, e_j\}$, $\forall 1 \le i \ne j \le n$. So we conclude that H_k^n has constant sectional curvature. For any close and bounded set U of H_k^n, we can choose a ball of L^{n+1} containing U, which is compact. So U is close in a compact set. Then it's compact. By Hopf-Rinow Theorem (Chapter 7 Theorem 2.8 b) $\Rightarrow$ d)), H_k^n is complete.

For e), by c) and d), H_{-1}^n is a complete Riemannian manifold with constant curvature -1. By Chapter 8 Theorem 4.1, it has universal covering $\widetilde{M}$ isometric to H^n. In addition, it's easy to see that H_{-1}^n is simple connected. So $H_{-1}^n = \widetilde{M}$ isometric to H^n, as desired.

Finally, for f), by d), choosing $X = Y = p \in H_k^n \cap x_0-$axis, these symmetries are linear and must map $\frac{X}{r} = \frac{p}{r}$ to $\frac{Y}{r} = \frac{p}{r}$ and map the orthonormal basis $v_1 = e_1, \cdots, v_n = e_n$ of $T_X(H_k^n) = T_p(H_k^n)$ to another orthonormal basis $w_1, \cdots, w_n$ of $T_Y(H_k^n) = T_p(H_k^n)$. By d), they are isometries of H_k^n. And for any geodesic $\gamma(t)$ of H_k^n passing $p = (r, 0, \cdots, 0)$, we can assume $\gamma(t) = \left(\sqrt{\sum_{i=1}^n x_i^2(t) + r + 2}, x_1(t), \cdots, x_n(t)\right)$ starting from p with $\gamma'(0) = v = (0, x_1, \cdots, x_n)$. Let P be the plane spanned by x_0-axis and v. By the geodesic equation:

$$\frac{d^2 x_k}{dt^2} + \sum_{i,j=1}^n \Gamma_{ij}^k \frac{dx_i}{dt}\frac{dx_j}{dt} = 0, \qquad 1 \le k \le n,$$

where $\Gamma_{ij}^k = 0$ (see the proof of Exercise 2.9 b)). Then we have

$$\frac{d^2 x_k}{dt^2} = 0, \qquad 1 \le k \le n,$$

which means $x_k(t)$ are polynomials with degree not greater than 1. Solving the system:

$$\begin{cases} \left(\sqrt{\sum_{i=1}^n x_i^2(0) + r + 2}, x_1(0), \cdots, x_n(0)\right) = \gamma(0) = p = (r, 0, \cdots, 0), \\ \left(\frac{\sum_{i=1}^n x_i(0)x_i'(0)}{\sqrt{\sum_{i=1}^n x_i^2(0) + r + 2}}, x_1'(0), \cdots, x_n'(0)\right) = \gamma'(0) = v = (0, x_1, \cdots, x_n), \end{cases}$$

we have

$$
\begin{aligned}
\gamma(t) &= \left(\sqrt{\sum_{i=1}^{n} t^2 x_i^2 + r^2}, t x_1, \cdots, t x_n\right) \\
&= \left(\sqrt{\sum_{i=1}^{n} t^2 x_i^2 + r^2}\right)(1, 0, \cdots, 0) + t\,(0, x_1, \cdots, x_n) \\
&= \left(\sqrt{\sum_{i=1}^{n} t^2 x_i^2 + r^2}\right)(1, 0, \cdots, 0) + t v \subset \operatorname{Span}\{x_0\text{−axis}, v\} = P.
\end{aligned}
$$

So we conclude $\gamma(t) \subset H_k^n \cap P$, as desired. $\square$

Exercise 8.4. Identify $\mathbb{R}^4$ with $\mathbb{C}^2$ by letting (x_1, x_2, x_3, x_4) correspond to $(x_1 + \mathrm{i}\, x_2, x_3 + \mathrm{i}\, x_4)$. Let

$$
S^3 = \left\{ (z_1, z_2) \in \mathbb{C}^2 \middle| \left|z_1\right|^2 + \left|z_2\right|^2 = 1 \right\},
$$

and let $h\colon S^3 \to S^3$ be given by

$$
h\left(z_1, z_2\right) = \left(\mathrm{e}^{\frac{2\pi\,\mathrm{i}}{q}} z_1, \mathrm{e}^{\frac{2\pi\,\mathrm{i}\, r}{q}} z_2\right), \qquad \forall\left(z_1, z_2\right) \in S^3,
$$

where q and r are relatively prime integers, $q > 2$.

a) Show that $G = \left\{\mathrm{Id}, h, \cdots, h^{q-1}\right\}$ is a group of isometries of the sphere S^3, with the usual metric, which operates in a totally discontinuous manner. The manifold S^3/G is called a *lens space*.

b) Consider S^3/G with the metric induced by the projection $p\colon S^3 \to S^3/G$. Show that all the geodesics of S^3/G are closed but can have different lengths.

Proof. For a), notice that $\forall\left(z_1, z_2\right) \in S^3$,

$$
\begin{aligned}
h^q\left(z_1, z_2\right) &= \left(\mathrm{e}^{\frac{2\pi\,\mathrm{i}}{q}} \cdots \mathrm{e}^{\frac{2\pi\,\mathrm{i}}{q}} z_1, \mathrm{e}^{\frac{2\pi\,\mathrm{i}\, r}{q}} \cdots \mathrm{e}^{\frac{2\pi\,\mathrm{i}\, r}{q}} z_2\right) \\
&= \left(\mathrm{e}^{\frac{2\pi\,\mathrm{i}}{q} + \cdots + \frac{2\pi\,\mathrm{i}}{q}} z_1, \mathrm{e}^{\frac{2\pi\,\mathrm{i}\, r}{q} + \cdots + \frac{2\pi\,\mathrm{i}\, r}{q}} z_2\right) = \left(\mathrm{e}^{2\pi\,\mathrm{i}} z_1, \mathrm{e}^{2\pi\,\mathrm{i}\, r} z_2\right) = \left(z_1, z_2\right),
\end{aligned}
$$

so $h^q = \mathrm{Id}$ and then h have order q and then $G = \left\{\mathrm{Id}, h, \cdots, h^{q-1}\right\}$ is a cyclic group of order q. We can extend h to $\mathbb{R}^4$ naturally. Then we have

$$
\begin{aligned}
h\left(x_1, x_2, x_3, x_4\right) = h\left(z_1, z_2\right) &= \left(\mathrm{e}^{\frac{2\pi\,\mathrm{i}}{q}} z_1, \mathrm{e}^{\frac{2\pi\,\mathrm{i}\, r}{q}} z_2\right) \\
&= \left(\left(\cos\left(\frac{2\pi}{q}\right) + \mathrm{i}\sin\left(\frac{2\pi}{q}\right)\right)\left(x_1 + \mathrm{i}\, x_2\right), \left(\cos\left(\frac{2\pi r}{q}\right) + \mathrm{i}\sin\left(\frac{2\pi r}{q}\right)\right)\left(x_3 + \mathrm{i}\, x_4\right)\right) \\
&= \left(\cos\left(\frac{2\pi}{q}\right) x_1 - \sin\left(\frac{2\pi}{q}\right) x_2 + \mathrm{i}\left(\sin\left(\frac{2\pi}{q}\right) x_1 + \cos\left(\frac{2\pi}{q}\right) x_2\right),\right. \\
&\qquad \left.\cos\left(\frac{2\pi r}{q}\right) x_3 - \sin\left(\frac{2\pi r}{q}\right) x_4 + \mathrm{i}\left(\sin\left(\frac{2\pi r}{q}\right) x_3 + \cos\left(\frac{2\pi r}{q}\right) x_4\right)\right) \\
&= \left(\cos\left(\frac{2\pi}{q}\right) x_1 - \sin\left(\frac{2\pi}{q}\right) x_2, \sin\left(\frac{2\pi}{q}\right) x_1 + \cos\left(\frac{2\pi}{q}\right) x_2,\right. \\
&\qquad \left.\cos\left(\frac{2\pi r}{q}\right) x_3 - \sin\left(\frac{2\pi r}{q}\right) x_4, \sin\left(\frac{2\pi r}{q}\right) x_3 + \cos\left(\frac{2\pi r}{q}\right) x_4\right),
\end{aligned}
$$

i.e.

$$
h = \begin{pmatrix}
\cos\left(\frac{2\pi}{q}\right) & -\sin\left(\frac{2\pi}{q}\right) & 0 & 0 \\
\sin\left(\frac{2\pi}{q}\right) & \cos\left(\frac{2\pi}{q}\right) & 0 & 0 \\
0 & 0 & \cos\left(\frac{2\pi r}{q}\right) & -\sin\left(\frac{2\pi r}{q}\right) \\
0 & 0 & \sin\left(\frac{2\pi r}{q}\right) & \cos\left(\frac{2\pi r}{q}\right)
\end{pmatrix} \in \mathrm{O}_4
$$

is an orthogonal transformation of $\mathbb{R}^4$. So we have $dh_x = h$, $\forall x \in \mathbb{R}^4$ and then for any $u, v \in T_x\mathbb{R}^4 \cong \mathbb{R}^4$,

$$\langle dh_x(u), dh_x(v)\rangle_x = \langle h(u), h(v)\rangle_{\mathbb{R}^4} = \left\langle u, h^T h(v)\right\rangle = \langle u, v\rangle ,$$

i.e. h is an isometry of $\mathbb{R}^4$. So it's naturally an isometry of the embedding submanifold S^3 of $\mathbb{R}^4$ and so do $h^2, \cdots, h^{q-1}, h^q = \mathrm{Id}$.

By the structure of orthogonal transformations, h is a rotation of angle $\frac{2\pi}{q}$ anticlockwise about the x_1-x_2 plane and a rotation of angle $\frac{2\pi r}{q}$ anticlockwise about the x_3-x_4 plane. So the orbit of any $x \in S^3$ are some finite discrete points. Then we can easily choose a neighborhood U of x such that $h^i(U) \cap U = \varnothing, \forall 1 \le i \le q-1$. We conclude that G operates on S^3 in a totally discontinuous manner. And so we get the lens space.

For b), for any $x \in S^3/G$ and $v \in T_x\left(S^3/G\right)$, let' consider one of their lifting $\overline{x} \in S^3$ and $\overline{v} \in T_{\overline{x}}\left(S^3/G\right)$. By Chapter 3 Example 2.11, the geodesic γ on S^3 starting at $\overline{x}$ with $\gamma'(0) = \overline{v}$ is a great circle, which is close. By the uniqueness of geodesic, $\rho(\gamma)$ is the geodesic of S^3/G starting from x with $\rho(\gamma)'(0) = v$ under the local isometric metric induced by the projection ρ (see Exercise 7.2), which is also close naturally.

In $S^3 \subset \mathbb{R}^4$, notice that $v \in T_x\left(S^3/G\right)$ if and only if $\langle x, v\rangle_{\mathbb{R}^4} = 0$. Now we can consider $p = (1, 0) = (1, 0, 0, 0) \in S^3/G$ and $v = (0, 1, 0, 0) \in T_p\left(S^3/G\right)$. We can consider their lifting $\overline{p} = (1, 0, 0, 0)$ and $\overline{v} = (0, 1, 0, 0)$ in S^3. It's easy to see that $\gamma(t) = (\mathrm{e}^{\mathrm{i}\,t}, 0)$ is the geodesic starting at $\gamma(0) = (1, 0) = \overline{p}$ with $\gamma'(0) = (i, 0) = (0, 1, 0, 0) = \overline{v}$. And $\gamma(t) = (\mathrm{e}^{\mathrm{i}\,t}, 0)$ is contained in the x_1-x_2 plane then have length 2π. Notice that h acts on $\gamma(t)$ is induced by the action of h restricted on x_1-x_2 plane, i.e. $\begin{pmatrix} \cos\left(\frac{2\pi}{q}\right) & -\sin\left(\frac{2\pi}{q}\right) \\ \sin\left(\frac{2\pi}{q}\right) & \cos\left(\frac{2\pi}{q}\right) \end{pmatrix}$ acts on x_1-x_2 plane. It's a rotation of angle $\frac{2\pi}{q}$ anticlockwise. So γ is a q-fold of $\rho(\gamma)$. Then $\rho(\gamma)$ is the geodesic starting at p with $\gamma'(0) = v$ and has length $\frac{2\pi}{q}$ under the induced metric by the projection ρ. But consider $h \in \mathrm{O}_4$, it has two linear independent eigenvectors of eigenvalue 1 of h, say u, w. Then h acts on the u-w plane fixed and so the great circle α of the intersection of this plane and S^3. So in S^3/G, $\rho(\alpha)$ is a geodesic with length 2π under the induced metric. Now we have found two close geodesics γ and α of S^3/G who has different lengths $\frac{2\pi}{q}$ and 2π, respectively. So the close geodesics of S^3/G can have different lengths, as desired. $\square$

Exercise 8.5 (Connections of conformal metrics)**.** Let M be a differentiable manifold. Two Riemannian metrics g and $\overline{g}$ on M are *conformal* if there exists a positive function $\mu\colon M \to \mathbb{R}$ such that $\overline{g}(X, Y) = \mu g(X, Y)$, for all $X, Y \in \mathfrak{X}(M)$. Let ∇ and $\overline{\nabla}$ be the Riemannian connections of g and $\overline{g}$, respectively. Prove that

$$\overline{\nabla}_X Y = \nabla_X Y + S(X, Y),$$

where

$$S(X, Y) = \frac{1}{2\mu}\left(X(\mu)Y + Y(\mu)X - g(X, Y)\operatorname{grad}\mu\right)$$

and $\operatorname{grad}\mu$ is calculated in the metric g, that is

$$X\left(\mu\right) = g\left(X, \operatorname{grad}\mu\right).$$

Proof. We prove it as follow: We firstly defined a operator $\widetilde{\nabla}$:

$$\widetilde{\nabla}_X Y = \nabla_X Y + S(X, Y), \quad \forall X, Y \in \mathfrak{X}(M)$$

and prove that it is the Riemannian connection of $\overline{g}$. Then by the uniqueness of the Riemannian connection of a Riemannian metric (see Chapter 2 Theorem 3.6), we have that

$$\overline{\nabla}_X Y = \widetilde{\nabla}_X Y = \nabla_X Y + S(X, Y), \quad \forall X, Y \in \mathfrak{X}(M),$$

as desired.

Firstly, notice that for any $\forall X, Y, Z \in \mathfrak{X}(M)$ and $f, h \in C(M)$, because ∇ is the Riemannian connection of g, we have

$$\begin{aligned}
\widetilde{\nabla}_{fX+hY}Z &= \nabla_{fX+hY}Z + S(fX+hY,Z) \\
&= f\nabla_X Z + h\nabla_Y Z + \frac{1}{2\mu}\left((fX+hY)(\mu)Z + Z(\mu)(fX+hY) - g(fX+hY,Z)\operatorname{grad}\mu\right) \\
&= f\nabla_X Z + \frac{1}{2\mu}\left(fX(\mu)Z + fZ(\mu)X - fg(X,Z)\operatorname{grad}\mu\right) \\
&\quad + h\nabla_Y Z + \frac{1}{2\mu}\left(hY(\mu)Z + hZ(\mu)Y - hg(Y,Z)\operatorname{grad}\mu\right) \\
&= f\left(\nabla_X Z + \frac{1}{2\mu}\left(X(\mu)Z + Z(\mu)X - g(X,Z)\operatorname{grad}\mu\right)\right) \\
&\quad + h\left(\nabla_Y Z + \frac{1}{2\mu}\left(Y(\mu)Z + Z(\mu)Y - g(Y,Z)\operatorname{grad}\mu\right)\right) \\
&= f\left(\nabla_X Z + S(X,Z)\right) + h\left(\nabla_Y Z + S(Y,Z)\right) \\
&= f\widetilde{\nabla}_X Y + h\widetilde{\nabla}_Y Z,
\end{aligned}$$

$$\begin{aligned}
\widetilde{\nabla}_X(Y+Z) &= \nabla_X(Y+Z) + S(X,Y+Z) \\
&= \nabla_X Y + \nabla_X Z + \frac{1}{2\mu}\left(X(\mu)(Y+Z) + (Y+Z)(\mu)X - g(X,Y+Z)\operatorname{grad}\mu\right) \\
&= \nabla_X Y + \frac{1}{2\mu}\left(X(\mu)Y + Y(\mu)X - g(X,Y)\operatorname{grad}\mu\right) \\
&\quad + \nabla_X Z + \frac{1}{2\mu}\left(X(\mu)Z + Z(\mu)X - g(X,Z)\operatorname{grad}\mu\right) \\
&= \left(\nabla_X Y + S(X,Y)\right) + \left(\nabla_X Z + S(X,Z)\right) \\
&= \widetilde{\nabla}_X Y + \widetilde{\nabla}_X Z,
\end{aligned}$$

$$\begin{aligned}
\widetilde{\nabla}_X(fY) &= \nabla_X(fY) + S(X,fY) \\
&= f\nabla_X Y + X(f)Y + \frac{1}{2\mu}\left(X(\mu)fY + fY(\mu)X - g(X,fY)\operatorname{grad}\mu\right) \\
&= f\left(\nabla_X Y + \frac{1}{2\mu}\left(X(\mu)Y + Y(\mu)X - g(X,Y)\operatorname{grad}\mu\right)\right) + X(f)Y \\
&= f\left(\nabla_X Y + S(X,Y)\right) + X(f)Y \\
&= f\widetilde{\nabla}_X Y + X(f)Y.
\end{aligned}$$

So by definition, $\widetilde{\nabla}$ is a affine connection of M.

Then notice that S is symmetric: for any $X, Y \in \mathfrak{X}(M)$,

$$\begin{aligned}
S(X,Y) &= \frac{1}{2\mu}\left(X(\mu)Y + Y(\mu)X - g(X,Y)\operatorname{grad}\mu\right) \\
&= \frac{1}{2\mu}\left(Y(\mu)X + X(\mu)Y - g(Y,X)\operatorname{grad}\mu\right) = S(Y,X),
\end{aligned}$$

because ∇ is Riemannian connection of g and then is symmetric, we have

$$\begin{aligned}
\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X &= \nabla_X Y + S(X,Y) - \left(\nabla_Y X + S(Y,X)\right) \\
&= \nabla_X Y - \nabla_Y X + \left(S(X,Y) - S(Y,X)\right) = \nabla_X Y - \nabla_Y X = [X,Y].
\end{aligned}$$

So $\widetilde{\nabla}$ is symmetric.

Finally we prove that $\widetilde{\nabla}$ is compatible with $\overline{g}$. By Chapter 2 Corollary 3.3, we need to prove

$$X\overline{g}(Y,Z) = \overline{g}\left(\widetilde{\nabla}_X Y, Z\right) + \overline{g}\left(Y, \widetilde{\nabla}_X Z\right) \qquad \forall X, Y, Z \in \mathfrak{X}(M).$$

For the left side, by definition of $\overline{g} = \mu g$ and because ∇ is compatible with g, by Chapter 2 Corollary 3.3 again, we have

$$\begin{aligned}
X\overline{g}(Y,Z) &= X\left(\mu g(Y,Z)\right) = X\left(\mu\right) g(Y,Z) + \mu X g(Y,Z) \\
&= X\left(\mu\right) g(Y,Z) + \mu\left(g\left(\nabla_X Y, Z\right) + g\left(Y, \nabla_X Z\right)\right) = X\left(\mu\right) g(Y,Z) + \mu g\left(\nabla_X Y, Z\right) + \mu g\left(Y, \nabla_X Z\right).
\end{aligned}$$

For the right side, for the first term, by definitions, we have

$$\begin{aligned}
\overline{g}\left(\widetilde{\nabla}_X Y, Z\right) &= \mu g\left(\nabla_X Y + S(X,Y), Z\right) = \mu g\left(\nabla_X Y, Z\right) + \mu g\left(S(X,Y), Z\right) \\
&= \mu g\left(\nabla_X Y, Z\right) + \mu g\left(\frac{1}{2\mu}\left(X(\mu)Y + Y(\mu)X - g(X,Y)\operatorname{grad}\mu\right), Z\right) \\
&= \mu g\left(\nabla_X Y, Z\right) + \frac{1}{2}\left(X(\mu) g\left(Y,Z\right) + Y(\mu) g\left(X,Z\right) - g(X,Y) g\left(\operatorname{grad}\mu, Z\right)\right) \\
&= \mu g\left(\nabla_X Y, Z\right) + \frac{1}{2}\left(X(\mu) g\left(Y,Z\right) + Y(\mu) g\left(X,Z\right) - g(X,Y) Z(\mu)\right).
\end{aligned}$$

Similarly, the second term will be

$$\overline{g}\left(Y, \widetilde{\nabla}_X Z\right) = \mu g\left(Y, \nabla_X Z\right) + \frac{1}{2}\left(X(\mu) g\left(Z,Y\right) + Z(\mu) g\left(X,Y\right) - g(X,Z) Y(\mu)\right).$$

So the right side will be

$$\begin{aligned}
\overline{g}\left(\widetilde{\nabla}_X Y, Z\right) + \overline{g}\left(Y, \widetilde{\nabla}_X Z\right) &= \mu g\left(\nabla_X Y, Z\right) + \frac{1}{2}\left(X(\mu) g\left(Y,Z\right) + \underline{Y(\mu) g\left(X,Z\right)} - \underline{\underline{g(X,Y) Z(\mu)}}\right) \\
&\quad + \mu g\left(Y, \nabla_X Z\right) + \frac{1}{2}\left(X(\mu) g\left(Z,Y\right) + \underline{\underline{Z(\mu) g\left(X,Y\right)}} - \underline{g(X,Z) Y(\mu)}\right) \\
&= \mu g\left(\nabla_X Y, Z\right) + \mu g\left(Y, \nabla_X Z\right) + X(\mu) g\left(Y, Z\right).
\end{aligned}$$

Now we have

$$\begin{aligned}
X\overline{g}(Y,Z) &= X\left(\mu\right) g(Y,Z) + \mu g\left(\nabla_X Y, Z\right) + \mu g\left(Y, \nabla_X Z\right) \\
&= \mu g\left(\nabla_X Y, Z\right) + \mu g\left(Y, \nabla_X Z\right) + X(\mu) g\left(Y, Z\right) = \overline{g}\left(\widetilde{\nabla}_X Y, Z\right) + \overline{g}\left(Y, \widetilde{\nabla}_X Z\right),
\end{aligned}$$

as desired. So $\widetilde{\nabla}$ is compatible with $\overline{g}$.

Above all, $\widetilde{\nabla}$ is a symmetric affine connection compatible with $\overline{g}$. By definition, we conclude that $\widetilde{\nabla}$ is the Riemannian connection of $\overline{g}$, which completes our proof. $\square$

Exercise 8.6 (Umbilic hypersurfaces of the hyperbolic space)**.** Let $\left(M^{n+1}, g\right)$ be a manifold with a Riemannian metric g and let ∇ be its Riemannian connection. We say an immersion $x\colon N^n \to M^{n+1}$ is (totally) *umbilic* if for all $p \in N$, the second fundamental form B of x at p satisfies

$$\left\langle B(X,Y), \eta\right\rangle(p) = \lambda(p)\left\langle X, Y\right\rangle, \qquad \lambda(p) \in \mathbb{R},$$

for all $X, Y \in \mathfrak{X}(N)$ and for a given unit field η normal to $x(N)$; here we using $\langle\ ,\ \rangle$ to denote the metric g on M and the metric induced by x on N.

a) Show that if M^{n+1} has constant sectional curvature, λ does not depend on p.

b) Use Exercise 8.5 to show that if we change the metric g to a metric $\overline{g} = \mu g$, conformal to g, the immersion $x\colon N^n \to \left(M^{n+1}, g\right)$ continues being umbilic, that is, if (using the notation of Exercise 8.5) $\left\langle \nabla_X \eta, Y\right\rangle_g = -\lambda\left\langle X, Y\right\rangle_g$, then
$$\left\langle \overline{\nabla}_X\left(\frac{\eta}{\sqrt{\mu}}\right), Y\right\rangle_{\overline{g}} = \frac{-2\lambda\mu + \eta\left(\mu\right)}{2\mu\sqrt{\mu}}\left\langle X, Y\right\rangle_{\overline{g}}.$$

c) Take $M^{n+1} = \mathbb{R}^{n+1}$ with the Euclidean metric. Show that if $x\colon N^n \to \mathbb{R}^{n+1}$ is umbilic, then $x(N)$ is contained in an n-plane or an n-sphere in $\mathbb{R}^{n+1}$.

d) Use b) and c) to establish that the umbilic hypersurfaces of the hyperbolic space, in the upper half-space model H^{n+1}, are the intersections with H^{n+1} of n-planes or n-spheres of $\mathbb{R}^{n+1}$. Therefore, the umbilic hypersurfaces of the hyperbolic space are the geodesic spheres, the horospheres and the hyperspheres. Conclude that such hypersurfaces have constant sectional curvature.

e) Calculate the mean curvature and the sectional curvature of the umbilic hypersurfaces of the hyperbolic space.

Proof. In this proof, we will not discriminate N and $x(N)$.

Firstly for a), $\forall X, Y \in \mathfrak{X}(N)$ and a given unit field η normal to $x(N)$, by Chapter 6 Proposition 2.3 and the definition of S_η, we have

$$\langle B(X,Y), \eta\rangle = \langle S_\eta(X), Y\rangle = \left\langle -\left(\nabla_X \eta\right)^T, Y\right\rangle.$$

And because $Y \in TN$, we have

$$\langle \nabla_X \eta, Y\rangle = \left\langle \left(\nabla_X \eta\right)^T + \left(\nabla_X \eta\right)^N, Y\right\rangle = \left\langle \left(\nabla_X \eta\right)^T, Y\right\rangle + \left\langle \left(\nabla_X \eta\right)^N, Y\right\rangle = \left\langle \left(\nabla_X \eta\right)^T, Y\right\rangle.$$

So we $\boxed{\text{conclude}}$ that

$$-\langle \nabla_X \eta, Y\rangle = -\left\langle \left(\nabla_X \eta\right)^T, Y\right\rangle = \left\langle -\left(\nabla_X \eta\right)^T, Y\right\rangle = \langle B(X,Y), \eta\rangle = \lambda\langle X, Y\rangle.$$

Differentiating it with respect to any vector field $Z \in \mathfrak{X}(N)$, for the left side, by Chapter 2 Corollary 3.3 and using our conclusion, we have

$$-Z\langle \nabla_X \eta, Y\rangle = -\langle \nabla_Z \nabla_X \eta, Y\rangle - \langle \nabla_X \eta, \nabla_Z Y\rangle = -\langle \nabla_Z \nabla_X \eta, Y\rangle + \lambda\langle X, \nabla_Z Y\rangle.$$

For the right side, by Chapter 2 Corollary 3.3 again, we have

$$\begin{aligned} Z\left(\lambda\langle X, Y\rangle\right) = Z(\lambda)\langle X, Y\rangle + \lambda Z\langle X, Y\rangle &= Z(\lambda)\langle X, Y\rangle + \lambda\left(\langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle\right) \\ &= \langle Z(\lambda)X, Y\rangle + \lambda\langle \nabla_Z X, Y\rangle + \lambda\langle X, \nabla_Z Y\rangle. \end{aligned}$$

So we have

$$-\langle \nabla_Z \nabla_X \eta, Y\rangle + \lambda\langle X, \nabla_Z Y\rangle = -Z\langle \nabla_X \eta, Y\rangle = Z\left(\lambda\langle X, Y\rangle\right) = \langle Z(\lambda)X, Y\rangle + \lambda\langle \nabla_Z X, Y\rangle + \lambda\langle X, \nabla_Z Y\rangle,$$

i.e.

$$-\langle \nabla_Z \nabla_X \eta, Y\rangle = \langle Z(\lambda)X, Y\rangle + \lambda\langle \nabla_Z X, Y\rangle.$$

Exchanging X, Z, we have

$$-\langle \nabla_X \nabla_Z \eta, Y\rangle = \langle X(\lambda)Z, Y\rangle + \lambda\langle \nabla_X Z, Y\rangle.$$

Subtracting the first equation from the second one, we have

$$\langle \nabla_Z \nabla_X \eta, Y\rangle - \langle \nabla_X \nabla_Z \eta, Y\rangle = \langle X(\lambda)Z, Y\rangle - \langle Z(\lambda)X, Y\rangle + \lambda\langle \nabla_X Z, Y\rangle - \lambda\langle \nabla_Z X, Y\rangle,$$

where for the last two terms, by the symmetry of $\langle\ ,\ \rangle$ and our conclusion, we have

$$\lambda\langle \nabla_X Z, Y\rangle - \lambda\langle \nabla_Z X, Y\rangle = \lambda\langle \nabla_X Z - \nabla_Z X, Y\rangle = \lambda\langle [X,Z], Y\rangle = -\left\langle \nabla_{[X,Z]}\eta, Y\right\rangle.$$

So we have

$$\langle \nabla_Z \nabla_X \eta, Y\rangle - \langle \nabla_X \nabla_Z \eta, Y\rangle = \langle X(\lambda)Z, Y\rangle - \langle Z(\lambda)X, Y\rangle - \left\langle \nabla_{[X,Z]}\eta, Y\right\rangle,$$

i.e.

$$\left\langle \nabla_Z \nabla_X \eta - \nabla_X \nabla_Z \eta + \nabla_{[X,Z]}\eta, Y\right\rangle = \langle X(\lambda)Z - Z(\lambda)X, Y\rangle,$$

where the left side is the curvature tensor of $\langle\ ,\ \rangle$. Because M has constant sectional curbature, say K, by Chapter 4 Corollary 3.5, we can choose an orthonormal basis $e_1, \cdots, e_n$ of TN and add $e_{n+1} = \eta$ to an orthonormal basis of TM, then we have

$$R_{ijkl} = K\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right).$$

Firstly fixing $X = e_i$ and $Z = e_j$, for any $Y = e_1, \cdots, e_n$, because $\eta = e_{n+1}$ can never be e_i or e_j, we have

$$\begin{aligned}\langle X(\lambda)Z - Z(\lambda)X, Y\rangle &= \left\langle \nabla_Z\nabla_X\eta - \nabla_X\nabla_Z\eta + \nabla_{[X,Z]}\eta, Y\right\rangle \\ &= R_{ijkl} = K\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right) = K\left(0\delta_{jl} - \delta_{il}0\right) = 0.\end{aligned}$$

Now we conclude that for any $Y = e_1, \cdots, e_n$ which is an orthonormal basis of TN, $\langle X(\lambda)Z - Z(\lambda)X, Y\rangle = 0$, we must have $X(\lambda)Z - Z(\lambda)X = 0$. Then we do this for all $X \neq Z = e_1, \cdots, e_n$, because X, Z are linear independent, we must have $X(\lambda) = 0$, for any $X = e_1, \cdots, e_n$, which is an orthonormal basis of TN. Notice that λ is a function on N, it follows that λ is a constant and then does not depend on $p \in N$, as desired.

For b), there is direct calculation. By Exercise 8.5, we have

$$\begin{aligned}\overline{\nabla}_X\left(\frac{\eta}{\sqrt{\mu}}\right) &= \nabla_X\left(\frac{\eta}{\sqrt{\mu}}\right) + S(X, \frac{\eta}{\sqrt{\mu}}) = \nabla_X\left(\frac{\eta}{\sqrt{\mu}}\right) + \frac{1}{2\mu}\left(X(\mu)\frac{\eta}{\sqrt{\mu}} + \frac{\eta}{\sqrt{\mu}}(\mu)X - \left\langle X, \frac{\eta}{\sqrt{\mu}}\right\rangle_g \operatorname{grad}\mu\right) \\ &= \frac{1}{\sqrt{\mu}}\nabla_X\eta + X\left(\frac{1}{\sqrt{\mu}}\right)\eta + \frac{1}{2\mu\sqrt{\mu}}\left(X(\mu)\eta + \eta(\mu)X - \langle X, \eta\rangle_g \operatorname{grad}\mu\right).\end{aligned}$$

Because $X, Y \in \mathfrak{X}(N)$ and η is normal to $x(N)$, and the conformal metric preserves orthonormal relation:

$$\langle U, V\rangle_{\overline{g}} = \mu\langle\ ,\ \rangle_g = 0 \ \Leftrightarrow\ \langle U, V\rangle_g,$$

we have

$$\begin{aligned}\left\langle\overline{\nabla}_X\left(\frac{\eta}{\sqrt{\mu}}\right), Y\right\rangle_{\overline{g}} &= \frac{1}{\sqrt{\mu}}\langle\nabla_X\eta, Y\rangle_{\overline{g}} + X\left(\frac{1}{\sqrt{\mu}}\right)\langle\eta, Y\rangle_{\overline{g}} \\ &\quad + \frac{1}{2\mu\sqrt{\mu}}\left(X(\mu)\langle\eta, Y\rangle_{\overline{g}} + \eta(\mu)\langle X, Y\rangle_{\overline{g}} - \langle X, \eta\rangle_g\langle\operatorname{grad}\mu, Y\rangle_{\overline{g}}\right) \\ &= \frac{1}{\sqrt{\mu}}\langle\nabla_X\eta, Y\rangle_{\overline{g}} + X\left(\frac{1}{\sqrt{\mu}}\right)0 + \frac{1}{2\mu\sqrt{\mu}}\left(X(\mu)0 + \eta(\mu)\langle X, Y\rangle_{\overline{g}} - 0\langle\operatorname{grad}\mu, Y\rangle_{\overline{g}}\right) \\ &= \frac{1}{\sqrt{\mu}}\langle\nabla_X\eta, Y\rangle_{\overline{g}} + \frac{1}{2\mu\sqrt{\mu}}\left(\eta(\mu)\langle X, Y\rangle_{\overline{g}}\right),\end{aligned}$$

where by our conclusion above,

$$\langle\nabla_X\eta, Y\rangle_{\overline{g}} = \mu\langle\nabla_X\eta, Y\rangle_g = -\mu\lambda\langle X, Y\rangle_g = -\lambda\left(\mu\langle X, Y\rangle_g\right) = -\lambda\langle X, Y\rangle_{\overline{g}}.$$

So we conclude

$$\begin{aligned}\left\langle\overline{\nabla}_X\left(\frac{\eta}{\sqrt{\mu}}\right), Y\right\rangle_{\overline{g}} &= \frac{1}{\sqrt{\mu}}\langle\nabla_X\eta, Y\rangle_{\overline{g}} + \frac{1}{2\mu\sqrt{\mu}}\left(\eta(\mu)\langle X, Y\rangle_{\overline{g}}\right) \\ &= -\frac{\lambda}{\sqrt{\mu}}\langle X, Y\rangle_{\overline{g}} + \frac{\eta(\mu)}{2\mu\sqrt{\mu}}\langle X, Y\rangle_{\overline{g}} = \frac{-2\lambda\mu + \eta(\mu)}{2\mu\sqrt{\mu}}\langle X, Y\rangle_{\overline{g}}.\end{aligned}$$

And by our beginning of the proof,

$$\left\langle B(X, Y), \frac{\eta}{\sqrt{\mu}}\right\rangle_{\overline{g}} = -\left\langle\left(\nabla_X\left(\frac{\eta}{\sqrt{\mu}}\right)\right)^T, Y\right\rangle_{\overline{g}} = -\left\langle\nabla_X\left(\frac{\eta}{\sqrt{\mu}}\right), Y\right\rangle_{\overline{g}} = -\frac{-2\lambda\mu + \eta(\mu)}{2\mu\sqrt{\mu}}\langle X, Y\rangle_{\overline{g}},$$

i.e.

$$\overline{\lambda} = -\frac{-2\lambda\mu + \eta(\mu)}{2\mu\sqrt{\mu}}$$

and then $x\colon N \to (M, \overline{g})$ is also umbilic.

For the special case c), because $M^{n+1} = \mathbb{R}^{n+1}$ has constant curvature 0, by a) and its proof, $\forall X, Y \in \mathfrak{X}(N)$ and a given unit vector η normal to $x(N)$, we have

$$-\langle\nabla_X\eta, Y\rangle = \lambda\langle X, Y\rangle,$$

where λ is a constant function on N. We can choose an orthonormal basis $e_1, \cdots, e_n$ of N and extend it by $e_{n+1} = \eta$ to an orthonormal basis of $M = \mathbb{R}^{n+1}$. We can seen η as a mapping from N to $M = \mathbb{R}^{n+1}$. Then let $X, Y = e_1, \cdots, e_n$, we have

$$-\langle \nabla_{e_i}\eta, e_j\rangle = \lambda \langle e_i, e_j\rangle = \lambda\delta_{ij}.$$

Because in the Euclidean space $M^{n+1} = \mathbb{R}^{n+1}$, the covariant derivative is the same as the usual derivative (see the end of Chapter 2), we have

$$\nabla_{e_i}\eta = \nabla_{e_i'}\eta = \frac{\eta(e_i(t))}{dt} = e_i(\eta).$$

So we have

$$\langle e_i(\eta), e_j\rangle = \langle \nabla_{e_i}\eta, e_j\rangle = e\lambda\delta_{ij}.$$

This means $e_i(\eta) = -\lambda e_i$, $\forall i = 1, \cdots, n$.

So if $\lambda = 0$, we have $e_i(\eta) - \lambda e_i = 0$, for any $i = 1, \cdots, n$. Because η is seen as a function on N and $e_1, \cdots, e_n$ is an orthonormal basis of TN, we must have η is a constant vector. It follows that $x(N)$ is normal to a constant vector, then $x(N)$ is contained in an n-plane. In fact, it is contained in the n-plane

$$\left\{ x = (x_0, \cdots, x_n) \in \mathbb{R}^{n+1} \,\middle|\, \langle x, \eta\rangle = 0,\ \sum_{i=0}^{n} t_i x_i = d,\ \sum_{i=0}^{n} t_i = 1,\ t_i \in \mathbb{R},\ i = 0, 1, \cdots, n \right\}$$

for some $d \in \mathbb{R}$.

If $\lambda \neq 0$, denote

$$y(p) = x(p) + \frac{\eta(p)}{\lambda} \in \mathbb{R}^{n+1}, \qquad \forall p \in N.$$

Notice that $x(N) \subset M = \mathbb{R}^{n+1}$, so x^{-1} can be seen as a parametrization of N naturally. And $x\colon N \to \mathbb{R}^{n+1}$ can be also seen as a multiple function on N. We have

$$e_i(x) = \frac{dx\left(x^{-1}\left(e_i(t)\right)\right)}{dt} = \frac{de_i(t)}{dt} = e_i, \qquad \forall i = 1, \cdots, n.$$

Then

$$e_i(y) = e_i(x) + e_i\left(\frac{\eta}{\lambda}\right) = e_i + \frac{1}{\lambda}e_i(\eta) = e_i + \frac{1}{\lambda}(-\lambda e_i) = e_i - e_i = 0, \qquad \forall i = 1, \cdots, n.$$

It follows that η is caonstant, say $x_0 \in M = \mathbb{R}^{n+1}$. So we conclude

$$x_0 = x(p) + \frac{\eta(p)}{\lambda}, \qquad \forall p \in N,$$

i.e.

$$x(p) - x_0 = -\frac{\eta(p)}{\lambda}, \qquad \forall p \in N.$$

So we have

$$|x(p) - x_0| = \left| -\frac{\eta(p)}{\lambda} \right| = \frac{1}{|\lambda|}, \qquad \forall p \in N.$$

This means $x(N)$ is contained in a sphere of center x_0 and radius $\frac{1}{|\lambda|}$.

And for another special case d), for any umbilic hypersurfaces $N \subset H^{n+1}$, because H^{n+1} has the metric $g_{ij} = \frac{1}{x_{n+1}^2}\delta_{ij}$ conformal to the Euclidean metric, by b), N is an umbilic hypersurfaces of $\mathbb{R}^{n+1}$ with the Euclidean metric. Then by c), N is contained in an n-plane or an n-sphere of $\mathbb{R}^{n+1}$. So N is the intersection with H^{n+1} of an n-plane or an n-sphere of $\mathbb{R}^{n+1}$.

And if N is in an n-sphere or an n-plane completely contained in H^{n+1}, it's a geodesic sphere; if the n-sphere is tangent to ∂H^{n+1}, N is a horosphere; if the n-sphere or n-plane cuts ∂H^{n+1} at an angle α, N is a hypersphere (see Page 178).

Now we compute the sectional curvature of $N \subset H^{n+1}$. Let $e_1, \cdots, e_n$ be an orthonormal basis of TN and $e_{n+1} = \eta \in (TN)^{\perp}$ such that $e_1, \cdots, e_{n+1}$ is an orthonormal basis of H^{n+1}. We will use the Gauss equation (Chapter 6 Proposition 3.1 (a)):

$$\left\langle \overline{R}(X, Y)Z, T\right\rangle = \langle R(X, Y)Z, T\rangle - \langle B(Y, T), B(X, Z)\rangle + \langle B(X, T), B(Y, Z)\rangle, \qquad \forall X, Y, Z, T \in \mathfrak{X}(N),$$

where $\overline{R}$ and R are the curvature tensor of H^{n+1} and N, respectively. By a), because H^{n+1} has constant curvature -1,

$$\langle B(X,Y),\eta\rangle = \lambda \langle X,Y\rangle,$$

where λ is a constant. By definition, $B(X,Y) \in (TN)^{\perp} = \mathrm{Span}\{\eta\}$, so we have

$$B(X,Y) = \lambda \langle X,Y\rangle \eta.$$

Then let $X = Z, Y = T \in \{e_1,\cdots,e_n\}$, the Gauss equation will be

$$\begin{aligned}
\left\langle \overline{R}(e_i,e_j)e_i,e_j\right\rangle &= \langle R(e_i,e_j)e_i,e_j\rangle - \langle B(e_j,e_j),B(e_i,e_i)\rangle + \langle B(e_i,e_j),B(e_j,e_i)\rangle \\
&= \langle R(e_i,e_j)e_i,e_j\rangle - \langle \lambda\langle e_j,e_j\rangle \eta,\lambda\langle e_j,e_j\rangle \eta\rangle + \langle \lambda\langle e_i,e_j\rangle \eta,\lambda\langle e_j,e_i\rangle \eta\rangle \\
&= \langle R(e_i,e_j)e_i,e_j\rangle - \langle \lambda 1\eta,\lambda 1\eta\rangle + \langle \lambda 0\eta,\lambda 0\eta\rangle \\
&= \langle R(e_i,e_j)e_i,e_j\rangle - \lambda^2, \qquad \forall i\neq j = 1,\cdots,n.
\end{aligned}$$

So we conclude that

$$R_{ijij} = \langle R(e_i,e_j)e_i,e_j\rangle = \left\langle \overline{R}(e_i,e_j)e_i,e_j\right\rangle + \lambda^2 = \overline{R}_{ijij} + \lambda^2 = -1+\lambda^2, \qquad \forall i\neq j = 1,\cdots,n,$$

i.e. the hypersurface N has constant sectional curvature $-1+\lambda^2$, as desired.

Finally, for e), we have already calculated the sectional curvature in d), so let's calculate the mean curvature of the hypersurface $N \subset H^{n+1}$. By d), let $N = S\cap H^{n+1}$ where S is an n-sphere or an n-plane of $\mathbb{R}^{n+1}$. Because N is umbilic, by definition,

$$\langle S_\eta(e_i),e_j\rangle = \langle B(e_i,e_j),\eta\rangle = \lambda\langle e_i,e_j\rangle,$$

i.e. $S_\eta(e_i) = \lambda e_i$, for all $i = 1,\cdots,n$. This means all the directions are principle and

$$H = \frac{1}{n}(\lambda + \cdots,\lambda) = \lambda.$$

So we only need to calculate for one direction, for example the curvature of the curve of the intersection of S and x_1x_{n+1}-plane. The curve is a circle. But I forget how to calculate the curvature of a curve in H^2.

See reference:
https://math.stackexchange.com/questions/2430495
https://math.stackexchange.com/questions/2344369
https://math.franklin.uga.edu/sites/default/files/inline-files/ShifrinDiffGeo.pdf □

Exercise 8.7. Define a "stereographic projection" $f\colon H^n_{-1}\to D^n$ from the model of the hyperbolic space H^n_{-1} of curvature -1 given in Exercise 8.3 onto the open ball

$$D^n = \left\{(x_0,\cdots,x_n)\,\middle|\, x_0 = 0,\ \sum_{\alpha=1}^n x_\alpha^2 < 1\right\}$$

in the following way: If $p\in H^n_{-1}\subset L^{n+1}$, join p to $p_0 = (-1,0,\cdots,0)$ be a line r; $f(p)$ is the intersection of r with D^n. Let $p = (x_0,\cdots,x_n)$ and $f(p) = (0,u_1,\cdots,u_n)$.

a) Prove that:

$$x_\alpha = \frac{2u_\alpha}{1-\sum_{\alpha=1}^n u_\alpha^2}, \qquad \alpha = 1,\cdots,n,$$
$$x_0 = \frac{2}{1-\sum_{\alpha=1}^n u_\alpha^2} - 1.$$

b) Show that

$$-(dx_0)^2 + (dx_1)^2 + \cdots + (dx_n)^2 = \frac{4\left((dx_1)^2+\cdots+(dx_n)^2\right)}{\left(1-\sum_{\alpha=1}^n u_\alpha^2\right)^2}.$$

Conclude that $f^{-1}\colon D^n\to H^n_{-1}$ induces on D the metric $g_{ij} = \frac{4\delta_{ij}}{\left(1-\sum_{\alpha=1}^n u_\alpha^2\right)^2}$. Therefore, D^n with the metric g_{ij} has constant curvature -1 (see Exercise 8.1 c)).

c) Show that the images by f of the non-empty intersections of affine hyperplanes P of L^{n+1} with H^n_{-1} are intersections with D^n of spheres (or planes, when P passes through p_0) contained in the hyperplane $x_0 = 0$. Conclude that the umbilic hypersurfaces of H^n_{-1} (see Exercise 8.6) are of the form $P \cap H^n_{-1}$.

Proof. Firstly, for a), because $p = (x_0, \cdots, x_n) \in H^n_{-1}$, we have

$$-x_0^2 + \sum_{i=1}^{n} x_i^2 = -1.$$

Let the straight line connecting p_0 and p be $(1-t)p_0 + tp = (1 - t + tx_0, tx_1, \cdots, tx_n)$. Then $f(p)$ is the intersection of this line with the plane $x_0 = 0$. So let $1 - t + tx_0 = 0$, i.e. $t = \frac{1}{1-x_0}$, then we have

$$(0, u_1, \cdots, u_n) = f(p) = (1 - t + tx_0, tx_1, \cdots, tx_n)|_{t=\frac{1}{1-x_0}} = \left(0, \frac{x_1}{1-x_0}, \cdots, \frac{x_n}{1-x_0}\right).$$

Solving the system

$$\begin{cases} -x_0^2 + \sum_{i=1}^n x_i^2 = -1 \\ u_i = \frac{x_i}{1-x_0}, \quad i = 1, \cdots, n, \end{cases}$$

we have that

$$\begin{cases} x_\alpha = \frac{2u_\alpha}{1-\sum_{\alpha=1}^n u_\alpha^2}, \quad \alpha = 1, \cdots, n, \\ x_0 = \frac{2}{1-\sum_{\alpha=1}^n u_\alpha^2} - 1, \end{cases}$$

as desired.

For b), calculating directly, we have

$$\frac{dx_i}{du_j} = \frac{4u_iu_j}{\left(1 - \sum_{k=1}^n u_k^2\right)^2}, \quad \forall i \neq j = 1, \cdots, n,$$

$$\frac{dx_i}{du_i} = \frac{2\left(1 - \sum_{k=1}^n u_k^2\right) + 4u_i^2}{\left(1 - \sum_{k=1}^n u_k^2\right)^2}, \quad \forall i = 1, \cdots, n,$$

$$\frac{dx_0}{du_j} = \frac{4u_j}{\left(1 - \sum_{k=1}^n u_k^2\right)^2}, \quad \forall j = 1, \cdots, n.$$

Then

$$\begin{aligned} dx_i &= \sum_{\substack{j\neq i \\ j=1}}^{n} \frac{4u_iu_j}{\left(1 - \sum_{k=1}^n u_k^2\right)^2} du_j + \frac{2\left(1 - \sum_{k=1}^n u_k^2\right) + 4u_i^2}{\left(1 - \sum_{k=1}^n u_k^2\right)^2} du_i \\ &= \sum_{j=1}^{n} \frac{4u_iu_j}{\left(1 - \sum_{k=1}^n u_k^2\right)^2} du_j + \frac{2}{1 - \sum_{k=1}^n u_k^2} du_i, \quad \forall i = 1, \cdots, n, \end{aligned}$$

$$dx_0 = \sum_{j=1}^{n} \frac{4u_j}{\left(1 - \sum_{k=1}^n u_k^2\right)^2} du_j.$$

Then we calculate the coefficients of du_idu_j, $i, j = 1, \cdots, n$ in

$$Q(x, x) = -\left(dx_0\right)^2 + \left(dx_1\right)r + \cdots + \left(dx_n\right)^2.$$

For $i = j$, the coefficients will be

$$\begin{aligned} &-\left(\frac{4u_i}{\left(1 - \sum_{k=1}^n u_k^2\right)^2}\right)^2 + \sum_{l=1}^{n}\left(\frac{4u_lu_i}{\left(1 - \sum_{k=1}^n u_k^2\right)^2}\right)^2 + \left(\frac{2}{1 - \sum_{k=1}^n u_k^2}\right)^2 + 2\frac{4u_iu_i}{\left(1 - \sum_{k=1}^n u_k^2\right)^2}\frac{2}{1 - \sum_{k=1}^n u_k^2} \\ =\ & \frac{-16u_i^2 + \sum_{l=1}^n 16u_l^2u_i^2 + 4\left(1 - \sum_{k=1}^n u_k^2\right)^2 + 16u_i^2\left(1 - \sum_{k=1}^n u_k^2\right)}{\left(1 - \sum_{k=1}^n u_k^2\right)^4} \\ =\ & \frac{16u_i^2\left(\sum_{l=1}^n u_l^2 - 1\right) + 4\left(1 - \sum_{k=1}^n u_k^2\right)^2 + 16u_i^2\left(1 - \sum_{k=1}^n u_k^2\right)}{\left(1 - \sum_{k=1}^n u_k^2\right)^4} = \frac{4\left(1 - \sum_{k=1}^n u_k^2\right)^2}{\left(1 - \sum_{k=1}^n u_k^2\right)^4} = \frac{4}{\left(1 - \sum_{k=1}^n u_k^2\right)^2}. \end{aligned}$$

Similarly, for $i \neq j$, the coefficients will be

$$\begin{aligned}
&-2\frac{4u_i}{\left(1-\sum_{k=1}^n u_k^2\right)^2}\frac{4u_j}{\left(1-\sum_{k=1}^n u_k^2\right)^2}+2\sum_{l=1}^n \frac{4u_lu_i}{\left(1-\sum_{k=1}^n u_k^2\right)^2}\frac{4u_lu_j}{\left(1-\sum_{k=1}^n u_k^2\right)^2}\\
&\quad+2\frac{4u_ju_i}{\left(1-\sum_{k=1}^n u_k^2\right)^2}\frac{2}{1-\sum_{k=1}^n u_k^2}+2\frac{4u_iu_j}{\left(1-\sum_{k=1}^n u_k^2\right)^2}\frac{2}{1-\sum_{k=1}^n u_k^2}\\
=&\,2\frac{-16u_iu_j+\sum_{l=1}^n 16u_l^2u_iu_j+16u_iu_j\left(1-\sum_{k=1}^n u_k^2\right)}{\left(1-\sum_{k=1}^n u_k^2\right)^4}\\
=&\,2\frac{16u_iu_j\left(\sum_{l=1}^n u_l^2-1\right)+16u_iu_j\left(1-\sum_{k=1}^n u_k^2\right)}{\left(1-\sum_{k=1}^n u_k^2\right)^4}=0.
\end{aligned}$$

So we conclude that

$$\begin{aligned}
Q&=(dx_0)^2+(dx_1)\,r+\cdots+(dx_n)^2\\
&=\sum_{i=1}^n \frac{4}{\left(1-\sum_{k=1}^n u_k^2\right)^2}(du_i)^2+\sum_{\substack{i\neq j\\ i,j=1}}^n 0du_idu_j=\frac{4\left((du_1)^2+\cdots+(du_n)^2\right)}{\left(1-\sum_{k=1}^n u_k^2\right)^2},
\end{aligned}$$

as desired. Then for any $i,j=1,\cdots,n$, under the metric $\langle\ ,\ \rangle$ induced by f^{-1}, by definition,

$$g_{ij}=\left\langle \frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j}\right\rangle=Q\left(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j}\right)=\frac{4\left((du_1)^2+\cdots+(du_n)^2\right)}{\left(1-\sum_{k=1}^n u_k^2\right)^2}\left(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j}\right)=\frac{4\delta_{ij}}{\left(1-\sum_{k=1}^n u_k^2\right)^2}.$$

And by Exercise 8.1 b),

$$F=\frac{1-\sum_{i=1}^n u_i^2}{2}=\sum_{i=1}^n\left(-\frac{1}{2}u_i^2+\frac{1}{2n}\right),$$

i.e.

$$G_i(u_i)=-\frac{1}{2}u_i^2+\frac{1}{2n},$$

i.e.

$$a=-\frac{1}{2},\qquad b_i=0,\qquad c_i=\frac{1}{2n}.$$

So D^n with the metric g_{ij} has constant curvature

$$K=\sum_{i=1}^n\left(4c_ia-b_i^2\right)=\sum_{i=1}^n\left(4\frac{1}{2n}\left(-\frac{1}{2}\right)-0\right)=\sum_{i=1}^n\left(-\frac{1}{n}\right)=-1;$$

whence b).

For c), let

$$P=\left\{(x_0,\cdots,x_n)\in L^{n+1}\,\middle|\,\sum_{i=0}^n t_ix_i=d,\ \text{where}\ \sum_{i=0}^n t_i=1\ \text{and}\ d\in\mathbb{R}\right\}$$

be a affine hyperplane in L^{n+1} has non-empty intersection of H_{-1}^n. Then the intersection under f is non-empty in D^n. By a), we have

$$d=\sum_{i=0}^n t_ix_i=t_0\left(\frac{2}{1-\sum_{k=1}^n u_k^2}-1\right)+\sum_{i=1}^n t_i\frac{2u_i}{1-\sum_{k=1}^n u_k^2}.$$

Arranging this equation, we have

$$(t_0+d)\sum_{k=1}^n u_i^2+\sum_{i=1}^n 2t_iu_i+t_0-d=0.$$

If P passes through $p_0 = (-1, 0, \cdots, 0)$, we have $-t_0 = d$. Then the equation will be

$$\sum_{i=1}^{n} 2t_i u_i + 2t_0 = 0.$$

This means the intersection under f is a plane in the hyperplane $x_0 = 0$, especially, in D^n. If P doesn't pass through p_0, we can continue to arrange this equation:

$$\sum_{i=1}^{n} \left(u_i + \frac{t_i}{t_0 + d} \right)^2 = \sum_{j=1}^{n} \frac{t_i^2}{(t_0 + d)^2} + \frac{t_0 - d}{t_0 + d}.$$

This means the intersection under f is a sphere in the hyperplane $x_0 = 0$, especially, in D^n.

Like the proof of Exercise 8.6, the metric $g_{ij} = \frac{4\delta_{ij}}{\left(1 - \sum_{k=1}^{n} u_k^2\right)^2}$ of D^n is conformal to the Euclidean metric δ_{ij}. So the umbilic hypersurfaces of D^n is the intersection of D^n with spheres or planes. And in fact, any sphere or plane in D^n under f^{-1} is an intersection of a plane P with H^n_{-1}. And the isometry f (by b), f is an isometry) preserve umbilic hypersurfaces. So we conclude that the umbilic hypersurfaces of H^n_{-1} are of the form $P \cap H^n_{-1}$ where P is a affine hyperplane of L^{n+1}, as desired. □

Exercise 8.8 (Riemannian submersions). A differentiable mapping $f\colon \overline{M}^{n+k} \to M^n$ is called *submersion* if f is surjective, and for all $\overline{p} \in \overline{M}$, $df_{\overline{p}}\colon T_{\overline{p}}\overline{M} \to T_{f(p)}M$ has rank n. In this case, for all $p \in M$, the fiber $f^{-1}(p) = F_p$ is a submanifold of $\overline{M}$ and a tangent vector of $\overline{M}$, tangent to some F_p, $p \in M$, is called a *vertical vector* of the submersion. If, in addition, $\overline{M}$ and M have Riemannian metrics, the submersion f is said to be *Riemannian* if, for all $p \in \overline{M}$, $df_p\colon T_p\overline{M} \to T_{f(p)}M$ preserves lengths of vectors orthogonal to F_p. Show that:

a) If $M_1 \times M_2$ is the Riemannian product, then the natural projections $\pi_i\colon M_1 \times M_2 \to M_i$, $i = 1, 2$, are Riemannian submersion.

b) If the tangent bundle TM is given the Riemannian metric as in Exercise 3.2, then the projection $\pi\colon TM \to M$ is a Riemannian submersion.

Proof. For a), obviously,

$$\pi_1\colon M_1 \times M_2 \to M_1$$

is surjective. And we have that

$$T_{(p,q)}(M_1 \times M_2) \cong T_pM_1 \oplus T_qM_2, \quad \forall p \in M_1, q \in M_2.$$

So we have

$$\begin{aligned} d\pi_1\colon T_{(p,q)}(M_1 \times M_2) \cong T_pM_1 \oplus T_qM_2 &\to T_pM_1 \\ v + w &\mapsto v + 0, \end{aligned}$$

i.e.

$$d\pi_{1(p,q)} = \begin{pmatrix} \mathrm{Id}_n \\ 0 \end{pmatrix}.$$

Then $d\pi_{1(p,q)}$ has rank n. So π_1 is a submersion. And noticing that $F_p = \{p\} \times M_2$, we have

$$\left(T_{(p,q)}F_p\right)^{\perp} = \left(T_{(p,q)}(\{p\} \times M_2)\right)^{\perp} = (T_qM_2)^{\perp} = T_pM_1.$$

Then for any $v \in \left(T_{(p,q)}F_p\right)^{\perp} = T_pM_1$,

$$d\pi_{1(p,q)}(v) = \mathrm{Id}_n(v) = v.$$

So we have

$$\left|d\pi_{1(p,q)}(v)\right| = |v|$$

naturally. This means $d\pi_{1(p,q)}$ preserves the lengths of vectors orthogonal to F_p. By the arbitrariness of $(p,q) \in M_1 \times M_2$, we conclude that π_1 is a Riemannian submersion. The case for $i = 2$ is simialr.

For b), it's obvious that $\pi\colon TM \to M$ is surjective. By Exercise 3.2, locally, we have

$$T_{(p,v)}(TM) = T_pM \oplus T_v(T_pM) \cong T_pM \oplus \mathbb{R}^n, \qquad \forall p \in M, v \in T_pM$$

and π can be seen as

$$\pi\colon TM \to M \cong M \times \{0\} \subset TM,$$

we have

$$\begin{aligned} d\pi_{(p,v)}\colon T_{(p,v)}(TM) \cong T_pM \oplus \mathbb{R}^n &\to T_pM \\ u + w &\mapsto u + 0. \end{aligned}$$

So

$$d\pi_{(p,v)} = \begin{pmatrix} \mathrm{Id}_n \\ 0 \end{pmatrix}$$

and then $d\pi_{(p,v)}$ has rank n. It follows that π is a submersion. And we have $F_p = T_pM$. Then locally,

$$\left(T_{(p,v)}F_p\right)^{\perp} \cong \left(T_{(p,v)}(T_pM)\right)^{\perp} \cong \left(T_v(T_pM)\right)^{\perp} \cong T_pM.$$

(In fact, for any $W \in T_{(p,v)}F_p \cong T_v(T_pM)$, we can choose $\beta(s) = (q(s), w(s))$ in TM where $q(s)$ is constant such that $W = \beta'(0) = (q'(0), w'(0)) = (0, w'(0))$. Then for any $V \in \left(T_{(p,v)}F_p\right)^{\perp}$, denote $\alpha(t) = (p(t), v(t))$ be a curve in TM such that $\alpha'(0) = V$, we have

$$0 = \langle V, W\rangle = \langle d\pi(V), d\pi(W)\rangle + \left\langle \frac{Dv}{dt}(0), \frac{Dw}{ds}\right\rangle(0) = \langle d\pi(V), 0\rangle + \left\langle \frac{Dv}{dt}(0), \frac{Dw}{ds}(0)\right\rangle, \qquad \forall W \in T_{(p,v)}F_p.$$

This means V has form $(p'(0), v'(0))$ where $\frac{Dv}{dt}(0) = 0$. So $\alpha(t) = (p(t), v(t))$ where $v(t)$ is the parallel transport of $v(0) \in T_pM$ along $p(t)$. So we have $\left(T_{(p,v)}F_p\right)^{\perp} \cong T_pM$.) Then for any $u \in \left(T_{(p,v)}F_p\right)^{\perp} = T_pM$,

$$d\pi_{(p,v)}(u) = \mathrm{Id}_n(u) = u.$$

So we have

$$\left|d\pi_{(p,v)}(u)\right| = |u|$$

naturally. This means $d\pi_{(p,v)}$ preserves the lengths of vectors orthogonal to F_p. By the arbitrariness of $(p, v) \in TM$, we conclude that π is a Riemannian submersion, as desired. ☐

Exercise 8.9 (Connection of a Riemannian submersion)**.** Let $f\colon \overline{M} \to M$ be a Riemannian submersion. A vector $\overline{x} \in T_{\overline{p}}\overline{M}$ is *horizontal* if it is orthogonal to the fiber. The tangent space $T_{\overline{p}}\overline{M}$ then admits a decomposition $T_{\overline{p}}\overline{M} = \left(T_{\overline{p}}\overline{M}\right)^h \oplus \left(T_{\overline{p}}\overline{M}\right)^v$, where $\left(T_{\overline{p}}\overline{M}\right)^h$ and $\left(T_{\overline{p}}\overline{M}\right)^v$ denote the subspaces of horizontal and vertical vectors, respectively. If $X \in \mathfrak{X}(M)$, the *horizontal lift* $\overline{X}$ of X is the horizontal field defined by $df_{\overline{p}}\left(\overline{X}(\overline{p})\right) = X(X(p))$.

a) Show that $\overline{X}$ is differentiable.

b) Let ∇ and $\overline{\nabla}$ be the Riemannian connections of M and $\overline{M}$ respectively. Show that

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \overline{(\nabla_X Y)} + \frac{1}{2}\left[\overline{X}, \overline{Y}\right]^v, \qquad X, Y \in \mathfrak{X}(M),$$

where Z^v is the vertical component of Z.

c) Show that $\left[\overline{X}, \overline{Y}\right]^v(\overline{p})$ depends only on $\overline{X}(\overline{p})$ and $\overline{Y}(\overline{p})$.

Proof. For a), because f is a Riemannian submersion, df has rank n. And df preverses the lengths of horizontal vectors. Then we can choose a basis, say $e_1 = \frac{\partial}{\partial x_1}, \cdots, e_{n+k} = \frac{\partial}{\partial x_{n+k}}$, such that df has the form

$$df = \begin{pmatrix} \mathrm{Id}_n \\ 0 \end{pmatrix}.$$

So under the local coordinates $x_1, \cdots, x_{n+k}$, f is the projection into the first n coordinates. Then for any $X = (u_1, \cdots, u_n)$, the horizontal lift is just the embedding $\overline{X} = (u_1, \cdots, u_n, 0, \cdots, 0)$, which is naturally differnetiable.

For b), by a), we know that TM and $\left(T\overline{M}\right)^h$ can be spanned by $e_1, \cdots, e_n$, $\left(T\overline{M}\right)^v$ can be spanned by $e_{n+1}, \cdots, e_{n+k}$. Then for any $X, Y, Z \in \mathfrak{X}(M)$ and $T \in \left(\mathfrak{X}(\overline{M})\right)^v \cong (TM)^{\perp}$, we have $\overline{X}, \overline{Y}, \overline{Z} \in \left(\mathfrak{X}(\overline{M})\right)^h \cong TM$. And then

$$\left\langle \overline{X}, T \right\rangle = 0, \quad \left\langle \overline{Y}, T \right\rangle = 0, \quad \left\langle \overline{Z}, T \right\rangle = 0. \tag{a}$$

And by a), X, Y, Z can be seen as identical embeddings to $\overline{X}, \overline{Y}, \overline{Z}$, so we have

$$\overline{X} \left\langle \overline{Y}, \overline{Z} \right\rangle = X \left\langle Y, Z \right\rangle \tag{b}$$

naturally. By the theory of differential form, we know that

$$df \left[\overline{X}, T\right] = \left[df\overline{X}, dfT\right] = \left[df\overline{X}, 0\right] = 0, \tag{c}$$

$$df \left[\overline{X}, \overline{Y}\right] = \left[df\overline{X}, df\overline{Y}\right] = [X, Y]. \tag{d}$$

And noticing that $\left\langle \overline{X}, \overline{Y} \right\rangle$ is a function of $x_1, \cdots, x_n$ and $T \in (TM)^{\perp}$, we have

$$T \left\langle \overline{X}, \overline{Y} \right\rangle = 0. \tag{e}$$

Now we conclude that

$$\langle [X, Y], Z \rangle = \left\langle df \left[\overline{X}, \overline{Y}\right], df\overline{Z} \right\rangle = \left\langle \left[\overline{X}, \overline{Y}\right], \overline{Z} \right\rangle \tag{f}$$

where the last equation is because $\left[\overline{X}, \overline{Y}\right]$ and $\overline{Z}$ are horizontal fields and f is a Riemannian submersion, then df preserves their lengths and then their inner product. Similarly, we also conclude

$$0 = \left\langle df \left[\overline{X}, T\right], df\overline{Y} \right\rangle = \left\langle \left[\overline{X}, T\right], \overline{Y} \right\rangle. \tag{g}$$

Above all, using the formula (9) in Page 55, we conclude that

$$\begin{aligned}
\left\langle \overline{\nabla}_{\overline{X}} \overline{Y}, \overline{Z} \right\rangle &= \frac{1}{2} \left(\overline{Y} \left\langle \overline{X}, \overline{Z} \right\rangle + \overline{X} \left\langle \overline{Z}, \overline{Y} \right\rangle - \overline{Z} \left\langle \overline{X}, \overline{Y} \right\rangle - \left\langle \left[\overline{Y}, \overline{Z}\right], \overline{X} \right\rangle - \left\langle \left[\overline{X}, \overline{Z}\right], \overline{Y} \right\rangle - \left\langle \left[\overline{Y}, \overline{X}\right], \overline{Z} \right\rangle \right) \\
&= \frac{1}{2} \left(Y \left\langle X, Z \right\rangle + X \left\langle Z, Y \right\rangle - Z \left\langle X, Y \right\rangle - \left\langle [Y, Z], X \right\rangle - \left\langle [X, Z], Y \right\rangle - \left\langle [Y, X], Z \right\rangle \right) = \left\langle \nabla_X Y, Z \right\rangle,
\end{aligned} \tag{h}$$

$$\begin{aligned}
\left\langle \overline{\nabla}_{\overline{X}} \overline{Y}, T \right\rangle &= \frac{1}{2} \left(\overline{Y} \left\langle \overline{X}, T \right\rangle + \overline{X} \left\langle T, \overline{Y} \right\rangle - T \left\langle \overline{X}, \overline{Y} \right\rangle - \left\langle \left[\overline{Y}, T\right], \overline{X} \right\rangle - \left\langle \left[\overline{X}, T\right], \overline{Y} \right\rangle - \left\langle \left[\overline{Y}, \overline{X}\right], T \right\rangle \right) \\
&= \frac{1}{2} \left(\overline{Y} 0 + \overline{X} 0 - 0 - \left\langle df \left[\overline{Y}, T\right], df\overline{X} \right\rangle - \left\langle df \left[\overline{X}, T\right], df\overline{Y} \right\rangle + \left\langle \left[\overline{X}, \overline{Y}\right], T \right\rangle \right) \\
&= \frac{1}{2} \left(- \left\langle 0, df\overline{X} \right\rangle - \left\langle 0, df\overline{Y} \right\rangle + \left\langle \left[\overline{X}, \overline{Y}\right], T \right\rangle \right) = \frac{1}{2} \left\langle \left[\overline{X}, \overline{Y}\right], T \right\rangle.
\end{aligned} \tag{i}$$

By the arbitrariness of $Z \in \mathfrak{X}(M)$, the two equations above means that the horizontal part of $\overline{\nabla}_{\overline{X}} \overline{Y}$ is the lift of $\nabla_X Y$ and the vertical part is vertical part of $\frac{1}{2} \left[\overline{X}, \overline{Y}\right]$, i.e.

$$\overline{\nabla}_{\overline{X}} \overline{Y} = \overline{(\nabla_X Y)} + \frac{1}{2} \left[\overline{X}, \overline{Y}\right]^v, \quad X, Y \in \mathfrak{X}(M),$$

as desired.

Finally for c), $\left[\overline{X}, \overline{Y}\right]^v (\overline{p})$ depends on

$$\left\langle \left[\overline{X}, \overline{Y}\right], T \right\rangle, \quad \forall T \in \left(\mathfrak{X}\left(\overline{M}\right)\right)^v.$$

Because the Riemannian connection $\overline{\nabla}$ is symmetric, we have

$$\left\langle \left[\overline{X}, \overline{Y}\right], T \right\rangle = \left\langle \overline{\nabla}_{\overline{X}} \overline{Y} - \overline{\nabla}_{\overline{Y}} \overline{X}, T \right\rangle.$$

And by Chapter 2 Remark 2.3, $\overline{\nabla}_{\overline{X}} \overline{Y} - \overline{\nabla}_{\overline{Y}} \overline{X}$ only depends on $\overline{X}(\overline{p})$ and $\overline{Y}(\overline{p})$. So we conclude that $\left[\overline{X}, \overline{Y}\right]^v (\overline{p})$ only depends on $\overline{X}(\overline{p})$ and $\overline{Y}(\overline{p})$. □

Exercise 8.10 (Curvature of a Riemannian submersion)**.** Let $f\colon \overline{M} \to M$ be a Riemannian submersion. Let $X, Y, Z, W \in \mathfrak{X}(M)$, $\overline{X}, \overline{Y}, \overline{Z}, \overline{W}$ be their horizontal lifts, and let R and $\overline{R}$ be the curvature tensors of M and $\overline{M}$ respectively. Prove that:

a)

$$\left\langle \overline{R}\left(\overline{X},\overline{Y}\right)\overline{Z},\overline{W}\right\rangle = \langle R(X,Y)Z,W\rangle - \frac{1}{4}\left\langle \left[\overline{X},\overline{Z}\right]^v, \left[\overline{Y},\overline{W}\right]^v\right\rangle + \frac{1}{4}\left\langle \left[\overline{Y},\overline{Z}\right]^v, \left[\overline{X},\overline{W}\right]^v\right\rangle - \frac{1}{2}\left\langle \left[\overline{Z},\overline{W}\right]^v, \left[\overline{X},\overline{Y}\right]^v\right\rangle.$$

b)

$$K(\sigma) = \overline{K}\left(\overline{\sigma}\right) + \frac{3}{4}\left|\left[\overline{X},\overline{Y}\right]^v\right|^2 \geq \overline{K}\left(\overline{\sigma}\right),$$

where σ is the plane generated by the orthonormal vectors $X, Y \in \mathfrak{X}(M)$ and $\overline{\sigma}$ is the plane generated by $\overline{X}, \overline{Y}$.

Proof. For a), by Exercise 8.9 b) and equation (b), we know that

$$\begin{aligned}\overline{X}\left\langle \overline{\nabla}_{\overline{Y}}\overline{Z},\overline{W}\right\rangle &= \overline{X}\left\langle \overline{(\nabla_X Y)} + \frac{1}{2}\left[\overline{X},\overline{Y}\right]^v, \overline{W}\right\rangle \\ &= \overline{X}\left\langle \overline{(\nabla_X Y)}, \overline{W}\right\rangle + \frac{1}{2}\overline{X}\left\langle \left[\overline{X},\overline{Y}\right]^v, \overline{W}\right\rangle = \overline{X}\left\langle \overline{(\nabla_X Y)}, \overline{W}\right\rangle + 0 = X\left\langle \nabla_Y Z, W\right\rangle.\end{aligned}$$

Then by Chapter 2 Corollary 3.3, we have

$$\left\langle \overline{\nabla}_{\overline{X}}\overline{\nabla}_{\overline{Y}}\overline{Z},\overline{W}\right\rangle = \overline{X}\left\langle \overline{\nabla}_{\overline{Y}}\overline{Z},\overline{W}\right\rangle - \left\langle \overline{\nabla}_{\overline{Y}}\overline{Z},\overline{\nabla}_{\overline{X}}\overline{W}\right\rangle.$$

Then by Exerxise 8.9 b) and equation (f), we have

$$\begin{aligned}\left\langle \overline{\nabla}_{\overline{Y}}\overline{Z},\overline{\nabla}_{\overline{X}}\overline{W}\right\rangle &= \left\langle \overline{\nabla}_{\overline{Y}}\overline{Z}, \overline{(\nabla_X W)} + \frac{1}{2}\left[\overline{X},\overline{W}\right]^v\right\rangle = \left\langle \overline{\nabla}_{\overline{Y}}\overline{Z}, \overline{(\nabla_X W)}\right\rangle + \left\langle \overline{\nabla}_{\overline{Y}}\overline{Z}, \frac{1}{2}\left[\overline{X},\overline{W}\right]^v\right\rangle \\ &= \langle \nabla_Y Z, \nabla_X W\rangle + \left\langle \overline{(\nabla_Y Z)} + \frac{1}{2}\left[\overline{Y},\overline{Z}\right]^v, \frac{1}{2}\left[\overline{X},\overline{W}\right]^v\right\rangle \\ &= \langle \nabla_Y Z, \nabla_X W\rangle + \left\langle \overline{(\nabla_Y Z)}, \frac{1}{2}\left[\overline{X},\overline{W}\right]^v\right\rangle + \left\langle \frac{1}{2}\left[\overline{Y},\overline{Z}\right]^v, \frac{1}{2}\left[\overline{X},\overline{W}\right]^v\right\rangle \\ &= \langle \nabla_Y Z, \nabla_X W\rangle + 0 + \left\langle \frac{1}{2}\left[\overline{Y},\overline{Z}\right]^v, \frac{1}{2}\left[\overline{X},\overline{W}\right]^v\right\rangle = \langle \nabla_Y Z, \nabla_X W\rangle + \frac{1}{4}\left\langle \left[\overline{Y},\overline{Z}\right]^v, \left[\overline{X},\overline{W}\right]^v\right\rangle.\end{aligned}$$

Now we conclude

$$\begin{aligned}\left\langle \overline{\nabla}_{\overline{X}}\overline{\nabla}_{\overline{Y}}\overline{Z},\overline{W}\right\rangle &= \overline{X}\left\langle \overline{\nabla}_{\overline{Y}}\overline{Z},\overline{W}\right\rangle - \left\langle \overline{\nabla}_{\overline{Y}}\overline{Z},\overline{\nabla}_{\overline{X}}\overline{W}\right\rangle \\ &= X\left\langle \nabla_Y Z, W\right\rangle - \langle \nabla_Y Z, \nabla_X W\rangle - \frac{1}{4}\left\langle \left[\overline{Y},\overline{Z}\right]^v, \left[\overline{X},\overline{W}\right]^v\right\rangle \\ &= \langle \nabla_X\nabla_Y Z, W\rangle - \frac{1}{4}\left\langle \left[\overline{Y},\overline{Z}\right]^v, \left[\overline{X},\overline{W}\right]^v\right\rangle, \qquad (1)\end{aligned}$$

where the last equation is by Chapter 2 Corollary 3.3 again. Exchanging $\overline{X}$ and $\overline{Y}$, we also have

$$\left\langle \overline{\nabla}_{\overline{Y}}\overline{\nabla}_{\overline{X}}\overline{Z},\overline{W}\right\rangle = \langle \nabla_Y\nabla_X Z, W\rangle - \frac{1}{4}\left\langle \left[\overline{X},\overline{Z}\right]^v, \left[\overline{Y},\overline{W}\right]^v\right\rangle. \qquad (2)$$

On the other hand, for any $T \in \left(\mathfrak{X}\left(\overline{M}\right)\right)^v$, because $\overline{\nabla}$ is symmetric, we have

$$\left\langle \overline{\nabla}_T\overline{X},\overline{Y}\right\rangle = \left\langle \overline{\nabla}_{\overline{X}}T + \left[T,\overline{X}\right], \overline{Y}\right\rangle = \left\langle \overline{\nabla}_{\overline{X}}T, \overline{Y}\right\rangle + \left\langle \left[T,\overline{X}\right], \overline{Y}\right\rangle,$$

where by equation (g) and (a), $\left\langle \left[T,\overline{X}\right], \overline{Y}\right\rangle = 0$ and $\left\langle T, \overline{Y}\right\rangle = 0$. Then by Chapter 2 Corollary 3.3, we have

$$\begin{aligned}\left\langle \overline{\nabla}_T\overline{X},\overline{Y}\right\rangle &= \left\langle \overline{\nabla}_{\overline{X}}T, \overline{Y}\right\rangle + \left\langle \left[T,\overline{X}\right], \overline{Y}\right\rangle \\ &= \overline{X}\left\langle T, \overline{Y}\right\rangle - \left\langle T, \overline{\nabla}_{\overline{X}}\overline{Y}\right\rangle + \left\langle \left[T,\overline{X}\right], \overline{Y}\right\rangle = \overline{X}0 - \left\langle T, \overline{\nabla}_{\overline{X}}\overline{Y}\right\rangle + 0 = -\left\langle T, \overline{\nabla}_{\overline{X}}\right\rangle. \qquad (\#)\end{aligned}$$

Therefore, we have

$$\left\langle\overline{\nabla}_{[\overline{X},\overline{Y}]}\overline{Z},\overline{W}\right\rangle=\left\langle\overline{\nabla}_{[\overline{X},\overline{Y}]^h}\overline{Z},\overline{W}\right\rangle+\left\langle\overline{\nabla}_{[\overline{X},\overline{Y}]^v}\overline{Z},\overline{W}\right\rangle,$$

where by equation (f), because $\left[\overline{X},\overline{Y}\right]^h,\overline{Z},\overline{W}$ are all horizontal, and by equation (d),

$$\left\langle\overline{\nabla}_{[\overline{X},\overline{Y}]^h}\overline{Z},\overline{W}\right\rangle=\left\langle\nabla_{df[\overline{X},\overline{Y}]^h}Z,W\right\rangle=\left\langle\nabla_{[X,Y]}Z,W\right\rangle,$$

and by equation (#) and (i),

$$\left\langle\overline{\nabla}_{[\overline{X},\overline{Y}]^v}\overline{Z},\overline{W}\right\rangle=-\left\langle\left[\overline{X},\overline{Y}\right]^v,\overline{\nabla}_{\overline{Z}}\overline{W}\right\rangle=-\frac{1}{2}\left\langle\left[\overline{X},\overline{Y}\right]^v,\left[\overline{Z},\overline{W}\right]^v\right\rangle.$$

So we conclude

$$\left\langle\overline{\nabla}_{[\overline{X},\overline{Y}]}\overline{Z},\overline{W}\right\rangle=\left\langle\overline{\nabla}_{[\overline{X},\overline{Y}]^h}\overline{Z},\overline{W}\right\rangle+\left\langle\overline{\nabla}_{[\overline{X},\overline{Y}]^v}\overline{Z},\overline{W}\right\rangle=\left\langle\nabla_{[X,Y]}Z,W\right\rangle-\frac{1}{2}\left\langle\left[\overline{X},\overline{Y}\right]^v,\left[\overline{Z},\overline{W}\right]^v\right\rangle. \tag{3}$$

Let equation (2)−(1)+(3), by definition, we have

$$\begin{aligned}
\left\langle\overline{R}\left(\overline{X},\overline{Y}\right)\overline{Z},\overline{W}\right\rangle&=\left\langle\overline{\nabla}_{\overline{Y}}\overline{\nabla}_{\overline{X}}\overline{Z}-\overline{\nabla}_{\overline{X}}\overline{\nabla}_{\overline{Y}}\overline{Z}+\overline{\nabla}_{[\overline{X},\overline{Y}]}\overline{Z},\overline{W}\right\rangle\\
&=\langle\overline{\nabla}_{\overline{Y}}\overline{\nabla}_{\overline{X}}\overline{Z},\overline{W}\rangle-\langle\overline{\nabla}_{\overline{X}}\overline{\nabla}_{\overline{Y}}\overline{Z},\overline{W}\rangle+\left\langle\overline{\nabla}_{[\overline{X},\overline{Y}]}\overline{Z},\overline{W}\right\rangle\\
&=\langle\nabla_Y\nabla_XZ,W\rangle-\frac{1}{4}\left\langle\left[\overline{X},\overline{Z}\right]^v,\left[\overline{Y},\overline{W}\right]^v\right\rangle-\left(\langle\nabla_X\nabla_YZ,W\rangle-\frac{1}{4}\left\langle\left[\overline{Y},\overline{Z}\right]^v,\left[\overline{X},\overline{W}\right]^v\right\rangle\right)\\
&\quad+\left\langle\nabla_{[X,Y]}Z,W\right\rangle-\frac{1}{2}\left\langle\left[\overline{X},\overline{Y}\right]^v,\left[\overline{Z},\overline{W}\right]^v\right\rangle\\
&=\langle\nabla_Y\nabla_XZ,W\rangle-\langle\nabla_X\nabla_YZ,W\rangle+\left\langle\nabla_{[X,Y]}Z,W\right\rangle\\
&\quad-\frac{1}{4}\left\langle\left[\overline{X},\overline{Z}\right]^v,\left[\overline{Y},\overline{W}\right]^v\right\rangle+\frac{1}{4}\left\langle\left[\overline{Y},\overline{Z}\right]^v,\left[\overline{X},\overline{W}\right]^v\right\rangle-\frac{1}{2}\left\langle\left[\overline{Z},\overline{W}\right]^v,\left[\overline{X},\overline{Y}\right]^v\right\rangle\\
&=\langle R(X,Y)Z,W\rangle-\frac{1}{4}\left\langle\left[\overline{X},\overline{Z}\right]^v,\left[\overline{Y},\overline{W}\right]^v\right\rangle\\
&\quad+\frac{1}{4}\left\langle\left[\overline{Y},\overline{Z}\right]^v,\left[\overline{X},\overline{W}\right]^v\right\rangle-\frac{1}{2}\left\langle\left[\overline{Z},\overline{W}\right]^v,\left[\overline{X},\overline{Y}\right]^v\right\rangle,
\end{aligned}$$

as desired.

Then for b), let $Z=X$ and $W=Y$. Because f is a Riemannian submersion, df preserves the length of $\overline{X}$ and $\overline{Y}$. So we have

$$\left|\overline{X}\right|=\left|df\overline{X}\right|=|X|=1$$

and $\left|\overline{Y}\right|=1$ similarly, i.e. $\overline{X}$ and $\overline{Y}$ are also orthonormal (because $\overline{X},\overline{Y}$ are embedded from X,Y, they are orthogonal) vectors of $\overline{M}$. Then by a), we have

$$\begin{aligned}
\overline{K}\left(\overline{\sigma}\right)&=\frac{\overline{R}(\overline{X},\overline{Y})\overline{X},\overline{Y}}{\left|\overline{X}\wedge\overline{Y}\right|}=\langle\overline{R}(\overline{X},\overline{Y})\overline{X},\overline{Y}\rangle\\
&=\langle R(X,Y)X,Y\rangle-\frac{1}{4}\left\langle\left[\overline{X},\overline{X}\right]^v,\left[\overline{Y},\overline{Y}\right]^v\right\rangle+\frac{1}{4}\left\langle\left[\overline{Y},\overline{X}\right]^v,\left[\overline{X},\overline{Y}\right]^v\right\rangle-\frac{1}{2}\left\langle\left[\overline{X},\overline{Y}\right]^v,\left[\overline{X},\overline{Y}\right]^v\right\rangle\\
&=\frac{\langle R(X,Y)X,Y\rangle}{|X\wedge Y|}-\frac{1}{4}\langle 0^v,0^v\rangle+\frac{1}{4}\left\langle-\left[\overline{X},\overline{Y}\right]^v,\left[\overline{X},\overline{Y}\right]^v\right\rangle-\frac{1}{2}\left\langle\left[\overline{X},\overline{Y}\right]^v,\left[\overline{X},\overline{Y}\right]^v\right\rangle\\
&=K(\sigma)-\frac{3}{4}\left|\left[\overline{X},\overline{Y}\right]^v\right|^2,
\end{aligned}$$

i.e.

$$K(\sigma)=\overline{K}\left(\overline{\sigma}\right)+\frac{3}{4}\left|\left[\overline{X},\overline{Y}\right]^v\right|^2\geq\overline{K}\left(\overline{\sigma}\right),$$

as desired. $\square$

Exercise 8.11 (The complex projective space)**.** Let

$$\mathbb{C}^{n+1} - \{0\} = \{(z_0, \cdots, z_n) = Z \neq 0,\ z_j = x_j + \mathrm{i}\, y_j,\ j = 0, \cdots, n\}$$

be the set of all non-zero $(n+1)$-tuples of complex numbers z_j. Define an equivalence relation on $\mathbb{C}^{n+1} - \{0\}$:

$$Z = (z_0, \cdots, z_n) \sim W = (w_0, \cdots, w_n) \text{ if } z_j = \lambda w_j,\ \lambda \in \mathbb{C}^*.$$

The equivalence class of Z will be denoted by $[Z]$ (= the complex line passing through the origin and through Z). The set of such classes is called, by analogy with the real case, *the complex projective space* $\mathbb{C}P^n$ of complex dimension n.

a) Show that $\mathbb{C}P^n$ has a differentiable structure of a manifold of real dimension $2n$ and that $\mathbb{C}P^1$ is diffeomorphic to S^2.

b) Let $(Z, W) = z_0\overline{w_0} + \cdots + z_n\overline{w_n}$ be the Hermitian product on $\mathbb{C}^{n+1}$, where the bar denotes complex conjugation. Identify $\mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2}$ by putting $z_j = x_j + \mathrm{i}\, y_j = (x_j, y_j)$. Show that

$$S^{2n+1} = \left\{ N \in \mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2} \middle| (N, N) = 1 \right\}$$

is the unit sphere in $\mathbb{R}^{2n+2}$.

c) Show that the equivalence relation $\sim$ induces on S^{2n+1} the following equivalence relation:

$$Z \sim W \text{ if } \mathrm{e}^{\mathrm{i}\,\theta} Z = W.$$

Establish that there exists a differentiable map (the Hopf fibering) $f\colon S^{2n+1} \to \mathbb{C}P^n$ such that

$$f^{-1}([Z]) = \left\{ \mathrm{e}^{\mathrm{i}\,\theta} N \in S^{2n+1} \middle| N \in [Z] \cap S^{2n+1},\ 0 \le \theta \le 2\pi \right\} = [Z] \cap S^{2n+1}.$$

d) Show that f is a submersion.

Proof. Firstly for a), just like the real case, the charts are given by

$$U_i = \{ [z_0, \cdots, z_n] |\, z_i \neq 0\}$$

and

$$\begin{aligned} \varphi_i^{-1}\colon U_i &\to \mathbb{R}^{2n} \\ [z_0, \cdots, z_n] &\mapsto \left(\frac{z_0}{z_i}, \cdots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \cdots, \frac{z_n}{z_i} \right) \in \mathbb{C}^n \cong \mathbb{R}^{2n}, \end{aligned}$$

for any $0 \le i \le n$. The translation maps are

$$\begin{aligned} \varphi_j^{-1} \circ \varphi_i (z_1, \cdots, z_n) &= \varphi_j^{-1} ([z_1, \cdots, z_{i-1}, 1, z_i, \cdots, z_n]) \\ &= \varphi_j^{-1} \left(\left[\frac{z_1}{z_j}, \cdots, \frac{z_{j-1}}{z_j}, 1, \frac{z_{j+1}}{z_j}, \cdots, \frac{z_{i-1}}{z_j}, \frac{1}{z_j}, \frac{z_i}{z_j}, \cdots, \frac{z_n}{z_j} \right] \right) \\ &= \left(\frac{z_1}{z_j}, \cdots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \cdots, \frac{z_{i-1}}{z_j}, \frac{1}{z_j}, \frac{z_i}{z_j}, \cdots, \frac{z_n}{z_j} \right), \end{aligned}$$

for any $(z_1, \cdots, z_n) \in \varphi_i^{-1} (U_i \cap U_j)$, which are differentiable. So we conclude that $\{U_i, \varphi_i\}_{i=0}^n$ is the differentiable charts of $\mathbb{C}P^n$, which makes it be a manifold with real dimension $2n$.

For the case $n = 0$, if $[z_0, z_1] \in \mathbb{C}P^1$, $z_0 \neq 0$, it can be rewritten as $\left[1, \frac{z_1}{z_0}\right]$ where z_1 can be any element of $\mathbb{C}$ and so $\frac{z_1}{z_0}$. We conclude that

$$\left\{ [z_0, z_1] \in \mathbb{C}P^1 \middle| z_0 \neq 0 \right\}$$

is diffeomorphic to $\mathbb{C}$. For $z_0 = 0$, any $[0, z_1] \in \mathbb{C}P^1$ can be rewritten as $\left[\frac{0}{z_1}, \frac{z_1}{z_1}\right] = [0, 1]$. So

$$\left\{ [z_0, z_1] \in \mathbb{C}P^1 \middle| z_0 = 0 \right\}$$

is a point. Now $\mathbb{C}P^1$ is given by attaching $\mathbb{C}$ a point diffeomorphicly, i.e. the infinity point ∞. We conclude that

$$\mathbb{C}P^1 \cong \mathbb{C} \cup \{\infty\} \cong S^2$$

is diffeomorphic to the Riemannian sphere.

And b) is just a trivial calculation:

$$N = (z_0, \cdots, z_n) \in S^{2n+1} = \left\{ N \in \mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2} \middle| (N, N) = 1 \right\},$$

if and only if

$$1 = (N, N) = z_0 \overline{z_0} + \cdots + z_n \overline{z_n} = a_0^2 + b_0^2 + \cdots + a_n^2 + b_n^2,$$

if and only if

$$N = (z_0, \cdots, z_n) = (a_0, b_0, \cdots, a_n, b_n) \in S^{2n+1} \subset \mathbb{R}^{2n+2}.$$

So we conclude that

$$S^{2n+1} = \left\{ N \in \mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2} \middle| (N, N) = 1 \right\}$$

is the unit sphere in $\mathbb{R}^{2n+2}$.

For c), if $Z, W \in S^{2n+1} \subset \mathbb{C}^{n+1} - \{0\}$ such that $Z \sim W$ in $\mathbb{C}^{n+1} - \{0\}$, there is $\lambda \in \mathbb{C}^*$ such that $\lambda Z = W$. By b), we have

$$1 = |W| = |\lambda Z| = |\lambda| \cdot |Z| = |\lambda| \cdot 1 = |\lambda|, \tag{4}$$

i.e. $\lambda = \mathrm{e}^{\mathrm{i}\,\theta}$ for some $0 \leq \theta < 2\pi$. So the equivalence relation $\sim$ on $\mathbb{C}^{n+1} - \{0\}$ induces on S^{2n+1} the equivalence relation:

$$Z \sim W \text{ if } \mathrm{e}^{\mathrm{i}\,\theta} Z = \lambda Z = W.$$

Denote the equivalence class of $Z \in S^{2n+1}$ by $\overline{Z}$.

Let's define

$$\begin{aligned} f \colon S^{2n+1} &\to \mathbb{C}P^n \\ Z &\to [Z]. \end{aligned}$$

Notice that $\varphi_i^{-1} \circ f$ has the form

$$(z_0, \cdots, z_n) = Z \mapsto [Z] = [z_0, \cdots, z_n] \mapsto \left(\frac{z_0}{z_i}, \cdots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \cdots, \frac{z_n}{z_i} \right), \tag{5}$$

so f is differentiable. And if $W \in f^{-1}([Z])$, i.e. $[W] = f(W) = [Z]$, we have $W \in [Z]$. Then there is some $\lambda \in \mathbb{C}^*$ such that $W = \lambda Z$. By equation (4) again, we have $\lambda = \mathrm{e}^{\mathrm{i}\,\theta}$ for some $0 \leq \theta < 2\pi$. So we conclude that $W = \mathrm{e}^{\mathrm{i}\,\theta} Z \in S^{2n+1}$ where $Z \in [Z] \cap S^{2n+1}$ naturally, i.e.

$$W \in \left\{ \mathrm{e}^{\mathrm{i}\,\theta} N \in S^{2n+1} \middle| N \in [Z] \cap S^{2n+1},\ 0 \leq \theta \leq 2\pi \right\}.$$

Then

$$f^{-1}([Z]) \subset \left\{ \mathrm{e}^{\mathrm{i}\,\theta} N \in S^{2n+1} \middle| N \in [Z] \cap S^{2n+1},\ 0 \leq \theta \leq 2\pi \right\}.$$

On the other hand, for any $\mathrm{e}^{\mathrm{i}\,\theta} N$ where $N \in [Z] \cap S^{2n+1}$ and $0 \leq \theta < 2\pi$,

$$f\left(\mathrm{e}^{\mathrm{i}\,\theta} N\right) = \left[ee^{\mathrm{i}\,\theta} N\right] = [N] = [Z],$$

i.e. $\mathrm{e}^{\mathrm{i}\,\theta} N \in f^{-1}([Z])$. This means

$$\left\{ \mathrm{e}^{\mathrm{i}\,\theta} N \in S^{2n+1} \middle| N \in [Z] \cap S^{2n+1},\ 0 \leq \theta \leq 2\pi \right\} \subset f^{-1}([Z]).$$

So now we conclude that

$$f^{-1}([Z]) = \left\{ \mathrm{e}^{\mathrm{i}\,\theta} N \in S^{2n+1} \middle| N \in [Z] \cap S^{2n+1},\ 0 \leq \theta \leq 2\pi \right\} = [Z] \cap S^{2n+1},$$

where the last equation is same as above.

Finally for d), for any $[Z] \in \mathbb{C}P^n$, $\left|\frac{Z}{|Z|}\right| = 1$ and then $\frac{Z}{|Z|} \in S^{2n+1}$. So we have

$$f\left(\frac{Z}{|Z|}\right) = \left[\frac{Z}{|Z|}\right] = [Z].$$

Now we conclude that f is surjective.

Then we firstly calculate df on $\mathbb{R}^{2n+2}$. By equation (5), f has the form

$$\begin{aligned}
f(a_0, b_0, \cdots, a_n, b_n) = f(z_0, \cdots, z_n) &= \left(\frac{z_0}{z_i}, \cdots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \cdots, \frac{z_n}{z_i}\right) \\
&= \left(\frac{a_0 a_i + b_0 b_i}{|z_i|}, \frac{a_i b_0 - a_0 b_i}{|z_i|}, \cdots, \frac{a_{i-1} a_i + b_{i-1} b_i}{|z_i|}, \frac{a_i b_{i-1} - a_{i-1} b_i}{|z_i|},\right. \\
&\quad \left.\frac{a_{i+1} a_i + b_{i+1} b_i}{|z_i|}, \frac{a_i b_{i+1} - a_{i+1} b_i}{|z_i|}, \cdots, \frac{a_n a_i + b_n b_i}{|z_i|}, \frac{a_i b_n - a_n b_i}{|z_i|}\right) \\
&= (x_0, y_0, \cdots, x_{i-1}, y_{i-1}, x_{i+1}, y_{i+1}, \cdots, x_n, y_n).
\end{aligned}$$

Then we have

$$\frac{x_j}{a_k} = \frac{a_i}{|z_i|}\delta_{jk}, \quad \frac{x_j}{b_k} = \frac{b_i}{|z_i|}\delta_{jk}, \quad \frac{y_j}{a_k} = -\frac{b_i}{|z_i|}\delta_{jk}, \quad \frac{y_j}{b_k} = \frac{a_i}{|z_i|}\delta_{jk}, \quad \forall k \neq i,$$

$$\frac{x_j}{a_i} = \frac{-a_j a_i^2 + a_j b_i^2 - 2a_i b_j b_i}{|z_i|^2}, \quad \frac{x_j}{b_i} = \frac{-b_j b_i^2 + b_j a_i^2 - 2a_j a_i b_i}{|z_i|^2},$$

$$\frac{y_j}{a_i} = \frac{-b_j a_i^2 + b_j b_i^2 + 2a_j a_i b_i}{|z_i|^2}, \quad \frac{x_j}{b_i} = \frac{a_j b_i^2 - a_j a_i^2 - 2a_i b_j b_i}{|z_i|^2}.$$

So df has A has $2n$ rows and $2n+2$ columns and is of the form $A + B$ where A's non-vanishing elements are 2-dimensional diagram blocks $\begin{pmatrix} \frac{a_i}{|z_i|} & \frac{b_i}{|z_i|} \\ -\frac{b_i}{|z_i|} & \frac{a_i}{|z_i|} \end{pmatrix}$, and B's possible non-vanishing elements are the $2i-1$th and $2i$th columns

$$\begin{pmatrix} \frac{x_0}{a_i} & \frac{x_0}{b_i} \\ \frac{y_0}{a_i} & \frac{y_0}{b_i} \\ \vdots & \vdots \\ \frac{x_n}{a_i} & \frac{x_n}{b_i} \\ \frac{y_n}{a_i} & \frac{y_n}{b_i} \end{pmatrix}.$$

Then A's diagram blocks are orthogonal matrices so A can be translated to Id_{2n} plusing two zero columns and B's $2i-1$th and $2i$th columns can be moved to the last two columns. So df can be shift to

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \frac{x_0}{a_i} & \frac{x_0}{b_i} \\ 0 & 1 & 0 & \cdots & 0 & \frac{y_0}{a_i} & \frac{y_0}{b_i} \\ \vdots & & \ddots & & \vdots & \vdots & \vdots \\ 0 & & 0 & 1 & 0 & \frac{x_n}{a_i} & \frac{x_n}{b_i} \\ 0 & 0 & \cdots & 0 & 1 & \frac{y_n}{a_i} & \frac{y_n}{b_i} \end{pmatrix}.$$

Because $S^{2n+1} \subset \mathbb{R}^{2n+2}$ has dimension $2n+1$, locally, we have

$$a_l = \pm\sqrt{1 - \sum_{\substack{i=1 \\ i \neq l}}^{n} a_i^2 - \sum_{\substack{i=1 \\ i \neq l}}^{n} a_i^2 - b_l}$$

or

$$b_l = \pm\sqrt{1 - \sum_{\substack{i=1 \\ i \neq l}}^{n} a_i^2 - \sum_{\substack{i=1 \\ i \neq l}}^{n} a_i^2 - a_l}.$$

So now, they are not variables. Then We just need to delete the $2i-1$th row and $2i-1$th column or the $2i$th row and $2i$th column, respectively, to get the exact df, by chain rule. Obviously, df also has rank $2n$ now. So we have that df has rank $2n$.

Above all, we conlcude that f is a submersion, as desired. $\square$

Exercise 8.12 (Curvature of the complex projective space)**.** Define a Riemannian metric on $\mathbb{C}^{n+1}-\{0\}$ in the following way: If $Z \in \mathbb{C}^{n+1}-\{0\}$ and $V, W \in T_Z\left(\mathbb{C}^{n+1}-\{0\}\right)$, let

$$\langle V, W\rangle_Z = \frac{\operatorname{Re}(V, W)}{(Z, Z)}.$$

Observe that the metric $\langle\ ,\ \rangle$ restricted to $S^{2n+1} \subset \mathbb{C}^{n+1}-\{0\}$ coincides with the metric induced from $\mathbb{R}^{2n+2}$.

a) Show that, for all $0 \leq \theta \leq 2\pi$, $\mathrm{e}^{\mathrm{i}\,\theta}\colon S^{2n+1} \to S^{2n+1}$ is an isometry, and that, therefore, it is possible to define a Riemannian metric on $\mathbb{C}P^n$ in such a way that the submersion f is Riemannian.

b) Show that, in this metric, the sectional curvature of $\mathbb{C}P^n$ is given by

$$K(\sigma) = 1 + 3\cos^2\varphi,$$

where σ is generated by the orthonormal pair X, Y, $\cos\varphi = \langle \overline{X}, \mathrm{i}\,\overline{Y}\rangle$, and $\overline{X}, \overline{Y}$ are the horizontal lifts of X and Y, respectively. In particular,

$$1 \leq K(\sigma) \leq 4.$$

Proof. For a), we firstly consider $\mathrm{e}^{\mathrm{i}\,\theta}$ on $\mathbb{C}^{n+1}$. We have

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}\,\theta}(z_0, \cdots, z_n) &= \left(\mathrm{e}^{\mathrm{i}\,\theta} z_0, \cdots, \mathrm{e}^{\mathrm{i}\,\theta} z_n\right) \\
&= \left((\cos\theta + \mathrm{i}\sin\theta)(a_0 + \mathrm{i}\,b_0), \cdots, (\cos\theta + \mathrm{i}\sin\theta)(a_n + \mathrm{i}\,b_n)\right) \\
&= \left((\cos\theta a_0 - \sin\theta b_0) + \mathrm{i}(\cos\theta b_0 + \sin\theta a_0), \cdots, (\cos\theta a_n - \sin\theta b_n) + \mathrm{i}(\cos\theta b_n + \sin\theta a_n)\right),
\end{aligned}$$

i.e.

$$\mathrm{e}^{\mathrm{i}\,\theta}(a_0, b_0, \cdots, a_n, b_n) = (\cos\theta a_0 - \sin\theta b_0, \cos\theta b_0 + \sin\theta a_0, \cdots, \cos\theta a_n - \sin\theta b_n, \cos\theta b_n + \sin\theta a_n).$$

So $d\,\mathrm{e}^{\mathrm{i}\,\theta}$ is linear and has the form

$$\begin{pmatrix}
\cos\theta & -\sin\theta & 0 & \cdots & 0 \\
\sin\theta & \cos\theta & 0 & \cdots & 0 \\
0 & & \ddots & & 0 \\
0 & \cdots & 0 & \cos\theta & -\sin\theta \\
0 & \cdots & 0 & \sin\theta & \cos\theta
\end{pmatrix}.$$

It's an orthogonal matrix and then $\mathrm{e}^{\mathrm{i}\,\theta}$ is an isometry on $\mathbb{C}^{n+1}$ naturally. So $\mathrm{e}^{\mathrm{i}\,\theta}$ is an isometry on the embedding submanifold S^{2n+1} of $\mathbb{C}^{n+1}$ naturally.

Then for any $[Z] \in \mathbb{C}P^n$ and $X, Y \in T_{[Z]}\mathbb{C}P^n$, we would assume that $Z \in S^{2n+1} \subset \mathbb{C}^{n+1}$. Let $\overline{X}, \overline{Y}$ be their horizontal lift (see Exercise 8.9). Then we define

$$\langle X, Y\rangle_{[Z]} = \langle \overline{X}, \overline{Y}\rangle_Z,$$

where $\langle\ ,\ \rangle_Z$ is the metric defined on $\mathbb{C}^{n+1}-\{0\}$ above and then a metric on $S^{2n+1} \subset \mathbb{C}^{n+1}-\{0\}$ naturally. It's easy to check that $\langle\ ,\ \rangle_{[Z]}$ is a Riemannian metric on $\mathbb{C}P^n$. And for any horizontal vector $\overline{X}$ of $T_Z\mathbb{C}P^n$,

$$\left|df_Z\overline{X}\right| = \sqrt{\langle df_Z\overline{X}, df_Z\overline{X}\rangle_{[Z]}} = \sqrt{\langle X, X\rangle_{[Z]}} = \sqrt{\langle \overline{X}, \overline{X}\rangle_Z} = \left|\overline{X}\right|.$$

So df_Z preserves the lengths of horizontal vectors, for any $Z \in S^{2n+1}$. Then f is a Riemannian submersion, as desired. (In fact, we need to define the metric on evary $U_i \subset \mathbb{C}P^n$ and use the partition of unity to get a global metric on $\mathbb{C}P^n$.)

Then for b), for any $Z \in S^{2n+1}$, let $\gamma(\theta) = \mathrm{e}^{\mathrm{i}\,\theta} Z \subset S^{2n+1}$. Notice that, by Exercise 8.11 c), the fiber $f^{-1}([Z]) = \left\{ \mathrm{e}^{\mathrm{i}\,\theta} N \in S^{2n+1} \middle| N \in [Z] \cap S^{2n+1},\ 0 \le \theta \le 2\pi \right\} = \gamma(\theta)$, we have

$$\frac{d}{d\theta} \mathrm{e}^{\mathrm{i}\,\theta} Z\bigg|_{t=0} = \mathrm{i}\, Z \in T_Z\left(f^{-1}([Z])\right)$$

is vertical. For any $X \in \mathfrak{X}(\mathbb{C}P^n)$, we can take a curve α in S^{2n+1} such that $\alpha(0) = Z$ and $\alpha'(0) = \overline{X}$, where $\overline{X}$ is the horizontal lift of X (see Exercise 8.9). Then

$$\overline{\nabla}_{\overline{X}}(\mathrm{i}\, Z) = \overline{\nabla}_{\overline{X}}\left(\gamma'(\theta)\right)(0) = \overline{\nabla}_{\alpha'}\left(\gamma'(\theta)\right)(0) = \frac{D\gamma'}{dt}(0).$$

And because the metric $\langle\ ,\ \rangle$ restricted to $S^{2n+1} \subset \mathbb{C}^{n+1} - \{0\}$ coincides with the usual metric of S^{2n+1} induced by the Euclidean metric, we have Why?

$$\overline{\nabla}_{\overline{X}}(\mathrm{i}\, Z) = \frac{D\gamma'}{dt}(0) = \frac{d}{dt}\gamma'(0)(\alpha(t))\bigg|_{t=0} = \frac{d}{dt}\,\mathrm{i}\, Z(\alpha(t))\bigg|_{t=0} = \frac{d}{dt}\,\mathrm{i}\,\alpha(t)\bigg|_{t=0} = \mathrm{i}\,\overline{X}. \tag{6}$$

Now for any $X, Y \in \mathfrak{X}(\mathbb{C}P^n)$, by Chapter 2 Corollary 3.3,

$$\left\langle \overline{\nabla}_{\overline{X}}\overline{Y}, \mathrm{i}\, Z\right\rangle = \overline{X}\left\langle \overline{Y}, \mathrm{i}\, Z\right\rangle - \left\langle \overline{Y}, \overline{\nabla}_{\overline{X}}(\mathrm{i}\, Z)\right\rangle,$$

where because $\overline{X}$ is horizontal and we have proved that $\mathrm{i}\, Z$ is vertical, $\left\langle \overline{Y}, \mathrm{i}\, Z\right\rangle = 0$. And by equation (6), we have

$$\left\langle \overline{\nabla}_{\overline{X}}\overline{Y}, \mathrm{i}\, Z\right\rangle = -\left\langle \overline{Y}, \overline{\nabla}_{\overline{X}}(\mathrm{i}\, Z)\right\rangle = -\left\langle \overline{Y}, \mathrm{i}\,\overline{X}\right\rangle.$$

Then because $\overline{\nabla}$ is symmetric,

$$\left\langle \left[\overline{X}, \overline{Y}\right], \mathrm{i}\, Z\right\rangle = \left\langle \overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X}, \mathrm{i}\, Z\right\rangle = \left\langle \overline{\nabla}_{\overline{X}}\overline{Y}, \mathrm{i}\, Z\right\rangle - \left\langle \overline{\nabla}_{\overline{Y}}\overline{X}, \mathrm{i}\, Z\right\rangle = -\left\langle \overline{Y}, \mathrm{i}\,\overline{X}\right\rangle + \left\langle \overline{X}, \mathrm{i}\,\overline{Y}\right\rangle,$$

where

$$-\left\langle \overline{Y}, \mathrm{i}\,\overline{X}\right\rangle = -\bar{\mathrm{i}}\left\langle \overline{Y}, \overline{X}\right\rangle = -(-\mathrm{i})\left\langle \overline{Y}, \overline{X}\right\rangle = \mathrm{i}\left\langle \overline{Y}, \overline{X}\right\rangle = \left\langle \mathrm{i}\,\overline{Y}, \overline{X}\right\rangle = \overline{\left\langle \overline{X}, \mathrm{i}\,\overline{Y}\right\rangle} = \left\langle \overline{X}, \mathrm{i}\,\overline{Y}\right\rangle,$$

where the last equation is because the metric $\langle\ ,\ \rangle$ we defined is a real function. Now we conclude

$$\left\langle \left[\overline{X}, \overline{Y}\right], \mathrm{i}\, Z\right\rangle = -\left\langle \overline{Y}, \mathrm{i}\,\overline{X}\right\rangle + \left\langle \overline{X}, \mathrm{i}\,\overline{Y}\right\rangle = \left\langle \overline{X}, \mathrm{i}\,\overline{Y}\right\rangle + \left\langle \overline{X}, \mathrm{i}\,\overline{Y}\right\rangle = 2\left\langle \overline{X}, \mathrm{i}\,\overline{Y}\right\rangle = 2\cos\varphi.$$

And by counting dimensions, we know that all vertical fields has dimension 1 and can be generated by $\mathrm{i}\, Z$. And because

$$\langle \mathrm{i}\, Z, \mathrm{i}\, Z\rangle = \mathrm{i}\,\bar{\mathrm{i}}\,\langle Z, Z\rangle_{\mathbb{R}^{2n+2}} = 1,$$

we have

$$\left[\overline{X}, \overline{Y}\right]^v = \frac{\left\langle \left[\overline{X}, \overline{Y}\right], \mathrm{i}\, Z\right\rangle}{\langle \mathrm{i}\, Z, \mathrm{i}\, Z\rangle}\,\mathrm{i}\, Z = \frac{2\cos\varphi}{1}\,\mathrm{i}\, Z = 2\cos\varphi\,\mathrm{i}\, Z.$$

Finally, we can use Exercise 8.10 b):

$$K(\sigma) = \overline{K}\left(\overline{\sigma}\right) + \frac{3}{4}\left|\left[\overline{X}, \overline{Y}\right]^v\right|^2.$$

Because S^{2n+1} has constant curvature 1, $\overline{K}\left(\overline{\sigma}\right) = 1$. And we have

$$\left|\left[\overline{X}, \overline{Y}\right]^v\right|^2 = \left\langle \left[\overline{X}, \overline{Y}\right]^v, \left[\overline{X}, \overline{Y}\right]^v\right\rangle = \langle 2\cos\varphi\,\mathrm{i}\, Z, 2\cos\varphi\,\mathrm{i}\, Z\rangle = 4\cos^2\varphi\,\langle \mathrm{i}\, Z, \mathrm{i}\, Z\rangle = 4\cos^2\varphi 1 = 4\cos^2\varphi.$$

So we conclude that

$$K(\sigma) = \overline{K}\left(\overline{\sigma}\right) + \frac{3}{4}\left|\left[\overline{X}, \overline{Y}\right]^v\right|^2 = 1 + \frac{3}{4}4\cos^2\varphi = 1 + 3\cos^2\varphi$$

and

$$1 \le K(\sigma) = 1 + 3\cos^2\varphi \le 4$$

particularly, as desired. □

Exercise 8.13. Let $p \in M$ and let $\sigma\colon M \to M$ be an isometry such that $\sigma(p) = p$ and $d\sigma_p(v) = -v$, for all $v \in T_pM$. Let X be a parallel field along a geodesic γ in M with $\gamma(0) = p$. Show that

$$d\sigma_{\gamma(t)}\left(X(\gamma(t))\right) = -\left(X(\gamma(-t))\right).$$

Proof. Firstly, notice because σ is an isometry, $\sigma\left(\gamma(t)\right)$ is also a geodesic, with initial velocity $d\sigma_p\left(\gamma'(0)\right) = -\gamma'(0)$. By the homogeneity of geodesic (Chapter 3 Lemma 2.6),

$$\sigma\left(\gamma(t)\right) = \gamma\left(t, p, -\gamma'(0)\right) = \gamma\left(-t, p, \gamma'(0)\right) = \gamma(-t).$$

Now $d\sigma_{\gamma(t)}X(\gamma(t))$ is a field along $\sigma(\gamma(t)) = \gamma(-t)$. And because σ is an isometry of M, it will not change the connection (covariant derivative) of M on $\sigma(M)$. So $d\sigma_{\gamma(t)}X(\gamma(t))$ is parallel along $\sigma(\gamma(t)) = \gamma(-t)$. We claim that a vector field is parallel along a curve if and only if it is parallel along the image of the curve. To see that, just notice that to get $\frac{DX}{dt}$, we only need to make a scale multiple on $\frac{DX}{ds}$, which is the ratio of the initial velocities of them. So we conclude that $d\sigma_{\gamma(t)}X(\gamma(t))$ is parallel along the image of $\gamma(-t)$, and then is parallel along $\gamma(t)$.

On the other side, denote $\gamma(s) = \sigma(\gamma(t)) = \gamma(-t)$, i.e. $s = -t$, then

$$\frac{D\left(-X(\gamma(-t))\right)}{dt} = \frac{D\left(-X(\gamma(s))\right)}{d(-s)} = \frac{DX(\gamma(s))}{ds} = 0.$$

This means $-X(\gamma(-t))$ is also parallel along $\gamma(t)$.

Finally, we have

$$d\sigma_{\gamma(t)}X(\gamma(t))\big|_{t=0} = d\sigma_p X(\gamma(0)) = -X(\gamma(0)) = -X(\gamma(-t))\big|_{t=0}.$$

So we conclude that $d\sigma_{\gamma(t)}X(\gamma(t))$ and $-X(\gamma(-t))$ are all parallel along $\gamma(t)$ with same initial velocities. By the uniqueness of parallel transport with initial velocity (Chapter 2 Proposition 2.6), we have

$$d\sigma_{\gamma(t)}X(\gamma(t)) = -X(\gamma(-t)),$$

for all well-defined t, as desired. □

Exercise 8.14 (Geometric characterization of locally symmetric spaces)**.** Let M be a Riemannian manifold. A *local symmetric* at $p \in M$ is a map $\sigma\colon B_\varepsilon(p) \to B_\varepsilon(p)$ of a normal geodesic ball centered at p such that $\sigma\left(\gamma(t)\right) = \gamma(-t)$, where γ is a radical geodesic ($\gamma(0) = p$) of $B_\varepsilon(p)$. Prove that: M is locally symmetric $\Leftrightarrow$ every local symmetry is an isometry.

Proof. ($\Rightarrow$): Let M be a local symmetric space, i.e. $\nabla R = 0$. For any local symmetry $\sigma\colon B_\varepsilon(p) \to B_\varepsilon(p)$, we need to proof that σ is an isometry. Let's choose the geodesic frame $e_1, \cdots, e_n$ of $B_\varepsilon(p)$ (see Exercise 3.7). For any $1 \le i, j, k, l, h \le n$, we have

$$\begin{aligned}
0 &= \nabla R\left(e_i, e_j, e_k, e_l, e_h\right) \\
&= e_h\left(R\left(e_i, e_j, e_k, e_l\right)\right) - R\left(\nabla_{e_h}e_i, e_j, e_k, e_l\right) - R\left(e_i, \nabla_{e_h}e_j, e_k, e_l\right) \\
&\quad - R\left(e_i, e_j, \nabla_{e_h}e_k, e_l\right) - R\left(e_i, e_j, e_k, \nabla_{e_h}e_l\right) \\
&= e_h\left(R_{ijkl}\right) - R\left(0, e_j, e_k, e_l\right) - R\left(e_i, 0, e_k, e_l\right) - R\left(e_i, e_j, 0, e_l\right) - R\left(e_i, e_j, e_k, 0\right) \\
&= e_h\left(R_{ijkl}\right),
\end{aligned}$$

where the third equation is because $e_1, \cdots, e_n$ are parallel along γ and $\nabla_{e_i}e_j(\gamma(0)) = \nabla_{e_i}e_j(p) = 0$ for any $1 \le i, j \le n$, then $\nabla_{e_i}e_j(\gamma(t)) = 0$ for any t. By the arbitrariness of h and linearity, for any geodesic γ in $B_\varepsilon(p)$,

$$\gamma'(t)\left(R_{ijkl}\right) = 0,$$

i.e.

$$\frac{R_{ijkl}(\gamma(t))}{dt} = \gamma'(t)\left(R_{ijkl}\right) = 0.$$

This means R_{ijkl} is constant along any geodesic γ in $B_\varepsilon(p)$, for any i, j, k, l. By linearity, R is constant along any geodesic in $B_\varepsilon(p)$.

Then we want to use Cartan's Theorem (Chapter 8 Theorem 2.1). In this theorem, because $\sigma\colon B_\varepsilon(p) \to B_\varepsilon(p) \subset M$, we have $\widetilde{R} = R$. And by Exercise 4.6 a), $R(x,y)u$ is also parallel along any geodesic γ, for any vector fields x, y, u parallel to that geodesic. Now because ϕ_t are generated by parallel transport and isometry of T_pM, and we have proved that R is constant along any geodesic, we have

$$\langle R(x,y),u,v\rangle = \langle R\left(\phi_t(x),\phi_t(y)\right)\phi_t(u),\phi_t(v)\rangle = \left\langle \widetilde{R}\left(\phi_t(x),\phi_t(y)\right)\phi_t(u),\phi_t(v)\right\rangle.$$

Then we cliam that

$$\sigma = \exp_p \circ i \circ \exp_p^{-1}$$

where

$$\begin{aligned} i\colon T_pM &\to T_pM \\ v &\mapsto -v \end{aligned}$$

is an isometry. To see that, for any $q \in B_\varepsilon(p)$, denote $q = \gamma(t) = \gamma(t,p,v)$. By the homogeneity of geodesic (Chapter 3 Lemma 2.6),

$$q = \gamma(t,p,v) = \gamma(1,p,tv)$$

and then $\exp_p^{-1}(q) = tv$. Then $i(tv) = -tv$. By the homogeneity of geodesic (Chapter 3 Lemma 2.6) again, we have

$$\exp_p(-tv) = \gamma(1,p,-tv) = \gamma(-t,p,v) = \gamma(-t).$$

Above all, we conclude that

$$\exp_p \circ i \circ \exp_p^{-1}(q) = \exp_p \circ i(tv) = \exp_p(-tv) = \gamma(-t) = \sigma(\gamma(t)) = \sigma(q).$$

So we have $\sigma = \exp_p \circ i \circ \exp_p^{-1}$. Now using Cartan's Theorem (Chapter 8 Theorem 2.1), we have that

$$\sigma = \exp_p \circ i \circ \exp_p^{-1}\colon B_\varepsilon(p) \to \sigma\left(B_\varepsilon(p)\right) \subset M$$

is a local isometry. And because $B_\varepsilon(p)$ is a normal neighborhood and $\sigma(\gamma(t)) = \gamma(-t)$ for any radical geodeisc γ, we have $\sigma\left(B_\varepsilon(p)\right) = B_\varepsilon(p)$. Then we conclude

$$\sigma\colon B_\varepsilon(p) \to B_\varepsilon(p)$$

is an isometry, as desired.

($\Leftarrow$): For any $p \in M$, denote $e_1, \cdots, e_n$ the geodesic frame (see Exercise 3.7) of M at p, for any $1 \le h \le n$, we can choose a geodesic γ in $B_\varepsilon(p)$ such that $\gamma(0) = p$ and $\gamma'(0) = e_h$. Put $R_{ijkl}(t) = R(e_i(t), e_j(t), e_k(t), e_l(t))$. Because $e_1(t), \cdots, e_n(t)$ are generated by parallel transports along geodesics (see Exercise 3.7), we have

$$R_{ijkl}(\gamma(t)) = R(e_i(\gamma(t)), e_j(\gamma(t)), e_k(\gamma(t)), e_l(\gamma(t))) = R(e_i(t), e_j(t), e_k(t), e_l(t)) = R_{ijkl}(t).$$

Then by definitions, we have

$$\begin{aligned} \nabla_{e_h} R\left(e_i,e_j,e_k,e_l\right)(p) &= e_h\left(R\left(e_i,e_j,e_k,e_l\right)\right)(p) - R\left(\nabla_{e_h}e_i,e_j,e_k,e_l\right)(p) - R\left(e_i,\nabla_{e_h}e_j,e_k,e_l\right)(p) \\ &\quad - R\left(e_i,e_j,\nabla_{e_h}e_k,e_l\right)(p) - R\left(e_i,e_j,e_k,\nabla_{e_h}e_l\right)(p) \\ &= e_h\left(R_{ijkl}\right)(p) - R\left(0,e_j,e_k,e_l\right)(p) - R\left(e_i,0,e_k,e_l\right)(p) \\ &\quad - R\left(e_i,e_j,0,e_l\right)(p) - R\left(e_i,e_j,e_k,0\right)(p) \\ &= e_h\left(R_{ijkl}\right)(p) = \gamma'\left(R_{ijkl}\right)(0) = \left.\frac{dR_{ijkl}(\gamma(t))}{dt}\right|_{t=0} = \left.\frac{dR_{ijkl}(t)}{dt}\right|_{t=0} \\ &= \lim_{t\to 0}\frac{R_{ijkl}(t) - R_{ijkl}(-t)}{2t}. \end{aligned}$$

Because $\sigma\colon B_\varepsilon(p) \to B_\varepsilon(p)$ is an isometry, it doesn't change the connection and then the curvature. Denote $\widetilde{R}$ the curvature of $\sigma(M)$ and then we have $\widetilde{R} = R$. Also because the geodesic frame $e_1, \cdots, e_n$ are generated by parallel transports along geodesics, we have $e_i(\gamma(t)) = e_i(t)$ for any $1 \leq i \leq n$. Then by Exrecise 8.13, we have

$$\begin{aligned} R_{ijkl}(t) &= R\left(e_i(t), e_j(t), e_k(t), e_l(t)\right) \\ &= \widetilde{R}\left(d\sigma_{\gamma(t)} e_i(\gamma(t)), d\sigma_{\gamma(t)} e_j(\gamma(t)), d\sigma_{\gamma(t)} e_k(\gamma(t)), d\sigma_{\gamma(t)} e_l(\gamma(t))\right) \\ &= \widetilde{R}\left(-e_i(\gamma(-t)), -e_j(\gamma(-t)), -e_k(\gamma(-t)), -e_l(\gamma(-t))\right) \\ &= \widetilde{R}\left(e_i(-t), e_j(-t), e_k(-t), e_l(-t)\right) \\ &= R\left(e_i(-t), e_j(-t), e_k(-t), e_l(-t)\right) \\ &= R_{ijkl}(-t), \end{aligned}$$

for any $t \in (-\varepsilon, \varepsilon)$. So we conclude that

$$\nabla_{e_h} R\left(e_i, e_j, e_k, e_l\right)(p) = \lim_{t \to 0} \frac{R_{ijkl}(t) - R_{ijkl}(-t)}{2t} = \lim_{t \to 0} \frac{0}{2t} = 0.$$

Now by definition,

$$\nabla R\left(e_i, e_j, e_k, e_l, e_h\right)(p) = \nabla_{e_h} R\left(e_i, e_j, e_k, e_l\right)(p) = 0,$$

for any $1 \leq i, j, k, l, h \leq n$. By linearity, we have

$$\nabla R(p) = 0.$$

And by the arbitrariness of $p \in M$, we have

$$\nabla R = 0$$

on M, i.e. M is locally symmetric, by definition, as desired. $\square$

9 Variations of Energy

Exercise 9.1. Let M be a complete Riemannian manifold, and let $N \subset M$ be a closed submanifold of M. Let $p_0 \in M$, $p_0 \notin N$, and let $d(p_0, N)$ be the distance from p_0 to N. Show that there exists a point $q_0 \in N$ such that $d(p_0, q_0) = d(p_0, N)$ and that a minimizing geodesic which joins p_0 to q_0 is orthogonal to N at q_0.

Proof. Denote

$$N_k = \{q \in N \,|d(p_0, q) \le k\} \subset N \subset M, \qquad \forall k \in \mathbb{Z}_+.$$

Then we have

$$N_1 \subset N_2 \subset \cdots .$$

Suppose that n is the minimal integral such that $N_n \neq \varnothing$. Because N_n is the preimage of a close inequality $d(p_0, \ ) \le n$ in a close subset N, N_n is close. And it's bounded. By Hopf-Rinow Theorem (Chapter 7 Theorem 2.8 d) $\Rightarrow$ b)), N_n is a bounded close subset of complete Riemannian manifold M, it's compact. Then the continue function $d(p_0, \ )$ on $N_n \subset N$ can achieve the minimum, say there exists a $q_0 \in N_n \subset N$, such that

$$d(p_0, q_0) = \inf_{q \in N_n} d(p_0, q) = \inf_{q \in N} d(p_0, q) = d(p_0, N),$$

as desired.

Let γ be the minimizing geodesic connecting p_0 and $q_0 = \gamma(t)$. If γ is not orthogonal to N, denote $\widetilde{\gamma}'(t)$ be the projection of $\gamma'(t)$ into $T_{q_0}N$. Then the included angle of $\gamma'(t)$ and $\widetilde{\gamma}'(t)$ is a acute angle. Let's go along the geodesic $\widetilde{\gamma}$ starting at q_0 with initial velocity $-\widetilde{\gamma}'(t)$ a few to $q_1 \in N$. Denote the minimizing geodesic connecting p_0 and q_1 be $\overline{\gamma}$. We can go along $\widetilde{\gamma}$ short enough such that the included angle between γ and $\overline{\gamma}$ at p_0 is small enough such that in the triangle formed by $\gamma, \widetilde{\gamma}, \overline{\gamma}$, the included angle between $\overline{\gamma}$ and $\widetilde{\gamma}$ is an obtuse angle. So we must have

$$d(p_0, q_1) = \ell(\overline{\gamma}) < \ell(\gamma) = d(p_0, q_0),$$

which is contradict to the fact that $d(p_0, q_0) = d(p_0, N)$, i.e. q_0 achieves the minimal distant among all $q \in N$. We conclude that γ is orthogonal to N. $\square$

Exercise 9.2. Introduce a complete Riemannian metric on $\mathbb{R}^2$. Prove that

$$\lim_{r \to \infty} \inf_{x^2 + y^2 \ge r^2} K(x, y) \le 0,$$

where $(x, y) \in \mathbb{R}^2$ and $K(x, y)$ is the Gaussian curvature of the given metric at (x, y).

Proof. As we all know, the Euclidean metric on $\mathbb{R}^2$ is complete. And under this metric, all kinds of curvatures are 0 at any point. So we have

$$\lim_{r \to \infty} \inf_{x^2 + y^2 \ge r^2} K(x, y) = \lim_{r \to \infty} 0 = 0 \le 0,$$

as desired.

I don't know why this exercise is here. $\square$

Exercise 9.3. Prove that following generalization of the Theorem of Bonnet-Myers: Let M^n be a complete Riemannian manifold. Suppose that there exist constant $a > 0$ and $c \ge 0$ such that for all pairs of points in M^n and for all minimizing geodesics $\gamma(s)$, parametrized by arc length s, joining these points, we have

$$\operatorname{Ric}(\gamma'(s)) \ge a + \frac{df}{ds}, \qquad \text{along } \gamma,$$

where f is a function of s, satisfying $|f(s)| \le c$ along γ. Then M^n is compact.

Calculate an estimate for the diameter of M^n, and observe that if $f \equiv 0$ and $c = 0$, we obtain the Theorem of Bonnet-Myers.

Proof. We follow the proof of Bonnet-Myers Theorem (Chapter 9 Theorem 3.1).

If M is unbounded, let γ be a minimizing with length ℓ. We follow the proof until

$$\frac{1}{2} \sum_{j=1}^{n-1} E_j''(0) = \int_0^1 \left\{ \sin^2(\pi t) \left((n-1)\pi^2 - (n-1)\ell^2 \operatorname{Ric}_{\gamma(t)}(e_n(t)) \right) \right\} \mathrm{d}t.$$

We adjust f to arc length along γ, say $g(t) = f(\ell t)$. Then we have

$$\frac{dg}{dt} = \ell \frac{df}{dt}.$$

We still use f to denote g. Then we have

$$(n-1)\ell^2 \operatorname{Ric}_{\gamma(t)}(e_n(t)) > (n-1)\ell^2 \left(a + \frac{1}{\ell}\frac{df}{dt}\right).$$

Calculating directly, we have

$$\begin{aligned}
&\int_0^1 \left\{\sin^2(\pi t)\left((n-1)\pi^2 - (n-1)\ell^2 \operatorname{Ric}_{\gamma(t)}(e_n(t))\right)\right\} \mathrm{d}t \\
<& \int_0^1 \left\{\sin^2(\pi t)\left((n-1)\pi^2 - (n-1)\ell^2 \left(a + \frac{1}{\ell}\frac{df}{dt}\right)\right)\right\} \mathrm{d}t \\
=& (n-1)\left(\pi^2 - a\ell^2\right)\int_0^1 \sin^2(\pi t)\,\mathrm{d}t - (n-1)\ell \int_0^1 \sin^2(\pi t)\frac{df}{dt}\,\mathrm{d}t,
\end{aligned}$$

where

$$\int_0^1 \sin^2(\pi t)\,\mathrm{d}t = \frac{1}{2}$$

and

$$\int_0^1 \sin^2(\pi t)\frac{df}{dt}\,\mathrm{d}t = \int_0^1 \sin^2(\pi t)\,\mathrm{d}f = \sin^2(\pi t) f(t)\Big|_0^1 - \int_0^1 2\pi \sin(\pi t)\cos(\pi t) f\,\mathrm{d}t = -\pi \int_0^1 \sin(2\pi t) f\,\mathrm{d}t.$$

So we conclude that

$$\begin{aligned}
&\int_0^1 \left\{\sin^2(\pi t)\left((n-1)\pi^2 - (n-1)\ell^2 \operatorname{Ric}_{\gamma(t)}(e_n(t))\right)\right\} \mathrm{d}t \\
<& (n-1)\left(\pi^2 - a\ell^2\right)\frac{1}{2} + (n-1)\ell\pi \int_0^1 \sin(2\pi t) f\,\mathrm{d}t,
\end{aligned}$$

where

$$\int_0^1 \sin(2\pi t) f\,\mathrm{d}t \le \int_0^1 \left|\sin(2\pi t) f\right|\,\mathrm{d}t \le \int_0^1 \left|\sin(2\pi t)\right| c\,\mathrm{d}t = 2c.$$

So finally, we conclude that

$$\begin{aligned}
\frac{1}{2}\sum_{j=1}^{n-1} E_j''(0) &= \int_0^1 \left\{\sin^2(\pi t)\left((n-1)\pi^2 - (n-1)\ell^2 \operatorname{Ric}_{\gamma(t)}(e_n(t))\right)\right\} \mathrm{d}t \\
&< (n-1)\left(\pi^2 - a\ell^2\right)\frac{1}{2} + (n-1)\ell\pi \int_0^1 \sin(2\pi t) f\,\mathrm{d}t \\
&< (n-1)\left(\pi^2 - a\ell^2\right)\frac{1}{2} + (n-1)\ell\pi 2c = \frac{1}{2}(n-1)\left(-a\ell^2 + 4\pi c\ell + \pi^2\right),
\end{aligned}$$

where the right side is a degree 2 polynomial of ℓ with negetive coefficient of ℓ^2. So we can let $\ell \to \infty$ until

$$\frac{1}{2}\sum_{j=1}^{n-1} E_j''(0) = \frac{1}{2}(n-1)\left(-a\ell^2 + 4\pi c\ell + \pi^2\right) < 0.$$

Then we get a contradiction like the original proof. So M is bounded. The rest of the proof is same as the beginning of the original proof to conclude that M is compact. $\square$

Exercise 9.4. Let M^n be an orientable Riemannian manifold with positive curvature and even dimension. Let γ be a closed geodesic in M, that is, γ is an immersion of the circle S^1 in M that is geodesic at all of its points. Prove that γ is homotopic to a closed curve whose length is strictly less than that of γ.

Proof. Firstly, at any $p \in \gamma$, for any $v \in TpM$, define a map

$$\varphi \colon T_pM \to T_pM$$

by $\varphi(v)$ is the parallel transport of v along the closed geodesic γ at $t = 2\pi$, i.e. $\varphi(v)$ is the parallel transport of v along γ until a whole round. And because φ fixes γ', it induces a map

$$\widetilde{\varphi} \colon \left(T_p\gamma\right)^{\perp} \to \left(T_p\gamma\right)^{\perp}.$$

And we know that $\widetilde{\varphi}$ is an orthogonal linear transformation. Counting the dimensions, $\left(T_p\gamma\right)^{\perp}$ has odd dimension. Because M is orientable, $\widetilde{\varphi}$ has positive determinant. By the structure of orthogonal matrices, we know that $\widetilde{\varphi}$ has at least one non-trivial fixed point, say V. This means that the parallel transport $V(t)$ of V along γ, goes back to the original vector after a lap, i.e.

$$V(2\pi) = V(0).$$

So $V(t)$ is a well-defined parallel vector field along the closed geodesic γ.

Now let's calculate the second variation $E_V''(0)$ along γ with variational field V. By Chapter 9 Proposition 2.8, we have

$$\frac{1}{2}E_V''(0) = -\int_0^a \left\langle V(t), \frac{D^2V}{dt^2} + R\left(\frac{d\gamma}{dt}, V\right)\frac{d\gamma}{dt}\right\rangle \mathrm{d}t - \sum_{i=1}^{k}\left\langle V\left(t_i\right), \frac{DV}{dt}\left(t_i^+\right) - \frac{DV}{dt}\left(t_i^-\right)\right\rangle,$$

where because V is parallel along γ,

$$\frac{DV}{dt} = 0, \qquad \frac{D^2V}{dt^2} = 0.$$

Then because M has positive curvature, we conclude that

$$\begin{aligned}
\frac{1}{2}E_V''(0) &= -\int_0^a \left\langle V(t), \frac{D^2V}{dt^2} + R\left(\frac{d\gamma}{dt}, V\right)\frac{d\gamma}{dt}\right\rangle \mathrm{d}t - \sum_{i=1}^{k}\left\langle V\left(t_i\right), \frac{DV}{dt}\left(t_i^+\right) - \frac{DV}{dt}\left(t_i^-\right)\right\rangle \\
&= -\int_0^a \left\langle V(t), 0 + R\left(\frac{d\gamma}{dt}, V\right)\frac{d\gamma}{dt}\right\rangle \mathrm{d}t - \sum_{i=1}^{k}\left\langle V\left(t_i\right), 0\right\rangle \\
&= -\int_0^a K\left(\frac{d\gamma}{dt}(t), V(t)\right) \mathrm{d}t < -\int_0^a 0\,\mathrm{d}t = 0.
\end{aligned}$$

This means that E_V near 0 is concave. So near γ, there is a curve c in the variation whose energy is strictly less then it of γ, and so does its length. And c is closed and homotopic to γ, where the homotopy function is given by letting $\gamma(t)$ go along the geodesic starting at $\gamma(t)$ with initial velocity $V(t)$ to $c(t)$. We conclude that c is as desired. ☐

Exercise 9.5. Let N_1 and N_2 be two closed disjoint submanifolds of a compact Riemannian manifold.

1. Show that the distance between N_1 and N_2 is assumed by a geodesic γ perpendicular to both N_1 and N_2.

2. Show that, for any orthogonal variation $h(t,s)$ of γ, with $h(0,s) \in N_1$ and $h(\ell,s) \in N_2$, we have the following expression for the formula for the second variation

$$\frac{1}{2}E''(0) = I_\ell(V,V) + \left\langle V(\ell), S^{(2)}_{\gamma'(\ell)}(V(\ell))\right\rangle - \left\langle V(0), S^{(1)}_{\gamma'(0)}(V(0))\right\rangle$$

where V is the variational vector and $S^{(i)}_{\gamma'}$ is the linear map associated to the second fundamental form of N_i in the direction γ', $i = 1, 2$.

Proof. For a), by Exercise 9.1, we define a function f on N_1 as follow: For any $p \in N_1$, let $f(p) = \ell_p$ be the length of the minimizing geodesic assuming the distance of $d\left(p, N_2\right)$. It's easy to see that f is continuous. And because N_1 is a closed submanifold of a compact manifold M, it is compact. Then there is $p_0 \in N_1$

realizing the minimal value of f and $f(p_0) > 0$. Denote γ the minimizing geodesic connecting p_0 and $q_0 \in N_2$ in Exercise 9.1. We have

$$d(N_1, N_2) = \inf_{p \in N_1, q \in N_2} d(p, q) = \inf_{p \in N_1} \inf_{p \in N_1} d(p, q) = \inf_{p \in N_1} d(p, N_2) = \min f = \ell_{p_0}.$$

So γ assumes the distance of N_1 and N_2. And by Exercise 9.1, notice that

$$d(p_0, N_2) = d(N_1, N_2) \leq d(p_0, N_2),$$

γ naturally assumes $d(p_0, N_2) = d(N_1, N_2)$, so γ is perpendicular to N_2 at q_0; symmetrically, γ also assumes $d(N_1, q_0)$, so γ is also perpendicular to N_1 at p_0, as desired.

Then for b), by Chapter 9 Remark 2.10, we have

$$\frac{1}{2}E''(0) = I_\ell(V, V) - \left\langle \frac{D}{ds}\frac{\partial h}{\partial s}, \gamma' \right\rangle (0, 0) + \left\langle \frac{D}{ds}\frac{\partial h}{\partial s}, \gamma' \right\rangle (\ell, 0).$$

And by Chapter 6 Proposition 2.8, we have

$$S^{(2)}_{\gamma'(\ell)}(V(\ell)) = -\left(\nabla_V \gamma'(\ell)\right)^T,$$

where ∇ is the Riemannian connection of M. By a), $\gamma(\ell) \perp T_{q_0}N_2$ and the variation is orthogonal, $V(\ell) \in T_{q_0}N_2$. So we have

$$\left\langle V(\ell), S^{(2)}_{\gamma'(\ell)}(V(\ell)) \right\rangle = \left\langle V(\ell), -\left(\nabla_V \gamma'(\ell)\right)^T \right\rangle = \left\langle V(\ell), -\left(\nabla_V \gamma'(\ell)\right)^T - \left(\nabla_V \gamma'(\ell)\right)^N \right\rangle = \left\langle V(\ell), -\nabla_V \gamma'(\ell) \right\rangle.$$

Then because $V(\ell) = \frac{\partial h}{\partial s}(\ell, 0)$ and by Chapter 2 Corollary 3.3, we have

$$\begin{aligned}
\left\langle V(\ell), S^{(2)}_{\gamma'(\ell)}(V(\ell)) \right\rangle &= \left\langle V(\ell), -\nabla_V \gamma'(\ell) \right\rangle = -\left\langle \frac{\partial h}{\partial s}(\ell, 0), \nabla_{\frac{\partial h}{\partial s}(\ell,0)} \gamma'(\ell) \right\rangle \\
&= -\left\langle \frac{\partial h}{\partial s}, \frac{D}{ds}\gamma' \right\rangle (\ell, 0) = -\frac{d}{ds}\left\langle \frac{\partial h}{\partial s}, \gamma' \right\rangle (\ell, 0) + \left\langle \frac{D}{ds}\frac{\partial h}{\partial s}, \gamma' \right\rangle (\ell, 0),
\end{aligned}$$

where because $\frac{\partial h}{\partial s} = V$ and the variation is orthogonal, $\frac{\partial h}{\partial s}$ is orthogonal to γ', i.e. $\left\langle \frac{\partial h}{\partial s}, \gamma' \right\rangle = 0$. So we conclude

$$\begin{aligned}
\left\langle V(\ell), S^{(2)}_{\gamma'(\ell)}(V(\ell)) \right\rangle &= -\frac{d}{ds}\left\langle \frac{\partial h}{\partial s}, \gamma' \right\rangle (\ell, 0) + \left\langle \frac{D}{ds}\frac{\partial h}{\partial s}, \gamma' \right\rangle (\ell, 0) \\
&= -\frac{d}{ds}0 + \left\langle \frac{D}{ds}\frac{\partial h}{\partial s}, \gamma' \right\rangle (\ell, 0) = \left\langle \frac{D}{ds}\frac{\partial h}{\partial s}, \gamma' \right\rangle (\ell, 0).
\end{aligned}$$

Similarly, we have

$$\left\langle V(0), S^{(1)}_{\gamma'(0)}(V(0)) \right\rangle = \left\langle \frac{D}{ds}\frac{\partial h}{\partial s}, \gamma' \right\rangle (0, 0).$$

Finally, we conclude that

$$\begin{aligned}
\frac{1}{2}E''(0) &= I_\ell(V, V) - \left\langle \frac{D}{ds}\frac{\partial h}{\partial s}, \gamma' \right\rangle (0, 0) + \left\langle \frac{D}{ds}\frac{\partial h}{\partial s}, \gamma' \right\rangle (\ell, 0) \\
&= I_\ell(V, V) - \left\langle V(0), S^{(1)}_{\gamma'(0)}(V(0)) \right\rangle + \left\langle V(\ell), S^{(2)}_{\gamma'(\ell)}(V(\ell)) \right\rangle \\
&= I_\ell(V, V) + \left\langle V(\ell), S^{(2)}_{\gamma'(\ell)}(V(\ell)) \right\rangle - \left\langle V(0), S^{(1)}_{\gamma'(0)}(V(0)) \right\rangle,
\end{aligned}$$

as desired. □

Exercise 9.6. Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with curvature $K \leq 0$. Let $\gamma\colon (-\infty, \infty) \to \widetilde{M}$ be a normalized geodesic and let $p \in \widetilde{M}$ be a point which does not belong to γ. Let $d(s) = d(p, \gamma(s))$.

a) Consider the minimizing geodesic $\sigma_s\colon [0, d(s)] \to \widetilde{M}$ joining p to $\gamma(s)$, that is, $\sigma_s(0) = p$, $\sigma_s(d(s)) = \gamma(s)$. Consider the variation $h(t, s) = \sigma_s(t)$, and show that:

   (i) $\frac{1}{2}E'(s) = \langle \gamma'(s), \sigma_s'(d(s)) \rangle \underline{+\sigma_s'(d(s)) \cdot d'(s)}_?$,

   (ii) $\frac{1}{2}E''(s_0) > 0$, if $d'(s_0) = 0$.

b) Conclude from (i) that s_0 is a critical point of d if and only if $\langle \gamma'(s_0), \sigma_s'(d(s_0)) \rangle = 0$. Conclude from (ii) that d has a unique critical point, which is a minimum.

c) From b), it follows that if $\widetilde{M}$ is complete, simply connected and has curvature $K \le 0$, then a point off the geodesic γ of $\widetilde{M}$ can be connected by a unique perpendicular to γ. Show by example that the condition on the curvature and the condition of simple connectivity are essential to the theorem.

Proof. For a), by equation (2) in the proof of Chapter 9 Proposition 2.4:

$$\frac{1}{2}\frac{dE}{ds} = \sum_{i=0}^{k} \left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle \Bigg|_{t_i}^{t_{i+1}} - \int_0^a \left\langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t} \right\rangle \mathrm{d}t,$$

noticing that

$$\frac{\partial h}{\partial t}(s,t) = \sigma_s'(t), \qquad \frac{\partial h}{\partial s}(0,t) = V(t), \qquad \frac{\partial h}{\partial s}(s,0) = 0, \qquad \frac{\partial h}{\partial s}(s,d(s)) = \gamma'(s),$$

we have

$$\begin{aligned}
\frac{1}{2}E'(s) &= \left\langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial t} \right\rangle (s, d(s)) - \left\langle \frac{\partial h}{\partial s} . \frac{\partial h}{\partial t} \right\rangle (s, 0) - \int_0^{d(s)} \left\langle \frac{\partial h}{\partial s}, \frac{D}{dt}\frac{\partial h}{\partial t} \right\rangle (s,t)\,\mathrm{d}t \\
&= \langle \gamma'(s), \sigma_s'(d(s)) \rangle - \langle 0 . \sigma_s'(0) \rangle - \int_0^{d(s)} \left\langle \frac{\partial h}{\partial s}(s,t), \frac{D}{dt}\sigma_s'(s,t) \right\rangle \mathrm{d}t,
\end{aligned} \tag{7}$$

where because σ_s a is geodesic, $\frac{D}{dt}\sigma_s' = 0$. So we conclude

$$\begin{aligned}
\frac{1}{2}E'(s) &= \langle \gamma'(s), \sigma_s'(d(s)) \rangle - \langle 0 . \sigma_s'(0) \rangle - \int_0^{d(s)} \left\langle \frac{\partial h}{\partial s}(s,t), \frac{D}{dt}\sigma_s'(s,t) \right\rangle \mathrm{d}t \\
&= \langle \gamma'(s), \sigma_s'(d(s)) \rangle - \langle 0 . \sigma_s'(0) \rangle - \int_0^{d(s)} \left\langle \frac{\partial h}{\partial s}(s,t), 0 \right\rangle \mathrm{d}t \\
&= \langle \gamma'(s), \sigma_s'(d(s)) \rangle ;
\end{aligned}$$

whence (i). Then for (ii), because for any s, σ_s is a normalized geodesic, the equations in this chapter proved for $s = 0$ is valid for all s. By Chapter 9 Proposition 2.8, we have

$$\frac{1}{2}E''(s) = -\int_0^{d(s)} \left\langle V(t), \frac{D^2V}{dt^2} + R\left(\frac{d\sigma_s}{dt}, V\right)\frac{d\sigma_s}{dt} \right\rangle \mathrm{d}t - \left\langle V(0), \frac{DV}{dt}(0) \right\rangle + \left\langle V(d(s)), \frac{DV}{dt}(d(s)) \right\rangle,$$

For the first term, by Chapter 2 Corollary 3.3,

$$\left\langle V, \frac{D^2V}{dt^2} \right\rangle = \frac{d}{dt}\left\langle V, \frac{DV}{dt} \right\rangle - \left\langle \frac{DV}{dt}, \frac{DV}{dt} \right\rangle = \frac{d}{dt}\left\langle V, \frac{DV}{dt} \right\rangle - \left| \frac{DV}{dt} \right|^2.$$

And because $V(0) = 0$, the second term is zero. Then

$$\begin{aligned}
\frac{1}{2}E''(s) &= -\int_0^{d(s)} \left\langle V(t), \frac{D^2V}{dt^2} - R\left(\frac{d\sigma_s}{dt}, V\right)\frac{d\sigma_s}{dt} \right\rangle \mathrm{d}t - \left\langle V(0), \frac{DV}{dt}(0) \right\rangle + \left\langle V(d(s)), \frac{DV}{dt}(d(s)) \right\rangle \\
&= -\int_0^{d(s)} \left\langle V(t), \frac{D^2V}{dt^2} \right\rangle \mathrm{d}t - \int_0^{d(s)} \left\langle V, R\left(\frac{d\sigma_s}{dt}, V\right)\frac{d\sigma_s}{dt} \right\rangle \mathrm{d}t + \left\langle V(d(s)), \frac{DV}{dt}(d(s)) \right\rangle \\
&= -\int_0^{d(s)} \frac{d}{dt}\left\langle V, \frac{DV}{dt} \right\rangle \mathrm{d}t + \int_0^{d(s)} \left| \frac{DV}{dt} \right|^2 \mathrm{d}t - \int_0^{d(s)} K\left(\frac{d\sigma_s}{dt}, V\right) \mathrm{d}t + \left\langle V(d(s)), \frac{DV}{dt}(d(s)) \right\rangle,
\end{aligned}$$

where

$$\int_0^{d(s)} \frac{d}{dt}\left\langle V, \frac{DV}{dt}\right\rangle \mathrm{dt} = \left\langle V, \frac{DV}{dt}\right\rangle\Big|_0^{d(s)} = \left\langle V(d(s)), \frac{DV}{dt}(d(s))\right\rangle - \left\langle V(0), \frac{DV}{dt}(0)\right\rangle$$
$$= \left\langle V(d(s)), \frac{DV}{dt}(d(s))\right\rangle - \left\langle 0, \frac{DV}{dt}(0)\right\rangle = \left\langle V(d(s)), \frac{DV}{dt}(d(s))\right\rangle.$$

So we conclude

$$\begin{aligned}
\frac{1}{2}E''(s) &= -\int_0^{d(s)} \frac{d}{dt}\left\langle V, \frac{DV}{dt}\right\rangle \mathrm{dt} + \int_0^{d(s)} \left|\frac{DV}{dt}\right|^2 \mathrm{dt} - \int_0^{d(s)} K\left(\frac{d\sigma_s}{dt}, V\right)\mathrm{dt} + \left\langle V(d(s)), \frac{DV}{dt}(d(s))\right\rangle \\
&= -\left\langle V(d(s)), \frac{DV}{dt}(d(s))\right\rangle + \int_0^{d(s)} \left|\frac{DV}{dt}\right|^2 \mathrm{dt} - \int_0^{d(s)} K\left(\frac{d\sigma_s}{dt}, V\right)\mathrm{dt} + \left\langle V(d(s)), \frac{DV}{dt}(d(s))\right\rangle \\
&= \int_0^{d(s)} \left|\frac{DV}{dt}\right|^2 \mathrm{dt} - \int_0^{d(s)} K\left(\frac{d\sigma_s}{dt}, V\right)\mathrm{dt} \\
&> \int_0^{d(s)} 0, \mathrm{dt} + \int_0^{d(s)} 0\,\mathrm{dt} = 0,
\end{aligned}$$

as desired.

And for b), s_0 is a critical point of d if and only if s_0 is a critical point of E, if and only if $E'(s_0) = 0$, by (i), if and only if

$$\left\langle \gamma'(s_0), \sigma'_{s_0}(d(s_0))\right\rangle = 0.$$

And by (ii), E is strictly convex, so it has a unique critical point, which is a minimum, so does d.

Finally for c), for any point p off the geodesic γ, let s_0 be the critical point of d, by b),

$$\left\langle \gamma'(s_0), \sigma'_{s_0}(d(s_0))\right\rangle = 0.$$

So the geodesic σ_{s_0} connecting p and γ and is perpendicular to γ. So we proved the existence. And if there is another geodesic connecting p and γ and is perpendicular to γ, by b), there will be another critical point of d, which is contradict to the uniqueness of critical point of d in b). So we proved the uniqueness.

Now we turn to give the counterexamples:

For the flat torus $T^2 \cong S^1 \times S^1$ which is not simple connected, we choose a longitude geodesic

$$\gamma(t) = \left(\mathrm{e}^{\mathrm{i}\,0}, \mathrm{e}^{\mathrm{i}\,t}\right).$$

Then for the point $\left(\mathrm{e}^{\mathrm{i}\,\pi}, \mathrm{e}^{\mathrm{i}\,0}\right)$ off γ, there are two geodesic

$$\gamma_1(t) = \left(\mathrm{e}^{\mathrm{i}(\pi-t)}, \mathrm{e}^{\mathrm{i}\,0}\right), \qquad t \in [0, \pi]$$

and

$$\gamma_2(t) = \left(\mathrm{e}^{\mathrm{i}(\pi+t)}, \mathrm{e}^{\mathrm{i}\,0}\right), \qquad t \in [0, \pi]$$

connecting p and γ which are perpendicular to γ.

For the sphere S^2 whose curvature is $K = 1 > 0$, under the polar coordinates (ρ, θ) we choose the weft geodesic great circle

$$\gamma(t) = \left(\frac{\pi}{2}, \mathrm{e}^{\mathrm{i}\,t}\right).$$

Then for the north point $\left(0, \mathrm{e}^{\mathrm{i}\,0}\right)$ off γ, any geodesic

$$\widetilde{\gamma}(t) = \left(\frac{\pi}{2}t, \mathrm{e}^{\mathrm{i}\,t_0}\right), \qquad t \in [0, 1]$$

connects p and γ and is perpendicular to γ. $\square$

10 The Rauch Comparison Theorem

Exercise 10.1 (Klingenberg's Lemma)**.** Let M be a complete Riemannian manifold with sectional curvature $K \leq K_0$, where K_0 is a positive constant. Let $p, q \in M$ and let γ_0 and γ_1 be two distinct geodesics joining p to q with $\ell(\gamma_0) \leq \ell(\gamma_1)$. Assume that γ_0 is homotopic to γ_1, that is, there exists a continuous family of curves α_t, $t \in [0,1]$ such that $\alpha_0 = \gamma_0$ and $\alpha_1 = \gamma_1$. Prove that there exists $t_0 \in [0,1]$ such that

$$\ell(\gamma_0) + \ell(\alpha_{t_0}) \geq \frac{2\pi}{\sqrt{K_0}}.$$

(Thus, the given homotopy has to pass through a "long" curve.)

Proof. We can assume that $\ell(\gamma_0) < \frac{\pi}{\sqrt{K_0}}$. If not, just choose $t_0 = 0$ and we have

$$\ell(\gamma_0) + \ell(\alpha_{t_0}) = 2\ell(\gamma_0) \geq \frac{2\pi}{\sqrt{K_0}},$$

as desired.

By Chapter 10 Proposition 2.4, in $B_{\frac{\pi}{\sqrt{K_0}}}(p)$, there is no conjugate point of p. By Chapter 5 Proposition 3.5,

$$\exp_p : T_pM \to M$$

has no critical point in $B_{\frac{\pi}{\sqrt{K_0}}}(0)$, i.e. $\exp_p$ is a diffeomorphism on it. Then because $\ell(\gamma_0) < \frac{\pi}{\sqrt{K_0}}$, for small t, α_t has a whole lift $\widetilde{\alpha_t}$ such that $\exp_p(\widetilde{\alpha_t}) = \alpha_t$. In fact, $\widetilde{\alpha_t}(s) = s\alpha_t'(0)$.

Denote $B = B_{\frac{\pi}{\sqrt{K_0}}}(0) \subset T_pM$. We claim that for all $\varepsilon > 0$, there is a $t(\varepsilon)$ such that $\alpha_{t(\varepsilon)}$ has a lift $\widetilde{\alpha_{t(\varepsilon)}}$ and $\widetilde{\alpha_{t(\varepsilon)}}$ contains points with distance $< \varepsilon$ from the boundary ∂B of B: If not, say for a $\varepsilon > 0$, for any α_t who has a lift $\widetilde{\alpha_t}$,

$$d(\widetilde{\alpha_t}, \partial B) \geq \varepsilon.$$

Denote $T \subset [0,1]$ the set of t's such that α_t has a lift. By continuity, any convergent sequence of T can not reach the boundary, so T is open. On the other hand, T is the preimage of some continuous maps (induced by the homotopy and lifts)

$$t \mapsto \alpha_t \mapsto \widetilde{\alpha_t}$$

and a closed inequality

$$d(\widetilde{\alpha_t}, \partial B) \geq \varepsilon,$$

Then T is close. So we conclude that T is open and close in $[0,1]$. Then we must have $T = [0,1]$. Especially, $1 \in T$, i.e. α_1 has a lift. But if that, say $\exp_p(\widetilde{\alpha_1}) = \alpha_1$, we have $\widetilde{\alpha_1}(s) = s\alpha_1'$. Then

$$\exp_p(\alpha_1') = \exp_p(\widetilde{\alpha_1}(1)) = \alpha_1(1) = q = \alpha_0(1) = \exp_p(\widetilde{\alpha_0}(1)) = \exp_p(\alpha_0').$$

This means we can construct a Jacobi field J in $\exp_p(B)$ between $\gamma_0 = \alpha_0$ and $\gamma_1 = \alpha_1$. In fact, let $f(s,t) = \exp_p(s\alpha_t'(0))$ and the following are standard calculations of Jaboci field (see Chapter 5). Notice that γ_0 and γ_1 has the same end point q, so $J(1) = 0$. So q is a conjugate point of p with distance of p

$$d \leq \ell(\gamma_0) < \frac{\pi}{\sqrt{K_0}},$$

which is contradict to our fact that p has no conjugate point in $B_{\frac{\pi}{\sqrt{K_0}}}(p)$, obtained at the beginning of our proof. Now we have proved our claim.

Then for any $\varepsilon > 0$, denote r be the point of $\alpha_{t(\varepsilon)}$ such that $d(r, \partial B) < \varepsilon$. We have $d(p,r) > \frac{\pi}{\sqrt{K_0}} - \varepsilon$. Then

$$\begin{aligned} \ell(\gamma_0) + \ell(\alpha_{t(\varepsilon)}) &= \ell(\gamma_0) + \ell\left(\alpha_{t(\varepsilon)}\big|_p^r\right) + \ell\left(\alpha_{t(\varepsilon)}\big|_r^q\right) \\ &\geq d(p,q) + d(p,r) + d(r,q) = \underline{d(p,q) + d(r,q)} + d(p,r) \geq 2d(p,r) > \frac{2\pi}{\sqrt{K_0}} - 2\varepsilon, \end{aligned}$$

where the penultimate inequality is by triangle inequality. Now we can choose a sequnece $\varepsilon_n \to 0$. And because $\{t(\varepsilon_n)\}_{n=1}^{\infty} \subset [0,1]$, it has a convergent subsequence. Denote its limit by $t_0 \in [0,1]$. Then because the homotopy $t \to \alpha_t$ is continuous, let $n \to \infty$ in the inequality

$$\ell(\gamma_0) + \ell\left(\alpha_{t(\varepsilon_n)}\right) > \frac{2\pi}{\sqrt{K_0}} - 2\varepsilon_n,$$

we have

$$\ell(\gamma_0) + \ell(\alpha_{t_0}) \geq \frac{2\pi}{\sqrt{K_0}},$$

i.e. we have proved the existence of $t_0 \in [0,1]$, as desired. □

Exercise 10.2. Use Klingenberg's Lemma (Exercise 10.1) for the proof of Hadamard's Theorem (Chapter 7 Theorem 3.1).

Proof. We firstly claim that for any $p, q \in M$, there exists unique geodesic connecting p and q: If not, say γ_0 and γ_1 are two geodesics connecting p and q. Because M is simply connected, they are homotopic. (Because $\gamma_0 \cdot \gamma_1^{-1}$ is a loop in M, it is homotopic to a point. Then we can cut off the homotopy to get a homotopy from γ_0 to γ_1.) Assume $\ell(\gamma_1) \leq \ell(\gamma_1)$.By Klingenberg's Lemma (Exercise 10.1), for any $n \in \mathbb{N}_+$, let $K_0 = \frac{1}{n}$, and then we have

$$K(p,\sigma) \leq 0 < \frac{1}{n} = K_0,$$

so there exists $t_n \in [0,1]$ such that in the homotopy α_t, $t \in [0,1]$,

$$\ell(\gamma_0) + \ell(\alpha_{t_n}) \geq \frac{2\pi}{\sqrt{K_0}} = 2\pi\sqrt{n},$$

i.e.

$$\ell(\alpha_{t_n}) \geq 2\pi\sqrt{n} - \ell(\gamma_0).$$

Now we get a sequence of curves $\{\alpha_{t_n}\}_{n=1}^{\infty}$ in the homotopy whose lengths become infinity when $n \to \infty$, which is impossible, because γ_0 and γ_1 have finite lengths. So we have proved our claim.

Fix $p \in M$. For any $q \in M$, by we can choose a minimizing geodesic γ form p to q. Because M is complete, we have $\exp_p(\ell(\gamma)\gamma'(0)) = q$. So $\exp_p$ is surjective. On the other hand, if there is another vector v which is not parallel to $\gamma'(0)$ such that $\exp_p(v) = q$, then because M is complete, $\exp_p(tv)$ is another geodesic connecting p and q, which is contradict to our claim. So $\exp_p$ is injective. Naturally, $\exp_p$ is differentiable. Finally, we conclude that $\exp_p\colon T_pM \to M$ is a diffeomorphism; whence Hadamard's Theorem (Chapter 7 Theorem 3.1). □

Exercise 10.3. Let M be a complete Riemannian manifold with non-positive sectional curvature. Prove that

$$\left|\left(d\exp_p\right)_v(w)\right| \geq |w|,$$

for all $p \in M$, all $v \in T_pM$ and all $w \in T_v(T_pM)$.

Proof. This is just a special case of Rauch Comparison Theorem for $\widetilde{M} = \mathbb{R}^n$.

For any $p \in M$, $v \in T_pM$, $w \in T_v(T_pM)$, let $\gamma(t) = \exp_p(tv)$ and $\widetilde{\gamma}(t) = \exp_{\widetilde{p}}(tv)$ be two geodesics in M and $\widetilde{M} = \mathbb{R}^n$, respectively. Now we can choose a curve $v(s)$ in T_pM such that $v(0) = v$ and $v'(0) = w$. Define $f(s,t) = \exp_p(tv(s))$. Then

$$J(t) = \frac{\partial f}{\partial s}(0,t) = \left(d\exp_p\right)_{tv}(tw)$$

is a Jacobi field of M along γ, with $J(0) = 0$ and $J'(0) = w$, by Chapter 5 Corollary 2.5. Similarly, we can construct a Jacobi field of M along γ

$$\widetilde{J}(t) = \frac{\partial \widetilde{f}}{\partial s}(0,t) = \left(d\exp_{\widetilde{p}}\right)_{tv}(tw),$$

with $\widetilde{J}(0) = 0$ and $\widetilde{J}'(0) = w$, where $\widetilde{f}(s,t) = \exp_{\widetilde{p}}(t\widetilde{v}(s))$, where $\widetilde{v}(s)$ is a curve in $T_{\widetilde{p}}\widetilde{M} = T_{\widetilde{p}}\mathbb{R}^n$ such that $\widetilde{v}(0) = v$ and $\widetilde{v}'(0) = w$.

Now we can use the Rauch Comparison Theorem (Chapter 10 Theorem 2.3). We already have

$$J(0) = \widetilde{J}(0) = 0.$$

Notice that

$$J'(0) = \widetilde{J}'(0) = w, \qquad \gamma'(0) = \widetilde{\gamma}'(0) = v,$$

we have

$$\langle J'(0), \gamma'(0) \rangle = \langle w, v \rangle = \left\langle \widetilde{J}'(0), \widetilde{\gamma}'(0) \right\rangle, \qquad |J'(0)| = |w| = \left| \widetilde{J}'(0) \right|.$$

Obviously,

$$\widetilde{f}(s,t) = \exp_{\widetilde{p}}\left(t\widetilde{v}(s)\right) = \widetilde{p} + t\widetilde{v}(s),$$

and then

$$\widetilde{J}(t) = \frac{\partial \widetilde{f}}{\partial s}(0,t) = tw.$$

So $\widetilde{\gamma} \subset \widetilde{M} = \mathbb{R}^n$ has no conjugate point. And because M has non-positive curvature and $\widetilde{M} = \mathbb{R}^n$ has zero curvature, we have

$$\widetilde{K}\left(\widetilde{x}, \widetilde{\gamma}'(t)\right) = 0 \geq K\left(x, \gamma'(t)\right)$$

for all t and all $x \in T_{\gamma(t)}M$, $\widetilde{x} \in T_{\widetilde{\gamma}(t)}\widetilde{M}$. By the Rauch Comparison Theorem (Chapter 10 Theorem 2.3), we have

$$\left|\widetilde{J}\right| \leq |J|$$

for all t. Especially, at $t = 1$, we have

$$|w| = \left|\widetilde{J}(1)\right| \leq |J(1)| = \left|\left(d\exp_p\right)_v(w)\right|,$$

i.e.

$$\left|\left(d\exp_p\right)_v(w)\right| \geq |w|,$$

as desired. □

Exercise 10.4 (Focal sets of plane curves)**.** a) Let $C \subset \mathbb{R}^2$ be a regular curve. Show that the focal set $F(C) \subset \mathbb{R}^2$ of C is obtained by taking, on the positive normal n at $p \in C$ a length equal to $\frac{1}{k}$, where k is the curvature of C at p.

b) Show that the focal set of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given by

$$\left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, (ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = \left(a^2 - b^2\right)^{\frac{2}{3}} \right\}.$$

c) Show that the focal setof the curve

$$t \mapsto (\cos t + t\sin t, -\sin t + t\cos t)$$

is the circle $t \mapsto (\cos t, -\sin t)$.

Proof. For a), with the same argument of Chapter 10 Example 4.6 until the last sentence. Because plane curves have dimension 1, they have only one type of curvature. So we conclude $\mathbf{x} + tn$ is a focal point if and only if $\frac{1}{t}$ is an eigenvalue of (h_{ij}), that is, $\frac{1}{t}$ is the curvature of C at p. This means the focal set of C is obtained by taking, on the positive normal n at $p \in C$ a length equal to $\frac{1}{k}$, where k is the curvature of C at p, as desired.

In the following proof, we will use C to denote plane curves.

Then for the case b), the ellipse can be written as

$$C(t) = (x(t), y(t)) = (a\cos t, b\sin t).$$

Then we have

$$v(t) = (-a\sin t, b\cos t), \qquad n(t) = \frac{1}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}(-b\cos t, -a\sin t).$$

Using the equation

$$k(t)=\frac{x'y''-x''y'}{\left((x')^2+(y')^2\right)^{\frac{3}{2}}},$$

the curvature of C will be

$$k(t)=\frac{ab}{\left((x')^2+(y')^2\right)^{\frac{3}{2}}}.$$

By a), we can calculate the focal set of C directly:

$$F(C(t))=(x(t),y(t))=\mathbf{x}(t)+\frac{1}{k(t)}n(t)=\left(\frac{\left(a^2-b^2\right)\cos^3 t}{a},-\frac{\left(a^2-b^2\right)\sin^3 t}{b}\right),$$

i.e.

$$\left(\frac{ax(t)}{a^2-b^2}\right)^{\frac{2}{3}}=\cos^2 t,\qquad \left(-\frac{by(t)}{a^2-b^2}\right)^{\frac{2}{3}}=\left(\frac{by(t)}{a^2-b^2}\right)^{\frac{2}{3}}=\sin^2 t.$$

So we have

$$\left(\frac{ax(t)}{a^2-b^2}\right)^{\frac{2}{3}}+\left(\frac{by(t)}{a^2-b^2}\right)^{\frac{2}{3}}=1,$$

i.e.

$$(ax(t))^{\frac{2}{3}}+(ax(t))^{\frac{2}{3}}=\left(a^2-b^2\right)^{\frac{2}{3}}.$$

Finally, we conclude that the focal set of C is

$$F(C)=\left\{(x,y)\in\mathbb{R}^2\,\middle|\,(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=\left(a^2-b^2\right)^{\frac{2}{3}}\right\},$$

as desired.

And for another case c), we can repeat the calculations of b) for the curve

$$C(t)=(\cos t+t\sin t,-\sin t+t\cos t)\,.$$

Firstly we have

$$v(t)=(t\cos t,-t\sin t)\,,\qquad n(t)=(-\sin t,-\cos t)\,,\qquad k(t)=\frac{1}{t}.$$

Then

$$F(C(t))=\mathbf{x}(t)+\frac{1}{k(t)}n(t)=(\cos t,-\sin t)\,.$$

We conclude that the focal set of C is the circle $t\mapsto(\cos t,-\sin t)$, as desired. $\square$

Exercise 10.5 (The Sturm Comparison Theorem)**.** In this exercise we present a direct proof of Rauch's Theorem in dimension two, without using material from the present chapter. We will indicate a proof of the Theorem of Sturm mentioned in the Introduction to the chapter.

Let

$$\begin{aligned} f''(t)+K(t)f(t)&=0, && f(0)=0, && t\in[0,\ell],\\ \widetilde{f}''(t)+\widetilde{K}(t)\widetilde{f}(t)&=0, && \widetilde{f}(0)=0, && t\in[0,\ell],\end{aligned}$$

be two ordinary differential equations. Suppose that $\widetilde{K}(t)\geq K(t)$ for $t\in[0,\ell]$, and that $f'(0)=\widetilde{f}'(0)=1$.

a) Show that for all $t\in[0,\ell]$,

$$\begin{aligned} 0&=\int_0^t\left(\widetilde{f}\,(f''+Kf)-f\left(\widetilde{f}''+\widetilde{K}\widetilde{f}\right)\right)\mathrm{d}s\\ &=\left(\widetilde{f}f'-f\widetilde{f}'\right)\Big|_0^t+\int_0^t\left(K-\widetilde{K}\right)f\widetilde{f}\,\mathrm{d}s. \end{aligned}\tag{8}$$

Conclude from this that the first zero of f does not occur before the first zero of $\widetilde{f}$ (that is, if $\widetilde{f}(t)>0$ on $(0,t_0)$ and $\widetilde{f}\,(t_0)=0$, then $f(t)>0$ on $(0,t_0)$).

b) Suppose that $\widetilde{f}(t) > 0$ on $(0, \ell]$. Use equation (8) and the fact that $f(t) > 0$ on $(0, \ell]$ to show that $f(t) \geq \widetilde{f}(t)$, $t \in [0, \ell]$, and that the equality is verified for $t = t_1 \in (0, \ell]$ if and only if $K(t) = \widetilde{K}(t)$, $t \in [0, t_1]$.

Verify that this is the Theorem of Rauch in dimension two.

Proof. Firstly for a), we have

$$\int_0^t \left(\widetilde{f}\,(f'' + Kf) - f\left(\widetilde{f}'' + \widetilde{K}\widetilde{f}\right)\right) \mathrm{d}s = \int_0^t \left(\widetilde{f} \cdot 0 - f \cdot 0\right) \mathrm{d}s = 0.$$

On the other side,

$$\int_0^t \left(\widetilde{f}\,(f'' + Kf) - f\left(\widetilde{f}'' + \widetilde{K}\widetilde{f}\right)\right) \mathrm{d}s = \int_0^t \widetilde{f}f'' \,\mathrm{d}s - \int_0^t f\widetilde{f}'' \,\mathrm{d}s + \int_0^t \left(K - \widetilde{K}\right) f\widetilde{f} \,\mathrm{d}s,$$

where for the first term, we have

$$\int_0^t \widetilde{f}f'' \,\mathrm{d}s = \widetilde{f}f'\Big|_0^t - \int_0^t \widetilde{f}'f' \,\mathrm{d}s$$

and similarly, for the second term, we have

$$\int_0^t f\widetilde{f}'' \,\mathrm{d}s = f\widetilde{f}'\Big|_0^t - \int_0^t f'\widetilde{f}' \,\mathrm{d}s.$$

Then we conclude

$$\begin{aligned}
\int_0^t \left(\widetilde{f}\,(f'' + Kf) - f\left(\widetilde{f}'' + \widetilde{K}\widetilde{f}\right)\right) \mathrm{d}s &= \int_0^t \widetilde{f}f'' \,\mathrm{d}s - \int_0^t f\widetilde{f}'' \,\mathrm{d}s + \int_0^t \left(K - \widetilde{K}\right) f\widetilde{f} \,\mathrm{d}s \\
&= \widetilde{f}f'\Big|_0^t - \int_0^t \widetilde{f}'f' \,\mathrm{d}s - f\widetilde{f}'\Big|_0^t + \int_0^t f'\widetilde{f}' \,\mathrm{d}s + \int_0^t \left(K - \widetilde{K}\right) f\widetilde{f} \,\mathrm{d}s \\
&= \left(\widetilde{f}f' - f\widetilde{f}'\right)\Big|_0^t + \int_0^t \left(K - \widetilde{K}\right) f\widetilde{f} \,\mathrm{d}s.
\end{aligned}$$

So we get equation (8). Then we suppose that $\widetilde{f}(t) > 0$ on $(0, t_0)$ and $s \in (0, t_0)$ is the first zero of f in $(0, t_0)$. Let $t = s$ in equation (8), we have

$$\begin{aligned}
\left(\widetilde{f}f' - f\widetilde{f}'\right)\Big|_0^t &= \left(\widetilde{f}(s)f'(s) - f(s)\widetilde{f}'(s)\right) - \left(\widetilde{f}(0)f'(0) - f(0)\widetilde{f}'(0)\right) \\
&= \left(\widetilde{f}(s)f'(s) - 0\widetilde{f}'(s)\right) - \left(0f'(0) - 0\widetilde{f}'(0)\right) = \widetilde{f}(s)f'(s).
\end{aligned}$$

Because s is the first zero of f and $f(0) = 0$ and $f'(0) = 1$, $f'(s) \leq 0$. Notice that $\widetilde{f}(s) > 0$, $\widetilde{K} \geq K$ and $f, \widetilde{f} \geq 0$ on $(0, s)$, we have

$$0 = \left(\widetilde{f}f' - f\widetilde{f}'\right)\Big|_0^s + \int_0^s \left(K - \widetilde{K}\right) f\widetilde{f} \,\mathrm{d}t = \widetilde{f}(s)f'(s) + \int_0^s \left(K - \widetilde{K}\right) f\widetilde{f} \,\mathrm{d}t \leq 0.$$

So the equation must hold. Then we must have $\widetilde{K} = K$ on $(0, s)$. By the uniqueness of the solution of ODEs, we have $\widetilde{f} = f$ on $(0, s)$, which is contradict to our assumption that $f(s) = 0$ and $\widetilde{f} > 0$ on $(0, t_0)$. We conclude that if $\widetilde{f}(t) > 0$ on $(0, t_0)$ and $\widetilde{f}\,(t_0) = 0$, then $f(t) > 0$ on $(0, t_0)$, i.e. the first zero of f does not occur before the first zero of $\widetilde{f}$, as desired.

Then for b), by equation (8) and a), for any $t \in (0, \ell]$,

$$\begin{aligned}
0 \leq - \int_0^t \left(K - \widetilde{K}\right) f\widetilde{f} \,\mathrm{d}s &= \left(\widetilde{f}f' - f\widetilde{f}'\right)\Big|_0^t \\
&= \left(\widetilde{f}(t)f'(t) - f(t)\widetilde{f}'(t)\right) - \left(\widetilde{f}(0)f'(0) - f(0)\widetilde{f}'(0)\right) \\
&= \left(\widetilde{f}(t)f'(t) - f(t)\widetilde{f}'(t)\right) - \left(0f'(0) - 0\widetilde{f}'(0)\right) = \widetilde{f}(t)f'(t) - f(t)\widetilde{f}'(t). \qquad (9)
\end{aligned}$$

Because $f, \widetilde{f} > 0$ on $(0, \ell]$ by a), we have

$$\frac{f'}{f} \geq \frac{\widetilde{f}'}{\widetilde{f}},$$

at any $(0, \ell]$. This means

$$(\ln f)' = \frac{f'}{f} \geq \frac{\widetilde{f}'}{\widetilde{f}} = \left(\ln \widetilde{f}\right)'.$$

Let $0 < s \leq t \leq \ell$ and integrating the equation above from s to t, we have

$$\ln f(t) - \ln f(s) = \int_s^t (\ln f)' \, \mathrm{d}u \geq \int_s^t \left(\ln \widetilde{f}\right)' \, \mathrm{d}u = \ln \widetilde{f}(t) - \ln \widetilde{f}(s),$$

i.e.

$$\frac{f(t)}{\widetilde{f}(t)} = \ln f(t) - \ln f(s) \geq \ln \widetilde{f}(t) - \ln \widetilde{f}(s) = \frac{f(s)}{\widetilde{f}(s)}.$$

This is means $\frac{f}{\widetilde{f}}$ is monotonic increasing on $(0, \ell]$. And by L'Hospital rule, we have

$$\lim_{s \to 0} \frac{f(s)}{\widetilde{f}(s)} = \lim_{s \to 0} \frac{f'(s)}{\widetilde{f}'(s)} = \frac{f'(0)}{\widetilde{f}'(0)} = \frac{1}{1} = 1.$$

So we conclude that $\frac{f}{\widetilde{f}} \geq 1$, i.e. $f \geq \widetilde{f}$ on $(0, \ell]$, because $\widetilde{f} > 0$ on $(0, \ell]$, as desired. In addition, if there exists $u \in (0, \ell]$ such that $f(u) = \widetilde{f}(u)$, by the monotonicity of $\frac{f}{\widetilde{f}}$, we have $\frac{f}{\widetilde{f}} = 1$ on $(0, u]$, i.e. $f = \widetilde{f}$ on $(0, u]$. Then back to equation (9), for any $t \in (0, u]$, we have

$$0 \leq -\int_0^t \left(K - \widetilde{K}\right) f \widetilde{f} \, \mathrm{d}s = \widetilde{f}(t) f'(t) - f(t) \widetilde{f}'(t) = 0.$$

Because $f, \widetilde{f} > 0$ on $(0, u]$, we must have $K - \widetilde{K} = 0$, i.e. $K = \widetilde{K}$, on $(0, u]$. The reverse direction is obviously.

In dimension 2, the ODEs are the Jacobi equations. So the solutions f and $\widetilde{f}$ are Jacobi fields, respectively. We verify that this theorem is a special case of the Rauch Comparison Theorem in dimension 2. ☐

Exercise 10.6 (The Oscillation Theorem of Sturm)**.** What follows is a slight generalization of Sturm's Comparison Theorem (Exrecise 10.5). We present the theorem in geometric form.

Let M^2 be a complete Riemannian manifold of dimension 2, and let $\gamma\colon [0, \infty) \to M^2$ be a geodesic. Let $J(t)$ be a Jacobi field along γ with $J(0) = J(t_0) = 0$, $t_0 \in (0, \infty)$, and $J(t) \neq 0$, $t \in (0, t_0)$. Then J is a field normal to γ and can be written $J(t) = f(t) e_2(t)$, where $e_2(t)$ is the parallel transport of a unit vector $e_2 \in T_{\gamma(0)} M$ with $e_2 \perp \gamma'(0)$. Because J is a Jacobi field,

$$f''(t) + K(t) f(t) = 0,$$

where K is the Gaussian curvature of M^2. Assume that

$$K(t) \leq L(t),$$

where L is adifferentiable function on $[0, \infty)$. Prove that any solution of the equation

$$\widetilde{f}''(t) + L(t) \widetilde{f}(t) = 0$$

has a zero on $[0, t_0]$, that is, there exists $t_1 \in [0, t_0]$ with $\widetilde{f}(t_1) = 0$.

Proof. If not, say $\widetilde{f} \neq 0$ on $[0, t_0]$. Because the proof of Exercise 10.5 a) is independent of the condition $f'(0) = \widetilde{f}'(0) = 1$, we still have equation (8). Let $t = t_0$, we have

$$\begin{aligned} 0 &= \left(\widetilde{f} f' - f \widetilde{f}'\right)\Big|_0^{t_0} + \int_0^{t_0} (K - L) f \widetilde{f} \, \mathrm{d}s \\ &= \left(\widetilde{f}(t_0) f'(t_0) - f(t_0) \widetilde{f}'(t_0)\right) - \left(\widetilde{f}(0) f'(0) - f(0) \widetilde{f}'(0)\right) + \int_0^{t_0} (K - L) f \widetilde{f} \, \mathrm{d}s. \end{aligned}$$

Because $0=J(0)=f(0)e_2(0)$, we have $f(0)=0$, and similarly, $f(t_0)=0$. So we conclude

$$\begin{aligned}
0 &= \left(\widetilde{f}(t_0) f'(t_0) - f(t_0)\widetilde{f}'(t_0)\right) - \left(\widetilde{f}(0) f'(0) - f(0)\widetilde{f}'(0)\right) + \int_0^{t_0} (K-L) f\widetilde{f}\,\mathrm{d}s \\
&= \left(\widetilde{f}(t_0) f'(t_0) - 0\widetilde{f}'(t_0)\right) - \left(\widetilde{f}(0) f'(0) - 0\widetilde{f}'(0)\right) + \int_0^{t_0} (K-L) f\widetilde{f}\,\mathrm{d}s \\
&= \widetilde{f}(t_0) f'(t_0) - \widetilde{f}(0) f'(0) + \int_0^{t_0} (K-L) f\widetilde{f}\,\mathrm{d}s. \qquad (10)
\end{aligned}$$

Then we need to discuss the following four cases:

1. $f>0$, $\widetilde{f}>0$ on $(0,t_0)$;
2. $f>0$, $\widetilde{f}<0$ on $(0,t_0)$;
3. $f<0$, $\widetilde{f}>0$ on $(0,t_0)$;
4. $f<0$, $\widetilde{f}<0$ on $(0,t_0)$.

For the first case, because $f(0)=f(t_0)=0$, $f\geq 0$ on $(0,t_0)$ and f has no zero in $(0,t_0)$, we have $f'(0)>0$ and $f'(t_0)<0$. Then for the right side of equation (10),

$$\widetilde{f}(t_0) f'(t_0) < 0, \qquad \widetilde{f}(0) f'(0) > 0, \qquad \int_0^{t_0} (K-L) f\widetilde{f}\,\mathrm{d}s \leq 0.$$

So we have

$$0 = \widetilde{f}(t_0) f'(t_0) - \widetilde{f}(0) f'(0) + \int_0^{t_0} (K-L) f\widetilde{f}\,\mathrm{d}s < 0,$$

which is a contradiction. For the other cases, we can get contradictions by similar $\pm$ argument. Finally, all posiible situations are impossible, so we conclude that there exists $t_1\in[0,t_0]$ such that $\widetilde{f}(t_1)=0$, i.e. $\widetilde{f}$ has a zero on $[0,t_0]$, as desired. □

Exercise 10.7 (Kneser's criterion for points conjugate in surfaces)**.** Let M^2 be a complete Riemannian manifold of dimension two and let $\gamma\colon[0,\infty)\to M^2$ be a geodesic with $\gamma(0)=p$. Let $K(s)$ be the Gaussian curvature of M^2 along γ. Assume that

$$\int_t^\infty K(s)\,\mathrm{d}s \leq \frac{1}{4(t+1)}, \qquad \forall t\geq 0, \qquad (9)$$

in the sense that the integral converges and has the bound indicated.

a) Define
$$w(t) = \int_t^\infty K(s)\,\mathrm{d}s + \frac{1}{4(t+1)},$$
and show that $w'(t)+w^2(t)\leq -K(t)$.

b) For $t\geq 0$, put $w'(t)+w^2(t)=-L(t)$ (hence $L(t)\geq K(t)$) and define
$$\widetilde{f}(t) = \exp\left(\int_0^t w(s)\,\mathrm{d}s\right), \qquad t\geq 0.$$
Show that
$$\widetilde{f}''(t) + L(t)\widetilde{f}(t) = 0, \qquad \widetilde{f}(0)=1.$$

c) Observe that $\widetilde{f}(t)>0$ and use the oscillation theorem of Sturm (Exercise 10.6) to show that there does not exist a Jacobi field $J(s)$ on $\gamma(s)$ with $J(0)=0$ and $J(s_0)=0$, for some $s_0\in(0,\infty)$. Therefore *the condition equation (9) implies that there do not exist conjugate points to p along γ.*

Proof. For a), firstly we have

$$w'(t) = -K(t) - \frac{1}{4(t+1)^2}.$$

Then to prove the desired inequality, we need to prove

$$\left(\int_t^\infty K(s)\,\mathrm{d}s + \frac{1}{4(t+1)}\right)^2 - K(t) - \frac{1}{4(t+1)^2} \le -K(t),$$

i.e.

$$\left(\int_t^\infty K(s)\,\mathrm{d}s\right)^2 + \frac{1}{2(t+1)}\int_t^\infty K(s)\,\mathrm{d}s + \frac{1}{16(t+1)^2} - K(t) - \frac{1}{4(t+1)^2} \le -K(t),$$

i.e.

$$\left(\int_t^\infty K(s)\,\mathrm{d}s\right)^2 + \frac{1}{2(t+1)}\int_t^\infty K(s)\,\mathrm{d}s - \frac{3}{16(t+1)^2} \le 0.$$

By condition (9), we have

$$\begin{aligned}\left(\int_t^\infty K(s)\,\mathrm{d}s\right)^2 + \frac{1}{2(t+1)}\int_t^\infty K(s)\,\mathrm{d}s - \frac{3}{16(t+1)^2} &\le \left(\int_t^\infty K(s)\,\mathrm{d}s\right)^2 + \frac{1}{2(t+1)}\frac{1}{4(t+1)} - \frac{3}{16(t+1)^2}\\ &= \left(\int_t^\infty K(s)\,\mathrm{d}s\right)^2 - \frac{1}{16(t+1)^2}.\end{aligned}$$

So we only need to prove

$$\left(\int_t^\infty K(s)\,\mathrm{d}s\right)^2 - \frac{1}{16(t+1)^2} \le 0,$$

i.e.

$$\left(\int_t^\infty K(s)\,\mathrm{d}s\right)^2 \le \frac{1}{16(t+1)^2}.$$

We want to use condition (9), but we need

$$\left|\int_t^\infty K(s)\,\mathrm{d}s\right| \le \frac{1}{4(t+1)}$$

instead of condition (9) and I don't know how to prove it.

For b), calculating directly, we have

$$\widetilde{f}'(t) = w(t)\exp\left(\int_0^t w(s)\,\mathrm{d}s\right) = w(t)\widetilde{f}(t).$$

Then

$$\widetilde{f}''(t) = w'(t)\widetilde{f}(t) + w(t)\widetilde{f}'(t) = w'(t)\widetilde{f}(t) + w^2(t)\widetilde{f}(t) = \left(w'(t) + w^2(t)\right)\widetilde{f}(t) = -L(t)\widetilde{f}(t),$$

i.e.

$$\widetilde{f}'' + L(t)\widetilde{f}(t) = 0,$$

as desired. In addition, we have

$$\widetilde{f}(0) = \exp\left(\int_0^0 w(s)\,\mathrm{d}s\right) = \exp(0) = 1.$$

Finally for c), use the same symbol of Exercise 10.6, let $J(t) = f(t)e_2(t)$ be a Jacobi field along γ. Then by the Jacobi equation, we have

$$f''(t) + K(t)f(t) = 0.$$

If there exists $s_0 \in (0,\infty)$ such that $J(s_0) = 0$, we have $f(s_0) = 0$. Because $L \ge K$, by the oscillation theorem of Sturm (Exercise 10.6), there must exist $t_0 \in [0, s_0]$ such that $\widetilde{f}(t_0) = 0$, which is contradict to the fact that $\widetilde{f} > 0$ on $(0,\infty)$. We conclude that there does not exist a Jacobi field $J(s)$ on $\gamma(s)$ with $J(0) = 0$ and $J(s_0) = 0$, for some $s_0 \in (0,\infty)$. This means there do not exist conjugate points to p along γ. $\square$

11 The Morse Index Theorem

Exercise 11.1. Prove the following version of the Theorem of Bonnet-Myers (Chapter 9 Theorem 3.1): *If M is complete and the sectional curvature K satisfies $K \geq \delta > 0$, then M is compact and* $\operatorname{diam} M \leq \frac{\pi}{\sqrt{\delta}}$, using the Comparison Theorem of Rauch (Chapter 10 Theorem 2.3) and the Jacobi theorem (Chapter 11 Corollary 2.9).

Proof. Choose $p \in M$. For any normalized geodesic γ starting at p, by Chapter 10 Proposition 2.4, there is a conjugate point of $\gamma(0) = p$ in $\left(0, \frac{\pi}{\sqrt{\delta}}\right]$. So we conclude that, in M, any geodesic of length greater than $\frac{\pi}{\sqrt{\delta}}$ contains a conjugate point. By Jacobi theorem (Chapter 11 Corollary 2.9), these geodesics are not minimizing. This means any minimizing geodesic of M has length less than $\frac{\pi}{\sqrt{\delta}}$. Then for any $p, q \in M$, we can choose a minimizing geodesic γ connecting p and q. Then we have

$$d(p,q) = \ell(\gamma) \leq \frac{\pi}{\sqrt{\delta}}.$$

Then

$$\operatorname{diam} M = \sup_{p,q \in M} d(p,q) \leq \frac{\pi}{\sqrt{\delta}},$$

as desired. And so M is bounded. Because M is complete and then naturally closed, by Hopf-Rinow Theorem (Chapter 7 Theorem 2.8 d) $\Rightarrow$ b)), M is compact. ☐

Exercise 11.2. Prove the following inequality on real function (Wirtinger's inequality). Let $f\colon [0,\pi] \to \mathbb{R}$ be a real function of class C^2 such that $f(0) = f(\pi) = 0$. Then

$$\int_0^\pi f^2 \,\mathrm{d}t \leq \int_0^\pi (f')^2 \,\mathrm{d}t,$$

and equality occurs if and only if $f(t) = c \sin t$, where c is a constant. (In the next exercise, we use this fact to prove an interesting geometric fact.)

Proof. We give a geometric proof following the hint.

Let $p, -p$ be two antipodal points of S^2 and γ a normalized geodesic connecting them. Let $v(t)$ be a unit parallel field along γ normal to $\gamma'(t)$. Denote $V = fv$. Because v is parallel along γ, $V' = f'v$, $V'' = f''v$. Then because $f(0) = f(\pi) = 0$, $V(0) = V(\pi) = 0$. So the variation of γ along V is proper. By Chapter 9 Remark 2.10 and Proposition 2.8, we have

$$\begin{aligned}
I_\pi(V,V) &= \frac{1}{2}E''(0) = -\int_0^\pi \langle V, V'' + R(\gamma', V)\gamma' \rangle \,\mathrm{d}t = -\int_0^\pi \langle V, V'' \rangle \,\mathrm{d}t - \int_0^\pi \langle R(\gamma', V)\gamma', V \rangle \,\mathrm{d}t \\
&= -\int_0^\pi \langle fv, f''v \rangle \,\mathrm{d}t - \int_0^\pi \langle R(\gamma', fv)\gamma', fv \rangle \,\mathrm{d}t = -\int_0^\pi ff'' \langle v, v \rangle \,\mathrm{d}t - \int_0^\pi f^2 \langle R(\gamma', v)\gamma', v \rangle \,\mathrm{d}t \\
&= -\int_0^\pi ff'' 1 \,\mathrm{d}t - \int_0^\pi f^2 1 \,\mathrm{d}t = -\int_0^\pi ff'' \,\mathrm{d}t - \int_0^\pi f^2 \,\mathrm{d}t.
\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}
I_\pi(V,V) &= -\int_0^\pi ff'' \,\mathrm{d}t - \int_0^\pi f^2 \,\mathrm{d}t \\
&= -ff'|_0^\pi + \int_0^\pi (f')^2 \,\mathrm{d}t - \int_0^\pi f^2 \,\mathrm{d}t = f(\pi)f'(\pi) - f(0)f'(0) + \int_0^\pi (f')^2 \,\mathrm{d}t - \int_0^\pi f^2 \,\mathrm{d}t \\
&= 0f'(\pi) - 0f'(0) + \int_0^\pi (f')^2 \,\mathrm{d}t - \int_0^\pi f^2 \,\mathrm{d}t = \int_0^\pi (f')^2 \,\mathrm{d}t - \int_0^\pi f^2 \,\mathrm{d}t.
\end{aligned}$$

As we all know, $\gamma(0) = p$ has no conjugate point before $\gamma(\pi) = -p$. Then by Morse Index Theorem (Chapter 11 Theorem 2.2), I_π has zero index. That's to say, I_π is semi-positive definite. Then $I_\pi(V,V) \geq 0$, i.e.

$$\int_0^\pi (f')^2 \,\mathrm{d}t - \int_0^\pi f^2 \,\mathrm{d}t = I_\pi(V,V) \geq 0,$$

i.e.

$$\int_0^\pi f^2 \, \mathrm{d}t \le \int_0^\pi (f')^2 \, \mathrm{d}t.$$

as desired. And if the quality holds, we have $I_\pi(V,V) = 0$. By Chapter 11 Proposition 2.3, V is a Jaboci field along γ. So V should satisfies the Jacobi equation

$$V'' + R(\gamma', V)\gamma' = 0,$$

i.e.

$$f''v + R(\gamma', fv)\gamma' = 0,$$

i.e.

$$f''v + fR(\gamma', v)\gamma' = 0.$$

In the case $M = S^2$,

$$\langle R(\gamma', v)\gamma', v\rangle = R(\gamma', v) = 1$$

and

$$\langle R(\gamma', v)\gamma', \gamma'\rangle = R(\gamma', v, \gamma', \gamma') = 0,$$

we have

$$R(\gamma', v)\gamma' = v.$$

Then the Jacobi equation will be

$$f''v + fv = 0,$$

i.e.

$$f'' + f = 0.$$

Solving the ODE with initial conditions $f(0) = f(\pi) = 0$, we have

$$f(t) = c \sin t,$$

where c is a constant, as desired. □

Exercise 11.3. Let M^n be a complete simply connected n-dimensional Riemannian manifold. Suppose that for each point $p \in M$, the locus $C(p)$ of (first) conjugate points of p reduces to a unique point $q \neq p$ and that $d(p, C(p)) = \pi$. Prove that, if the sectional curvature K of M satisfies $K \le 1$, then M is isometric to the sphere S^n with constant curvature 1.

Proof. Let $\gamma \colon [0,\pi] \to M$ be a normalized geodesic of M joining p and q and $J(t)$ a Jacobi field along $\gamma(t)$ with $J(0) = J(\pi) = 0$ and $\langle J, \gamma'\rangle = 0$. Extend $\gamma'(t)$ to a orthonormal basis $e_1(t), \cdots, e_{n-1}(t), \gamma'(t)$ of $T\gamma$. Then we can suppose $J(t) = \sum_{i=1}^{n-1} a_i(t)e_i(t)$, where $a_i(0) = a_i(\pi) = 0$ for all $1 \le i \le n-1$ and then $J'(t) = \sum_{i=1}^{n-1} a_i'(t)e_i(t)$, $J''(t) = \sum_{i=1}^{n-1} a_i''(t)e_i(t)$. Denote $K(t) = K(\gamma'(t), J(t))$ along γ. Because J is a Jacobi field along γ, by Chapter 11 Proposition 2.3, Chapter 9 Remark 2.10 and Proposition 2.8,

$$0 = I_\pi(J,J) = -\int_0^\pi \langle J, J'' + R(\gamma', J)\gamma'\rangle \, \mathrm{d}t = -\int_0^\pi \langle J, J''\rangle \, \mathrm{d}t - \int_0^\pi \langle R(\gamma', J)\gamma', J\rangle \, \mathrm{d}t.$$

For the first term, by linearity, we have

$$\langle J, J''\rangle = \sum_{i,j=1}^{n-1} a_i a_j'' \langle e_i, e_j\rangle = \sum_{i=1}^{n-1} a_i a_i''.$$

Then integrating by parts, we have

$$\begin{aligned}
\int_0^\pi \langle J, J''\rangle \, \mathrm{d}t &= \sum_{i=1}^{n-1} \int_0^\pi a_i a_i'' \, \mathrm{d}t = \sum_{i=1}^{n-1} \left(a_i a_i' |_0^\pi - \int_0^\pi (a_i')^2 \, \mathrm{d}t \right) \\
&= \sum_{i=1}^{n-1} \left(a_i(\pi)a_i'(\pi) - a_i(0)a_i'(0) - \int_0^\pi (a_i')^2 \, \mathrm{d}t \right) \\
&= \sum_{i=1}^{n-1} \left(0a_i'(\pi) - 0a_i'(0) - \int_0^\pi (a_i')^2 \, \mathrm{d}t \right) = -\sum_{i=1}^{n-1} \int_0^\pi (a_i')^2 \, \mathrm{d}t.
\end{aligned}$$

And for the second term, notice that

$$|\gamma' \wedge J| = |\gamma'|^2 |J|^2 - \langle \gamma', J\rangle = 1 \sum_{i=1}^{n-1} a_i^2 - 0 = \sum_{i=1}^{n-1} a_i^2,$$

we have

$$\int_0^\pi \langle R(\gamma', J)\gamma', J\rangle \, \mathrm{d}t = \int_0^\pi |\gamma' \wedge J| K(\gamma', J) \, \mathrm{d}t = \sum_{i=1}^{n-1} \int_0^\pi a_i^2 K(t) \, \mathrm{d}t.$$

So we conclude that

$$0 = -\int_0^\pi \langle J, J''\rangle \, \mathrm{d}t - \int_0^\pi \langle R(\gamma', J)\gamma', J\rangle \, \mathrm{d}t = \sum_{i=1}^{n-1} \int_0^\pi (a_i')^2 \, \mathrm{d}t - \sum_{i=1}^{n-1} \int_0^\pi a_i^2 K(t) \, \mathrm{d}t.$$

Then by Exercise 11.2, because $K(t) \leq 1$, we have

$$0 = \sum_{i=1}^{n-1} \int_0^\pi (a_i')^2 \, \mathrm{d}t - \sum_{i=1}^{n-1} \int_0^\pi a_i^2 K(t) \, \mathrm{d}t \geq \sum_{i=1}^{n-1} \int_0^\pi a_i^2 \, \mathrm{d}t - \sum_{i=1}^{n-1} \int_0^\pi a_i^2 K(t) \, \mathrm{d}t = \sum_{i=1}^{n-1} \int_0^\pi a_i^2 (1 - K(t)) \, \mathrm{d}t \geq 0.$$

So all the equalities hold. Because $J \neq 0$, there exists some $a_i \neq 0$. Then we must have $1 - K(t) = 0$, i.e. $K \equiv 1$ on $(0, \pi)$. By the arbitrariness of $p \in M$, M has constant curvature 1. Then by Chapter 8 Theorem 4.1, M has universal covering $\widetilde{M}$ isometric to S^n. And because M is simply connected, we conclude that $M = \widetilde{M}$ is isometric to S^n, as desired. ☐

Exercise 11.4. Let $a\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function with $a(t) \geq 0$, $t \in \mathbb{R}$, and $a(0) > 0$. Prove that the solution to the differential equation

$$\frac{d^2\varphi}{dt^2} + a\varphi = 0$$

with initial conditions $\varphi(0) = 1$, $\varphi'(0) = 0$, has at least one positive zero and one negative zero.

Proof. For the positive side, if not, because $\varphi(0) = 1 > 0$, we say $\varphi > 0$ on $(0, \infty)$. Because $a \geq 0$, we have

$$\varphi'' = -a\varphi \leq 0$$

on $(0, \infty)$. This means φ' is not increasing on $(0, \infty)$. And because $\varphi'(0) = 0$, we have $\varphi' \leq 0$ on $(0, \infty)$. Notice that $\varphi(0) = 1 > 0$, there is a neighborhood $[0, \varepsilon)$ of 0 such that $\varphi > 0$ on it. Now we calculate, for any $T > \varepsilon$,

$$\varphi(T) = \varphi(0) + \int_0^T \varphi'(t) \, \mathrm{d}t = \varphi(0) + \int_0^\varepsilon \varphi'(t) \, \mathrm{d}t + \int_\varepsilon^T \varphi'(t) \, \mathrm{d}t \leq 1 + 0 + \int_\varepsilon^T \varphi'(s) \, \mathrm{d}t = 1 + \varphi'(s)(T - \varepsilon).$$

Because $a(0) = 0$,

$$\varphi''(0) = -a(0)\varphi(0) = -a(0) < 0.$$

Then φ' strictly decreases at 0. Because $\varphi'(0) = 0$, we can shrink ε such that $\varphi'(\varepsilon) < 0$. Therefore,

$$\lim_{T \to \infty} \varphi(T) = \lim_{T \to \infty} \varphi(T) \left(1 + \varphi'(s)(T - \varepsilon)\right) = -\infty < 0,$$

which is contradict to our assumption that $\varphi > 0$ on $(0, \infty)$. We conclude that φ has at least one positive zero. The negative side can be deduced by similar arguments. ☐

Exercise 11.5. Suppose that M^n is a complete Riemannian manifold with sectional curvature strictly positive and let $\gamma\colon (-\infty, \infty) \to M$ be a normalized geodesic in M. Show that there exists $t_0 \in \mathbb{R}$ such that the segment $\gamma([-t_0, t_0])$ has index greater or equal to $n - 1$.

Proof. For any unit vector field V parallel along γ and orthogonal to γ', we denote

$$\varphi_V = \langle V, R(\gamma', V)\gamma'\rangle = \frac{\langle V, R(\gamma', V)\gamma'\rangle}{|\gamma', V|} = K(\gamma', V)$$

and

$$K(t) = \inf_{\substack{V\perp\gamma' \\ |V|=1}} \varphi_V(t).$$

Because $K(t)$ is strictly positive, we can find a differentiable function on $\mathbb{R}$ such that

$$0 \le a(t) < K(t), \qquad \forall t \in \mathbb{R}$$

and

$$0 < a(0) < K(0).$$

Let φ be the solution of the ODE

$$\varphi'' + a\varphi = 0$$

with initial condition $\varphi(0) = 1$ and $\varphi'(0) = 0$. By Exercise 11.4, there exsits $t_1, t_2 > 0$ such that $\varphi(-t_1) = \varphi(t_2) = 0$. Then for any unit vector field Y parallel along γ and orthogonal to γ', denote $X = \varphi Y$, we have $X' = \varphi' Y$, $X'' = \varphi'' Y$. By Chapter 9 Remark 2.10 and Proposition 2.8,

$$\begin{aligned} I_{\gamma([-t_1,t_2])}(X,X) &= -\int_{-t_1}^{t_2} \langle X, X'' + R(\gamma', X)\gamma'\rangle \,\mathrm{d}t = -\int_{-t_1}^{t_2} \langle X, X''\rangle \,\mathrm{d}t - \int_{-t_1}^{t_2} \langle X, R(\gamma', X)\gamma'\rangle \,\mathrm{d}t \\ &= -\int_{-t_1}^{t_2} \varphi\varphi'' \langle Y, Y''\rangle \,\mathrm{d}t - \int_{-t_1}^{t_2} \varphi^2 \langle Y, R(\gamma', Y)\gamma'\rangle \,\mathrm{d}t = -\int_{-t_1}^{t_2} \varphi\varphi'' \,\mathrm{d}t - \int_{-t_1}^{t_2} \varphi^2\varphi_Y \,\mathrm{d}t. \end{aligned}$$

By definition, $\varphi_V \ge K$, then we have

$$I_{\gamma([-t_1,t_2])}(X,X) = -\int_{-t_1}^{t_2} \varphi\varphi'' \,\mathrm{d}t - \int_{-t_1}^{t_2} \varphi^2\varphi_Y \,\mathrm{d}t \le -\int_{-t_1}^{t_2} \varphi\varphi'' \,\mathrm{d}t - \int_{-t_1}^{t_2} \varphi^2 K \,\mathrm{d}t = -\int_{-t_1}^{t_2} \left(\varphi\varphi'' + \varphi^2 K\right) \mathrm{d}t.$$

By continuity, there is a neighborhood $(-\varepsilon, \varepsilon)$ of 0 such that $K > a$ on it. So we have

$$\begin{aligned} I_{\gamma([-t_1,t_2])}(X,X) &\le -\int_{-t_1}^{t_2} \left(\varphi\varphi'' + \varphi^2 K\right) \mathrm{d}t = -\left(\int_{-t_1}^{-\varepsilon} + \int_{-\varepsilon}^{\varepsilon} + \int_{\varepsilon}^{t_2}\right) \left(\varphi\varphi'' + \varphi^2 K\right) \mathrm{d}t \\ &< -\left(\int_{-t_1}^{-\varepsilon} + \int_{-\varepsilon}^{\varepsilon} + \int_{\varepsilon}^{t_2}\right) \left(\varphi\varphi'' + \varphi^2 a\right) \mathrm{d}t = -\int_{-t_1}^{t_2} \left(\varphi\varphi'' + \varphi^2 a\right) \mathrm{d}t = -\int_{-t_1}^{t_2} 0 \,\mathrm{d}t = 0. \end{aligned}$$

Then we can choose $t_0 > 0$ such that $[-t_1, t_2] \subset [-t_0, t_0]$ and extend X to $[-t_0, t_0]$ by vanishing X outside $[-t_1, t_2]$. Then we alos have

$$I_{\gamma([-t_0,t_0])}(X,X) < 0,$$

i.e. X is in the negative-definite subspace of $I_{\gamma([-t_0,t_0])}$. By the arbitrariness of Y and definition of index, we know that the index of $\gamma([-t_0, t_0])$ is not less than the dimension of the space of all unit vector fields Y parallel to γ and orthogonal to γ, which is equal to $n-1$. So we conclude that $\gamma([-t_0, t_0])$ has index greater or equal to $n-1$, as desired. □

Exercise 11.6. A *line* in a complete Riemannian manifold is a geodesic

$$\gamma\colon (-\infty, \infty) \to M$$

which minimizes the arc length between any two of its points. Show that if the sectional curvature K of M is strictly positive, M does not have any lines. By an example show that the theorem if false if $K \ge 0$.

Proof. For any geodesic $\gamma\colon (-\infty, \infty) \to M$, by Exercise 11.5, there is a positive number t such that the index of $\gamma([-t, t])$ is not less than $n-1$. By Morse Index Theorem (Chapter 11 Theorem 2.2), $\gamma(0)$ has at least one conjugate point in $\gamma([-t, t])$. And by Chapter 11 Corollary 2.9, $\gamma([-t, t])$ is not minimizing. So γ is not a line. We conclude that M does not have any lines, as desired.

For $M = \mathbb{R}^n$ under Euclidean metric with $K = 0$, as we all know, any geodesic, which is a straight line, minimizes the arc length between any two of its points. So it is a line. So the Theorem is false if $K \ge 0$. □