# Homotopy Theory and Characteristic Classes

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#### Abstract

This is the notes of a course given by Prof. Ma Langte in 25spring at Shanghai Jiaotong University.

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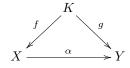
# Part I

# Homotopy Theory

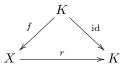
Let **TOP** be the category of topological spaces. Then we can take a quotient of **TOP** and get the homotopy category  $h-\mathbf{TOP}$ . The quotient may bring more algebraic structures. For example, Mor  $(S^1, X)$ , the homotopy classes of maps from  $S^1$  to X, is the fundamental group of X. Our goal is to study functors from hmotopy category to some algebraic categories.

Let  $\mathbf{TOP}^o$  be the pointed topological category, where the sum is wedge sum  $(X, x_0) \wedge (Y, y_0) =$  $X \sqcup Y/x_0 \sim y_0$  and the product is the smash product  $(X, x_0) \lor (Y, y_0) = X \times Y/\{x_0\} \times Y \cup X \times \{y_0\}$ . Similarly, we can take a quotient to get  $h - \mathbf{TOP}^o$ .

Let  $\mathbf{TOP}(2)$  be the category of pairs and  $h - \mathbf{TOP}(2)$  be its quotient. Fix  $K \in \mathrm{Ob}(\mathbf{TOP})$ . Let's consider  $\mathbf{TOP}^K$ , the category of spaces under K. Its objects are maps  $f: K \to X$  and morphisms are maps  $\alpha: X \to Y$  such that  $\alpha \circ f = g$ .



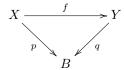
If  $K = \{*\}$  is a single point set, then  $\mathbf{TOP}^{\{*\}} = \mathbf{TOP}^o$  is the pointed topological category. Take X = K. A morphism from  $f: K \to X$  to id:  $K \to K$  is  $r: X \to K$  such that  $r \circ f = \mathrm{id}$ .



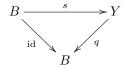
When  $K \subset X$ ,  $f = i : K \hookrightarrow X$ , we say that r is a retraction.

We have  $r: X \to K$  is a deformation retraction, if and only if  $i \circ r \simeq \mathrm{id}_X$  rel K, if and only if  $r: X \to K$  is a homotopy equivalence in  $\mathbf{TOP}^K$ .

Fix  $B \in \text{Ob}(\mathbf{TOP})$ . Let's consider  $\mathbf{TOP}_B$ , the category of spaces over B, where the objects are  $p: X \to B$  and morphisms are  $f: X \to Y$  such that  $p = q \circ f$ .



Take X = B. A morphism from id:  $B \to B$  to  $q: Y \to B$  is  $s: B \to Y$  such that  $q \circ s = \mathrm{id}_B$ .



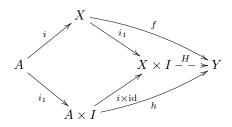
Then s is called a section of q.

Similarly, we can define  $h - \mathbf{TOP}^K$  and  $h - \mathbf{TOP}_B$ .

## 1 Cofibrations and Fibrations

#### 1.1 Cofibrations

**Definition 1.1.** A map  $i: A \to X$  has the homotopy extension property (HEP) for a space Y if for all homotopy  $h: A \times I \to Y$  and  $f: X \to Y$  with  $f \circ i(a) = h(a, 1)$ , there exists  $H: X \times I \to Y$  satisfies



We say  $i: A \to X$  is a cofibration if it has HEP for each  $Y \in \text{Ob}(\mathbf{TOP})$ .

Recall the mapping cylinder: if  $i: A \to X$  is a map, then  $Z(i) := (A \times I) \sqcup X/(a,1) \sim i(a)$ .

**Proposition 1.2.** Given a map  $i: A \to X$ . The followings are equivalent:

- 1.  $i: A \to X$  is a cofibration.
- 2. i has HEP for Z(i).

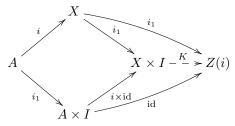
#### 3. The map

$$s \colon Z(i) \to X \times I$$
$$(a,t) \mapsto (f(a),t),$$
$$x \mapsto (x,1)$$

has a retraction.

*Proof.*  $(1)\Longrightarrow(2)$  is only by definition.

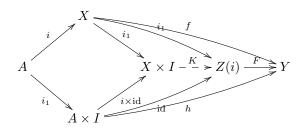
(2) $\Longrightarrow$ (1): By definition, there exists  $K\colon X\times I\to Z(i)$  such that the following diagram is commutative.



For any Y and homotopy  $h: A \times I \to Y$  and  $f: X \to Y$  with  $f \circ i(a) = h(a, 1)$ , we define

$$F: Z(i) \to Y$$
  
 $(a,t) \mapsto h(a,t)$   
 $x \mapsto f(x).$ 

Then  $F \circ K$  is as desired.

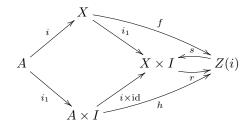


(2) $\Longrightarrow$ (3): We can easily check that the extension  $K: X \times I \to Z(i)$  in the proof of (2) $\Longrightarrow$ (1) is a retraction of s.

(3) $\Longrightarrow$ (2): Let r be a retraction of s. For any homotopy  $h: A \times I \to Z(i)$  and  $f: X \to Z(i)$  with  $f \circ i(a) = h(a, 1)$ , we define

$$\begin{split} \sigma \colon Z(i) &\to Z(i) \\ (a,t) &\mapsto h(a,t) \\ x &\mapsto f(x). \end{split}$$

Then we can verify that  $H = \sigma \circ r \colon X \times I \to Z(i)$  extends h.



**Corollary 1.3.** When  $A \subset X$  is a close subset,  $i: A \hookrightarrow X$  is the inclusion map. Then  $i: A \to X$  is a cofibration  $\iff Z(i) = A \times I \cup X \times \{1\}$  is a retraction of  $X \times I$ .

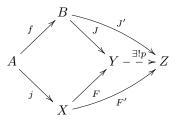
Therefore, we can construct many cofibrations. For example, let (X, A) be a manifold with boundary, then  $i \colon A \hookrightarrow X$  is a cofibration.

#### **Push-Out of Cofibration**

Given a commutative diagram,

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow j & & \downarrow J \\
X & \xrightarrow{F} & Y
\end{array}$$

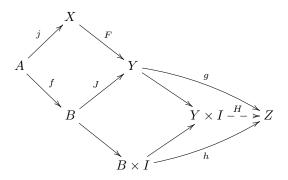
the push-out of j along f is the initial object of this diagram, i.e.  $j: B \to Y, F: X \to Y$ , s.t.  $\forall Z$  with  $J': B \to Z, F': X \to Z$  satisfying  $J' \circ f = F' \circ j$ ,  $\exists !$  map  $p: Y \to Z$  such that the diagram is commutative.



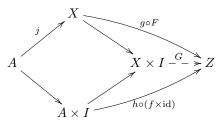
In our setting, we can construct  $Y = X \sqcup B/f(a) \sim j(a)$  directly.

**Proposition 1.4.** If  $j: A \to X$  is a cofibration, then the push-out of j along  $f: B \to Y$  is also a cofibration.

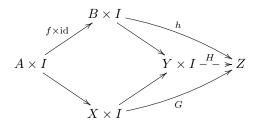
*Proof.* For any  $Z, g: Y \to Z, h: B \times I \to Z$  such that  $g \circ J = h \circ (i_1 \times id)$ , we need to find  $H: Y \times I \to Z$  such that the following diagram is commutative.



Because  $j \colon A \to X$  is a cofibration, we have  $G \colon X \times I \to Z$  such that the following diagram is commutative.



Using the fact that  $J \times \text{id} : B \times I \to Y \times I$  is also the push-out of  $j \times \text{id} : A \times I \to X \times I$  along  $f \times \text{id} : A \times I \to B \times I$ , we have unique  $H : Y \times I \to Z$  such that the following diagram is commutative.

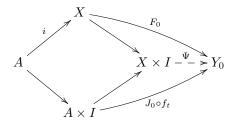


The  $H: Y \times I \to Z$  is the extension of  $h: B \times I \to Z$ , as desired.

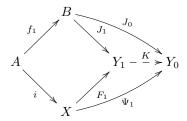
In terms of categorical language, let  $\Pi(A, B)$  be a category, whose objects are continue maps from A to B and morphisms are homotopy of maps from A to B. Consider  $\mathbf{COF}^B \subset \mathbf{TOP}^B$  the subcategory of cofibrations under B (i.e.  $J \colon B \to Y$ ). Then we have homotopy category  $h - \mathbf{COF}^B$ . Given a cofibration  $i \colon A \to X$ , we get a contravariant functor

$$\beta \colon \Pi(A,B) \to h - \mathbf{COF}^B$$
.

In fact, we only need to check that if  $f_0 \simeq f_1 \colon A \to B$ , then we get a morphism from  $J_0 \colon B \to Y_0$  to  $J_1 \colon B \to Y_1$ . Firstly, consider the homotopy  $J_0 \circ f_t \colon A \times I \to Y_0$ , we get its extension  $\Psi \colon X \times I \to Y_0$ .



Then by the universal property of the push-out  $J_1: B \to Y_1$  of i along  $f_1$  for  $J_0: B \to Y_0$  and  $\Psi_1: X \to Y_0$ , we get a map  $K: Y_1 \to Y_0$ , as desired.



## Replacing a Map by a Cofibration

Given a map  $f: X \to Y$ , consider the mapping cylinder Z(f). We can notice that Z(f) is the push-out.

$$X \xrightarrow{f} Y$$

$$\downarrow s$$

$$X \times I \xrightarrow{a} Z(f)$$

We also have a map

$$q \colon Z(f) \to Y$$
  
 $(x,t) \mapsto f(x).$ 

Note that by Proposition 1.2,  $i_1: X \hookrightarrow X \times I$  is a cofibration  $\iff X \times \{1\} \times I \cup X \times I \times \{1\}$  is a retraction of  $X \times I \times I$ , we have  $s: Y \to Z(f)$  is a cofibration.

#### Proposition 1.5. Let

$$j \colon X \to Z(f)$$
  
 $x \mapsto (x,0),$ 

we have

- 1.  $j: X \to Z(f)$  is a cofibration.
- 2.  $s \circ q \simeq \mathrm{id}_{Z(f)}$  rel Y.
- 3. If f is a cofibration, then  $q: Z(f) \to Y$  is a homotopy equicalence in  $\mathbf{TOP}^X$ .

*Proof.* (1). We construct a retraction  $R: Z(f) \times I \to X \times I \cup Z(f) \times \{1\}$  as follow. Let  $R': I \times I \to I \times \{1\} \cup \{0\} \times I$  be a retraction. Then we define

$$R \colon Z(f) \times I \to X \times I \cup Z(f) \times \{1\}$$
$$((x,s),t) \mapsto (x,R'(s,t))$$
$$(y,t) \mapsto (y,1)$$

is as desired. By Proposition 1.2,  $j: X \to Z(f)$  is a cofibration.

(2). The homotopy

$$h_t \colon Z(f) \to Z(f)$$
  
 $(x, \sigma) \mapsto (x, (1-t)\sigma + t)$ 

is as desired.

(3). By Proposition 1.2, there is a retraction  $r: Y \times I \to Z(f)$ . Define

$$g\colon Y\to Z(f)$$
 
$$y\mapsto r(y,1).$$

One can verifies that g is the homotopy inverse of g.

**Summery 1.** Any map  $f: X \to Y$  factors into

$$X \xrightarrow{j} Z \xrightarrow{q} Y$$

where  $j\colon X\to Z$  is a cofibration and  $q\colon Z\to Y$  is a homotopy equivalence. Moreover, such a factorization is unique up to homotopy equivalence. In particular, we can choose Z=Z(f). We define  $C_f=Z(f)/\operatorname{im} j$  as the homotopy cofibre of f, i.e.  $C_f=X\times I\sqcup Y/(x,0)\sim *,(x,1)\sim f(x)$ , is called the mapping cone of f.

$$X \xrightarrow{f} Y \xrightarrow{s} C_f$$

# The Cofibre Sequence (Puppe's Sequence)

To get finer structure, we work in **TOP**<sup>o</sup>. Given a map  $f: (X, x_0) \to (Y, y_0)$ , we get an induced map

$$f^* : [Y, B]^o \to [X, B]^o$$
  
 $[\alpha] \mapsto [f \circ \alpha],$ 

where  $[X, B]^o$  is the homotopy class of basepoint preserving maps. In particular, we have the constant map

$$[*]: X \to B$$
  
 $x \mapsto b_0.$ 

**Definition 1.6.** We say a sequence

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$$

in  $\mathbf{TOP}^o$  is h-coexact if  $\forall (B, b_0) \in \mathrm{Ob}(\mathbf{TOP}^o)$ ,

$$[Z,B]^o \xrightarrow{g^*} [Y,B]^o \xrightarrow{f^*} [X,B]^o$$

is exact, i.e.  $(f^*)^{-1}([*]) = \operatorname{im} g^*$ .

In **TOP**<sup>o</sup>, we consider the reduced mapping cone  $CX := X \times I/X \times \{0\} \cup \{x_0\} \times I$ . The basepoint of CX is  $X \times \{0\} \cup \{x_0\} \times I$ . And we consider the reduced mapping cone: For  $f : (X, x_0) \to (Y, y_0)$ ,  $C(f) := CX \vee Y/(x, 1) \sim f(x)$ . It is equivalent to the following push-out diagram.

$$X \xrightarrow{f} Y$$

$$\downarrow_{i_1} \qquad \qquad \downarrow_{f_1}$$

$$CX \longrightarrow C(f)$$

We will also use symbol X instead of  $(X, x_0)$  in  $\mathbf{TOP}^o$  for short.

Proposition 1.7. The sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} C(f)$$

is h-coexact.

*Proof.*  $f_1 \circ f \colon X \to C(f)$  is null-homotopic.

# Part II

# Generalized Homology

# Part III

# Characteristic Classes