Related Topics in Geometric Group Theory

CUI Jiaqi
East China Normal University

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Abstract

This is a note of discussions with my tutor WAN Renxing.

Contents

1 Bounded Cohomology

1.1 Definitions

For a group G, denote

$$C_b^n(G,\mathbb{R}) \coloneqq \{\varphi\colon G^n \to \mathbb{R} : \sup|\varphi| < \infty\}$$

where φ is just a map instead of a homomorphism. Define the boundary operator $\delta\colon C_b^n(G,\mathbb{R}) \to C_b^{n+1}(G,\mathbb{R})$ as follow: For any $\varphi \in C_b^n(G,\mathbb{R})$, let

$$\delta\varphi\left(g_0,\cdots,g_n\right) \coloneqq \varphi\left(g_1,\cdots,g_n\right) + \sum_{i=1}^{n}(-1)^i\varphi\left(g_0,\cdots,g_{i-1}g_i,\cdots,g_n\right) + (-1)^{n+1}\varphi\left(g_0,\cdots,g_{n-1}\right).$$

It's easy to check that $\delta\varphi \in C_b^{n+1}(G,\mathbb{R})$ and $\delta^2 = 0$. So $(C_b^*(G,\mathbb{R}),\delta)$ is a cochain complex.

Definition 1.1. The *bounded cohomology* of G is defined by

$$H_b^*(G,\mathbb{R}) \coloneqq \frac{\ker\delta^*}{\operatorname{im}\delta^{*-1}}.$$

Fact 1.2. (1) For any group G, $H_b^1(G,\mathbb{R}) = 0$. It's because $\varphi \in \ker\delta^1$, if and only if

$$0 = \delta\varphi(g,h) = \varphi(gh) - \varphi(g) - \varphi(h), \quad \forall g,h \in G,$$

if and only if φ is a homomorphism. But a bounded homomorphism to $\mathbb{R}$ must be zero.

(2) For any solvable group G, $H_b^n(G,\mathbb{R}) = 0$, $\forall n > 0$.

(3) For any hyperbolic group G, $H_b^2(G,\mathbb{R})$ has infinite dimension.

(4) For free group F_n, $\forall n > 0$, $H_b^3(F_n,\mathbb{R})$ has infinite dimension.

(5) For amenable group G, $H_b^n(G) = 0$, $\forall n \geq 1$.

Question 1.3. What about $H_b^n(F_n,\mathbb{R})$ for $n \geq 4$?

1.2 Quasimorphism

Definition 1.4. For a group G, a map $\varphi\colon G \to \mathbb{R}$ is a *quasimorphism* if $\exists D > 0$ such that

$$|\varphi(gh) - \varphi(g) - \varphi(h)| \leq D, \quad \forall g, h \in G.$$

Example 1.5. (1) The integer function $\mathbb{R} \to \mathbb{R}$, $x \mapsto \lfloor x \rfloor$ is a quasimorphism.

(2) For a manifold M with a 1-form ω, $\varphi_\omega\colon \pi_1(M) \to \mathbb{R}$, $\varphi_\omega(\alpha) \coloneqq \int_\alpha \omega$ is a quasimorphism.

Example 1.6 (Brooks Counting Quasimorphism)**.** For any free group, for example, $F_2 = \langle a, b\rangle$, and any reduced word w on it, define $C_w\colon F_2 \to \mathbb{Z}$ by

$$C_w(g) \coloneqq \text{the number of occurences of } w \text{ in } g, \quad \forall g = s_1 s_2 \cdots s_n \in F_2,\ s_i \in \{\pm a, \pm b\}.$$

Define the counting function $h_w\colon F_2 \to \mathbb{Z}$ by

$$h_w(g) \coloneqq C_w(g) - C_{w^{-1}}(g).$$

Then h_w is a quasimorphism. Especially, h_w is a homomorphism if $|w| = 1$.

Remark 1.7. Under a suitable topology on the space of all quasimorphisms of F_n, the space of all Brooks counting quasimorphisms is dense.

1.3 The 2nd Bounded Cohomology of Free Groups

Lemma 1.8. Let $\varphi\colon G \to \mathbb{R}$ be a quasimorphism, then $[\delta\varphi] \in H_b^2(G,\mathbb{R})$. Especially, if φ is unbounded, $[\delta\varphi] \neq 0$.

Proof. It follows by definition that

$$|\delta\varphi(g,h)| = |\varphi(g) + \varphi(h) - \varphi(gh)| \leq D < \infty.$$

So $[\delta\varphi] \in H_b^2(G,\mathbb{R})$. And if φ is unbounded, $\varphi \notin C_b^1(G,\mathbb{R})$. Therefore, $[\delta\varphi] \notin \operatorname{im}\delta^1$ and then $[\delta\varphi] \neq 0$. ☐

Theorem 1.9. For free group F_2, $H_b^2(F_2,\mathbb{R})$ has infinite dimension.

Proof. Choose two non-conjugate elements g_1, g_2 of F_2 and let $w_i = g_1^{l_i} g_2^{m_i} g_1^{n_i} g_2^{k_i}$ for $i \geq 1$ where $l_1 \ll m_1 \ll n_1 \ll k_1 \ll l_2 \ll m_2 \ll n_2 \ll k_2 \ll$
cdots. We claim that

(1) For any $j > i$, $h_{w_i}(w_j) = 0$.

(2) For any $i, n \geq 1$, $h_{w_i}(w_i^n) \geq n$.

Then we prove that $\{\delta h_{w_i}\}$ is linear independent. Suppose that $\sum_{i=1}^{\infty} a_i \delta h_{w_i} = 0$, where the infinite sum is well defined by our claim (1). This means that there exists a bounded map b such that

$$\sum_{i=1}^{\infty} a_i h_{w_i} + b = 0.$$

Operating on w_1^n, we have

$$0 = a_1 h_{w_1}\left(w_1^n\right) + b\left(w_1^n\right) \geq a_i n + b\left(w_1^n\right)$$

by claim (2). Because b is bounded, let $n \to \pm\infty$, we must have $a_1 = 0$. Then doing the same things for $i = 2$, by induction, we have $a_i = 0$, $\forall i \geq 1$.

Finally, by claim (2), $\{\delta h_{w_i}\}$ are all unbounded. Then by our lemma above, linear independent $\{\delta h_{w_i}\}$ give independent classes $\{[\delta h_{w_i}]\}$ in $H_b^2\left(F_2, \mathbb{R}\right)$. So we conclude that $\dim H_b^2\left(F_2, \mathbb{R}\right) = \infty$, as desired. $\square$

1.4 Generalization

Epstein and Fujiwara generalized Brooks counting function for any group and proved that $H_b^2\left(G, \mathbb{R}\right)$ has infinite dimension for any group G acting on a Gromov-hyperbolic space properly and discontinuously [1].

Let X be a metric space and G be a group acting on X isometrically. Fix a finite directed path w in X. For any path γ in X, define

$$|\gamma|_w := \text{the number of occurences of } w \text{ in } \gamma,$$

where "occurence" means that there is $g \in G$ such that $gw \subset \gamma$. Then for any $x, y \in X$ and $0 < W \leq |w|$, define

$$c_{w,W}([x, y]) := d(x, y) - \inf_{\alpha}\left(|\alpha| - W\,|\alpha|_w\right),$$

where $[x, y]$ denotes the geodesic connecting x, y, the infimum ranges over all paths in X connecting x, y and $|\alpha|$ denote the length of α in X.

They proved that $h_{w,W} = c_{w,W} - c_{w^{-1},W}$ is also a quaismorphism if X is a Gromov hyperbolic space and G contains F_2 a subgroup, which promises that the proof for free groups above is valid for these G, especially, for hyperbolic groups (just let hyperbolic groups act on their Cayley Graph).

Question 1.10. If G can act on a two hyperbolic spaces X, Y isometrically and coboundedly (and non-properly), which induce a proper action of G on $X \times Y$ with ℓ^1-norm, what can we say about $H_b^2(G)$?

We can consider the classification of unbounded isometric actions on Gromov Hyperbolic spaces [2]:

1. horocyclic (parabolic): if there is no hyperbolic elements;
2. lineal: if all hyperbolic elements have the same fixed points;
3. focal (quasi-parabolic): if all hyperbolic elements have exactly one common fixed point;
4. of general type: if there are two independent hyperbolic elements,

where hyperbolic element g means that $g^{+\infty} \neq g^{-\infty}$, and independent hyperbolic elements g_1, g_2 means that $g_1^{+\infty} \neq g_2^{\pm\infty}$ and $g_1^{-\infty} \neq g_2^{\pm\infty}$. We found that if one of the actions on X or Y is of general type, the proof of [1] is valid. And it's well-known that any group have a horocyclic action on a hyperbolic space. So there are three cases left:

1. two actions are lineal;
2. one action is lineal and another is focal;
3. two actions are focal.

Theorem 1.11. If the two actions $G \curvearrowright X, Y$ are both lineal, then G is virtually abelian. Therefore, the second bounded cohomology of G vanishes.

Proof. Consider a subgroup with index 2, we can assume that the two lineal actions are orientable. Then all the conmmutators of G act on X and Y uniformly boundedly hence on $X \times Y$. And because the action on $X \times Y$ is proper, G must have only finite commutators. Therefore, G is virtually abelian. In fact, G must be isometric to trivial group, $\mathbb{Z}$ or $\mathbb{Z}^2$. $\square$

The goal is to classify the groups satisfying Case (2) and Case (3).

Example 1.12. Let's consider the Baumslag–Solitar Group $BS(1,2) = \langle a, t : tat^{-1} = a^2 \rangle$.

1. There is a focal action of $BS(1,2)$ on $\mathbb{H}^2$: notice that $\operatorname{Isom}(\mathbb{H}^2) = \mathrm{PSL}(2,\mathbb{R})$, let $a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $t \mapsto \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$. It isn't proper. Just consider $\infty \in \mathbb{H}^2$. For any z near ∞, use t to let z be close to 0 and use a to let z go to the infinity.

2. There is a focal action of $BS(1,2)$ on T, where T is the corresponding Bass-Serre tree of degree 3: $BS(1,2)$ can be written as a HNN-extension:

$$BS(1,2) = \langle a, t : tat^{-1} = a^2 \rangle = \langle a \rangle_{a \sim a^2}.$$

   It's also non-proper. Consider the vertex $[t^{-1}]$, we have $a[t^{-1}] = [at^{-1}] = [t^{-1}a^2] = [t^{-1}]$.

But the induced action $BS(1,2) \curvearrowright \mathbb{H}^2 \times T$ is proper. In fact, by Bass-Serre theory, the vertex stablizers of $G \curvearrowright T$ is $\langle a \rangle$, while $\langle a \rangle \curvearrowright \mathbb{H}^2$ is proper. And because T is a tree, other elements of G must move all vertexes away. So the product action is proper. And because $BS(1,2) = \mathbb{Z}\left[\frac{1}{2}\right] \rtimes \mathbb{Z}$ is solvable hence amenable, $H_b^2(BS(1,2),\mathbb{R}) = 0$. This gives an example of Case (3).

2 Quasi-homomorphism

Definition 2.1 (Ulam)**.** We say $\phi\colon G \to H$ is a quasi-homomorphism, if the set

$$\left\{ (\phi(x))^{-1}(\phi(y))^{-1}\phi(yx) : x, y \in G \right\}$$

is finite.

Fujiwara and Kapovich proved that every quasi-homomorphism is constructible [3], that's to say,

Theorem 2.2. Any quasi-homomorphism $\phi\colon G \to H$ can be constructed by the following operations:

1. Lift (if possible):

$$\begin{tikzcd} & & G \arrow[d, "\phi"'] \arrow[dr, "\overline{\phi}"] & & \\ 1 \arrow[r] & A \arrow[r] & H \arrow[r] & \overline{H} \arrow[r] & 1 \end{tikzcd}$$

   where A is an abelian group.

2. Product:

$$\phi = (\phi_1, \cdots, \phi_n) : G \to \prod_{i=1}^{n} H_i$$

   where $\phi_i\colon G \to H_i$ is a quasi-homomorphism.

3. Composition: $\phi\colon G \to H$ fits into

$$G \xrightarrow{\phi_1} K \xrightarrow{\phi_2} H$$

   where ϕ_1, ϕ_2 are quasi-homomorphism.

4. Extension from a finite index subgroup (if possible): $\phi\colon G \to H$ is induced by $G_0 \leq G$ such that $[G, G_0] < \infty$ and a quasi-homomorphism $\phi_0\colon G_0 \to H$.

5. Bounded perturbation (if possible): $\phi\colon G \to H$ is induced by $\phi'\colon G \to H$ where

$$\operatorname{dist}(\phi, \phi') \coloneqq \sup_{g \in G} d(\phi(g), \phi'(g)) < \infty$$

under the word metric of H.

Remark 2.3. Their proof is only "group theoretical" without any "geometric argument".

There is another definition of quasi-homomorphism by Hartnick and Schweitzer [4]:

Definition 2.4. We say $\phi\colon G \to H$ is a quasi-homomorphism, if $f \circ \phi\colon G \to \mathbb{R}$ is a quasimorphism for any quasimorphism $f\colon H \to \mathbb{R}$.

Remark 2.5. There are examples saying that the two kinds of definitions are not equivalent.

Question 2.6.
1. Under what conditions, a quasi-homomorphism from a finite index subgroup can be extended to the whole group?
2. Under what conditions, a quasi-homomorphism from a quotient group can be lifted to the whole group?
3. Under what conditions, a quasi-homomorphism can be constructed by a bounded perturbation from a quasi-homomorphism?

Question 2.7. A map $r\colon G \to H$ is a quasi-retraction if r is a quasi-homomorphism and $r|_H = \mathrm{id}_H$. Under what conditions, a finite index subgroup is a quasi-retraction of G?

3 Girth Alternative

Definition 3.1. Let G be a finitely generated group. Denote by $X(G)$ the set of finite non-empty subsets of G which generate the whole group. The girth of $X \in X(G)$, denoted by $U(X, G)$, is the length of shortest relation among the elements of X in G. The girth of G then is defined as $U(G) = \sup\{U(X, G) : X \in X(G)\}$.

In [5], Akhmedov proved the Girth Alternative of hyperbolic groups (and of one-relator groups and linear groups), using the idea of the proof of their Tits Alternative by Tits [6].

Theorem 3.2 (Tits Alternative)**.** For any hyperbolic (or one-relator or linear) group G, it either contains free group of rank two or is virtually solvable.

Theorem 3.3 (Girth Alternative)**.** For any hyperbolic (or one-relator or linear) group G, the property of containing non-abelian free subgroup and the property of having infinite girth coincide.

We want to generalize this theorem just like in Section 1:

Question 3.4.
1. Does Girth Alternative hold for a group acting properly on a Gromov Hyperbolic space with at least 3 limit points?
2. Does Girth Alternative hold for a group acting properly on a product of two Gromov Hyperbolic space with cobounded action on each factor?

References

[1] Koji Fujiwara. The second bounded cohomology of a group acting on a Gromov-hyperbolic space. *Proc. London Math. Soc. (3)*, 76(1):70–94, 1998.

[2] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

[3] Koji Fujiwara and Michael Kapovich. On quasihomomorphisms with noncommutative targets. *Geom. Funct. Anal.*, 26(2):478–519, 2016.

[4] Tobias Hartnick and Pascal Schweitzer. On quasioutomorphism groups of free groups and their transitivity properties. *J. Algebra*, 450:242–281, 2016.

[5] Azer Akhmedov. The girth of groups satisfying Tits alternative. *J. Algebra*, 287(2):275–282, 2005.

[6] J. Tits. Free subgroups in linear groups. *J. Algebra*, 20:250–270, 1972.