Introduction to Bass-Serre Theory

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Abstract

We will give a brief introduction to J.P.Serre's original work of GGT – Bass-Serre Theory. This is the final paper of the author's course: Research Training (H) in ECNU.

Contents

1 Introduction

Geometric Group Theory (GGT) is the art of studying groups without algebra. It is about using geometry (i.e. drawing pictures) to help us understand groups, which can otherwise be fairly dry algebraic objects (i.e. a bunch of letters on a piece of paper).

The starting point of Serre's work has been the theorem of Ihara, according to which every torsion-free subgroup G of $\mathrm{SL}_2(\mathbb{Q}_p)$ is a $free$ group. This striking result was at the time (1966) the only one known concerning the structure of the discrete subgroups of p-adic groups.

We will study "amalgams" and "trees", and how "a group acting on a tree" determines the structure of the group. Finally, we will study the relations between "amalgams" and "fixed points".

2 Basic Definitions

2.1 Amalgams

2.1.1 Direct limits

Definition 2.1. *Let $(G_i)_{i\in I}$ be a family of groups, and, for each pair (i,j), Let F_{ij} be a set of homomorphisms of G_i into G_j.*

We seek a group $G = \varinjlim G_i$ and a family of homomorphisms $f_i : G_i \to G$ such that $h_i \circ f = f_i$ for all $f \in F_{ij}$, the group and the family being universal in the following sense:

(*) If H is a group and if $h_i : G_i \to H$ is a family of homomorphisms such that $h_j \circ f = h_i$ for all $f \in F_{ij}$, then there is exactly one homomorphism $h : G \to H$ such that $h_i = h \circ f_i$.

We then say that G is the direct limit of the G_i, relative to the F_{ij}.

Proposition 2.2. *The pair consisting of G and the family $(f_i)_{i\in I}$ exists and is unique up to unique isomorphism.*

Proof. Uniqueness follows in the usual manner from the universal property. Existence is easy. One can define G by generators and relations; one takes the generating family to be the disjoint union of those for the G_i; as relations, on the one hand the xyz^{-1} where x, y, z belong to the same G_i and $z = xy$ in G_i, on the other hand the xy^{-1} where $x \in G_i$, $f \in G_j$ and $y = f(x)$ for at least one $f \in F_{ij}$. □

2.1.2 Structure of amalgams

Definition 2.3. *Suppose we are given a group A, a family of groups $(G_i)_{i\in I}$ and, for each $i \in I$, an injective homomorphism $A \to G_i$. We identify A with its image in each of the G_i. We denote by $*_A G_i$ the direct limit of the family (A, G_i) with respect to these homomorphisms, and call it the sum of G_i with A amalgamated.*

Definition 2.4. *For all $i \in I$ we choose a set S_i of right coset representatives of G_i modulo A, and assume $1 \in S_i$; the map $(a, s) \mapsto as$ is then a bijection of $A \times S_i$ onto G_i mapping $A \times (S_i - \{1\})$ onto $G_i' := G_i - A$.*

Let $\boldsymbol{i} = (i_1, \cdots, i_n)$ be a sequence of elements of I satisfying the following condition:

$$i_m \neq i_{m+1} \text{ for } 1 \leq m \leq n-1. \tag{T}$$

A reduced word of type $\boldsymbol{i}$ is any family

$$m = (a; s_1, \cdots, s_n)$$

where $a \in A$, $s_1 \in S_{i_1}, \cdots, s_n \in S_{i_n}$ and $s_j \neq 1$ for all j.

Notation 2.5. *We denote by f (resp. f_i) the canonical homomorphism of A (resp. G_i) into the group $G = *_A G_i$.*

Theorem 2.6. *For all $g \in G$ there is a sequence $\boldsymbol{i}$ satisfying (T) and a reduced word $m = (a; s_1, \cdots, s_n)$ of type $\boldsymbol{i}$ such that*

$$g = f(a) f_{i_1}(s_1) \cdots f_{i_n}(s_n). \tag{*}$$

Furthermore, $\boldsymbol{i}$ and m are unique.

Theorem 2.7. *The maps f and f_i define a bijection of the disjoint union of A and the G'_i onto G.*

2.1.3 Consequences of the structure theorem

Definition 2.8. *For $g = f(a) f_{i_1}(s_1) \cdots f_{i_n}(s_n)$, the integer n is called the length of g; we denote it by $\mathrm{l}(g)$. We have $\mathrm{l}(g) \leq 1$ if and only if g belongs to one of the G_i.*

Finally, an element g of length ≥ 2 is called cyclically reduced if its type $\boldsymbol{i} = (i_1, \cdots, i_n)$ is such that $i_1 \neq i_n$.

Proposition 2.9. *a) Every element g of G is conjugate to a cyclically reduced element, or an element of one of the G_i.*

b) Every cyclically reduced element is of infinite order.

Proof. We prove a) by induction on $\mathrm{l}(g)$. If $\mathrm{l}(g) \geq 2$ and if g is not cyclically reduced, let $\boldsymbol{i} = (i_1, \cdots, i_n)$ be its type. We have $i_1 = i_n$. We can write g in the form

$$g = g_1 \cdots g_n \text{ with } g_1 \in G'_{i_1}, \cdots, g_n \in G'_{i_n}$$

and then

$$g_1^{-1} g g_1 = g_2 \cdots, g_{n-1}(g_n g_1) \text{ with } g_n g_1 \in G_{i_i}.$$

We conclude that $g_1^{-1} g g_1$ is of legnth $n-1$ (if $g_n g_1 \notin A$) or of length $n-2$ (if $g_n g_1 \in A$). In view of the induction hypothesis, this element is conjugate to a cyclically reduced element or an element of one of the G_i, so the same is true of g.

As for b), if g is cyclically reduced of type $\boldsymbol{i} = (i_1, \cdots, i_n)$ it is clear that g^2 is of type $2\boldsymbol{i} = (i_1, \cdots, i_n, i_1, \cdots, i_n)$ and hence of length $2n$; more generally g^k $(k \geq 1)$ is of length kn, and therefore not equal to 1. □

As corollaries, we have

Corollary 2.10. *Every element of G of finite order is conjugate to an element of one of the G_i.*

Corollary 2.11. *If the G_i are torsion-free, so is G.*

Proposition 2.12. *For all $i \in I$, let H_i be a subgroup of G_i. Suppose that $B = H_i \cap A$ is independent of i. Then the homomorphism $*_B H_i \to *_A G_i$ induced by the injections $H_i \to G_i$ is injective.*

Proof. Let $i \in I$ and let T_i be a system of right coset representatives of H_i modulo B, including 1. Because $A \cap H_i = B$ we can extend T_i to a system S_i of right coset representatives of G_i modulo A. It is then clear that every reduced decomposition in $*_B H_i$ relative to T_i gives a reduced decomposition in $*_A G_i$ relative to S_i; whence the proposition. □

Corollary 2.13. *If $H_i \cap A = \{1\}$ for all $i \in I$, the subgroup of G generated by the H_i can be identified with the free product $*H_i$.*

We now give an application to free product:

Proposition 2.14. *Let A and B be two groups, and let R be the kernel of the canonical homomorphism $A*B \to A\times B$. The group R is a free group, with free basis the set X of commutators $[a,b] = a^{-1}b^{-1}ab$ with $a \in A-\{1\}$, $b \in B-\{1\}$.*

Proof. Let $S = \langle X\rangle \le A*B$. If $a' \in A$ and $[a,b]\in X$ we have

$$a'^{-1}[a,b]a' = [aa',b]\cdot[a',b]^{-1} \in S$$

whence $a'^{-1}Sa' = S$. In the same way one shows that $b'^{-1}Sb' = S$ if $b' \in B$, whence the fact that S is normal in $A*B$. It is clear that $(A*B)/S$ has the universal property which characterizes $A\times B$; so $S = R$.

It remains to see that X is a free subset of R. Now we have the following lemma:

Lemma 2.15. *Let X be a subset of a group R. The following two conditions are equivalent:*

(i) X is a free subset of R.

(ii) For every finite sequence $x_1,\cdots,x_n$ of elements of $\{\pm1\}$ such that we do not have $x_i = x_{i+1}$ and $\varepsilon_i = -\varepsilon_{i+1}$ for any i, the product $x_1^{\varepsilon_1}\cdots x_n^{\varepsilon_n}$ is not equal to 1.

In view of this, it will suffice to prove that the set X of the $[a,b]$ satisfies (ii). For

$$g = [a_1,b_1]^{\varepsilon_1}\cdots[a_n,b_n]^{\varepsilon_n}$$

where we do not simultaneously have $a_i = a_{i+1}$, $b_i = b_{i+1}$ and $\varepsilon_i = -\varepsilon_{i+1}$, we have to prove that $g \ne 1$. To be more precise, we have

(1) $\mathrm{l}(g) \ge n+3$.

(2) If $\varepsilon_n = +1$ (resp. -1) the reduced decomposition of g terminates in a_nb_n (resp. in b_na_n).

This claim can be proceed by induction on n. □

2.1.4 Constructions using amalgams

Amalgams are often used to prove the non-triviality of groups defined by generators and relations. We give two elementary examples:

Proposition 2.16 (G. Higman, B.H. Neumann, H. Neumann)**.** *Let A be a subgroup of a group G and $\theta : A \to G$ an injective homomorphism. Then there is a group G' containing G and an element s of G' such that $\theta(a) = sas^{-1}$ for all $a \in A$. Furthermore, if G is denumerable (or finitely generated, or torsion-free) we can choose G' te be a group with the same property.*

Corollary 2.17. *Every group G can be embedded in a group K enjoying the following property:*

(*) All the elements of K of the same order are conjugate.

Furthermore, if G is denumerable (or torsion-free) we can choose K to be denumerable (or torsion-free).

Remark 2.18. *If K is torsion-free, condition (*) just says that the elements of K other than 1 are conjugate to each other. In particular, K is a simple group.*

Proposition 2.19 (G. Higman)**.** *Let G be the group defined by the four generators x_1, x_2, x_3, x_4 and the four relations:*

$$x_2 x_1 x_2^{-1} = x_1^2, \quad x_3 x_2 x_3^{-1} = x_2^2, \quad x_4 x_3 x_4^{-1} = x_3^2, \quad x_1 x_4 x_1^{-1} = x_4^2.$$

a) Each subgroup of finite index in G is equal to G.

b) G is infinite.

Corollary 2.20. *There exists an infinite simple group generated by 4 elements.*

2.1.5 Examples

Example 2.21. *The infinite dihedral group D_∞, isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$.*

Example 2.22. *The "trefoil knot" group, defined by $\langle a, b; aba = bab \rangle$, is the sum of two copies of $\mathbb{Z}$ amalgamated by $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ and $\mathbb{Z} \xrightarrow{3} \mathbb{Z}$; in other words, it can be defined by $\langle x, y; x^2 = y^3 \rangle$.*

Example 2.23. *The group $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_3$. The group $\mathrm{SL}_2(\mathbb{Z})$ is isomorphic to $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$.*

2.2 Trees

2.2.1 Graphs

Definition 2.24. *A graph Γ consists of a set $X = \mathrm{vert}\,\Gamma$, a set $Y = \mathrm{edge}\,\Gamma$ and two maps*

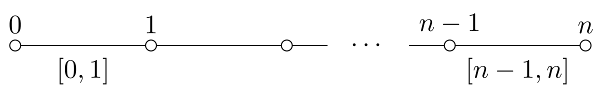
$$Y \to X \times X, \quad y \mapsto (\mathrm{o}(y), \mathrm{t}(y))$$

and

$$Y \to Y, \quad y \mapsto \overline{y}$$

which satisfy the following condition: for each $y \in Y$ we have $\overline{\overline{y}} = y$, $\overline{y} \neq y$ and $\mathrm{o}(y) = \mathrm{t}(\overline{y})$.

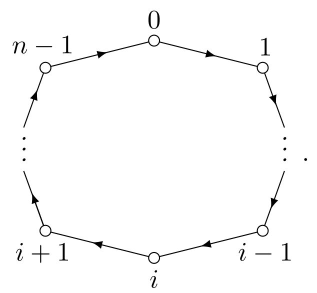
Definition 2.25. *A path (of length n) in a graph Γ is a morphism c of*

Path_n :


into Γ.

Definition 2.26. *A circuit (of length n) in a graph is any subgraph isomorphic to*

Circ_n :



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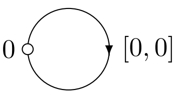
A circuit of length 1 *is called a loop.*

Circ_1 :



Definition 2.27. *A graph is called combinatorial if it has no circuit of length* ≤ 2.

Definition 2.28. *Let G be a group and let S be a subset of G. We let $\Gamma = \Gamma(G, S)$ denote the oriented graph having G as its set of vertices, $G \times S = (\mathrm{edge}\,\Gamma)_+$ as its orientation, with*

$$\mathrm{o}(g, s) = g \quad \textit{and} \quad \mathrm{t}(g, s) = gs \quad \textit{for each edge} \quad (g, s) \in G \times S,$$

And Γ is called the Cayley graph of G with respect to S.

The left multiplication by the elements of G defines an action of G on Γ which preserves orientation. Furthermore, G acts freely on the vertices and on edges.

Proposition 2.29. *Let $\Gamma = \Gamma(G, S)$ be the graph defined by a group G and a subset S of G.*

(a) For Γ to be connected, it is necessary and sufficient that S generate G.

(b) For Γ to contain a loop, it is necessary and sufficient that 1 *belong to S.*

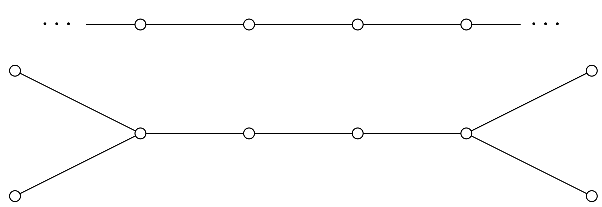
(c) For Γ to be a combinatorial graph, it is necessary and sufficient that $S \cap S^{-1} = \varnothing$.

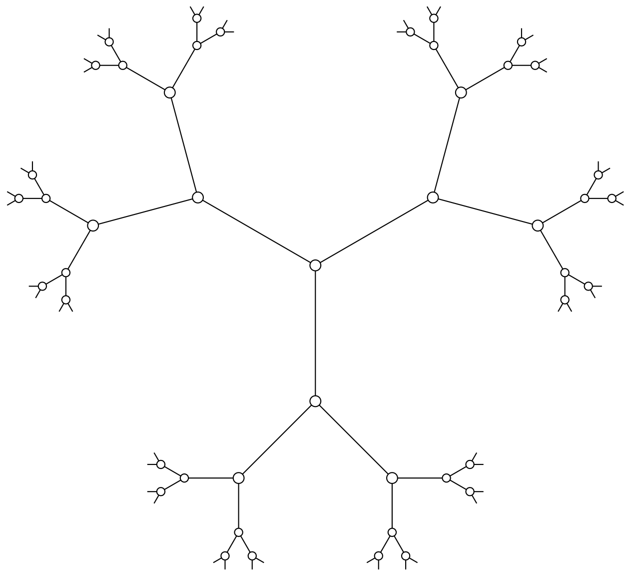
2.2.2 Trees

Definition 2.30. *A tree is a connected non-empty graph without circuits.*

In particular, a tree is a combinatorial graph.

Example 2.31.





Definition 2.32. *A geodesic in a tree is a path without backtracking.*

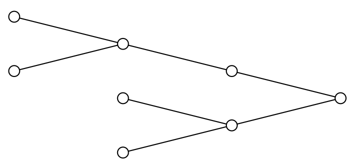
Proposition 2.33. *Let P and Q be two vertices in a tree Γ. There is exactly one geodesic from P to Q and it is an injective path.*

The length of the geodesic from P to Q is called the distance from P to Q, and is denoted by $\mathrm{l}(P, Q)$.

Definition 2.34. *Let P be a vertex of a tree Γ. For each integer $n \geq 0$ let X_n be the set of vertices Q of Γ such that $\mathrm{l}(P, Q) = n$. If $Q \in X_n$, with $n \geq 1$, there is a single vertex Q' at distance $< n$ from P to which Q is adjacent; it is the vertex $\mathrm{o}(y_n)$ where $(y_1, \cdots, y_n)$ is the geodesic from P to Q. This defines a map $f_n : Q \mapsto Q'$ of X_n into X_{n-1}, and hence an inverse system*

$$\cdots \longrightarrow X_n \xrightarrow{f_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = \{P\}\ . \qquad (S_p)$$

Knowledge of this system permits the reconstruction of the tree Γ; indeed, the set of vertices of Γ is the union X of the X_n, and the geometric edges are the $\{Q, f_n(Q)\}$ for $n \geq 1$ and $Q \in X_n$. Furthermore, every inverse system indexed by integers ≥ 1 can be obtained in this way. We therefore have an equivalence between pointed trees and inverse systems of sets indexed by integers ≥ 1:



$$X_3 \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0$$

Definition 2.35. *Let Γ be a tree and let X' be a subset of $X = \operatorname{vert}\Gamma$. Every subtree of Γ containing X' contains the geodesic with extremities in X'. Conversely, the vertices and edges of these geodesic form a subtree Γ' of Γ containing X'; it is the subtree generated by X'.*

Definition 2.36. *Let P be a vertex of a tree Γ. If Q is a vertex at distance n from P, the subtree $\Gamma(P, Q)$ generated by $\{P, Q\}$ is isomorphic to* Path_n*, and canonically so if we make P correspond to the vertex 0 of* Path_n*. We can identify* $\mathrm{real}(\mathrm{Path}_n)$ *with the interval $[0, n]$. Consider the contraction of* $\mathrm{real}(\Gamma(P, Q))$ *which corresponds to the contrction $x \mapsto tx$ $(0 \leq t \leq 1)$ of the interval $0, n$ to 0.*

Because Γ is the union of the subtrees $\Gamma(P, Q)$ $(Q \in \mathrm{vert}\,\Gamma)$ the realization of Γ is also the union of the subspaces $\mathrm{real}(\Gamma(P, Q))$*. Furthermore, it is easy to see that the contractions of these subspaces we have just defined are compatible. Hence we have a contraction of* $\mathrm{real}(\Gamma)$*: the realization of a tree is contractible.*

Definition 2.37. *Let Γ be a graph and let $X = \mathrm{vert}\,\Gamma$, $Y = \mathrm{edge}\,\Gamma$. Let P be a vertex and let Y_P be the set of edges y such that $P = \mathrm{t}(y)$. The cardical n of Y_P is called the index of P. If $n = 0$ one says that P is isolated. If $n \leq 1$ one says that P is a terminal vertex.*

Proposition 2.38. *Let P be a non-isolated terminal vertex of a graph Γ.*

(a) Γ is connected if and only if $\Gamma - P$ is connected.

(b) Every circuit of Γ is contained in $\Gamma - P$.

(c) Γ is a tree if and only if $\Gamma - P$ is a tree.

Definition 2.39. *The set of vertices if a tree Γ is a metric space under the distance* l*, whence the notions of the diameter of a tree of a bounded tree.*

Proposition 2.40. *Let Γ be a tree of diameter $n < \infty$.*

(a) The set $\mathrm{t}(\Gamma)$ of terminal vertices of Γ is non-empty.

(b) If $n \geq 2$, $\mathrm{vert}\,\Gamma - \mathrm{t}(\Gamma)$ is the vertex set of a subtree of diameter $n - 2$.

(c) If $n = 0$ we have $\Gamma \simeq \mathrm{Path}_0$ (diagram : ○) and if $n = 1$ we have $\Gamma \simeq \mathrm{Path}_1$ (diagram : ○———○).

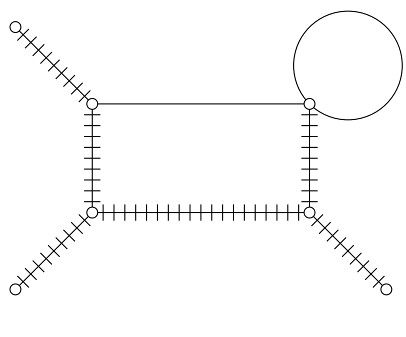
Corollary 2.41. *A tree of even finite diameter (resp. odd finite diameter), has a vertex (resp. geometric edge), which is invariant under all automorphisms.*

2.2.3 Subtrees of a graph

Definition 2.42. *Let Γ be a non-empty graph. The set of subgraphs of Γ which are trees, ordered by inclusion, is evidently directed. By Zorn's lemma it has a maximal element; such an element is called a maximal tree of Γ.*

Proposition 2.43. *Let Λ be a maximal tree of a connected non-empty graph Γ. Then Λ contains all the vertices of Γ.*

Example 2.44.



Proposition 2.45. *Let Γ be a connected graph with a finite number of vertices. Put*
$$s = \mathrm{Card}(\mathrm{vert}\,\Gamma), \quad a = \frac{1}{2}\mathrm{Card}(\mathrm{edge}\,\Gamma).$$
Then $a \geq s-1$ and equality holds if and only if Γ is a tree.

Remark 2.46. *The Betti numbers B_i of the graph Γ are $B_0 = 1$, $B_1 = a(\Gamma) - a(\Gamma')$ and $B_i = 0$ for $i \geq 2$, if Γ is non-empty; if not $B_i = 0$ for all $i \geq 0$. The formula $a(\Gamma) = s(\Gamma) - 1 + (a(\Gamma) - a(\Gamma'))$ can then be written:*
$$s(\Gamma) - a(\Gamma) = \sum_i (-1)^i B_i$$
which is a special case of the Euler-Poincaré formula.

Definition 2.47. *Let Γ be aconnected non-empty graph, and let Λ be a subgraph of Γ which is a disjoint union of a family Λ_i $(i \in I)$ of trees. We are going to define a graph Γ/Λ such that* $\mathrm{real}(\Gamma/\Lambda)$ *is the quotient space of* $\mathrm{real}(\Gamma)$ *obtained by identification of each subspace* $\mathrm{real}(\Lambda_i)$ *to a point.*

More precisely, the set of vertices of Γ/Λ is the quotient of $\mathrm{vert}\,\Gamma$ *by the equivalence relation whose classes are the sets* $\mathrm{vert}\,\Lambda_i$ *and the elements of* $\mathrm{vert}\,\Gamma - \mathrm{vert}\,\Lambda$. *Its edge is* $\mathrm{edge}\,\Gamma - \mathrm{edge}\,\Lambda$, *with the involution $y \mapsto \overline{y}$ induced by that on* $\mathrm{edge}\,\Gamma$. *Finally,*
$$\mathrm{edge}(\Gamma/\Lambda) \to \mathrm{vert}(\Gamma/\Lambda) \times \mathrm{vert}(\Gamma/\Lambda)$$
is induced by
$$\mathrm{edge}\,\Gamma \to \mathrm{vert}\,\Gamma \times \mathrm{vert}\,\Gamma$$
by passing to quotients. It is easy to see that $\mathrm{real}(\Gamma/\Lambda)$ *has the desired property.*

Proposition 2.48. *The canonical projection* $\mathrm{real}(\Gamma) \to \mathrm{real}(\Gamma/\Lambda)$ *is a homotopy equivalence.*

Corollary 2.49. *Let Γ be a connected non-empty graph. Then* $\mathrm{real}(\Gamma)$ *has the homotopy type of a bouquet of circles. Furthermore, Γ is a tree if and only if* $\mathrm{real}(\Gamma)$ *is contractible.*

Corollary 2.50. *Under the hypotheses of Proposition 2.48, Γ is a tree if and only if Γ/Λ is one.*

3 Groups Acting on Trees

What can we said of a group G acting on a tree X when we know the quotient graph $G\backslash X$ as well as the stabilizers G_x $(x \in \mathrm{vert}\,X)$ and G_y $(y \in \mathrm{edge}\,X)$ of the vertices and edges? We first treat two special cases:

3.1 Trees and Free Groups

Definition 3.1. *We say that a group G acts freely on a graph X if it acts without inversion and no element $g \neq 1$ of G leaves a vertex of X fixed.*

Theorem 3.2. *Let G be a group and let S be a subset of G. Then $\Gamma(G,S)$ is a tree on which G acts freely if and only if $S = X \sqcup X^{-1}$ and $G = F(X)$.(See Definition 2.28.)*

Proof. ($\Leftarrow$): A cycle would correspond to a reduced word $w = 1$ in $F(X)$.

($\Rightarrow$): G acts freely, so $\forall s \in S$, $\left|\{s, s^{-1}\}\right| = 2$. For every such pair, pick one and together let these form X. X generates G and so there exists an onto homomorphism $\varphi : F(X) \to G$. Suppose $w \in F(X)$, $w \in \operatorname{Ker}\varphi$. Since w is reduced as a word in X, it is also reduced as a word in S. So if $w \neq w_\varnothing$ then w gives a cycle in $\Gamma(G,S)$. So $\operatorname{Ker}\varphi = \{w_\varnothing\}$. □

In fact, we have a general theorem:

Theorem 3.3. *Let G be a group. Then G is free if and only if G acts freely by isometries on a tree T.*

Proof. ($\Rightarrow$): This direction is given by Theorem 3.35.

($\Leftarrow$): We need a lemma:

Lemma 3.4. *There exists $X \subset T$, X a tree, such that X contains exactly one vertex from each orbit.*

Choose an orientation E^+ on the edges of T that is G-invariant. Let

$$S = \left\{g \in G : \exists e \in E^+, \mathrm{o}(e) \in X, \mathrm{t}(e) \in g(X)\right\}.$$

We will prove that $G = F(S)$.

If $gX \cap X \neq \varnothing$ then there exists $v \in X$ such that $gv = v$ and so $g = 1$ by the freeness of the action. Hence if $g_1 \neq g_2$ then $g_1X \cap g_2X = \varnothing$.

We have $\{g_1, g_2\}$ is an edge of $\Gamma(G, S \cup S^{-1})$ if and only if there exists an edge of T with one endpoint in g_1X and the other in g_2X.

Because T is connected so is $\Gamma(G, S \cup S^{-1})$. It is a tree because it is a Cayley graph. And if $\Gamma(G, S \cup S^{-1})$ contains a cycle then so does T. So $G = F(S)$. □

More precisely, we have proved:

Theorem 3.5. *Let G be a group which acts freely on a tree X. Choose a tree T of representatives of $X \operatorname{mod} G$ and an orientation $Y_+ \subset \operatorname{edge} X$ preserved by G.*

a) Let S be the set of elements $g \neq 1$ in G for which there is an edge $y \in Y_+$ with origin in T and terminus in gT. Then S is a basis for G.

b) If $X^* = G\backslash X$ has a finite number s of vertices, and if $\operatorname{Card}(\operatorname{edge} X^*) = 2a$ we have $\operatorname{Card}(S) - 1 = a - s$.

As applications, we have the famous Schreier's theorem:

Theorem 3.6 (Schreier)**.** *Every subgroup of a free group is free.*

Corollary 3.7 (Schreier index formula)**.** *Let G be a free group and H a subgroup of finite index n in G. Then*

$$r_H - 1 = n(r_G - 1)$$

where r_G is the rank of the free group G.

Proposition 3.8 (Explicit form of Schreier's theorem). *Let G be a free group woth basis S, and let H be a subgroup of G.*

a) One can choose a set T of representatives of $H\backslash G$ satisfying the following condition:

(*) If $t \in T$ has the reduced decomposition

$$t = s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}. \ (s_i \in S, \ \varepsilon_i = \pm 1 \text{ and } \varepsilon_i = \varepsilon_{i+1} \ if \ s_i = s_{i+1})$$

then all the partial products $1, s_1^{\varepsilon_1}, s_1^{\varepsilon_1} s_2^{\varepsilon_2}, s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}$ belong to T.

b) Let T be as above and let $W = \{(t, s) \in T \times S, ts \notin T\}$. If $(t, s) \in W$, set $h_{t,s} = tsu^{-1}$ where $u \in T$ is such that $Hts = Hu$. Then

$$R = \{h_{t,s}, (t, s) \in W\}$$

is a basis for H.

Example 3.9. *Let $G = F(x, y)$ be the free group with basis $S = \{x, y\}$ with $x \neq y$, and let H be the kernel of the projection $G \to \mathbb{Z}_2 \times \mathbb{Z}_2$. By the Schreier index formula, we have $r_H = 1+4(r_G-1) = 5$. If we take the set if representatives $T = \{1, x, y, xy\}$, then Proposition 3.8 shows that H has the basis $\{h_1, h_2, h_3, h_4, h_5\}$ define by the formulae:*

$$xx = h_1 \cdot 1, \ yx = h_2 \cdot xy, \ yy = h_3 \cdot 1, \ xy \cdot x = h_4 \cdot y, \ xy \cdot y = h_5 \cdot x.$$

We therefore obtain the basis

$$\{x^2, yxy^{-1}x^{-1}, y^2, xyxy^{-1}, xy^2x^{-1}\}.$$

3.2 Trees and Amalgams

Suppose G is a group acting without inversions on a tree T.

Definition 3.10. *A subtree $S \subseteq T$ is a fundamental domain if it intersects the orbit $G \cdot v$ of every vertex v of T, and it intersects the orbit of every edge exactly once.*

3.2.1 The case of two factors

Theorem 3.11. *The group $G = A *_H B$ acts on a tree T with fundamental domain an edge $[P, Q]$ such that* $\mathrm{Stab}(P) = A$, $\mathrm{Stab}(Q) = B$, $\mathrm{Stab}([P, Q]) = H$.

Proof. Let $\mathrm{vert}(T) = (G/A) \sqcup (G/B)$.

Let $\mathrm{edge}(T) = \{(gA, gB), g \in G\}$.

$$g_1 A = gA, \ g_1 B = gB \iff g^{-1} g_1 \in A \cap B = H.$$

So the set is exactly gH. We label the edge (gA, gB) by gH and the edge (gB, gA) by $g\overline{H}$. Clearly G acts transitively on the edges and there are two orbits of vertices.

T is connected: For each edge $\{gA, gB\}$, $g = hs_1 \cdots s_n$, we will prove it is connected by an edge path to $\{A, B\}$ by induction on n. Moreover, the length of the edge path (including $\{A, B\}$ and $\{gA, gB\}$) is $n+1$. The $n = 0$ case is obvious. Induction: if $s_n \in A - \{1\}$ then

$$gA = hs_1 \cdots s_{n-1}A$$

and $\{gA, gB\}$ shares a common endpoint with $\{g'A, g'B\}$ where $g' = hs_1 \cdots s_{n-1}$. Similarly, if $s_n \in B - \{1\}$ then $gB = hs_1 \cdots s_{n-1}B$ and $\{gA, gB\}$ shares a common endpoint with $\{g'A, g'B\}$.

T is a tree: A path without spikes in T of origin A and even length $2n$ has vertices of the form:

$$A = a_1A, a_1B, a_1b_1A, \cdots, a_1b_1 \cdots a_nb_nA$$

where $a_i \notin H$ and $b_i \notin H$. An easy indection on n shows that the reduced form of $a_1b_1 \cdots a_nb_n$ is $ha'_1b'_1 \cdots a'_nb'_n$: for $n = 1$ we have

$$a_1b_1 = a_1(hb'_1) = h'a'_1b'_1 \textit{ where } a'_1, b'_1 \neq 1.$$

Likewise,

$$a_1b_1a_2b_2 \cdots a_{n+1}b_{n+1} = a_1b_1ha'_2b'_2 \cdots a'_{n+1}b'_{n+1} = h'a'_1b'_1 \cdots a'_{n+1}b'_{n+1}.$$

In particular we connot have $a_1b_1 \cdots a_nb_nA = A$ otherwise

$$ha'_1b'_1 \cdots a'_nb'_n = h'a'.$$

The left side is of length $2n$ while the right side is of length 0 or 1. So there is no cycle through A and so there is no cycle in T (every cycle must contain one vertex in G/A and so can be G-translated to a cycle through A). $\square$

Corollary 3.12. *If $F \leq A *_H B$ is such that $F \cap gAg^{-1} = \{1\}$ and $F \cap gBg^{-1} = \{1\}$ for every $g \in G$, then F is free.*

Theorem 3.13. *Suppose $G \curvearrowright T$ with fundamental domain an edge $e = [P, Q]$. If $A = \mathrm{Stab}(P)$, $B = \mathrm{Stab}(Q)$, $H = \mathrm{Stab}(e)$ then $G = A *_H B$.*

Proof. Since we have $\alpha_1 : A \to G$, $\beta_1 : B \to G$ agreeing on H, there exists some $\varphi : A *_H B \to G$, by the Universal Property of $A *_H B$.

Step 1: $G = \langle A, B\rangle$, that is, φ is onto.

For all $g \in G$, ge is joined to e by a unique path of length n. We will prove that $g \in \langle A, B\rangle$ by induction on n. If $n = 1$ then $g \in H$.

Assume true for n, and let ge be joined to e by a path of length $n + 1$. Let $g'e$ be the previous edge on the path. Then either $gP = g'P$ or $gQ = g'Q$ and so $g^{-1}g' \in A \cup B$. Since $g' \in \langle A, B\rangle$ we are done.

Step 2: φ is injective.

Let $hs_1 \cdots s_n \in \operatorname{Ker} \varphi$. We can prove by induction on n that $hs_1 \cdots s_ne$ can be joined to e by an edge path with no spikes of length $n + 1$. Hence $hs_1 \cdots s_n \neq 1$ in G. $\square$

Proposition 3.14. *Let Γ be a group acting on a tree X. The following conditions are equivalent:*

(a) For every bounded subset A of $\operatorname{vert} X$*, $\Gamma \cdot A$ is bounded.*

(b) There is $P \in \operatorname{vert} X$ such that $\Gamma \cdot P$ is bounded.

(c) There is a vertex of X invariant under Γ.

Corollary 3.15. *Every bounded subgroup of $G = A *_H B$ is contained in a conjugate of A or B.*

Corollary 3.16. *Every finite subgroup of G is contained in a conjugate of A or B.*

3.2.2 General case

Definition 3.17. *Let Y be an oriented graph such that the corresponding unoriented graph is connected and each of its edges appears with both orientations in Y.*

A graph of groups is a pair (G, Y), where G is a map that assigns a group G_v to each vertex $v \in \operatorname{vert}(Y)$ and a group G_e to each edge $e \in \operatorname{edge}(Y)$ such that

(a) $G_e = G_{\overline{e}}$

(b) for all edges e, there exists an injective homomorphism $\alpha_e : G_e \to G_{\mathrm{t}(e)}$.

Graph of groups appear naturally when G acts on a graph X without inversions. When this happens,we define:

Definition 3.18. *The quotient graph $Y = X/G$ and the projection $p : X \to Y$ as follows:*

- *Vertices are orbits Gv, $v \in X$*
- *Gv, Gw are joined of there exists an edge $[v_1, w_1]$ such that $v_1 \in Gv$, $w_1 \in Gw$.*

We define $p : X \to X/G$ by $p(v) = Gv$, $p(e) = \{G\,\mathrm{o}(e), G\,\mathrm{t}(e)\}$.

In this case,

- *$\forall v \in Y$, define $G_v = \operatorname{Stab}(\hat{v})$ where $\hat{v}$ is some element of $p^{-1}(v)$*
- *$\forall e \in Y$, define $G_e = \operatorname{Stab}(\hat{e})$ where $\hat{e}$ is some element of $p^{-1}(e)$*

take care that, whenever we can, $\hat{v}$ is an endpoint of $\hat{e}$ such that $G_e \subseteq G_v$.

For some edges, we might have to define α_e not as an inclusion, but as an inclusion composed with a conjugation.

Theorem 3.19. *Let (G, T) be a tree of groups. There is a graph X containing T and an action of $G_T = \varinjlim(G, T)$ on X which is characterized (up to isomorphism) by the following property:*

T is a fundamental domian for $X \operatorname{mod} G_T$ and, for all $P \in \operatorname{vert} T$ (resp. all $y \in \operatorname{edge} T$) the stabilizer of P (resp. y) in G_T is G_P (resp. G_y).

Moreover X is a tree.

3.3 Structure of a Group Acting on a Tree

Two special cases have been treated in the preceding sections: that of a free action and that where the quotient graph $G \backslash X$ is a tree. In this section, we will study the general case.

3.3.1 Fundamental group of a graph of groups

Definition 3.20. *Let $V = \mathrm{vert}(Y)$. The path group of the graph of groups (G, Y) is*

$$F(G,Y) = \left\langle \bigcup_{v\in V} G_v \cup \mathrm{edge}(Y) \middle| \overline{e} = e^{-1}, e\alpha_e(g)e^{-1} = \alpha_{\overline{e}}(g), \forall e \in \mathrm{edge}(Y), g \in G_e \right\rangle.$$

If $G_v = \langle S_v | R_v \rangle$ then

$$F(G,Y) = \left\langle \bigcup_{v\in V} S_v \cup \mathrm{edge}(Y) \middle| \bigcup_{v\in V} R_v, \overline{e} = e^{-1}, e\alpha_e(g)e^{-1} = \alpha_{\overline{e}}(g) \right\rangle.$$

Remark 3.21. *(1) If all $G_v = \{1\}$ then $F(G,Y) = F(E^+(Y))$.*

(2) There exists an epimorphism $F(G,Y) \to F(E^+(Y))$ defined by sending each G_v to $\{1\}$.

(3) If all $G_e = 1$ then

$$F(G,Y) = *_{v\in\mathrm{vert}(Y)} G_v * F(E^+(Y)).$$

Definition 3.22. *A path in (G, Y) is a sequence*

$$c = (g_0, e_1, g_1, e_2, \cdots, g_{n-1}, e_n, g_n)$$

such that $\mathrm{t}(e_i) = \mathrm{o}(e_{i+1})$ and $g_i \in G_{\mathrm{t}(e_i)} = G_{\mathrm{o}(e_{i+1})}$. If $v_0 = \mathrm{o}(e_1)$, $v_n = \mathrm{t}(e_n)$ then we call this a path from v_0 to v_n. We call

$$v_0, v_1 = \mathrm{t}(e_1) = \mathrm{o}(e_2), \cdots, v_i = \mathrm{t}(e_i) = \mathrm{o}(e_{i+1}), \cdots, v_n$$

its sequence of vertices. We define $|c|$ to be the element of the path group $g_0e_1g_1 \cdots e_ng_n$. If $a_0, a_1 \in \mathrm{vert}(Y)$ then we define

$$\pi[a_0, a_1] = \{|c| : c \ a \ path \ from \ a_0 \ to \ a_1\}.$$

Proposition 3.23. *Let (G, Y) be a graph of groups and suppose $a_0 \in \mathrm{vert}(Y)$. The set $\pi[a_0, a_0]$ is a subgroup of $F(G, Y)$.*

Definition 3.24. *We call the subgroup $\pi[a_0, a_0]$ the fundamental group of the graph of groups (G, Y) with basepoint a_0 and denote it $\pi_1(G, Y, a_0)$.*

Definition 3.25. *Let (G, Y) be a graph of groups, and let T be a maximal subtree of Y. The fundamental group of (G, Y) with respect to T, denoted $\pi_1(G, Y, T)$, is*

$$F(G,Y)/\langle\langle \{e : e \in T\} \rangle\rangle.$$

Proposition 3.26. *Let $q : F(G,Y) \to \pi_1(G,Y,T)$ be the quotient map. Then $q|_{\pi_1(G,Y,a_0)}$ is an isomorphism to $\pi_1(G, Y, T)$.*

Proof. We define a homomorphism $f : \pi_1(G,Y,T) \to \pi_1(G,Y,a_0)$ as follows.

$\forall a \in \mathrm{vert}(Y)$, there exists a unique geodesic path $e_1, e_2, \cdots, e_n$ in T from a_0 to a. Set

$$g_a := e_1 \cdots e_n \in F(G,Y), \ g_{a_0} = 1.$$

We first define $\hat{f} : F(G,Y) \to \pi_1(G,Y,a_0)$:

- $\forall g \in G_a$, set $\hat{f}(g) = g_a g g_a^{-1} \in \pi_1(G, Y, a_0)$
- $\forall e \in E^+(Y)$ with $\mathrm{o}(e) = a$, $\mathrm{t}(e) = b$, set $\hat{f}(e) = g_a e g_b^{-1} \in \pi_1(G, Y, a_0)$.

The definition of $\hat{f}$ is consistent with the relations $e\alpha_e(g)e^{-1} = \alpha_{\overline{e}}(g)$ since if $e = [P, Q]$:

$$\begin{aligned}\hat{f}(e\alpha_e(g)\overline{e}) &= (g_P e g_Q^{-1})(g_Q \alpha_e(g) g_Q^{-1})(g_Q \overline{e} g_P^{-1}) \\ &= g_P(e\alpha_e(g)e^{-1})g_P^{-1} \\ &= g_P \alpha_{\overline{e}}(g) g_P^{-1}.\end{aligned}$$

So $\hat{f}$ is define on $F(G, Y)$. Now, for every $e = [P, Q] \in T$, $\hat{f}(e) = g_P e g_Q^{-1} = 1$. Hence $\hat{f}$ defines $f : \pi_1(G, Y, T) \to \pi_1(G, Y, a_0)$.

Consider $q \circ f : \pi_1(G, Y, T) \to \pi_1(G, Y, T)$. For all $g \in G_a$,

$$q \circ f(g) = q(g_a g g_a^{-1}) = g.$$

For all $e \notin T$,

$$q \circ f(e) = q(g_P e g_Q^{-1}) = e.$$

Hence $q \circ f = \mathrm{Id}$.

Consider now $f \circ q : \pi_1(G, Y, a_0) \to \pi_1(G, Y, a_0)$. If $g_0 e_1 \cdots e_n g_n$ arbitrary in $\pi_1(G, Y, a_0)$ and $e_i = [P_{i-1}, P_i]$,

$$\begin{aligned}f \circ q(g_0 e_1 \cdots e_n g_n) &= g_0(e_1 g_{P_1}^{-1})(g_{P_1} g_1 g_{P_1}^{-1})(g_{P_1} e_2 g_{P_2}^{-1}) \cdots g_{P_{n-1}}^{-1}(g_{P_{n-1}} e_n) g_n \\ &= g_0 e_1 \cdots e_n g_n.\end{aligned}$$

If $e_i \in T$ then

$$\begin{aligned}f \circ q(g_{i-1} e_i g_i) &= f(g_{i-1} g_i) \\ &= g_{P_{i-1}} g_{i-1} g_{P_{i-1}}^{-1} g_{P_i} g_i g_{P_i}^{-1} \\ &= g_{P_{i-1}} g_{i-1} e_i g_i g_{P_i}^{-1}.\end{aligned}$$

□

Corollary 3.27. *The fundamental group $\pi_1(G, Y, a_0)$ of the graph of groups (G, Y) does not depend on the choice of basepoint a_0.*

Corollary 3.28. *The quotient $\pi_1(G, Y, T)$ of the path group does not depend on the choice of the tree T.*

3.3.2 Reduced words

Definition 3.29. *Let (G, Y) be a graph of groups. A path*

$$c = (g_0, e_1, g_1, e_2, \cdots, g_{n-1}, e_n, g_n)$$

is reduced if

(1) $g_0 \neq 1$ if $n = 0$;

(2) If $e_{i+1} = \overline{e_i}$ then $g_i \notin \alpha_{e_i}(G_{e_i})$.

We say that $g_0 e_1 \cdots e_n g_n$ is a reduced word.

Theorem 3.30. *If c is a reduced path then $|c| \neq 1$ in $F(G,Y)$. In particular, $G_v \hookrightarrow F(G,Y)$ is injective for every $v \in \mathrm{vert}(Y)$.*

Proof. First assume that Y is finite. We will argue by induction on the number of edges in Y. If there are no edges, then the theorem holds. So assume the theorem is true for graph with n edges, and suppose that Y has $n+1$ edges.

Case 1: $Y = Y' \cup \{e\}$, $\mathrm{o}(e) \in \mathrm{vert}(Y')$, $v = \mathrm{t}(e) \notin \mathrm{vert}(Y')$. Then

$$F(G,Y) = (F(G,Y') * G_v)*_{\alpha_e(G_e)}$$

and a reduced word containing e corresponds to a reduced word in the HNN extension that is $\neq 1$.

Case 2: $Y = Y' \cup \{e\}$, $\{\mathrm{o}(e).\, \mathrm{t}(e)\} \subseteq \mathrm{vert}(Y')$. Then

$$F(G,Y) = F(G,Y')*_{\alpha_e(G_e)}$$

and the comment above holds again.

Now suppose that Y is infinite. Any reduced path c involves finitely many orbits of vertices and edges and so c lies within a finite subgraph Y_1 of Y. c is a reduced path in $F(G,Y_1)$ and so $c \neq 1$ in $F(G,Y_1)$. $\square$

Corollary 3.31. *For every $v \in \mathrm{vert}(Y)$, the homomorphism $G_v \to \pi_1(G,Y,T)$ is injective.*

Example 3.32. *One can easily see that*

(1) If Y has 2 vertices and one edge then

$$\pi_1(G,Y,T) = G_u *_{G_e} G_v.$$

(2) If Y has 1 vertices and 1 edge with stable letter 'e' then

$$\pi_1(G,Y,T) = G_v *_{\alpha_e(G_e)}$$

and $\theta : \alpha_e(G_e) \to \alpha_{\overline{e}}(G_e) \in G_v$, $\theta(g) = \alpha_{\overline{e}} \circ \alpha_e^{-1}$.

(3) If $Y = Y' \cup \{e\}$ and $\mathrm{t}(e) = v \notin Y'$ then

$$\pi_1(G,Y,T) = \pi_1(G,Y,T') *_{G_e} G_v.$$

(4) If $Y = Y' \cup \{e\}$ and $\mathrm{t}(e) = v \in Y'$ then

$$\pi_1(G,Y,T) = \pi_1(G,Y,T') *_{\alpha_e(G_e)}.$$

3.3.3 Universal covering relative to a graph of groups

Definition 3.33. *We will find a choice of representatives of elements in $F(G,Y)$, where (G,Y) is a graph of groups. For each edge $e \in \mathrm{edge}(Y)$, pick a set S_e of left coset representatives of $\alpha_{\overline{e}}(G_e)$ in $G_{\mathrm{o}(e)}$, with $1 \in S_e$.*

An S-reduced path is a path $(s_1, e_1, \cdots, s_n, e_n, g)$ with

- *$s_i \in S_{e_i}$, $\forall i$;*

- $s_i \neq 1$ *if* $e_i = \overline{e_{i-1}}$;
- $g \in G_{\mathrm{t}(e_n)}$.

Lemma 3.34. *Given* $a, b \in \mathrm{vert}(Y)$*, every element in* $\pi[a,b]$ *is represented by a unique* S*-reduced path.*

Proof. **Existence:** Let $\gamma \in \pi[a,b]$ and consider the path

$$c = (g_0, e_1, g_1, e_2, \cdots, g_{n-1}, e_n, g_n)$$

such that $\mathrm{t}(e_i) = \mathrm{o}(e_{i+1})$, $g_i \in G_{\mathrm{t}(e_i)} = G_{\mathrm{o}(e_{i+1})}$ and $\gamma = |c|$.

We will prove by induction on n that γ can be represented by an S-reduced path. For $n = 0$ it is obvious. For $n = 1$,

$$\gamma = g_0 e_1 g_1 = s_0 \alpha_{\overline{e_1}}(h_0) e_1 g_1 = s_0 e_1 \alpha_{e_1}(h_0) g_1 = s_0 e_1 g_1'.$$

A similar argument holds for the inductive step.

Uniqueness: Consider two reduced paths

$$c = (s_1, e_1, \cdots, s_n, e_n, g)$$
$$c' = (\sigma_1, \eta_1, \cdots, \sigma_k, \eta_k, \gamma)$$

such that $|c| = |c'|$. Then

$$\gamma^{-1} \eta_k^{-1} \sigma_k^{-1} \cdots \eta_1^{-1} \sigma_1^{-1} s_1 e_1 \cdots s_n e_n g = 1.$$

We will prove that $c = c'$ by induction on the length. The above word cannot be reduced hence $\eta_1^{-1} = e_1^{-1}$ and $\sigma_1^{-1} s_1 \in \alpha_{\overline{e_1}}(G_{e_1})$. So $\sigma_1 = s_1$. And so we can apply the inductive assumption. $\square$

Theorem 3.35. $H = \pi_1(G, Y, a_0)$ *acts on a tree* T *without inversion and such that*

(1) The quotient graph $H \backslash T$ *can be identified with* Y*;*

(2) Let $q : T \to Y$ *be the quotient map:*

   (a) For all $v \in \mathrm{vert}(T)$, $\mathrm{Stab}_H(v)$ *is a conjugate in* H *of* $G_{q(v)}$*;*

   (b) For all $e \in \mathrm{edge}(T)$, $\mathrm{Stab}_H(e)$ *is a conjugate in* H *of* $G_{q(e)}$.

Proof. For all $a \in \mathrm{vert}(Y)$, we define an equivalence relation on $\pi[a_0, a]$ by

$$|c_1| \sim |c_2| \iff |c_1| = |c_2|\, g \ for\ some\ g \in G_a.$$

Vertices of the tree:

$$\mathrm{vert}(T) = \bigsqcup_{a \in \mathrm{vert}(Y)} \pi[a_0, a]/\sim.$$

By the lemma, every element of $\pi[a_0, a]/\sim$ has a unique representative corresponding to an S-reduced path of the form $(s_1, e_1, \cdots, s_n, e_n)$, $\mathrm{o}(e_1) = a_0$, $\mathrm{t}(e_n) = a$. Thus $\mathrm{vert}(T)$ can also be identified with S-reduced paths as above.

Edges of the tree: $\{(s_1, e_1, \cdots, s_n, e_n), (s_1, e_1, \cdots, s_n, e_n, s_{n+1}, e_{n+1})\}$. Connectedness is obvious. By our definition of edges, a cycle/circuit gives an S-reduced path with corresponding element $1 \in \pi[a_0, a]$ contradicting the uniqueness of the representation of a reduced path.

Action of $H = \pi_1(G, Y, a_0) = \pi[a_0, a]$ on T: For all $h \in \pi[a_0, a]$ and for all $[g] \in \mathrm{vert}(T)$ define the action

$$h \cdot [g] = [hg].$$

- If $[g_1], [g_2]$ are such that $h \cdot [g_1] = [g_2]$ then $a_1 = a_2$ where $g_i \in \pi[a_0, a_i]$.
- Conversely, if $[g_1], [g_2] \in \pi[a_0, a]$ then $h = g_2 g_1^{-1} \in \pi[a_0, a_0]$ and $h[g_1] = [g_2]$.

Thus $H \backslash \mathrm{vert}(T)$ can be identified with $\mathrm{vert}(Y)$. And likewise $H \backslash \mathrm{edge}(T)$ can be identified with $\mathrm{edge}(Y)$.

Stabilisers of vertices: For all $[v] \in \mathrm{vert}(T)$ with $v \in \pi[a_0, b]$,

$$\begin{aligned} h \in \mathrm{Stab}([v]) \iff hv \sim v &\iff hv = vg_b \; for\ some\ g_b \in G_b \\ &\iff h = vg_b v^{-1} \; for\ some\ g_b \in G_b. \end{aligned}$$

Thus $\mathrm{Stab}([v]) = vG_b v^{-1}$.

Stabilisers of edges: Every edge in $\mathrm{edge}(T)$ is of the form $\delta = [[v], [vge]]$, $v \in \pi[a_0, a]$, $g \in G_a$, $\delta = [a, b]$. Then

$$\begin{aligned} \mathrm{Stab}(\delta) = \mathrm{Stab}(v) \cap \mathrm{Stab}(vge) &= vG_a v^{-1} \cap (vge) G_b (vge)^{-1} \\ &= vg(G_a \cap eG_b e^{-1}) g^{-1} v^{-1} = vg(\alpha_{\overline{e}}(G_e)) g^{-1} v^{-1}. \end{aligned}$$

□

Definition 3.36. *We denote the tree thus obtained $\widetilde{X}(G, Y, a_0)$ and we call it the universal covering tree or the Bass-Serre tree of the graph of groups (G, Y).*

3.3.4 Structure theorem

Recall Theorem 3.35:

Theorem. *$H = \pi_1(G, Y, a_0)$ acts on a tree T without inversion and such that*

(1) The quotient graph $H \backslash T$ can be identified with Y;

(2) Let $q : T \to Y$ be the quotient map:

  (a) For all $v \in \mathrm{vert}(T)$, $\mathrm{Stab}_H(v)$ is a conjugate in H of $G_{q(v)}$;

  (b) For all $e \in \mathrm{edge}(T)$, $\mathrm{Stab}_H(e)$ is a conjugate in H of $G_{q(e)}$.

Conversely, if a group Γ acts on a tree T with quotient Y then there exists a graph of groups (G, Y) such that $\Gamma \simeq \pi_1(G, Y, a_0)$. Indeed, suppose $\Gamma \hookrightarrow T$, $Y = T/\Gamma$ and $p : T \to Y$. Let $X \subset S \subset T$ be such that $p(X)$ is a maximal tree of Y, $p(S) = Y$ and $p|_{\mathrm{edge}\, S}$ is 1-to-1.

Notation: If v is a vertex of Y and e is an edge of Y then let v^X be the vertex of X such that $p(v^X) = v$ and similarly let e^S be the edge of S such that $p(e^S) = e$.

A graph of groups with graph Y:

(1) The map G:

- Let $G_v = \mathrm{Stab}_\Gamma(v^X)$;
- Let $G_e = \mathrm{Stab}_\Gamma(e^S)$.

(2) For each edge e, we define $\alpha_e : G_e \to G_{\mathrm{t}(e)}$: For all $x \in \mathrm{vert}(S)$, define

$$g_x = \begin{cases} 1 & if\ x \in \mathrm{vert}(X) \\ some\ g_x\ such\ that\ g_x x \in \mathrm{vert}(X) & othewise. \end{cases}$$

Define $\alpha_e : G_e \to G_{\mathrm{t}(e)}$, $\alpha_e(g) = g_{\mathrm{t}(e)} g g_{\mathrm{t}(e)}^{-1}$.

We can define a homomorphism $\varphi : F(G, Y) \to \Gamma$ by:

- $\forall a \in \mathrm{vert}(Y)$, $\varphi|_{G_a} = \mathrm{incl}_{G_a}$;
- $\forall e \in \mathrm{edge}(Y)$, $e = [y, x]$, $\varphi(e) = g_y g_x^{-1}$.

It satisfies the relations:

$$\varphi(e\alpha_e(g)e^{-1}) = (g_y g_x^{-1})(g_x g g_x^{-1})(g_x g_y^{-1}) = g_y g g_y^{-1} = \varphi(\alpha_{\overline{e}}(g)).$$

Also, $\forall e \in p(X)$, $\varphi(e) = 1$. Hence, φ defines a homomorphism

$$\overline{\varphi} : \pi_1(G, Y, p(X)) \simeq \pi_1(G, Y, a_0) \to \Gamma.$$

So we have:

Theorem 3.37. *The homomorphism $\overline{\varphi}$ is an isomorphism. If $\widetilde{T}$ is the universal covering tree of (G, Y) then there exists s a graph isomorphism $f : \widetilde{T} \to T$ such that $\forall g \in \pi_1(G, Y, a_0)$, $\forall v \in \mathrm{vert}(\widetilde{T})$,*

$$f(g \cdot v) = \overline{\varphi}(g) \cdot f(v).$$

As applications, we have:

Theorem 3.38. *Let $\Gamma = \pi_1(G, Y, a_0)$. If $B \leq \Gamma$ then there exists (H, Z) a graph of groups such that $B = \pi_1(H, Z, b_0)$ and*

- *for all $v \in \mathrm{vert}(Z)$, $H_v \leq gG_a g^{-1}$ for some $a \in \mathrm{vert}(Y)$, $g \in \Gamma$;*
- *for all $e \in \mathrm{edge}(Z)$, $H_e \leq \gamma G_y \gamma^{-1}$ for some $y \in \mathrm{edge}(Y)$, $\gamma \in \Gamma$.*

Proof. Γ acts on a tree T with quotient a graph of groups (G, Y). The subgroup B acts on T, $\mathrm{Stab}_B(v) \leq \mathrm{Stab}_\Gamma(v)$ for all $v \in \mathrm{vert}(T)$ and $\mathrm{Stab}_B(e) \leq \mathrm{Stab}_\Gamma(e)$ for all $e \in \mathrm{edge}(T)$. □

Theorem 3.39 (Kurosh)**.** *Suppose $G = G_1 * \cdots * G_n$. If $H \leq G$ then*

$$H = (*_{i \in I} H_i) * F$$

where I is finite or countable, F is a free group and the H_i are subgroups of conjugates of G_j.

4 Amalgams and Fixed Points

4.1 The fixed point property for groups acting on trees

We say that a group G is an amalgam if it can be written $G \simeq G_1 *_A G_2$ with $G_1 \neq A \neq G_2$.

Let G be a group acting without inversion on a tree X. The set X^G of fixed points of G in X is asubgraph of X; if P and Q are two vertices of X^G, the geodesic joining P to Q is fixed by G and therefore contained in X^G; it follows that, if X^G is non-empty, it is a tree. We are interested in the groups G with the property:

(FA) $X^G \neq \varnothing$ for any tree X on which G acts.

This property is "almost" equivalent to that of not being an amalgam. More precisely:

Theorem 4.1. *Suppose that G is denumerable. Then G has property (FA) if and only if the following three conditions are satisfied:*

(i) G is not an amalgam;

(ii) G has no quotient isomorphic to $\mathbb{Z}$;

(iii) G is finitely generated.

There are some consequences and examples of property (FA).

Proposition 4.2. *Let G be a group with property (FA). If G is contained in an amalgam $G_1 *_A G_2$, then G is contained in a conjugate of G_1 or of G_2.*

Example 4.3. *A finitely generated torsion group has property (FA).*

Example 4.4. *If G has property (FA), so has every quotient of G.*

Example 4.5. *Let H be a normal subgroup of G. If H and G/H have property (FA), then so has G.*

Example 4.6. *Let G' be a subgroup of finite index in G. If G acts on a tree X and if $X^{G'} \neq \varnothing$, then $X^G \neq \varnothing$.*

4.2 Fixed points of an automorphism of a tree

4.2.1 Automorphisms with fixed points

Proposition 4.7. *Suppose that s has a fixed point. Let $P \in \operatorname{vert} X$, let n be the distance between P and X^s, and let $P - P'$ be the geodesic joining P to X^s. Then the geodesic $P - sP$ is obtained by juxtaposing the geodesic $P - P'$ and $P' - sP = s(P' - P)$.*

Corollary 4.8. $\mathrm{l}(P, sP) = 2n$.

Corollary 4.9. *The midpoint of the geodesic $P - sP$ is fixed by s.*

Corollary 4.10. *Suppose $n \geq 1$. Let P_1, P_2 be the vertices of $P - sP$ at distance 1 from P, sP respectively. Then $sP_1 = P_2$.*

4.2.2 Automorphisms without fixed points

Proposition 4.11 (Tits)**.** *Suppose the automorphism s hs no fixed points. Put*

$$m = \inf_{P \in \operatorname{vert} X} \mathrm{l}(P, sP) \text{ and } T = \{P \in \operatorname{vert} X | \mathrm{l}(P, sP) = m\}.$$

Then:

(i) T is the vertex set of a straight path of X.

(ii) s induces a translation of T of amplitude m.

(iii) Every subtree of X stable under s and s^{-1} contains T.

(iv) If a vertex Q of X is at a distance d from T then $\mathrm{l}(Q, sQ) = m + 2s$.

Corollary 4.12. *Let $y \in \operatorname{edge} X$. Then y and sy are coherent if and only if y is an edge of the straight path T associated with s.*

Combining the propositions above, we have:

Proposition 4.13. *Let s be an automorphism of a tree X. The following properties are equivalent:*

(a) s has no fixed points;

(b) there is an edge y of X such that y and sy are coherent and distinct;

(c) there is a straight path X stable under s, on which s induces a translation of non-zero amplitude.

4.3 Auxiliary results

Corollary 4.14. *Let a, b and c be three automorphisms of a tree such that $abs = 1$. If a, b, c have fixed points, then they have a common fixed point.*

Corollary 4.15. *Suppose that G is generated by a finite number of elements $s_1, \cdots, s_m$ such that the s_j and the $s_i s_j$ have fixed points. Then G has a fixed point.*

Corollary 4.16. *If G is finitely generated, and if each of its elements has fixed points, then so has G.*

4.3.1 The case of nilpotent groups

Proposition 4.17. *Let G be a finitely generated nilpotent group acting on a tree X. Only two cases are possible:*

(a) G has a fixed point.

(b) There is a straight path T stable under G on which G acts by translations by means of a non-trivial homomorphism $G \to \mathbb{Z}$.

Corollary 4.18. *If G is generated by elements which have fixed points, then G has a fixed point.*

Corollary 4.19. *Let G' be the commutator subgroup of G, and let s be an element of G such that $s^n \in G'$ for some integer $n \geq 1$. Then s has a fixed point.*

4.4 The case of $\mathrm{SL}_3(\mathbb{Z})$

Finally, as application, we study the case of $\mathrm{SL}_3(\mathbb{Z})$.

Theorem 4.20. *The group* $\mathrm{SL}_3(\mathbb{Z})$ *has property (FA).*

Corollary 4.21. *The group* $\mathrm{SL}_3(\mathbb{Z})$ *is not an amalgam.*

Theorem 4.22 (Margulis-Tits)**.** *Every subgroup of finite index in* $\mathrm{SL}_3(\mathbb{Z})$ *has property (FA).*