Homotopy Theory and Characteristic Classes

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Abstract

This is the notes of a course given by Prof. Ma Langte in 25spring at Shanghai Jiaotong University. The textbook is *Algebraic Topology* by Tammo tom Dieck.

Contents

Part I
Homotopy Theory

Let **TOP** be the category of topological spaces. Then we can take a quotient of **TOP** and get the homotopy category $h-\mathbf{TOP}$. The quotient may bring more algebraic structures. For example, Mor $\left(S^1, X\right)$, the homotopy classes of maps from S^1 to X, is the fundamental group of X. Our goal is to study functors from hmotopy category to some algebraic categories.

Let $\mathbf{TOP}^o$ be the pointed topological category, where the sum is wedge sum $(X, x_0) \wedge (Y, y_0) = X \sqcup Y/x_0 \sim y_0$ and the product is the smash product $(X, x_0) \vee (Y, y_0) = X \times Y/\left\{x_0\right\} \times Y \cup X \times \left\{y_0\right\}$. Similarly, we can take a quotient to get $h-\mathbf{TOP}^o$.

Let $\mathbf{TOP}(2)$ be the category of pairs and $h-\mathbf{TOP}(2)$ be its quotient.

Fix $K \in \mathrm{Ob}(\mathbf{TOP})$. Let's consider $\mathbf{TOP}^K$, the category of spaces under K. Its objects are maps $f\colon K \to X$ and morphisms are maps $\alpha\colon X \to Y$ such that $\alpha \circ f = g$.

$$\begin{tikzcd} & K \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, "\alpha"] & & Y \end{tikzcd}$$

If $K = \{*\}$ is a single point set, then $\mathbf{TOP}^{\{*\}} = \mathbf{TOP}^o$ is the pointed topological category. Take $X = K$. A morphism from $f\colon K \to X$ to $\mathrm{id}\colon K \to K$ is $r\colon X \to K$ such that $r \circ f = \mathrm{id}$.

$$\begin{tikzcd} & K \arrow[dl, "f"'] \arrow[dr, "\mathrm{id}"] & \\ X \arrow[rr, "r"] & & K \end{tikzcd}$$

When $K \subset X$, $f = i\colon K \hookrightarrow X$, we say that r is a retraction.

We have $r\colon X \to K$ is a deformation retraction, if and only if $i \circ r \simeq \mathrm{id}_X$ rel K, if and only if $r\colon X \to K$ is a homotopy equivalence in $\mathbf{TOP}^K$.

Fix $B \in \mathrm{Ob}(\mathbf{TOP})$. Let's consider $\mathbf{TOP}_B$, the category of spaces over B, where the objects are $p\colon X \to B$ and morphisms are $f\colon X \to Y$ such that $p = q \circ f$.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

Take $X = B$. A morphism from $\mathrm{id}\colon B \to B$ to $q\colon Y \to B$ is $s\colon B \to Y$ such that $q \circ s = \mathrm{id}_B$.

$$\begin{tikzcd} B \arrow[rr, "s"] \arrow[dr, "\mathrm{id}"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

Then s is called a section of q.

Similarly, we can define $h-\mathbf{TOP}^K$ and $h-\mathbf{TOP}_B$.

1 Cofibrations and Fibrations

1.1 Cofibrations

Definition 1.1. A map $i\colon A \to X$ has the homotopy extension property (HEP) for a space Y if for all homotopy $h\colon A \times I \to Y$ and $f\colon X \to Y$ with $f \circ i(a) = h(a, 1)$, there exists $H\colon X \times I \to Y$ satisfies

$$\begin{tikzcd}
 & X \arrow[dr, "i_1"] \arrow[drr, bend left, "f"] & & \\
A \arrow[ur, "i"] \arrow[dr, "i_1"'] & & X \times I \arrow[r, dashed, "H"] & Y \\
 & A \times I \arrow[ur, "i\times\mathrm{id}"] \arrow[urr, bend right, "h"'] & &
\end{tikzcd}$$

We say $i\colon A \to X$ is a cofibration if it has HEP for each $Y \in \mathrm{Ob}(\mathbf{TOP})$.

Recall the mapping cylinder: if $i\colon A \to X$ is a map, then $Z(i) := (A \times I) \sqcup X/(a, 1) \sim i(a)$.

Proposition 1.2. Given a map $i\colon A \to X$. The followings are equivalent:

1. $i\colon A \to X$ is a cofibration.
2. i has HEP for $Z(i)$.
3. The map

$$\begin{aligned}
s\colon Z(i) &\to X \times I \\
(a, t) &\mapsto (i(a), t), \\
x &\mapsto (x, 1)
\end{aligned}$$

   has a retraction.

Proof. (1)$\Longrightarrow$(2) is only by definition.

(2)$\Longrightarrow$(1): By definition, there exists $K\colon X \times I \to Z(i)$ such that the following diagram is commutative.

$$\begin{tikzcd}
 & X \arrow[dr, "i_1"] \arrow[drr, bend left, "i_1"] & & \\
A \arrow[ur, "i"] \arrow[dr, "i_1"'] & & X \times I \arrow[r, dashed, "K"] & Z(i) \\
 & A \times I \arrow[ur, "i\times\mathrm{id}"] \arrow[urr, bend right, "\mathrm{id}"'] & &
\end{tikzcd}$$

For any Y and homotopy $h\colon A \times I \to Y$ and $f\colon X \to Y$ with $f \circ i(a) = h(a, 1)$, we define

$$\begin{aligned}
F\colon Z(i) &\to Y \\
(a, t) &\mapsto h(a, t) \\
x &\mapsto f(x).
\end{aligned}$$

Then $F \circ K$ is as desired.

$$\begin{tikzcd}
 & X \arrow[dr, "i_1"] \arrow[drr, bend left, "i_1"] \arrow[drrr, bend left, "f"] & & & \\
A \arrow[ur, "i"] \arrow[dr, "i_1"'] & & X \times I \arrow[r, dashed, "K"] & Z(i) \arrow[r, "F"] & Y \\
 & A \times I \arrow[ur, "i\times\mathrm{id}"] \arrow[urr, bend right, "\mathrm{id}"'] \arrow[urrr, bend right, "h"'] & & &
\end{tikzcd}$$

(2)$\Longrightarrow$(3): We can easily check that the extension $K\colon X \times I \to Z(i)$ in the proof of (2)$\Longrightarrow$(1) is a retraction of s.

(3)$\Longrightarrow$(2): Let r be a retraction of s. For any homotopy $h\colon A \times I \to Z(i)$ and $f\colon X \to Z(i)$ with $f \circ i(a) = h(a, 1)$, we define

$$\begin{aligned} \sigma\colon Z(i) &\to Z(i) \\ (a,t) &\mapsto h(a,t) \\ x &\mapsto f(x). \end{aligned}$$

Then we can verify that $H = \sigma \circ r\colon X \times I \to Z(i)$ extends h.

$$\begin{tikzcd} & X \arrow[dr, "i_1"] \arrow[drr, bend left, "f"] & & \\ A \arrow[ur, "i"] \arrow[dr, "i_1"'] & & X \times I \arrow[r, shift right, "r"'] & Z(i) \arrow[l, shift right, "s"'] \\ & A \times I \arrow[ur, "i\times\mathrm{id}"] \arrow[urr, bend right, "h"'] & & \end{tikzcd}$$

$\square$

Corollary 1.3. When $A \subset X$ is a close subset, $i\colon A \hookrightarrow X$ is the inclusion map. Then $i\colon A \to X$ is a cofibration $\iff Z(i) = A \times I \cup X \times \{1\}$ is a retraction of $X \times I$.

Therefore, we can construct many cofibrations. For example, let (X, A) be a manifold with boundary, then $i\colon A \hookrightarrow X$ is a cofibration.

1.1.1 Push-Out of Cofibration

Given a commutative diagram,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & B \arrow[d, "J"] \\ X \arrow[r, "F"] & Y \end{tikzcd}$$

the push-out of j along f is the initial object of this diagram, i.e. $j\colon B \to Y$, $F\colon X \to Y$, s.t. $\forall Z$ with $J'\colon B \to Z$, $F'\colon X \to Z$ satisfying $J' \circ f = F' \circ j$, $\exists!$ map $p\colon Y \to Z$ such that the diagram is commutative.

$$\begin{tikzcd} & B \arrow[dr, "J"] \arrow[drr, bend left, "J'"] & & \\ A \arrow[ur, "f"] \arrow[dr, "j"'] & & Y \arrow[r, dashed, "\exists! p"] & Z \\ & X \arrow[ur, "F"] \arrow[urr, bend right, "F'"'] & & \end{tikzcd}$$

In our setting, we can construct $Y = X \sqcup B/f(a) \sim j(a)$ directly.

Proposition 1.4. If $j\colon A \to X$ is a cofibration, then the push-out of j along f $J\colon B \to Y$ is also a cofibration.

Proof. For any Z, $g\colon Y \to Z$, $h\colon B\times I \to Z$ such that $g\circ J = h\circ(i_1 \times \mathrm{id})$, we need to find $H\colon Y\times I \to Z$ such that the following diagram is commutative.

$$\begin{tikzcd}
& X \arrow[dr, "F"] & & & \\
A \arrow[ur, "j"] \arrow[dr, "f"'] & & Y \arrow[dr] \arrow[drr, bend left, "g"] & & \\
& B \arrow[ur, "J"] \arrow[dr] & & Y \times I \arrow[r, dashed, "H"] & Z \\
& & B \times I \arrow[ur] \arrow[urr, bend right, "h"'] & &
\end{tikzcd}$$

Because $j\colon A \to X$ is a cofibration, we have $G\colon X \times I \to Z$ such that the following diagram is commutative.

$$\begin{tikzcd}
& X \arrow[dr] \arrow[drr, bend left, "g\circ F"] & & \\
A \arrow[ur, "j"] \arrow[dr] & & X \times I \arrow[r, dashed, "G"] & Z \\
& A \times I \arrow[ur] \arrow[urr, bend right, "h\circ(f\times\mathrm{id})"'] & &
\end{tikzcd}$$

Using the fact that $J \times \mathrm{id}\colon B \times I \to Y \times I$ is also the push-out of $j \times \mathrm{id}\colon A \times I \to X \times I$ along $f\times\mathrm{id}\colon A\times I \to B\times I$, we have unique $H\colon Y\times I \to Z$ such that the following diagram is commutative.

$$\begin{tikzcd}
& B \times I \arrow[dr] \arrow[drr, bend left, "h"] & & \\
A \times I \arrow[ur, "f\times\mathrm{id}"] \arrow[dr] & & Y \times I \arrow[r, dashed, "H"] & Z \\
& X \times I \arrow[ur] \arrow[urr, bend right, "G"'] & &
\end{tikzcd}$$

The $H\colon Y \times I \to Z$ is the extension of $h\colon B \times I \to Z$, as desired. ☐

In terms of categorical language, let $\Pi(A, B)$ be a category, whose objects are continue maps from A to B and morphisms are homotopy of maps from A to B. Consider $\mathbf{COF}^B \subset \mathbf{TOP}^B$ the subcategory of cofibrations under B (i.e. $J\colon B \to Y$). Then we have homotopy category $h - \mathbf{COF}^B$. Given a cofibration $i\colon A \to X$, we get a contravariant functor

$$\beta\colon \Pi(A, B) \to h - \mathbf{COF}^B.$$

In fact, we only need to check that if $f_0 \simeq f_1\colon A \to B$, then we get a morphism from $J_0\colon B \to Y_0$ to $J_1\colon B \to Y_1$. Firstly, consider the homotopy $J_0 \circ f_t\colon A \times I \to Y_0$, we get its extension $\Psi\colon X \times I \to Y_0$.

$$\begin{tikzcd}
& X \arrow[dr] \arrow[drr, bend left, "F_0"] & & \\
A \arrow[ur, "i"] \arrow[dr] & & X \times I \arrow[r, dashed, "\Psi"] & Y_0 \\
& A \times I \arrow[ur] \arrow[urr, bend right, "J_0\circ f_t"'] & &
\end{tikzcd}$$

Then by the universal property of the push-out $J_1\colon B \to Y_1$ of i along f_1 for $J_0\colon B \to Y_0$ and $\Psi_1\colon X \to Y_0$, we get a map $K\colon Y_1 \to Y_0$, as desired.

$$\begin{tikzcd}
 & B \arrow[dr, "J_1"] \arrow[drr, bend left, "J_0"] & & \\
A \arrow[ur, "f_1"] \arrow[dr, "i"'] & & Y_1 \arrow[r, dashed, "K"] & Y_0 \\
 & X \arrow[ur, "F_1"] \arrow[urr, bend right, "\Psi_1"'] & &
\end{tikzcd}$$

1.1.2 Replacing a Map by a Cofibration

Given a map $f\colon X \to Y$, consider the mapping cylinder $Z(f)$. We can notice that $Z(f)$ is the push-out.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i_1"'] & Y \arrow[d, "s"] \\
X \times I \arrow[r, "a"] & Z(f)
\end{tikzcd}$$

We also have a map

$$\begin{aligned} q\colon Z(f) &\to Y \\ (x,t) &\mapsto f(x). \end{aligned}$$

Note that by Proposition 1.2, $i_1\colon X \hookrightarrow X \times I$ is a cofibration $\iff X \times \{1\} \times I \cup X \times I \times \{1\}$ is a retraction of $X \times I \times I$, we have $s\colon Y \to Z(f)$ is a cofibration.

Proposition 1.5. Let

$$\begin{aligned} j\colon X &\to Z(f) \\ x &\mapsto (x,0), \end{aligned}$$

we have

1. $j\colon X \to Z(f)$ is a cofibration.
2. $s \circ q \simeq \mathrm{id}_{Z(f)}$ rel Y.
3. If f is a cofibration, then $q\colon Z(f) \to Y$ is a homotopy equicalence in $\mathbf{TOP}^X$.

Proof. (1). We construct a retraction $R\colon Z(f) \times I \to X \times I \cup Z(f) \times \{1\}$ as follow. Let $R'\colon I \times I \to I \times \{1\} \cup \{0\} \times I$ be a retraction. Then we define

$$\begin{aligned} R\colon Z(f) \times I &\to X \times I \cup Z(f) \times \{1\} \\ ((x,s),t) &\mapsto (x, R'(s,t)) \\ (y,t) &\mapsto (y,1) \end{aligned}$$

is as desired. By Proposition 1.2, $j\colon X \to Z(f)$ is a cofibration.

(2). The homotopy

$$\begin{aligned} h_t\colon Z(f) &\to Z(f) \\ (x,\sigma) &\mapsto (x,(1-t)\sigma + t) \end{aligned}$$

is as desired.

(3). By Proposition 1.2, there is a retraction $r\colon Y \times I \to Z(f)$. Define

$$\begin{aligned} g\colon Y &\to Z(f) \\ y &\mapsto r(y,1). \end{aligned}$$

One can verifies that g is the homotopy inverse of q. $\square$

Summery 1. Any map $f\colon X \to Y$ factors into

$$X \xrightarrow{j} Z \xrightarrow{q} Y$$

where $j\colon X \to Z$ is a cofibration and $q\colon Z \to Y$ is a homotopy equivalence. Moreover, such a factorization is unique up to homotopy equivalence. In particular, we can choose $Z = Z(f)$. We define $C_f = Z(f)/\operatorname{im} j$ as the homotopy cofibre of f, i.e. $C_f = X \times I \sqcup Y/(x,0) \sim *, (x,1) \sim f(x)$, is called the mapping cone of f.

$$X \xrightarrow{f} Y \xrightarrow{s} C_f$$

1.1.3 The Cofibre Sequence (Puppe's Sequence)

To get finer structure, we work in $\mathbf{TOP}^o$. Given a map $f\colon (X,x_0) \to (Y,y_0)$, we get an induced map

$$\begin{aligned} f^*\colon [Y,B]^o &\to [X,B]^o \\ [\alpha] &\mapsto [f \circ \alpha], \end{aligned}$$

where $[X,B]^o$ is the homotopy class of basepoint preserving maps. In particular, we have the constant map

$$\begin{aligned} [*]\colon X &\to B \\ x &\mapsto b_0. \end{aligned}$$

Definition 1.6. We say a sequence

$$(X,x_0) \xrightarrow{f} (Y,y_0) \xrightarrow{g} (Z,z_0)$$

in $\mathbf{TOP}^o$ is h-coexact if $\forall\,(B,b_0) \in \operatorname{Ob}(\mathbf{TOP}^o)$,

$$[Z,B]^o \xrightarrow{g^*} [Y,B]^o \xrightarrow{f^*} [X,B]^o$$

is exact, i.e. $(f^*)^{-1}([*]) = \operatorname{im} g^*$.

In $\mathbf{TOP}^o$, we consider the reduced mapping cone $CX \coloneqq X \times I/X \times \{0\} \cup \{x_0\} \times I$. The basepoint of CX is $X \times \{0\} \cup \{x_0\} \times I$. And we consider the reduced mapping cone: For $f\colon (X,x_0) \to (Y,y_0)$, $C(f) \coloneqq CX \vee Y/(x,1) \sim f(x)$. It is equivalent to the following push-out diagram.q

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_1"'] & Y \arrow[d, "f_1"] \\ CX \arrow[r] & C(f) \end{tikzcd}$$

In fact, f_1 maps y to $(y,1)$.

We will also use symbol X instead of (X,x_0) in $\mathbf{TOP}^o$ for short.

Proposition 1.7. The sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} C(f)$$

is h-coexact.

Proof. Consider the following sequence

$$[C(f), B]^o \xrightarrow{f_1^*} [Y, B]^o \xrightarrow{f^*} [X, B]^o$$

for any (B, b_0).

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr] & Y \arrow[r, "f_1"] \arrow[d, "\alpha"] & C(f) \arrow[dl] \\
& B &
\end{tikzcd}$$

Assume that $[\alpha] \in [Y, B]^o$ s.t. $[\alpha \circ f] = [*] \in [X, B]^o$, i.e. $\alpha \circ f$ is null-homotopic. This is equivalent that there exists a map $h\colon CX \to B$. The mapping cone $C(f)$ is the push-out of

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i_1"'] & Y \arrow[d, "f_1"] \\
CX \arrow[r] & C(f)
\end{tikzcd}$$

Using the universal property of push-out, we have the following commutative diagram,

$$\begin{tikzcd}
& Y \arrow[dr, "f_1"] \arrow[drr, bend left, "\alpha"] & & \\
X \arrow[ur, "f"] \arrow[dr, "i_1"'] & & C(f) \arrow[r, dashed, "\exists\beta"] & B \\
& CX \arrow[ur] \arrow[urr, bend right, "h"'] & &
\end{tikzcd}$$

i.e. $\alpha = \beta \circ f_1$. Therefore $[\alpha] = f_1^*[\beta]$ and this proposition follows. $\square$

Iterate the procedure, we get a long h-coexact sequence:

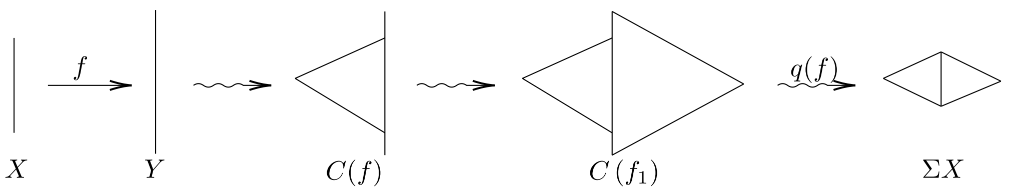
$$X \xrightarrow{f} Y \xrightarrow{f_1} C(f) \xrightarrow{f_2} C(f_1) \xrightarrow{f_3} C(f_2) \longrightarrow \cdots$$

Consider the injection $j_1\colon CY \to C(f_1)$, we have that

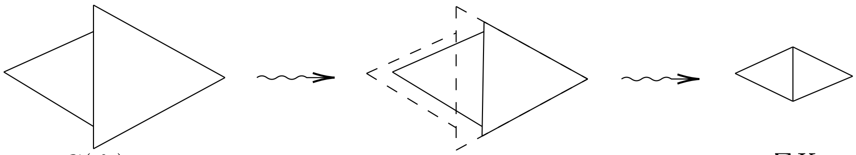
$$C(f_1)/j_1(CY) = X \times I / X \times \partial I \cup \{x_0\} \times I = \Sigma X$$

is the reduced suspension of X. Then we get a quotient map

$$q(f)\colon C(f_1) \to \Sigma X.$$



Claim 1. $q(f)$ is a homotopy equivalence.



$C(f_1)$ ΣX

Denote by $s(f)\colon \Sigma X \to C(f_1)$ the homotopy inverse of $q(f)$. Then our original sequence becomes

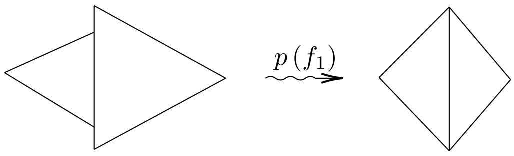
$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "f_1"] & C(f) \arrow[r, "f_2"] \arrow[dr, "q(f)\circ f_2"'] & C(f_1) \arrow[r, "f_3"] \arrow[d, "q(f)"] & C(f_2) \\
 & & & \Sigma X &
\end{tikzcd}$$

Consider the following diagram.

$$\begin{tikzcd}
C(f_1) \arrow[r, "f_3"] \arrow[d, shift right, "q(f)"'] & C(f_2) \arrow[d, "q(f_1)"] \\
\Sigma X \arrow[u, shift right, "s(f)"'] \arrow[r, dashed, "q(f_1)\circ f_3\circ s(f)"'] & \Sigma Y
\end{tikzcd}$$

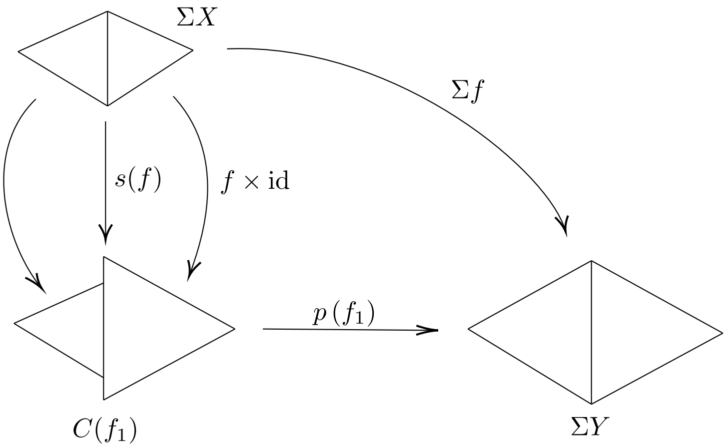
Claim 2. Consider $\tau\colon \Sigma X \to \Sigma X$ which maps (x,t) to $(x,1-t)$, we have $q(f_1)\circ f_3\circ s(f) \simeq \Sigma f\circ\tau$

To prove it, denote $p(f_1) = q(f_1)\circ f_3$. In fact, $p(f_1)$ retracts the left triangle, i.e. CX to a point.



$C(f_1)$ ΣY

In the following diagram, $s(f)$ is the union of id and $f\times\mathrm{id}$, i.e. id maps the left triangel of ΣX to the left triangle of $C(f_1)$, $f\times\mathrm{id}$ maps the right triangel of ΣX to the right triangle of $C(f_1)$. Then $\Sigma f = p(f_1)\circ s(f)$ naturally. Notice that τ flips ΣX left and right. Therefore, by symmetry, we have $p(f_1)\circ s(f)\simeq \Sigma f\circ\tau$, as desired.



$C(f_1)$ ΣY

Now we get

$$X \xrightarrow{f} Y \xrightarrow{f_1} C(f) \xrightarrow{p(f)} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{(\Sigma f)_1} C(\Sigma f)$$

Claim 3. There is a homeomorphism $\tau_1 \colon C(\Sigma f) \to \Sigma C(f)$ such that the following diagram is commutative.

$$\begin{tikzcd}
\Sigma Y \arrow[r, "(\Sigma f)_1"] \arrow[dr, "\Sigma f_1"'] & C(\Sigma f) \arrow[d, "\tau_1"] \\
& \Sigma C(f)
\end{tikzcd}$$

In fact, regard both $C(\Sigma f)$ and $\Sigma C(f)$ as the quotient spaces of $X \times I \times I$ unioned with Y, τ_1 is induced from interchanging the two I-factors.

As conclusion, we have

Theorem 1.8 (Puppe's Sequence)**.** The sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} C(f) \xrightarrow{p(f)} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma f_1} \Sigma C(f) \xrightarrow{p(\Sigma f)} \Sigma^2 X \longrightarrow \Sigma^2 Y \longrightarrow \cdots$$

is h-coexact.

1.2 Fibrations

Definition 1.9. A map $p \colon E \to B$ has the homotopy lifting property (HLP) for the space X if $\forall$ homotopy $h \colon X \times I \to B$ and $a \colon X \to E$ s.t. $p \circ a(x) = h(x, 0)$, there exists a homotopy $H \colon X \times I \to E$ s.t. $p \circ H = h$. H is called a lifting of h.

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
X \times I \arrow[r, "h"] \arrow[ur, dashed, "H"] & B
\end{tikzcd}$$

A map $p \colon E \to B$ is called a fibration if it has HLP for all spaces X.

Definition 1.10. Given maps $f \colon A \to B$ and $p \colon E \to B$. The pull-back of p along f is the terminal object of the following diagram,

$$\begin{tikzcd}
f^*E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

i.e. for any C, $g \colon C \to E$, $h \colon C \to A$, there exists unique r such that the following diagram is commutative.

$$\begin{tikzcd}
& & E \arrow[dr, "p"] & \\
C \arrow[r, dashed, "r"] \arrow[urr, bend left, "g"] \arrow[drr, bend right, "h"'] & f^*E \arrow[ur] \arrow[dr] & & B \\
& & A \arrow[ur, "f"'] &
\end{tikzcd}$$

Explicity,

$$f^*E = \{(a, e) \in A \times E : f(a) = p(e)\}$$

and $\pi \colon f^*E \to A$ is the projection.

Denote $B^I = \mathrm{Map}(I, B)$. Consider the pull-back

$$W(p) \coloneqq \{(x, w) \in E \times B^I : p(x) = w(0)\}$$

which is given by the pull-back

$$\begin{tikzcd}
W(p) \arrow[r, "k"] \arrow[d, "b"'] & B^I \arrow[d, "e^0"] \\
E \arrow[r, "p"'] & B
\end{tikzcd}$$

where e^0 maps w to $w(0)$.

Proposition 1.11. Given a map $p\colon E \to B$, the followings are equivalence:

1. $p\colon E \to B$ is a fibration.
2. p has HLP for $W(p)$.
3. $$\begin{aligned} r\colon E^I &\to W(p) \\ \alpha &\mapsto (\alpha(0), p \circ \alpha) \end{aligned}$$

   admits a section.

Proof. (1)$\Longrightarrow$(2) is by definition.

(2)$\Longrightarrow$(3): Because $W(p)$ is a pull-back, by its universal property, we have the following diagram and we want to find s such that $r \circ s = \mathrm{id}$.

$$\begin{tikzcd}
 & & B^I \arrow[dr, "e^0"] & \\
E^I \arrow[r, shift right, "r"'] \arrow[urr, bend left, "p^I"] \arrow[drr, bend right, "e^0"'] & W(p) \arrow[l, dashed, shift right, "s"'] \arrow[ur, "k"] \arrow[dr, "b"] & & B \\
 & & E \arrow[ur, "p"'] &
\end{tikzcd}$$

Notice that $\mathrm{Map}\left(W(p), E^I\right) = \mathrm{Map}\left(W(p) \times I, E\right)$, because p has HLP for $W(p)$, we have the following commutative diagram.

$$\begin{tikzcd}
W(p) \arrow[r, "b"] \arrow[d] & E \arrow[d, "p"] \\
W(p) \times I \arrow[r, "k"] \arrow[ur, dashed, "s"] & B
\end{tikzcd}$$

We have $b \circ r \circ s = e^0 \circ s = b$ and $k \circ r \circ s = p^I s = k$. Using the universal property (uniqueness) of pull-back $W(p)$ for $W(p)$, we must have $r \circ s = \mathrm{id}$, i.e. s is a section of r.

(3)$\Longrightarrow$(1): Let s be the section of r. For any X, a, h as in the definition of fibration, we want to find H such that the following diagram is commutative.

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
X \times I \arrow[r, "h"] \arrow[ur, dashed, "H"] & B
\end{tikzcd}$$

Using the universal property of pull-back $W(p)$, we have unique f such that the following diagram is commutative, where $h\colon X \to B^I$ is the same as $h\colon X \times I \to B$.

$$\begin{tikzcd}
 & & B^I \arrow[dr, "e^0"] & \\
X \arrow[r, dashed, "\exists! f"] \arrow[urr, bend left, "h"] \arrow[drr, bend right, "a"'] & W(p) \arrow[ur, "k"] \arrow[dr, "b"] & & B \\
 & & E \arrow[ur, "p"] &
\end{tikzcd}$$

Then because $\mathrm{Map}\left(W(p), E^I\right) = \mathrm{Map}\left(W(p) \times I, E\right)$, one can check that $H = s \circ f$ is as desired. In fact,

$$p \circ H(x,t) = (p \circ H(x))(t) = (k \circ r \circ s \circ f(x))(t) = (k \circ \mathrm{id} \circ f(x))(t) = h(x,t)$$

and $H \circ i_0 = a$ is similar. $\square$

1.2.1 Pull-back of Fibration

Proposition 1.12. If $p\colon E \to B$ is a fibration, then $f^*E \to A$ is also a fibration.

Proof. In the following diagram, F is induced by HLP for fibration $p\colon E \to B$ and then H is induced by universal property of pull-back f^*E.

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "i_0"'] & f^*E \arrow[r] \arrow[d, "\pi"] & E \arrow[d, "p"] \\
X \times I \arrow[r, "h"] \arrow[ur, dashed, "H"] \arrow[urr, dashed, "F"] & A \arrow[r, "f"] & B
\end{tikzcd}$$

$\square$

1.2.2 Replacing Maps by Fibration

Proposition 1.13. The evaluation $e^1\colon Y^I \to Y$, $w \mapsto w(1)$ is a fibration.

Proof. We can define H directly:

$$\begin{aligned}
H\colon X \times I &\to Y^I \\
(x,s) &\mapsto \begin{cases} [t \mapsto a|_X((1+s)t)], & \textit{when } 0 \le (1+s)t \le 1 \\ [t \mapsto h(x,(1+s)t-1)], & \textit{when } (1+s)t \ge 1. \end{cases}
\end{aligned}$$

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "i_0"'] & Y^I \arrow[d, "e^1"] \\
X \times I \arrow[r, "h"] \arrow[ur, dashed, "H"] & Y
\end{tikzcd}$$

$\square$

Given $f\colon X \to Y$, consider the following pull-back.

$$\begin{tikzcd}
W(f) = f^*Y^I \arrow[r] \arrow[d, "i_0"'] & Y^I \arrow[d, "e^1"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

In fact,

$$W(f) = \left\{(x, w) \in X \times Y^I : f(x) = w(1)\right\}.$$

Denote $p\colon W(f) \to Y$, $(x, w) \mapsto w(0)$ and $s\colon X \to W(f)$, $x \mapsto \left(x, k_{f(x)}\right)$ where $k_{f(x)}$ is a constant path at $f(x)$, and $q\colon W(f) \to X$, $(x, w) \mapsto x$. We can check that the following diagram is commutative.

$$\begin{tikzcd}
W(f) = f^*Y^I \arrow[r] \arrow[d, shift right, "i_0"'] \arrow[dr, "p"] & Y^I \arrow[d, "e^1"] \\
X \arrow[u, shift right, "s"'] \arrow[r, "f"'] & Y
\end{tikzcd}$$

Theorem 1.14. In the following commutative diagram,

$$\begin{tikzcd}
X \arrow[rr, "s"] \arrow[dr, "f"'] & & W(f) \arrow[dl, "p"] \\
& Y &
\end{tikzcd}$$

s is a homotopy equivalence and p is a fibration.

Proof. Consider the following fibration

$$\begin{tikzcd}
(f \times \mathrm{id})^*Y^I \arrow[r] \arrow[d, "{(q,p)}"'] & Y^I \arrow[d, "{\left(e^1, e^0\right)}"] \\
X \times Y \arrow[r, "f \times \mathrm{id}"] & Y \times Y
\end{tikzcd}$$

Claim 4. $(f \times \mathrm{id})^*Y^I = W(f)$.

To see that, notice that

$$(f \times \mathrm{id})^*Y^I = \left\{(x, y, w) \in X \times Y \times Y^I : f(x) = w(1), y = w(0)\right\},$$

we can construct a map from $W(f)$ to $(f \times \mathrm{id})^*Y^I$ that maps (x, w) to $()x, w(0), w)$. It's one to one.

Then $p\colon W(f) \to Y$ is a fibration if and only if $(f \times \mathrm{id})^*Y^I \xrightarrow{(q,p)} X \times Y \xrightarrow{p_2} Y$ is a fibration. It's a composition of two fibration and then a fibration, as desired.

Claim 5. q is a homotopy inverse of s.

□

By this theorem, given any $f\colon X \to Y$, we can replace it by a fibration $p\colon W(f) \to Y$ homotopically. Then we can define the homotopy fibre at y_0 of $f\colon X \to Y$ to be

$$F(f) := p^{-1}(y_0) = \left\{(x, w) \in X \times Y^I : f(x) = w(1), y_0 = w(0)\right\}.$$

Remark 1.15. Apply HLP again, we can prove the factorization $f = s \circ p\colon X \to Y$ such that $s\colon X \to W$ is a homotopy equivalence and $p\colon W \to Y$ is a fibration. And this factorization is unique up to homotopy equivalence.

Theorem 1.16. Let $p\colon E \to B$ be a fibration and B is path-connected. Then all fibres $p^{-1}(b)$ are homotopy equivalent.

Proof. Given a path $\alpha\colon I \to B$, $\alpha(0) = b_0$ and $\alpha(1) = b_1$. Consider HLP property:

$$\begin{tikzcd}
p^{-1}(b_0) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
p^{-1}(b_0) \times I \arrow[r, "h"] \arrow[ur, "H"] & B
\end{tikzcd}$$

where $h(x,t) = \alpha(t)$. Consider $H_1\colon p^{-1}(b_0) \to p^{-1}(b_1)$ the restriction of H at $t = 1$. Similarly, consider the reversed path $\overline{\alpha}$ of α, we get $\overline{H_1}\colon p^{-1}(b_1) \to p^{-1}(b_0)$.

Claim 6. $\overline{H_1} \circ H_1 \simeq \mathrm{id}$.

It's by applying homotopy lifting to the homotopy from $\overline{\alpha}\alpha$ to k_{b_0}. Therefore, all fibres $p^{-1}(b)$ are homotopy equivalent. □

1.2.3 Fibre Exact Sequence (Puppe's Sequence)

Definition 1.17. We say a sequence of pointed maps

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$$

is h-coexact if $\forall\,(B, b_0)$, the induced sequence

$$[B, X]^o \xrightarrow{f_*} [B, Y]^o \xrightarrow{g_*} [B, Z]^o$$

is exact, i.e. $g_*^{-1}([c_{z_0}]) = \operatorname{im} f_*$.

Recall the homotopy fibre of $f\colon X \to Y$ is

$$F(f) \coloneqq p^{-1}(y_0) = \left\{(x, w) \in X \times Y^I : f(x) = w(1), y_0 = w(0)\right\}.$$

Denote $f^1\colon F(f) \to X$, $(x, w) \mapsto x$.

Proposition 1.18. For any $f\colon (X, x_0) \to (Y, y_0)$, the sequence

$$F(f) \xrightarrow{f^1} X \xrightarrow{f} Y$$

is h-coexact.

Proof. Assume $\alpha\colon B \to X$ satisfies $f \circ \alpha\colon B \to Y$ is null-homotopic and $f_*[\alpha] = [c_{y_0}]$. Apply HLP property:

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & FY = \left\{w \in Y^I : w(0) = y_0\right\} \arrow[d, "e^1"] \\
B \times I \arrow[r, "h"'] \arrow[ur, "H"] & Y
\end{tikzcd}$$

where h is a null-homotopy from $f \circ \alpha$ to c_{y_0}. Notice that $H_0\colon B \times \{1\} \to FY$ satisfies

$$\begin{tikzcd}
 & & FY \arrow[dr] & \\
B \arrow[r, dashed, "\beta"] \arrow[urr, bend left, "H_0"] \arrow[drr, bend right, "\alpha"'] & F(f) \arrow[ur] \arrow[dr, "f^1"] & & Y \\
 & & X \arrow[ur] &
\end{tikzcd}$$

where β is induced by the universal property of the pull-back $F(f)$, such that $f^1 \circ \beta = \alpha$. Therefore, $f^1_*([\beta]) = [\alpha]$. □

Iterate the procedure, we get a long h-exact sequence

$$\cdots \longrightarrow F\left(f^2\right) \xrightarrow{f^3} F\left(f^1\right) \xrightarrow{f^2} F(f) \xrightarrow{f^1} X \longrightarrow Y .$$

Question 1.19. How to understand $F\left(f^n\right) \xrightarrow{f^{n+1}} F\left(f^{n-1}\right)$?

We consider the loop space

$$\Omega Y := \left\{w \in Y^I : w(0) = w(1) = y_0\right\}.$$

Notice that

$$\left(f^1\right)^{-1}(x_0) = \left\{(x_0, w) \in X \times Y^I : w(0) = y_0, w(1) = f(x_0) = y_0\right\},$$

we have $\Omega Y = \left(f^1\right)^{-1}(x_0)$. We write $i(f)\colon \Omega Y \to F(f)$ for the inclusion.

Theorem 1.20 (The puppe's fibre sequence)**.** The sequence

$$\Omega^k F(f) \xrightarrow{\Omega^k f^1} \Omega^k X \xrightarrow{\Omega^k f} \Omega^k Y \xrightarrow{i\left(\Omega^{k-1} f\right)} \cdots \longrightarrow \Omega X \longrightarrow \Omega Y \longrightarrow F(f) \xrightarrow{f^1} X \longrightarrow Y$$

is h-exact.

Proof. Step 1:

$$\begin{aligned} F\left(f^1\right) &= \left\{(x, w, v) \in X \times Y^I \times X^I : w(0) = y_0, v(0) = x_0, w(1) = f(x), v(1) = x\right\} \\ &= \left\{(w, v) \in Y^I \times X^I : w(0) = g_0, v(0) = x_0, w(1) = f(v(1))\right\}. \end{aligned}$$

Define $j(f)\colon \Omega Y \to F\left(f^1\right)$, $w \mapsto (w, k_{x_0})$.

Claim 7. $j(f)$ is a homotopy equivalence.

In fact, define $r(f)\colon F\left(f^1\right) \to \Omega Y$, $(w, v) \mapsto w * \overline{(f \circ v)}$, then $r(f) \circ j(f) = \mathrm{id}$. The homotopy from $\mathrm{id}_{F(f^1)}$ to $j(f) \circ r(f)$ is $h_t(w, v) = \left(h_t^1, h_t^2\right)$, where $h_t^1(s) = \begin{cases} w(s(1+t)), & s(1+t) \leq 1, \\ f(v(2-(1+t)s)), & s(1+t) \geq 1 \end{cases}$ and $h_t^2(s) = v(s(1-t))$.

Step 2: From $F\left(f^1\right) \xrightarrow{f^2} F(f) \xrightarrow{f^1} X$, we get

$$\begin{tikzcd} F\left(f^2\right) \arrow[r, "f^3"] & F\left(f^1\right) \\ \Omega X \arrow[u, "j\left(f^1\right)"] \arrow[r, "\Omega f"'] \arrow[ur, "i\left(f^1\right)"'] & \Omega Y \arrow[u, "j(f)"'] \end{tikzcd}$$

Because $j\left(f^1\right)$ is a homotopy equivalence, we have $i\left(f^1\right) \simeq j(f) \circ \Omega f$.

Step 3: Now we have $\Omega X \xrightarrow{\Omega f} \Omega Y i(f) \longrightarrow F(f)$. Then we get $F\Omega f \longrightarrow \Omega X \xrightarrow{\Omega f} \Omega Y$.

Claim 8. $F(\Omega f)$ is homotopy equivalent to $\Omega F(f)$.

To see that, notice that $F(\Omega f)$ and $\Omega F(f)$ are all quotient of $\mathrm{Map}(I \times I, Y)$.

Finally, we get the h-exact sequence

$$\Omega F(f) \longrightarrow \Omega X \longrightarrow \Omega Y \longrightarrow F(f) \longrightarrow X \longrightarrow Y .$$

$\square$

1.3 Duality of Cofibration and Fibration

1.3.1 Duality of Reduced Suspension and Loop Space

Write $Y^X = \mathrm{Map}(X, Y)$ equipped with compact-open topology. We define the adjunction

$$\begin{aligned}\alpha\colon Z^{X\times Y} &\to \left(Z^Y\right)^X\\ f &\mapsto [x \mapsto f(x, \cdot)].\end{aligned}$$

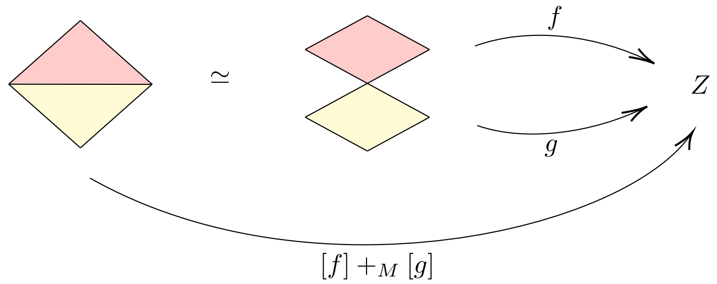
Theorem 1.21. Suppose that X and Y are locally compact. Then α is a homeomorphism.

In the pointed version, we replace $X \times Y$ by $X \wedge Y = X \times Y / \{x_0\} \times Y \cup X \times \{y_0\}$ and $\mathrm{Map}^o(X, Y)$ is the space of basepoint preserving maps. Then $\alpha^o\colon \mathrm{Map}^o(X \wedge Y, Z) \to \mathrm{Map}^o(X, \mathrm{Map}^o(Y, Z))$ is a homeomorphism. Therefore, α^o induces a bijection $\alpha^o_*\colon [X \wedge Y, Z]^o \to [X, \mathrm{Map}^o(Y, Z)]^o$.

Choose $Y = S^1 = I/\partial I$, then $X \wedge Y = X \times I / X \times \partial I \cup \{x_0\} \times I = \Sigma X$ is the reduced suspension of X and $\mathrm{Map}^o(Y, Z) = \Omega Z$ is the loop space of Z. Therefore, we get a bijection $\alpha^o_*\colon [\Sigma X, Z]^o \to [X, \Omega Z]^o$.

On $[\Sigma X, Z]^o$, we have a group structure:

$$[f] +_M [g]\colon (x, t) \mapsto \begin{cases} f(x, 2t), & t \le \frac{1}{2},\\ g(x, 2t-1), & t \ge \frac{1}{2}.\end{cases}$$



Let τ be the inversion of ΣX. For any $[f]$, $-[f] = [f \circ \tau]$.

On $[X, \Omega Z]^o$, we have

$$\begin{aligned} m\colon \Omega Z \times \Omega Z &\to \Omega Z\\ (u, v) &\mapsto u * v.\end{aligned}$$

Define

$$[f] +_m [g] \coloneqq [m \circ (f \times g) \circ d],$$

where

$$\begin{aligned} d\colon X &\to X \times X\\ x &\mapsto (x, x)\end{aligned}$$

is the diagonal embedding.

One can verify that

$$\alpha^o_*([f] +_M [g]) = \alpha^o_*([f]) +_m \alpha^o_*([g]).$$

Then the adjunction map $\alpha^o_*\colon [\Sigma X, Z]^o \to [X, \Omega Z]^o$ is an isomorphism. In categorical language, this means $\mathrm{Mor}(\Sigma X, Z) = \mathrm{Mor}(X, \Omega Z)$ in $\mathbf{TOP}^o$. As conclusion, $\Sigma\colon \mathbf{TOP}^o \to \mathbf{TOP}^o$ and $\Omega\colon \mathbf{TOP}^o \to \mathbf{TOP}^o$ are dual functors.

1.3.2 Duality of HLP and HEP

Given a homotopy lifting diagram,

$$\begin{tikzcd}
X \times \{0\} \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
X \times I \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

notice that $\mathrm{Map}(X \times I, Z) = \mathrm{Map}\left(X, Z^I\right)$, it is equivalent to

$$\begin{tikzcd}
E & E^I \arrow[l, "e^0"'] \arrow[d] \\
X \arrow[u] \arrow[r] \arrow[ur, dashed] & B^I
\end{tikzcd}$$

Dualize it, also by, $\mathrm{Map}(X \times I, Z) = \mathrm{Map}\left(X, Z^I\right)$, we have

$$\begin{tikzcd}
E' \arrow[r, "i_0"] \arrow[d] & E' \times I \arrow[dl, dashed] \\
X' & B' \times I \arrow[l] \arrow[u]
\end{tikzcd}$$

It is equivalent to

$$\begin{tikzcd}
 & E' \arrow[dr] \arrow[drr, bend left] & & \\
B' \arrow[ur] \arrow[dr] & & E' \times I \arrow[r, dashed] & X' \\
 & B' \times I \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

which is the homotopy extension diagram.

1.3.3 Duality of Two Puppe's Sequences

Notice that $[\mathrm{id}] \in [\Sigma X, \Sigma X]^o$, it induces $\alpha_*^o[\mathrm{id}] = \eta\colon X \to \Omega\Sigma X$. For each map $f\colon X \to Y$, it induces

$$\begin{aligned}
\eta\colon F(f) &\to \Omega C(f) \\
(x, w) &\mapsto \begin{cases} (x, 2t), \ t \leq \frac{1}{2}, \\ w(2-2t), \ t \geq \frac{1}{2}, \end{cases}
\end{aligned}$$

where $C(f) = X \times I \sqcup Y / \{x_0\} \times I, f(x) \sim (x, 1)$ is the reduced cone of f. Then we get a diagram commutative up to homotopy.

$$\begin{tikzcd}
\Omega Y \arrow[r] \arrow[d, "\mathrm{id}"'] & F(f) \arrow[r] \arrow[d] & X \arrow[d] \\
\Omega Y \arrow[r] & \Omega C(f) \arrow[r] & \Omega\Sigma X
\end{tikzcd}$$

2 Homotopy Groups

2.1 Definitions and Properties

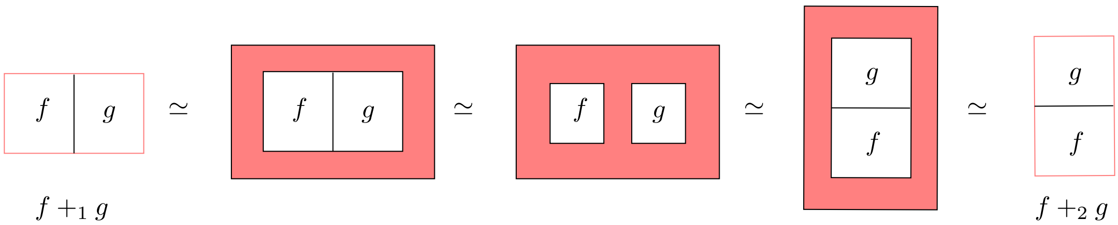
Given (X, x_0), define n-th homotopy group

$$\pi_n(X, x_0) := [(I^n, \partial I^n), (X, x_0)],$$

where the identity element is the constant map and $[f] + [g]$ can be represented by

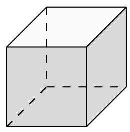
$$f +_i g \colon (t_1, \cdots, t_n) \mapsto \begin{cases} f(t_1, \cdots, 2t_i, \cdots, t_n), & t_i \leq \frac{1}{2} \\ g(t_1, \cdots, 2t_i - 1, \cdots, t_n), & t_i \geq \frac{1}{2} \end{cases}$$

for any i. The following picture shows that $f +_i g$ and $f +_j g$ are homotopy equivalent for any $i \neq j$, where the red parts are mapped into the base point so the homotopies work. Sometimes, we write $\pi_n(X)$ for short.



$f +_1 g$ $f +_2 g$

Given a pair (X, A, x_0), $J^n = \partial I^n \times I \cup I^n \times \{0\} = I^n - I^n \times \{1\} \subset I^{n+1}$,



define the $n+1$-th relative homotopy group to be

$$\pi_{n+1}(X, A, x_0) := \left[\left(I^{n+1}, \partial I^{n+1}, J^n\right), (X, A, x_0)\right].$$

Similarly, we sometimes use $\pi_{n+1}(X, A)$ for short.

Proposition 2.1. When $n \geq 2$, $\pi_n(X, x_0)$ and $\pi_{n+1}(X, A, x_0)$ are both abelian.

Proof. Exchanging f and g in the picture after the definition of $\pi_n(X, x_0)$, we can know that $\pi_n(X, x_0)$ is abelian for $n \geq 2$. For the relative case, we can not process homotopy in the top red region. But for $n \geq 3$, the squares of f and g should be cubes, then we can place the cubes in front and behind to get new homotopy. Therefore, $\pi_n(X, A, x_0)$ is abelian for $n \geq 3$. $\square$

Theorem 2.2 (Exact Homotopy Sequence)**.** Given a pair (X, A), we have a long exact sequence

$$\longrightarrow \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \longrightarrow \cdots \longrightarrow \pi_0(A, x_0) \xrightarrow{i_*} \pi_0(X, x_0),$$

where $j \colon (X, x_0, x_0) \to (X, A, x_0)$ is the inclusion and ∂ is induced from the restriction of I^n on $I^{n-1} \times \{1\}$.

Proof. Notice that each map $f \colon (I^n, \partial I^n) \to (X, x_0)$ induces a map

$$\begin{aligned} \overline{f_k} \colon I^{n-k} &\to \Omega^k(X, x_0) \\ (u_1, \cdots, u_{n-k}) &\mapsto [(t_1, \cdots, t_k) \mapsto f(t_1, \cdots, t_k, u_1, \cdots, u_{n-k})]. \end{aligned}$$

Then we get an isomorphism $\pi_n(X, x_0) \to \pi_{n-k}\left(\Omega^k X, c_{x_0}\right)$. This is because $\pi_n(X, x_0) = [S^n, X]^o$ and $\Sigma S^{n-1} = S^n$, then $[S^n, X]^o = \left[\Sigma S^{n-1}, X\right]^o \cong \left[S^{n-1}, \Omega X\right]^o \cong \left[S^{n-k}, \Omega^k X\right]^o$ by duality (Section 1.3.1).

Given a pair (X, A), the homotopy fibre of $\iota\colon A \hookrightarrow X$ is

$$F(\iota) = \left\{(a, w) \in A \times X^I : w(0) = x_0, w(1) = a\right\} = \left\{w \in X^I : w(0) = x_0, w(1) \in A\right\} \coloneqq F(X, A).$$

Each map $f\colon \left(I^{n+1}, \partial I^{n+1}, J^n\right) \to (X, A, x_0)$ induces a map

$$\begin{aligned} \hat{f}\colon I^n &\to F(X, A) \\ (t_1, \cdots, t_n) &\mapsto [t \mapsto f(t_1, \cdots, t_n, t)], \end{aligned}$$

induces an isomorphism $\pi_{n+1}(X, A, x_0) \to \pi_n(F(X, A), x_0)$.

The fibre sequence of $\iota\colon A \hookrightarrow X$ is

$$\Omega^n F(\iota) \longrightarrow \Omega^n A \longrightarrow \Omega^n X \longrightarrow \cdots \longrightarrow F(\iota) \longrightarrow A \xrightarrow{\iota} X .$$

Appling $\left[S^1, \cdot\right]^o$, we have

$$\left[S^1, \Omega^n F(\iota)\right]^o = \pi_1(\Omega^n F(\iota)) = \pi_{n+1}(F(\iota)) = \pi_{n+2}(X, A),$$

$$\left[S^1, \Omega^n A\right]^o = \pi_1(\Omega^n A) = \pi_{n+1}(A),$$

$$\left[S^1, \Omega^n X\right]^o = \pi_1(\Omega^n X) = \pi_{n+1}(X).$$

Then we get exact sequence

$$\pi_{n+2}(X, A) \longrightarrow \pi_{n+1}(A) \longrightarrow \pi_{n+1}(X) \longrightarrow \cdots \longrightarrow \pi_1(X) \longrightarrow \pi_1(X, A) \longrightarrow \pi_0(A) \longrightarrow \pi_0(X) ,$$

where the exactness of the last a few places is straightforward to verify. □

2.2 Change of Basepoint

Assume $v\colon I \to X$ is a continuous path with $v(0) = x_0$ and $v(1) = x_1$. We regard v as a homotopy

$$\begin{aligned} \hat{v}_t\colon I^n &\to X \\ u &\mapsto v(t). \end{aligned}$$

Note that $\partial I^n \hookrightarrow I^n$ is a cofibration (by Corollary 1.3), by HEP, we have the following commutative diagram,

$$\begin{tikzcd} & \partial I^n \times I \arrow[dr] \arrow[drr, bend left, "\hat{v}_t"] & & \\ \partial I^n \arrow[ur] \arrow[dr] & & I^n \times I \arrow[r, dashed, "V"'] & X \\ & I^n \arrow[ur] \arrow[urr, bend right, "f"'] & & \end{tikzcd}$$

where $[f] \in \pi_n(X, x_0)$.

Proposition 2.3. The map

$$\begin{aligned} v_\sharp\colon \pi_n(X, x_0) &\to \pi_n(X, x_1) \\ [v_0] &\mapsto [v_1] \end{aligned}$$

only depends on the homotopy class of v rel ∂_1 and defines an isomorphism.

Proof. Use HEP again. $\square$

Proposition 2.4. Suppose $f\colon (X, A) \to (Y, B)$ is a homotopy equivalence. Then $f_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, f(x_0))$ is an isomorphism.

Proof. We only prove that homotopic maps induce isomorphic maps on π_n. Assume we have a homotopy $g_t\colon (X, A) \to (Y, B)$, we get a path in Y

$$
\begin{aligned}
w\colon I &\to Y \\
t &\mapsto g_t(x_0).
\end{aligned}
$$

Then we have the following commutative diagram by HEP.

$$
\begin{tikzcd}
 & \pi_n(Y, B, g_0(x_0)) \arrow[dd, "w_\sharp"] \\
\pi_n(X, A, x_0) \arrow[ur, "g_{0,*}"] \arrow[dr, "g_{1,*}"'] & \\
 & \pi_n(Y, B, g_1(x_0))
\end{tikzcd}
$$

$\square$

Remark 2.5. By the proposition, we get a right action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$.

2.3 Serre Fibration

Definition 2.6. We say $p\colon E \to B$ is a Serre fibration, if it has HLP for all cube I^n, $\forall n \geq 0$.

Theorem 2.7. Let $p\colon E \to B$ be a Serre fibration. Fix $b_0 \in B$ and $e_0 \in E$ such that $p(e_0) = b_0$. Given $B_0 \subset B$, write $E_0 = p^{-1}(B_0)$. Then $p_*\colon \pi_n(E, E_0, e_0) \to \pi_n(B, B_0, b_0)$ is an isomorphism for all $n \geq 1$.

Proof. **Surjectivity**: Given $h\colon \left(I^n, \partial I^n, J^{n-1}\right) \to (B, B_0, b_0)$. Consider the lifting problem.

$$
\begin{tikzcd}
I^{n-1} \times \{0\} \cup \partial I^{n-1} \times I \arrow[r, "c_{e_0}"] \arrow[d] & E \arrow[d, "p"] \\
I^{n-1} \times I \arrow[r, "h"] \arrow[ur, dashed, "H"] & B
\end{tikzcd}
$$

Notice that $I^{n-1} \times \{0\} \cup \partial I^{n-1} \times I \cong I^{n-1} \times \{0\}\}$, the map of the first line is c_{e_0}. Then we have the lifting $H\colon I^n \to E$ such that $H(\partial I^n) \subset E_0 = p^{-1}(B_0)$ and $H\left(J^{n-1}\right) = e_0$.

Injectivity: Assume $p_*[f_0] = p_*(f_1]$. We get a homotopy $\phi_t\colon \left(I^n, \partial I^n, J^{n-1}\right) \to (B, B_0, b_0)$. Consider the lifting problem.

$$
\begin{tikzcd}
I^n \times \partial I \cup J^{n-1} \times I \arrow[r] \arrow[d] & E \arrow[d] \\
I^n \times I \arrow[r, "\phi_t"] \arrow[ur, dashed, "\phi"] & B
\end{tikzcd}
$$

Notice that $I^n \times \partial I \cup J^{n-1} \times I \cong I^n$, we have the lifting ϕ. $\square$

Corollary 2.8. Given a Serre fibration $F \hookrightarrow E \xrightarrow{p} B$ where F is a regular fibre, we have a long exact sequence

$$
\pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots \longrightarrow pi_0(E) \longrightarrow \pi_0(B).
$$

Proof. Consider the pair (E, F). By Theorem 2.2, we have exact sequence

$$\pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots.$$

Choose $B_0 = b_0$ and $F = E_{b_0}$, we have $\pi_n(E, F, b_0) \cong \pi_n(E, b_0, b_0) \cong \pi_n(B, b_0)$ and this corollary follows. □

Proposition 2.9. Every fibre bundle is a Serre fibration.

Proof. Given the lifting problem.

$$\begin{tikzcd} I^n \times \{0\} \arrow[r, "a"] \arrow[d] & E \arrow[d] \\ I^n \times I \arrow[r, "h"'] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

We choose an open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of B such that finitely many U_α's cover $\operatorname{im} h$ and over each U_α, $E|_{U_\alpha}$ is trivialized. Choose a subdivision $\left\{I^n_\beta\right\}$ of I^n and partition $\{I_\lambda\}$ of I, such that $\forall \beta, \lambda$, $h\left(I^n_\beta \times I_\lambda\right) \subset U_\alpha$ for some α. Over each $I^n_\beta \times I_\lambda$, we consider

$$\begin{tikzcd} I^n_\beta \times \partial I_\lambda \cup \partial I^n_\beta \times I_\lambda \arrow[r] \arrow[d] & U_\alpha \times F \arrow[d] \\ I^n_\beta \times I_\lambda \arrow[r, "h"'] \arrow[ur, dashed, "H_{\beta,\lambda}"] & U_\alpha \end{tikzcd}$$

where $I^n_\beta \times \partial I_\lambda \cup \partial I^n_\beta \times I_\lambda \cong I^n_\beta \times \{0\}$ and $U_\alpha \times F \cong E|_{U_\alpha}$. We construct the lifting of h inductively on β and λ. □

2.4 Higher Connectivity

Proposition 2.10. Let (X, A) be a pair, and $f\colon (I^n, \partial I^n) \to (X, A)$ a pointed map. The follwoings are equivalent.

1. f is null-homotopic.

2. f is homotopic rel ∂I^n to a map in A.

Proof. (1) $\implies$ (2): Consider a surjective continuous map $\lambda\colon I^n \times I \to I^n \times I$ such that $\lambda|_{\partial I^n \times I}\colon (x, t) \mapsto (x, 0)$ and $\lambda|_{I\{0\}} = \mathrm{id}_{I^n}$. Consider a null-homotopy $F\colon I^n \times I \to X$ of f, we let $H = F \circ \lambda\colon I^n \times I \to X$. Then H is a homotopy of f such that $H|_{\partial I^n \times \{t\}} = \mathrm{id}_{\partial I^n}$ and $H_1(I^n) \subset A$.

(2) $\implies$ (1): We may assume $f(I^n) \subset A$. J^{n-1} is a deformation retract of I^n. This is equivalent to that we get a homotopy $h_t\colon I^n \to I^n$ such that $\operatorname{im} h_1 = J^{n-1}$ and $h_0 = \mathrm{id}$. Then $f \circ h_t$ is a homotopy from f to c_{x_0}. □

Remark 2.11. By (2), $\pi_n(A, A) \to \pi_n(X, A)$ is trivial.

Definition 2.12. We say a pair (X, A) is n-connected if $\pi_q(X, A) = 0$, $\forall 1 \le q \le n$ and $\pi_0(A) \to \pi_0(X)$ is surjective. Note that $\pi_q(X, A) = 0$ is computed for all basepoints.

Proposition 2.13. The followings are equivalent.

1. (X, A) is n-connected.

2. $j_*\colon \pi_q(A, *) \to \pi_q(X, *)$ is an isomorphism for $q < n$ and is an epimorphism for $q = n$.

Proof. The proof follows from exact sequence of the pair (X, A) (Proposition 2.2). □

Definition 2.14. We say $f\colon X \to Y$ is n-connected if $f_*\colon \pi_k(X) \to \pi_k(Y)$ is an isomorphism for $1 \le k \le n-1$ and is an epimorphism for $k = n$.

Proposition 2.15. $f\colon X \to Y$ is n-connected if and only if $(Z(f), X)$ is n-connected.

Proof. The proof follows from exact sequence of the pair $(Z(f), X)$ (Proposition 2.2) and $Z(f) \simeq Y$. □

2.5 Excision and Suspension

Theorem 2.16 (Blaskers-Massey)**.** Let $Y = Y_1 \cup Y_2$ be union of two open subsets and $Y_0 = Y_1 \cap Y_2 \neq \varnothing$. Suppose $\pi_i(Y_1, Y_0) = 0$ for any $0 < i < p$, $p \ge 1$ and $\pi_j(Y_2, Y_0) = 0$ for any $0 < j < q$, $q \ge 1$. Then the map $\iota\colon \pi_n(Y_2, Y_0) \to \pi_n(Y, Y_1)$ is an isomorphism for $1 \le n \le p+q-3$ and is an epimorphism for $n = p+q-2$.

Proof. See textbook § 6.7. □

Proposition 2.17. Let $j\colon A \hookrightarrow X$ be a cofibration. Consider a push-out diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "j"'] & B \arrow[d, "J"] \\ X \arrow[r, "F"] & Y \end{tikzcd}$$

where $Y = X \sqcup B/f(a) \sim j(a)$. Suppose $\pi_i(X, A) = 0$, $\forall 0 < i < p$ and $\pi_i(Z(f), A) = 0$, $\forall 0 < i < q$. Then the induced map $(F, f)_*\colon \pi_n(X, A) \to \pi_n(Y, B)$ is an isomorphism for $1 \le n \le p+q-3$ and is an epimorphism for $n = p+q-2$.

Proof. Replace f by a cofibration

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, hook, "j"'] & Z(f) \arrow[r, "p"] \arrow[d] & B \arrow[d] \\ X \arrow[r, "K"] & Z \arrow[r, "P"] & Y \end{tikzcd}$$

where $Z = Z(f) \sqcup X/(a, 0) \sim j(a)$, $f = p \circ k$, $F = P \circ K$. Since $p\colon Z(f) \to B$ is a homotopy equivelence and $P\colon Z \to Y$ is given by push-out, P is also a homotopy equivalence. Let $Z = Z_1 \cup Z_2$ where $Z_2 = X \sqcup A \times (\varepsilon, 1]/\sim$ and $Z_1 = B \sqcup A \times [0, \varepsilon)/\sim$. Then $Z_1 \cap Z_2 = A \times (\varepsilon, 1-\varepsilon)$. Applying excision (Theorem 2.16),

$$\pi_n(X, A) \cong \pi_n(Z_2, Z_0) \to \pi_n(Z, Z_1) \cong \pi_n(Y, B)$$

has desired properties. □

Theorem 2.18 (Quotient)**.** Let $A \hookrightarrow X$ be a cofibration. Suppose $\pi_i(CA, A) = 0$ for $0 < i < p$ and $\pi_i(X, A) = 0$ for $0 < i < q$. Then $p_*\colon \pi_n(X, A) \to \pi_n(X/A, *)$ is an isomorphism for $1 \le n \le p+q-3$ and is an epimorphism for $n = p+q-2$.

Proof. Note $X \cup CA$ fits into the following push-out diagram.

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & CA \arrow[d] \\ X \arrow[r] & X \cup CA \end{tikzcd}$$

Then we get the result for

$$\pi_n(X, A) \to \pi_n(X \cup CA, CA).$$

Since $A \hookrightarrow X$ is a cofibration, $CA \hookrightarrow X \cup CA$ is also a cofibration. Notice that because CA is contractible, $X \cup CA \to X \cup CA/CA$ is a homotopy equivalence (This is left as an exercise). Then

$$\pi_n(X, A) \to \pi_n(X \cup CA, CA) \cong \pi_n(X \cup CA/CA, *) \cong \pi_n(X/A, *)$$

has desired properties. □

Definition 2.19. We say (X, x_0) is well-pointed if $x_0 \hookrightarrow X$ is a cofibration.

Example 2.20.
- For any CW-complex or manifold, it is well-pointed for any point.
- $X = \left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\} \cup \{0\}$, $x_0 = 0$ is not well-pointed.

Theorem 2.21 (Freudenthal Suspension)**.** Let (X, x_0) be a well-pointed n-connected space. Then $\Sigma_* \colon \pi_j(X) \to \pi_{j+1}(\Sigma X)$ is an isomorphism for $0 \le j \le 2n$ and is an epimorphism for $j = 2n + 1$.

Proof. The suspension map is given by

$$\pi_j(X) = \left[S^j, X\right]^o \xrightarrow{\Sigma_*} \left[S^{j+1}, \Sigma X\right]^o = \pi_{j+1}(X) \ .$$

We factor Σ_* into

$$\begin{tikzcd} \Sigma_* \colon \pi_j(X) & \pi_{j+1}(CX, X) \arrow[l, "\partial"', "\cong"] \arrow[d, "p_*"] \\ & \pi_{j+1}(\Sigma X) \end{tikzcd}$$

To use Theorem 2.18, we verify $X \hookrightarrow CX$ is a cofibration. Consider the push-out diagram

$$\begin{tikzcd} X \times \partial I \cup \{x_0\} \times I \arrow[r] \arrow[d] & X \arrow[d] \\ X \times I \arrow[r] & CX \end{tikzcd}$$

where $CX = X \times I / X \times \{0\} \cup \{x_0\} \times I$. Because $\partial I \hookrightarrow I$ and $x_0 \hookrightarrow X$ are cofibrations, we have $\{x_0\} \times I \cup X \times \partial X \hookrightarrow X \times I$ is also a cofibration. By push-out diagram, $X \hookrightarrow CX$ is a cofibration. Now we have exact sequence

$$\begin{tikzcd} \pi_j(CX, X)\pi_{j-1}(X) \arrow[r] & 0 \\ \pi_j(CX) = 0 \arrow[u] & \\ \pi_j(X) \arrow[u] & \end{tikzcd}$$

Then (CX, X) is $(n+1)$-connected. And $p_* \colon \pi_j(CX, X) \to \pi_j(\Sigma X)$ is isomorphism for $j \le 2n - 1$ and is an epimorphism for $j = 2n$. Then we apply Theorem 2.18 with $p = q = n + 2$ and get the desired properties for $\Sigma_* \colon \pi_{j-1}(X) \to \pi_j(X)$. $\square$

2.6 Computation of Homotopy Groups

Example 2.22.

$$\pi_k\left(S^n\right) \cong \begin{cases} 0, k < n \\ \mathbb{Z}, k = n \end{cases}.$$

$$\pi_1\left(S^1\right) \cong \mathbb{Z}, \qquad \pi_1\left(S^n\right) \cong 0, \ \forall n \ge 2.$$

To compute $\pi_2\left(S^2\right)$, consider the Hopf fibration

$$S^1 \hookrightarrow S^3 \longrightarrow S^2 \ .$$

This is given by the fibre bundle

$$S^2 = \mathbb{CP}^1 = \mathbb{C}^2 - \{0\}/\mathbb{C}^* = S^3/S^1.$$

We have the following fibre sequence

$$\begin{tikzcd}
\pi_2\left(S^1\right) \arrow[r] \arrow[d, equal] & \pi_2\left(S^3\right) \arrow[r] & \pi_2\left(S^2\right) \arrow[r, "\partial"] & \pi_1\left(S^1\right) \arrow[r] \arrow[d, equal] & \pi_1\left(S^3\right) \arrow[d, equal] \\
0 & & & \mathbb{Z} & 0
\end{tikzcd}$$

Because S^1 is 0-connected, by Suspension Theorem, $\pi_1\left(S^1\right) \to \pi_2\left(S^2\right)$ is an epimorphism. Then $\pi_2\left(S^2\right) \cong \mathbb{Z}$ and $\pi_2\left(S^3\right) = 0$.

For $n \geq 2$, assume S^n is $(n-1)$-connected, by Freudenthal's Suspension, $\pi_j\left(S^n\right) \to \pi_{j+1}\left(S^{n+1}\right)$ is an isomorphism for $j \leq n \leq 2n$. By induction, $\pi_n\left(S^n\right) \cong \mathbb{Z}$ and $\pi_j\left(S^n\right) = 0$ for $j < n$.

Example 2.23. Notice that

$$\mathbb{CP}^n = \mathbb{C}^{n+1} - \{0\}/\mathbb{C}^* = S^{2n+1}/U(1)$$

for $n \geq 2$, we get a fibre bundle

$$\begin{tikzcd}
U(1) \arrow[r, hook] & S^{2n+1} \arrow[r] & \mathbb{CP}^n
\end{tikzcd}.$$

Then we have fibre sequence

$$\begin{tikzcd}
\pi_j\left(S^{2n+1}\right) \arrow[r] & \pi_j\left(\mathbb{CP}^n\right) \pi_{j-1}(U(1)) \arrow[r] & \pi_{j-1}\left(S^{2n+1}\right)
\end{tikzcd}.$$

Then when $j = 2$, $\pi_2\left(\mathbb{CP}^n\right) \cong \mathbb{Z}$. When $2 \neq j \leq 2n$, $\pi_j\left(\mathbb{CP}^n\right) = 0$.

Consider $\mathbb{CP}^\infty = \bigcup_{n \geq 1} \mathbb{CP}^n$,

$$\begin{tikzcd}
\mathbb{CP}^n \arrow[r, hook] & \mathbb{CP}^{n+1} \\
S^{2n+1} \arrow[r, hook] \arrow[u] & S^{2n+3} \arrow[u] \\
U(1) \arrow[u, hook] & U(1) \arrow[u, hook]
\end{tikzcd}$$

is induced from Five-Lemma. Then $i_* \colon \pi_2\left(\mathbb{CP}^n\right) \to \pi_2\left(\mathbb{CP}^{n+1}\right)$ is an isomorphism. As conclusion,

$$\pi_n\left(\mathbb{CP}^\infty\right) \cong \begin{cases} \mathbb{Z}, & n = 2 \\ 0, & n \neq 2. \end{cases}$$

Example 2.24. We have the following fibre bundle by transitive group action.

$$\begin{tikzcd}
\mathrm{O}(n) \arrow[r, hook, "j"] & \mathrm{O}(n+1) \arrow[r] & S^n
\end{tikzcd}.$$

Since S^n is $(n-1)$-connected, the homotopy exact sequence for fibrations show $j \colon \mathrm{O}(n) \hookrightarrow \mathrm{O}(n+1)$ is $(n-1)$-connected.

Write $\mathrm{O}(\infty) = \bigcup_{n=1}^{\infty} \mathrm{O}(n)$.

Theorem 2.25 (Bott-Periodicity)**.**

$$\pi_k(\mathrm{O}(\infty)) \cong \pi_{k+8}(\mathrm{O}(\infty)).$$

Example 2.26 (Stiefel Manifolds)**.** Denote $\mathrm{V}_k(\mathbb{R}^n)$ be the orthogonal k-frames in $\mathbb{R}^n$. Then we have

$$\mathrm{V}_k(\mathbb{R}^n) = \mathrm{O}(n)/\,\mathrm{O}(n-k).$$

Then we get a fibration

$$\mathrm{O}(n-k) \hookrightarrow \mathrm{O}(n) \longrightarrow \mathrm{V}_k(\mathbb{R}^n)\ .$$

Notice that in

$$\mathrm{O}(n-k) \xhookrightarrow{j} \mathrm{O}(n-k+1) \hookrightarrow \cdots \hookrightarrow \mathrm{O}(n)\ ,$$

j is $(n-k-1)$-connected, then

$$\pi_i(\mathrm{O}(n-k)) \xrightarrow{\cong} \pi_i(\mathrm{O}(n)) \longrightarrow \pi_i(\mathrm{V}_k(\mathbb{R}^n))$$

for $i \leq n-k-2$. Therefore, $\pi_i(\mathrm{V}_k(\mathbb{R}^n)) = 0$ when $i \leq n-k-1$.

Claim 9. $\mathrm{V}_k(\mathbb{R}^n)$ is $(n-k-1)$-connected.

Consider the projection

$$\begin{aligned} p\colon \mathrm{V}_{k+1}\left(\mathbb{R}^{n+1}\right) &\to \mathrm{V}_1\left(\mathbb{R}^{n+1}\right) \cong S^n \\ (v_1,\cdots,v_{k+1}) &\mapsto v_{k+1}. \end{aligned}$$

The fibre is $\mathrm{V}_k(\mathbb{R}^n)$. We know S^n is $(n-1)$-connected, then $j\colon \mathrm{V}_k(\mathbb{R}^n) \to \mathrm{V}_{k+1}\left(\mathbb{R}^{n+1}\right)$ is $(n-1)$-connected. Therefore, we have $\pi_{n-k}(\mathrm{V}_k(\mathbb{R}^n)) \cong \pi_{n-k}\left(\mathrm{V}_2\left(\mathbb{R}^{n-k+2}\right)\right)$. We know that $\pi_1\left(\mathrm{V}_2\left(\mathbb{R}^{n-k+2}\right)\right) = 0$. By Hurewicz Theorem, $H_i\left(\mathrm{V}_2\left(\mathbb{R}^{n-k+2}\right)\right) \cong \pi_i\left(\mathrm{V}_2\left(\mathbb{R}^{n-k+2}\right)\right)$ for $2 \leq i \leq n-k$, which is non-trivial. We will do these calculations later.

Part II
Generalized Homology

3 Homology Theory and CW-Complexes

3.1 Homology Theory

Denote $R-\mathbf{MOD}$ be the categpry of left R-modules and $\mathbf{TOP}(2)$ be the categpry of pairs (X,A) and

$$\begin{aligned} k\colon \mathbf{TOP}(2) &\to \mathbf{TOP}(2) \\ (X,A) &\mapsto (A,\varnothing) \end{aligned}$$

be the forgetful functor.

Definition 3.1 (Eilenberg-Steenrod Axioms)**.** A homology theory on $\mathbf{TOP}(2)$ consists

1. a family of functors $h_n\colon \mathbf{TOP}(2) \to R-\mathbf{MOD}$,
2. a family of natural transformations $\partial_n\colon h_n \to h_{n-1}\circ k$ such that
   (a) Homotopy invariance: $h_n(f_0) = h_n(f_1)$ for $f_0 \simeq f_1$.
   (b) Exact sequence:
   $$\cdots \longrightarrow h_{n+1}(X,A) \xrightarrow{\partial_{n+1}} h_n(A) \longrightarrow h_n(X) \longrightarrow h_n(X,A) \longrightarrow \cdots$$
   for any pair (X,A).
   (c) Excison: Given a pair (X,A), for any $U\subset A$ such that $\overline{U}\subset \mathrm{Int}(A)$, then inclusion induces an isomorphism $h_n(X-U, A-U)\to h_n(X,A)$.

Proposition 3.2. Given two pairs (X_i,A_i), $i=1,2$, we get an isomorphism

$$\oplus_{i=1}^2 h_n(X_i,A_i) \to h_n(X_1\sqcup X_2, A_1\sqcup A_2).$$

Proof. Consider the commutative diagram for $A_i=\varnothing$.

$$\begin{tikzcd}
h_n(X_1\sqcup X_2, X_2) & & & & h_n(X_1\sqcup X_2, X_1) \\
& & h_n(X_1\sqcup X_2) \arrow[ull, "j_1"'] \arrow[urr, "j_2"] & & \\
h_n(X_1) \arrow[uu, "a_1"] \arrow[urr, "i_1"'] & & & & h_n(X_2) \arrow[uu, "a_2"', "\cong"] \arrow[ull, "i_2"]
\end{tikzcd}$$

Injectivity of $i_1\oplus i_2$ is easy to check. For its surjectivity, take $c\in h_n(X_1\sqcup X_2)$, we have $j_1(c) = j_1\circ i_1\circ a_1^{-1}(j_1(c))$. Then $c - i_1\circ a_1^{-1}(j_1(c))\in \ker j_1$. Therefore, there exists $x\in h_n(X_2)$ such that $i_2(x) = c - i_1\left(a_1^{-1}\circ j_1(c)\right)$. Then $c = i_1(y)+i_2(x)$ where $y = a_1^{-1}\circ j_1(c)\in h_n(X_1)$.

The general case will be proved later. □

Let $A=*$ be a single point. Define $\widetilde{h}(X) \coloneqq h(X,*)$.

Assume there is a map $r\colon X\to A$ such that $r\circ i\simeq \mathrm{id}$. Then $i_*\colon h_n(A)\to h_n(X)$ is injective. We get short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & h_n(A) \arrow[r, shift left, "i_*"] & h_n(X) \arrow[l, shift left, "r_*"] \arrow[r] & h_n(X,A) \arrow[r] & 0 .
\end{tikzcd}$$

Then we have splitting $h_n(X) \cong h_n(A) \oplus h_n(X,A)$ and $h_n(X,A) = \ker r_*$. When $A = *$, take $r = c\colon X \to *$, then $\widetilde{h_n}(X) = h_n(X,*) = \ker(c_*\colon h_n(X) \to h_n(*))$.

Proposition 3.3. Let $A \hookrightarrow X$ be a cofibration. Then the quotient map induces an isomorphism $j_*\colon h_n(X,A) \to h_n(X/A,*)$.

Proof. Apply excision to $(X \cup CA, CA)$ for $U =$ the cone point of CA, we have $h_n(X,A) \cong h_n(X \cup CA, CA)$. When $A \hookrightarrow X$ is a cofibration, $CA \hookrightarrow X \cup CA$ is a cofibre. Since CA is contractible, $X \cup CA/CA \simeq X \cup CA$. Then $h_n(X \cup CA, CA) \cong h_n(X/A,*)$. □

Proposition 3.4. Let $(X,*)$ and $(Y,*)$ be well-pointed spaces and $f\colon X \to Y$ is a pointed map. Then the cofibre sequence $X \xrightarrow{f} Y \xrightarrow{f^1} C(f)$ induces an exact sequence

$$\widetilde{h_n}(X) \xrightarrow[f_*]{} \widetilde{h_n}(Y) \xrightarrow{f^1_*} \widetilde{h_n}(C(f)) \ .$$

Proof. The proof follows the commutative diagrams

$$\begin{tikzcd}
\widetilde{h_n}(X) \arrow[r] \arrow[d, "\cong"'] & \widetilde{h_n}(Z(f)) \arrow[r] \arrow[d, "\cong"'] & \widetilde{h_n}(Z(f),X) \arrow[d, "\cong"'] \\
\widetilde{h_n}(X) \arrow[r] & \widetilde{h_n}(Y) \arrow[r] & \widetilde{h_n}(C(f))
\end{tikzcd}$$

and

$$\begin{tikzcd}
X \times \partial I \arrow[r, "(\mathrm{id},f)"] \arrow[d] & X \sqcup Y \arrow[d] \\
X \times I \arrow[r] & Z(f)
\end{tikzcd}$$

□

Proposition 3.5. Given a triple (X,A,B). Assume $B \hookrightarrow X$ is a cofibration, we get an exact sequence

$$\cdots \longrightarrow h_n(A,B) \longrightarrow h_n(X,B) \longrightarrow h_n(X,A) \xrightarrow{\partial} h_{n-1}(A,B) \longrightarrow \cdots .$$

Proof. Applying excision, we know that (X,A,B) and $(X \cup CB, A \cup CB, CB)$ have the same sequence. Applying homotopy equivalence, $(X \cup CB, A \cup CB, CB)$ and $(X,A,*)$ have the same sequence. The triple sequence of $(X,A,*)$ is the reduced pair sequence of (X,A). □

3.1.1 Suspension Isomorphism

Given a pair (X,A), we have the suspension isomorphism

$$\sigma\colon h_n(X,A) \to h_n(\partial I \times X \cup I \times A, \{0\} \times X \cup I \times A)$$

by excision for $U = (0,1] \times A \cup \{0\} \times X$. Consider the boundary map $\partial_{n+1}\colon h_{n+1}(I \times X, \partial I \times X \cup I \times A) \to h_n(\partial I \times X \cup I \times A, \{0\} \times X \cup I \times A)$. Notice that $X \simeq I \times X \simeq \{0\} \times X \cup I \times A$, we have the exact sequence

$$h_{n+1}(I \times X, \partial I \times X \cup I \times A) \xrightarrow{\partial_{n+1}} h_n(\partial I \times X \cup I \times A, \{0\} \times X \cup I \times A) \longrightarrow h_n(I \times X, \{0\} \times X \cup I \times A) = 0 \ .$$

Then ∂_{n+1} is an isomorphism and so is ∂^{-1}_{n+1}. We get isomorphisms

$$h_n(x,A) \xrightarrow{\sigma} h_n(\partial I \times X \cup I \times A, \{0\} \times X \cup I \times A) \xrightarrow{\partial^{-1}_{n+1}} h_{n+1}((I,\partial I) \times (X,A)) \ .$$

Choose $A = *$, define the suspension isomorphism by

$$\begin{tikzcd}
h_n(X,*) \arrow[r] \arrow[d, "\cong"'] & h^{\sigma}_{n+1}(X \times I, \partial I \times X \cup I \times *) \arrow[d, "\text{quotient}"] \\
\widetilde{h_n}(X) \arrow[r, "\widetilde{\sigma}"] & \widetilde{h_{n+1}}(\Sigma X)
\end{tikzcd}$$

Assume $(X,*)$ is well-pointed, by Hurwicz map, we have the commutative diagram

$$\begin{tikzcd}
\pi_n(X) \arrow[r, "\Sigma_*"] \arrow[d] & \pi_n(\Sigma X) \arrow[d] \\
\widetilde{h_n}(X) \arrow[r, "\widetilde{\sigma}"] & \widetilde{h_{n+1}}(X)
\end{tikzcd}$$

3.2 CW-Complex

Definition 3.6. We say X is obtained from A by attaching an n-cell if there exists a push-out diagram

$$\begin{tikzcd}
S^{n-1} \arrow[r, "\varphi"] \arrow[d] & A \arrow[d] \\
D^n \arrow[r, "\Phi"] & X
\end{tikzcd}$$

where φ is called attaching map and Φ is called characteristic map.

A CW-decomposition of (X, A) is a filtration $A = X^{-1} \subset X^0 \subset \cdots \subset X$ such that

1. $X = \bigcup_{n \geq -1} X^n$,
2. X^n is obtained from X^{n-1} by attaching n-cells,
3. X carries the colimit topology (weak topology).

Proposition 3.7. Let (Y, B) be an n-connected pair, (X, A) be a relative CW-complex of dimension $\leq n$. Then each map $(F, f)\colon (X, A) \to (Y, B)$ is homotopic rel. A to a map into B. When dimension $< n$, the homotopy class rel. A of maps $X \to B$ is unique.

Proof. Consider

$$\begin{tikzcd}
\bigsqcup_k S_k^{q-1} \arrow[r] \arrow[d, hook] & A \arrow[r] \arrow[d] & B \arrow[d] \\
\bigsqcup_k D_k^q \arrow[r, "\Phi^q"] & X^q \arrow[r, "F^q"] & Y
\end{tikzcd}$$

For any $q \leq n$, $\pi_q(Y, B) = 0$. Then $F^q \circ \Phi^q$ can be homotoped into B rel. $\bigsqcup_k S_k^{q-1}$.

$$\begin{tikzcd}
\bigsqcup_k S_k^{q-1} \arrow[r] \arrow[d, hook] & A \arrow[r] \arrow[d] & B \arrow[d] \\
\bigsqcup_k D_k^q \arrow[r, "\Phi^q"] \arrow[urr, dashed] & X^q \arrow[r, "F^q"] \arrow[ur, dashed] & Y
\end{tikzcd}$$

When dimension of $(X, A) < n$, apply the argument to $(X \times I, X \times \partial I \cup A \times I)$ which is a relative CW-complex of dimension $< n + 1$. $\square$

Theorem 3.8. Suppose $h\colon B \to Y$ is n-connected. Then for a CW-complex X, $h_*\colon [X, B] \to [X, Y]$ is bijective when $\dim X < n$ and surjective when $\dim X = n$.

Proof. We map replace Y by $Z(h)$: $B \longrightarrow Z(h) \xrightarrow{\cong} Y$.

Surjectivity: Let $A = \varnothing$. Apply Proposition 3.7 to $(X, \varnothing) \to (Z(h), B)$.

Injectivity: Apply Proposition 3.7 to $(X \times I, X \times \partial I)$. $\square$

Theorem 3.9 (Whitehead)**.** Let $f\colon Y \to Z$ be a map between CW-complexes with $\dim Y, \dim Z \leq n \leq \infty$. If $f_*\colon \pi_q(Y) \to \pi_q(Z)$ is an isomorphism for $0 \leq q \leq n$, then f is a homotopy equivalence.

Proof. The map $f\colon Y \to Z$ is n-connected. By Theorem 3.8, $f_*\colon [Z, Y] \to [Z, Z]$ is surjective. Then there exists $g\colon Z \to Y$ such that $f \circ g \simeq \mathrm{id}_Z$ and g is n-connected. Use Theorem 3.8 again, there exists $h\colon Y \to Z$ such that $g \circ h \simeq \mathrm{id}_Y$. Therefore, g is a homotopy equivalence. $\square$

Theorem 3.10 (Suspension Theorem)**.** Suppose Y is n-connected and X is a CW-complex. Then $\Sigma_*\colon [X, Y]^o \to [\Sigma X, \Sigma Y]^o$ is bijective if $\dim X \leq 2n$ and is surjective if $\dim X = 2n + 1$.

Proof. We know that $[\Sigma X, \Sigma Y]^o \cong [X, \Omega\Sigma Y]^o$. By Freudethal's Suspension Theorem, $\Sigma_*\colon \left[S^k, Y\right]^o \to \left[S^{k-1}, \Sigma Y\right]^o$ is an isomorphism when $k \leq 2n$ and epimorphism if $k = 2n + 1$. Notice that $\pi_{k+1}(\Sigma Y) \cong \pi_k(\Omega\Sigma Y)$, $\sigma_*\colon \left[S^k, Y\right]^o \to \left[S^k, \Omega\Sigma Y\right]^o$ is adjoint to Σ_* and is reduced from

$$\begin{aligned} \sigma\colon Y &\to \Omega\Sigma Y \\ y &\mapsto [t \mapsto (y, t)]. \end{aligned}$$

Therefore, σ is $(2n + 1)$-connected. Apply Theorem 3.8 to $\sigma_*\colon [X, Y]^o \to [X, \Omega\Sigma Y]^o$. $\square$

3.3 CW-Approximation

Proposition 3.11. Suppose X is obtained from A by attaching $(n+1)$-cell. Then (X, A) is n-connected.

Proof. Consider the push-out diagram

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & A \arrow[d] \\ D^{n+1} \arrow[r] & X \end{tikzcd}$$

The Excision Theorem of push-out shows that $\pi_0(X, A) = 0$ and $\pi_q\left(D^{n+1}, S^n\right) = 0$ for any $1 \leq q \leq n$. Then $(\Phi, \varphi)\colon \left(D^{n+1}, S^n\right) \to (X, A)$ is $(n-1)$-connected. When $k \leq n - 1$, $0 = \pi_k\left(D^{n+1}, S^n\right) \to \pi_n(X, A)$ is an isomorphism. $\square$

Theorem 3.12. Let $f\colon A \to Y$ be a k-connected map. Then for each $n > k$, there exists a relative CW-complex (X, A) with cells in dim $\in \{k+1, \cdots, n\}$ and an n-connected extension $F\colon X \to Y$ of f.

Proof. When $n = 1$, $k = 0$, the proof is trivial. Consider $k = n - 1$, $n \geq 2$. Assume $f\colon A \to Y$ is $(n-1)$-connected. Replace Y by $Z(f)$:

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & Z(f) \arrow[r] & Y \\ X \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

Assume $f\colon A \to Y$ is an inclusion. Let $(\Phi_j, \varphi_j)\colon \left(D^n, S^{n-1}\right) \to (Y, A)$ be a set of generators of $\pi_n(Y, A)$. Attach n-cells on A using φ_j. Regard Φ_j as a null-homotopy of $f \circ \varphi_j$. F is obtained by push-out property.

$$\begin{tikzcd} S^{n-1} \arrow[r, "\varphi_j"] \arrow[d] & A \arrow[r, "f"] \arrow[d, "\Phi_j"'] & Y \\ D^n \arrow[r] \arrow[urr] & X \arrow[ur, dashed] & \end{tikzcd}$$

And then $F_*\colon \pi_n(X,A) \to \pi_n(Y,A)$ is an epimorphism.

Cnosider the diagram

$$
\begin{tikzcd}
\pi_n(A) \arrow[r] \arrow[d, "\cong"] & \pi_n(X) \arrow[r] \arrow[d, "F_*"] & \pi_n(X,A) \arrow[r] \arrow[d, "F_*"] & \pi_{n-1}(A) \arrow[r] \arrow[d, "\cong"] & \pi_{n-1}(X) \arrow[r] \arrow[d, "F_*"] & 0 \arrow[d] \\
\pi_n(A) \arrow[r] & \pi_n(Y) \arrow[r] & \pi_n(Y,A) \arrow[r] & \pi_{n-1}(A) \arrow[r, "f_*"'] & \pi_{n-1}(Y) \arrow[r] & 0
\end{tikzcd}
$$

Notice that $F_*\colon \pi_n(X) \to \pi_n(Y)$ is also an epimorphism. Then by chasing diagram, we know that $F_*\colon \pi_{n-1}(X) \to \pi_{n-1}(Y)$ is an isomorphism. □

Corollary 3.13. Given any space Y, there exists a CW-complex X and a map $F\colon X \to Y$ such that $F_*\colon \pi_n(X) \to \pi_n(Y)$ is an isomorphism for any $n \geq 0$. Such X is called a CW-approximation of Y.

Theorem 3.14. Let Y be a k-connected CW-complex. Then there exists a CW-complex X such that

1. X is homotopy equivalent to Y;
2. $X^k = \{*\}$.

Proof. Apply Theorem3.12 to $A = \{*\} \hookrightarrow Y$ which is a k-connected map. □

3.4 Eilenberg-MacLane Space

3.4.1 Remarks about Compactly Generated Spaces

Definition 3.15. A Hausdorff space X is said to be compactly generated if for any compact subset K, a subset $A \subset X$ satisfies $A \cap K$ is closed, then A is closed in X.

Example 3.16. There spaces are compactly generated spaces:

- locally compact Hausdorff spaces,
- metric spaces,
- CW-complexes with finite cells in each dimension.

Given a Hausdorff space X, we can put a new topology $\mathcal{T}$ on X by imposing:

$$A \subset X \text{ is } \mathcal{T}\text{-closed} \iff A \cap K \text{ is closed for any compact subset } K \subset X$$

such that X is compactly generated under $\mathcal{T}$.

Fact 3.17. If X, Y are both compactly generated spaces, then $X \times Y$ needs not to be compactly generated.

Definition 3.18. We deonte by $X \times_k Y$ the product with compactly generated topology. We denote by $kF(X,Y)$ the space of continuous maps from X to Y, equipped the compactly generated topology.

Theorem 3.19. Let X, Y, Z be compactly generated spaces. Then

1. The evaluation map

$$\begin{aligned} kF(Y,Z) \times_k Y &\to Z \\ (f,g) &\mapsto f(g) \end{aligned}$$

is continuous.

2. The adjoint map
$$kF(X, kF(Y,Z)) \to kF\left(X \times_k Y, Z\right)$$
is a homeomorphism.

Proposition 3.20. Suppose $\pi_j(Y) = 0$ for $j > n$. Let X be obtained from A by attaching cells of $\dim \geq n+2$. Then $\iota_*\colon [X,Y] \to [A,Y]$ is a bijection.

Proof. **Surjectivity**: Given $f\colon A \to Y$ and attaching map $\varphi\colon S^k \to A$, $k \geq n+1$. Then $f \circ \varphi\colon S^k \to Y$ is null-homotopic which can be extended over X.

Injectivity: Apply the argument to $(X \times I, X \times \partial I \cup A \times I)$. □

Definition 3.21. Let π be an abelian group. An Eilenberg-MacLane space of type $K(\pi, n)$ is a CW-complex such that
$$\pi_j(X) = \begin{cases} \pi, & i = j; \\ 0, & n \neq j. \end{cases}$$

Proposition 3.22. Suppose X_1, X_2 are $(n-1)$-connected CW-complex with $n \geq 2$. Then
$$\pi_n\left(X_1\right) \oplus \pi_n\left(X_2\right) \to \pi_n\left(X_1 \vee X_2\right)$$
is an isomorphism.

Proof. We can assume $X_i^{n-1} = \{*\}$ by CW-approximation. Therefore, cells in $X_1 \times X_2$ have dimension $0, n, \geq 2n$. Then $X_1 \times X_2$ is obtained from $X_1 \vee X_2$ by attaching cells of dim $\geq 2n$. We have $\pi_n\left(X_1 \vee X_2\right) \to \pi_n\left(X_1 \times X_2\right) = \pi_n\left(X_1\right) \oplus \pi_n\left(X_2\right)$ is an isomorphism. □

Theorem 3.23. Let X be a $(n-1)$-connected CW-complex. Suppose Y satisfies $\pi_j(Y) = 0, \forall j > n \geq 2$. Then the map $h_*\colon [X,Y]^o \to \operatorname{Hom}\left(\pi_n(X), \pi_n(Y)\right)$ is a bijection.

Proof. We can assume $X^{n-1} = \{*\}$ by Proposition 3.20. Then $[X,Y]^o = \left[X^{n+1}, Y\right]^o$. Notice that $\pi_n\left(X^{n+1}\right) = \pi_n(X)$, we only need to prove $h_X\colon \left[X^{n+1}, Y\right]^o \to \operatorname{Hom}\left(\pi_n\left(X^{n+1}\right), \pi_n(Y)\right)$ is a bijection.

We know $X^n = \bigvee_j S_j^n := B$. Applying homotopy, we may assume all attaching maps of $(n+1)$-cells are cased. Then X^{n+1} is the mapping cone $f\colon A := \bigvee_k S_k^n \to \bigvee S_j^n = B$.

We have the cofibre sequence
$$[A,Y]^o \longleftarrow [B,Y]^o \longleftarrow \left[X^{n+1}, Y\right]^o \longleftarrow [\Sigma A, Y]^o \longleftarrow \cdots.$$
Notice that
$$[\Sigma A, Y]^o = \left[\Sigma \bigvee_k S_k^n, Y\right]^o = \left[\bigvee_k \Sigma S_k^n, Y\right]^o = \left[\bigvee_k S_k^{n+1}, Y\right]^o = 0$$
because $[h] = \sum_k [h_k]$ and $\pi_{n+1}(Y) = 0$.

Claim 10.
$$\pi_n(A) \xrightarrow{f_*} \pi_n(B) \longrightarrow \pi_n\left(X^{n+1}\right) \longrightarrow 0$$
is exact.

Proof of Claim. Consider the push-out diagram:
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow \\ CA & \longrightarrow & X^{n+1} \end{array}$$

We know

$$\begin{array}{ccccccccc}
\pi_m(A) & \longrightarrow & \pi_m(CA) & \longrightarrow & \pi_m(CA,A) & \xrightarrow{\cong} & \pi_{m-1}(A) & \longrightarrow & 0 \\
 & & \| & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

Then $\pi_m(CA,A)=0$ for any $m\leq n$. We know f is $(n-1)$-connected. Applying excision, $\pi_m(CA,A)\to\pi_m\left(X^{n+1},B\right)$ is an isomorphism for $m\leq 2n-1$. We have an exact sequence

$$\begin{array}{ccccccc}
\pi_m(B) & \longrightarrow & \pi_m\left(X^{n+1}\right) & \longrightarrow & \pi_m\left(X^{n+1},B\right) & \longrightarrow & \pi_{m-1}(B)0 \\
 & & & & \| & & \\
 & & & & 0 & &
\end{array}$$

when $m\leq n$. Then

$$\begin{array}{ccccccccc}
\pi_{n+1}(CA,A) & \xrightarrow[\cong]{\text{excision}} & \pi_{n+1}\left(X^{n+1},B\right) & \longrightarrow & \pi_n(B) & \longrightarrow & \pi_n(X) & \longrightarrow & 0 \\
\partial\big\downarrow\cong & & & & & & & & \\
\pi_n(A) & & & & & & & &
\end{array}$$

$\square$

Apply $\operatorname{Hom}(-,\pi_n(Y))$, we get an exact sequence

$$\begin{array}{ccccccc}
\operatorname{Hom}(\pi_n(A),\pi_n(Y)) & \longleftarrow & \operatorname{Hom}(\pi_n(B),\pi_n(Y)) & \longleftarrow & \operatorname{Hom}\left(\pi_n\left(X^{n+1}\right),\pi_n(Y)\right) & \longleftarrow & 0 \\
h_A\big\uparrow & & h_B\big\uparrow & & h_X\big\uparrow & & \big\uparrow\cong \\
[A,Y]^o & \longleftarrow & [B,Y]^o & \longleftarrow & \left[X^{n+1},Y\right]^o & \longleftarrow & 0
\end{array}$$

Claim 11. h_A and h_B are bijections.

Proof of Claim. We have

$$\begin{aligned}
\operatorname{Hom}(\pi_n(A),\pi_n(Y)) &= \operatorname{Hom}\left(\pi_n\left(\bigvee_j S_j^n\right),\pi_n(Y)\right) = \operatorname{Hom}\left(\bigoplus_j \pi_n\left(S_j^n\right),\pi_n(Y)\right) \\
&= \prod_j \operatorname{Hom}\left(\pi_n\left(S_j^n\right),\pi_n(Y)\right) \cong \prod_j \pi_n(Y)
\end{aligned}$$

and

$$[A,Y]^o = \left[\bigvee_j S_j^n, Y\right]^o = \prod_j \left[S_j^n,Y\right]^o = \prod_j \pi_n(Y).$$

$\square$

Finally, by claim that $\left[X^{n+1},Y\right]^o\to[B,Y]^o$ is injective, we get our conclusion by something like Five Lemma. $\square$

Theorem 3.24. Let π be an abelian group and $n\geq 2$. Then the Eilenberg-MacLane space $K(\pi,n)$ exists and is unique up to homotopy.

Proof. **Uniqueness**: Assume X, Y are both $K(\pi, n)$. Then by Theorem 3.23,

$$h_X \colon [X, Y]^o \to \operatorname{Hom}(\pi_n(X), \pi_n(Y)) = \operatorname{Hom}(\pi, \pi)$$

is a bijection. Choose $f \colon X \to Y$ such that $h_X([f]) = \mathrm{id}$. Then f is a weak homotopy equivalence. Whitehead Theorem gives us that f is in fact a homotopy equivalence.

Existence: Consider a free resolution

$$F_1 \longrightarrow F_0 \longrightarrow \pi \longrightarrow 0$$

with relators F_1 and generators F_0. Construct X^{n+1} as the mapping cone of $g \colon F_1 \hookrightarrow \bigvee_k S^n_k \to \bigvee_j S^n_j \hookleftarrow F_0$. Therefore, X^{n+1} is $(n+1)$-connected and $\pi_n\left(X^{n+1}\right) = \pi$. We attach cells of dim $\geq n+2$ to eliminate $\pi_m(X)$ for $m \geq n+1$, by Zorn's Lemma, we finish our construction. □

Definition 3.25. $K(\pi, 0) \coloneqq \pi$ equipped with discrete topology. $K(\pi, 1)$ is constructed similar to Theorem 3.24, but the uniqueness will be proved later.

3.5 Spectral Homology

In this section, we assume that π is finitely generated and X is compactly generated.

Definition 3.26. A spectrum is a sequence of pairs $\{(E_n, e_n)\}_{n \geq 0}$ where $E(n)$ is a pointed space, $e_n \colon \Sigma E(n) \to E(n+1)$ is a pointed map. We say a spectrum is an Ω-spectrum if $\varepsilon_n \colon E(n) \to \Omega E(n+1)$ is a homotopy equivalence, where ε_n is the adjoint of e_n.

Example 3.27.

1. Sphere Spectrum: $E(n) = S^n$, $e_n \colon \Sigma S^n \to S^{n+1}$ is the identity map

$$\Sigma S^n = S^n \wedge S^1 \cong S^{n+1}$$
$$\mathbb{R}^{n+1} \times I \hookrightarrow \mathbb{R}^{n+2}.$$

2. Eilenberg-MacLane Spectrum: Fix an abelian group π. Let $E(n) = K(\pi, n)$. Construct $e_n \colon \Sigma K(\pi, n) \to K(\pi, n+1)$ as follows:
   (a) Milnor: $\Omega K(\pi, n+1)$ is a CW-complex. Then $\left[S^k, \Omega K(\pi, n+1)\right]^o = \left[S^{k+1}, K(\pi, n+1)\right]^o$ and then $\Omega K(\pi, n+1) \cong K(\pi, n)$. Define $e_n \colon \Sigma K(\pi, n) \to K(\pi, n+1)$ as the adjoint map; or
   (b) Notice that $\pi_k\left(\Sigma K(\pi, n)\right) = \begin{cases} 0, & k \leq n \\ \pi, & k = n+1 \end{cases}$ because $\pi_k(K(\pi, n)) \to \pi_{k+1}(\Sigma K(\pi, n))$ is an isomorphism when $k \leq 2n-2$. Then $K(\pi, n+1)$ is obtained from $\Sigma K(\pi, n)$ by attaching cells of dim $\geq n+3$. Take $e_n \colon \Sigma K(\pi, n) \to K(\pi, n+1)$ to be the inclusion map.

Definition 3.28. A reduced homology theory consists of a family of functors $\widetilde{h}_n \colon \mathbf{TOP}^o \to R-\mathbf{MOD}$ and isomorphisms $\sigma_n \colon \widetilde{h}_n \to \widetilde{h}_{n+1} \circ \Sigma$ that satisfy

1. Homotopy invariance: $\widetilde{h}_n(f_0) = \widetilde{h}_n(f_1)$ if $f_0 \simeq f_1$.
2. Exactness: each cofibre sequence

$$X \xrightarrow{f} Y \xrightarrow{f'} C(f)$$

   induces an exact sequence

$$\widetilde{h}_*(X) \longrightarrow \widetilde{h}_*(Y) \longrightarrow \widetilde{h}_*(C(f)) .$$

Remark 3.29. Unreduced theory $\Longleftrightarrow$ reduced theory. To see that, define $h_n(X)=\widetilde{h}_n(X\sqcup\{*\})$ and $h_n(X,A)=\widetilde{h}_n(C(X,A))$.

Let $E=\{(E(n),e_n)\}$ be a spectrum. We get suspension maps

$$\left[S^{n+k},E(n)\wedge X\right]^o=\pi_{n+k}(E(n)\wedge X)\to\pi_{n+k+1}(E(n+1)\wedge X)=\left[S^{n+k+1},E(n+1)\wedge X\right]^o$$

and

$$\Sigma(E(n)\wedge X)=S^1\wedge(E(n)\wedge X)=\Sigma E(n)\wedge X.$$

Define $E_n(X)\coloneqq\operatorname{colim}_{k\to\infty}\pi_{n+k}(E(k)\wedge X)$, and $\sigma_n\colon E_n(X)\to E_{n+1}(\Sigma X)$ is defined via $\left[S^{n+k},E(n)\wedge X\right]\to\left[S^{n+k+1},E(n)\wedge\Sigma X\right]$.

Theorem 3.30. $\{(E_n(X),\sigma_n)\}$ defines a reduced homology theory.

Proof. Homotopy invariance is by definitions.

Injectivity of σ_n: Suppose $x\in\ker\sigma_n$, there exists $[f]\in\left[S^{n+k},E(k)\wedge X\right]^o$ such that $[f]$ represents x and $f\wedge\mathrm{id}_{S^1}\colon S^{n+k}\wedge S^1\to(E(k)\wedge X)\wedge S^1$ is null-homotopic. Then

$$S^{n+k}\wedge S^1\xrightarrow{f\wedge\mathrm{id}}E(k)\wedge\Sigma X\xrightarrow{e_k\wedge\mathrm{id}}E(k+1)\wedge X$$

is null-homotopic. Note that $[(e_k\wedge\mathrm{id})\circ(f\wedge\mathrm{id})]$ represents x as well. We must have $x=0$.

Surjectivity of σ_n: Given $g\colon S^{n+k+1}\to E(k)\wedge X\wedge S^1$. Then define

$$f\colon\ S^{n+k+1}\xrightarrow{g}E(k)\wedge X\wedge S^1\xrightarrow{e_k}E(k+1)\wedge X$$

and we have $\sigma_n([f])=[g]$.

Exactness of Cofibre Sequence: Consider

$$E_n(X)\xrightarrow{f_n}E_n(Y)\xrightarrow{f_n'}E_n(C(f))\ .$$

Suppose $z\in\ker f_n'$ and write $h\colon S^{n+k}\to E(k)\wedge Y$ to represent z. Then $\left(\mathrm{id}_{E(k)}\wedge f'\right)\circ h\colon S^{n+k}\to E(k)\wedge C(f)$ is null-homotopic. Consider cofibre sequences:

$$\begin{tikzcd}
S^{n+k} \arrow[r] \arrow[d, "h"'] & C(\mathrm{id}) \arrow[r] \arrow[d, "H"'] & S^{n+k}\wedge S^1 \arrow[r] \arrow[d, "\beta"'] & S^{n+k}\wedge S^1 \arrow[d, "h\wedge\mathrm{id}"'] \\
E(k)\wedge Y \arrow[r] & C(\mathrm{id}\wedge f) \arrow[r] \arrow[d, "\simeq"] & E(k+1)\wedge X \arrow[r, "\mathrm{id}\wedge f"] & E(k+1)\wedge Y \\
& E(k)\wedge C(f) & &
\end{tikzcd}$$

where H is given by null-homotopy of $(\mathrm{id}\wedge f')\circ h$ and β is the quotient of H and the first two squares are commutative. These indece $h\wedge\mathrm{id}$ such that the last square is commutaive up to homotopy. Therefore, under $\operatorname{colim}_{k\to\infty}$, we have

$$f_*[\beta]=[(\mathrm{id}\wedge f)\circ\beta]=[h\wedge\mathrm{id}]=[h].$$

$\square$

Remark 3.31. In Example 3.27,

1. When $E=\{(S^n,\Sigma)\}_{n\geq 0}$,

$$E_n(X)=\operatorname{colim}_{k\to\infty}\pi_{n+k}\left(S^k\wedge X\right)=\operatorname{colim}_{k\to\infty}\pi_{n+k}\left(\Sigma^kX\right)=\pi_n^s(X),$$

which is the stable homotopy group.

2. When $E = \{(K(\mathbb{Z}, n), \sigma_n)\}_{n \geq 0}$,

$$E_n(X) = \operatorname{colim}_{k \to \infty} \pi_{n+k} \left(K(\mathbb{Z}, n) \wedge X \right) \cong \widetilde{H}_n(X, \mathbb{Z}),$$

which is the reduced singular homology.

Theorem 3.32 (Brown's Representation Theory)**.** Let $\{(h_n, \partial_n)\}$ be a homology theory. Then there exists a sectrum $E = \{(E(n), e_n)\}$ and natural isomorphisms $h_n(X, A) \cong \operatorname{colim}_{k \to \infty} \pi_{n+k} \left(E(k) \wedge (X^+/A^+) \right)$ for all finite CW-complexes (X, A), where $X^+ = X \sqcup \{*\}$ and $A^+ = A \sqcup \{*\}$.

4 Cohomology

4.1 Axiomatic Cohomology

Definition 4.1. A cohomology theory consists of

1. a family of contravariant functors $h^n \colon \mathbf{TOP}(2) \to R - \mathbf{MOD}$,
2. a family of natural transformations $\delta^n \colon h^{n-1} \circ K \to h^n$, where $K \colon (X, A) \to (A, \varnothing)$ is the restriction, that satisfy
   (a) H-Invariance: $h^n(f_0) = h^n(f_1)$ if $f_0 \simeq f_1$.
   (b) Exact Sequence: Given (X, A),

$$\cdots \longrightarrow h^{n-1}(A) \xrightarrow{\delta} h^n(X, A) \longrightarrow h^n(X) \longrightarrow h^n(A)$$

   is exact.
   (c) Excision: Given a pair (X, A) with $U \subset A$ and $\overline{U} \subset \operatorname{Int}(A)$, then the restriction $h^n(X, A) \to h^n(X - U, A - U)$ is an isomorphism for any n.

Definition 4.2. A reduced cohomology theory is given by $\widetilde{h}^n(X) \coloneqq \ker\left(h^n(X) \to h^n(\{*\})\right)$ which fits into a splitting exact sequence

$$0 \longrightarrow h^n(X, *) \longrightarrow h^n(X) \longrightarrow h^n(*) \longrightarrow 0 .$$

And we have $\widetilde{h}^n(X) \cong h^n(X, *)$.

4.1.1 Mayer-Vietoris Sequence

Definition 4.3. Given $A, B \subset X$, we say the pair (A, B) is excisive if the restriction $h^*(A \cup B, A) \to h^*(B, A \cap B)$ is an isomorphism.

Lemma 4.4. The followings are equivalent:

1. (A, B) is excisive.
2. (B, A) is excisive.

Proof. The proof is given by chasing the following diagram, where the "crossing" diagram is given by the exact sequences of triples $(A \cup B, A, A \cap B)$ and $(A \cup B, B, A \cap B)$.

$$\begin{tikzcd}
h^*(A \cup B, A) \arrow[rr, "a"] \arrow[dr, "\alpha"'] & & h^*(B, A \cap B) \\
& h^*(A \cup B, A \cap B) \arrow[ur, "f"'] \arrow[dr, "g"] & \\
h^*(A \cup B, B) \arrow[rr, "b"] \arrow[ur, "\beta"] & & h^*(A, A \cap B)
\end{tikzcd}$$

Assume a is an isomorphism.

Injectivity of b: Assume $b(x)=0$. Then $g\circ\beta(x)=b(x)=0$. Therefore, there is y such that $\alpha(y)=\beta(x)$. Then $a(y)=f\circ\alpha(y)=f\circ\beta(x)=0$. Note that a is an isomorphism, $y=0$. Therefore $\beta(x)=\alpha(y)=0$.

Then there is z such that $\eta(z)=x$ where $\eta\colon h^*(B,A\cap B)\to h^*(A\cup B,B)$. Note that $z=a\left(a^{-1}(z)\right)=f\circ\alpha\left(a^{-1}(z)\right)$. Then we have $x=\eta\circ f\left(\alpha\left(a^{-1}(z)\right)\right)=0$.

Surjectivity of b: Take $x\in h^*(A,A\cap B)$. Note that $a(\delta(x))=f\circ\alpha\circ\delta(X)=0$ where $\delta\colon h^*(A,A\cap B)\to h^*(A\cup B,A)$. Then $\delta(x)=0$ and then there exists y such that $g(y)=x$. Note that $f\left(y-\alpha\circ a^{-1}\circ f(y)\right)=f(y)-f(y)=0$. Then there exists $z\in h^*(A\cup B,B)$ such that $\beta(z)=y-\alpha\circ a^{-1}\circ f(y)$. Therefore $b(z)=g\circ\beta(z)=g\left(y-\alpha\circ a^{-1}\circ f(y)\right)=g(y)=x$. □

Assume (X_0,X_1) is an excisive pair such that $X=X_0\cup X_1$. We get a connecting map

$$\Delta\colon h^{n-1}\left(X_0\cap X_1\right)\to h^n\left(X_0,X_0\cap X_1\right)\cong h^n\left(X,X_1\right)\to h^n(X).$$

Then we have the Mayer-Vietoris exact sequence

$$\begin{tikzcd}
\arrow[l, "\Delta"'] h^n\left(X_0,X_1\right) & h^n\left(X_0\right)\oplus h^n\left(X_1\right) \arrow[l] & h^n(X) \arrow[l] & h^{n-1}\left(X_0,X_1\right) \arrow[l, "\Delta"'] & \arrow[l] \\
& i_0^*x_0-i_1^*x_1 & \left(x_0,x_1\right) \arrow[l, mapsto] &
\end{tikzcd}$$

4.1.2 Multiplicative Structure

Definition 4.5. A cup product on (h^*,δ^*) consists of a family of R-linear maps

$$h^m(X,A)\otimes_R h^n(X,B)\to h^{m+n}(X,A\cup B)$$

for excisive pairs (A,B), which satisfies

1. Naturality: $f^*(x\cup y)=f^*(x)\cup f^*(y)$.
2. Stability: $\delta(a)\cup x=S_A\left(a\cup\tau_Ax\right)$ where $S_A\colon h^m(A,A\cap B)\overset{\cong}{\to}h^m(A\cup B,B)\overset{\delta}{\to}h^{r+1}(X,A\cap B)$ and $\tau_A\colon h^n(X,B)\to h^n(A,A\cap B)$.
3. Unity: There is $1\in h^0(\{*\})$ with $1_X=c^*(1)$, where $c\colon X\to\{*\}$ is contraction map, satisfies
$$1_X\cup x=x\cup 1_X=x.$$
4. Associativity: $(x\cup y)\cup z=x\cup(y\cup z)$.
5. Commutativity: $x\cup y=(-1)^{|x|\cdot|y|}y\cup x$.

Definition 4.6. A cross product consists of R-linear maps

$$h^m(X,A)\otimes_R h^n(Y,B)\overset{\times}{\to}h^{m+n}((X,A)\times(Y,B))$$

that satisfies

1. Naturality: $(f\times g)^*(a\times b)=f^*a\times g^*b$.
2. Stability: $\delta x\times y=\delta'(x\times y)$ where $x\in h^*(A))$ and $y\in h^*(Y,B)$ and $\delta'\colon h^k(A\times(Y,B))\overset{\cong}{\to}h^k(A\times Y\cup X\times B,X\times B)\overset{\delta}{\to}h^k((X,A)\times(Y,B))$.
3. Unity: There is $1\in h^0(\{*\})$ such that $1\times x=x\times 1=x$.
4. Associativity: $(x\times y)\times z=x\times(y\times z)$.

5. Commutativity: $x \times y = (-1)^{|x|\cdot|y|}\tau^*(y \times x)$ where $\tau\colon X \times Y \to Y \times X$, $(x, y) \mapsto (y, x)$.

In fact, the two products are equivalent. If we have a cup product, we can get a cross product by

$$x \times y := \mathrm{pr}_1^*(x) \cup \mathrm{pr}_2^* y, \qquad x \in h^m(X, Z), y \in h^n(Y, B)$$

where pr_i is the projection map. If we have a cross product, let $d\colon X \to X \times X$ be the diagonal map. We can define

$$x \cup y := d^*(x \times y).$$

When either (1) or (2) is imposed, we say the cohomology theory (h^*, δ^*) is multiplicative.

4.2 The Thom Isomorphism

Denote $h^* := h^*(\{*\})$. The coefficient group $h^a st(-)$ is additive and multiplicative cohomology. Then $h^*(X, A)$ is a h^*-module given by

$$a \cdot x := c^*(a) \cup X,$$

where $c\colon X \to \{*\}$ is the contraction.

Theorem 4.7 (Leray-Hirsch)**.** Let $(E, E') \xrightarrow{p} B$ be relative filtration over a CW-complex B. Assume there are finitely many elements $t_j \in h^*(E, E')$ such that $t_j|_b \in h^*(E_b, E'_b)$ forms a basis as h^*-modules for any $b \in B$. Then $h^*(E, E')$ is a free $h^*(B)$-module with basis $\{t_j\}$ given by $a \cdot x \mapsto p^*(a) \cup x$.

Proof. Given $C \subset B$, we write $h^*(C)\langle t\rangle$ for the free $h^*(C)$-module generated by formal variables $\{t_j\}$. We get a R-linear map

$$\begin{aligned} \varphi(C)\colon h^*(C)\langle t\rangle &\to h^*(E|_C, E'|_C) \\ \sum a_j t_j &\mapsto \sum p^*(a_j) \cup t_j. \end{aligned}$$

Notice that the results holds for B^0. Assume the result holds on B^{n-1}. Decompose $B^n = U \cup V$ where $U = B^n -$ **one point from each n-cell** and V is the union of all open n-cells.

Notice that $U \cap V$ is disjoint unions of S^{n-1}, $\varphi(U \cap V)\colon h^*(U \cap V)\langle t\rangle \to h^*(E_{U\cap V}, E'|_{U\cap V})$ is an isomorphism by induction.

Notice that U deformation retracts into B^{n-1}, $\varphi(U)\colon h^*(U)\langle t\rangle \to h^*(E|_U, E'|_V)$ is an isomorphism. Similarly, because V deformation retracts onto disjoint of points, $\varphi(V)$ is also an isomorphism.

Appling Mayer-Virtoris sequence

$$\begin{tikzcd} h^*(U \cup V)\langle t\rangle \arrow[r, "\varphi"] \arrow[d] & h^*(E|_{U\cup V}, E'|_{U\cup V}) \arrow[d] \\ h^*(U)\langle t\rangle \oplus h^*(V)\langle t\rangle \arrow[r, "\varphi", "\cong"'] \arrow[d] & h^*(E|_U) \oplus h^*(E'|_V) \arrow[d] \\ h^*(U \cup V)\langle t\rangle \arrow[r, "\varphi", "\cong"'] & h^*(E|_{U\cap V}, E|_{U\cap V}) \end{tikzcd}$$

we know that $\varphi(U \cup V)$ is an isomorphism and $h^*(U \cup V)\langle t\rangle$ is a free module. $\square$

Definition 4.8. Given a relative filtration $p\colon (E, E') \to B$, we say $t(p) \in h^n(E, E')$ is a Thom class if $t(p)|_b$ generates $h^n(E_b, E'_b)$ for each $b \in B$.

Theorem 4.9 (Thom Isomorphism)**.** Let $p\colon (E, E') \to B$ be a relative filtration. Suppose $t(p) \in h^n(E, E')$ is a Thom class. Then

$$\begin{aligned} \Phi\colon h^k(B) &\to h^{k+n}(E, E') \\ b &\mapsto p^*(b) \cup t(p) \end{aligned}$$

is an isomorphism.

Proof. Apply Leray-Hirsch Theorem (Theorem 4.7) to $\{t_j\} = t(p)$. $\square$

Definition 4.10. We further assume $p^*\colon h^*(B) \to h^*(E)$ is an isomorphism. We define the Euler class $e(p) \in h^(B)$ by

$$\begin{tikzcd}
h^n(E, E') \arrow[r] & h^n(E) \arrow[r, "(p^*)^{-1}"] & h^*(B) .
\end{tikzcd}$$

$$t(p) \longmapsto e(p)$$

Theorem 4.11 (Gysin Sequence)**.** Assume $t(p) \in h^n(E, E')$ is a Thom class and $p^*\colon h^*(B) \to h^*(E)$ is an isomorphism. Then we have the Gysin's sequence

$$\longrightarrow h^{k-1}(E') \longrightarrow h^{k-n}(B) \xrightarrow{\cup e(p)} h^k(B) \xrightarrow{p^*} h^k(E') \longrightarrow$$

Proof. Consider the exact sequence of pair (E, E')

$$\begin{tikzcd}
h^{k-1}(E') \arrow[r, "\delta"] & h^k(E, E') \arrow[r, "j"] & h^k(E) \arrow[r] & h^k(E') \arrow[r] & {} \\
& h^{k-n}(B) \arrow[u, "\cong"', "\Phi"] \arrow[r, "\cup e(p)"] & h^k(B). \arrow[u, "\cong"', "p^*"] & &
\end{tikzcd}$$

For any $b \in h^{k-n}(B)$,

$$j(\Phi(b)) = j\left(p^*(b) \cup t(p)\right) = p^*(b) \cup p^*(e(p)).$$

$\square$

Let $\xi\colon E \to B$ be a real vector bundle of rank n, E^0 = complement of zero section of E. Then $\left(E_b, E_b^0\right) = (\mathbb{R}^n, \mathbb{R}^n - \{0\}) = \left(D^n, S^{n-1}\right)$.

Proposition 4.12. Assume $\xi\colon E \to B$ admits a nowhere vanishing section. Then $e(\xi) = 0$.

Proof. Take $s\colon B \to E^0$. The Euler class factors through $p \circ s = \mathrm{id}$. Chasing the diagram,

$$\begin{tikzcd}
h^n\left(E, E^0\right) \arrow[r, "j_1"] & h^n(E) \arrow[rr, "(p^*)^{-1}"] \arrow[dr, "j_2"'] & & h^n(B) \\
& & h^n\left(E^0\right) \arrow[ur, "s^*"'] &
\end{tikzcd}$$

$$t(s) \longmapsto e(s)$$

$j_2 \circ j_1 = 0$. Then $e(\xi) = 0$. $\square$

4.3 Singular Cohomology

Let (X, A) be a pair of spaces. Then we have singular chain complexes $S_*(X)$ and $S_*(X, A) \coloneqq S_*(X)/S_*(A)$. Given an R-module M. We define

$$S^n(X, A; M) \coloneqq \mathrm{Hom}_R\left(S_n(X, A), M\right).$$

We have the cohomology map

$$\begin{aligned} \delta\colon S^n(X, A) &\mapsto S^{n+1}(X, A) \\ \varphi &\mapsto (-1)^{n+1} \varphi \circ \partial. \end{aligned}$$

Since $\partial^2 = 0$, $\delta^2 = 0$. Define $H^n(X, A; M) \coloneqq \ker \delta / \operatorname{im} \delta$.

Theorem 4.13 (Universal Coefficient Theorem)**.** We have exact sequences:

1. $$0 \longrightarrow \operatorname{Ext}\left(H_{n-1}(X,A;R),M\right) \longrightarrow H^n(X,A,M) \longrightarrow \operatorname{Hom}_R\left(H_n(X,A),M\right) \longrightarrow 0\ .$$

   It splits but does not split naturally.

2. $$0 \longrightarrow H^n\left(X,A;R\right)\otimes M \longrightarrow H^n(X,A,M) \longrightarrow \operatorname{Tor}\left(H^{n+1}(X,A;R),M\right) \longrightarrow 0\ .$$

   It splits but does not split naturally.

On the cochain level, we define

$$\begin{gathered} S^k(X,R)\otimes S^l(S;R) \to S^{k+l}(X;R) \\ \varphi\otimes\psi \mapsto \varphi\cup\psi \end{gathered}$$

by

$$\varphi\cup\psi(\sigma) \coloneqq (-1)^{kl}\varphi\left(\sigma|_{[e_0,\cdots,e_k]}\right)\cdot\psi\left(\sigma|_{[e_{k+1},\cdots,e_{k+l}]}\right)$$

for any simplex $\sigma\colon \Delta^{k+l}\to X$.

Claim 12. $\delta(\varphi\cup\psi) = \delta\varphi\cup\psi + (-1)^{|\varphi|}\varphi\cup\delta\psi$.

Proof. This claim can be checked by definition. □

This Claim shows that cup product descends to the cohomology level: The homomorphism

$$\cup\colon H^k(X;R)\otimes H^l(X;R)\to H^{k+l}(X;R)$$

is well-defined.

Fact 4.14. When (A,B) is an excisive pair, we get a chain equivalence:

$$S_*(A)+S_*(B)\to S_*(A\cup B).$$

We can define relative cohomology:

$$S^*(X,A)\otimes S^*(X,B) \xrightarrow{\cup} \operatorname{Hom}\left(S_*(X)/S_*(A)+S_*(B),R\right) \longrightarrow \operatorname{Hom}\left(S_*(X)/S_*(A\cup B),R\right) = S^*(X,A\cup B)\ .$$

Then we have a well-defined homomorphism

$$\cup\colon H^k(X,A)\otimes H^l(X,B)\to H^{k+l}(X,A\cup B).$$

We need to check that singular cohomology satisfies cohomology axioms. It is only non-trivial to verify

$$[\varphi]\cup[\psi] = (-1)^{|\varphi|\cdot|\psi|}\cdot[\psi]\cup[\varphi].$$

Consider

$$\begin{aligned} \rho\colon S_n(X) &\to S_n(X) \\ \sigma &\mapsto (-1)^{\frac{n(n+1)}{2}}\overline{\sigma}, \end{aligned}$$

where $\overline{\sigma} = \sigma|_{[e_n,\cdots,e_0]}$.

Fact 4.15. ρ is chain homotopic to id.

Denote $\rho^\vee\colon S^n(X;R)\to S^n(X;R)$ for the map induced by ρ. Then we have $\rho^\vee(\varphi\cup\psi) = (-1)^{|\varphi|\cdot|\psi|}\cdot\psi\cup\varphi$.

4.3.1 Existence of Thom Class

Recall $p\colon (E,E') \to B$ is a relative fibration over a CW-complex. Suppose $t \in H^n(E,E')$ restricts to a basis of the $H^*(\{*\})$-module $H^n(E_b,E'_b)$, $\forall b \in B$. Then we say $t \in H^n(E,E')$ is a Thom class.

For singular cohomology, $H^*(\{*\},R) = R$. A necessary condition for the existence of t is $H^n(E_b,E'_b) \cong R$.

Given a path $\gamma\colon I \to B$ from b_0 to b_1. We get a transport map

$$\gamma^\sharp\colon\ H^n\left(E_{b_0},E'_{b_0}\right) \xleftarrow[\cong]{i^*_{b_0}} H^n\left(\gamma^*E,\gamma^*E'\right) \xrightarrow[\cong]{i_{b^*_1}} H^n\left(E_{b_1},E'_{b_1}\right).$$

Proposition 4.16. Assume $H^n(E_b,E'_b) \cong R$. Then a Thom class $t \in H^n(E,E')$ exists if and only if the transport map $\gamma^\sharp$ is independent of γ.

Proof. Assume $t \in H^n(E,E')$ is a Thom class. Then $\gamma^\sharp(t|_{b_0}) = t|_{b_1}$ which is independent of the choices of γ.

Conversely, if $\gamma^\sharp$ is independent of γ,we can apply the argument of Leray-Hirsch Theorem (Theorem 4.7). It is ensured by fixing a generator/basis t_0 of $H^n\left(E_{b_0},E'_{b_0}\right)$. For any $b \in B$, we get a $t_b = \gamma^\sharp(t_0) \in H^n(E_b,E'_b)$ where γ connects from b_0 to b. Then use Mayer-Vietoris sequence to glue t. □

4.3.2 Orientation

Suppose $\Sigma \hookrightarrow V$ is a linearly embeded n-simplex with ordered vertices $A_0,\cdots,A_n$. Define the orientation of V by $v_1 = A_1 - A_0, v_2 = A_2 - A_1, \cdots, v_n - A_n - A_0$.

Fix Δ^n as the standard n-simplex. Choose a linear embedding $f\colon \Delta^n \to V$ such that f sends the barycenter of Δ^n to $o \in V$. Then $[f] \in H_n\left(V,V^0;\mathbb{Z}\right)$ is a generator where $V^0 = V - \{0\}$. In fact, we have

$$\text{generator of } H_n\left(V,V^0,\mathbb{Z}\right) \overset{1:1}{\longleftrightarrow} \text{orientation of } V.$$

Given an orientation generator $o_V \in H_n\left(V,V^0,\mathbb{Z}\right)$, we get a generator $u_V \in H^n\left(V,V^0,\mathbb{Z}\right)$ such that $u_V(o_V) = 1$. Then we get

$$\text{generator of } H^n\left(V,V^0,\mathbb{Z}\right) \overset{1:1}{\longleftrightarrow} \text{orientation of } V.$$

Let $\xi\colon E \to B$ be a real vector bundle of rank n. An orienting bundle atlas on ξ consists $\{(U_\alpha,\varphi_\alpha)\}$ with $\varphi_\alpha\colon \xi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ such that the transition maps $g_{\beta\alpha}\colon U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{R})$ have positive determinant.

After fixing an orientation on $\mathbb{R}^n$, an orienting atlas induces an orientation on $\xi\colon E \to B$.

Definition 4.17. An orientation on ξ is an assignment of orientations on E_b such that for any $b \in B$, there is a neighborhood U and a trivialization $\varphi\colon \xi^{-1}(U) \to U \times \mathbb{R}^n$ which is fibrewise orientation-preserving.

Proposition 4.18. Let $\xi\colon E \to B$ be a real vector bundle. Then ξ is orientable if and only if ξ admits a Thom class $t(\xi) \in H^n\left(E,E^0,\mathbb{Z}\right)$.

Proof. Given an orienting atlas. We define $t_{U_\alpha} = \varphi^*_\alpha(t_\alpha)$, where $t_\alpha \in H^n(U_\alpha \times (\mathbb{R}^n,\mathbb{R}^n - \{0\}))$, $t_\alpha = p^* t_{\mathbb{R}^n}$, $p\colon U_\alpha \times \mathbb{R}^n \to \mathbb{R}^n$ is the projection, $t_{\mathbb{R}^n} \in H^n(\mathbb{R}^n,\mathbb{R}^n - \{0\};\mathbb{Z})$ is a fixed generator. Then $t_{U_\alpha}|_b = t_{U_\beta}\big|_b$ for any $b \in U_\alpha \cap U_\beta$. Mayer-Vietoris sequence glues these t_{U_α} to a Thom class $t(\xi)$. The proof of another direction is more straightforeward. □

Motivated by this proposition, we have

Definition 4.19. Given a ring R, we define an R-orientation of $\xi\colon E \to B$ to be a Thom class $t(\xi) \in H^n\left(E,E^0;R\right)$.

4.4 Homology and Homotopy

4.4.1 Hurewicz Theorem

We fix generators $z_n \in H_n(S^n;\mathbb{Z})$ and $\widetilde{z}_n \in H_n\left(D^n, S^{n-1};\mathbb{Z}\right)$ such that $\partial\widetilde{z}_n = z_{n-1}$ and $q_*\widetilde{z}_n = z_n$ where $q\colon D^n \to D^n/S^{n-1} \cong S^n$ is the quotient map. Define the Hurewicz homomorphisms

$$\begin{aligned} h\colon \pi_n(X,*) &\to H_n(X,\mathbb{Z}) \\ [f] &\mapsto f_* z_n, \end{aligned}$$

and

$$\begin{aligned} h\colon \pi_n(X,A,*) &\to H_n(X,A,\mathbb{Z}) \\ [f] &\mapsto f_* \widetilde{z}_n. \end{aligned}$$

Recall that we have a left action of $\pi_1(A,*)$ on $\pi_n(X,A,*)$: Any path $v\colon I \to A$ 1:1 corresponds to a homotopy $J^{n-1} \to v(t)$ of constant maps. Then $J^{n-1} \hookrightarrow \partial I^n$ is a cofibration and $\partial I^n \hookrightarrow I^n$ is a fibration. We extends this homotopy to $V\colon \left(I^n, \partial I^n, J^{n-1}\right) \times I \to (X, A, *)$. Given $\alpha = [v] \in \pi_1(A,*)$, define $[f]\cdot\alpha = [v_1]$ where $v_0 = f$. Suppose $[g] = [f]\cdot\alpha$, then $g \simeq f$.

Define $\pi_n^\sharp(X,A,*) := \pi_n(X,A,*)/\pi_1(A,*) = \pi_n(X,A,*)/\left\{x - x\cdot\alpha : \alpha\in\pi_1(A)\right\}$. Then the Hurewicz map descends to

$$h^\sharp\colon \pi_n^\sharp(X,A,*) \to H_n(X,A,\mathbb{Z}).$$

Theorem 4.20 (Hurewicz Theorem)**.** Assume X is $(n-1)$-connected, $n \geq 1$. Then $h^\sharp\colon \pi_n^\sharp(X,*) \to H_n(X,\mathbb{Z})$ is an isomorphism.

Proof. When $n = 1$, for any $\alpha, x \in \pi_1(X,*)$, $x\cdot\alpha := \alpha^{-1}x\alpha$. Then by definition, $\pi_1^\sharp(X,*)$ is the abelianization of $\pi_1(X,*)$, where is isomorphism to $H_1(X,\mathbb{Z})$.

When $n \geq 2$, X is simply-connected, we know $\pi_n(X,*) = \pi_n^\sharp(X,*)$.

Fact 4.21. A weak homotopy equivalence induces isomorphism on homology groups.

We may assume X is a CW-complex such that $X^{n-1} = \{*\}$. Then X^{n+1} is the cone of a map $\varphi\colon \bigvee S_j^n \to \bigvee S_k^n$. The conclusion holds for spheres. Additivity of π_n and H_n shows that h is an isomorphism for $\bigvee S_k^n$. we get exact sequence

$$\begin{tikzcd} \pi_n\left(\bigvee S_j^n\right) \arrow[r, "\varphi_*"] \arrow[d, "\cong"] & \pi_n\left(\bigvee S_k^n\right) \arrow[r] \arrow[d, "\cong"] & \pi_n(X) \arrow[r] \arrow[d] & 0 \arrow[d, "\cong"] \\ H_n\left(\bigvee S_j^n\right) \arrow[r] & H_n\left(\bigvee S_k^n\right) \arrow[r] & \pi_n(X) \arrow[r] & 0 \end{tikzcd}.$$

Therefore h is an isomorphism for X^{n+1}.

Since X is obtained from X^{n+1} by attaching cells of dim $\geq n+2$, $\pi_n(X) \cong \pi_n\left(X^{n+1}\right)$ and $H_n(X) \cong H_n\left(X^{n+1}\right)$. Then h is an isomorphism for X. Let ☐

Corollary 4.22. Let (X,A) be a pair of simply-connected CW-complexes. Suppose $H_i(X,A) = 0$ for any $i < n$, $n \geq 2$. Then $\pi_i(X,A) = 0$ for any $i < n$ and $h\colon \pi_n(X,A) \to H_n(X,A)$ is an isomorphism.

Proof. Apply induction on n: When $n \geq 2$, we have

$$\begin{tikzcd} \pi_n(X,A) \arrow[r, "\cong"] \arrow[d, "h"'] & \pi_n(X/A) \arrow[d, "\cong"'] \\ H_n(X,A) \arrow[r, "\cong"] & H_n(X/A) \end{tikzcd}.$$

☐

Theorem 4.23 (Whitehead)**.** Suppose X, Y are simply-connected. If $f\colon X \to Y$ induces isomorphisms on H_*, then f is a weak homotopy equivalence.

Proof. We may assume X, Y are CW-complexes. Apply Corollary 4.22 to $(Z(f), X)$. $\square$

4.4.2 Singular Cohomology and Eilenberg-MacLane Spaces

Let G be an abelian group and $n \geq 1$. Denote $K \coloneqq K(G,n)$. Define a natural transformation $\lambda\colon [-, K(G,n)] \to H^n(-;G)$ as follows: We have a sequence of isomorphisms

$$H^n(K;G) \cong \operatorname{Hom}\left(H_n(K), G\right) \cong \operatorname{Hom}\left(\pi_n(K), G\right) = \operatorname{Hom}(G,G)$$

where the first isomorphism is by Universal Coefficient Theorem and the second is by Hurewicz Theorem. Suppose $\mathrm{id} \in \operatorname{Hom}(G,G)$ corresponds to $\iota_n \in H^n(K;G)$. Define

$$\begin{aligned} \lambda(X)\colon [X, K(G,n)] &\to H^n(X;G) \\ [f] &\mapsto f^*\iota_n. \end{aligned}$$

Notice that $K(G,n) = \Omega K(G,n+1)$, $\lambda(X)$ is a homomorphism.

Theorem 4.24. Let $(X,*)$ be a based CW-complexes. Then $\lambda(X)\colon [X, K(G,n)]^o \to \widetilde{H^n}(X;G)$ is an isomorphism.

Proof. Note that the conclusion holds for spheres: If $m \neq n$, $[S^m, K(G,n)]^o = 0$ and $\widetilde{H^n}\left(S^m;G\right) = 0$. For n, it follows from definition.

Consider cofibre sequence

$$\bigvee S_j^{k-1} \longrightarrow X^{k-1} \longrightarrow X^k \longrightarrow X^k/X^{k-1} \longrightarrow \Sigma X^{k-1}\ .$$

Apply $[-, K(G,n)]^o$ and $\widetilde{H^n}$, we get corresponding exact sequences. Use induction on k to conclude (to be continue...) $\square$

4.5 Homology with Local Coefficient

Let X be a path-connected space. Recall that its fundamental groupoid $\Pi(X)$ is a category whose objects are points in X and morphisms $x \to y$ are homotopy class of path $\gamma\colon I \to X$, $\gamma(0) = x, \gamma(1) = y$ rel ∂I.

Definition 4.25. A local coefficient system is a functor $\mathcal{E}\colon \Pi(X) \to \mathcal{A}\lfloor$ such that $\mathcal{E}(x) \cong \mathcal{E}(y)$ for any $x, y \in X$.

Definition 4.26 (Homology with Coefficient $\mathcal{E}$)**.** We define the *Homology with Coefficient* $\mathcal{E}$ as follows:

The chain complex $S_k(X;\mathcal{E})$ consists of formed sums $\sum_{i=1}^m a_i\sigma_i$ where $\sigma_i\colon \Delta^k \to X$ is a k-simplex and $a_i \in \mathcal{E}\left(\sigma_i\left(e_0\right)\right)$, $e_0 = (1, 0 \cdots, 0) \in \Delta^k$.

Recall that we have the face maps:

$$\begin{aligned} f_m^k\colon \Delta^{k-1} &\to \Delta^k \\ (t_0, \cdots, t_{k-1}) &\mapsto (t_0, \cdots, t_{m-1}, 0, t_m, \cdots, t_{k-1})\,. \end{aligned}$$

Then we have $f_i^k(1, 0, \cdots, 0) = (1, 0, \cdots, 0)$ for any $i \geq 1$ and $f_0^k(1, 0, \cdots, 0) = (0, 1, 0, \cdots, 0) \coloneqq e_1$.

Given a simplex $\sigma\colon \Delta^k \to X$, we get a path

$$\begin{aligned} \gamma_\sigma\colon [0,1] &\to X \\ t &\mapsto \sigma(t, 1-t, 0, \cdots, 0) \end{aligned}$$

with $\gamma_\sigma(0) = \sigma(e_1)$ and $\gamma_\sigma(1) = \sigma(e_0)$. Define

$$\begin{aligned}\partial \colon S_k(X;\mathcal{E}) &\to S_{k-1}(X;\mathcal{E})\\ a\cdot\sigma &\mapsto \mathcal{E}(\gamma_\sigma)^{-1}(a)\cdot\left(\sigma\circ f_0^k\right) + \sum_{m=1}^{k}(-1)^m a\cdot\left(\sigma\circ f_m^k\right),\end{aligned}$$

where $\mathcal{E}(\gamma_\sigma) : \mathcal{E}(\sigma(e_1)) \to \mathcal{E}(\sigma(e_1))$.

Claim 13. $\partial^2 = 0$.

We define $H_*(X;\mathcal{E}) := H_*(S_*(X;\mathcal{E}),\partial)$.

Definition 4.27 (Cohomology with Coefficient $\mathcal{E}$)**.** We define the *Cohomology with Coefficient $\mathcal{E}$* as follows:

The cochain complex $S^k(X;\mathcal{E})$ is generated by

$$c\colon \{\text{singular } k\text{-simplex}\} \to \oplus_{x\in X}\mathcal{E}(x)$$

such that $c(\sigma) \in \mathcal{E}(\sigma(e_0))$. The coboundary map $\delta\colon S^k(X;\mathcal{E}) \to S^{k+1}(X;\mathcal{E})$ is defined by

$$\delta(c)(\sigma) := (-1)^k\left(\mathcal{E}(x_0)\cdot c\left(\sigma\circ f_0^{k+1}\right) + \sum_{m=1}^{k+1}(-1)^m c\left(\sigma\circ f_m^{k+1}\right)\right).$$

Claim 14. $\delta^2 = 0$.

We define $H^*(X;\mathcal{E}) := H^*(S^*(X;\mathcal{E}),\delta)$.

Proposition 4.28. (Co)homology with local coefficients is a (co)homology theory.

4.5.1 An Equivalent Definition

Assume X admits a universal cover $\widetilde{X}$. Given a local system $\mathcal{E}$ on X. We fix an abelian group A such that $A \cong \mathcal{E}(x)$, $\forall x \in X$. We get a representation

$$\rho_{\mathcal{E}}\colon \pi_1(X,*) \to \mathrm{Aut}(\mathcal{E}(x)) \cong \mathrm{Aut}(A).$$

Write $\pi = \pi_1(X,*)$. We can regard A as left $\mathbb{Z}[\pi]$-module by

$$\gamma\cdot a := \rho_{\mathcal{E}}(\gamma)(a).$$

Note that $\pi_1(X,*)$ have a right action on $\widetilde{X}$, $S_*\left(\widetilde{X}\right)$ is a right $\mathbb{Z}[\pi]$-module. Define the chain complex

$$S_*(X;\rho_{\mathcal{E}}) := S_*\left(\widetilde{X}\right)\otimes_{\mathbb{Z}[\pi]} A$$

and the boundary map

$$\partial_{\mathcal{E}}\sigma\otimes a \mapsto (\partial\sigma)\otimes a.$$

We claim that $\partial_{\mathcal{E}}^2 = 0$. Then we define $H_*(X;\rho_{\mathcal{E}}) := H_*(S_*(X;\rho_{\mathcal{E}}),\partial_{\mathcal{E}})$.

Regard $S_k\left(\widetilde{X}\right)$ as left $\mathbb{Z}[\pi]$-module by $\gamma\cdot\sigma := \sigma\cdot\gamma^{-1}$. Define

$$S^*(X;\rho_{\mathcal{E}}) := \mathrm{Hom}_{\mathbb{Z}[\pi]}\left(S_k\left(\widetilde{X}\right), A\right)$$

and

$$\delta_{\mathcal{E}}c(\sigma) := c(\rho_{\mathcal{E}}\sigma).$$

We claim that $\delta_{\mathcal{E}}^2 = 0$. Then we define $H^*(X;\rho_{\mathcal{E}}) := H^*(S^*(X;\rho_{\mathcal{E}}),\delta_{\mathcal{E}})$.

Theorem 4.29. Let X be a connected space admitting a universal cover $\widetilde{X}$. Assume $\mathcal{E}$ is a local system on X and $\rho_{\mathcal{E}} \colon \pi_1(X) \to \mathrm{Aut}(A)$. Then $H_*(X;\mathcal{E}) \cong H_*(X;\rho_{\mathcal{E}})$ and $H^*(X;\mathcal{E}) \cong H^*(X;\rho_{\mathcal{E}})$.

Proof. We have pairwise 1 to 1 corresponding of the following three items:

1. local system $\mathcal{E}$,
2. covering space $\widetilde{X}_{\mathcal{E}}$ with fibre A,
3. representation $\rho\colon \pi_1(X) \to \mathrm{Aut}(A)$.

(1) $\Rightarrow$ (3) is by our construction.
For (2) $\Rightarrow$ (3), ρ is given by the endpoint of lifts of π_1.
For (3) $\Rightarrow$ (2), $\widetilde{X}_{\mathcal{E}} = \widetilde{X} \times A/\left(\widetilde{x}, a\right) \sim \left(\widetilde{x}\cdot\gamma^{-1}, \rho(\gamma)a\right)$ is called Borel construction.
For (2) $\Rightarrow$ (1), lifts give the morphisms. For $a\cdot\sigma \in S_k(X,\mathcal{E})$, we lift the simplex $a\cdot\sigma$ to a π-equivalent simplex on $S_*\left(\widetilde{X}\right)$. $\square$

Example 4.30.

1. The local system is $\mathcal{E}$ trivial ($\mathcal{E}\colon \mathcal{E}(x_0) \to \mathcal{E}(x_1)$ is independent of γ). Then we have
$$S_*\left(X;\rho_{\mathcal{E}}\right) = S_*\left(\widetilde{X}\right) \otimes_{\mathbb{Z}[\pi]} A = S_*(X) \otimes A.$$
Therefore $H_*\left(X;\rho_{\mathcal{E}}\right) = H_*(X;A)$ is the singular homology.

2. Let $A = \mathbb{Z}[\pi]$. We have a tautological representation
$$\begin{aligned} \rho\colon \pi_1(X,*) &\mapsto \mathrm{Aut}(\pi) \\ g &\mapsto L_g. \end{aligned}$$
Then $S_*(X;\rho) = S_*\left(\widetilde{X}\right) \otimes_{\mathbb{Z}[\pi]} A = S_*\left(\widetilde{X}\right)$ and $H_*(X;\rho) = H_*\left(\widetilde{X}\right)$. Generally,

   Lemma 4.31. Let $H < \pi_1(X,*)$ be a subgroup. Choose $A = \mathbb{Z}[\pi/H]$ which admits a representation $\rho_A\colon \pi \to \mathrm{Aut}(A)$ be left multiplication. Then $H_*\left(X;\rho_A\right) \cong H_*\left(\widetilde{X}_H\right)$ where $\widetilde{X}_H \to X$ is the H-covering of X.

3. Let X be a connected and closed n-manifold. Then we get the orientation local system $\mathcal{O}_X$. Its objects are orientations $\mathcal{O}_X(x) = H_n(X, X-\{x\}) \cong \mathbb{Z}$ and morphisms are orientation transport: $H_n(X,X-\{x\}) \cong H_n(X,X-U) \cong H_n(X,X-\{x'\})$ for $x,x' \in U$ where U is a small neighborhood. We have $\mathcal{O}_X$ is trivial if and only if X is orientable, if and only if $w_1\colon \pi_1(X,*) \to \mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$ is trivial, if and only if $w_1 = 0 \in H^1\left(X;\mathbb{Z}_2\right)$. In fact, w_1 is the first Stefeil-Whitney class of X. Then we know $H_n\left(X;w_1\right) \cong \mathbb{Z}$.
   - If w_1 is trivial, then $H_n\left(X;w_1\right) = H_n(X,\mathbb{Z}) = \mathbb{Z}$.
   - If w_1 is non-trivial, then $H_n\left(X;w_1\right) \cong H_n\left(\widetilde{X}_{w_1},\mathbb{Z}\right) \cong \mathbb{Z}$ where $\widetilde{X}_{w_1} \to X$ is the orientable double cover..

4.6 Obstruction

4.6.1 Obstruction of Extension

Question 4.32. Suppose $f\colon A \to Y$ and X is obtained by attaching cells on A. Is there an extension g of f on X?

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & Y \\ X \arrow[ur, dashed, "\exists ? g"'] & \end{tikzcd}$$

Assume $\pi_1(Y) = 1$ and (X, A) is a relative CW-complex.

For each $(n+1)$-cell e^{n+1}, we have

$$\begin{tikzcd} S^n \arrow[r, "\varphi"] \arrow[d] & X^n \arrow[d, hook] \\ D^n \arrow[r, "\Phi"] & X \end{tikzcd}$$

Assume we have a map $g\colon X^n \to Y$. We wish to extend g over X^{n+1}.

For each $(n+1)$-cell e_i^{n+1}, g extends over e_i^{n+1} if and only if $S^n \xrightarrow{\varphi_i} X^n \xrightarrow{g} Y$ is null-homotopic in $[S^n, Y] = \pi_n(Y)$. For each g, we get a cochain $\theta^{n+1}(g) \in C^{n+1}(X, A; \pi_n(Y)) =: H^{n+1}(X^{n+1}, X^n; \pi_n(Y))$ by setting $\theta^{n+1}(g)(e_i^{n+1}) = [g \circ \varphi_i] \in \pi_n(Y)$.

Lemma 4.33. g extends over X^{n+1} if and only if $\theta^{n+1}(g) = 0$.

We will give an algebraic definition of $\theta^{n+1}(g)$.

Lemma 4.34. We have the factorization

$$\begin{tikzcd} \pi_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial"] \arrow[d, "j"'] & \pi_n(X^n) \arrow[r, "g_*"] & \pi_n(Y) \\ \pi^{\sharp}_{n+1}(X^{n+1}, X^n) \arrow[urr, "\overline{g_* \circ \partial}"'] & & \end{tikzcd}$$

Proof. Recall that

$$\pi^{\sharp}_{n+1}(X^{n+1}, X^n) = \pi_{n+1}(X^{n+1}, X^n) / \langle x - x \cdot \alpha : \alpha \in \pi_1(X^n) \rangle .$$

Then

$$g_* \circ \partial(x \cdot \alpha) = g_*((\partial(x)) \cdot \alpha) = (g_* \circ \partial(x)) \cdot g_*(\alpha) = g_* \circ \partial(x).$$

□

Since (X^{n+1}, X^n) is n-connected, $h\colon \pi^{\sharp}_{n+1}(X^{n+1}, X^n) \xrightarrow{\cong} H_{n+1}(X^{n+1}, X^n; \mathbb{Z}) := C_{n+1}(X^{n+1}, X^n)$. Define

$$\theta^{n+1}(g) := \overline{g_* \circ \partial} \circ h^{-1} \in \mathrm{Hom}_{\mathbb{Z}}(C_{n+1}(X^{n+1}, X^n), \pi_n(Y)) = C^{n+1}(X^{n+1}, X^n; \pi_1(Y)).$$

Lemma 4.35. $\theta^{n+1}(g)$ is a cocycle.

Proof. Consider the commutative diagram

$$\begin{tikzcd} \pi_{n+2}(X^{n+2}, X^{n+1}) \arrow[r, "\partial"] \arrow[d] & \pi_{n+1}(X^{n+1}) \arrow[r, "i"] \arrow[d] & \pi_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial"] \arrow[d] & \pi_n(X) \arrow[d, "g_*"] \\ H_{n+2}(X^{n+2}, X^{n+1}) \arrow[r, "\partial"] & H_{n+1}(X^{n+1}) \arrow[r, "i"] & H_{n+1}(X^{n+1}, X^n) \arrow[r, "\theta^{n+1}(g)"] & \pi_n(Y) \end{tikzcd}$$

Then $\delta\theta^{n+1}(g) = 0$ if and only if $\theta^{n+1}(g) \circ i \circ \partial = 0$. Note that the vertical arrows are surjective. Since $g_* \circ (\partial \circ i) \circ \alpha = g_* \circ (0) \circ \alpha = 0$. Then $\theta^{n+1}(g) \circ i \circ \partial = 0$. □

We call $[\theta^{n+1}(g)] \in H^{n+1}(X, A; \pi_n(Y))$ the obstruction class of $g\colon X^n \to Y$.

Question 4.36. What kind of geometric information does $[\theta^{n+1}(g)] = 0$ tell us?

Answer. $[\theta^{n+1}(g)] = 0$ if and only if $g|_{X^{n-1}}\colon X^{n-1} \to Y$ extends over X^{n+1}.

Lemma 4.37. Suppose $g_0, g_1\colon X^n \to Y$ are two maps such that $g_0|_{X^{n-1}} \simeq g_1|_{X^{n-1}}$. Then each homotopy $G\colon X^{n-1}\times I \to Y$ determines a cochain $d(g_0, G, g_1) \in C^n(X, A; \pi_n(Y))$ such that $\theta^{n+1}(g_0) - \theta^{n+1}(g_1) = \delta d(g_0, G, g_1)$.

Proof. The maps g_0, G, g_1 combine to a map

$$\widetilde{G}\colon X^{n-1}\times I \cup X^n \times \partial I \to Y.$$

Note that $X^{n-1}\times I \cup X^n \times \partial I$ is the n-skeleton of $X \times I$. Consider $\theta^{n+1}(\widetilde{G}) \in H^{n+1}((X, A)\times I, \pi_n(Y))$. We define

$$d(g_0, G, g_1)(e_i^n) := \theta^{n+1}(\widetilde{G})(e_i^n \times I) \in \pi_n(Y).$$

Then

$$\begin{aligned}
&\left(\delta d(g_0, G, g_1) + \theta^{n+1}(g_1) - \theta^{n+1}(g_0)\right)(e_i^{n+1})\\
&= d(g_0, G, g_1)(\partial e_i^{n+1}) + \theta^{n+1}(g_1)(e_i^{n+1}) - \theta^{n+1}(g_0)(e_i^{n+1})\\
&= \theta^{n+1}(\widetilde{G})(\partial e_i^{n+1}\times I) + \theta^{n+1}(\widetilde{G})(e_i^{n+1}\times \partial I)\\
&= \delta\theta^{n+1}(\widetilde{G})(e_i^{n+1}\times I)\\
&= 0.
\end{aligned}$$

□

Lemma 4.38. Given a map $g_0\colon X^n \to Y$ and a homotopy $G\colon X^{n-1}\times I \to Y$ such that $G|_{X^{n-1}\times\{0\}} = g_0|_{X^{n-1}}$, then for any $d \in C^n(X, A; \pi_n(Y))$, we can find $g_1\colon X^n \to Y$ such that

1. $g_1|_{X^{n-1}} = G|_{X^{n-1}\times\{1\}}$,
2. $\delta d = \theta^{n+1}(g_0) - \theta^{n+1}(g_1)$.

Proof. Let e_i^n be a n-cell of X.

$$\begin{tikzcd}
S^{n_1} \arrow[r, "\varphi_i"] \arrow[d, hook] & X^{n-1} \arrow[d] \\
D^n \arrow[r, "\Phi_i"] & X
\end{tikzcd}$$

Composing with g_0 and G, we get a map $f_i\colon D^n\times\{0\}\cup S^{n-1}\times I \to Y$.

Wirte $[k_i] := d(e_i^n) \in \pi_n(Y) = [\partial(D^{n+1}), Y] = [\partial(D^n \times I), Y]$. Then k_i can be seen as a map $k_i\colon \partial(D^n\times I) \to Y$. Since $D^n\times\{0\}\cup S^{n-1}\times I$ is contractible, then $f_i \simeq k_i|_{D^n\times\{0\}\cup S^{n-1}\times I}$. Since $D^n\times\{0\}\cup S^{n-1}\times I \hookrightarrow \partial(D^n\times I)$ is a cofibration, we get an extension $H_i\colon \partial(D^n\times I)\times I \to Y$. Define $g_1\colon X^n \to Y$ by

$$g_1(\Phi_i(x)) := H_i(x, 1, 1) \in \partial(D^n\times I).$$

Then

$$d(g_0, G, g_1)(e_i^n) = \theta^{n+1}(\widetilde{G})(e_i^n\times I) = [k_i] = d(e_i^n).$$

□

Theorem 4.39 (Obstruction Theorem)**.** Let $g\colon X^n \to Y$ be a continuous map (with $\pi_1(Y) = 1$). Then $[\theta^{n+1}(g)] = 0$ if and only if $g|_{X^{n-1}}$ extends over X^{n+1}.

Proof. ($\Rightarrow$): If $\theta^{n+1}(g) = \delta d$, then apply Lemma 4.38 to $(g, g|_{x^{n-1}}\times \mathrm{id}, d)$, we get $g'\colon X^n \to Y$ such that $g'|_{X^{n-1}} = g$ and $\theta^{n+1}(g) - \theta^{n+1}(g') = \delta d$. Then $\theta^{n+1}(g') = 0$, i.e. g' extends over X^{n+1}.

($\Leftarrow$): Write $g'\colon X^{n+1} \to Y$ such that $g'|_{X^{n-1}} = g|_{X^{n-1}}$. Denote by $\widetilde{g} = g'|_{X^n}$. Then $\theta^{n+1}(\widetilde{g}) = 0$ and $\theta^{n+1}(g) - \theta^{n+1}(g') = \delta d$. □

When Y is n-connected, then any map $f\colon A \to Y$ can be extends to $g\colon X^{n+1} \to Y$, since $H^{n+1}(X, A; \pi_k(Y)) = 0$, $\forall k \le n$. In this case, we define

$$\alpha(f) \coloneqq \theta^{n+2}(g) \in H^{n+2}(X, A; \pi_{n+1}(Y))$$

as the primary obstruction of f.

For homotopy extension, suppose $f_0, f_1\colon X^n \to Y$ such that $f_0|_{X^{n-1}} \simeq f_1|_{X^{n-1}}$. We apply the discussion to $(X^n \times I, X^n \times \partial I \cup X^{n-1} \times I)$. We get an obstruction $d^n(f_0, f_1) \in H^n(X, A; \pi_n(Y))$.

4.6.2 Obstruction of Lifting

Question 4.40. Suppose $f\colon X \to B$ and $p\colon E \to B$ is a fibration. Is there a lifting g of f on X?

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "\exists ? g"] \arrow[r, "f"'] & B \end{tikzcd}$$

Proposition 4.41. Let $p\colon E \to B$ be a fibration with fibre F such that $\pi_1(F) = 1$. Then for each $n \ge 0$, we get a local system

$$\rho_n\colon \pi_1(B) \to \mathrm{Aut}(\pi_n(F)).$$

Proof. $\pi_1(B)$ contains homotopy classes of homotopy self-equivalence of F. They induce $\mathrm{Aut}(\pi_n(F))$. $\square$

Assume we have a lift on X^n.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "g"] \arrow[r, "f"] & B \end{tikzcd}$$

Given an $(n+1)$-cell e_i^{n+1}, we get $g \circ \varphi_i\colon S^n \to E$. Note that $p \circ g \circ \varphi_i = f \circ \varphi_i$ extends over X^{n+1}. Then $f \circ \varphi_i$ is null-homotopic: $f \circ \varphi_i \simeq \mathbf{const}_{f \circ \Phi_i(0)}$. Apply homotopy lifting, we know $g \circ \varphi_i\colon S^n \to E$ is homotopic to $f_i\colon S^n \to E_{f\circ\Phi_i(0)}$. Define $\theta^{n+1}(g)(e_i^{n+1}) \coloneqq [f_i] \in \pi_n(E_{f\circ\Phi_i(0)})$. Then $\theta^{n+1}(g) \in C^{n+1}(X; f^*\rho_n)$.

Then $[\theta^{n+1(g)}] = 0 \in H^{n+1}(X; f^*\rho_n)$ if and only if $g|_{X^{n-1}}$ extends over X^{n+1}, which deduces that f lifts over X^{n+1}.

5 Principal Bundle and Characteristic classes

5.1 Principal Bundle and Classifying Space

Let G be a topological group and B a C_2 Haussdorff space.

Definition 5.1. A principal G-bundle $\pi\colon P \to B$ is a fibre bundle with a right G-action such that near each $b \in B$, $\exists$ a neighborhood $U \subset B$ and a trivialization

$$\begin{aligned} \varphi\colon \pi^{-1}(U) &\to U \times G \\ p &\mapsto (\pi(p), \alpha(p)) \end{aligned}$$

such that $\varphi(p, g) = (\pi(p), \alpha(p)g)$.

We can generate fibre bundles using principal bundle. Assume F is a left G-space or we have a representation $\rho\colon G \to \mathrm{Aut}(F)$. We set $E = P \times_G F = P \times F/(p, f) \sim (p \cdot g, g^{-1} f)$. Then $E \to B$ is a fibre bundle with fibre F. We call this construction Borel construction.

To get vector bundles, consider $\rho\colon G \to \mathrm{GL}_n(\mathbb{F})$. We get vector bundles over $\mathbb{F}$ of rank n.

Denote by $\mathcal{B}(B, G)$ the set of isomorphism class of principal G-bundles ober B.

Definition 5.2. We say EG is a universal (right) G-space if $\forall$ right G-space admits, up to G-homology, a unique G-map to EG.

For any $\Phi\colon P \to EG$, take quotients,

$$\begin{tikzcd} P \arrow[r, "\Phi"] \arrow[d] & EG \arrow[d] \\ B \arrow[r, "\overline{\Phi}"] & BG \end{tikzcd} .$$

Denote by $BG = EG/G$. We define

$$\begin{aligned} K_B\colon \mathcal{B}(B,G) &\to [B,BG] \\ [P] &\mapsto [\overline{\Phi}]. \end{aligned}$$

We have another map

$$\begin{aligned} \iota_B\colon [B,BG] &\to \mathcal{B}(B,G) \\ [f] &\mapsto f^*EG \end{aligned}$$

under the assumption that G acts freely and locally trivially on EG, where "locally trivially" means that for any $x \in EG$, there is a neighborhood U and a G-map $\varphi\colon U \to G$ (Then we have $U \cong U/G \times G$ by $x \mapsto (\overline{x}, \varphi(x))$).

Theorem 5.3. K_B and ι_B are inverse to each other. BG is called the classifying space of G.

Proof. $K_B \circ \iota_B = \mathrm{id}$**:**

Start with $f\colon B \to BG$, $\iota_B[f]$ is represented by f^*EG. Then we have

$$\begin{tikzcd} f^*EG \arrow[r, "F"] \arrow[d] & EG \arrow[d] \\ B \arrow[r, "f"] & BG \end{tikzcd}$$

Since EG is universal, F the unique map up to G-homotopy, then $[f] = K_B(f^*EG)$.

$\iota_B \circ K_B = \mathrm{id}$**:**

Given $P \to B$ representing $[P] \in \mathcal{B}(B,G)$, we get

$$\begin{tikzcd} P \arrow[r, "\Phi"] \arrow[d] & EG \arrow[d] \\ B \arrow[r, "\overline{\Phi}"] & BG \end{tikzcd}$$

Then $\iota_B \circ K_B([p]) = [\overline{\Phi}^* EG]$. By universal property of pull-back, we have

$$\begin{tikzcd} P \arrow[r, "\overline{\Psi}"] \arrow[rr, bend left, "\Phi"] \arrow[d] & \overline{\Phi}^* EG \arrow[r, "\Phi'"] \arrow[d] & EG \arrow[d] \\ B \arrow[r, "\mathrm{id}"] & B \arrow[r, "\overline{\Phi}"] & BG \end{tikzcd}$$

where $\overline{\Psi}$ is given by

$$\begin{aligned} &\overline{\Psi}\colon U \times G \to U \times G \\ &\overline{\Psi}(x, g \cdot h) = \overline{\Psi}(xg) \cdot h = (x, \Psi(g) \cdot h). \end{aligned}$$

Then $\overline{\Psi}$ is an isomorphism due to G-equivalence. $\square$

Definition 5.4. Let X_1, X_2 be two spaces. The joint of X_1, X_2 is defined by

$$X_1 * X_2 \coloneqq X_1 \times X_2 \times [0,1]/(x_1, x_2, 0) \sim (x_1, x_2', 0), (x_1, x_2, 1) \sim (x_1', x_2, 1).$$

Then the embedding $X_1 \hookrightarrow X_1 * X_2$ is given by

$$x_1 \mapsto [x_1, x_2, 0] = [(1, x_1), (0, x_2)].$$

Definition 5.5. The Milnor space if G is defined to be

$$EG \coloneqq \operatorname{colim}_{\underset{n}{\rightarrow}} G * \cdots * G = \left\{ \prod_{i \geq 1} (t_i, g_i) : \sum t_i = 1, \text{only finitely } t_i\text{'s are non-zero} \right\}.$$

Define $G \curvearrowright EG$ by

$$g \cdot \left[\prod_{i \geq 1} (t_i, g_i) \right] = \left[\prod_{i \geq 1} (t_i, g_i g) \right].$$

Claim 15. G acts freely and locally trivially on EG.

Reason: Choose an atlas $\{V_j\}_{j \geq 1}$ of EG as follows:

$$V_j \coloneqq \left\{ \left[\prod_{i \geq 1} (t_i, g_i) \right] : t_j \in (0, 1] \right\}.$$

Define $p_j \colon V_j \to G$ by

$$\left[\prod_{i \geq 1} (t_i, g_i) \right] \mapsto g_j.$$

Then $\pi_G \colon EG \to BG = EG/G$ is a principal G-bundle.

Proposition 5.6. The Milnor space EG is contractible.

Proof. **Step 1:**
We start with the map

$$\begin{gathered} \alpha_0 \colon EG \to EG \\ [(t_1, g_1), (t_2, g_2), \cdots] \mapsto [(t_1, g_1), (0, e), (t_2, g_2), \cdots]. \end{gathered}$$

Consider the homotopy

$$\begin{gathered} \alpha_t \colon EG \to EG \\ [(t_1, g_1), (t_2, g_2), \cdots] \mapsto [(t_1, g_1), (tt_2, g_2), ((1-t)t_2, g_2), (t_3, g_3), \cdots]. \end{gathered}$$

Then

$$\begin{gathered} \alpha_1 \colon EG \to EG \\ [(t_1, g_1), (t_2, g_2), \cdots] \mapsto [(t_1, g_1), (t_2, g_2), (0, e), (t_3, g_3), \cdots]. \end{gathered}$$

Inductivilt, we get α_n which inserts $(0, e)$ into the $(n+1)$-th entry. Then take $\alpha_\infty = \operatorname{colim}_{\rightarrow} \alpha_n = \mathrm{id}$. Hence we have $\alpha_0 \simeq \alpha_\infty$.

Step 2:
Consider

$$f_t \colon [(t_1, g_1), \cdots] \mapsto [((1-t)t_1, g_1), (t, e), ((1-t)t_2, g_2), ((1-t)t_3, g_3), \cdots].$$

Then $f_0 = \alpha_0$, f_1 is the contraction onto $[(0, e), (1, e), (0, e), \cdots]$. ∎

Theorem 5.7. The Milnor space $\pi_G\colon EG \to BG$ is the universal G-bundle.

Proof. Let $\pi\colon P \to B$ be a principal G-bundle with contractible trivialization charts

$$\begin{aligned}\pi^{-1}(U_i) &\to U_i \times G\\ p &\mapsto (\pi(p), \varphi_i(p)).\end{aligned}$$

Choose a partition of unity $\rho_i\colon U_i \to [0,1]$ subordinate to $\{U_i\}$. Write $v_i = \rho_i \circ \pi$ which is G-invariant. We define

$$\begin{aligned}\Phi\colon P &\to EG\\ p &\mapsto \left[\prod_{i\geq 1}(v_i(p), \varphi_i(p))\right].\end{aligned}$$

Then $\sum_{i\geq 1} v_i(p) = 1$ hence φ_i is a G-map.

We need to show $\Phi\colon P \to EG$ is defined uniquely up to G-homotopy. Suppose $\Psi\colon P \to EG$ is another G-map. Write

$$\begin{aligned}\Phi([\prod(t_i, g_i)]) &= [\prod(\Phi^1(t_i), \Phi^2(g_i))],\\ \Psi([\prod(t_i, g_i)]) &= [\prod(\Psi^n(t_i), \Psi^n(g_i))].\end{aligned}$$

By Proposition 5.6, Φ is homotopic to

$$\Phi_1\colon [(t_1, g_1), (t_2, g_2), \cdots] \mapsto [(\Phi^1(t_1), \Phi^2(g_1)), (0, e), (\Phi^1(t_2), \Phi^2(g_2)), (0, e), (\Phi^1(t_3), \Phi^2(g_3)), (0, e), \cdots].$$

Apply the same procedure to odd entries, we get a homotopy Ψ to

$$\Psi_1\colon [(t_1, g_1), (t_2, g_2), \cdots] \mapsto [(\Psi^1(t_1), \Psi^2(g_1)), (0, e), (\Psi^1(t_2), \Psi^2(g_2)), (0, e), (\Psi^1(t_3), \Psi^2(g_3)), (0, e), \cdots].$$

We know Φ_1 is homotopic to Ψ_1 by

$$\prod(t_i, g_i) \mapsto [(((1-t)\Phi^1(t_i), \Phi^2(g_i)), (t\Psi^1(t_i), \Psi^2(g_i))].$$

Therefore Φ is G-homotopic to Ψ. $\square$

Theorem 5.8. Let $\pi_G\colon E \to X$ be a principal G-bundle such that E is contractible. Then $\pi_G\colon E \to X$ is the universal G-bundle.

Proof. Let $\pi\colon P \to B$ be a principal G-bundle with B a CW-complex. We consider the fibre bundle $p \times_G E \to B$ with contractible fibres.

Consider Obstruction Theory.

$$\begin{tikzcd} & P \times_G E \arrow[d] \\ B \arrow[ur, dashed, "\exists ? s"] \arrow[r, "\mathrm{id}"] & B \end{tikzcd}$$

The obstruction of lifting on B^{n+1} lies in $H^{n+1}(B, \{\pi(\text{fibre})\}) = 0$. Then $P \times_G E \to B$ admits sections.

Take a section

$$\begin{aligned}s\colon B &\to P \times_G E = P \times E/(p, x) \sim (pg, xg)\\ b &\mapsto [p, s^p(p)].\end{aligned}$$

Then we get a G-map $S^p\colon P \to E$.

Suppose we have two G-maps $S_1^p, S_2^p\colon P \to E$. We get two sections $s_1, s_2\colon B \to P \times_G E$. Apply obstruction theory to $(P \times_G E) \times I \to B \times I$. We have $s_1 \simeq s_2$ hence s_1^p is G-homotopic to s_2^p. $\square$

5.1.1 Functorial Property of BG

Let $\alpha\colon H \to G$ be a continuous homomorphism. We regard G as a left H-space by $h \cdot g := \alpha(h)g$. We get principal G-bundle $EH \times_\alpha G \to BH$. By universal property of EG, we have

$$\begin{tikzcd} EH \times_\alpha G \arrow[r, "E(\alpha)"] \arrow[d] & EG \arrow[d] \\ BH \arrow[r, "B(\alpha)"] & BG \end{tikzcd}$$

where $B(\alpha)$ is defined uniquely up to homotopy. We have $B(\alpha) \circ B(\beta) \simeq B(\alpha \circ \beta)$. Now we get a functor $B(-)$.

When $H < G$ is a subgroup, we get $B(i)\colon BH \to BG$. We can regard EG as a H-space. H acts freely and locally trivially on EG.Therefore $EG \to EG/H$ is a principal H-bundle. Note that EG is contractible. EG is a universal H-bundle. This means $BH \simeq EG/H$. Hence $B(i)\colon EG/H \to EG/G$ is a fibre bundle with fibre G/H.

Proposition 5.9. Let $H < G$ be a subgroup such that G/H admits a CW-complex. Suppose $i\colon H \hookrightarrow G$ is a homotopy equivalence. Then $B(i)\colon BH \to BG$ is a homotopy equivalence.

Proof. $i\colon H \to G$ is a homotopy equivalence, then G/H is weakly contractible. This is given by some directly calculations of homotopy exact sequence of $H \hookrightarrow G \to G/H$. Then G/H is contractible. Hence $BH \simeq BG$. □

Example 5.10.

H	G
$\mathrm{O}(n)$	$\mathrm{GL}_n(\mathbb{R})$
$\mathrm{SO}(n)$	$\mathrm{GL}_n^*(\mathbb{R})$
$\mathrm{U}(n)$	$\mathrm{GL}_n(\mathbb{C})$
$\mathrm{SU}(n)$	$\mathrm{SL}_n^*(\mathbb{C})$

This means the classification of $\mathbb{R}^n$-bundles is equivalent to the classification of $\mathrm{O}(n)$-bundles.

Definition 5.11. Let R be a ring. Then each non-zero element $c \in H^*(BG; R)$ is called a universal characteristic class.

Given any principal G-bundle $\pi\colon P \to B$, define the corresponding characteristic class to be $\overline{\Phi}^* c \in H^*(B; R)$ by

$$\begin{tikzcd} P \arrow[r, "\Phi"] \arrow[d] & EG \arrow[d] \\ B \arrow[r, "\overline{\Phi}"] & BG \end{tikzcd}$$

. $\overline{\Phi}^*$ is called the classifying map.

5.2 Stiefel-Whitney Classes