

# Homotopy Theory and Characteristic Classes

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## Abstract

This is the notes of a course given by Prof. Ma Langte in 25spring at Shanghai Jiaotong University. The main textbook is *Algebraic Topology* by Tammo tom Dieck.

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# Part I

## Homotopy Theory

Let  $\mathbf{TOP}$  be the category of topological spaces. Then we can take a quotient of  $\mathbf{TOP}$  and get the homotopy category  $h\text{-}\mathbf{TOP}$ . The quotient may bring more algebraic structures. For example,  $\text{Mor}(S^1, X)$ , the homotopy classes of maps from  $S^1$  to  $X$ , is the fundamental group of  $X$ . Our goal is to study functors from homotopy category to some algebraic categories.

Let  $\mathbf{TOP}^o$  be the pointed topological category, where the sum is wedge sum  $(X, x_0) \wedge (Y, y_0) = X \sqcup Y / x_0 \sim y_0$  and the product is the smash product  $(X, x_0) \vee (Y, y_0) = X \times Y / \{x_0\} \times Y \cup X \times \{y_0\}$ . Similarly, we can take a quotient to get  $h\text{-}\mathbf{TOP}^o$ .

Let  $\mathbf{TOP}(2)$  be the category of pairs and  $h\text{-}\mathbf{TOP}(2)$  be its quotient.

Fix  $K \in \text{Ob}(\mathbf{TOP})$ . Let's consider  $\mathbf{TOP}^K$ , the category of spaces under  $K$ . Its objects are maps  $f: K \rightarrow X$  and morphisms are maps  $\alpha: X \rightarrow Y$  such that  $\alpha \circ f = g$ .

$$\begin{array}{ccc} & K & \\ f \swarrow & & \searrow g \\ X & \xrightarrow{\alpha} & Y \end{array}$$

If  $K = \{*\}$  is a single point set, then  $\mathbf{TOP}^{\{*\}} = \mathbf{TOP}^o$  is the pointed topological category. Take  $X = K$ . A morphism from  $f: K \rightarrow X$  to  $\text{id}: K \rightarrow K$  is  $r: X \rightarrow K$  such that  $r \circ f = \text{id}$ .

$$\begin{array}{ccc} & K & \\ f \swarrow & & \searrow \text{id} \\ X & \xrightarrow{r} & K \end{array}$$

When  $K \subset X$ ,  $f = i: K \hookrightarrow X$ , we say that  $r$  is a retraction.

We have  $r: X \rightarrow K$  is a deformation retraction, if and only if  $i \circ r \simeq \text{id}_X \text{ rel } K$ , if and only if  $r: X \rightarrow K$  is a homotopy equivalence in  $\mathbf{TOP}^K$ .

Fix  $B \in \text{Ob}(\mathbf{TOP})$ . Let's consider  $\mathbf{TOP}_B$ , the category of spaces over  $B$ , where the objects are  $p: X \rightarrow B$  and morphisms are  $f: X \rightarrow Y$  such that  $p = q \circ f$ .

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow p & \swarrow q \\ & B & \end{array}$$

Take  $X = B$ . A morphism from  $\text{id}: B \rightarrow B$  to  $q: Y \rightarrow B$  is  $s: B \rightarrow Y$  such that  $q \circ s = \text{id}_B$ .

$$\begin{array}{ccc} B & \xrightarrow{s} & Y \\ & \searrow \text{id} & \swarrow q \\ & B & \end{array}$$

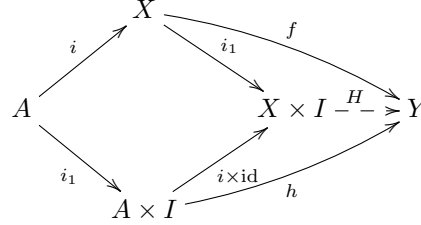
Then  $s$  is called a section of  $q$ .

Similarly, we can define  $h\text{-}\mathbf{TOP}^K$  and  $h\text{-}\mathbf{TOP}_B$ .

# 1 Cofibrations and Fibrations

## 1.1 Cofibrations

**Definition 1.1.** A map  $i: A \rightarrow X$  has the homotopy extension property (HEP) for a space  $Y$  if for all homotopy  $h: A \times I \rightarrow Y$  and  $f: X \rightarrow Y$  with  $f \circ i(a) = h(a, 1)$ , there exists  $H: X \times I \rightarrow Y$  satisfies



We say  $i: A \rightarrow X$  is a cofibration if it has HEP for each  $Y \in \text{Ob}(\mathbf{TOP})$ .

Recall the mapping cylinder: if  $i: A \rightarrow X$  is a map, then  $Z(i) := (A \times I) \sqcup X / (a, 1) \sim i(a)$ .

**Proposition 1.2.** Given a map  $i: A \rightarrow X$ . The followings are equivalent:

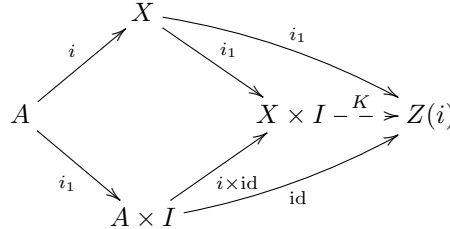
1.  $i: A \rightarrow X$  is a cofibration.
2.  $i$  has HEP for  $Z(i)$ .
3. The map

$$\begin{aligned} s: Z(i) &\rightarrow X \times I \\ (a, t) &\mapsto (i(a), t), \\ x &\mapsto (x, 1) \end{aligned}$$

has a retraction.

*Proof.* (1) $\implies$ (2) is only by definition.

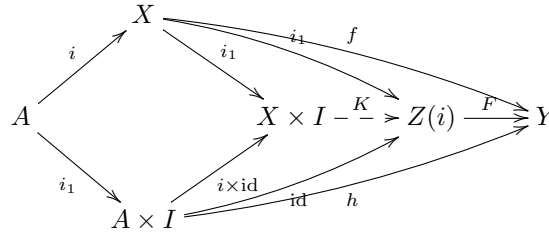
(2) $\implies$ (1): By definition, there exists  $K: X \times I \rightarrow Z(i)$  such that the following diagram is commutative.



For any  $Y$  and homotopy  $h: A \times I \rightarrow Y$  and  $f: X \rightarrow Y$  with  $f \circ i(a) = h(a, 1)$ , we define

$$\begin{aligned} F: Z(i) &\rightarrow Y \\ (a, t) &\mapsto h(a, t) \\ x &\mapsto f(x). \end{aligned}$$

Then  $F \circ K$  is as desired.

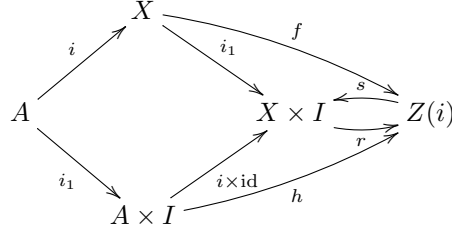


(2) $\implies$ (3): We can easily check that the extension  $K: X \times I \rightarrow Z(i)$  in the proof of (2) $\implies$ (1) is a retraction of  $s$ .

(3) $\implies$ (2): Let  $r$  be a retraction of  $s$ . For any homotopy  $h: A \times I \rightarrow Z(i)$  and  $f: X \rightarrow Z(i)$  with  $f \circ i(a) = h(a, 1)$ , we define

$$\begin{aligned}\sigma: Z(i) &\rightarrow Z(i) \\ (a, t) &\mapsto h(a, t) \\ x &\mapsto f(x).\end{aligned}$$

Then we can verify that  $H = \sigma \circ r: X \times I \rightarrow Z(i)$  extends  $h$ .



□

**Corollary 1.3.** When  $A \subset X$  is a close subset,  $i: A \hookrightarrow X$  is the inclusion map. Then  $i: A \rightarrow X$  is a cofibration  $\iff Z(i) = A \times I \cup X \times \{1\}$  is a retraction of  $X \times I$ .

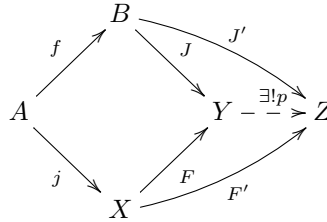
Therefore, we can construct many cofibrations. For example, let  $(X, A)$  be a manifold with boundary, then  $i: A \hookrightarrow X$  is a cofibration.

### 1.1.1 Push-Out of Cofibration

Given a commutative diagram,

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ j \downarrow & & \downarrow J \\ X & \xrightarrow{F} & Y \end{array}$$

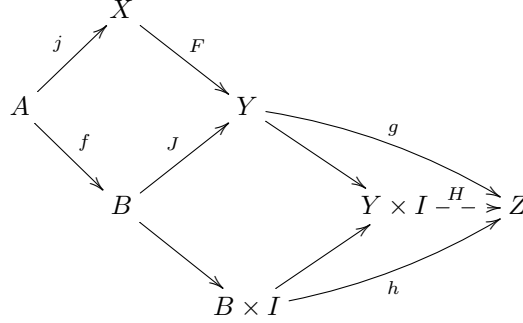
the push-out of  $j$  along  $f$  is the initial object of this diagram, i.e.  $j: B \rightarrow Y$ ,  $F: X \rightarrow Y$ , s.t.  $\forall Z$  with  $J': B \rightarrow Z$ ,  $F': X \rightarrow Z$  satisfying  $J' \circ f = F' \circ j$ ,  $\exists!$  map  $p: Y \rightarrow Z$  such that the diagram is commutative.



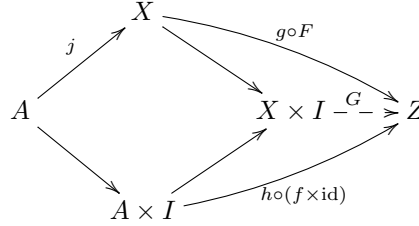
In our setting, we can construct  $Y = X \sqcup B / f(a) \sim j(a)$  directly.

**Proposition 1.4.** If  $j: A \rightarrow X$  is a cofibration, then the push-out of  $j$  along  $f: A \rightarrow B$   $J: B \rightarrow Y$  is also a cofibration.

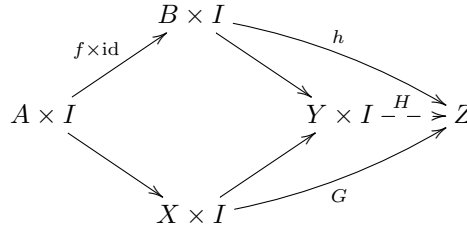
*Proof.* For any  $Z, g: Y \rightarrow Z, h: B \times I \rightarrow Z$  such that  $g \circ J = h \circ (i_1 \times \text{id})$ , we need to find  $H: Y \times I \rightarrow Z$  such that the following diagram is commutative.



Because  $j: A \rightarrow X$  is a cofibration, we have  $G: X \times I \rightarrow Z$  such that the following diagram is commutative.



Using the fact that  $J \times \text{id}: B \times I \rightarrow Y \times I$  is also the push-out of  $j \times \text{id}: A \times I \rightarrow X \times I$  along  $f \times \text{id}: A \times I \rightarrow B \times I$ , we have unique  $H: Y \times I \rightarrow Z$  such that the following diagram is commutative.

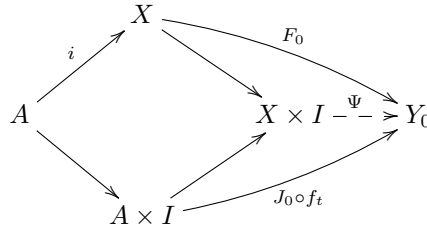


The  $H: Y \times I \rightarrow Z$  is the extension of  $h: B \times I \rightarrow Z$ , as desired.  $\square$

In terms of categorical language, let  $\Pi(A, B)$  be a category, whose objects are continue maps from  $A$  to  $B$  and morphisms are homotopy of maps from  $A$  to  $B$ . Consider  $\mathbf{COF}^B \subset \mathbf{TOP}^B$  the subcategory of cofibrations under  $B$  (i.e.  $J: B \rightarrow Y$ ). Then we have homotopy category  $h - \mathbf{COF}^B$ . Given a cofibration  $i: A \rightarrow X$ , we get a contravariant functor

$$\beta: \Pi(A, B) \rightarrow h - \mathbf{COF}^B.$$

In fact, we only need to check that if  $f_0 \simeq f_1: A \rightarrow B$ , then we get a morphism from  $J_0: B \rightarrow Y_0$  to  $J_1: B \rightarrow Y_1$ . Firstly, consider the homotopy  $J_0 \circ f_t: A \times I \rightarrow Y_0$ , we get its extension  $\Psi: X \times I \rightarrow Y_0$ .



Then by the universal property of the push-out  $J_1: B \rightarrow Y_1$  of  $i$  along  $f_1$  for  $J_0: B \rightarrow Y_0$  and  $\Psi_1: X \rightarrow Y_0$ , we get a map  $K: Y_1 \rightarrow Y_0$ , as desired.

$$\begin{array}{ccccc}
 & & B & & \\
 & f_1 \nearrow & & \searrow J_1 & \\
 A & & & & Y_1 \xrightarrow{K} Y_0 \\
 & i \searrow & & \nearrow F_1 & \\
 & & X & & 
 \end{array}
 \quad
 \begin{array}{c}
 \text{curved arrow } J_0 \text{ from } B \text{ to } Y_0 \\
 \text{curved arrow } \Psi_1 \text{ from } X \text{ to } Y_0
 \end{array}$$

### 1.1.2 Replacing a Map by a Cofibration

Given a map  $f: X \rightarrow Y$ , consider the mapping cylinder  $Z(f)$ . We can notice that  $Z(f)$  is the push-out.

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 i_1 \downarrow & & \downarrow s \\
 X \times I & \xrightarrow{a} & Z(f)
 \end{array}$$

We also have a map

$$\begin{aligned}
 q: Z(f) &\rightarrow Y \\
 (x, t) &\mapsto f(x).
 \end{aligned}$$

Note that by Proposition 1.2,  $i_1: X \hookrightarrow X \times I$  is a cofibration  $\iff X \times \{1\} \times I \cup X \times I \times \{1\}$  is a retraction of  $X \times I \times I$ , we have  $s: Y \rightarrow Z(f)$  is a cofibration.

**Proposition 1.5.** Let

$$\begin{aligned}
 j: X &\rightarrow Z(f) \\
 x &\mapsto (x, 0),
 \end{aligned}$$

we have

1.  $j: X \rightarrow Z(f)$  is a cofibration.
2.  $s \circ q \simeq \text{id}_{Z(f)} \text{ rel } Y$ .
3. If  $f$  is a cofibration, then  $q: Z(f) \rightarrow Y$  is a homotopy equivalence in  $\mathbf{TOP}^X$ .

*Proof.* (1). We construct a retraction  $R: Z(f) \times I \rightarrow X \times I \cup Z(f) \times \{1\}$  as follow. Let  $R': I \times I \rightarrow I \times \{1\} \cup \{0\} \times I$  be a retraction. Then we define

$$\begin{aligned}
 R: Z(f) \times I &\rightarrow X \times I \cup Z(f) \times \{1\} \\
 ((x, s), t) &\mapsto (x, R'(s, t)) \\
 (y, t) &\mapsto (y, 1)
 \end{aligned}$$

is as desired. By Proposition 1.2,  $j: X \rightarrow Z(f)$  is a cofibration.

(2). The homotopy

$$\begin{aligned}
 h_t: Z(f) &\rightarrow Z(f) \\
 (x, \sigma) &\mapsto (x, (1-t)\sigma + t)
 \end{aligned}$$

is as desired.

(3). By Proposition 1.2, there is a retraction  $r: Y \times I \rightarrow Z(f)$ . Define

$$\begin{aligned} g: Y &\rightarrow Z(f) \\ y &\mapsto r(y, 1). \end{aligned}$$

One can verify that  $g$  is the homotopy inverse of  $q$ . □

**Summery 1.** Any map  $f: X \rightarrow Y$  factors into

$$X \xrightarrow{j} Z \xrightarrow{q} Y$$

where  $j: X \rightarrow Z$  is a cofibration and  $q: Z \rightarrow Y$  is a homotopy equivalence. Moreover, such a factorization is unique up to homotopy equivalence. In particular, we can choose  $Z = Z(f)$ . We define  $C_f = Z(f)/\text{im } j$  as the homotopy cofibre of  $f$ , i.e.  $C_f = X \times I \sqcup Y/(x, 0) \sim *, (x, 1) \sim f(x)$ , is called the mapping cone of  $f$ .

$$X \xrightarrow{f} Y \xrightarrow{s} C_f$$

### 1.1.3 The Cofibre Sequence (Puppe's Sequence)

To get finer structure, we work in  $\mathbf{TOP}^o$ . Given a map  $f: (X, x_0) \rightarrow (Y, y_0)$ , we get an induced map

$$\begin{aligned} f^*: [Y, B]^o &\rightarrow [X, B]^o \\ [\alpha] &\mapsto [f \circ \alpha], \end{aligned}$$

where  $[X, B]^o$  is the homotopy class of basepoint preserving maps. In particular, we have the constant map

$$\begin{aligned} [*]: X &\rightarrow B \\ x &\mapsto b_0. \end{aligned}$$

**Definition 1.6.** We say a sequence

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$$

in  $\mathbf{TOP}^o$  is h-coexact if  $\forall (B, b_0) \in \text{Ob}(\mathbf{TOP}^o)$ ,

$$[Z, B]^o \xrightarrow{g^*} [Y, B]^o \xrightarrow{f^*} [X, B]^o$$

is exact, i.e.  $(f^*)^{-1}([*]) = \text{im } g^*$ .

In  $\mathbf{TOP}^o$ , we consider the reduced mapping cone  $CX := X \times I / X \times \{0\} \cup \{x_0\} \times I$ . The basepoint of  $CX$  is  $X \times \{0\} \cup \{x_0\} \times I$ . And we consider the reduced mapping cone: For  $f: (X, x_0) \rightarrow (Y, y_0)$ ,  $C(f) := CX \vee Y/(x, 1) \sim f(x)$ . It is equivalent to the following push-out diagram.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i_1 \downarrow & & \downarrow f_1 \\ CX & \longrightarrow & C(f) \end{array}$$

In fact,  $f_1$  maps  $y$  to  $(y, 1)$ .

We will also use symbol  $X$  instead of  $(X, x_0)$  in  $\mathbf{TOP}^o$  for short.



**Proposition 1.7.** The sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} C(f)$$

is h-coexact.

*Proof.* Consider the following sequence

$$[C(f), B]^o \xrightarrow{f_1^*} [Y, B]^o \xrightarrow{f^*} [X, B]^o$$

for any  $(B, b_0)$ .

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{f_1} & C(f) \\ & \searrow & \downarrow \alpha & \swarrow & \\ & & B & & \end{array}$$

Assume that  $[\alpha] \in [Y, B]^o$  s.t.  $[\alpha \circ f] = [*] \in [X, B]^o$ , i.e.  $\alpha \circ f$  is null-homotopic. This is equivalent that there exists a map  $h: CX \rightarrow B$ . The mapping cone  $C(f)$  is the push-out of

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i_1 \downarrow & & \downarrow f_1 \\ CX & \longrightarrow & C(f) \end{array}$$

Using the universal property of push-out, we have the following commutative diagram,

$$\begin{array}{ccccc} & & Y & & \\ & \nearrow f & & \searrow f_1 & \\ X & & & & C(f) \xrightarrow{\exists \beta} B \\ & \searrow i_1 & \nearrow & \searrow h & \\ & & CX & & \end{array}$$

i.e.  $\alpha = \beta \circ f_1$ . Therefore  $[\alpha] = f_1^*[\beta]$  and this proposition follows. □

Iterate the procedure, we get a long h-coexact sequence:

$$X \xrightarrow{f} Y \xrightarrow{f_1} C(f) \xrightarrow{f_2} C(f_1) \xrightarrow{f_3} C(f_2) \longrightarrow \dots$$

Consider the injection  $j_1: CY \rightarrow C(f_1)$ , we have that

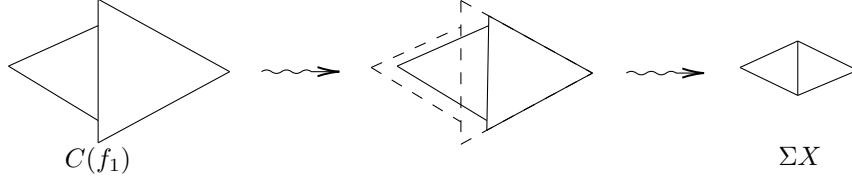
$$C(f_1)/j_1(CY) = X \times I/X \times \partial I \cup \{x_0\} \times I = \Sigma X$$

is the reduced suspension of  $X$ . Then we get a quotient map

$$q(f): C(f_1) \rightarrow \Sigma X.$$

$$\begin{array}{ccccccc} \begin{array}{c} | \\ X \end{array} & \xrightarrow{f} & \begin{array}{c} | \\ Y \end{array} & \rightsquigarrow & \begin{array}{c} \triangle \\ C(f) \end{array} & \rightsquigarrow & \begin{array}{c} \triangle \\ C(f_1) \end{array} & \xrightarrow{q(f)} & \begin{array}{c} \triangle \\ \Sigma X \end{array} \end{array}$$

**Claim 1.**  $q(f)$  is a homotopy equivalence.



Denote by  $s(f): \Sigma X \rightarrow C(f_1)$  the homotopy inverse of  $q(f)$ . Then our original sequence becomes

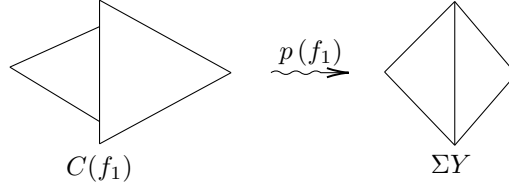
$$\begin{array}{ccccccc}
 X & \xrightarrow{f} & Y & \xrightarrow{f_1} & C(f) & \xrightarrow{f_2} & C(f_1) \xrightarrow{f_3} C(f_2) \\
 & & & & \searrow q(f) \circ f_2 & & \downarrow q(f) \\
 & & & & & & \Sigma X
 \end{array}$$

Consider the following diagram.

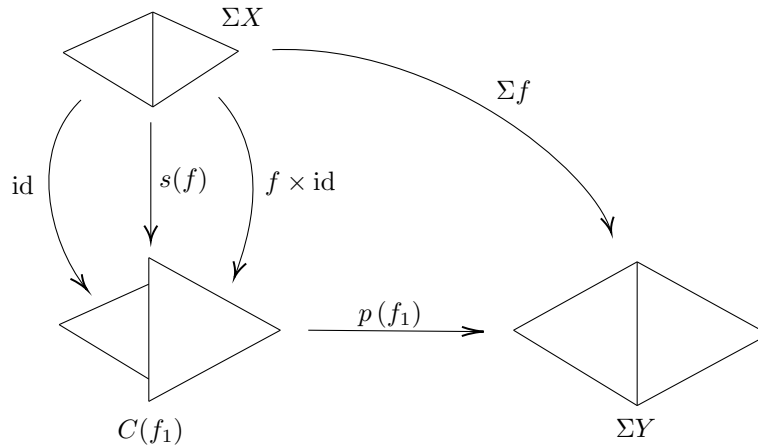
$$\begin{array}{ccc}
 C(f_1) & \xrightarrow{f_3} & C(f_2) \\
 q(f) \downarrow & \uparrow s(f) & \downarrow q(f_1) \\
 \Sigma X & \xrightarrow{q(f_1) \circ f_3 \circ s(f)} & \Sigma Y
 \end{array}$$

**Claim 2.** Consider  $\tau: \Sigma X \rightarrow \Sigma X$  which maps  $(x, t)$  to  $(x, 1 - t)$ , we have  $q(f_1) \circ f_3 \circ s(f) \simeq \Sigma f \circ \tau$

To prove it, denote  $p(f_1) = q(f_1) \circ f_3$ . In fact,  $p(f_1)$  retracts the left triangle, i.e.  $CX$  to a point.



In the following diagram,  $s(f)$  is the union of  $\text{id}$  and  $f \times \text{id}$ , i.e.  $\text{id}$  maps the left triangle of  $\Sigma X$  to the left triangle of  $C(f_1)$ ,  $f \times \text{id}$  maps the right triangle of  $\Sigma X$  to the right triangle of  $C(f_1)$ . Then  $\Sigma f = p(f_1) \circ s(f)$  naturally. Notice that  $\tau$  flips  $\Sigma X$  left and right. Therefore, by symmetry, we have  $p(f_1) \circ s(f) \simeq \Sigma f \circ \tau$ , as desired.



Now we get

$$X \xrightarrow{f} Y \xrightarrow{f_1} C(f) \xrightarrow{p(f)} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{(\Sigma f)_1} C(\Sigma f)$$

**Claim 3.** There is a homeomorphism  $\tau_1: C(\Sigma f) \rightarrow \Sigma C(f)$  such that the following diagram is commutative.

$$\begin{array}{ccc} \Sigma Y & \xrightarrow{(\Sigma f)_1} & C(\Sigma f) \\ & \searrow \Sigma f_1 & \downarrow \tau_1 \\ & & \Sigma C(f) \end{array}$$

In fact, regard both  $C(\Sigma f)$  and  $\Sigma C(f)$  as the quotient spaces of  $X \times I \times I$  unioned with  $Y$ ,  $\tau_1$  is induced from interchanging the two  $I$ -factors.

As conclusion, we have

**Theorem 1.8** (Puppe's Sequence). The sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} C(f) \xrightarrow{p(f)} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma f_1} \Sigma C(f) \xrightarrow{p(\Sigma f)} \Sigma^2 X \longrightarrow \Sigma^2 Y \longrightarrow \dots$$

is h-coexact.

## 1.2 Fibrations

**Definition 1.9.** A map  $p: E \rightarrow B$  has the homotopy lifting property (HLP) for the space  $X$  if  $\forall$  homotopy  $h: X \times I \rightarrow B$  and  $a: X \rightarrow E$  s.t.  $p \circ a(x) = h(x, 0)$ , there exists a homotopy  $H: X \times I \rightarrow E$  s.t.  $p \circ H = h$ .  $H$  is called a lifting of  $h$ .

$$\begin{array}{ccc} X & \xrightarrow{a} & E \\ i_0 \downarrow & \nearrow H & \downarrow p \\ X \times I & \xrightarrow{h} & B \end{array}$$

A map  $p: E \rightarrow B$  is called a fibration if it has HLP for all spaces  $X$ .

**Definition 1.10.** Given maps  $f: A \rightarrow B$  and  $p: E \rightarrow B$ . The pull-back of  $p$  along  $f$  is the terminal object of the following diagram,

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \downarrow & & \downarrow p \\ A & \xrightarrow{f} & B \end{array}$$

i.e. for any  $C$ ,  $g: C \rightarrow E$ ,  $h: C \rightarrow A$ , there exists unique  $r$  such that the following diagram is commutative.

$$\begin{array}{ccccc} & & E & & \\ & \nearrow g & & \searrow p & \\ C & \xrightarrow{r} f^*E & & & B \\ & \searrow & & \nearrow f & \\ & & A & & \end{array}$$

Explicitly,

$$f^*E = \{(a, e) \in A \times E : f(a) = p(e)\}$$

and  $\pi: f^*E \rightarrow A$  is the projection.

Denote  $B^I = \text{Map}(I, B)$ . Consider the pull-back

$$W(p) := \{(x, w) \in E \times B^I : p(x) = w(0)\}$$

which is given by the pull-back

$$\begin{array}{ccc} W(p) & \xrightarrow{k} & B^I \\ b \downarrow & & \downarrow e^0 \\ E & \xrightarrow{p} & B \end{array}$$

where  $e^0$  maps  $w$  to  $w(0)$ .

**Proposition 1.11.** Given a map  $p: E \rightarrow B$ , the followings are equivalence:

1.  $p: E \rightarrow B$  is a fibration.
2.  $p$  has HLP for  $W(p)$ .
- 3.

$$\begin{aligned} r: E^I &\rightarrow W(p) \\ \alpha &\mapsto (\alpha(0), p \circ \alpha) \end{aligned}$$

admits a section.

*Proof.* (1) $\implies$ (2) is by definition.

(2) $\implies$ (3): Because  $W(p)$  is a pull-back, by its universal property, we have the following diagram and we want to find  $s$  such that  $r \circ s = \text{id}$ .

$$\begin{array}{ccccc} & & & B^I & \\ & & p^I \nearrow & & \searrow e^0 \\ E^I & \xrightleftharpoons[r]{s} & W(p) & \xrightarrow{k} & B \\ & \searrow e^0 & \downarrow b & & \nearrow p \\ & & E & & \end{array}$$

Notice that  $\text{Map}(W(p), E^I) = \text{Map}(W(p) \times I, E)$ , because  $p$  has HLP for  $W(p)$ , we have the following commutative diagram.

$$\begin{array}{ccc} W(p) & \xrightarrow{b} & E \\ \downarrow & \nearrow s & \downarrow p \\ W(p) \times I & \xrightarrow{k} & B \end{array}$$

We have  $b \circ r \circ s = e^0 \circ s = b$  and  $k \circ r \circ s = p^I s = k$ . Using the universal property (uniqueness) of pull-back  $W(p)$  for  $W(p)$ , we must have  $r \circ s = \text{id}$ , i.e.  $s$  is a section of  $r$ .

(3) $\implies$ (1): Let  $s$  be the section of  $r$ . For any  $X, a, h$  as in the definition of fibration, we want to find  $H$  such that the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{a} & E \\ i_0 \downarrow & \nearrow H & \downarrow p \\ X \times I & \xrightarrow{h} & B \end{array}$$

Using the universal property of pull-back  $W(p)$ , we have unique  $f$  such that the following diagram is commutative, where  $h: X \rightarrow B^I$  is the same as  $h: X \times I \rightarrow B$ .

$$\begin{array}{ccccc}
 & & B^I & & \\
 & \nearrow h & & \searrow e^0 & \\
 X & \xrightarrow{\exists! f} & W(p) & \xrightarrow{k} & B \\
 & \searrow a & \downarrow b & & \uparrow p \\
 & & E & & 
 \end{array}$$

Then because  $\text{Map}(W(p), E^I) = \text{Map}(W(p) \times I, E)$ , one can check that  $H = s \circ f$  is as desired. In fact,

$$p \circ H(x, t) = (p \circ H(x))(t) = (k \circ r \circ s \circ f(x))(t) = (k \circ \text{id} \circ f(x))(t) = h(x, t)$$

and  $H \circ i_0 = a$  is similar.  $\square$

### 1.2.1 Pull-back of Fibration

**Proposition 1.12.** If  $p: E \rightarrow B$  is a fibration, then  $f^*E \rightarrow A$  is also a fibration.

*Proof.* In the following diagram,  $F$  is induced by HLP for fibration  $p: E \rightarrow B$  and then  $H$  is induced by universal property of pull-back  $f^*E$ .

$$\begin{array}{ccccc}
 X & \xrightarrow{a} & f^*E & \xrightarrow{\pi} & E \\
 i_0 \downarrow & \nearrow H & \nearrow F & \searrow \pi & \downarrow p \\
 X \times I & \xrightarrow{h} & A & \xrightarrow{f} & B
 \end{array}$$

$\square$

### 1.2.2 Replacing Maps by Fibration

**Proposition 1.13.** The evaluation  $e^1: Y^I \rightarrow Y$ ,  $w \mapsto w(1)$  is a fibration.

*Proof.* We can define  $H$  directly:

$$\begin{aligned}
 H: X \times I &\rightarrow Y^I \\
 (x, s) &\mapsto \begin{cases} [t \mapsto a|_X((1+s)t)], & \text{when } 0 \leq (1+s)t \leq 1 \\ [t \mapsto h(x, (1+s)t - 1)], & \text{when } (1+s)t \geq 1. \end{cases}
 \end{aligned}$$

$$\begin{array}{ccc}
 X & \xrightarrow{a} & Y^I \\
 i_0 \downarrow & \nearrow H & \downarrow e^1 \\
 X \times I & \xrightarrow{h} & Y
 \end{array}$$

$\square$

Given  $f: X \rightarrow Y$ , consider the following pull-back.

$$\begin{array}{ccc}
 W(f) = f^*Y^I & \xrightarrow{\quad} & Y^I \\
 i_0 \downarrow & & \downarrow e^1 \\
 X & \xrightarrow{f} & Y
 \end{array}$$

In fact,

$$W(f) = \{(x, w) \in X \times Y^I : f(x) = w(1)\}.$$

Denote  $p: W(f) \rightarrow Y$ ,  $(x, w) \mapsto w(1)$  and  $s: X \rightarrow W(f)$ ,  $x \mapsto (x, k_{f(x)})$  where  $k_{f(x)}$  is a constant path at  $f(x)$ , and  $q: W(f) \rightarrow X$ ,  $(x, w) \mapsto x$ . We can check that the following diagram is commutative.

$$\begin{array}{ccc} W(f) = f^*Y^I & \xrightarrow{\quad} & Y^I \\ i_0 \downarrow \uparrow s & \searrow p & \downarrow e^1 \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

**Theorem 1.14.** In the following commutative diagram,

$$\begin{array}{ccc} X & \xrightarrow{\quad s \quad} & W(f) \\ & \searrow f & \swarrow p \\ & & Y \end{array}$$

$s$  is a homotopy equivalence and  $p$  is a fibration.

*Proof.* Consider the following fibration

$$\begin{array}{ccc} (f \times \text{id})^*Y^I & \xrightarrow{\quad} & Y^I \\ (q, p) \downarrow & & \downarrow (e^1, e^0) \\ X \times Y & \xrightarrow{\quad f \times \text{id} \quad} & Y \times Y \end{array}$$

**Claim 4.**  $(f \times \text{id})^*Y^I = W(f)$ .

To see that, notice that

$$(f \times \text{id})^*Y^I = \{(x, y, w) \in X \times Y \times Y^I : f(x) = w(1), y = w(0)\},$$

we can construct a map from  $W(f)$  to  $(f \times \text{id})^*Y^I$  that maps  $(x, w)$  to  $(x, w(0), w)$ . It's one to one.

Then  $p: W(f) \rightarrow Y$  is a fibration if and only if  $(f \times \text{id})^*Y^I \xrightarrow{(q, p)} X \times Y \xrightarrow{p_2} Y$  is a fibration. It's a composition of two fibration and then a fibration, as desired.

**Claim 5.**  $q$  is a homotopy inverse of  $s$ .

□

By this theorem, given any  $f: X \rightarrow Y$ , we can replace it by a fibration  $p: W(f) \rightarrow Y$  homotopically. Then we can define the homotopy fibre at  $y_0$  of  $f: X \rightarrow Y$  to be

$$F(f) := p^{-1}(y_0) = \{(x, w) \in X \times Y^I : f(x) = w(1), y_0 = w(0)\}.$$

**Remark 1.15.** Apply HLP again, we can prove the factorization  $f = s \circ p: X \rightarrow Y$  such that  $s: X \rightarrow W$  is a homotopy equivalence and  $p: W \rightarrow Y$  is a fibration. And this factorization is unique up to homotopy equivalence.

**Theorem 1.16.** Let  $p: E \rightarrow B$  be a fibration and  $B$  is path-connected. Then all fibres  $p^{-1}(b)$  are homotopy equivalent.

*Proof.* Given a path  $\alpha: I \rightarrow B$ ,  $\alpha(0) = b_0$  and  $\alpha(1) = b_1$ . Consider HLP property:

$$\begin{array}{ccc} p^{-1}(b_0) & \xrightarrow{\quad} & E \\ \downarrow & \nearrow H & \downarrow p \\ p^{-1}(b_0) \times I & \xrightarrow{h} & B \end{array}$$

where  $h(x, t) = \alpha(t)$ . Consider  $H_1: p^{-1}(b_0) \rightarrow p^{-1}(b_1)$  the restriction of  $H$  at  $t = 1$ . Similarly, consider the reversed path  $\bar{\alpha}$  of  $\alpha$ , we get  $\bar{H}_1: p^{-1}(b_1) \rightarrow p^{-1}(b_0)$ .

**Claim 6.**  $\bar{H}_1 \circ H_1 \simeq \text{id}$ .

It's by applying homotopy lifting to the homotopy from  $\bar{\alpha}\alpha$  to  $k_{b_0}$ . Therefore, all fibres  $p^{-1}(b)$  are homotopy equivalent.  $\square$

### 1.2.3 Fibre Exact Sequence (Puppe's Sequence)

**Definition 1.17.** We say a sequence of pointed maps

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$$

is h-coexact if  $\forall (B, b_0)$ , the induced sequence

$$[B, X]^o \xrightarrow{f_*} [B, Y]^o \xrightarrow{g_*} [B, Z]^o$$

is exact, i.e.  $g_*^{-1}([c_{z_0}]) = \text{im } f_*$ .

Recall the homotopy fibre of  $f: X \rightarrow Y$  is

$$F(f) := p^{-1}(y_0) = \{(x, w) \in X \times Y^I : f(x) = w(1), y_0 = w(0)\}.$$

Denote  $f^1: F(f) \rightarrow X$ ,  $(x, w) \mapsto x$ .

**Proposition 1.18.** For any  $f: (X, x_0) \rightarrow (Y, y_0)$ , the sequence

$$F(f) \xrightarrow{f^1} X \xrightarrow{f} Y$$

is h-coexact.

*Proof.* Assume  $\alpha: B \rightarrow X$  satisfies  $f \circ \alpha: B \rightarrow Y$  is null-homotopic and  $f_*[\alpha] = [c_{y_0}]$ . Apply HLP property:

$$\begin{array}{ccc} B & \xrightarrow{\quad} & FY = \{w \in Y^I : w(0) = y_0\} \\ \downarrow & \nearrow H & \downarrow e^1 \\ B \times I & \xrightarrow{h} & Y \end{array}$$

where  $h$  is a null-homotopy from  $f \circ \alpha$  to  $c_{y_0}$ . Notice that  $H_0: B \times \{1\} \rightarrow FY$  satisfies

$$\begin{array}{ccccc} & & FY & & \\ & \nearrow H_0 & & \searrow & \\ B & \xrightarrow{\beta} & F(f) & \xrightarrow{f^1} & X \\ & \searrow \alpha & & \nearrow & \\ & & X & & Y \end{array}$$

where  $\beta$  is induced by the universal property of the pull-back  $F(f)$ , such that  $f^1 \circ \beta = \alpha$ . Therefore,  $f_*^1([\beta]) = [\alpha]$ .  $\square$

Iterate the procedure, we get a long h-exact sequence

$$\cdots \longrightarrow F(f^2) \xrightarrow{f^3} F(f^1) \xrightarrow{f^2} F(f) \xrightarrow{f^1} X \longrightarrow Y.$$

**Question 1.19.** How to understand  $F(f^n) \xrightarrow{f^{n+1}} F(f^{n-1})$  ?

We consider the loop space

$$\Omega Y := \{w \in Y^I : w(0) = w(1) = y_0\}.$$

Notice that

$$(f^1)^{-1}(x_0) = \{(x_0, w) \in X \times Y^I : w(0) = y_0, w(1) = f(x_0) = y_0\},$$

we have  $\Omega Y = (f^1)^{-1}(x_0)$ . We write  $i(f) : \Omega Y \rightarrow F(f)$  for the inclusion.

**Theorem 1.20** (The puppe's fibre sequence). The sequence

$$\Omega^k F(f) \xrightarrow{\Omega^k f^1} \Omega^k X \xrightarrow{\Omega^k f} \Omega^k Y \xrightarrow{i(\Omega^{k-1} f)} \cdots \longrightarrow \Omega X \longrightarrow \Omega Y \longrightarrow F(f) \xrightarrow{f^1} X \longrightarrow Y$$

is h-exact.

*Proof.* Step 1:

$$\begin{aligned} F(f^1) &= \{(x, w, v) \in X \times Y^I \times X^I : w(0) = y_0, v(0) = x_0, w(1) = f(x), v(1) = x\} \\ &= \{(w, v) \in Y^I \times X^I : w(0) = y_0, v(0) = x_0, w(1) = f(v(1))\}. \end{aligned}$$

Define  $j(f) : \Omega Y \rightarrow F(f^1)$ ,  $w \mapsto (w, k_{x_0})$ .

**Claim 7.**  $j(f)$  is a homotopy equivalence.

In fact, define  $r(f) : F(f^1) \rightarrow \Omega Y$ ,  $(w, v) \mapsto w * \overline{(f \circ v)}$ , then  $r(f) \circ j(f) = \text{id}$ . The homotopy from  $\text{id}_{F(f^1)}$  to  $j(f) \circ r(f)$  is  $h_t(w, v) = (h_t^1, h_t^2)$ , where  $h_t^1(s) = \begin{cases} w(s(1+t)), & s(1+t) \leq 1, \\ f(v(2 - (1+t)s)), & s(1+t) \geq 1 \end{cases}$  and  $h_t^2(s) = v(s(1-t))$ .

Step 2: From  $F(f^1) \xrightarrow{f^2} F(f) \xrightarrow{f^1} X$ , we get

$$\begin{array}{ccc} F(f^2) & \xrightarrow{f^3} & F(f^1) \\ j(f^1) \uparrow & \nearrow i(f^1) & \uparrow j(f) \\ \Omega X & \xrightarrow{\Omega f} & \Omega Y \end{array}$$

Because  $j(f^1)$  is a homotopy equivalence, we have  $i(f^1) \simeq j(f) \circ \Omega f$ .

Step 3: Now we have  $\Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{i(f)} F(f)$ . Then we get  $F \Omega f \longrightarrow \Omega X \xrightarrow{\Omega f} \Omega Y$ .

**Claim 8.**  $F(\Omega f)$  is homotopy equivalent to  $\Omega F(f)$ .

To see that, notice that  $F(\Omega f)$  and  $\Omega F(f)$  are all quotient of  $\text{Map}(I \times I, Y)$ .

Finally, we get the h-exact sequence

$$\Omega F(f) \longrightarrow \Omega X \longrightarrow \Omega Y \longrightarrow F(f) \longrightarrow X \longrightarrow Y.$$

□



### 1.3 Duality of Cofibration and Fibration

#### 1.3.1 Duality of Reduced Suspension and Loop Space

Write  $Y^X = \text{Map}(X, Y)$  equipped with compact-open topology. We define the adjunction

$$\begin{aligned} \alpha: Z^{X \times Y} &\rightarrow (Z^Y)^X \\ f &\mapsto [x \mapsto f(x, \cdot)]. \end{aligned}$$

**Theorem 1.21.** Suppose that  $X$  and  $Y$  are locally compact. Then  $\alpha$  is a homeomorphism.

In the pointed version, we replace  $X \times Y$  by  $X \wedge Y = X \times Y / \{x_0\} \times Y \cup X \times \{y_0\}$  and  $\text{Map}^o(X, Y)$  is the space of basepoint preserving maps. Then  $\alpha^o: \text{Map}^o(X \wedge Y, Z) \rightarrow \text{Map}^o(X, \text{Map}^o(Y, Z))$  is a homeomorphism. Therefore,  $\alpha^o$  induces a bijection  $\alpha_*^o: [X \wedge Y, Z]^o \rightarrow [X, \text{Map}^o(Y, Z)]^o$ .

Choose  $Y = S^1 = I/\partial I$ , then  $X \wedge Y = X \times I / X \times \partial I \cup \{x_0\} \times I = \Sigma X$  is the reduced suspension of  $X$  and  $\text{Map}^o(Y, Z) = \Omega Z$  is the loop space of  $Z$ . Therefore, we get a bijection  $\alpha_*^o: [\Sigma X, Z]^o \rightarrow [X, \Omega Z]^o$ .

On  $[\Sigma X, Z]^o$ , we have a group structure:

$$[f] +_M [g]: (x, t) \mapsto \begin{cases} f(x, 2t), & t \leq \frac{1}{2}, \\ g(x, 2t - 1), & t \geq \frac{1}{2}. \end{cases}$$

Let  $\tau$  be the inversion of  $\Sigma X$ . For any  $[f]$ ,  $-[f] = [f \circ \tau]$ .

On  $[X, \Omega Z]^o$ , we have

$$\begin{aligned} m: \Omega Z \times \Omega Z &\rightarrow \Omega Z \\ (u, v) &\mapsto u * v. \end{aligned}$$

Define

$$[f] +_m [g] := [m \circ (f \times g) \circ d],$$

where

$$\begin{aligned} d: X &\rightarrow X \times X \\ x &\mapsto (x, x) \end{aligned}$$

is the diagonal embedding.

One can verify that

$$\alpha_*^o([f] +_M [g]) = \alpha_*^o([f]) +_m \alpha_*^o([g]).$$

Then the adjunction map  $\alpha_*^o: [\Sigma X, Z]^o \rightarrow [X, \Omega Z]^o$  is an isomorphism. In categorical language, this means  $\text{Mor}(\Sigma X, Z) = \text{Mor}(X, \Omega Z)$  in  $\mathbf{TOP}^o$ . As conclusion,  $\Sigma: \mathbf{TOP}^o \rightarrow \mathbf{TOP}^o$  and  $\Omega: \mathbf{TOP}^o \rightarrow \mathbf{TOP}^o$  are dual functors.

### 1.3.2 Duality of HLP and HEP

Given a homotopy lifting diagram,

$$\begin{array}{ccc} X \times \{0\} & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow p \\ X \times I & \longrightarrow & B \end{array}$$

notice that  $\text{Map}(X \times I, Z) = \text{Map}(X, Z^I)$ , it is equivalent to

$$\begin{array}{ccc} E & \xleftarrow{e^0} & E^I \\ \uparrow & \nearrow & \downarrow \\ X & \longrightarrow & B^I \end{array}$$

Dualize it, also by,  $\text{Map}(X \times I, Z) = \text{Map}(X, Z^I)$ , we have

$$\begin{array}{ccc} E' & \xrightarrow{i_0} & E' \times I \\ \downarrow & \nearrow & \uparrow \\ X' & \longleftarrow & B' \times I \end{array}$$

It is equivalent to

$$\begin{array}{ccccc} & & E' & & \\ & \nearrow & & \searrow & \\ B' & & & & X' \\ & \searrow & & \nearrow & \\ & & B' \times I & & \end{array}$$

$E' \times I \dashrightarrow X'$

which is the homotopy extension diagram.

### 1.3.3 Duality of Two Puppe's Sequences

Notice that  $[\text{id}] \in [\Sigma X, \Sigma X]^o$ , it induces  $\alpha_*^o[\text{id}] = \eta: X \rightarrow \Omega \Sigma X$ . For each map  $f: X \rightarrow Y$ , it induces

$$\eta: F(f) \rightarrow \Omega C(f)$$

$$(x, w) \mapsto \begin{cases} (x, 2t), & t \leq \frac{1}{2}, \\ w(2 - 2t), & t \geq \frac{1}{2}, \end{cases}$$

where  $C(f) = X \times I \sqcup Y / \{x_0\} \times I, f(x) \sim (x, 1)$  is the reduced cone of  $f$ . Then we get a diagram commutative up to homotopy.

$$\begin{array}{ccccc} \Omega Y & \longrightarrow & F(f) & \longrightarrow & X \\ \text{id} \downarrow & & \downarrow & & \downarrow \\ \Omega Y & \longrightarrow & \Omega C(f) & \longrightarrow & \Omega \Sigma X \end{array}$$

## 2 Homotopy Groups

### 2.1 Definitions and Properties

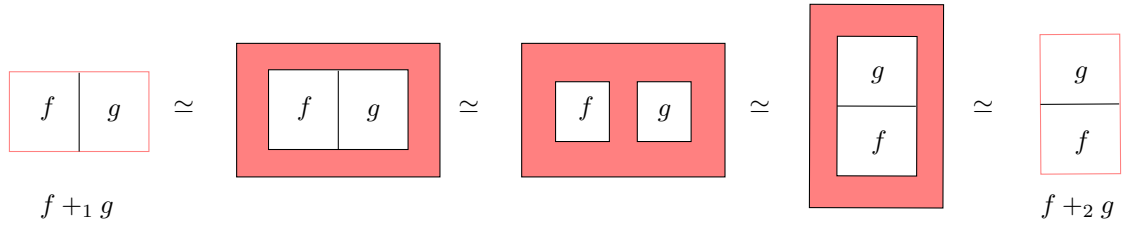
Given  $(X, x_0)$ , define  $n$ -th homotopy group

$$\pi_n(X, x_0) := [(I^n, \partial I^n), (X, x_0)],$$

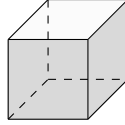
where the identity element is the constant map and  $[f] + [g]$  can be represented by

$$f +_i g: (t_1, \dots, t_n) \mapsto \begin{cases} f(t_1, \dots, 2t_i, \dots, t_n), & t_i \leq \frac{1}{2} \\ g(t_1, \dots, 2t_i - 1, \dots, t_n), & t_i \geq \frac{1}{2} \end{cases}$$

for any  $i$ . The following picture shows that  $f +_i g$  and  $f +_j g$  are homotopy equivalent for any  $i \neq j$ , where the red parts are mapped into the base point so the homotopies work. Sometimes, we write  $\pi_n(X)$  for short.



Given a pair  $(X, A, x_0)$ ,  $J^n = \partial I^n \times I \cup I^n \times \{0\} = I^n - I^n \times \{1\} \subset I^{n+1}$ ,



define the  $n + 1$ -th relative homotopy group to be

$$\pi_{n+1}(X, A, x_0) := [(I^{n+1}, \partial I^{n+1}, J^n), (X, A, x_0)].$$

Similarly, we sometimes use  $\pi_{n+1}(X, A)$  for short.

**Proposition 2.1.** When  $n \geq 2$ ,  $\pi_n(X, x_0)$  and  $\pi_{n+1}(X, A, x_0)$  are both abelian.

*Proof.* Exchanging  $f$  and  $g$  in the picture after the definition of  $\pi_n(X, x_0)$ , we can know that  $\pi_n(X, x_0)$  is abelian for  $n \geq 2$ . For the relative case, we can not process homotopy in the top red region. But for  $n \geq 3$ , the squares of  $f$  and  $g$  should be cubes, then we can place the cubes in front and behind to get new homotopy. Therefore,  $\pi_n(X, A, x_0)$  is abelian for  $n \geq 3$ .  $\square$

**Theorem 2.2** (Exact Homotopy Sequence). Given a pair  $(X, A)$ , we have a long exact sequence

$$\longrightarrow \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \longrightarrow \cdots \longrightarrow \pi_0(A, x_0) \xrightarrow{i_*} \pi_0(X, x_0),$$

where  $j: (X, x_0, x_0) \rightarrow (X, A, x_0)$  is the inclusion and  $\partial$  is induced from the restriction of  $I^n$  on  $I^{n-1} \times \{1\}$ .

*Proof.* Notice that each map  $f: (I^n, \partial I^n) \rightarrow (X, x_0)$  induces a map

$$\begin{aligned} \overline{f_k}: I^{n-k} &\rightarrow \Omega^k(X, x_0) \\ (u_1, \dots, u_{n-k}) &\mapsto [(t_1, \dots, t_k) \mapsto f(t_1, \dots, t_k, u_1, \dots, u_{n-k})]. \end{aligned}$$

Then we get an isomorphism  $\pi_n(X, x_0) \rightarrow \pi_{n-k}(\Omega^k X, c_{x_0})$ . This is because  $\pi_n(X, x_0) = [S^n, X]^o$  and  $\Sigma S^{n-1} = S^n$ , then  $[S^n, X]^o = [\Sigma S^{n-1}, X]^o \cong [S^{n-1}, \Omega X]^o \cong [S^{n-k}, \Omega^k X]^o$  by duality (Section 1.3.1).

Given a pair  $(X, A)$ , the homotopy fibre of  $\iota: A \hookrightarrow X$  is

$$F(\iota) = \{(a, w) \in A \times X^I : w(0) = x_0, w(1) = a\} = \{w \in X^I : w(0) = x_0, w(1) \in A\} := F(X, A).$$

Each map  $f: (I^{n+1}, \partial I^{n+1}, J^n) \rightarrow (X, A, x_0)$  induces a map

$$\begin{aligned} \hat{f}: I^n &\rightarrow F(X, A) \\ (t_1, \dots, t_n) &\mapsto [t \mapsto f(t_1, \dots, t_n, t)], \end{aligned}$$

induces an isomorphism  $\pi_{n+1}(X, A, x_0) \rightarrow \pi_n(F(X, A), x_0)$ .

The fibre sequence of  $\iota: A \hookrightarrow X$  is

$$\Omega^n F(\iota) \longrightarrow \Omega^n A \longrightarrow \Omega^n X \longrightarrow \dots \longrightarrow F(\iota) \longrightarrow A \xrightarrow{\iota} X.$$

Applying  $[S^1, \cdot]^o$ , we have

$$\begin{aligned} [S^1, \Omega^n F(\iota)]^o &= \pi_1(\Omega^n F(\iota)) = \pi_{n+1}(F(\iota)) = \pi_{n+2}(X, A), \\ [S^1, \Omega^n A]^o &= \pi_1(\Omega^n A) = \pi_{n+1}(A), \\ [S^1, \Omega^n X]^o &= \pi_1(\Omega^n X) = \pi_{n+1}(X). \end{aligned}$$

Then we get exact sequence

$$\pi_{n+2}(X, A) \longrightarrow \pi_{n+1}(A) \longrightarrow \pi_{n+1}(X) \longrightarrow \dots \longrightarrow \pi_1(X) \longrightarrow \pi_1(X, A) \longrightarrow \pi_0(A) \longrightarrow \pi_0(X),$$

where the exactness of the last a few places is straightforward to verify.  $\square$

## 2.2 Change of Basepoint

Assume  $v: I \rightarrow X$  is a continuous path with  $v(0) = x_0$  and  $v(1) = x_1$ . We regard  $v$  as a homotopy

$$\begin{aligned} \hat{v}_t: I^n &\rightarrow X \\ u &\mapsto v(t). \end{aligned}$$

Note that  $\partial I^n \hookrightarrow I^n$  is a cofibration (by Corollary 1.3), by HEP, we have the following commutative diagram,

$$\begin{array}{ccccc} & & \partial I^n \times I & & \\ & \nearrow & & \searrow \hat{v}_t & \\ \partial I^n & & & & I^n \times I \xrightarrow[-V]{} X \\ & \searrow & \nearrow f & \nearrow & \\ & & I^n & & \end{array}$$

where  $[f] \in \pi_n(X, x_0)$ .

**Proposition 2.3.** The map

$$\begin{aligned} v_\#: \pi_n(X, x_0) &\rightarrow \pi_n(X, x_1) \\ [v_0] &\mapsto [v_1] \end{aligned}$$

only depends on the homotopy class of  $v$  rel  $\partial_1$  and defines an isomorphism.

*Proof.* Use HEP again.  $\square$

**Proposition 2.4.** Suppose  $f: (X, A) \rightarrow (Y, B)$  is a homotopy equivalence. Then  $f_*: \pi_n(X, A, x_0) \rightarrow \pi_n(Y, B, f(x_0))$  is an isomorphism.

*Proof.* We only prove that homotopic maps induce isomorphic maps on  $\pi_n$ . Assume we have a homotopy  $g_t: (X, A) \rightarrow (Y, B)$ , we get a path in  $Y$

$$\begin{aligned} w: I &\rightarrow Y \\ t &\mapsto g_t(x_0). \end{aligned}$$

Then we have the following commutative diagram by HEP.

$$\begin{array}{ccc} & & \pi_n(Y, B, g_0(x_0)) \\ & \nearrow^{g_{0,*}} & \downarrow w_* \\ \pi_n(X, A, x_0) & & \\ & \searrow_{g_{1,*}} & \downarrow \\ & & \pi_n(Y, B, g_1(x_0)) \end{array}$$

$\square$

**Remark 2.5.** By the proposition, we get a right action of  $\pi_1(X, x_0)$  on  $\pi_n(X, x_0)$ .

### 2.3 Serre Fibration

**Definition 2.6.** We say  $p: E \rightarrow B$  is a Serre fibration, if it has HLP for all cube  $I^n, \forall n \geq 0$ .

**Theorem 2.7.** Let  $p: E \rightarrow B$  be a Serre fibration. Fix  $b_0 \in B$  and  $e_0 \in E$  such that  $p(e_0) = b_0$ . Given  $B_0 \subset B$ , write  $E_0 = p^{-1}(B_0)$ . Then  $p_*: \pi_n(E, E_0, e_0) \rightarrow \pi_n(B, B_0, b_0)$  is an isomorphism for all  $n \geq 1$ .

*Proof.* **Surjectivity:** Given  $h: (I^n, \partial I^n, J^{n-1}) \rightarrow (B, B_0, b_0)$ . Consider the lifting problem.

$$\begin{array}{ccc} I^{n-1} \times \{0\} \cup \partial I^{n-1} \times I & \xrightarrow{c_{e_0}} & E \\ \downarrow & \nearrow H & \downarrow p \\ I^{n-1} \times I & \xrightarrow{h} & B \end{array}$$

Notice that  $I^{n-1} \times \{0\} \cup \partial I^{n-1} \times I \cong I^{n-1} \times \{0\}$ , the map of the first line is  $c_{e_0}$ . Then we have the lifting  $H: I^n \rightarrow E$  such that  $H(\partial I^n) \subset E_0 = p^{-1}(B_0)$  and  $H(J^{n-1}) = e_0$ .

**Injectivity:** Assume  $p_*[f_0] = p_*[f_1]$ . We get a homotopy  $\phi_t: (I^n, \partial I^n, J^{n-1}) \rightarrow (B, B_0, b_0)$ . Consider the lifting problem.

$$\begin{array}{ccc} I^n \times \partial I \cup J^{n-1} \times I & \xrightarrow{\quad} & E \\ \downarrow & \nearrow \phi & \downarrow \\ I^n \times I & \xrightarrow{\phi_t} & B \end{array}$$

Notice that  $I^n \times \partial I \cup J^{n-1} \times I \cong I^n$ , we have the lifting  $\phi$ .  $\square$

**Corollary 2.8.** Given a Serre fibration  $F \hookrightarrow E \xrightarrow{p} B$  where  $F$  is a regular fibre, we have a long exact sequence

$$\pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots \longrightarrow \pi_0(E) \longrightarrow \pi_0(B).$$

*Proof.* Consider the pair  $(E, F)$ . By Theorem 2.2, we have exact sequence

$$\pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots$$

Choose  $B_0 = b_0$  and  $F = E_{b_0}$ , we have  $\pi_n(E, F, b_0) \cong \pi_n(E, b_0, b_0) \cong \pi_n(B, b_0)$  and this corollary follows.  $\square$

**Proposition 2.9.** Every fibre bundle is a Serre fibration.

*Proof.* Given the lifting problem.

$$\begin{array}{ccc} I^n \times \{0\} & \xrightarrow{a} & E \\ \downarrow & \nearrow H & \downarrow \\ I^n \times I & \xrightarrow{h} & B \end{array}$$

We choose an open cover  $\{U_\alpha\}_{\alpha \in \Lambda}$  of  $B$  such that finitely many  $U_\alpha$ 's cover  $\text{im } h$  and over each  $U_\alpha$ ,  $E|_{U_\alpha}$  is trivialized. Choose a subdivision  $\{I_\beta^n\}$  of  $I^n$  and partition  $\{I_\lambda\}$  of  $I$ , such that  $\forall \beta, \lambda, h(I_\beta^n \times I_\lambda) \subset U_\alpha$  for some  $\alpha$ . Over each  $I_\beta^n \times I_\lambda$ , we consider

$$\begin{array}{ccc} I_\beta^n \times \partial I_\lambda \cup \partial I_\beta^n \times I_\lambda & \longrightarrow & U_\alpha \times F \\ \downarrow & \nearrow H_{\beta, \lambda} & \downarrow \\ I_\beta^n \times I_\lambda & \xrightarrow{h} & U_\alpha \end{array}$$

where  $I_\beta^n \times \partial I_\lambda \cup \partial I_\beta^n \times I_\lambda \cong I_\beta^n \times \{0\}$  and  $U_\alpha \times F \cong E|_{U_\alpha}$ . We construct the lifting of  $h$  inductively on  $\beta$  and  $\lambda$ .  $\square$

## 2.4 Higher Connectivity

**Proposition 2.10.** Let  $(X, A)$  be a pair, and  $f: (I^n, \partial I^n) \rightarrow (X, A)$  a pointed map. The followings are equivalent.

1.  $f$  is null-homotopic.
2.  $f$  is homotopic rel  $\partial I^n$  to a map in  $A$ .

*Proof.* (1)  $\implies$  (2): Consider a surjective continuous map  $\lambda: I^n \times I \rightarrow I^n \times I$  such that  $\lambda|_{\partial I^n \times I}: (x, t) \mapsto (x, 0)$  and  $\lambda|_{I \times \{0\}} = \text{id}_{I^n}$ . Consider a null-homotopy  $F: I^n \times I \rightarrow X$  of  $f$ , we let  $H = F \circ \lambda: I^n \times I \rightarrow X$ . Then  $H$  is a homotopy of  $f$  such that  $H|_{\partial I^n \times \{t\}} = \text{id}_{\partial I^n}$  and  $H_1(I^n) \subset A$ .

(2)  $\implies$  (1): We may assume  $f(I^n) \subset A$ .  $J^{n-1}$  is a deformation retract of  $I^n$ . This is equivalent to that we get a homotopy  $h_t: I^n \rightarrow I^n$  such that  $\text{im } h_1 = J^{n-1}$  and  $h_0 = \text{id}$ . Then  $f \circ h_t$  is a homotopy from  $f$  to  $c_{x_0}$ .  $\square$

**Remark 2.11.** By (2),  $\pi_n(A, A) \rightarrow \pi_n(X, A)$  is trivial.

**Definition 2.12.** We say a pair  $(X, A)$  is  $n$ -connected if  $\pi_q(X, A) = 0$ ,  $\forall 1 \leq q \leq n$  and  $\pi_0(A) \rightarrow \pi_0(X)$  is surjective. Note that  $\pi_q(X, A) = 0$  is computed for all basepoints.

**Proposition 2.13.** The followings are equivalent.

1.  $(X, A)$  is  $n$ -connected.
2.  $j_*: \pi_q(A, *) \rightarrow \pi_q(X, *)$  is an isomorphism for  $q < n$  and is an epimorphism for  $q = n$ .

*Proof.* The proof follows from exact sequence of the pair  $(X, A)$  (Proposition 2.2).  $\square$

**Definition 2.14.** We say  $f: X \rightarrow Y$  is  $n$ -connected if  $f_*: \pi_k(X) \rightarrow \pi_k(Y)$  is an isomorphism for  $1 \leq k \leq n-1$  and is an epimorphism for  $k = n$ .

**Proposition 2.15.**  $f: X \rightarrow Y$  is  $n$ -connected if and only if  $(Z(f), X)$  is  $n$ -connected.

*Proof.* The proof follows from exact sequence of the pair  $(Z(f), X)$  (Proposition 2.2) and  $Z(f) \simeq Y$ .  $\square$

## 2.5 Excision and Suspension

**Theorem 2.16** (Blaskers-Massey). Let  $Y = Y_1 \cup Y_2$  be union of two open subsets and  $Y_0 = Y_1 \cap Y_2 \neq \emptyset$ . Suppose  $\pi_i(Y_1, Y_0) = 0$  for any  $0 < i < p$ ,  $p \geq 1$  and  $\pi_j(Y_2, Y_0) = 0$  for any  $0 < j < q$ ,  $q \geq 1$ . Then the map  $\iota: \pi_n(Y_2, Y_0) \rightarrow \pi_n(Y, Y_1)$  is an isomorphism for  $1 \leq n \leq p+q-3$  and is an epimorphism for  $n = p+q-2$ .

*Proof.* See textbook § 6.7.  $\square$

**Proposition 2.17.** Let  $j: A \hookrightarrow X$  be a cofibration. Consider a push-out diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ j \downarrow & & \downarrow J \\ X & \xrightarrow{F} & Y \end{array}$$

where  $Y = X \sqcup B/f(a) \sim j(a)$ . Suppose  $\pi_i(X, A) = 0$ ,  $\forall 0 < i < p$  and  $\pi_i(Z(f), A) = 0$ ,  $\forall 0 < i < q$ . Then the induced map  $(F, f)_*: \pi_n(X, A) \rightarrow \pi_n(Y, B)$  is an isomorphism for  $1 \leq n \leq p+q-3$  and is an epimorphism for  $n = p+q-2$ .

*Proof.* Replace  $f$  by a cofibration

$$\begin{array}{ccccc} A & \xrightarrow{k} & Z(f) & \xrightarrow{p} & B \\ j \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{K} & Z & \xrightarrow{P} & Y \end{array}$$

where  $Z = Z(f) \sqcup X/(a, 0) \sim j(a)$ ,  $f = p \circ k$ ,  $F = P \circ K$ . Since  $p: Z(f) \rightarrow B$  is a homotopy equivalence and  $P: Z \rightarrow Y$  is given by push-out,  $P$  is also a homotopy equivalence. Let  $Z = Z_1 \cup Z_2$  where  $Z_2 = X \sqcup A \times (\varepsilon, 1]/\sim$  and  $Z_1 = B \sqcup A \times [0, \varepsilon]/\sim$ . Then  $Z_1 \cap Z_2 = A \times (\varepsilon, 1 - \varepsilon)$ . Applying excision (Theorem 2.16),

$$\pi_n(X, A) \cong \pi_n(Z_2, Z_0) \rightarrow \pi_n(Z, Z_1) \cong \pi_n(Y, B)$$

has desired properties.  $\square$

**Theorem 2.18** (Quotient). Let  $A \hookrightarrow X$  be a cofibration. Suppose  $\pi_i(CA, A) = 0$  for  $0 < i < p$  and  $\pi_i(X, A) = 0$  for  $0 < i < q$ . Then  $p_*: \pi_n(X, A) \rightarrow \pi_n(X/A, *)$  is an isomorphism for  $1 \leq n \leq p+q-3$  and is an epimorphism for  $n = p+q-2$ .

*Proof.* Note  $X \cup CA$  fits into the following push-out diagram.

$$\begin{array}{ccc} A & \longrightarrow & CA \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \cup CA \end{array}$$

Then we get the result for

$$\pi_n(X, A) \rightarrow \pi_n(X \cup CA, CA).$$

Since  $A \hookrightarrow X$  is a cofibration,  $CA \hookrightarrow X \cup CA$  is also a cofibration. Notice that because  $CA$  is contractible,  $X \cup CA \rightarrow X \cup CA/CA$  is a homotopy equivalence (This is left as an exercise). Then

$$\pi_n(X, A) \rightarrow \pi_n(X \cup CA, CA) \cong \pi_n(X \cup CA/CA, *) \cong \pi_n(X/A, *)$$

has desired properties.  $\square$

**Definition 2.19.** We say  $(X, x_0)$  is well-pointed if  $x_0 \hookrightarrow X$  is a cofibration.

**Example 2.20.** • For any CW-complex or manifold, it is well-pointed for any point.

- $X = \{\frac{1}{n} : n \in \mathbb{Z}^+\} \cup \{0\}$ ,  $x_0 = 0$  is not well-pointed.

**Theorem 2.21** (Freudenthal Suspension). Let  $(X, x_0)$  be a well-pointed  $n$ -connected space. Then  $\Sigma_* : \pi_j(X) \rightarrow \pi_{j+1}(\Sigma X)$  is an isomorphism for  $0 \leq j \leq 2n$  and is an epimorphism for  $j = 2n + 1$ .

*Proof.* The suspension map is given by

$$\pi_j(X) = [S^j, X]^o \xrightarrow{\Sigma_*} [S^{j+1}, \Sigma X]^o = \pi_{j+1}(X) .$$

We factor  $\Sigma_*$  into

$$\begin{array}{ccc} \Sigma_* : \pi_j(X) & \xleftarrow[\cong]{\partial} & \pi_{j+1}(CX, X) \\ & & \downarrow p_* \\ & & \pi_{j+1}(\Sigma X) \end{array}$$

To use Theorem 2.18, we verify  $X \hookrightarrow CX$  is a cofibration. Consider the push-out diagram

$$\begin{array}{ccc} X \times \partial I \cup \{x_0\} \times I & \longrightarrow & X \\ \downarrow & & \downarrow \\ X \times I & \longrightarrow & CX \end{array}$$

where  $CX = X \times I / X \times \{0\} \cup \{x_0\} \times I$ . Because  $\partial I \hookrightarrow I$  and  $x_0 \hookrightarrow X$  are cofibrations, we have  $\{x_0\} \times I \cup X \times \partial I \hookrightarrow X \times I$  is also a cofibration. By push-out diagram,  $X \hookrightarrow CX$  is a cofibration. Now we have exact sequence

$$\begin{array}{ccc} \pi_j(CX, X) & \xrightarrow{\partial} & \pi_{j-1}(X) \longrightarrow 0 \\ \uparrow & & \uparrow \\ \pi_j(CX) & = & 0 \\ \uparrow & & \uparrow \\ \pi_j(X) & & \end{array}$$

Then  $(CX, X)$  is  $(n+1)$ -connected. And  $p_* : \pi_j(CX, X) \rightarrow \pi_j(\Sigma X)$  is isomorphism for  $j \leq 2n-1$  and is an epimorphism for  $j = 2n$ . Then we apply Theorem 2.18 with  $p = q = n+2$  and get the desired properties for  $\Sigma_* : \pi_{j-1}(X) \rightarrow \pi_j(X)$ .  $\square$

## 2.6 Computation of Homotopy Groups

**Example 2.22.**

$$\pi_k(S^n) \cong \begin{cases} 0, & k < n \\ \mathbb{Z}, & k = n \end{cases} .$$

$$\pi_1(S^1) \cong \mathbb{Z}, \quad \pi_1(S^n) \cong 0, \quad \forall n \geq 2.$$

To compute  $\pi_2(S^2)$ , consider the Hopf fibration

$$S^1 \hookrightarrow S^3 \twoheadrightarrow S^2 .$$



This is given by the fibre bundle

$$S^2 = \mathbb{CP}^1 = \mathbb{C}^2 - \{0\}/\mathbb{C}^* = S^3/S^1.$$

We have the following fibre sequence

$$\begin{array}{ccccccc} \pi_2(S^1) & \longrightarrow & \pi_2(S^3) & \longrightarrow & \pi_2(S^2) & \xrightarrow{\partial} & \pi_1(S^1) \longrightarrow \pi_1(S^3) \\ \parallel & & & & & & \parallel & & \parallel \\ 0 & & & & & & \mathbb{Z} & & 0 \end{array}$$

Because  $S^1$  is 0-connected, by Suspension Theorem,  $\pi_1(S^1) \rightarrow \pi_2(S^2)$  is an epimorphism. Then  $\pi_2(S^2) \cong \mathbb{Z}$  and  $\pi_2(S^3) = 0$ .

For  $n \geq 2$ , assume  $S^n$  is  $(n-1)$ -connected, by Freudenthal's Suspension,  $\pi_j(S^n) \rightarrow \pi_{j+1}(S^{n+1})$  is an isomorphism for  $j \leq n \leq 2n$ . By induction,  $\pi_n(S^n) \cong \mathbb{Z}$  and  $\pi_j(S^n) = 0$  for  $j < n$ .

**Example 2.23.** Notice that

$$\mathbb{CP}^n = \mathbb{C}^{n+1} - \{0\}/\mathbb{C}^* = S^{2n+1}/U(1)$$

for  $n \geq 2$ , we get a fibre bundle

$$U(1) \hookrightarrow S^{2n+1} \longrightarrow \mathbb{CP}^n.$$

Then we have fibre sequence

$$\pi_j(S^{2n+1}) \longrightarrow \pi_j(\mathbb{CP}^n) \longrightarrow \pi_{j-1}(U(1)) \longrightarrow \pi_{j-1}(S^{2n+1}).$$

Then when  $j = 2$ ,  $\pi_2(\mathbb{CP}^n) \cong \mathbb{Z}$ . When  $2 \neq j \leq 2n$ ,  $\pi_j(\mathbb{CP}^n) = 0$ .

Consider  $\mathbb{CP}^\infty = \bigcup_{n \geq 1} \mathbb{CP}^n$ ,

$$\begin{array}{ccc} \mathbb{CP}^n & \hookrightarrow & \mathbb{CP}^{n+1} \\ \uparrow & & \uparrow \\ S^{2n+1} & \hookrightarrow & S^{2n+3} \\ \uparrow & & \uparrow \\ U(1) & & U(1) \end{array}$$

is induced from Five-Lemma. Then  $i_*: \pi_2(\mathbb{CP}^n) \rightarrow \pi_2(\mathbb{CP}^{n+1})$  is an isomorphism. As conclusion,

$$\pi_n(\mathbb{CP}^\infty) \cong \begin{cases} \mathbb{Z}, & n = 2 \\ 0, & n \neq 2. \end{cases}$$

**Example 2.24.** We have the following fibre bundle by transitive group action.

$$O(n) \xrightarrow{j} O(n+1) \longrightarrow S^n.$$

Since  $S^n$  is  $(n-1)$ -connected, the homotopy exact sequence for fibrations show  $j: O(n) \hookrightarrow O(n+1)$  is  $(n-1)$ -connected.

Write  $O(\infty) = \bigcup_{n=1}^\infty O(n)$ .

**Theorem 2.25** (Bott-Periodicity).

$$\pi_k(O(\infty)) \cong \pi_{k+8}(O(\infty)).$$

**Example 2.26** (Stiefel Manifolds). Denote  $V_k(\mathbb{R}^n)$  be the orthogonal  $k$ -frames in  $\mathbb{R}^n$ . Then we have

$$V_k(\mathbb{R}^n) = O(n) / O(n-k).$$

Then we get a fibration

$$O(n-k) \hookrightarrow O(n) \twoheadrightarrow V_k(\mathbb{R}^n).$$

Notice that in

$$O(n-k) \xrightarrow{j} O(n-k+1) \hookrightarrow \cdots \hookrightarrow O(n),$$

$j$  is  $(n-k-1)$ -connected, then

$$\pi_i(O(n-k)) \xrightarrow{\cong} \pi_i(O(n)) \twoheadrightarrow \pi_i(V_k(\mathbb{R}^n))$$

for  $i \leq n-k-2$ . Therefore,  $\pi_i(V_k(\mathbb{R}^n)) = 0$  when  $i \leq n-k-1$ .

**Claim 9.**  $V_k(\mathbb{R}^n)$  is  $(n-k-1)$ -connected.

Consider the projection

$$\begin{aligned} p: V_{k+1}(\mathbb{R}^{n+1}) &\rightarrow V_1(\mathbb{R}^{n+1}) \cong S^n \\ (v_1, \dots, v_{k+1}) &\mapsto v_{k+1}. \end{aligned}$$

The fibre is  $V_k(\mathbb{R}^n)$ . We know  $S^n$  is  $(n-1)$ -connected, then  $j: V_k(\mathbb{R}^n) \rightarrow V_{k+1}(\mathbb{R}^{n+1})$  is  $(n-1)$ -connected. Therefore, we have  $\pi_{n-k}(V_k(\mathbb{R}^n)) \cong \pi_{n-k}(V_2(\mathbb{R}^{n-k+2}))$ . We know that  $\pi_1(V_2(\mathbb{R}^{n-k+2})) = 0$ . By Hurewicz Theorem,  $H_i(V_2(\mathbb{R}^{n-k+2})) \cong \pi_i(V_2(\mathbb{R}^{n-k+2}))$  for  $2 \leq i \leq n-k$ , which is non-trivial. We will do these calculations later.

## Part II

# Generalized Homology

### 3 Homology Theory and CW-Complexes

#### 3.1 Homology Theory

Denote  $R - \mathbf{MOD}$  be the category of left  $R$ -modules and  $\mathbf{TOP}(2)$  be the category of pairs  $(X, A)$  and

$$\begin{aligned} k: \mathbf{TOP}(2) &\rightarrow \mathbf{TOP}(2) \\ (X, A) &\mapsto (A, \emptyset) \end{aligned}$$

be the forgetful functor.

**Definition 3.1** (Eilenberg-Steenrod Axioms). A homology theory on  $\mathbf{TOP}(2)$  consists

1. a family of functors  $h_n: \mathbf{TOP}(2) \rightarrow R - \mathbf{MOD}$ ,
2. a family of natural transformations  $\partial_n: h_n \rightarrow h_{n-1} \circ k$  such that
  - (a) Homotopy invariance:  $h_n(f_0) = h_n(f_1)$  for  $f_0 \simeq f_1$ .
  - (b) Exact sequence:

$$\cdots \longrightarrow h_{n+1}(X, A) \xrightarrow{\partial_{n+1}} h_n(A) \longrightarrow h_n(X) \longrightarrow h_n(X, A) \longrightarrow \cdots$$

for any pair  $(X, A)$ .

- (c) Excision: Given a pair  $(X, A)$ , for any  $U \subset A$  such that  $\bar{U} \subset \text{Int}(A)$ , then inclusion induces an isomorphism  $h_n(X - U, A - U) \rightarrow h_n(X, A)$ .

**Proposition 3.2.** Given two pairs  $(X_i, A_i)$ ,  $i = 1, 2$ , we get an isomorphism

$$\bigoplus_{i=1}^2 h_n(X_i, A_i) \rightarrow h_n(X_1 \sqcup X_2, A_1 \sqcup A_2).$$

*Proof.* Consider the commutative diagram for  $A_i = \emptyset$ .

$$\begin{array}{ccccc} h_n(X_1 \sqcup X_2, X_2) & & & & h_n(X_1 \sqcup X_2, X_1) \\ & \nwarrow j_1 & & \nearrow j_2 & \\ & & h_n(X_1 \sqcup X_2) & & \\ & \nearrow i_1 & & \nwarrow i_2 & \\ h_n(X_1) & & & & h_n(X_2) \end{array}$$

$\begin{array}{c} \uparrow a_1 \\ \downarrow a_2 \cong \end{array}$

Injectivity of  $i_1 \oplus i_2$  is easy to check. For its surjectivity, take  $c \in h_n(X_1 \sqcup X_2)$ , we have  $j_1(c) = j_1 \circ i_1 \circ a_1^{-1}(j_1(c))$ . Then  $c - i_1 \circ a_1^{-1}(j_1(c)) \in \ker j_1$ . Therefore, there exists  $x \in h_n(X_2)$  such that  $i_2(x) = c - i_1(a_1^{-1} \circ j_1(c))$ . Then  $c = i_1(y) + i_2(x)$  where  $y = a_1^{-1} \circ j_1(c) \in h_n(X_1)$ .

The general case will be proved later. □

Let  $A = *$  be a single point. Define  $\tilde{h}(X) := h(X, *)$ .

Assume there is a map  $r: X \rightarrow A$  such that  $r \circ i \simeq \text{id}$ . Then  $i_*: h_n(A) \rightarrow h_n(X)$  is injective. We get short exact sequences

$$0 \longrightarrow h_n(A) \xrightleftharpoons[r_*]{i_*} h_n(X) \longrightarrow h_n(X, A) \longrightarrow 0.$$

Then we have splitting  $h_n(X) \cong h_n(A) \oplus h_n(X, A)$  and  $h_n(X, A) = \ker r_*$ . When  $A = *$ , take  $r = c: X \rightarrow *$ , then  $\widetilde{h}_n(X) = h_n(X, *) = \ker(c_*: h_n(X) \rightarrow h_n(*))$ .

**Proposition 3.3.** Let  $A \hookrightarrow X$  be a cofibration. Then the quotient map induces an isomorphism  $j_*: h_n(X, A) \rightarrow h_n(X/A, *)$ .

*Proof.* Apply excision to  $(X \cup CA, CA)$  for  $U =$  the cone point of  $CA$ , we have  $h_n(X, A) \cong h_n(X \cup CA, CA)$ . When  $A \hookrightarrow X$  is a cofibration,  $CA \hookrightarrow X \cup CA$  is a cofibre. Since  $CA$  is contractible,  $X \cup CA/CA \simeq X \cup CA$ . Then  $h_n(X \cup CA, CA) \cong h_n(X/A, *)$ .  $\square$

**Proposition 3.4.** Let  $(X, *)$  and  $(Y, *)$  be well-pointed spaces and  $f: X \rightarrow Y$  is a pointed map. Then the cofibre sequence  $X \xrightarrow{f} Y \xrightarrow{f^1} C(f)$  induces an exact sequence

$$\widetilde{h}_n(X) \xrightarrow{f_*} \widetilde{h}_n(Y) \xrightarrow{f_*^1} \widetilde{h}_n(C(f)) .$$

*Proof.* The proof follows the commutative diagrams

$$\begin{array}{ccccc} \widetilde{h}_n(X) & \longrightarrow & \widetilde{h}_n(Z(f)) & \longrightarrow & \widetilde{h}_n(Z(f), X) \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ \widetilde{h}_n(X) & \longrightarrow & \widetilde{h}_n(Y) & \longrightarrow & \widetilde{h}_n(C(f)) \end{array}$$

and

$$\begin{array}{ccc} X \times \partial I & \xrightarrow{(\text{id}, f)} & X \sqcup Y \\ \downarrow & & \downarrow \\ X \times I & \longrightarrow & Z(f) \end{array}$$

$\square$

**Proposition 3.5.** Given a triple  $(X, A, B)$ . Assume  $B \hookrightarrow X$  is a cofibration, we get an exact sequence

$$\cdots \longrightarrow h_n(A, B) \longrightarrow h_n(X, B) \longrightarrow h_n(X, A) \xrightarrow{\partial} h_{n-1}(A, B) \longrightarrow \cdots .$$

*Proof.* Applying excision, we know that  $(X, A, B)$  and  $(X \cup CB, A \cup CB, CB)$  have the same sequence. Applying homotopy equivalence,  $(X \cup CB, A \cup CB, CB)$  and  $(X, A, *)$  have the same sequence. The triple sequence of  $(X, A, *)$  is the reduced pair sequence of  $(X, A)$ .  $\square$

### 3.1.1 Suspension Isomorphism

Given a pair  $(X, A)$ , we have the suspension isomorphism

$$\sigma: h_n(X, A) \rightarrow h_n(\partial I \times X \cup I \times A, \{0\} \times X \cup I \times A)$$

by excision for  $U = (0, 1] \times A \cup \{0\} \times X$ . Consider the boundary map  $\partial_{n+1}: h_{n+1}(I \times X, \partial I \times X \cup I \times A) \rightarrow h_n(\partial I \times X \cup I \times A, \{0\} \times X \cup I \times A)$ . Notice that  $X \simeq I \times X \simeq \{0\} \times X \cup I \times A$ , we have the exact sequence

$$h_{n+1}(I \times X, \partial I \times X \cup I \times A) \xrightarrow{\partial_{n+1}} h_n(\partial I \times X \cup I \times A, \{0\} \times X \cup I \times A) \longrightarrow h_n(I \times X, \{0\} \times X \cup I \times A) = 0 .$$

Then  $\partial_{n+1}$  is an isomorphism and so is  $\partial_{n+1}^{-1}$ . We get isomorphisms

$$h_n(x, A) \longrightarrow h_n(\partial I \times X \cup I \times A, \{0\} \times X \cup I \times A) \xrightarrow{\partial_{n+1}^{-1}} h_{n+1}((I, \partial I) \times (X, A)) .$$

Choose  $A = *$ , define the suspension isomorphism by

$$\begin{array}{ccc} h_n(X, *) & \longrightarrow & h_{n+1}^\sigma(X \times I, \partial I \times X \cup I \times *) \\ \cong \downarrow & & \downarrow \text{quotient} \\ \widetilde{h}_n(X) & \xrightarrow{\tilde{\sigma}} & \widetilde{h}_{n+1}(\Sigma X) \end{array}$$

Assume  $(X, *)$  is well-pointed, by Hurwicz map, we have the commutative diagram

$$\begin{array}{ccc} \pi_n(X) & \xrightarrow{\Sigma_*} & \pi_n(\Sigma X) \\ \downarrow & & \downarrow \\ \widetilde{h}_n(X) & \xrightarrow{\tilde{\sigma}} & \widetilde{h}_{n+1}(X) \end{array}$$

### 3.2 CW-Complex

**Definition 3.6.** We say  $X$  is obtained from  $A$  by attaching an  $n$ -cell if there exists a push-out diagram

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\varphi} & A \\ \downarrow & & \downarrow \\ D^n & \xrightarrow{\Phi} & X \end{array}$$

where  $\varphi$  is called attaching map and  $\Phi$  is called characteristic map.

A CW-decomposition of  $(X, A)$  is a filtration  $A = X^{-1} \subset X^0 \subset \dots \subset X$  such that

1.  $X = \bigcup_{n \geq -1} X^n$ ,
2.  $X^n$  is obtained from  $X^{n-1}$  by attaching  $n$ -cells,
3.  $X$  carries the colimit topology (weak topology).

**Proposition 3.7.** Let  $(Y, B)$  be an  $n$ -connected pair,  $(X, A)$  be a relative CW-complex of dimension  $\leq n$ . Then each map  $(F, f): (X, A) \rightarrow (Y, B)$  is homotopic rel.  $A$  to a map into  $B$ . When dimension  $< n$ , the homotopy class rel.  $A$  of maps  $X \rightarrow B$  is unique.

*Proof.* Consider

$$\begin{array}{ccccc} \bigsqcup_k S_k^{q-1} & \longrightarrow & A & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ \bigsqcup_k D_k^q & \xrightarrow{\Phi^q} & X^q & \xrightarrow{F^q} & Y \end{array}$$

For any  $q \leq n$ ,  $\pi_q(Y, B) = 0$ . Then  $F^q \circ \Phi^q$  can be homotoped into  $B$  rel.  $\bigsqcup_k S_k^{q-1}$ .

$$\begin{array}{ccccc} \bigsqcup_k S_k^{q-1} & \longrightarrow & A & \longrightarrow & B \\ \downarrow & & \downarrow & \nearrow \text{dashed} & \downarrow \\ \bigsqcup_k D_k^q & \xrightarrow{\Phi^q} & X^q & \xrightarrow{F^q} & Y \end{array}$$

When dimension of  $(X, A) < n$ , apply the argument to  $(X \times I, X \times \partial I \cup A \times I)$  which is a relative CW-complex of dimension  $< n + 1$ .  $\square$

**Theorem 3.8.** Suppose  $h: B \rightarrow Y$  is  $n$ -connected. Then for a CW-complex  $X$ ,  $h_*: [X, B] \rightarrow [X, Y]$  is bijective when  $\dim X < n$  and surjective when  $\dim X = n$ .

*Proof.* We map replace  $Y$  by  $Z(h)$ :  $B \longrightarrow Z(h) \xrightarrow{\cong} Y$ .

**Surjectivity:** Let  $A = \emptyset$ . Apply Proposition 3.7 to  $(X, \emptyset) \rightarrow (Z(h), B)$ .

**Injectivity:** Apply Proposition 3.7 to  $(X \times I, X \times \partial I)$ .  $\square$

**Theorem 3.9** (Whitehead). Let  $f: Y \rightarrow Z$  be a map between CW-complexes with  $\dim Y, \dim Z \leq n \leq \infty$ . If  $f_*: \pi_q(Y) \rightarrow \pi_q(Z)$  is an isomorphism for  $0 \leq q \leq n$ , then  $f$  is a homotopy equivalence.

*Proof.* The map  $f: Y \rightarrow Z$  is  $n$ -connected. By Theorem 3.8,  $f_*: [Z, Y] \rightarrow [Z, Z]$  is surjective. Then there exists  $g: Z \rightarrow Y$  such that  $f \circ g \simeq \text{id}_Z$  and  $g$  is  $n$ -connected. Use Theorem 3.8 again, there exists  $h: Y \rightarrow Z$  such that  $g \circ h \simeq \text{id}_Y$ . Therefore,  $g$  is a homotopy equivalence.  $\square$

**Theorem 3.10** (Suspension Theorem). Suppose  $Y$  is  $n$ -connected and  $X$  is a CW-complex. Then  $\Sigma_*: [X, Y]^o \rightarrow [\Sigma X, \Sigma Y]^o$  is bijective if  $\dim X \leq 2n$  and is surjective if  $\dim X = 2n + 1$ .

*Proof.* We know that  $[\Sigma X, \Sigma Y]^o \cong [X, \Omega \Sigma Y]^o$ . By Freudenthal's Suspension Theorem,  $\Sigma_*: [S^k, Y]^o \rightarrow [S^{k-1}, \Sigma Y]^o$  is an isomorphism when  $k \leq 2n$  and epimorphism if  $k = 2n + 1$ . Notice that  $\pi_{k+1}(\Sigma Y) \cong \pi_k(\Omega \Sigma Y)$ ,  $\sigma_*: [S^k, Y]^o \rightarrow [S^k, \Omega \Sigma Y]^o$  is adjoint to  $\Sigma_*$  and is reduced from

$$\begin{aligned} \sigma: Y &\rightarrow \Omega \Sigma Y \\ y &\mapsto [t \mapsto (y, t)]. \end{aligned}$$

Therefore,  $\sigma$  is  $(2n + 1)$ -connected. Apply Theorem 3.8 to  $\sigma_*: [X, Y]^o \rightarrow [X, \Omega \Sigma Y]^o$ .  $\square$

### 3.3 CW-Approximation

**Proposition 3.11.** Suppose  $X$  is obtained from  $A$  by attaching  $(n+1)$ -cell. Then  $(X, A)$  is  $n$ -connected.

*Proof.* Consider the push-out diagram

$$\begin{array}{ccc} S^n & \longrightarrow & A \\ \downarrow & & \downarrow \\ D^{n+1} & \longrightarrow & X \end{array}$$

The Excision Theorem of push-out shows that  $\pi_0(X, A) = 0$  and  $\pi_q(D^{n+1}, S^n) = 0$  for any  $1 \leq q \leq n$ . Then  $(\Phi, \varphi): (D^{n+1}, S^n) \rightarrow (X, A)$  is  $(n - 1)$ -connected. When  $k \leq n - 1$ ,  $0 = \pi_k(D^{n+1}, S^n) \rightarrow \pi_k(X, A)$  is an isomorphism.  $\square$

**Theorem 3.12.** Let  $f: A \rightarrow Y$  be a  $k$ -connected map. Then for each  $n > k$ , there exists a relative CW-complex  $(X, A)$  with cells in  $\dim \in \{k + 1, \dots, n\}$  and an  $n$ -connected extension  $F: X \rightarrow Y$  of  $f$ .

*Proof.* When  $n = 1$ ,  $k = 0$ , the proof is trivial. Consider  $k = n - 1$ ,  $n \geq 2$ . Assume  $f: A \rightarrow Y$  is  $(n - 1)$ -connected. Replace  $Y$  by  $Z(f)$ :

$$\begin{array}{ccccc} A & \longrightarrow & Z(f) & \longrightarrow & Y \\ \downarrow & \nearrow & \nearrow & \nearrow & \\ X & & & & \end{array}$$

Assume  $f: A \rightarrow Y$  is an inclusion. Let  $(\Phi_j, \varphi_j): (D^n, S^{n-1}) \rightarrow (Y, A)$  be a set of generators of  $\pi_n(Y, A)$ . Attach  $n$ -cells on  $A$  using  $\varphi_j$ . Regard  $\Phi_j$  as a null-homotopy of  $f \circ \varphi_j$ .  $F$  is obtained by push-out property.

$$\begin{array}{ccccc} S^{n-1} & \xrightarrow{\varphi_j} & A & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \Phi_j & \nearrow & \\ D^n & \longrightarrow & X & \xrightarrow{\quad} & \end{array}$$

And then  $F_*: \pi_n(X, A) \rightarrow \pi_n(Y, A)$  is an epimorphism.

Consider the diagram

$$\begin{array}{ccccccccc}
\pi_n(A) & \longrightarrow & \pi_n(X) & \longrightarrow & \pi_n(X, A) & \longrightarrow & \pi_{n-1}(A) & \longrightarrow & \pi_{n-1}(X) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow F_* & & \downarrow F_* & & \downarrow \cong & & \downarrow F_* & & \downarrow \\
\pi_n(A) & \longrightarrow & \pi_n(Y) & \longrightarrow & \pi_n(Y, A) & \longrightarrow & \pi_{n-1}(A) & \xrightarrow{f_*} & \pi_{n-1}(Y) & \longrightarrow & 0
\end{array}$$

Notice that  $F_*: \pi_n(X) \rightarrow \pi_n(Y)$  is also an epimorphism. Then by chasing diagram, we know that  $F_*: \pi_{n-1}(X) \rightarrow \pi_{n-1}(Y)$  is an isomorphism.  $\square$

**Corollary 3.13.** Given any space  $Y$ , there exists a CW-complex  $X$  and a map  $F: X \rightarrow Y$  such that  $F_*: \pi_n(X) \rightarrow \pi_n(Y)$  is an isomorphism for any  $n \geq 0$ . Such  $X$  is called a CW-approximation of  $Y$ .

**Theorem 3.14.** Let  $Y$  be a  $k$ -connected CW-complex. Then there exists a CW-complex  $X$  such that

1.  $X$  is homotopy equivalent to  $Y$ ;
2.  $X^k = \{*\}$ .

*Proof.* Apply Theorem 3.12 to  $A = \{*\} \hookrightarrow Y$  which is a  $k$ -connected map.  $\square$

### 3.4 Eilenberg-MacLane Space

#### 3.4.1 Remarks about Compactly Generated Spaces

**Definition 3.15.** A Hausdorff space  $X$  is said to be compactly generated if for any compact subset  $K$ , a subset  $A \subset X$  satisfies  $A \cap K$  is closed, then  $A$  is closed in  $X$ .

**Example 3.16.** These spaces are compactly generated spaces:

- locally compact Hausdorff spaces,
- metric spaces,
- CW-complexes with finite cells in each dimension.

Given a Hausdorff space  $X$ , we can put a new topology  $\mathcal{T}$  on  $X$  by imposing:

$$A \subset X \text{ is } \mathcal{T}\text{-closed} \iff A \cap K \text{ is closed for any compact subset } K \subset X$$

such that  $X$  is compactly generated under  $\mathcal{T}$ .

**Fact 3.17.** If  $X, Y$  are both compactly generated spaces, then  $X \times Y$  needs not to be compactly generated.

**Definition 3.18.** We denote by  $X \times_k Y$  the product with compactly generated topology. We denote by  $kF(X, Y)$  the space of continuous maps from  $X$  to  $Y$ , equipped the compactly generated topology.

**Theorem 3.19.** Let  $X, Y, Z$  be compactly generated spaces. Then

1. The evaluation map

$$\begin{aligned}
kF(Y, Z) \times_k Y &\rightarrow Z \\
(f, g) &\mapsto f(g)
\end{aligned}$$

is continuous.

2. The adjoint map

$$kF(X, kF(Y, Z)) \rightarrow kF(X \times_k Y, Z)$$

is a homeomorphism.

**Proposition 3.20.** Suppose  $\pi_j(Y) = 0$  for  $j > n$ . Let  $X$  be obtained from  $A$  by attaching cells of  $\dim \geq n + 2$ . Then  $\iota_*: [X, Y] \rightarrow [A, Y]$  is a bijection.

*Proof. Surjectivity:* Given  $f: A \rightarrow Y$  and attaching map  $\varphi: S^k \rightarrow A$ ,  $k \geq n + 1$ . Then  $f \circ \varphi: S^k \rightarrow Y$  is null-homotopic which can be extended over  $X$ .

*Injectivity:* Apply the argument to  $(X \times I, X \times \partial I \cup A \times I)$ .  $\square$

**Definition 3.21.** Let  $\pi$  be an abelian group. An Eilenberg-MacLane space of type  $K(\pi, n)$  is a CW-complex such that

$$\pi_j(X) = \begin{cases} \pi, & j = n; \\ 0, & j \neq n. \end{cases}$$

**Proposition 3.22.** Suppose  $X_1, X_2$  are  $(n - 1)$ -connected CW-complex with  $n \geq 2$ . Then

$$\pi_n(X_1) \oplus \pi_n(X_2) \rightarrow \pi_n(X_1 \vee X_2)$$

is an isomorphism.

*Proof.* We can assume  $X_i^{n-1} = \{*\}$  by CW-approximation. Therefore, cells in  $X_1 \times X_2$  have dimension  $0, n, \geq 2n$ . Then  $X_1 \times X_2$  is obtained from  $X_1 \vee X_2$  by attaching cells of  $\dim \geq 2n$ . We have  $\pi_n(X_1 \vee X_2) \rightarrow \pi_n(X_1 \times X_2) = \pi_n(X_1) \oplus \pi_n(X_2)$  is an isomorphism.  $\square$

**Theorem 3.23.** Let  $X$  be a  $(n - 1)$ -connected CW-complex. Suppose  $Y$  satisfies  $\pi_j(Y) = 0, \forall j > n \geq 2$ . Then the map  $h_*: [X, Y]^o \rightarrow \text{Hom}(\pi_n(X), \pi_n(Y))$  is a bijection.

*Proof.* We can assume  $X^{n-1} = \{*\}$  by Proposition 3.20. Then  $[X, Y]^o = [X^{n+1}, Y]^o$ . Notice that  $\pi_n(X^{n+1}) = \pi_n(X)$ , we only need to prove  $h_X: [X^{n+1}, Y]^o \rightarrow \text{Hom}(\pi_n(X^{n+1}), \pi_n(Y))$  is a bijection.

We know  $X^n = \bigvee_j S_j^n := B$ . Applying homotopy, we may assume all attaching maps of  $(n + 1)$ -cells are cased. Then  $X^{n+1}$  is the mapping cone  $f: A := \bigvee_k S_k^n \rightarrow \bigvee S_j^n = B$ .

We have the cofibre sequence

$$[A, Y]^o \longleftarrow [B, Y]^o \longleftarrow [X^{n+1}, Y]^o \longleftarrow [\Sigma A, Y]^o \longleftarrow \dots$$

Notice that

$$[\Sigma A, Y]^o = \left[ \Sigma \bigvee_k S_k^n, Y \right]^o = \left[ \bigvee_k \Sigma S_k^n, Y \right]^o = \left[ \bigvee_k S_k^{n+1}, Y \right]^o = 0$$

because  $[h] = \sum_k [h_k]$  and  $\pi_{n+1}(Y) = 0$ .

**Claim 10.**

$$\pi_n(A) \xrightarrow{f_*} \pi_n(B) \longrightarrow \pi_n(X^{n+1}) \longrightarrow 0$$

is exact.

*Proof of Claim.* Consider the push-out diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow i & & \downarrow \\ CA & \longrightarrow & X^{n+1} \end{array}$$



We know

$$\begin{array}{ccccccc} \pi_m(A) & \longrightarrow & \pi_m(CA) & \longrightarrow & \pi_m(CA, A) & \xrightarrow{\cong} & \pi_{m-1}(A) \longrightarrow 0 \\ & & \parallel & & & & \\ & & 0 & & & & \end{array}$$

Then  $\pi_m(CA, A) = 0$  for any  $m \leq n$ . We know  $f$  is  $(n-1)$ -connected. Applying excision,  $\pi_m(CA, A) \rightarrow \pi_m(X^{n+1}, B)$  is an isomorphism for  $m \leq 2n-1$ . We have an exact sequence

$$\begin{array}{ccccccc} \pi_m(B) & \longrightarrow & \pi_m(X^{n+1}) & \longrightarrow & \pi_m(X^{n+1}, B) & \longrightarrow & \pi_{m-1}(B)0 \\ & & & & \parallel & & \\ & & & & 0 & & \end{array}$$

when  $m \leq n$ . Then

$$\begin{array}{ccccccc} \pi_{n+1}(CA, A) & \xrightarrow[\cong]{\text{excision}} & \pi_{n+1}(X^{n+1}, B) & \longrightarrow & \pi_n(B) & \longrightarrow & \pi_n(X) \longrightarrow 0 \\ \partial \downarrow \cong & & & & & & \\ \pi_n(A) & & & & & & \end{array}$$

□

Apply  $\text{Hom}(-, \pi_n(Y))$ , we get an exact sequence

$$\begin{array}{ccccccc} \text{Hom}(\pi_n(A), \pi_n(Y)) & \longleftarrow & \text{Hom}(\pi_n(B), \pi_n(Y)) & \longleftarrow & \text{Hom}(\pi_n(X^{n+1}), \pi_n(Y)) & \longleftarrow & 0 \\ h_A \uparrow & & h_B \uparrow & & h_X \uparrow & & \cong \uparrow \\ [A, Y]^o & \longleftarrow & [B, Y]^o & \longleftarrow & [X^{n+1}, Y]^o & \longleftarrow & 0 \end{array}$$

**Claim 11.**  $h_A$  and  $h_B$  are bijections.

*Proof of Claim.* We have

$$\begin{aligned} \text{Hom}(\pi_n(A), \pi_n(Y)) &= \text{Hom}\left(\pi_n\left(\bigvee_j S_j^n\right), \pi_n(Y)\right) = \text{Hom}\left(\bigoplus_j \pi_n(S_j^n), \pi_n(Y)\right) \\ &= \prod_j \text{Hom}(\pi_n(S_j^n), \pi_n(Y)) \cong \prod_j \pi_n(Y) \end{aligned}$$

and

$$[A, Y]^o = \left[\bigvee_j S_j^n, Y\right]^o = \prod_j [S_j^n, Y]^o = \prod_j \pi_n(Y).$$

□

Finally, by claim that  $[X^{n+1}, Y]^o \rightarrow [B, Y]^o$  is injective, we get our conclusion by something like Five Lemma. □

**Theorem 3.24.** Let  $\pi$  be an abelian group and  $n \geq 2$ . Then the Eilenberg-MacLane space  $K(\pi, n)$  exists and is unique up to homotopy.

*Proof. Uniqueness:* Assume  $X, Y$  are both  $K(\pi, n)$ . Then by Theorem 3.23,

$$h_X: [X, Y]^o \rightarrow \text{Hom}(\pi_n(X), \pi_n(Y)) = \text{Hom}(\pi, \pi)$$

is a bijection. Choose  $f: X \rightarrow Y$  such that  $h_X([f]) = \text{id}$ . Then  $f$  is a weak homotopy equivalence. Whitehead Theorem gives us that  $f$  is in fact a homotopy equivalence.

**Existence:** Consider a free resolution

$$F_1 \longrightarrow F_0 \longrightarrow \pi \longrightarrow 0$$

with relators  $F_1$  and generators  $F_0$ . Construct  $X^{n+1}$  as the mapping cone of  $g: F_1 \hookrightarrow \bigvee_k S_k^n \rightarrow \bigvee_j S_j^n \hookrightarrow F_0$ . Therefore,  $X^{n+1}$  is  $(n+1)$ -connected and  $\pi_n(X^{n+1}) = \pi$ . We attach cells of  $\dim \geq n+2$  to eliminate  $\pi_m(X)$  for  $m \geq n+1$ , by Zorn's Lemma, we finish our construction.  $\square$

**Definition 3.25.**  $K(\pi, 0) := \pi$  equipped with discrete topology.  $K(\pi, 1)$  is constructed similar to Theorem 3.24, but the uniqueness will be proved later.

### 3.5 Spectral Homology

In this section, we assume that  $\pi$  is finitely generated and  $X$  is compactly generated.

**Definition 3.26.** A spectrum is a sequence of pairs  $\{(E_n, e_n)\}_{n \geq 0}$  where  $E(n)$  is a pointed space,  $e_n: \Sigma E(n) \rightarrow E(n+1)$  is a pointed map. We say a spectrum is an  $\Omega$ -spectrum if  $\varepsilon_n: E(n) \rightarrow \Omega E(n+1)$  is a homotopy equivalence, where  $\varepsilon_n$  is the adjoint of  $e_n$ .

**Example 3.27.** 1. Sphere Spectrum:  $E(n) = S^n$ ,  $e_n: \Sigma S^n \rightarrow S^{n+1}$  is the identity map

$$\begin{aligned} \Sigma S^n &= S^n \wedge S^1 \cong S^{n+1} \\ \mathbb{R}^{n+1} \times I &\hookrightarrow \mathbb{R}^{n+2}. \end{aligned}$$

2. Eilenberg-MacLane Spectrum: Fix an abelian group  $\pi$ . Let  $E(n) = K(\pi, n)$ . Construct  $e_n: \Sigma K(\pi, n) \rightarrow K(\pi, n+1)$  as follows:

(a) Milnor:  $\Omega K(\pi, n+1)$  is a CW-complex. Then  $[S^k, \Omega K(\pi, n+1)]^o = [S^{k+1}, K(\pi, n+1)]^o$  and then  $\Omega K(\pi, n+1) \cong K(\pi, n)$ . Define  $e_n: \Sigma K(\pi, n) \rightarrow K(\pi, n+1)$  as the adjoint map; or

(b) Notice that  $\pi_k(\Sigma K(\pi, n)) = \begin{cases} 0, & k \leq n \\ \pi, & k = n+1 \end{cases}$  because  $\pi_k(K(\pi, n)) \rightarrow \pi_{k+1}(\Sigma K(\pi, n))$  is an isomorphism when  $k \leq 2n-2$ . Then  $K(\pi, n+1)$  is obtained from  $\Sigma K(\pi, n)$  by attaching cells of  $\dim \geq n+3$ . Take  $e_n: \Sigma K(\pi, n) \rightarrow K(\pi, n+1)$  to be the inclusion map.

**Definition 3.28.** A reduced homology theory consists of a family of functors  $\tilde{h}_n: \mathbf{TOP}^o \rightarrow R\text{-MOD}$  and isomorphisms  $\sigma_n: \tilde{h}_n \rightarrow \tilde{h}_{n+1} \circ \Sigma$  that satisfy

1. Homotopy invariance:  $\tilde{h}_n(f_0) = \tilde{h}_n(f_1)$  if  $f_0 \simeq f_1$ .
2. Exactness: each cofibre sequence

$$X \xrightarrow{f} Y \xrightarrow{f'} C(f)$$

induces an exact sequence

$$\tilde{h}_*(X) \longrightarrow \tilde{h}_*(Y) \longrightarrow \tilde{h}_*(C(f)).$$

**Remark 3.29.** Unreduced theory  $\iff$  reduced theory. To see that, define  $h_n(X) = \tilde{h}_n(X \sqcup \{*\})$  and  $h_n(X, A) = \tilde{h}_n(C(X, A))$ .

Let  $E = \{(E(n), e_n)\}$  be a spectrum. We get suspension maps

$$[S^{n+k}, E(n) \wedge X]^o = \pi_{n+k}(E(n) \wedge X) \rightarrow \pi_{n+k+1}(E(n+1) \wedge X) = [S^{n+k+1}, E(n+1) \wedge X]^o$$

and

$$\Sigma(E(n) \wedge X) = S^1 \wedge (E(n) \wedge X) = \Sigma E(n) \wedge X.$$

Define  $E_n(X) := \text{colim}_{k \rightarrow \infty} \pi_{n+k}(E(k) \wedge X)$ , and  $\sigma_n: E_n(X) \rightarrow E_{n+1}(\Sigma X)$  is defined via  $[S^{n+k}, E(n) \wedge X] \rightarrow [S^{n+k+1}, E(n) \wedge \Sigma X]$ .

**Theorem 3.30.**  $\{(E_n(X), \sigma_n)\}$  defines a reduced homology theory.

*Proof.* Homotopy invariance is by definitions.

**Injectivity of  $\sigma_n$ :** Suppose  $x \in \ker \sigma_n$ , there exists  $[f] \in [S^{n+k}, E(k) \wedge X]^o$  such that  $[f]$  represents  $x$  and  $f \wedge \text{id}_{S^1}: S^{n+k} \wedge S^1 \rightarrow (E(k) \wedge X) \wedge S^1$  is null-homotopic. Then

$$S^{n+k} \wedge S^1 \xrightarrow{f \wedge \text{id}} E(k) \wedge \Sigma X \xrightarrow{e_k \wedge \text{id}} E(k+1) \wedge X$$

is null-homotopic. Note that  $[(e_k \wedge \text{id}) \circ (f \wedge \text{id})]$  represents  $x$  as well. We must have  $x = 0$ .

**Surjectivity of  $\sigma_n$ :** Given  $g: S^{n+k+1} \rightarrow E(k) \wedge X \wedge S^1$ . Then define

$$f: S^{n+k+1} \xrightarrow{g} E(k) \wedge X \wedge S^1 \xrightarrow{e_k} E(k+1) \wedge X$$

and we have  $\sigma_n([f]) = [g]$ .

**Exactness of Cofibre Sequence:** Consider

$$E_n(X) \xrightarrow{f_n} E_n(Y) \xrightarrow{f'_n} E_n(C(f)).$$

Suppose  $z \in \ker f'_n$  and write  $h: S^{n+k} \rightarrow E(k) \wedge Y$  to represent  $z$ . Then  $(\text{id}_{E(k)} \wedge f') \circ h: S^{n+k} \rightarrow E(k) \wedge C(f)$  is null-homotopic. Consider cofibre sequences:

$$\begin{array}{ccccccc} S^{n+k} & \longrightarrow & C(\text{id}) & \longrightarrow & S^{n+k} \wedge S^1 & \longrightarrow & S^{n+k} \wedge S^1 \\ \downarrow h & & \downarrow H & & \downarrow \beta & & \downarrow h \wedge \text{id} \\ E(k) \wedge Y & \longrightarrow & C(\text{id} \wedge f) & \longrightarrow & E(k+1) \wedge X & \xrightarrow{\text{id} \wedge f} & E(k+1) \wedge Y \\ & & \downarrow \simeq & & & & \\ & & E(k) \wedge C(f) & & & & \end{array}$$

where  $H$  is given by null-homotopy of  $(\text{id} \wedge f') \circ h$  and  $\beta$  is the quotient of  $H$  and the first two squares are commutative. These induce  $h \wedge \text{id}$  such that the last square is commutative up to homotopy. Therefore, under  $\text{colim}_{k \rightarrow \infty}$ , we have

$$f_*[\beta] = [(\text{id} \wedge f) \circ \beta] = [h \wedge \text{id}] = [h].$$

□

**Remark 3.31.** In Example 3.27,

1. When  $E = \{(S^n, \Sigma)\}_{n \geq 0}$ ,

$$E_n(X) = \text{colim}_{k \rightarrow \infty} \pi_{n+k}(S^k \wedge X) = \text{colim}_{k \rightarrow \infty} \pi_{n+k}(\Sigma^k X) = \pi_n^s(X),$$

which is the stable homotopy group.

2. When  $E = \{(K(\mathbb{Z}, n), \sigma_n)\}_{n \geq 0}$ ,

$$E_n(X) = \operatorname{colim}_{k \rightarrow \infty} \pi_{n+k}(K(\mathbb{Z}, n) \wedge X) \cong \tilde{H}_n(X, \mathbb{Z}),$$

which is the reduced singular homology.

**Theorem 3.32** (Brown's Representation Theory). Let  $\{(h_n, \partial_n)\}$  be a homology theory. Then there exists a spectrum  $E = \{(E(n), e_n)\}$  and natural isomorphisms  $h_n(X, A) \cong \operatorname{colim}_{k \rightarrow \infty} \pi_{n+k}(E(k) \wedge (X^+/A^+))$  for all finite CW-complexes  $(X, A)$ , where  $X^+ = X \sqcup \{*\}$  and  $A^+ = A \sqcup \{*\}$ .

## 4 Cohomology

### 4.1 Axiomatic Cohomology

**Definition 4.1.** A cohomology theory consists of

1. a family of contravariant functors  $h^n: \mathbf{TOP}(2) \rightarrow R\text{-MOD}$ ,
2. a family of natural transformations  $\delta^n: h^{n-1} \circ K \rightarrow h^n$ , where  $K: (X, A) \rightarrow (A, \emptyset)$  is the restriction, that satisfy
  - (a) H-Invariance:  $h^n(f_0) = h^n(f_1)$  if  $f_0 \simeq f_1$ .
  - (b) Exact Sequence: Given  $(X, A)$ ,

$$\cdots \longrightarrow h^{n-1}(A) \xrightarrow{\delta} h^n(X, A) \longrightarrow h^n(X) \longrightarrow h^n(A)$$

is exact.

- (c) Excision: Given a pair  $(X, A)$  with  $U \subset A$  and  $\bar{U} \subset \operatorname{Int}(A)$ , then the restriction  $h^n(X, A) \rightarrow h^n(X - U, A - U)$  is an isomorphism for any  $n$ .

**Definition 4.2.** A reduced cohomology theory is given by  $\tilde{h}^n(X) := \ker(h^n(X) \rightarrow h^n(\{*\}))$  which fits into a splitting exact sequence

$$0 \longrightarrow h^n(X, *) \longrightarrow h^n(X) \longrightarrow h^n(*) \longrightarrow 0.$$

And we have  $\tilde{h}^n(X) \cong h^n(X, *)$ .

#### 4.1.1 Mayer-Vietoris Sequence

**Definition 4.3.** Given  $A, B \subset X$ , we say the pair  $(A, B)$  is excisive if the restriction  $h^*(A \cup B, A) \rightarrow h^*(B, A \cap B)$  is an isomorphism.

**Lemma 4.4.** The followings are equivalent:

1.  $(A, B)$  is excisive.
2.  $(B, A)$  is excisive.

*Proof.* The proof is given by chasing the following diagram, where the “crossing” diagram is given by the exact sequences of triples  $(A \cup B, A, A \cap B)$  and  $(A \cup B, B, A \cap B)$ .

$$\begin{array}{ccc}
h^*(A \cup B, A) & \xrightarrow{a} & h^*(B, A \cap B) \\
& \searrow \alpha & \nearrow f \\
& h^*(A \cup B, A \cap B) & \\
& \nearrow \beta & \searrow g \\
h^*(A \cup B, B) & \xrightarrow{b} & h^*(A, A \cap B)
\end{array}$$

Assume  $a$  is an isomorphism.

**Injectivity of  $b$ :** Assume  $b(x) = 0$ . Then  $g \circ \beta(x) = b(x) = 0$ . Therefore, there is  $y$  such that  $\alpha(y) = \beta(x)$ . Then  $a(y) = f \circ \alpha(y) = f \circ \beta(x) = 0$ . Note that  $a$  is an isomorphism,  $y = 0$ . Therefore  $\beta(x) = \alpha(y) = 0$ .

Then there is  $z$  such that  $\eta(z) = x$  where  $\eta: h^*(B, A \cap B) \rightarrow h^*(A \cup B, B)$ . Note that  $z = a(a^{-1}(z)) = f \circ \alpha(a^{-1}(z))$ . Then we have  $x = \eta \circ f(\alpha(a^{-1}(z))) = 0$ .

**Surjectivity of  $b$ :** Take  $x \in h^*(A, A \cap B)$ . Note that  $a(\delta(x)) = f \circ \alpha \circ \delta(x) = 0$  where  $\delta: h^*(A, A \cap B) \rightarrow h^*(A \cup B, A)$ . Then  $\delta(x) = 0$  and then there exists  $y$  such that  $g(y) = x$ . Note that  $f(y - \alpha \circ a^{-1} \circ f(y)) = f(y) - f(y) = 0$ . Then there exists  $z \in h^*(A \cup B, B)$  such that  $\beta(z) = y - \alpha \circ a^{-1} \circ f(y)$ . Therefore  $b(z) = g \circ \beta(z) = g(y - \alpha \circ a^{-1} \circ f(y)) = g(y) = x$ .  $\square$

Assume  $(X_0, X_1)$  is an excisive pair such that  $X = X_0 \cup X_1$ . We get a connecting map

$$\Delta: h^{n-1}(X_0 \cap X_1) \rightarrow h^n(X_0, X_0 \cap X_1) \cong h^n(X, X_1) \rightarrow h^n(X).$$

Then we have the Mayer-Vietoris exact sequence

$$\longleftarrow h^n(X_0, X_1) \longleftarrow h^n(X_0) \oplus h^n(X_1) \longleftarrow h^n(X) \xleftarrow{\Delta} h^{n-1}(X_0, X_1) \longleftarrow$$

$$i_0^*x_0 - i_1^*x_1 \longleftarrow (x_0, x_1)$$

#### 4.1.2 Multiplicative Structure

**Definition 4.5.** A cup product on  $(h^*, \delta^*)$  consists of a family of  $R$ -linear maps

$$h^m(X, A) \otimes_R h^n(X, B) \rightarrow h^{m+n}(X, A \cup B)$$

for excisive pairs  $(A, B)$ , which satisfies

1. Naturality:  $f^*(x \cup y) = f^*(x) \cup f^*(y)$ .
2. Stability:  $\delta(a) \cup x = S_A(a \cup \tau_A x)$  where  $S_A: h^m(A, A \cap B) \xrightarrow{\cong} h^m(A \cup B, B) \xrightarrow{\delta} h^{r+1}(X, A \cap B)$  and  $\tau_A: h^n(X, B) \rightarrow h^n(A, A \cap B)$ .
3. Unity: There is  $1 \in h^0(\{*\})$  with  $1_X = c^*(1)$ , where  $c: X \rightarrow \{*\}$  is contraction map, satisfies

$$1_X \cup x = x \cup 1_X = x.$$

4. Associativity:  $(x \cup y) \cup z = x \cup (y \cup z)$ .
5. Commutativity:  $x \cup y = (-1)^{|x| \cdot |y|} y \cup x$ .

**Definition 4.6.** A cross product consists of  $R$ -linear maps

$$h^m(X, A) \otimes_R h^n(Y, B) \xrightarrow{\times} h^{m+n}((X, A) \times (Y, B))$$

that satisfies

1. Naturality:  $(f \times g)^*(a \times b) = f^*a \times g^*b$ .
2. Stability:  $\delta x \times y = \delta'(x \times y)$  where  $x \in h^*(A)$  and  $y \in h^*(Y, B)$  and  $\delta': h^k(A \times (Y, B)) \xrightarrow{\cong} h^k(A \times Y \cup X \times B, X \times B) \xrightarrow{\delta} h^k((X, A) \times (Y, B))$ .
3. Unity: There is  $1 \in h^0(\{*\})$  such that  $1 \times x = x \times 1 = x$ .
4. Associativity:  $(x \times y) \times z = x \times (y \times z)$ .

5. Commutativity:  $x \times y = (-1)^{|x| \cdot |y|} \tau^*(y \times x)$  where  $\tau: X \times Y \rightarrow Y \times X$ ,  $(x, y) \mapsto (y, x)$ .

In fact, the two products are equivalent. If we have a cup product, we can get a cross product by

$$x \times y := \text{pr}_1^*(x) \cup \text{pr}_2^* y, \quad x \in h^m(X, Z), y \in h^n(Y, B)$$

where  $\text{pr}_i$  is the projection map. If we have a cross product, let  $d: X \rightarrow X \times X$  be the diagonal map. We can define

$$x \cup y := d^*(x \times y).$$

When either (1) or (2) is imposed, we say the cohomology theory  $(h^*, \delta^*)$  is multiplicative.

## 4.2 The Thom Isomorphism

Denote  $h^* := h^*(\{*\})$ . The coefficient group  $h^*st(-)$  is additive and multiplicative cohomology. Then  $h^*(X, A)$  is a  $h^*$ -module given by

$$a \cdot x := c^*(a) \cup x,$$

where  $c: X \rightarrow \{*\}$  is the contraction.

**Theorem 4.7** (Leray-Hirsch). Let  $(E, E') \xrightarrow{p} B$  be relative filtration over a CW-complex  $B$ . Assume there are finitely many elements  $t_j \in h^*(E, E')$  such that  $t_j|_b \in h^*(E_b, E'_b)$  forms a basis as  $h^*$ -modules for any  $b \in B$ . Then  $h^*(E, E')$  is a free  $h^*(B)$ -module with basis  $\{t_j\}$  given by  $a \cdot x \mapsto p^*(a) \cup x$ .

*Proof.* Given  $C \subset B$ , we write  $h^*(C) \langle t \rangle$  for the free  $h^*(C)$ -module generated by formal variables  $\{t_j\}$ . We get a  $R$ -linear map

$$\begin{aligned} \varphi(C): h^*(C) \langle t \rangle &\rightarrow h^*(E|_C, E'|_C) \\ \sum a_j t_j &\mapsto \sum p^*(a_j) \cup t_j. \end{aligned}$$

Notice that the results holds for  $B^0$ . Assume the result holds on  $B^{n-1}$ . Decompose  $B^n = U \cup V$  where  $U = B^n - \text{one point from each } n\text{-cell}$  and  $V$  is the union of all open  $n$ -cells.

Notice that  $U \cap V$  is disjoint unions of  $S^{n-1}$ ,  $\varphi(U \cap V): h^*(U \cap V) \langle t \rangle \rightarrow h^*(E_{U \cap V}, E'|_{U \cap V})$  is an isomorphism by induction.

Notice that  $U$  deformation retracts into  $B^{n-1}$ ,  $\varphi(U): h^*(U) \langle t \rangle \rightarrow h^*(E|_U, E'|_U)$  is an isomorphism. Similarly, because  $V$  deformation retracts onto disjoint of points,  $\varphi(V)$  is also an isomorphism.

Applying Mayer-Vietoris sequence

$$\begin{array}{ccc} h^*(U \cup V) \langle t \rangle & \xrightarrow{\varphi} & h^*(E|_{U \cup V}, E'|_{U \cup V}) \\ \downarrow & & \downarrow \\ h^*(U) \langle t \rangle \oplus h^*(V) \langle t \rangle & \xrightarrow[\cong]{\varphi} & h^*(E|_U) \oplus h^*(E'|_V) \\ \downarrow & & \downarrow \\ h^*(U \cup V) \langle t \rangle & \xrightarrow[\cong]{\varphi} & h^*(E|_{U \cap V}, E'|_{U \cap V}) \end{array}$$

we know that  $\varphi(U \cup V)$  is an isomorphism and  $h^*(U \cup V) \langle t \rangle$  is a free module. □

**Definition 4.8.** Given a relative filtration  $p: (E, E') \rightarrow B$ , we say  $t(p) \in h^n(E, E')$  is a Thom class if  $t(p)|_b$  generates  $h^n(E_b, E'_b)$  for each  $b \in B$ .

**Theorem 4.9** (Thom Isomorphism). Let  $p: (E, E') \rightarrow B$  be a relative filtration. Suppose  $t(p) \in h^n(E, E')$  is a Thom class. Then

$$\begin{aligned} \Phi: h^k(B) &\rightarrow h^{k+n}(E, E') \\ b &\mapsto p^*(b) \cup t(p) \end{aligned}$$

is an isomorphism.

*Proof.* Apply Leray-Hirsch Theorem (Theorem 4.7) to  $\{t_j\} = t(p)$ .  $\square$

**Definition 4.10.** We further assume  $p^*: h^*(B) \rightarrow h^*(E)$  is an isomorphism. We define the Euler class  $e(p) \in h^*(B)$  by

$$h^n(E, E') \longrightarrow h^n(E) \xrightarrow{(p^*)^{-1}} h^*(B).$$

$$t(p) \longmapsto e(p)$$

**Theorem 4.11** (Gysin Sequence). Assume  $t(p) \in h^n(E, E')$  is a Thom class and  $p^*: h^*(B) \rightarrow h^*(E)$  is an isomorphism. Then we have the Gysin's sequence

$$\longrightarrow h^{k-1}(E') \longrightarrow h^{k-n}(B) \xrightarrow{\cup e(p)} h^k(B) \xrightarrow{p^*} h^k(E') \longrightarrow$$

*Proof.* Consider the exact sequence of pair  $(E, E')$

$$\begin{array}{ccccccc} h^{k-1}(E') & \xrightarrow{\delta} & h^k(E, E') & \xrightarrow{j} & h^k(E) & \longrightarrow & h^k(E') \longrightarrow \\ & & \uparrow \cong \Phi & & \uparrow \cong p^* & & \\ & & h^{k-n}(B) & \xrightarrow{\cup e(p)} & h^k(B) & & \end{array}$$

For any  $b \in h^{k-n}(B)$ ,

$$j(\Phi(b)) = j(p^*(b) \cup t(p)) = p^*(b) \cup p^*(e(p)).$$

$\square$

Let  $\xi: E \rightarrow B$  be a real vector bundle of rank  $n$ ,  $E^0 =$  complement of zero section of  $E$ . Then  $(E_b, E_b^0) = (\mathbb{R}^n, \mathbb{R}^n - \{0\}) = (D^n, S^{n-1})$ .

**Proposition 4.12.** Assume  $\xi: E \rightarrow B$  admits a nowhere vanishing section. Then  $e(\xi) = 0$ .

*Proof.* Take  $s: B \rightarrow E^0$ . The Euler class factors through  $p \circ s = \text{id}$ . Chasing the diagram,

$$\begin{array}{ccccc} h^n(E, E^0) & \xrightarrow{j_1} & h^n(E) & \xrightarrow{(p^*)^{-1}} & h^n(B) \\ & & \searrow j_2 & & \nearrow s^* \\ & & h^n(E^0) & & \end{array}$$

$$t(s) \longmapsto e(s)$$

$j_2 \circ j_1 = 0$ . Then  $e(\xi) = 0$ .  $\square$

### 4.3 Singular Cohomology

Let  $(X, A)$  be a pair of spaces. Then we have singular chain complexes  $S_*(X)$  and  $S_*(X, A) := S_*(X)/S_*(A)$ . Given an  $R$ -module  $M$ . We define

$$S^n(X, A; M) := \text{Hom}_R(S_n(X, A), M).$$

We have the cohomology map

$$\begin{aligned} \delta: S^n(X, A) &\mapsto S^{n+1}(X, A) \\ \varphi &\mapsto (-1)^{n+1} \varphi \circ \partial. \end{aligned}$$

Since  $\partial^2 = 0$ ,  $\delta^2 = 0$ . Define  $H^n(X, A; M) := \ker \delta / \text{im } \delta$ .

**Theorem 4.13** (Universal Coefficient Theorem). We have exact sequences:

1.

$$0 \longrightarrow \text{Ext}(H_{n-1}(X, A; R), M) \longrightarrow H^n(X, A, M) \longrightarrow \text{Hom}_R(H_n(X, A), M) \longrightarrow 0.$$

It splits but does not split naturally.

2.

$$0 \longrightarrow H^n(X, A; R) \otimes M \longrightarrow H^n(X, A, M) \longrightarrow \text{Tor}(H^{n+1}(X, A; R), M) \longrightarrow 0.$$

It splits but does not split naturally.

On the cochain level, we define

$$\begin{aligned} S^k(X, R) \otimes S^l(S; R) &\rightarrow S^{k+l}(X; R) \\ \varphi \otimes \psi &\mapsto \varphi \cup \psi \end{aligned}$$

by

$$\varphi \cup \psi(\sigma) := (-1)^{kl} \varphi(\sigma|_{[e_0, \dots, e_k]}) \cdot \psi(\sigma|_{[e_{k+1}, \dots, e_{k+l}]})$$

for any simplex  $\sigma: \Delta^{k+l} \rightarrow X$ .

**Claim 12.**  $\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^{|\varphi|} \varphi \cup \delta\psi$ .

*Proof.* This claim can be checked by definition. □

This Claim shows that cup product descends to the cohomology level: The homomorphism

$$\cup: H^k(X; R) \otimes H^l(X; R) \rightarrow H^{k+l}(X; R)$$

is well-defined.

**Fact 4.14.** When  $(A, B)$  is an excisive pair, we get a chain equivalence:

$$S_*(A) + S_*(B) \rightarrow S_*(A \cup B).$$

We can define relative cohomology:

$$S^*(X, A) \otimes S^*(X, B) \xrightarrow{\cup} \text{Hom}(S_*(X)/S_*(A) + S_*(B), R) \longrightarrow \text{Hom}(S_*(X)/S_*(A \cup B), R) = S^*(X, A \cup B).$$

Then we have a well-defined homomorphism

$$\cup: H^k(X, A) \otimes H^l(X, B) \rightarrow H^{k+l}(X, A \cup B).$$

We need to check that singular cohomology satisfies cohomology axioms. It is only non-trivial to verify

$$[\varphi] \cup [\psi] = (-1)^{|\varphi| \cdot |\psi|} \cdot [\psi] \cup [\varphi].$$

Consider

$$\begin{aligned} \rho: S_n(X) &\rightarrow S_n(X) \\ \sigma &\mapsto (-1)^{\frac{n(n+1)}{2}} \bar{\sigma}, \end{aligned}$$

where  $\bar{\sigma} = \sigma|_{[e_n, \dots, e_0]}$ .

**Fact 4.15.**  $\rho$  is chain homotopic to id.

Denote  $\rho^\vee: S^n(X; R) \rightarrow S^n(X; R)$  for the map induced by  $\rho$ . Then we have  $\rho^\vee(\varphi \cup \psi) = (-1)^{|\varphi| \cdot |\psi|} \cdot \psi \cup \varphi$ .



#### 4.3.1 Existence of Thom Class

Recall  $p: (E, E') \rightarrow B$  is a relative fibration over a CW-complex. Suppose  $t \in H^n(E, E')$  restricts to a basis of the  $H^*(\{*\})$ -module  $H^n(E_b, E'_b)$ ,  $\forall b \in B$ . Then we say  $t \in H^n(E, E')$  is a Thom class.

For singular cohomology,  $H^*(\{*\}, R) = R$ . A necessary condition for the existence of  $t$  is  $H^n(E_b, E'_b) \cong R$ .

Given a path  $\gamma: I \rightarrow B$  from  $b_0$  to  $b_1$ . We get a transport map

$$\gamma^\sharp: H^n(E_{b_0}, E'_{b_0}) \xleftarrow[\cong]{i_{b_0}^*} H^n(\gamma^*E, \gamma^*E') \xrightarrow[\cong]{i_{b_1}^*} H^n(E_{b_1}, E'_{b_1}).$$

**Proposition 4.16.** Assume  $H^n(E_b, E'_b) \cong R$ . Then a Thom class  $t \in H^n(E, E')$  exists if and only if the transport map  $\gamma^\sharp$  is independent of  $\gamma$ .

*Proof.* Assume  $t \in H^n(E, E')$  is a Thom class. Then  $\gamma^\sharp(t|_{b_0}) = t|_{b_1}$  which is independent of the choices of  $\gamma$ .

Conversely, if  $\gamma^\sharp$  is independent of  $\gamma$ , we can apply the argument of Leray-Hirsch Theorem (Theorem 4.7). It is ensured by fixing a generator/basis  $t_0$  of  $H^n(E_{b_0}, E'_{b_0})$ . For any  $b \in B$ , we get a  $t_b = \gamma^\sharp(t_0) \in H^n(E_b, E'_b)$  where  $\gamma$  connects from  $b_0$  to  $b$ . Then use Mayer-Vietoris sequence to glue  $t$ .  $\square$

#### 4.3.2 Orientation

Suppose  $\Sigma \hookrightarrow V$  is a linearly embedded  $n$ -simplex with ordered vertices  $A_0, \dots, A_n$ . Define the orientation of  $V$  by  $v_1 = A_1 - A_0, v_2 = A_2 - A_1, \dots, v_n = A_n - A_0$ .

Fix  $\Delta^n$  as the standard  $n$ -simplex. Choose a linear embedding  $f: \Delta^n \rightarrow V$  such that  $f$  sends the barycenter of  $\Delta^n$  to  $o \in V$ . Then  $[f] \in H_n(V, V^0; \mathbb{Z})$  is a generator where  $V^0 = V - \{o\}$ . In fact, we have

$$\text{generator of } H_n(V, V^0, \mathbb{Z}) \xleftrightarrow{1:1} \text{orientation of } V.$$

Given an orientation generator  $o_V \in H_n(V, V^0, \mathbb{Z})$ , we get a generator  $u_V \in H^n(V, V^0, \mathbb{Z})$  such that  $u_V(o_V) = 1$ . Then we get

$$\text{generator of } H^n(V, V^0, \mathbb{Z}) \xleftrightarrow{1:1} \text{orientation of } V.$$

Let  $\xi: E \rightarrow B$  be a real vector bundle of rank  $n$ . An orienting bundle atlas on  $\xi$  consists  $\{(U_\alpha, \varphi_\alpha)\}$  with  $\varphi_\alpha: \xi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^n$  such that the transition maps  $g_{\beta\alpha}: U_\alpha \cap U_\beta \rightarrow \text{GL}_n(\mathbb{R})$  have positive determinant.

After fixing an orientation on  $\mathbb{R}^n$ , an orienting atlas induces an orientation on  $\xi: E \rightarrow B$ .

**Definition 4.17.** An orientation on  $\xi$  is an assignment of orientations on  $E_b$  such that for any  $b \in B$ , there is a neighborhood  $U$  and a trivialization  $\varphi: \xi^{-1}(U) \rightarrow U \times \mathbb{R}^n$  which is fibrewise orientation-preserving.

**Proposition 4.18.** Let  $\xi: E \rightarrow B$  be a real vector bundle. Then  $\xi$  is orientable if and only if  $\xi$  admits a Thom class  $t(\xi) \in H^n(E, E^0, \mathbb{Z})$ .

*Proof.* Given an orienting atlas. We define  $t_{U_\alpha} = \varphi_\alpha^*(t_\alpha)$ , where  $t_\alpha \in H^n(U_\alpha \times (\mathbb{R}^n, \mathbb{R}^n - \{0\}))$ ,  $t_\alpha = p^*t_{\mathbb{R}^n}$ ,  $p: U_\alpha \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the projection,  $t_{\mathbb{R}^n} \in H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z})$  is a fixed generator. Then  $t_{U_\alpha}|_b = t_{U_\beta}|_b$  for any  $b \in U_\alpha \cap U_\beta$ . Mayer-Vietoris sequence glues these  $t_{U_\alpha}$  to a Thom class  $t(\xi)$ . The proof of another direction is more straightforward.  $\square$

Motivated by this proposition, we have

**Definition 4.19.** Given a ring  $R$ , we define an  $R$ -orientation of  $\xi: E \rightarrow B$  to be a Thom class  $t(\xi) \in H^n(E, E^0; R)$ .

## 4.4 Homology and Homotopy

### 4.4.1 Hurewicz Theorem

We fix generators  $z_n \in H_n(S^n; \mathbb{Z})$  and  $\tilde{z}_n \in H_n(D^n, S^{n-1}; \mathbb{Z})$  such that  $\partial \tilde{z}_n = z_{n-1}$  and  $q_* \tilde{z}_n = z_n$  where  $q: D^n \rightarrow D^n/S^{n-1} \cong S^n$  is the quotient map. Define the Hurewicz homomorphisms

$$\begin{aligned} h: \pi_n(X, *) &\rightarrow H_n(X, \mathbb{Z}) \\ [f] &\mapsto f_* z_n, \end{aligned}$$

and

$$\begin{aligned} h: \pi_n(X, A, *) &\rightarrow H_n(X, A, \mathbb{Z}) \\ [f] &\mapsto f_* \tilde{z}_n. \end{aligned}$$

Recall that we have a left action of  $\pi_1(A, *)$  on  $\pi_n(X, A, *)$ : Any path  $v: I \rightarrow A$  1:1 corresponds to a homotopy  $J^{n-1} \rightarrow v(t)$  of constant maps. Then  $J^{n-1} \hookrightarrow \partial I^n$  is a cofibration and  $\partial I^n \hookrightarrow I^n$  is a fibration. We extend this homotopy to  $V: (I^n, \partial I^n, J^{n-1}) \times I \rightarrow (X, A, *)$ . Given  $\alpha = [v] \in \pi_1(A, *)$ , define  $[f] \cdot \alpha = [v_1]$  where  $v_0 = f$ . Suppose  $[g] = [f] \cdot \alpha$ , then  $g \simeq f$ .

Define  $\pi_n^\#(X, A, *) := \pi_n(X, A, *) / \pi_1(A, *) = \pi_n(X, A, *) / \{x - x \cdot \alpha : \alpha \in \pi_1(A)\}$ . Then the Hurewicz map descends to

$$h^\#: \pi_n^\#(X, A, *) \rightarrow H_n(X, A, \mathbb{Z}).$$

**Theorem 4.20** (Hurewicz Theorem). Assume  $X$  is  $(n-1)$ -connected,  $n \geq 1$ . Then  $h^\#: \pi_n^\#(X, *) \rightarrow H_n(X, \mathbb{Z})$  is an isomorphism.

*Proof.* When  $n = 1$ , for any  $\alpha, x \in \pi_1(X, *)$ ,  $x \cdot \alpha := \alpha^{-1} x \alpha$ . Then by definition,  $\pi_1^\#(X, *)$  is the abelianization of  $\pi_1(X, *)$ , where is isomorphism to  $H_1(X, \mathbb{Z})$ .

When  $n \geq 2$ ,  $X$  is simply-connected, we know  $\pi_n(X, *) = \pi_n^\#(X, *)$ .

**Fact 4.21.** A weak homotopy equivalence induces isomorphism on homology groups.

We may assume  $X$  is a CW-complex such that  $X^{n-1} = \{*\}$ . Then  $X^{n+1}$  is the cone of a map  $\varphi: \bigvee S_j^n \rightarrow \bigvee S_k^n$ . The conclusion holds for spheres. Additivity of  $\pi_n$  and  $H_n$  shows that  $h$  is an isomorphism for  $\bigvee S_k^n$ . we get exact sequence

$$\begin{array}{ccccccc} \pi_n(\bigvee S_j^n) & \xrightarrow{\varphi_*} & \pi_n(\bigvee S_k^n) & \longrightarrow & \pi_n(X) & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \cong \\ H_n(\bigvee S_j^n) & \longrightarrow & H_n(\bigvee S_k^n) & \longrightarrow & \pi_n(X) & \longrightarrow & 0 \end{array}$$

Therefore  $h$  is an isomorphism for  $X^{n+1}$ .

Since  $X$  is obtained from  $X^{n+1}$  by attaching cells of  $\dim \geq n+2$ ,  $\pi_n(X) \cong \pi_n(X^{n+1})$  and  $H_n(X) \cong H_n(X^{n+1})$ . Then  $h$  is an isomorphism for  $X$ . Let  $\square$

**Corollary 4.22.** Let  $(X, A)$  be a pair of simply-connected CW-complexes. Suppose  $H_i(X, A) = 0$  for any  $i < n$ ,  $n \geq 2$ . Then  $\pi_i(X, A) = 0$  for any  $i < n$  and  $h: \pi_n(X, A) \rightarrow H_n(X, A)$  is an isomorphism.

*Proof.* Apply induction on  $n$ : When  $n \geq 2$ , we have

$$\begin{array}{ccc} \pi_n(X, A) & \xrightarrow{\cong} & \pi_n(X/A) \\ \downarrow h & & \downarrow \cong \\ H_n(X, A) & \xrightarrow{\cong} & H_n(X/A) \end{array}$$

$\square$

**Theorem 4.23** (Whitehead). Suppose  $X, Y$  are simply-connected. If  $f: X \rightarrow Y$  induces isomorphisms on  $H_*$ , then  $f$  is a weak homotopy equivalence.

*Proof.* We may assume  $X, Y$  are CW-complexes. Apply Corollary 4.22 to  $(Z(f), X)$ .  $\square$

#### 4.4.2 Singular Cohomology and Eilenberg-MacLane Spaces

Let  $G$  be an abelian group and  $n \geq 1$ . Denote  $K := K(G, n)$ . Define a natural transformation  $\lambda: [-, K(G, n)] \rightarrow H^n(-; G)$  as follows: We have a sequence of isomorphisms

$$H^n(K; G) \cong \text{Hom}(H_n(K), G) \cong \text{Hom}(\pi_n(K), G) = \text{Hom}(G, G)$$

where the first isomorphism is by Universal Coefficient Theorem and the second is by Hurewicz Theorem. Suppose  $\text{id} \in \text{Hom}(G, G)$  corresponds to  $\iota_n \in H^n(K; G)$ . Define

$$\begin{aligned} \lambda(X): [X, K(G, n)] &\rightarrow H^n(X; G) \\ [f] &\mapsto f^* \iota_n. \end{aligned}$$

Notice that  $K(G, n) = \Omega K(G, n+1)$ ,  $\lambda(X)$  is a homomorphism.

**Theorem 4.24.** Let  $(X, *)$  be a based CW-complexes. Then  $\lambda(X): [X, K(G, n)]^o \rightarrow \widetilde{H}^n(X; G)$  is an isomorphism.

*Proof.* Note that the conclusion holds for spheres: If  $m \neq n$ ,  $[S^m, K(G, n)]^o = 0$  and  $\widetilde{H}^n(S^m; G) = 0$ . For  $n$ , it follows from definition.

Consider cofibre sequence

$$\bigvee S_j^{k-1} \longrightarrow X^{k-1} \longrightarrow X^k \longrightarrow X^k/X^{k-1} \longrightarrow \Sigma X^{k-1}.$$

Apply  $[-, K(G, n)]^o$  and  $\widetilde{H}^n$ , we get corresponding exact sequences. Use induction on  $k$  to conclude (to be continue...)  $\square$

#### 4.5 Homology with Local Coefficient

Let  $X$  be a path-connected space. Recall that its fundamental groupoid  $\Pi(X)$  is a category whose objects are points in  $X$  and morphisms  $x \rightarrow y$  are homotopy class of path  $\gamma: I \rightarrow X$ ,  $\gamma(0) = x, \gamma(1) = y$  rel  $\partial I$ .

**Definition 4.25.** A local coefficient system is a functor  $\mathcal{E}: \Pi(X) \rightarrow \mathcal{A}$  such that  $\mathcal{E}(x) \cong \mathcal{E}(y)$  for any  $x, y \in X$ .

**Definition 4.26** (Homology with Coefficient  $\mathcal{E}$ ). We define the *Homology with Coefficient  $\mathcal{E}$*  as follows:

The chain complex  $S_k(X; \mathcal{E})$  consists of formed sums  $\sum_{i=1}^m a_i \sigma_i$  where  $\sigma_i: \Delta^k \rightarrow X$  is a  $k$ -simplex and  $a_i \in \mathcal{E}(\sigma_i(e_0))$ ,  $e_0 = (1, 0, \dots, 0) \in \Delta^k$ .

Recall that we have the face maps:

$$\begin{aligned} f_m^k: \Delta^{k-1} &\rightarrow \Delta^k \\ (t_0, \dots, t_{k-1}) &\mapsto (t_0, \dots, t_{m-1}, 0, t_m, \dots, t_{k-1}). \end{aligned}$$

Then we have  $f_i^k(1, 0, \dots, 0) = (1, 0, \dots, 0)$  for any  $i \geq 1$  and  $f_0^k(1, 0, \dots, 0) = (0, 1, 0, \dots, 0) := e_1$ .

Given a simplex  $\sigma: \Delta^k \rightarrow X$ , we get a path

$$\begin{aligned} \gamma_\sigma: [0, 1] &\rightarrow X \\ t &\mapsto \sigma(t, 1-t, 0, \dots, 0) \end{aligned}$$

with  $\gamma_\sigma(0) = \sigma(e_1)$  and  $\gamma_\sigma(1) = \sigma(e_0)$ . Define

$$\partial: S_k(X; \mathcal{E}) \rightarrow S_{k-1}(X; \mathcal{E})$$

$$a \cdot \sigma \mapsto \mathcal{E}(\gamma_\sigma)^{-1}(a) \cdot (\sigma \circ f_0^k) + \sum_{m=1}^k (-1)^m a \cdot (\sigma \circ f_m^k),$$

where  $\mathcal{E}(\gamma_\sigma): \mathcal{E}(\sigma(e_1)) \rightarrow \mathcal{E}(\sigma(e_0))$ .

**Claim 13.**  $\partial^2 = 0$ .

We define  $H_*(X; \mathcal{E}) := H_*(S_*(X; \mathcal{E}), \partial)$ .

**Definition 4.27** (Cohomology with Coefficient  $\mathcal{E}$ ). We define the *Cohomology with Coefficient  $\mathcal{E}$*  as follows:

The cochain complex  $S^k(X; \mathcal{E})$  is generated by

$$c: \{\text{singular } k\text{-simplex}\} \rightarrow \oplus_{x \in X} \mathcal{E}(x)$$

such that  $c(\sigma) \in \mathcal{E}(\sigma(e_0))$ . The coboundary map  $\delta: S^k(X; \mathcal{E}) \rightarrow S^{k+1}(X; \mathcal{E})$  is defined by

$$\delta(c)(\sigma) := (-1)^k \left( \mathcal{E}(x_0) \cdot c(\sigma \circ f_0^{k+1}) + \sum_{m=1}^{k+1} (-1)^m c(\sigma \circ f_m^{k+1}) \right).$$

**Claim 14.**  $\delta^2 = 0$ .

We define  $H^*(X; \mathcal{E}) := H^*(S^*(X; \mathcal{E}), \delta)$ .

**Proposition 4.28.** (Co)homology with local coefficients is a (co)homology theory.

#### 4.5.1 An Equivalent Definition

Assume  $X$  admits a universal cover  $\tilde{X}$ . Given a local system  $\mathcal{E}$  on  $X$ . We fix an abelian group  $A$  such that  $A \cong \mathcal{E}(x)$ ,  $\forall x \in X$ . We get a representation

$$\rho_\mathcal{E}: \pi_1(X, *) \rightarrow \text{Aut}(\mathcal{E}(x)) \cong \text{Aut}(A).$$

Write  $\pi = \pi_1(X, *)$ . We can regard  $A$  as left  $\mathbb{Z}[\pi]$ -module by

$$\gamma \cdot a := \rho_\mathcal{E}(\gamma)(a).$$

Note that  $\pi_1(X, *)$  have a right action on  $\tilde{X}$ ,  $S_*\left(\tilde{X}\right)$  is a right  $\mathbb{Z}[\pi]$ -module. Define the chain complex

$$S_*(X; \rho_\mathcal{E}) := S_*\left(\tilde{X}\right) \otimes_{\mathbb{Z}[\pi]} A$$

and the boundary map

$$\partial_\mathcal{E} \sigma \otimes a \mapsto (\partial \sigma) \otimes a.$$

We claim that  $\partial_\mathcal{E}^2 = 0$ . Then we define  $H_*(X; \rho_\mathcal{E}) := H_*(S_*(X; \rho_\mathcal{E}), \partial_\mathcal{E})$ .

Regard  $S_k\left(\tilde{X}\right)$  as left  $\mathbb{Z}[\pi]$ -module by  $\gamma \cdot \sigma := \sigma \cdot \gamma^{-1}$ . Define

$$S^*(X; \rho_\mathcal{E}) := \text{Hom}_{\mathbb{Z}[\pi]} \left( S_k\left(\tilde{X}\right), A \right)$$

and

$$\delta_\mathcal{E} c(\sigma) := c(\rho_\mathcal{E} \sigma).$$

We claim that  $\delta_\mathcal{E}^2 = 0$ . Then we define  $H^*(X; \rho_\mathcal{E}) := H^*(S^*(X; \rho_\mathcal{E}), \delta_\mathcal{E})$ .

**Theorem 4.29.** Let  $X$  be a connected space admitting a universal cover  $\tilde{X}$ . Assume  $\mathcal{E}$  is a local system on  $X$  and  $\rho_{\mathcal{E}}: \pi_1(X) \rightarrow \text{Aut}(A)$ . Then  $H_*(X; \mathcal{E}) \cong H_*(X; \rho_{\mathcal{E}})$  and  $H^*(X; \mathcal{E}) \cong H^*(X; \rho_{\mathcal{E}})$ .

*Proof.* We have pairwise 1 to 1 corresponding of the following three items:

1. local system  $\mathcal{E}$ ,
2. covering space  $\tilde{X}_{\mathcal{E}}$  with fibre  $A$ ,
3. representation  $\rho: \pi_1(X) \rightarrow \text{Aut}(A)$ .

(1)  $\Rightarrow$  (3) is by our construction.

For (2)  $\Rightarrow$  (3),  $\rho$  is given by the endpoint of lifts of  $\pi_1$ .

For (3)  $\Rightarrow$  (2),  $\tilde{X}_{\mathcal{E}} = \tilde{X} \times A / (\tilde{x}, a) \sim (\tilde{x} \cdot \gamma^{-1}, \rho(\gamma)a)$  is called Borel construction.

For (2)  $\Rightarrow$  (1), lifts give the morphisms. For  $a \cdot \sigma \in S_k(X, \mathcal{E})$ , we lift the simplex  $a \cdot \sigma$  to a  $\pi$ -equivalent simplex on  $S_*(\tilde{X})$ .  $\square$

**Example 4.30.** 1. The local system is  $\mathcal{E}$  trivial ( $\mathcal{E}: \mathcal{E}(x_0) \rightarrow \mathcal{E}(x_1)$  is independent of  $\gamma$ ). Then we have

$$S_*(X; \rho_{\mathcal{E}}) = S_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi]} A = S_*(X) \otimes A.$$

Therefore  $H_*(X; \rho_{\mathcal{E}}) = H_*(X; A)$  is the singular homology.

2. Let  $A = \mathbb{Z}[\pi]$ . We have a tautological representation

$$\begin{aligned} \rho: \pi_1(X, *) &\mapsto \text{Aut}(\pi) \\ g &\mapsto L_g. \end{aligned}$$

Then  $S_*(X; \rho) = S_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi]} A = S_*(\tilde{X})$  and  $H_*(X; \rho) = H_*(\tilde{X})$ . Generally,

**Lemma 4.31.** Let  $H < \pi_1(X, *)$  be a subgroup. Choose  $A = \mathbb{Z}[\pi/H]$  which admits a representation  $\rho_A: \pi \rightarrow \text{Aut}(A)$  be left multiplication. Then  $H_*(X; \rho_A) \cong H_*(\tilde{X}_H)$  where  $\tilde{X}_H \rightarrow X$  is the  $H$ -covering of  $X$ .

3. Let  $X$  be a connected and closed  $n$ -manifold. Then we get the orientation local system  $\mathcal{O}_X$ . Its objects are orientations  $\mathcal{O}_X(x) = H_n(X, X - \{x\}) \cong \mathbb{Z}$  and morphisms are orientation transport:  $H_n(X, X - \{x\}) \cong H_n(X, X - U) \cong H_n(X, X - \{x'\})$  for  $x, x' \in U$  where  $U$  is a small neighborhood. We have  $\mathcal{O}_X$  is trivial if and only if  $X$  is orientable, if and only if  $w_1: \pi_1(X, *) \rightarrow \text{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$  is trivial, if and only if  $w_1 = 0 \in H^1(X; \mathbb{Z}_2)$ . In fact,  $w_1$  is the first Steifel-Whitney class of  $X$ . Then we know  $H_n(X; w_1) \cong \mathbb{Z}$ .

- If  $w_1$  is trivial, then  $H_n(X; w_1) = H_n(X, \mathbb{Z}) = \mathbb{Z}$ .
- If  $w_1$  is non-trivial, then  $H_n(X; w_1) \cong H_n(\tilde{X}_{w_1}, \mathbb{Z}) \cong \mathbb{Z}$  where  $\tilde{X}_{w_1} \rightarrow X$  is the orientable double cover..

## 4.6 Obstruction

### 4.6.1 Obstruction of Extension

**Question 4.32.** Suppose  $f: A \rightarrow Y$  and  $X$  is obtained by attaching cells on  $A$ . Is there a extension  $g$  of  $f$  on  $X$ ?

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ \downarrow & \nearrow \exists ? g & \\ X & & \end{array}$$

Assume  $\pi_1(Y) = 1$  and  $(X, A)$  is a relative CW-complex.  
For each  $(n+1)$ -cell  $e^{n+1}$ , we have

$$\begin{array}{ccc} S^n & \xrightarrow{\varphi} & X^n \\ \downarrow & & \downarrow \\ D^n & \xrightarrow{\Phi} & X \end{array}$$

Assume we have a map  $g: X^n \rightarrow Y$ . We wish to extend  $g$  over  $X^{n+1}$ .

For each  $(n+1)$ -cell  $e_i^{n+1}$ ,  $g$  extends over  $e_i^{n+1}$  if and only if  $S^n \xrightarrow{\varphi_i} X^n \xrightarrow{g} Y$  is null-homotopic in  $[S^n, Y] = \pi_n(Y)$ . For each  $g$ , we get a cochain  $\theta^{n+1}(g) \in C^{n+1}(X, A; \pi_n(Y)) =: H^{n+1}(X^{n+1}, X^n; \pi_n(Y))$  by setting  $\theta^{n+1}(g)(e_i^{n+1}) = [g \circ \varphi_i] \in \pi_n(Y)$ .

**Lemma 4.33.**  $g$  extends over  $X^{n+1}$  if and only if  $\theta^{n+1}(g) = 0$ .

We will give an algebraic definition of  $\theta^{n+1}(g)$ .

**Lemma 4.34.** We have the factorization

$$\begin{array}{ccccc} \pi_{n+1}(X^{n+1}, X^n) & \xrightarrow{\partial} & \pi_n(X^n) & \xrightarrow{g_*} & \pi_n(Y) \\ \downarrow j & & \nearrow \overline{g_* \circ \partial} & & \\ \pi_{n+1}^\#(X^{n+1}, X^n) & & & & \end{array}$$

*Proof.* Recall that

$$\pi_{n+1}^\#(X^{n+1}, X^n) = \pi_{n+1}(X^{n+1}, X^n) / \langle x - x \cdot \alpha : \alpha \in \pi_1(X^n) \rangle.$$

Then

$$g_* \circ \partial(x \cdot \alpha) = g_*((\partial(x)) \cdot \alpha) = (g_* \circ \partial(x)) \cdot g_*(\alpha) = g_* \circ \partial(x).$$

□

Since  $(X^{n+1}, X^n)$  is  $n$ -connected,  $h: \pi_{n+1}^\#(X^{n+1}, X^n) \xrightarrow{\cong} H_{n+1}(X^{n+1}, X^n; \mathbb{Z}) := C_{n+1}(X^{n+1}, X^n)$ . Define

$$\theta^{n+1}(g) := \overline{g_* \circ \partial} \circ h^{-1} \in \text{Hom}_{\mathbb{Z}}(C_{n+1}(X^{n+1}, X^n), \pi_n(Y)) = C^{n+1}(X^{n+1}, X^n; \pi_1(Y)).$$

**Lemma 4.35.**  $\theta^{n+1}(g)$  is a cocycle.

*Proof.* Consider the commutative diagram

$$\begin{array}{ccccccc} \pi_{n+2}(X^{n+2}, X^{n+1}) & \xrightarrow{\partial} & \pi_{n+1}(X^{n+1}) & \xrightarrow{i} & \pi_{n+1}(X^{n+1}, X^n) & \xrightarrow{\partial} & \pi_n(X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow g_* \\ H_{n+2}(X^{n+2}, X^{n+1}) & \xrightarrow{\partial} & H_{n+1}(X^{n+1}) & \xrightarrow{i} & H_{n+1}(X^{n+1}, X^n) & \xrightarrow{\theta^{n+1}(g)} & \pi_n(Y) \end{array}$$

Then  $\delta\theta^{n+1}(g) = 0$  if and only if  $\theta^{n+1}(g) \circ i \circ \partial = 0$ . Note that the vertical arrows are surjective. Since  $g_* \circ (\partial \circ i) \circ \alpha = g_* \circ (0) \circ \alpha = 0$ . Then  $\theta^{n+1}(g) \circ i \circ \partial = 0$ . □

We call  $[\theta^{n+1}(g)] \in H^{n+1}(X, A; \pi_n(Y))$  the obstruction class of  $g: X^n \rightarrow Y$ .

**Question 4.36.** What kind of geometric information does  $[\theta^{n+1}(g)] = 0$  tell us?

**Answer.**  $[\theta^{n+1}(g)] = 0$  if and only if  $g|_{X^{n-1}}: X^{n-1} \rightarrow Y$  extends over  $X^{n+1}$ .

**Lemma 4.37.** Suppose  $g_0, g_1: X^n \rightarrow Y$  are two maps such that  $g_0|_{X^{n-1}} \simeq g_1|_{X^{n-1}}$ . Then each homotopy  $G: X^{n-1} \times I \rightarrow Y$  determines a cochain  $d(g_0, G, g_1) \in C^n(X, A; \pi_n(Y))$  such that  $\theta^{n+1}(g_0) - \theta^{n+1}(g_1) = \delta d(g_0, G, g_1)$ .

*Proof.* The maps  $g_0, G, g_1$  combine to a map

$$\tilde{G}: X^{n-1} \times I \cup X^n \times \partial I \rightarrow Y.$$

Note that  $X^{n-1} \times I \cup X^n \times \partial I$  is the  $n$ -skeleton of  $X \times I$ . Consider  $\theta^{n+1}(\tilde{G}) \in H^{n+1}((X, A) \times I, \pi_n(Y))$ . We define

$$d(g_0, G, g_1)(e_i^n) := \theta^{n+1}(\tilde{G})(e_i^n \times I) \in \pi_n(Y).$$

Then

$$\begin{aligned} & (\delta d(g_0, G, g_1) + \theta^{n+1}(g_1) - \theta^{n+1}(g_0))(e_i^{n+1}) \\ &= d(g_0, G, g_1)(\partial e_i^{n+1}) + \theta^{n+1}(g_1)(e_i^{n+1}) - \theta^{n+1}(g_0)(e_i^{n+1}) \\ &= \theta^{n+1}(\tilde{G})(\partial e_i^{n+1} \times I) + \theta^{n+1}(\tilde{G})(e_i^{n+1} \times \partial I) \\ &= \delta \theta^{n+1}(\tilde{G})(e_i^{n+1} \times I) \\ &= 0. \end{aligned}$$

□

**Lemma 4.38.** Given a map  $g_0: X^n \rightarrow Y$  and a homotopy  $G: X^{n-1} \times I \rightarrow Y$  such that  $G|_{X^{n-1} \times \{0\}} = g_0|_{X^{n-1}}$ , then for any  $d \in C^n(X, A; \pi_n(Y))$ , we can find  $g_1: X^n \rightarrow Y$  such that

1.  $g_1|_{X^{n-1}} = G|_{X^{n-1} \times \{1\}}$ ,
2.  $\delta d = \theta^{n+1}(g_0) - \theta^{n+1}(g_1)$ .

*Proof.* Let  $e_i^n$  be a  $n$ -cell of  $X$ .

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\varphi_i} & X^{n-1} \\ \downarrow & & \downarrow \\ D^n & \xrightarrow{\Phi_i} & X \end{array}$$

Composing with  $g_0$  and  $G$ , we get a map  $f_i: D^n \times \{0\} \cup S^{n-1} \times I \rightarrow Y$ .

Write  $[k_i] := d(e_i^n) \in \pi_n(Y) = [\partial(D^{n+1}), Y] = [\partial(D^n \times I), Y]$ . Then  $k_i$  can be seen as a map  $k_i: \partial(D^n \times I) \rightarrow Y$ . Since  $D^n \times \{0\} \cup S^{n-1} \times I$  is contractible, then  $f_i \simeq k_i|_{D^n \times \{0\} \cup S^{n-1} \times I}$ . Since  $D^n \times \{0\} \cup S^{n-1} \times I \hookrightarrow \partial(D^n \times I)$  is a cofibration, we get an extension  $H_i: \partial(D^n \times I) \times I \rightarrow Y$ . Define  $g_1: X^n \rightarrow Y$  by

$$g_1(\Phi_i(x)) := H_i(x, 1, 1) \in \partial(D^n \times I).$$

Then

$$d(g_0, G, g_1)(e_i^n) = \theta^{n+1}(\tilde{G})(e_i^n \times I) = [k_i] = d(e_i^n).$$

□

**Theorem 4.39** (Obstruction Theorem). Let  $g: X^n \rightarrow Y$  be a continuous map (with  $\pi_1(Y) = 1$ ). Then  $[\theta^{n+1}(g)] = 0$  if and only if  $g|_{X^{n-1}}$  extends over  $X^{n+1}$ .

*Proof.* ( $\Rightarrow$ ): If  $\theta^{n+1}(g) = \delta d$ , then apply Lemma 4.38 to  $(g, g|_{X^{n-1}} \times \text{id}, d)$ , we get  $g': X^n \rightarrow Y$  such that  $g'|_{X^{n-1}} = g$  and  $\theta^{n+1}(g) - \theta^{n+1}(g') = \delta d$ . Then  $\theta^{n+1}(g') = 0$ , i.e.  $g'$  extends over  $X^{n+1}$ .

( $\Leftarrow$ ): Write  $g': X^{n+1} \rightarrow Y$  such that  $g'|_{X^n} = g$ . Denote by  $\tilde{g} = g'|_{X^n}$ . Then  $\theta^{n+1}(\tilde{g}) = 0$  and  $\theta^{n+1}(g) - \theta^{n+1}(g') = \delta d$ . □

When  $Y$  is  $n$ -connected, then any map  $f: A \rightarrow Y$  can be extends to  $g: X^{n+1} \rightarrow Y$ , since  $H^{n+1}(X, A; \pi_k(Y)) = 0, \forall k \leq n$ . In this case, we define

$$\alpha(f) := \theta^{n+2}(g) \in H^{n+2}(X, A; \pi_{n+1}(Y))$$

as the primary obstruction of  $f$ .

For homotopy extension, suppose  $f_0, f_1: X^n \rightarrow Y$  such that  $f_0|_{X^{n-1}} \simeq f_1|_{X^{n-1}}$ . We apply the discussion to  $(X^n \times I, X^n \times \partial I \cup X^{n-1} \times I)$ . We get an obstruction  $d^n(f_0, f_1) \in H^n(X, A; \pi_n(Y))$ .

#### 4.6.2 Obstruction of Lifting

**Question 4.40.** Suppose  $f: X \rightarrow B$  and  $p: E \rightarrow B$  is a fibration. Is there a lifting  $g$  of  $f$  on  $X$ ?

$$\begin{array}{ccc} & & E \\ & \nearrow \exists?g & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

**Proposition 4.41.** Let  $p: E \rightarrow B$  be a fibration with fibre  $F$  such that  $\pi_1(F) = 1$ . Then for each  $n \geq 0$ , we get a local system

$$\rho_n: \pi_1(B) \rightarrow \text{Aut}(\pi_n(F)).$$

*Proof.*  $\pi_1(B)$  contains homotopy classes of homotopy self-equivalence of  $F$ . They induce  $\text{Aut}(\pi_n(F))$ .  $\square$

Assume we have a lift on  $X^n$ .

$$\begin{array}{ccc} & & E \\ & \nearrow g & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

Given an  $(n+1)$ -cell  $e_i^{n+1}$ , we get  $g \circ \varphi_i: S^n \rightarrow E$ . Note that  $p \circ g \circ \varphi_i = f \circ \varphi_i$  extends over  $X^{n+1}$ . Then  $f \circ \varphi_i$  is null-homotopic:  $f \circ \varphi_i \simeq \text{const}_{f \circ \Phi_i(0)}$ . Apply homotopy lifting, we know  $g \circ \varphi_i: S^n \rightarrow E$  is homotopic to  $f_i: S^n \rightarrow E_{f \circ \Phi_i(0)}$ . Define  $\theta^{n+1}(g)(e_i^{n+1}) := [f_i] \in \pi_n(E_{f \circ \Phi_i(0)})$ . Then  $\theta^{n+1}(g) \in C^{n+1}(X; f^* \rho_n)$ .

Then  $[\theta^{n+1}(g)] = 0 \in H^{n+1}(X; f^* \rho_n)$  if and only if  $g|_{X^{n-1}}$  extends over  $X^{n+1}$ , which deduces that  $f$  lifts over  $X^{n+1}$ .

## 5 Principal Bundle and Characteristic classes

### 5.1 Principal Bundle and Classifying Space

Let  $G$  be a topological group and  $B$  a  $C_2$  Hausdorff space.

**Definition 5.1.** A principal  $G$ -bundle  $\pi: P \rightarrow B$  is a fibre bundle with a right  $G$ -action such that near each  $b \in B$ ,  $\exists$  a neighborhood  $U \subset B$  and a trivialization

$$\begin{aligned} \varphi: \pi^{-1}(U) &\rightarrow U \times G \\ p &\mapsto (\pi(p), \alpha(p)) \end{aligned}$$

such that  $\varphi(p, g) = (\pi(p), \alpha(p)g)$ .

We can generate fibre bundles using principal bundle. Assume  $F$  is a left  $G$ -space or we have a representation  $\rho: G \rightarrow \text{Aut}(F)$ . We set  $E = P \times_G F = P \times F / (p, f) \sim (p \cdot g, g^{-1}f)$ . Then  $E \rightarrow B$  is a fibre bundle with fibre  $F$ . We call this construction Borel construction.

To get vector bundles, consider  $\rho: G \rightarrow \text{GL}_n(\mathbb{F})$ . We get vector bundles over  $\mathbb{F}$  of rank  $n$ .

Denote by  $\mathcal{B}(B, G)$  the set of isomorphism class of principal  $G$ -bundles ober  $B$ .



**Definition 5.2.** We say  $EG$  is a universal (right)  $G$ -space if  $\forall$  right  $G$ -space admits, up to  $G$ -homology, a unique  $G$ -map to  $EG$ .

For any  $\Phi: P \rightarrow EG$ , take quotients,

$$\begin{array}{ccc} P & \xrightarrow{\Phi} & EG \\ \downarrow & & \downarrow \\ B & \xrightarrow{\bar{\Phi}} & BG \end{array}$$

Denote by  $BG = EG/G$ . We define

$$\begin{aligned} K_B: \mathcal{B}(B, G) &\rightarrow [B, BG] \\ [P] &\mapsto [\bar{\Phi}]. \end{aligned}$$

We have another map

$$\begin{aligned} \iota_B: [B, BG] &\rightarrow \mathcal{B}(B, G) \\ [f] &\mapsto f^*EG \end{aligned}$$

under the assumption that  $G$  acts freely and locally trivially on  $EG$ , where “locally trivially” means that for any  $x \in EG$ , there is a neighborhood  $U$  and a  $G$ -map  $\varphi: U \rightarrow G$  (Then we have  $U \cong U/G \times G$  by  $x \mapsto (\bar{x}, \varphi(x))$ ).

**Theorem 5.3.**  $K_B$  and  $\iota_B$  are inverse to each other.  $BG$  is called the classifying space of  $G$ .

*Proof.*  $K_B \circ \iota_B = \text{id}$ :

Start with  $f: B \rightarrow BG$ ,  $\iota_B[f]$  is represented by  $f^*EG$ . Then we have

$$\begin{array}{ccc} f^*EG & \xrightarrow{F} & EG \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & BG \end{array}$$

Since  $EG$  is universal,  $F$  the unique map up to  $G$ -homotopy, then  $[f] = K_B(f^*EG)$ .

$\iota_B \circ K_B = \text{id}$ :

Given  $P \rightarrow B$  representing  $[P] \in \mathcal{B}(B, G)$ , we get

$$\begin{array}{ccc} P & \xrightarrow{\Phi} & EG \\ \downarrow & & \downarrow \\ B & \xrightarrow{\bar{\Phi}} & BG \end{array}$$

Then  $\iota_B \circ K_B([p]) = [\bar{\Phi}^*EG]$ . By universal property of pull-back, we have

$$\begin{array}{ccccc} P & \xrightarrow{\bar{\Psi}} & \bar{\Phi}^*EG & \xrightarrow{\Phi'} & EG \\ \downarrow & & \downarrow & & \downarrow \\ B & \xrightarrow{\text{id}} & B & \xrightarrow{\bar{\Phi}} & BG \end{array}$$

where  $\bar{\Psi}$  is given by

$$\begin{aligned} \bar{\Psi}: U \times G &\rightarrow U \times G \\ \bar{\Psi}(x, g \cdot h) &= \bar{\Psi}(xg) \cdot h = (x, \Psi(g) \cdot h). \end{aligned}$$

Then  $\bar{\Psi}$  is an isomorphism due to  $G$ -equivalence. □

**Definition 5.4.** Let  $X_1, X_2$  be two spaces. The joint of  $X_1, X_2$  is defined by

$$X_1 * X_2 := X_1 \times X_2 \times [0, 1] / (x_1, x_2, 0) \sim (x_1, x_2', 0), (x_1, x_2, 1) \sim (x_1', x_2, 1).$$

Then the embedding  $X_1 \hookrightarrow X_1 * X_2$  is given by

$$x_1 \mapsto [x_1, x_2, 0] = [(1, x_1), (0, x_2)].$$

**Definition 5.5.** The Milnor space if  $G$  is defined to be

$$EG := \operatorname{colim}_n G * \cdots * G = \left\{ \prod_{i \geq 1} (t_i, g_i) : \sum t_i = 1, \text{ only finitely } t_i \text{'s are non-zero} \right\}.$$

Define  $G \curvearrowright EG$  by

$$g \cdot \left[ \prod_{i \geq 1} (t_i, g_i) \right] = \left[ \prod_{i \geq 1} (t_i, g_i g) \right].$$

**Claim 15.**  $G$  acts freely and locally trivially on  $EG$ .

Reason: Choose an atlas  $\{V_j\}_{j \geq 1}$  of  $EG$  as follows:

$$V_j := \left\{ \left[ \prod_{i \geq 1} (t_i, g_i) \right] : t_j \in (0, 1] \right\}.$$

Define  $p_j: V_j \rightarrow G$  by

$$\left[ \prod_{i \geq 1} (t_i, g_i) \right] \mapsto g_j.$$

Then  $\pi_G: EG \rightarrow BG = EG/G$  is a principal  $G$ -bundle.

**Proposition 5.6.** The Milnor space  $EG$  is contractible.

*Proof.* **Step 1:**

We start with the map

$$\begin{aligned} \alpha_0: EG &\rightarrow EG \\ [(t_1, g_1), (t_2, g_2), \dots] &\mapsto [(t_1, g_1), (0, e), (t_2, g_2), \dots]. \end{aligned}$$

Consider the homotopy

$$\begin{aligned} \alpha_t: EG &\rightarrow EG \\ [(t_1, g_1), (t_2, g_2), \dots] &\mapsto [(t_1, g_1), (tt_2, g_2), ((1-t)t_2, g_2), (t_3, g_3), \dots]. \end{aligned}$$

Then

$$\begin{aligned} \alpha_1: EG &\rightarrow EG \\ [(t_1, g_1), (t_2, g_2), \dots] &\mapsto [(t_1, g_1), (t_2, g_2), (0, e), (t_3, g_3), \dots]. \end{aligned}$$

Inductivilt, we get  $\alpha_n$  which inserts  $(0, e)$  into the  $(n+1)$ -th entry. Then take  $\alpha_\infty = \operatorname{colim}_n \alpha_n = \operatorname{id}$ . Hence we have  $\alpha_0 \simeq \alpha_\infty$ .

**Step 2:**

Consider

$$f_t: [(t_1, g_1), \dots] \mapsto [((1-t)t_1, g_1), (t, e), ((1-t)t_2, g_2), ((1-t)t_3, g_3), \dots].$$

Then  $f_0 = \alpha_0$ ,  $f_1$  is the contraction onto  $[(0, e), (1, e), (0, e), \dots]$ . □

**Theorem 5.7.** The Milnor space  $\pi_G: EG \rightarrow BG$  is the universal  $G$ -bundle.

*Proof.* Let  $\pi: P \rightarrow B$  be a principal  $G$ -bundle with contractible trivialization charts

$$\begin{aligned}\pi^{-1}(U_i) &\rightarrow U_i \times G \\ p &\mapsto (\pi(p), \varphi_i(p)).\end{aligned}$$

Choose a partition of unity  $\rho_i: U_i \rightarrow [0, 1]$  subordinate to  $\{U_i\}$ . Write  $v_i = \rho_i \circ \pi$  which is  $G$ -invariant. We define

$$\begin{aligned}\Phi: P &\rightarrow EG \\ p &\mapsto \left[ \prod_{i \geq 1} (v_i(p), \varphi_i(p)) \right].\end{aligned}$$

Then  $\sum_{i \geq 1} v_i(p) = 1$  hence  $\varphi_i$  is a  $G$ -map.

We need to show  $\Phi: P \rightarrow EG$  is defined uniquely up to  $G$ -homotopy. Suppose  $\Psi: P \rightarrow EG$  is another  $G$ -map. Write

$$\begin{aligned}\Phi([\prod (t_i, g_i)]) &= [\prod (\Phi^1(t_i), \Phi^2(g_i))], \\ \Psi([\prod (t_i, g_i)]) &= [\prod (\Psi^n(t_i), \Psi^n(g_i))].\end{aligned}$$

By Proposition 5.6,  $\Phi$  is homotopic to

$$\Phi_1: [(t_1, g_1), (t_2, g_2), \dots] \mapsto [(\Phi^1(t_1), \Phi^2(g_1)), (0, e), (\Phi^1(t_2), \Phi^2(g_2)), (0, e), (\Phi^1(t_3), \Phi^2(g_3)), (0, e), \dots].$$

Apply the same procedure to odd entries, we get a homotopy  $\Psi$  to

$$\Psi_1: [(t_1, g_1), (t_2, g_2), \dots] \mapsto [(\Psi^1(t_1), \Psi^2(g_1)), (0, e), (\Psi^1(t_2), \Psi^2(g_2)), (0, e), (\Psi^1(t_3), \Psi^2(g_3)), (0, e), \dots].$$

We know  $\Phi_1$  is homotopic to  $\Psi_1$  by

$$\prod (t_i, g_i) \mapsto [(((1-t)\Phi^1(t_i), \Phi^2(g_i)), (t\Psi^1(t_i), \Psi^2(g_i)))].$$

Therefore  $\Phi$  is  $G$ -homotopic to  $\Psi$ . □

**Theorem 5.8.** Let  $\pi_G: E \rightarrow X$  be a principal  $G$ -bundle such that  $E$  is contractible. Then  $\pi_G: E \rightarrow X$  is the universal  $G$ -bundle.

*Proof.* Let  $\pi: P \rightarrow B$  be a principal  $G$ -bundle with  $B$  a CW-complex. We consider the fibre bundle  $p \times_G E \rightarrow B$  with contractible fibres.

Consider Obstruction Theory.

$$\begin{array}{ccc} & & P \times_G E \\ & \nearrow \exists ? s & \downarrow \\ B & \xrightarrow{\text{id}} & B \end{array}$$

The obstruction of lifting on  $B^{n+1}$  lies in  $H^{n+1}(B, \{\pi(\text{fibre})\}) = 0$ . Then  $P \times_G E \rightarrow B$  admits sections.

Take a section

$$\begin{aligned}s: B &\rightarrow P \times_G E = P \times E / (p, x) \sim (pg, xg) \\ b &\mapsto [p, s^p(p)].\end{aligned}$$

Then we get a  $G$ -map  $S^p: P \rightarrow E$ .

Suppose we have two  $G$ -maps  $S_1^p, S_2^p: P \rightarrow E$ . We get two sections  $s_1, s_2: B \rightarrow P \times_G E$ . Apply obstruction theory to  $(P \times_G E) \times I \rightarrow B \times I$ . We have  $s_1 \simeq s_2$  hence  $s_1^p$  is  $G$ -homotopic to  $s_2^p$ . □

### 5.1.1 Functorial Property of $BG$

Let  $\alpha: H \rightarrow G$  be a continuous homomorphism. We regard  $G$  as a left  $H$ -space by  $h \cdot g := \alpha(h)g$ . We get principal  $G$ -bundle  $EH \times_\alpha G \rightarrow BH$ . By universal property of  $EG$ , we have

$$\begin{array}{ccc} EH \times_\alpha G & \xrightarrow{E(\alpha)} & EG \\ \downarrow & & \downarrow \\ BH & \xrightarrow{B(\alpha)} & BG \end{array}$$

where  $B(\alpha)$  is defined uniquely up to homotopy. We have  $B(\alpha) \circ B(\beta) \simeq B(\alpha \circ \beta)$ . Now we get a functor  $B(-)$ .

When  $H < G$  is a subgroup, we get  $B(i): BH \rightarrow BG$ . We can regard  $EG$  as a  $H$ -space.  $H$  acts freely and locally trivially on  $EG$ . Therefore  $EG \rightarrow EG/H$  is a principal  $H$ -bundle. Note that  $EG$  is contractible.  $EG$  is a universal  $H$ -bundle. This means  $BH \simeq EG/H$ . Hence  $B(i): EG/H \rightarrow EG/G$  is a fibre bundle with fibre  $G/H$ .

**Proposition 5.9.** Let  $H < G$  be a subgroup such that  $G/H$  admits a CW-complex. Suppose  $i: H \hookrightarrow G$  is a homotopy equivalence. Then  $B(i): BH \rightarrow BG$  is a homotopy equivalence.

*Proof.*  $i: H \rightarrow G$  is a homotopy equivalence, then  $G/H$  is weakly contractible. This is given by some directly calculations of homotopy exact sequence of  $H \hookrightarrow G \rightarrow G/H$ . Then  $G/H$  is contractible. Hence  $BH \simeq BG$ .  $\square$

**Example 5.10.**

$H$	$G$
$O(n)$	$GL_n(\mathbb{R})$
$SO(n)$	$GL_n^*(\mathbb{R})$
$U(n)$	$GL_n(\mathbb{C})$
$SU(n)$	$SL_n^*(\mathbb{C})$

This means the classification of  $\mathbb{R}^n$ -bundles is equivalent to the classification of  $O(n)$ -bundles.

**Definition 5.11.** Let  $R$  be a ring. Then each non-zero element  $c \in H^*(BG; R)$  is called a universal characteristic class.

Given any principal  $G$ -bundle  $\pi: P \rightarrow B$ , define the corresponding characteristic class to be  $\overline{\Phi}^* c \in H^*(B; R)$  by

$$\begin{array}{ccc} P & \xrightarrow{\Phi} & EG \\ \downarrow & & \downarrow \\ B & \xrightarrow{\overline{\Phi}} & BG \end{array}$$

.  $\overline{\Phi}^*$  is called the classifying map.

## 5.2 Stiefel-Whitney Classes

Denote by  $Gr_n(\mathbb{R}^N)$  the Grassmannian manifold consists of  $n$ -planes in  $\mathbb{R}^N$ , which is topologized as follows:

We have the Stiefel manifold  $V_n(\mathbb{R}^N)$  consisting of  $n$ -planes in  $\mathbb{R}^N$ , i.e.

1. either identify  $V_n(\mathbb{R}^N) := O(N)/O(N-n)$ ,
2. or regard as  $V_n(\mathbb{R}^N) \subset (\mathbb{R}^N)^n$ .

Then we get a quotient map

$$q: V_n(\mathbb{R}^N) \rightarrow \text{Gr}_n(\mathbb{R}^N)$$

with fibre  $GL_n(\mathbb{R})$ . We have  $\dim \text{Gr}_n(\mathbb{R}^N) = n(N - n)$ .

Denote by

$$V_n(\mathbb{R}^\infty) := \text{colim}_{\vec{N}} V_n(\mathbb{R}^N)$$

and then

$$\text{Gr}_n(\mathbb{R}^\infty) := \text{colim}_{\vec{N}} \text{Gr}_n(\mathbb{R}^N)$$

is principal  $GL_n(\mathbb{R})$ -bundle. Recall that  $V_n(\mathbb{R}^N)$  is at least  $(N - n - 1)$ -connected, then  $V_n(\mathbb{R}^\infty)$  is contractible. Therefore,  $V_n(\mathbb{R}^\infty)$  is the universal  $GL_n(\mathbb{R})$ -bundle, and  $\text{Gr}_n(\mathbb{R}^\infty)$  is the classifying space  $BGL_n(\mathbb{R}) \simeq BO(n)$ .

### 5.2.1 Construction of Stiefel-Whitney Classes

Let  $\pi: E \rightarrow B$  be a real vector bundle of rank  $n$ . Consider the projectivization of  $E$ :

$$\mathbb{P}(E) := \{(b, v) \in E : v \neq 0 \in E_b\} / (b, v) \sim (b, \lambda v), \lambda \neq 0.$$

Write  $\rho: \mathbb{P}(E) \rightarrow B$  for the projection. Then we get a tautological line bundle

$$S(E) = \{((b, v), u) \in \mathbb{P}(E) \times E : u \in l_v\} \rightarrow \mathbb{P}(E).$$

The classifying space for line bundles is  $\text{Gr}_1(\mathbb{R}^\infty) = \mathbb{R}P^\infty$ .

The cohomology of  $\mathbb{R}P^\infty$  is  $H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[a]$ . Write  $f: \mathbb{P}(E) \rightarrow \mathbb{R}P^\infty$  for the classifying map. Define  $a_E := f^*a \in H^1(\mathbb{P}(E); \mathbb{Z}_2)$ .

**Proposition 5.12.**  $\{1, a_E, \dots, a_E^{n-1}\}$  is a free  $H^*(B; \mathbb{Z}_2)$ -basis for  $H^*(\mathbb{P}(E); \mathbb{Z}_2)$ .

*Proof.* By Leray-Hirsch Theorem, we only need to prove  $\{1, a_E, \dots, a_E^{n-1}\}|_{\mathbb{P}(E_b)}$  generates  $H^*(\mathbb{P}(E_b); \mathbb{Z}_2)$  for any  $b \in B$ .

Write  $i_b: \mathbb{P}(E_b) \hookrightarrow \mathbb{P}(E)$  for the inclusion.  $S(E)|_{\mathbb{P}(E_b)} \rightarrow \mathbb{P}(E_b) \simeq \mathbb{R}P^{n-1}$  is the tautological line bundle. Then  $f \circ i_b: \mathbb{P}(E_b) \cong \mathbb{R}P^{n-1} \rightarrow \mathbb{R}P^\infty$  satisfies  $(f \circ i_b)^*E_1(\mathbb{R}^\infty) = S(E)|_{\mathbb{P}(E_b)}$ . Note that  $E_1(\mathbb{R}^\infty) \rightarrow \mathbb{R}P^\infty$  is the tautological line bundle. By the uniqueness of classifying maps,  $f \circ i_b \simeq i: \mathbb{R}P^{n-1} \hookrightarrow \mathbb{R}P^\infty$  and  $a_E^k|_{\mathbb{P}(E_b)} = (f \circ i_b)^*a^k = a^k \in H^k(\mathbb{R}P^{n-1}, \mathbb{Z}_2)$ .  $\square$

**Definition 5.13.** We define Stiefel-Whitney classes  $w_k(E) \in H^k(B; \mathbb{Z}_2)$  by

$$a_E^n = \sum_{k=1}^n w_k(E) \cup a_E^{n-k}$$

and  $w_1(E) = 1$ ,  $w_i(E) = 0$  for  $i > n$ .

Let  $\pi: L \rightarrow B$  be a line bundle. Then  $\mathbb{P}(L) = B$  and  $S(E) = L$ . Write  $f: B \rightarrow \mathbb{R}P^\infty$  for the classifying map of  $L \rightarrow B$ . Then  $a_L = f^*a = w_1(L) \in H^1(B, \mathbb{Z}_2)$ . The transition maps of  $\pi: L \rightarrow B$   $\{g_{\alpha\beta}\}$  form a cocycle in Čech cohomology. Then  $[L] \in H^1(B; \mathbb{Z}_2)$ .

**Claim 16.**  $[L] = w_1(L)$ .

By the claim, we have

**Proposition 5.14.** 1. Let  $L_1, L_2$  be two line bundles over  $B$ . Then

$$w_1(L_1 \otimes L_2) = w_1(L_1) + w_1(L_2).$$

2. Let  $L^*$  be the dual bundle of  $L$ . Then

$$w_1(L^*) = w_1(L).$$

Now we discuss why the claim is true.

**Proposition 5.15** (Naturality of  $w_k$ ).  $w_k(f^*E) = f^*w_k(E)$ , where  $f: B' \rightarrow B$  is a map.

*Proof.* Write  $E' = f^*E \rightarrow B'$ . Then  $a_{E'} = F^*a_E$  where  $F: \mathbb{P}(E') \rightarrow \mathbb{P}(E)$ . Then relations specifying  $a_E^n$  pulls back to  $a_{E'}^n$ .  $\square$

Then we consider  $w_1$  of  $E_1(\mathbb{R}^2) \rightarrow \mathbb{R}P^1$ . Note that  $H^1(\mathbb{R}P^1; \mathbb{Z}_2) = \langle a \rangle$ ,  $\mathbb{P}(E_1(\mathbb{R}^2)) = \mathbb{R}P^1$ ,  $S(E_1(\mathbb{R}^2)) = E_1(\mathbb{R}^2)$ . Then the classifying space of  $E_1(\mathbb{R}^2)$  is the inclusion  $i: \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^\infty$ . Therefore,  $w_1(E_1(\mathbb{R}^2)) = a \in H^1(\mathbb{R}P^1; \mathbb{Z}_2)$ .

Let  $L \rightarrow B$  be a line bundle,  $f: B' \rightarrow B$  be a map. Then  $[f^*L] = f^*[L] \in H^1(B; \mathbb{Z}_2)$  and  $E_1(\mathbb{R}^2) \rightarrow \mathbb{R}P^1 = S^1$  is non-trivial. Therefore  $[E_1(\mathbb{R}^2)] = a \neq 0 \in H^1(\mathbb{R}P^1; \mathbb{Z}_2)$ . Then  $w_1(L) = f^*a \in H^1(B; \mathbb{Z}_2)$  and  $f: B \rightarrow \mathbb{R}P^\infty$  is the classifying map. We need  $[E_1(\mathbb{R}^\infty)] = a \in H^1(\mathbb{R}P^\infty; \mathbb{Z}_2) = \mathbb{Z}_2$ . This is because  $E_1(\mathbb{R}^\infty)|_{\mathbb{R}P^1}$  is non-trivial.

Now we consider the Whitney sum of line bundles. Write  $w(E) = 1 + w_1(E) + \dots + w_n(E)$  for the total Stiefel-Whitney class of  $E$ .

**Proposition 5.16.** Suppose  $E = E_1 \oplus E_2$ . Then

$$w(E) = w(E_1) \cup w(E_2).$$

**Lemma 5.17** (Splitting Principle). Let  $\pi: E \rightarrow B$  be a real vector bundle. Then there exists a space  $F(E)$  and a map  $\tau: F(E) \rightarrow B$  such that

1.  $\tau^*E = L_1 \oplus \dots \oplus L_n$  where  $L_i \rightarrow F(E)$  is a line bundle.
2.  $\tau^*: H^*(B; \mathbb{Z}_2) \rightarrow H^*(F(E); \mathbb{Z}_2)$  is injective.

*Proof.* We apply induction on the rank of  $E$ . When  $\text{rank}(E) = 1$ , there is nothing to prove. When  $\text{rank}(E) = n$ , we have the projectivization  $\rho: \mathbb{P}(E) \rightarrow B$ . Then  $\rho^*E = S(E) \oplus Q(E)$  where  $Q(E)$  is an orthogonla complement of  $S(E)$ , where  $\text{rank}(S(E)) = 1$  and  $\text{rank}(Q(E)) = n - 1$ .

By induction, there exists  $\tau_p: F(E) \rightarrow \mathbb{P}(E)$  such that  $\tau_p^*Q(E) = L_1 \oplus \dots \oplus L_{n-1}$ ,  $\tau_p^*: H^*(\mathbb{P}(E); \mathbb{Z}_2) \rightarrow H^*(F(E); \mathbb{Z}_2)$  is injective. Then let  $\tau = \rho \circ \tau_p$ , we have  $\tau^*E = \tau_p^*S(E) \oplus L_1 \oplus \dots \oplus L_{n-1}$ .  $\square$

*Proof of Proposition 5.16.* We can find a  $\tau: F(E) \rightarrow B$  such that  $\tau^*E_1 = L_1 \oplus \dots \oplus L_k$  and  $\tau^*E_2 = M_1 \oplus \dots \oplus M_j$ . Then  $w(E) = w(E_1) \cup w(E_2)$  if and only if  $w(\tau^*E) = w(\tau^*E_1) \cup w(\tau^*E_2)$ .

We only need to prove if  $E = L_1 \oplus L_2$ , then  $w(E) = (1 + w_1(L_1)) \cup (1 + w_1(L_2))$ . Consider the exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & S(E) & \longrightarrow & \rho^*E & \longrightarrow & Q(E) \longrightarrow 0 \\ & & & & \parallel & & \\ & & & & \rho^*L_1 \oplus \rho^*L_2 & & \end{array}$$

Tensoring with  $S^*(E)$ ,

$$0 \longrightarrow \mathbb{R} \longrightarrow S^*(E) \otimes \rho^*L_1 \oplus S^*(E) \otimes \rho^*L_2 \longrightarrow \dots$$

$$1 \longmapsto (s_1, s_2)$$

Then

$$w_1(S^*(E) \otimes \rho^*L_1) \cup w_1(S^*(E) \otimes \rho^*L_2) = 0,$$

i.e.

$$w_1(S(E) + \rho^*w_1(L_1)) \cup w_1(S(E) + \rho^*w_1(L_2)) = 0.$$

Note that  $w_1(S(E)) = a_E$ . We have

$$a_E^2 + \rho^*(w_1(L_1) + w_1(L_2)) \cdot a_E + \rho^*(w_1(L_1) \cup w_1(L_2)) = 0,$$

i.e.

$$w_1(E) = w_1(L_1) + w_1(L_2)$$

and

$$w_2(E) = w_1(L_1) \cup w_1(L_2).$$

□

Recall that  $BO(n) = Gr_n(\mathbb{R}^n)$ . Our goal is  $H^*(Gr_n(\mathbb{R}^\infty); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \dots, w_n]$ .

**Proposition 5.18.** For the tautological bundle  $E_n(\mathbb{R}^\infty) \rightarrow Gr_n(\mathbb{R}^\infty)$ , the Stiefel-Whitney classes  $w_1, \dots, w_n$  are algebraically independent.

*Proof.* By naturality, we prove the algebraic independence on the bundle  $E = \oplus_{i=1}^n p_i^* E_1(\mathbb{R}^\infty) \rightarrow \mathbb{R}P^\infty \times \dots \times \mathbb{R}P^\infty$ .

Write  $a_i = p_i^* a \in H^1(\mathbb{R}P^\infty \times \dots \times \mathbb{R}P^\infty; \mathbb{Z}_2)$ . Then

$$w(E) = 1 + w_1(E) + \dots + w_n(E) = \prod_{i=1}^n (1 + a_i) = \sum_{i=0}^n \sigma_i(a_1, \dots, a_n)$$

where  $\sigma_i$  is the  $i$ -th elementary symmetric polynomial. Therefore,  $w_i(E) = \sigma_i(a_1, \dots, a_n)$  which are algebraically independent. □

By Milnor-Stasheff,  $Gr_n(\mathbb{R}^\infty)$  admits a CW-decomposition such that the number of  $k$ -cells = the number of partition of  $k$  into integers in  $[1, n]$  = the number of  $(p_1, \dots, p_n)$  such that  $p_1 + 2p_2 + \dots + np_n = k \Leftrightarrow w_1^{p_1} \cup w_2^{p_2} \cup \dots \cup w_n^{p_n} \in H^k(Gr_n(\mathbb{R}^\infty); \mathbb{Z}_2)$ . Note that the number of  $k$ -cells in  $Gr_n(\mathbb{R}^\infty)$  = the rank of  $C^k(Gr_n(\mathbb{R}^\infty); \mathbb{Z}_2) \geq$  the rank of  $H^k(Gr_n(\mathbb{R}^\infty); \mathbb{Z}_2) \geq$  the number of  $k$ -cells in  $Gr_n(\mathbb{R}^\infty)$ . Hence the equality holds and then  $H^k(Gr_n(\mathbb{R}^\infty); \mathbb{Z}_2) = \text{Span}\{w_1^{p_1} \cup \dots \cup w_n^{p_n} : \sum p_i = k\}$ .

### 5.2.2 Construction of $w_i$ from obstruction theory

Recall that  $V_k(\mathbb{R}^n)$  is  $(n - k - 1)$ -connected.

**Fact 5.19.**

$$\pi_{n-k}(V_k(\mathbb{R}^n)) = \begin{cases} \mathbb{Z}_2, & \text{if } n - k \text{ odd and } k \geq 2; \\ \mathbb{Z}, & \text{otherwise.} \end{cases}$$

$$\pi_1(Gr_k(\mathbb{R}^n)) = \mathbb{Z}_2.$$

$$Gr_k(\mathbb{R}^n) \cong O(n)/O(k) \times O(n - k).$$

Let  $\pi: E \rightarrow B$  be an  $\mathbb{R}^n$ -bundle. Then we get a  $V_k(\mathbb{R}^n)$ -bundle  $\pi^k: E_k \rightarrow B$ . A section of  $E_k \rightarrow B$  is equivalent to a global  $k$ -frame of  $E$ . The primary obstruction of finding a section of  $E_k$  is  $\tilde{\alpha}_{n-k+1} \in H^{n-k+1}(B; \{\pi_{n-k}(E_k|_b)\})$ . The local coefficient is

$$\begin{array}{ccccc} \rho_*: \pi_1(Gr_k(\mathbb{R}^n)) & \longrightarrow & \text{Aut}(\pi_{n-k}(V_k(\mathbb{R}^n))) & \longrightarrow & \text{Aut}(\mathbb{Z}_2) \\ \parallel & & \parallel & & \parallel \\ \mathbb{Z}_2 & & \mathbb{Z} \text{ or } \mathbb{Z}_2 & & \{1\} \end{array}$$

Therefore  $\{\pi_{n-k}(E_k|_b)\} \bmod 2$  is trivial. We get  $\alpha_{n-k+1} \in H^{n-k+1}(B; \mathbb{Z}_2)$ .

**Theorem 5.20.** Let  $E \rightarrow B$  be a  $\mathbb{R}^n$ -bundle. Then

$$\alpha_k(E) = w_k(E) \in H^k(B; \mathbb{Z}_2).$$

*Proof.* Write  $w_{k,n} = w_k(E_n(\mathbb{R}^n))$ ,  $\alpha_{k,n} = \alpha_k(E_n(\mathbb{R}^\infty)) \in H^k(\text{Gr}_n(\mathbb{R}^\infty); \mathbb{Z}_2)$ . Then  $\alpha_{k,n} = p_{k,n}(w_{1,n}, \dots, w_{k-1,n}) + \lambda_{k,n} \cdot w_{k,n}$ .

Consider  $E = E_{k-1}(\mathbb{R}^\infty) \oplus \mathbb{R}^{n-k+1}$ . We have

$$w_i(E) = \begin{cases} w_{i,n}|_{E_{n-1}(\mathbb{R}^\infty)}, & 1 \leq i \leq k-1; \\ 0, & i = k. \end{cases}$$

Because  $\alpha_k(E) = 0$ , we have  $p_{k,n} \equiv 0$ . Then  $\alpha_{k,n} = \lambda_{k,n} \cdot w_{k,n}$ .

Consider  $E' = E_1(\mathbb{R}^{k+1}) \oplus \mathbb{R}^{n-1} \rightarrow \mathbb{R}P^k$ .

**Claim 17.**

$$\alpha_k(E') \neq 0.$$

Therefore  $\lambda_{k,n} = 1$ . □

In conclusion, Stiefel-Whitney classes

1. generate  $H^*(BO(n); \mathbb{Z}_2)$ ;
2. constructed from projectivization  $H^*(\mathbb{P}(E); \mathbb{Z}_2)$  admitting a free  $H^*(B; \mathbb{Z}_2)$  basis;
3. primary obstruction of mod 2 global frames.

### 5.3 Euler Class

Let  $\pi: E \rightarrow B$  be an oriented  $\mathbb{R}^n$ -bundle. Recall that the Euler class  $e(E)$  is defined by Thom isomorphism

$$\begin{aligned} \Phi_E: H^j(B; \mathbb{Z}) &\rightarrow H^{n+j}(E, E_0; \mathbb{Z}) \\ x &\mapsto x \cup t_E \end{aligned}$$

where  $t_E$  is the Thom class, through

$$H^n(E, E_0; \mathbb{Z}) \longrightarrow H^n(E; \mathbb{Z}) \xrightarrow[(\pi^*)^{-1}]{\cong} H^n(B; \mathbb{Z})$$

$$t_E \longmapsto e(E)$$

Applying Thom isomorphism and Exact sequence of  $(E, E_0)$ , we get Gysin sequence

$$H^j(B; \mathbb{Z}) \xrightarrow{\cup e(E)} H^{j+n}(B; \mathbb{Z}) \xrightarrow{\pi_0^*} H^{j+n}(E_0; \mathbb{Z}) \xrightarrow{\Phi_E^{-1} \circ \delta} H^{j+1}(B; \mathbb{Z})$$

**Proposition 5.21.** 1.  $f^*e(E) = e(f^*E)$ ;

2.  $e(E_1 \oplus E_2) = e(E_1) \cup e(E_2)$ .

3. Suppose  $-E \rightarrow B$  is bundle obtained from reversing the orientation of  $E$ . Then  $e(-E) = -e(E)$ .

*Proof.* They can be deduced from properties of  $t_E$ . □

**Proposition 5.22.** If rank  $E$  is odd, then  $2e(E) = 0$



*Proof.* Consider

$$\begin{aligned}\tau: E &\rightarrow -E \\ (b, v) &\mapsto (b, -v).\end{aligned}$$

Note that  $\tau$  is orientation-preserving, we have

$$e(E) = e(-E) = -e(E).$$

□

**Theorem 5.23.** Let  $\pi: E \rightarrow B$  be an oriented bundle of rank  $n$ . Then  $w_n(E) \equiv e(E) \pmod{2}$ .

*Proof.* Applying Splitting Principle and naturality, it suffices to consider the line bundle  $E_1(\mathbb{R}^2) \rightarrow \mathbb{R}P^1$ . Note that  $w_1(E_1(\mathbb{R}^\infty))|_{\mathbb{R}P^1} = w_1(E_1(\mathbb{R}^2))$  by the inclusion  $\mathbb{R}P^1 \hookrightarrow \mathbb{R}P^\infty$ , we have  $w_1(E_1(\mathbb{R}^2)) = a \in H^1(\mathbb{R}P^1; \mathbb{Z}_2)$ . We can find a section of  $E_1(\mathbb{R}^2) \rightarrow \mathbb{R}P^1$  with a simple zero. Then  $e(E_1(\mathbb{R}P^2)) = 1 \in H^1(\mathbb{R}P^1; \mathbb{Z}) \cong H^1(S^1; \mathbb{Z})$ . □

### 5.3.1 Euler class as obstruction

Recall that  $\pi_1(\text{Gr}_k(\mathbb{R}^n)) = \mathbb{Z}_2$ . We consider the universal cover  $\widetilde{\text{Gr}}_k(\mathbb{R}^n) \rightarrow \text{Gr}_k(\mathbb{R}^n)$  which is called the oriented Grassmannian consisting of oriented  $k$ -plane in  $\mathbb{R}^n$ . We get a principal bundle  $\widetilde{\text{V}}_k(\mathbb{R}^n) \rightarrow \widetilde{\text{Gr}}_k(\mathbb{R}^n)$  and a group action  $\text{GL}_k^+(\mathbb{R}^n) \curvearrowright \widetilde{\text{V}}_k(\mathbb{R}^n)$ . Then  $\widetilde{\text{V}}_k(\mathbb{R}^\infty)$  is contractible. Therefore,  $\widetilde{\text{V}}_k(\mathbb{R}^\infty) \rightarrow \widetilde{\text{Gr}}_k(\mathbb{R}^\infty)$  is the universal  $\text{SO}(k)$ -bundle. Define  $o(E) \in H^n(B; \{\pi_{n-1}(\widetilde{\text{V}}_1(E_b))\}) \cong H^n(B; \mathbb{Z})$  to be the primary obstruction of a global non-vanishing section because  $\pi_{n-1}(\widetilde{\text{V}}_1(E_b)) \cong \pi_{n-1}(\widetilde{\text{V}}_1(\mathbb{R}^n)) \cong \pi_{n-1}(S^{n-1}) \cong \mathbb{Z}$ .

**Theorem 5.24.**  $o(E) = e(E) \in H^n(B; \mathbb{Z})$ .

*Proof.* Denote  $\pi_0: E_0 \rightarrow B$  for the restricted projection. Recall that  $\pi_0^*E = \{(v, v) \in E_0 \times E : \pi_0(v) = \pi(v)\}$ . Then the  $\mathbb{R}^n$ -bundle  $\pi_0^*E \rightarrow E_0$  admits a tautological non-vanishing section. Then  $\pi_0^*o(E) = o(\pi_0^*E) = 0$ . Recall the Gysin sequence

$$H^0(B; \mathbb{Z}) \xrightarrow{\cup e(E)} H^n(B; \mathbb{Z}) \xrightarrow{\pi_0^*} H^n(E_0, \mathbb{Z})$$

$$\lambda_E \longmapsto o(E)$$

Therefore we have  $o(E) = \lambda_E \cup e(E) = \lambda_E e(E)$ . And by naturality,  $\lambda_E$  only depends on  $n$ .

When  $\text{rank}(E) = n$  is odd,  $o(E) \equiv w_n(E) \pmod{2}$ . Note that we have commutative diagram

$$\begin{array}{ccc}\pi_{n-1}(\widetilde{\text{V}}_1(\mathbb{R}^n)) & \longrightarrow & \pi_{n-1} \text{V}_1(\mathbb{R}^n) \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}_2\end{array}$$

Then  $e(E) \equiv w_n(E) \pmod{2}$ . Note that  $2e(E) = 0$ ,  $2o(E) = 0$ , we have  $e(E) = o(E)$ .

When  $\text{rank}(E) = n = 2m$  is even, one can compute  $e(TS^n) = o(TS^n) = 2$ . □

**Remark 5.25.**  $e(E) = 0$  does not imply there exists a non-vanishing section, except for, for example,  $\text{rank}(E) \geq \dim B$ . Especially, for  $I$ -bundle, it is right. This is because  $\pi_k(S^{n-1})$  can be non-trivial for  $k > n - 1$ .

## 5.4 Chern Classes

In this section, we consider  $U(n)$  instead of  $O(n)$ . The universal bundle of  $U(n)$  is similarly given by  $V_n(\mathbb{C}^\infty) \rightarrow \text{Gr}_n(\mathbb{C}^\infty)$ .

**Fact 5.26.**

$$\begin{aligned}\pi_1(\text{Gr}_n(\mathbb{C}^N)) &= 1. \\ \pi_{2(n-k)+1}(V_k(\mathbb{C}^n)) &= \mathbb{Z}. \\ V_k(\mathbb{C}^n) &\text{ is } 2(n-k)\text{-connected.}\end{aligned}$$

### 5.4.1 Obstruction construction of Chern classes

Let  $\pi: E \rightarrow B$  be a  $\mathbb{C}^n$ -bundle. The primary obstruction of finding a  $(n-k+1)$ -frame is  $\alpha_k(E) \in H^{2k}(B; \{\pi_{2k-1}(V_{n-k+1}(E_b))\})$ . Note that  $V_{n-k+1}(E_b)$  is  $2(k-1)$ -connected.  $\alpha_k(E) \in H^{2k}(B; \{\pi_{2k-1}(V_{n-k+1}(E_b))\}) \cong H^{2k}(B; \mathbb{Z})$  is the  $k$ -th Chern class of  $E$ .

### 5.4.2 Projectivization construction of Chern class

Note that

$$\text{Gr}_1(\mathbb{C}^n) = \mathbb{C}P^n = S^{2n+1}/U(1) = \mathbb{C}^{n+1} - \{0\}/\mathbb{C}^*.$$

Denote by  $c \in H^2(\mathbb{C}P^n; \mathbb{Z})$  for the Euler class of the Hopf fibration  $S^{2n+1} \rightarrow \mathbb{C}P^n$ . Consider Gysin sequence

$$H^{k+1}(S^{2n+1}) \longrightarrow H^k(\mathbb{C}P^n) \xrightarrow{\cup c} H^{k+2}(\mathbb{C}P^n) \longrightarrow H^{k+2}(S^{2n+1})$$

When  $k \leq 2n-2$ ,  $H^k(\mathbb{C}P^n) \xrightarrow{\cup c} H^{k+2}(\mathbb{C}P^n)$  is an isomorphism. We have

$$\begin{array}{c} H^0(\mathbb{C}P^n) \xrightarrow{\cup c} H^2(\mathbb{C}P^n) \xrightarrow{\cup c} \dots \xrightarrow{\cup c} H^{2n}(\mathbb{C}P^n) \\ \parallel \\ \mathbb{Z} \end{array}$$

and

$$\begin{array}{c} H^1(\mathbb{C}P^n) \xrightarrow{\cup c} H^3(\mathbb{C}P^n) \xrightarrow{\cup c} \dots \xrightarrow{\cup c} H^{2n+1}(\mathbb{C}P^n) \\ \parallel \\ 0 \end{array}$$

Therefore  $H^*(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}[i]/c^{n+1}$ . Then  $H^*(\text{Gr}_1(\mathbb{C}^\infty)) = H^*(\mathbb{C}P^\infty) = \mathbb{Z}[c]$ .

Given a  $\mathbb{C}^n$ -bundle  $\pi: E \rightarrow B$ . Denote  $\rho: \mathbb{P}(E) \rightarrow B$  for the projectivization. We have a line bundle

$$S(E) = \{(b, [v], u) \in \mathbb{P}(E) \times E : u \in l_v\} \rightarrow \mathbb{P}(E).$$

There exists a classifying map  $f: \mathbb{P}(E) \rightarrow \mathbb{C}P^\infty$ . Denote  $c_E = f^*c \in H^2(\mathbb{P}(E); \mathbb{Z})$ . By Leray-Hirsch Theorem,  $1, c_E, \dots, c_E^{n-1}$  is a free  $H^*(B; \mathbb{Z})$ -basis of  $H^*(\mathbb{P}(E); \mathbb{Z})$ . Define Chern classes  $c_k(E) \in H^{2k}(B; \mathbb{Z})$  by

$$\sum_{k=0}^n (-1)^k \rho^*(c_k(E)) \cup c_E^{n-k} = 0.$$

We require  $c_0(E) = 1$  and  $c_k(E) = 0$  when  $k > n$ .

### 5.4.3 $c_1$ from Čech cohomology

Let  $\pi: E \rightarrow B$  be a  $\mathbb{C}^n$ -bundle. We have a class  $[E] \in H^1(B; \mathbb{U}(n))$ , called the Čech class of  $E$ . Consider the exact sequences

$$1 \longrightarrow \mathrm{SU}(n) \longrightarrow \mathrm{U}(n) \xrightarrow{\det} \mathrm{U}(1) \longrightarrow 1$$

and

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathrm{U}(1) \longrightarrow 1$$

Consider  $\mathbb{R}$  as a locally constant sheaf, it is a fine sheaf. Then  $H^*(B; \mathbb{R}) = 0$ . Then we have a connecting map  $\delta: H^1(B; \mathrm{U}(1)) \xrightarrow{\cong} H^2(B; \mathbb{Z})$ . We define

$$c_1(E) := \delta \circ \det_*[E] \in H^2(B; \mathbb{Z}).$$

Then we have

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$$

$$c_1(L_1 \oplus L_2) = c_1(L_1) + c_1(L_2)$$

and

$$c_1(L_1) = -c_1(\overline{L_1})$$

for any line bundles  $L_1, L_2 \rightarrow B$ , where  $\overline{L_1}$  is the conjugate bundle of  $L_1$  which is defined by  $\overline{L_1} \otimes L_1 = \mathbb{C}$ .

Finally, we list some facts about chern classes here.

Denote  $c(E) = 1 + c_1(E) + \cdots + c_n(E)$ , we have

$$c(E_1 \oplus E_2) = c(E_1) \cup c(E_2).$$

Its proof is similar to Proposition 5.16.

For the Hopf fibration

$$\mathrm{U}(1) \hookrightarrow S^3 = H(1) \rightarrow \mathbb{C}P^1,$$

we have

$$\langle c_1(H(1)), [\mathbb{C}P^1] \rangle = -1.$$

Denote  $E_k$  for the  $\mathbb{R}^{2n}$ -bundle corresponding to  $E$ . We have

$$c_n(E) = e(E) \in H^{2n}(B; \mathbb{Z}),$$

$$c_k(E) \equiv w_{2n}(E_{\mathbb{R}}) \pmod{2}.$$

## 5.5 Pontrjagin Classes

**Definition 5.27.** Let  $\pi: E \rightarrow B$  be an  $\mathbb{R}^n$ -bundle. Denote by  $E_{\mathbb{C}} := E \otimes \mathbb{C}$  the complexification of  $E$ , which is a  $\mathbb{C}^*$ -bundle. Define the  $k$ -th Pontrjagin class

$$p_k(E) := (-1)^k c_{2k}(E_{\mathbb{C}}) \in H^{4k}(B; \mathbb{Z}).$$

For the oriented Grassmannian  $\widetilde{\mathrm{Gr}}_n(\mathbb{R}^{\infty}) = B\mathrm{SO}(n)$ , we have

$$H^*(\widetilde{\mathrm{Gr}}_n(\mathbb{R}^{\infty}); \mathbb{Z}) = \begin{cases} \mathbb{Z}[\frac{1}{2}][p_1, \dots, p_k], & n = 2k + 1; \\ \mathbb{Z}[\frac{1}{2}][p_1, \dots, p_{k-1}, e], & n = 2k, \end{cases}$$

where  $e = c_{2k} = (-1)^k p_k$ .

Denote  $\tilde{\Omega}_*$  for the oriented cobordism ring. It is the set of closed oriented smooth manifolds defined by taking a quotient of a equivalent relation  $\sim$ , which is defined by

$$M_1 \sim M_2 \Leftrightarrow -M_1 \sqcup M_2 = \partial W$$

for some manifold  $W$ . Its ring structure is defined by

$$\begin{aligned} [pt.] &= 1, \\ [M_1] + [M_2] &= [M_1 \sqcup M_2], \\ -[M] &= [\overline{M}], \\ [M_1] \times [M_2] &= [M_1 \times M_2], \end{aligned}$$

where  $\overline{M}$  is  $M$  with reversed orientation. Then  $\tilde{\Omega}_n$  is an abelian group and  $\tilde{\Omega}_*$  is a commutative graded ring.

**Fact 5.28.**

$$\begin{aligned} \tilde{\Omega}_0 &= \mathbb{Z}, \\ \tilde{\Omega}_1 &= 0, \\ \tilde{\Omega}_2 &= 0, \\ \tilde{\Omega}_3 &= 0 \end{aligned}$$

because any 3-manifold are obtained from  $S^3$  by integer surgery.

**Question 5.29.** What is  $\tilde{\Omega}_4$ ?

Consider  $\tilde{\Omega}_{4k}$ . Given a partition  $I = (i_1, \dots, i_r)$  for  $k: i_1 + 2i_2 + \dots + ri_r = k$ . Pontrjagin and Thom show that

$$\begin{aligned} P_I: \tilde{\Omega}_{4k} &\rightarrow \mathbb{Z} \\ [M] &\mapsto \langle p_1^{i_1} \cup \dots \cup p_r^{i_r}, [M] \rangle, \end{aligned}$$

where  $p_j = p_j(TM)$ , is a group homomorphism. Thom shows that when  $k = 1$ ,

$$\begin{aligned} P_I: \tilde{\Omega}_4 &\rightarrow \mathbb{Z} \\ [M] &\mapsto \langle p_1(M), [M] \rangle \end{aligned}$$

is an isomorphism. Therefore  $\tilde{\Omega}_4 = \langle [\mathbb{C}P^2] \rangle$ .

Recall that the signature  $\sigma: \tilde{\Omega}_4 \rightarrow \mathbb{Z}$  is also a homomorphism. Hence  $p_1(M) = \lambda\sigma$ . We have  $\lambda = 3$ , i.e., for any 4-manifold  $M$ ,  $\frac{1}{3}p_1(M) = \sigma(M)$ .

When  $k = 2$ , Hirzebruch shows that  $\sigma(M) = \frac{1}{45}(7p_2 - p_1)^2$ .

**Question 5.30.** Is there a general relation between  $p_i$  and  $\sigma$ ?

For a formal power series  $f(x) = 1 + a_1x + \dots + a_nx^n + \dots \in \mathbb{Q}[[x]]$ , denote

$$\prod_{i=1}^n f(x_i) = 1 + F_1(\sigma_1) + \dots + F_n(\sigma_1, \dots, \sigma_n) + \dots$$

where  $\sigma_i$  is the  $i$ -th elementary symmetric polynomial. We have  $F_n(t\sigma_1, t^2\sigma_2, \dots, t^n\sigma_n) = t^n F_n(\sigma_1, \dots, \sigma_n)$ . We get a corresponding

$$f \longleftrightarrow \{F_n\}_{n \geq 1} := \vec{F}$$

Define the  $\vec{F}$ -genus

$$\vec{F}[M] := \langle F_n(p_1, \dots, p_n), [M] \rangle$$

if  $\dim M = 4n$ .

**Theorem 5.31** (Hirzebruch Signature Theorem). Take  $f(x) = \frac{\sqrt{x}}{\tanh \sqrt{x}} \in \mathbb{Q}[[x]]$ . Then

$$\sigma(M) = \vec{F}[M]$$

when  $\dim M = 4n$ .

**Theorem 5.32** (Atiyah-Singer). Take  $f(x) = \frac{\sqrt{x}}{2} / \sinh \frac{\sqrt{x}}{2} \in \mathbb{Q}[[x]]$ . Assume  $M$  is spin. Then

$$\vec{F}[M] = \text{index } D$$

where  $D$  is the Dirac operator.