On the second bounded cohomology of groups acting on a product of two Gromov-hyperbolic spaces

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Abstract

We want to classify the second bounded cohomology of groups acting on a product of two Gromov-Hyperbolic spaces.

Contents

1 Introduction

In [1], K.Fujiwara proved that, if a group G acts on a Gromov-hyperbolic space X isometrically and properly, with limit set $|L(G)| \geq 3$, then $H_b^2(G;\mathbb{R})$ has infinite dimension. We found that his proof is also valid in a more general setting (Theorem 3.1).

For a group G, denote

$$C_b^n(G,\mathbb{R}) \coloneqq \left\{\varphi\colon G^n \to \mathbb{R} : \sup_{g1,\cdots,g_n\in G} |\varphi(g_1,\cdots,g_n)| < \infty\right\}$$

where φ is just a map instead of a homomorphism. Define the boundary operator $\delta\colon C_b^n(G,\mathbb{R}) \to C_b^{n+1}(G,\mathbb{R})$ as follow: For any $\varphi \in C_b^n(G,\mathbb{R})$, let

$$\delta\varphi(g_0,\cdots,g_n) \coloneqq \varphi(g_1,\cdots,g_n) + \sum_{i=1}^{n}(-1)^i\varphi(g_0,\cdots,g_{i-1}g_i,\cdots,g_n) + (-1)^{n+1}\varphi(g_0,\cdots,g_{n-1}).$$

It's easy to check that $\delta\varphi \in C_b^{n+1}(G,\mathbb{R})$ and $\delta^2 = 0$. So $(C_b^*(G,\mathbb{R}),\delta)$ is a cochain complex.

Definition 1.1. The *bounded cohomology* of G is defined by

$$H_b^*(G,\mathbb{R}) \coloneqq \frac{\ker\delta^*}{\operatorname{im}\delta^{*-1}}.$$

It is well known that $H_b^1(G;\mathbb{R})$ is trivial for any group , and that $H_b^n(G;\mathbb{R})$ is trivial for all $n \geq 1$ if G is amenable. See [2] as general reference.

Motivated by this example:

Example 1.2. Let's consider the Baumslag–Solitar Group $BS(1,2) = \langle a,t : tat^{-1} = a^2\rangle$.

1. There is a non-proper action of $BS(1,2)$ on $\mathbb{H}^2$: notice that $\operatorname{Isom}(\mathbb{H}^2) = \mathrm{PSL}(2,\mathbb{R})$, let $a \mapsto \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $t \mapsto \begin{pmatrix}\sqrt{2} & 0\\ 0 & \frac{1}{\sqrt{2}}\end{pmatrix}$.

2. There is a non-proper action of $BS(1,2)$ on T, where T is the corresponding Bass-Serre tree of degree 3: $BS(1,2)$ can be written as a HNN-extension:

$$BS(1,2) = \left\langle a, t : tat^{-1} = a^2 \right\rangle = \langle a \rangle_{a \sim a^2}.$$

But the induced action $BS(1,2) \curvearrowright \mathbb{H}^2 \times T$ is proper.

we asked this question:

Question 1.3. If G can act on two hyperbolic spaces X, Y isometrically and coboundedly (and non-properly), which induce a proper action of G on $X \times Y$ with ℓ^1-norm, what can we say about $H^2_b(G;\mathbb{R})$?

2 Preliminaries

In this section, we collect some definitions and facts on hyperbolic spaces and groups acting on them. See [3] as general reference.

If any two points in X are joined by a geodesic, then we say X is *geodesic*. For $\delta \geq 0$, if any side if a triangle is contained in the δ-neighborhood of the union if the other two sides, then the triangle is called *δ-thin*. If every geodesic triangle in X is δ-thin, we say that X is *δ-hyperbolic*. A *Gromov-hyperbolic space* is a space which is δ-hyperbolic for some $\delta \geq 0$.

Definition 2.1. Let α be a path in a geodesic space X. If we have, for some K and ε,

$$\frac{|t-s|}{K} - \varepsilon \leq d(\alpha(t), \alpha(s))$$

for all t and s, then α is called a *(K, ε)-quasi-geodesic.*

Now we recall the definition of the boundary of a hyperbolic space. Let X be a Gromov-hyperbolic space. Take a base point $x_0 \in X$. Given $x, y \in X$, we define the *Gromov product*

$$(x \cdot y) \coloneqq \frac{1}{2}\left(d\left(x_0, x\right) + d\left(x_0, y\right) - d(x, y)\right).$$

A sequence $\left(x_i\right)_i$ of points in X is called *convergent at infinity* if

$$\lim_{i,j \to \infty}\left(x_i \cdot x_j\right) = \infty.$$

We define a relation denoted by "$\sim$" on the set of sequences which are convergent at infinity by

$$\left(x_i\right) \sim \left(y_j\right) \Leftrightarrow \lim_{i,j \to \infty}\left(x_i \cdot y_j\right) = \infty.$$

This relation is an equivalence relation if X is Gromov-hyperbolic. The *boundary* ∂X is the set if the equivalence classes of sequences in X which are convergent at infinity. If $\left(x_i\right)_i$ is in a class $a \in \partial X$, we write $\lim_i x_i = a$. Let α be a quasi-geodesic. Then the sequence $(\alpha(i))_i$ and $(\alpha(-i))_i$ are convergent at infinity. We write

$$\alpha(\pm\infty) \coloneqq \lim_{i \to \pm\infty} \alpha(i).$$

Suppose a group G acts on X isometrically. Then G acts on ∂X by $g \cdot (x_i) = (g \cdot x_i)$. The *limit set* $L(G) \subset \partial X$ of the action is defined by

$$L(G) = \left\{\left[\left(x_i\right)_i\right] \in \partial X : x_i = g_i \cdot x_0, g_i \in G, 1 \leq i < \infty\right\}.$$

It is well-known that the number of the points in $L(G)$ is $0, 1, 2$ or ∞.

Definition 2.2. Suppose G acts on X and $x_0 \in X$. Let $g \in G$. If $\left\{g^i \cdot x_0\right\}_{i \in \mathbb{Z}}$ is quasi-isometric to $\mathbb{Z}$ with its standard word metric, then g is called a *hyperbolic isometry (element).*

Definition 2.3. Suppose $g \in G$ is a hyperbolic isometry of a Gromov-hyperbolic space X. Let $x_0 \in X$. We define $g^{\pm\infty} \in \partial X$ by

$$g^{\pm\infty} \coloneqq \lim_{i \to \pm\infty} g^i \cdot x_0.$$

Note that the result does not depend on the choice of x_0.

3 Discussion

We can consider the classification of unbounded isometric actions on Gromov Hyperbolic spaces [3]:

1. horocyclic (parabolic): if there is no hyperbolic elements;
2. lineal: if all hyperbolic elements have the same fixed points;
3. focal (quasi-parabolic): if all hyperbolic elements have exactly one common fixed point;
4. of general type: if there are two independent hyperbolic elements,

where independent hyperbolic elements g_1, g_2 means that $g_1^{+\infty} \neq g_2^{\pm\infty}$ and $g_1^{-\infty} \neq g_2^{\pm\infty}$.

It's well-known that any group has a horocyclic action on a Gromov-hyperbolic space. So the first case is trivial. That's why we suppose that $G \curvearrowright X, Y$ are cobounded.

We found that, in [1], the conditions "$|L(G)| \geq 3$" and "proper action" are only used to prove that there are two independent hyperbolic elements $g_1, g_2 \in G$. So in our setting, we have

Theorem 3.1. Suppose G acts on two hyperbolic spaces X, Y isometrically and coboundedly, which induce a proper action of G on $X \times Y$ with ℓ^1-norm. If one of the action $G \curvearrowright X, Y$ is of general type, then $H_b^2(G;\mathbb{R})$ has infinite dimension.

For Question 1.3, there are three cases left:

1. two actions $G \curvearrowright X, Y$ are both lineal;
2. one of the action $G \curvearrowright X, Y$ is lineal and the other is focal;
3. two actions $G \curvearrowright X, Y$ are both focal.

Theorem 3.2. If the two actions $G \curvearrowright X, Y$ are both lineal, then G is virtually abelian. Therefore, the second bounded cohomology of G vanishes.

Proof. Consider a subgroup with index 2, we can assume that the two lineal actions are orientable. Then all the conmmutators of G act on X and Y uniformly boundedly hence on $X \times Y$. And because the action on $X \times Y$ is proper, G must have only finite commutators. Therefore, G is virtually abelian. In fact, G must be isometric to trivial group, $\mathbb{Z}$ or $\mathbb{Z}^2$. $\square$

The goal is to classify the groups satisfying Case (2) and Case (3).

Example 3.3. In Example 1.2, the two actions are both focal. And because $BS(1,2) = \mathbb{Z}\left[\frac{1}{2}\right] \rtimes \mathbb{Z}$ is solvable hence amenable, $H_b^2(BS(1,2),\mathbb{R}) = 0$. This gives an example of Case (3).

References

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