Homotopy Theory and Characteristic Classes

CUI Jiaqi
East China Normal University

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Abstract

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Contents

Part I
Homotopy Theory

Let **TOP** be the category of topological spaces. Then we can take a quotient of **TOP** and get the homotopy category $h-\mathbf{TOP}$. The quotient may bring more algebraic structures. For example, $\operatorname{Mor}\left(S^1, X\right)$, the homotopy classes of maps from S^1 to X, is the fundamental group of X. Our goal is to study functors from hmotopy category to some algebraic categories.

Let $\mathbf{TOP}^o$ be the pointed topological category, where the sum is wedge sum $(X, x_0) \wedge (Y, y_0) = X \sqcup Y / x_0 \sim y_0$ and the product is the smash product $(X, x_0) \vee (Y, y_0) = X \times Y / \{x_0\} \times Y \cup X \times \{y_0\}$. Similarly, we can take a quotient to get $h - \mathbf{TOP}^o$.

Let $\mathbf{TOP}(2)$ be the category of pairs and $h - \mathbf{TOP}(2)$ be its quotient.

Fix $K \in \operatorname{Ob}(\mathbf{TOP})$. Let's consider $\mathbf{TOP}^K$, the category of spaces under K. Its objects are maps $f\colon K \to X$ and morphisms are maps $\alpha\colon X \to Y$ such that $\alpha \circ f = g$.

$$\begin{tikzcd} & K \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, "\alpha"] & & Y \end{tikzcd}$$

If $K = \{*\}$ is a single point set, then $\mathbf{TOP}^{\{*\}} = \mathbf{TOP}^o$ is the pointed topological category. Take $X = K$. A morphism from $f\colon K \to X$ to $\mathrm{id}\colon K \to K$ is $r\colon X \to K$ such that $r \circ f = \mathrm{id}$.

$$\begin{tikzcd} & K \arrow[dl, "f"'] \arrow[dr, "\mathrm{id}"] & \\ X \arrow[rr, "r"] & & K \end{tikzcd}$$

When $K \subset X$, $f = i\colon K \hookrightarrow X$, we say that r is a retraction.

We have $r\colon X \to K$ is a deformation retraction, if and only if $i \circ r \simeq \mathrm{id}_X$ rel K, if and only if $r\colon X \to K$ is a homotopy equivalence in $\mathbf{TOP}^K$.

Fix $B \in \mathrm{Ob}(\mathbf{TOP})$. Let's consider $\mathbf{TOP}_B$, the category of spaces over B, where the objects are $p\colon X \to B$ and morphisms are $f\colon X \to Y$ such that $p = q \circ f$.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

Take $X = B$. A morphism from $\mathrm{id}\colon B \to B$ to $q\colon Y \to B$ is $s\colon B \to Y$ such that $q \circ s = \mathrm{id}_B$.

$$\begin{tikzcd} B \arrow[rr, "s"] \arrow[dr, "\mathrm{id}"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

Then s is called a section of q.

Similarly, we can define $h - \mathbf{TOP}^K$ and $h - \mathbf{TOP}_B$.

1 Cofibrations and Fibrations

1.1 Cofibrations

Definition 1.1. A map $i\colon A \to X$ has the homotopy extension property (HEP) for a space Y if for all homotopy $h\colon A \times I \to Y$ and $f\colon X \to Y$ with $f \circ i(a) = h(a, 1)$, there exists $H\colon X \times I \to Y$ satisfies

$$\begin{tikzcd} & X \arrow[dr, "i_1"] \arrow[drr, bend left, "f"] & & \\ A \arrow[ur, "i"] \arrow[dr, "i_1"'] & & X \times I \arrow[r, dashed, "H"] & Y \\ & A \times I \arrow[ur, "i\times\mathrm{id}"] \arrow[urr, bend right, "h"'] & & \end{tikzcd}$$

We say $i\colon A \to X$ is a cofibration if it has HEP for each $Y \in \mathrm{Ob}(\mathbf{TOP})$.

Recall the mapping cylinder: if $i\colon A \to X$ is a map, then $Z(i) \coloneqq (A \times I) \sqcup X/(a, 1) \sim i(a)$.

Proposition 1.2. Given a map $i\colon A \to X$. The followings are equivalent:

1. $i\colon A \to X$ is a cofibration.

2. i has HEP for $Z(i)$.

3. The map

$$\begin{aligned} s\colon Z(i) &\to X \times I \\ (a,t) &\mapsto (f(a),t), \\ x &\mapsto (x,1) \end{aligned}$$

has a retraction.

Proof. (1)$\Longrightarrow$(2) is only by definition.

(2)$\Longrightarrow$(1): By definition, there exists $K\colon X \times I \to Z(i)$ such that the following diagram is commutative.

$$\begin{tikzcd}
 & X \arrow[dr, "i_1"] \arrow[drr, bend left, "i_1"] & & \\
A \arrow[ur, "i"] \arrow[dr, "i_1"'] & & X \times I \arrow[r, dashed, "K"] & Z(i) \\
 & A \times I \arrow[ur, "i\times\mathrm{id}"] \arrow[urr, bend right, "\mathrm{id}"'] & &
\end{tikzcd}$$

For any Y and homotopy $h\colon A \times I \to Y$ and $f\colon X \to Y$ with $f \circ i(a) = h(a,1)$, we define

$$\begin{aligned} F\colon Z(i) &\to Y \\ (a,t) &\mapsto h(a,t) \\ x &\mapsto f(x). \end{aligned}$$

Then $F \circ K$ is as desired.

$$\begin{tikzcd}
 & X \arrow[dr, "i_1"] \arrow[drr, bend left, "i_1"] \arrow[drrr, bend left, "f"] & & & \\
A \arrow[ur, "i"] \arrow[dr, "i_1"'] & & X \times I \arrow[r, dashed, "K"] & Z(i) \arrow[r, "F"] & Y \\
 & A \times I \arrow[ur, "i\times\mathrm{id}"] \arrow[urr, bend right, "\mathrm{id}"'] \arrow[urrr, bend right, "h"'] & & &
\end{tikzcd}$$

(2)$\Longrightarrow$(3): We can easily check that the extension $K\colon X \times I \to Z(i)$ in the proof of (2)$\Longrightarrow$(1) is a retraction of s.

(3)$\Longrightarrow$(2): Let r be a retraction of s. For any homotopy $h\colon A \times I \to Z(i)$ and $f\colon X \to Z(i)$ with $f \circ i(a) = h(a,1)$, we define

$$\begin{aligned} \sigma\colon Z(i) &\to Z(i) \\ (a,t) &\mapsto h(a,t) \\ x &\mapsto f(x). \end{aligned}$$

Then we can verify that $H = \sigma \circ r\colon X \times I \to Z(i)$ extends h.

$$\begin{tikzcd}
 & X \arrow[dr, "i_1"] \arrow[drr, bend left, "f"] & & \\
A \arrow[ur, "i"] \arrow[dr, "i_1"'] & & X \times I \arrow[r, bend right, "r"'] & Z(i) \arrow[l, bend right, "s"'] \\
 & A \times I \arrow[ur, "i\times\mathrm{id}"] \arrow[urr, bend right, "h"'] & &
\end{tikzcd}$$

$\square$

Corollary 1.3. When $A \subset X$ is a close subset, $i\colon A \hookrightarrow X$ is the inclusion map. Then $i\colon A \to X$ is a cofibration $\iff Z(i) = A \times I \cup X \times \{1\}$ is a retraction of $X \times I$.

Therefore, we can construct many cofibrations. For example, let (X, A) be a manifold with boundary, then $i\colon A \hookrightarrow X$ is a cofibration.

Push-Out of Cofibration

Given a commutative diagram,

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] & B \arrow[d, "J"] \\
X \arrow[r, "F"] & Y
\end{tikzcd}$$

the push-out of j along f is the initial object of this diagram, i.e. $j\colon B \to Y$, $F\colon X \to Y$, s.t. $\forall Z$ with $J'\colon B \to Z$, $F'\colon X \to Z$ satisfying $J' \circ f = F' \circ j$, $\exists!$ map $p\colon Y \to Z$ such that the diagram is commutative.

$$\begin{tikzcd}
 & B \arrow[dr, "J"] \arrow[drr, bend left, "J'"] & & \\
A \arrow[ur, "f"] \arrow[dr, "j"'] & & Y \arrow[r, dashed, "\exists! p"] & Z \\
 & X \arrow[ur, "F"] \arrow[urr, bend right, "F'"'] & &
\end{tikzcd}$$

In our setting, we can construct $Y = X \sqcup B / f(a) \sim j(a)$ directly.

Proposition 1.4. If $j\colon A \to X$ is a cofibration, then the push-out of j along f $J\colon B \to Y$ is also a cofibration.

Proof. For any Z, $g\colon Y \to Z$, $h\colon B \times I \to Z$ such that $g \circ J = h \circ (i_1 \times \mathrm{id})$, we need to find $H\colon Y \times I \to Z$ such that the following diagram is commutative.

$$\begin{tikzcd}
 & X \arrow[dr, "F"] & & & \\
A \arrow[ur, "j"] \arrow[dr, "f"'] & & Y \arrow[dr] \arrow[drr, bend left, "g"] & & \\
 & B \arrow[ur, "J"] \arrow[dr] & & Y \times I \arrow[r, dashed, "H"] & Z \\
 & & B \times I \arrow[ur] \arrow[urr, bend right, "h"'] & &
\end{tikzcd}$$

Because $j\colon A \to X$ is a cofibration, we have $G\colon X \times I \to Z$ such that the following diagram is commutative.

$$\begin{tikzcd}
 & X \arrow[dr] \arrow[drr, bend left, "g \circ F"] & & \\
A \arrow[ur, "j"] \arrow[dr] & & X \times I \arrow[r, dashed, "G"] & Z \\
 & A \times I \arrow[ur] \arrow[urr, bend right, "h \circ (f \times \mathrm{id})"'] & &
\end{tikzcd}$$

Using the fact that $J \times \mathrm{id}\colon B \times I \to Y \times I$ is also the push-out of $j \times \mathrm{id}\colon A \times I \to X \times I$ along $f \times \mathrm{id}\colon A \times I \to B \times I$, we have unique $H\colon Y \times I \to Z$ such that the following diagram is commutative.

$$\begin{tikzcd}
 & B \times I \arrow[dr] \arrow[drr, bend left, "h"] & & \\
A \times I \arrow[ur, "f \times \mathrm{id}"] \arrow[dr] & & Y \times I \arrow[r, dashed, "H"] & Z \\
 & X \times I \arrow[ur] \arrow[urr, bend right, "G"'] & &
\end{tikzcd}$$

The $H\colon Y \times I \to Z$ is the extension of $h\colon B \times I \to Z$, as desired. $\square$

In terms of categorical language, let $\Pi(A, B)$ be a category, whose objects are continue maps from A to B and morphisms are homotopy of maps from A to B. Consider $\mathbf{COF}^B \subset \mathbf{TOP}^B$ the subcategory of cofibrations under B (i.e. $J\colon B \to Y$). Then we have homotopy category $h - \mathbf{COF}^B$. Given a cofibration $i\colon A \to X$, we get a contravariant functor

$$\beta\colon \Pi(A, B) \to h - \mathbf{COF}^B.$$

In fact, we only need to check that if $f_0 \simeq f_1\colon A \to B$, then we get a morphism from $J_0\colon B \to Y_0$ to $J_1\colon B \to Y_1$. Firstly, consider the homotopy $J_0 \circ f_t\colon A \times I \to Y_0$, we get its extension $\Psi\colon X \times I \to Y_0$.

$$\begin{tikzcd}
 & X \arrow[dr] \arrow[drr, bend left, "F_0"] & & \\
A \arrow[ur, "i"] \arrow[dr] & & X \times I \arrow[r, dashed, "\Psi"] & Y_0 \\
 & A \times I \arrow[ur] \arrow[urr, bend right, "J_0 \circ f_t"'] & &
\end{tikzcd}$$

Then by the universal property of the push-out $J_1\colon B \to Y_1$ of i along f_1 for $J_0\colon B \to Y_0$ and $\Psi_1\colon X \to Y_0$, we get a map $K\colon Y_1 \to Y_0$, as desired.

$$\begin{tikzcd}
 & B \arrow[dr, "J_1"] \arrow[drr, bend left, "J_0"] & & \\
A \arrow[ur, "f_1"] \arrow[dr, "i"'] & & Y_1 \arrow[r, dashed, "K"] & Y_0 \\
 & X \arrow[ur, "F_1"'] \arrow[urr, bend right, "\Psi_1"'] & &
\end{tikzcd}$$

Replacing a Map by a Cofibration

Given a map $f\colon X \to Y$, consider the mapping cylinder $Z(f)$. We can notice that $Z(f)$ is the push-out.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i_1"'] & Y \arrow[d, "s"] \\
X \times I \arrow[r, "a"] & Z(f)
\end{tikzcd}$$

We also have a map

$$\begin{aligned} q\colon Z(f) &\to Y \\ (x, t) &\mapsto f(x). \end{aligned}$$

Note that by Proposition 1.2, $i_1\colon X \hookrightarrow X \times I$ is a cofibration $\iff X \times \{1\} \times I \cup X \times I \times \{1\}$ is a retraction of $X \times I \times I$, we have $s\colon Y \to Z(f)$ is a cofibration.

Proposition 1.5. Let

$$\begin{aligned} j\colon X &\to Z(f) \\ x &\mapsto (x, 0), \end{aligned}$$

we have

1. $j\colon X \to Z(f)$ is a cofibration.
2. $s \circ q \simeq \mathrm{id}_{Z(f)}$ rel Y.
3. If f is a cofibration, then $q\colon Z(f) \to Y$ is a homotopy equicalence in $\mathbf{TOP}^X$.

Proof. (1). We construct a retraction $R\colon Z(f) \times I \to X \times I \cup Z(f) \times \{1\}$ as follow. Let $R'\colon I \times I \to I \times \{1\} \cup \{0\} \times I$ be a retraction. Then we define

$$\begin{aligned} R\colon Z(f) \times I &\to X \times I \cup Z(f) \times \{1\} \\ ((x, s), t) &\mapsto (x, R'(s, t)) \\ (y, t) &\mapsto (y, 1) \end{aligned}$$

is as desired. By Proposition 1.2, $j\colon X \to Z(f)$ is a cofibration.

(2). The homotopy

$$\begin{aligned} h_t\colon Z(f) &\to Z(f) \\ (x, \sigma) &\mapsto (x, (1-t)\sigma + t) \end{aligned}$$

is as desired.

(3). By Proposition 1.2, there is a retraction $r\colon Y \times I \to Z(f)$. Define

$$\begin{aligned} g\colon Y &\to Z(f) \\ y &\mapsto r(y, 1). \end{aligned}$$

One can verifies that g is the homotopy inverse of q. ☐

Summery 1. Any map $f\colon X \to Y$ factors into

$$X \xrightarrow{j} Z \xrightarrow{q} Y$$

where $j\colon X \to Z$ is a cofibration and $q\colon Z \to Y$ is a homotopy equivalence. Moreover, such a factorization is unique up to homotopy equivalence. In particular, we can choose $Z = Z(f)$. We define $C_f = Z(f)/\operatorname{im} j$ as the homotopy cofibre of f, i.e. $C_f = X \times I \sqcup Y/(x, 0) \sim *, (x, 1) \sim f(x)$, is called the mapping cone of f.

$$X \xrightarrow{f} Y \xrightarrow{s} C_f$$

The Cofibre Sequence (Puppe's Sequence)

To get finer structure, we work in $\mathbf{TOP}^o$. Given a map $f\colon (X, x_0) \to (Y, y_0)$, we get an induced map

$$\begin{aligned} f^*\colon [Y, B]^o &\to [X, B]^o \\ [\alpha] &\mapsto [f \circ \alpha], \end{aligned}$$

where $[X, B]^o$ is the homotopy class of basepoint preserving maps. In particular, we have the constant map

$$\begin{aligned} [*]\colon X &\to B \\ x &\mapsto b_0. \end{aligned}$$

Definition 1.6. We say a sequence

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$$

in $\mathbf{TOP}^o$ is h-coexact if $\forall\,(B, b_0) \in \mathrm{Ob}\,(\mathbf{TOP}^o)$,

$$[Z, B]^o \xrightarrow{g^*} [Y, B]^o \xrightarrow{f^*} [X, B]^o$$

is exact, i.e. $(f^*)^{-1}([*]) = \operatorname{im} g^*$.

In $\mathbf{TOP}^o$, we consider the reduced mapping cone $CX \coloneqq X \times I / X \times \{0\} \cup \{x_0\} \times I$. The basepoint of CX is $X \times \{0\} \cup \{x_0\} \times I$. And we consider the reduced mapping cone: For $f\colon (X, x_0) \to (Y, y_0)$, $C(f) \coloneqq CX \vee Y/(x, 1) \sim f(x)$. It is equivalent to the following push-out diagram.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_1"'] & Y \arrow[d, "f_1"] \\ CX \arrow[r] & C(f) \end{tikzcd}$$

We will also use symbol X instead of (X, x_0) in $\mathbf{TOP}^o$ for short.

Proposition 1.7. The sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} C(f)$$

is h-coexact.

Proof. $f_1 \circ f\colon X \to C(f)$ is null-homotopic. $\square$

Part II
Generalized Homology

Part III
Characteristic Classes