Homotopy Theory and Characteristic Classes

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Abstract

This is the notes of a course given by Prof. Ma Langte in 25spring at Shanghai Jiaotong University.

Contents

Part I
Homotopy Theory

Let **TOP** be the category of topological spaces. Then we can take a quotient of **TOP** and get the homotopy category $h-\mathbf{TOP}$. The quotient may bring more algebraic structures. For example, Mor $\left(S^1, X\right)$, the homotopy classes of maps from S^1 to X, is the fundamental group of X. Our goal is to study functors from hmotopy category to some algebraic categories.

Let $\mathbf{TOP}^o$ be the pointed topological category, where the sum is wedge sum $(X, x_0) \wedge (Y, y_0) = X \sqcup Y / x_0 \sim y_0$ and the product is the smash product $(X, x_0) \vee (Y, y_0) = X \times Y / \{x_0\} \times Y \cup X \times \{y_0\}$. Similarly, we can take a quotient to get $h - \mathbf{TOP}^o$.

Let $\mathbf{TOP}(2)$ be the category of pairs and $h - \mathbf{TOP}(2)$ be its quotient.

Fix $K \in \mathrm{Ob}(\mathbf{TOP})$. Let's consider $\mathbf{TOP}^K$, the category of spaces under K. Its objects are maps $f\colon K \to X$ and morphisms are maps $\alpha\colon X \to Y$ such that $\alpha \circ f = g$.

$$\begin{tikzcd} & K \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, "\alpha"] & & Y \end{tikzcd}$$

If $K = \{*\}$ is a single point set, then $\mathbf{TOP}^{\{*\}} = \mathbf{TOP}^o$ is the pointed topological category. Take $X = K$. A morphism from $f\colon K \to X$ to $\mathrm{id}\colon K \to K$ is $r\colon X \to K$ such that $r \circ f = \mathrm{id}$.

$$\begin{tikzcd} & K \arrow[dl, "f"'] \arrow[dr, "\mathrm{id}"] & \\ X \arrow[rr, "r"] & & K \end{tikzcd}$$

When $K \subset X$, $f = i\colon K \hookrightarrow X$, we say that r is a retraction.

We have $r\colon X \to K$ is a deformation retraction, if and only if $i \circ r \simeq \mathrm{id}_X$ rel K, if and only if $r\colon X \to K$ is a homotopy equivalence in $\mathbf{TOP}^K$.

Fix $B \in \mathrm{Ob}(\mathbf{TOP})$. Let's consider $\mathbf{TOP}_B$, the category of spaces over B, where the objects are $p\colon X \to B$ and morphisms are $f\colon X \to Y$ such that $p = q \circ f$.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

Take $X = B$. A morphism from $\mathrm{id}\colon B \to B$ to $q\colon Y \to B$ is $s\colon B \to Y$ such that $q \circ s = \mathrm{id}_B$.

$$\begin{tikzcd} B \arrow[rr, "s"] \arrow[dr, "\mathrm{id}"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

Then s is called a section of q.

Similarly, we can define $h - \mathbf{TOP}^K$ and $h - \mathbf{TOP}_B$.

1 Cofibrations and Fibrations

1.1 Cofibrations

Definition 1.1. A map $i\colon A \to X$ has the homotopy extension property (HEP) for a space Y if for all homotopy $h\colon A \times I \to Y$ and $f\colon X \to Y$ with $f \circ i(a) = h(a,1)$, there exists $H\colon X \times I \to Y$ satisfies

$$\begin{tikzcd} & X \arrow[dr, "i_1"] \arrow[drr, bend left, "f"] & & \\ A \arrow[ur, "i"] \arrow[dr, "i_1"'] & & X \times I \arrow[r, dashed, "H"] & Y \\ & A \times I \arrow[ur, "i \times \mathrm{id}"] \arrow[urr, bend right, "h"'] & & \end{tikzcd}$$

We say $i\colon A \to X$ is a cofibration if it has HEP for each $Y \in \mathrm{Ob}(\mathbf{TOP})$.

Recall the mapping cylinder: if $i\colon A \to X$ is a map, then $Z(i) \coloneqq (A \times I) \sqcup X/(a,1) \sim i(a)$.

Proposition 1.2. Given a map $i\colon A \to X$. The followings are equivalent:

1. $i\colon A \to X$ is a cofibration.
2. i has HEP for $Z(i)$.
3. The map

$$\begin{aligned} s\colon Z(i) &\to X \times I \\ (a,t) &\mapsto (i(a),t), \\ x &\mapsto (x,1) \end{aligned}$$

   has a retraction.

Proof. (1)$\Longrightarrow$(2) is only by definition.

(2)$\Longrightarrow$(1): By definition, there exists $K\colon X \times I \to Z(i)$ such that the following diagram is commutative.

$$\begin{tikzcd}
& X \arrow[dr, "i_1"] \arrow[drr, bend left, "i_1"] & & \\
A \arrow[ur, "i"] \arrow[dr, "i_1"'] & & X \times I \arrow[r, dashed, "K"] & Z(i) \\
& A \times I \arrow[ur, "i\times\mathrm{id}"] \arrow[urr, bend right, "\mathrm{id}"'] & &
\end{tikzcd}$$

For any Y and homotopy $h\colon A \times I \to Y$ and $f\colon X \to Y$ with $f \circ i(a) = h(a,1)$, we define

$$\begin{aligned} F\colon Z(i) &\to Y \\ (a,t) &\mapsto h(a,t) \\ x &\mapsto f(x). \end{aligned}$$

Then $F \circ K$ is as desired.

$$\begin{tikzcd}
& X \arrow[dr, "i_1"] \arrow[drr, bend left, "i_1"] \arrow[drrr, bend left, "f"] & & & \\
A \arrow[ur, "i"] \arrow[dr, "i_1"'] & & X \times I \arrow[r, dashed, "K"] & Z(i) \arrow[r, "F"] & Y \\
& A \times I \arrow[ur, "i\times\mathrm{id}"] \arrow[urr, bend right, "\mathrm{id}"'] \arrow[urrr, bend right, "h"'] & & &
\end{tikzcd}$$

(2)$\Longrightarrow$(3): We can easily check that the extension $K\colon X \times I \to Z(i)$ in the proof of (2)$\Longrightarrow$(1) is a retraction of s.

(3)$\Longrightarrow$(2): Let r be a retraction of s. For any homotopy $h\colon A \times I \to Z(i)$ and $f\colon X \to Z(i)$ with $f \circ i(a) = h(a,1)$, we define

$$\begin{aligned} \sigma\colon Z(i) &\to Z(i) \\ (a,t) &\mapsto h(a,t) \\ x &\mapsto f(x). \end{aligned}$$

Then we can verify that $H = \sigma \circ r \colon X \times I \to Z(i)$ extends h.

$$
\begin{tikzcd}
 & X \arrow[dr, "i_1"] \arrow[drr, bend left, "f"] & & \\
A \arrow[ur, "i"] \arrow[dr, "i_1"'] & & X \times I \arrow[r, shift right, "r"'] & Z(i) \arrow[l, shift right, "s"'] \\
 & A \times I \arrow[ur, "i \times \mathrm{id}"] \arrow[urr, bend right, "h"'] & &
\end{tikzcd}
$$

$\square$

Corollary 1.3. When $A \subset X$ is a close subset, $i \colon A \hookrightarrow X$ is the inclusion map. Then $i \colon A \to X$ is a cofibration $\iff Z(i) = A \times I \cup X \times \{1\}$ is a retraction of $X \times I$.

Therefore, we can construct many cofibrations. For example, let (X, A) be a manifold with boundary, then $i \colon A \hookrightarrow X$ is a cofibration.

1.1.1 Push-Out of Cofibration

Given a commutative diagram,

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] & B \arrow[d, "J"] \\
X \arrow[r, "F"] & Y
\end{tikzcd}
$$

the push-out of j along f is the initial object of this diagram, i.e. $j \colon B \to Y$, $F \colon X \to Y$, s.t. $\forall Z$ with $J' \colon B \to Z$, $F' \colon X \to Z$ satisfying $J' \circ f = F' \circ j$, $\exists!$ map $p \colon Y \to Z$ such that the diagram is commutative.

$$
\begin{tikzcd}
 & B \arrow[dr, "J"] \arrow[drr, bend left, "J'"] & & \\
A \arrow[ur, "f"] \arrow[dr, "j"'] & & Y \arrow[r, dashed, "\exists! p"] & Z \\
 & X \arrow[ur, "F"] \arrow[urr, bend right, "F'"'] & &
\end{tikzcd}
$$

In our setting, we can construct $Y = X \sqcup B / f(a) \sim j(a)$ directly.

Proposition 1.4. If $j \colon A \to X$ is a cofibration, then the push-out of j along f $J \colon B \to Y$ is also a cofibration.

Proof. For any Z, $g \colon Y \to Z$, $h \colon B \times I \to Z$ such that $g \circ J = h \circ (i_1 \times \mathrm{id})$, we need to find $H \colon Y \times I \to Z$ such that the following diagram is commutative.

$$
\begin{tikzcd}
 & X \arrow[dr, "F"] & & & \\
A \arrow[ur, "j"] \arrow[dr, "f"'] & & Y \arrow[dr] \arrow[drr, bend left, "g"] & & \\
 & B \arrow[ur, "J"] \arrow[dr] & & Y \times I \arrow[r, dashed, "H"] & Z \\
 & & B \times I \arrow[ur] \arrow[urr, bend right, "h"'] & &
\end{tikzcd}
$$

Because $j\colon A \to X$ is a cofibration, we have $G\colon X \times I \to Z$ such that the following diagram is commutative.

$$\begin{tikzcd}
 & X \arrow[dr] \arrow[drr, bend left, "g\circ F"] & & \\
A \arrow[ur, "j"] \arrow[dr] & & X \times I \arrow[r, dashed, "G"] & Z \\
 & A \times I \arrow[ur] \arrow[urr, bend right, "h\circ(f\times\mathrm{id})"'] & &
\end{tikzcd}$$

Using the fact that $J \times \mathrm{id}\colon B \times I \to Y \times I$ is also the push-out of $j \times \mathrm{id}\colon A \times I \to X \times I$ along $f \times \mathrm{id}\colon A \times I \to B \times I$, we have unique $H\colon Y \times I \to Z$ such that the following diagram is commutative.

$$\begin{tikzcd}
 & B \times I \arrow[dr] \arrow[drr, bend left, "h"] & & \\
A \times I \arrow[ur, "f\times\mathrm{id}"] \arrow[dr] & & Y \times I \arrow[r, dashed, "H"] & Z \\
 & X \times I \arrow[ur] \arrow[urr, bend right, "G"'] & &
\end{tikzcd}$$

The $H\colon Y \times I \to Z$ is the extension of $h\colon B \times I \to Z$, as desired. $\square$

In terms of categorical language, let $\Pi(A, B)$ be a category, whose objects are continue maps from A to B and morphisms are homotopy of maps from A to B. Consider $\mathbf{COF}^B \subset \mathbf{TOP}^B$ the subcategory of cofibrations under B (i.e. $J\colon B \to Y$). Then we have homotopy category $h - \mathbf{COF}^B$. Given a cofibration $i\colon A \to X$, we get a contravariant functor

$$\beta\colon \Pi(A, B) \to h - \mathbf{COF}^B.$$

In fact, we only need to check that if $f_0 \simeq f_1\colon A \to B$, then we get a morphism from $J_0\colon B \to Y_0$ to $J_1\colon B \to Y_1$. Firstly, consider the homotopy $J_0 \circ f_t\colon A \times I \to Y_0$, we get its extension $\Psi\colon X \times I \to Y_0$.

$$\begin{tikzcd}
 & X \arrow[dr] \arrow[drr, bend left, "F_0"] & & \\
A \arrow[ur, "i"] \arrow[dr] & & X \times I \arrow[r, dashed, "\Psi"] & Y_0 \\
 & A \times I \arrow[ur] \arrow[urr, bend right, "J_0\circ f_t"'] & &
\end{tikzcd}$$

Then by the universal property of the push-out $J_1\colon B \to Y_1$ of i along f_1 for $J_0\colon B \to Y_0$ and $\Psi_1\colon X \to Y_0$, we get a map $K\colon Y_1 \to Y_0$, as desired.

$$\begin{tikzcd}
 & B \arrow[dr, "J_1"] \arrow[drr, bend left, "J_0"] & & \\
A \arrow[ur, "f_1"] \arrow[dr, "i"'] & & Y_1 \arrow[r, dashed, "K"] & Y_0 \\
 & X \arrow[ur, "F_1"] \arrow[urr, bend right, "\Psi_1"'] & &
\end{tikzcd}$$

1.1.2 Replacing a Map by a Cofibration

Given a map $f\colon X \to Y$, consider the mapping cylinder $Z(f)$. We can notice that $Z(f)$ is the push-out.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_1"'] & Y \arrow[d, "s"] \\ X \times I \arrow[r, "a"] & Z(f) \end{tikzcd}$$

We also have a map

$$\begin{aligned} q\colon Z(f) &\to Y \\ (x,t) &\mapsto f(x). \end{aligned}$$

Note that by Proposition 1.2, $i_1\colon X \hookrightarrow X \times I$ is a cofibration $\iff X \times \{1\} \times I \cup X \times I \times \{1\}$ is a retraction of $X \times I \times I$, we have $s\colon Y \to Z(f)$ is a cofibration.

Proposition 1.5. Let

$$\begin{aligned} j\colon X &\to Z(f) \\ x &\mapsto (x,0), \end{aligned}$$

we have

1. $j\colon X \to Z(f)$ is a cofibration.
2. $s \circ q \simeq \mathrm{id}_{Z(f)}$ rel Y.
3. If f is a cofibration, then $q\colon Z(f) \to Y$ is a homotopy equicalence in $\mathbf{TOP}^X$.

Proof. (1). We construct a retraction $R\colon Z(f) \times I \to X \times I \cup Z(f) \times \{1\}$ as follow. Let $R'\colon I \times I \to I \times \{1\} \cup \{0\} \times I$ be a retraction. Then we define

$$\begin{aligned} R\colon Z(f) \times I &\to X \times I \cup Z(f) \times \{1\} \\ ((x,s),t) &\mapsto (x, R'(s,t)) \\ (y,t) &\mapsto (y,1) \end{aligned}$$

is as desired. By Proposition 1.2, $j\colon X \to Z(f)$ is a cofibration.

(2). The homotopy

$$\begin{aligned} h_t\colon Z(f) &\to Z(f) \\ (x,\sigma) &\mapsto (x,(1-t)\sigma + t) \end{aligned}$$

is as desired.

(3). By Proposition 1.2, there is a retraction $r\colon Y \times I \to Z(f)$. Define

$$\begin{aligned} g\colon Y &\to Z(f) \\ y &\mapsto r(y,1). \end{aligned}$$

One can verifies that g is the homotopy inverse of q. $\square$

Summery 1. Any map $f\colon X \to Y$ factors into

$$X \xrightarrow{j} Z \xrightarrow{q} Y$$

where $j\colon X \to Z$ is a cofibration and $q\colon Z \to Y$ is a homotopy equivalence. Moreover, such a factorization is unique up to homotopy equivalence. In particular, we can choose $Z = Z(f)$. We define $C_f = Z(f)/\operatorname{im} j$ as the homotopy cofibre of f, i.e. $C_f = X \times I \sqcup Y/(x,0) \sim *, (x,1) \sim f(x)$, is called the mapping cone of f.

$$X \xrightarrow{f} Y \xrightarrow{s} C_f$$

1.1.3 The Cofibre Sequence (Puppe's Sequence)

To get finer structure, we work in $\mathbf{TOP}^o$. Given a map $f\colon (X,x_0)\to(Y,y_0)$, we get an induced map

$$\begin{aligned} f^*\colon [Y,B]^o &\to [X,B]^o \\ [\alpha] &\mapsto [f\circ\alpha], \end{aligned}$$

where $[X,B]^o$ is the homotopy class of basepoint preserving maps. In particular, we have the constant map

$$\begin{aligned} [*]\colon X &\to B \\ x &\mapsto b_0. \end{aligned}$$

Definition 1.6. We say a sequence

$$(X,x_0) \xrightarrow{f} (Y,y_0) \xrightarrow{g} (Z,z_0)$$

in $\mathbf{TOP}^o$ is h-coexact if $\forall\,(B,b_0)\in \mathrm{Ob}\,(\mathbf{TOP}^o)$,

$$[Z,B]^o \xrightarrow{g^*} [Y,B]^o \xrightarrow{f^*} [X,B]^o$$

is exact, i.e. $(f^*)^{-1}([*]) = \operatorname{im} g^*$.

In $\mathbf{TOP}^o$, we consider the reduced mapping cone $CX \coloneqq X\times I/X\times\{0\}\cup\{x_0\}\times I$. The basepoint of CX is $X\times\{0\}\cup\{x_0\}\times I$. And we consider the reduced mapping cone: For $f\colon (X,x_0)\to(Y,y_0)$, $C(f)\coloneqq CX\vee Y/(x,1)\sim f(x)$. It is equivalent to the following push-out diagram.q

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_1"'] & Y \arrow[d, "f_1"] \\ CX \arrow[r] & C(f) \end{tikzcd}$$

In fact, f_1 maps y to $(y,1)$.

We will also use symbol X instead of (X,x_0) in $\mathbf{TOP}^o$ for short.

Proposition 1.7. The sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} C(f)$$

is h-coexact.

Proof. Consider the following sequence

$$[C(f),B]^o \xrightarrow{f_1^*} [Y,B]^o \xrightarrow{f^*} [X,B]^o$$

for any (B,b_0).

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & Y \arrow[r, "f_1"] \arrow[d, "\alpha"] & C(f) \arrow[dl] \\ & B & \end{tikzcd}$$

Assume that $[\alpha]\in[Y,B]^o$ s.t. $[\alpha\circ f]=[*]\in[X,B]^o$, i.e. $\alpha\circ f$ is null-homotopic. This is equivalent that there exists a map $h\colon CX\to B$. The mapping cone $C(f)$ is the push-out of

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_1"'] & Y \arrow[d, "f_1"] \\ CX \arrow[r] & C(f) \end{tikzcd}$$

Using the universal property of push-out, we have the following commutative diagram,

$$
\begin{tikzcd}
 & Y \arrow[dr, "f_1"] \arrow[drr, bend left, "\alpha"] & & \\
X \arrow[ur, "f"] \arrow[dr, "i_1"'] & & C(f) \arrow[r, dashed, "\exists\beta"] & B \\
 & CX \arrow[ur] \arrow[urr, bend right, "h"'] & &
\end{tikzcd}
$$

i.e. $\alpha = \beta \circ f_1$. Therefore $[\alpha] = f_1^*[\beta]$ and this proposition follows. $\square$

Iterate the procedure, we get a long h-coexact sequence:

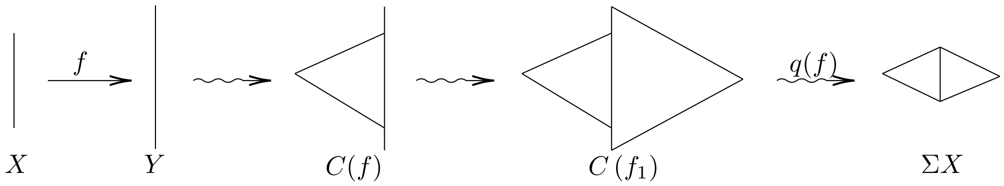
$$X \xrightarrow{f} Y \xrightarrow{f_1} C(f) \xrightarrow{f_2} C(f_1) \xrightarrow{f_3} C(f_2) \longrightarrow \cdots$$

Consider the injection $j_1 \colon CY \to C(f_1)$, we have that

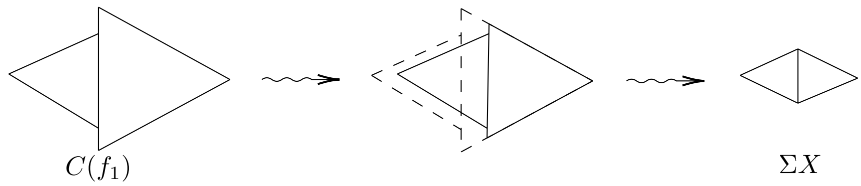
$$C(f_1)/j_1(CY) = X \times I / X \times \partial I \cup \{x_0\} \times I = \Sigma X$$

is the reduced suspension of X. Then we get a quotient map

$$q(f) \colon C(f_1) \to \Sigma X.$$



Claim 1. $q(f)$ is a homotopy equivalence.



Denote by $s(f) \colon \Sigma X \to C(f_1)$ the homotopy inverse of $q(f)$. Then our original sequence becomes

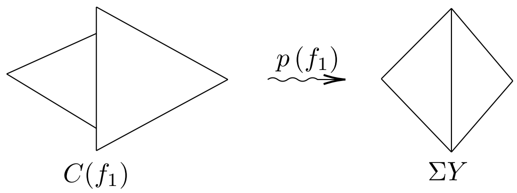
$$
\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "f_1"] & C(f) \arrow[r, "f_2"] \arrow[dr, "q(f)\circ f_2"'] & C(f_1) \arrow[r, "f_3"] \arrow[d, "q(f)"] & C(f_2) \\
 & & & \Sigma X &
\end{tikzcd}
$$

Consider the following diagram.

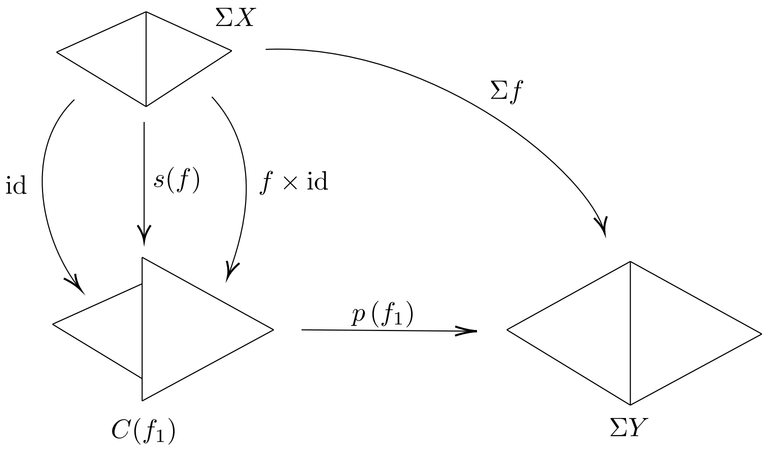
$$
\begin{tikzcd}
C(f_1) \arrow[r, "f_3"] \arrow[d, shift right, "q(f)"'] & C(f_2) \arrow[d, "q(f_1)"] \\
\Sigma X \arrow[u, shift right, "s(f)"'] \arrow[r, dashed, "q(f_1)\circ f_3 \circ s(f)"'] & \Sigma Y
\end{tikzcd}
$$

Claim 2. Consider $\tau \colon \Sigma X \to \Sigma X$ which maps (x,t) to $(x, 1-t)$, we have $q(f_1) \circ f_3 \circ s(f) \simeq \Sigma f \circ \tau$

To prove it, denote $p(f_1) = q(f_1) \circ f_3$. In fact, $p(f_1)$ retracts the left triangle, i.e. CX to a point.



In the following diagram, $s(f)$ is the union of id and $f \times \mathrm{id}$, i.e. id maps the left triangel of ΣX to the left triangle of $C(f_1)$, $f \times \mathrm{id}$ maps the right triangel of ΣX to the right triangle of $C(f_1)$. Then $\Sigma f = p(f_1) \circ s(f)$ naturally. Notice that τ flips ΣX left and right. Therefore, by symmetry, we have $p(f_1) \circ s(f) \simeq \Sigma f \circ \tau$, as desired.



Now we get

$$X \xrightarrow{f} Y \xrightarrow{f_1} C(f) \xrightarrow{p(f)} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{(\Sigma f)_1} C(\Sigma f)$$

Claim 3. There is a homeomorphism $\tau_1 \colon C(\Sigma f) \to \Sigma C(f)$ such that the following diagram is commutative.

$$\begin{tikzcd} \Sigma Y \arrow[r, "(\Sigma f)_1"] \arrow[dr, "\Sigma f_1"'] & C(\Sigma f) \arrow[d, "\tau_1"] \\ & \Sigma C(f) \end{tikzcd}$$

In fact, regard both $C(\Sigma f)$ and $\Sigma C(f)$ as the quotient spaces of $X \times I \times I$ unioned with Y, τ_1 is induced from interchanging the two I-factors.

As conclusion, we have

Theorem 1.8 (Puppe's Sequence)**.** The sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} C(f) \xrightarrow{p(f)} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma f_1} \Sigma C(f) \xrightarrow{p(\Sigma f)} \Sigma^2 X \longrightarrow \Sigma^2 Y \longrightarrow \cdots$$

is h-coexact.

1.2 Fibrations

Definition 1.9. A map $p \colon E \to B$ has the homotopy lifting property (HLP) for the space X if $\forall$ homotopy $h \colon X \times I \to B$ and $a \colon X \to E$ s.t. $p \circ a(x) = h(x, 0)$, there exists a homotopy $H \colon X \times I \to E$

s.t. $p \circ H = h$. H is called a lifting of h.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "h"] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

A map $p\colon E \to B$ is called a fibration if it has HLP for all spaces X.

Definition 1.10. Given maps $f\colon A \to B$ and $p\colon E \to B$. The pull-back of p along f is the terminal object of the following diagram,

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

i.e. for any C, $g\colon C \to E$, $h\colon C \to A$, there exists unique r such that the following diagram is commutative.

$$\begin{tikzcd} & & E \arrow[dr, "p"] & \\ C \arrow[r, dashed, "r"] \arrow[urr, bend left, "g"] \arrow[drr, bend right, "h"'] & f^*E \arrow[ur] \arrow[dr] & & B \\ & & A \arrow[ur, "f"'] & \end{tikzcd}$$

Explicity,

$$f^*E = \{(a, e) \in A \times E : f(a) = p(e)\}$$

and $\pi\colon f^*E \to A$ is the projection.

Denote $B^I = \mathrm{Map}(I, B)$. Consider the pull-back

$$W(p) := \left\{(x, w) \in E \times B^I : p(x) = w(0)\right\}$$

which is given by the pull-back

$$\begin{tikzcd} W(p) \arrow[r, "k"] \arrow[d, "b"'] & B^I \arrow[d, "e^0"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

where e^0 maps w to $w(0)$.

Proposition 1.11. Given a map $p\colon E \to B$, the followings are equivalence:

1. $p\colon E \to B$ is a fibration.
2. p has HLP for $W(p)$.
3. $$\begin{aligned} r\colon E^I &\to W(p) \\ \alpha &\mapsto (\alpha(0), p \circ \alpha) \end{aligned}$$

   admits a section.

Proof. (1)$\Longrightarrow$(2) is by definition.

(2)$\Longrightarrow$(3): Because $W(p)$ is a pull-back, by its universal property, we have the following diagram and we want to find s such that $r \circ s = \mathrm{id}$.

$$
\begin{tikzcd}
 & & B^I \arrow[dr, "e^0"] & \\
E^I \arrow[urr, bend left, "p^I"] \arrow[r, shift right, "r"'] \arrow[drr, bend right, "e^0"'] & W(p) \arrow[l, dashed, shift right, "s"'] \arrow[ur, "k"] \arrow[dr, "b"] & & B \\
 & & E \arrow[ur, "p"'] &
\end{tikzcd}
$$

Notice that $\mathrm{Map}\left(W(p), E^I\right) = \mathrm{Map}\left(W(p) \times I, E\right)$, because p has HLP for $W(p)$, we have the following commutative diagram.

$$
\begin{tikzcd}
W(p) \arrow[r, "b"] \arrow[d] & E \arrow[d, "p"] \\
W(p) \times I \arrow[r, "k"] \arrow[ur, dashed, "s"] & B
\end{tikzcd}
$$

We have $b \circ r \circ s = e^0 \circ s = b$ and $k \circ r \circ s = p^I s = k$. Using the universal property (uniqueness) of pull-back $W(p)$ for $W(p)$, we must have $r \circ s = \mathrm{id}$, i.e. s is a section of r.

(3)$\Longrightarrow$(1): Let s be the section of r. For any X, a, h as in the definition of fibration, we want to find H such that the following diagram is commutative.

$$
\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
X \times I \arrow[r, "h"] \arrow[ur, dashed, "H"] & B
\end{tikzcd}
$$

Using the universal property of pull-back $W(p)$, we have unique f such that the following diagram is commutative, where $h \colon X \to B^I$ is the same as $h \colon X \times I \to B$.

$$
\begin{tikzcd}
 & & B^I \arrow[dr, "e^0"] & \\
X \arrow[urr, bend left, "h"] \arrow[r, dashed, "\exists! f"] \arrow[drr, bend right, "a"'] & W(p) \arrow[ur, "k"] \arrow[dr, "b"] & & B \\
 & & E \arrow[ur, "p"'] &
\end{tikzcd}
$$

Then because $\mathrm{Map}\left(W(p), E^I\right) = \mathrm{Map}\left(W(p) \times I, E\right)$, one can check that $H = s \circ f$ is as desired. In fact,

$$p \circ H(x,t) = (p \circ H(x))(t) = (k \circ r \circ s \circ f(x))(t) = (k \circ \mathrm{id} \circ f(x))(t) = h(x,t)$$

and $H \circ i_0 = a$ is similar. $\square$

Proposition 1.12. If $p \colon E \to B$ is a fibration, then $f^*E \to A$ is also a fibration.

Proof. In the following diagram, F is induced by HLP for fibration $p \colon E \to B$ and then H is induced by universal property of pull-back f^*E.

$$
\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "i_0"'] & f^*E \arrow[r] \arrow[d, "\pi"] & E \arrow[d, "p"] \\
X \times I \arrow[r, "h"] \arrow[ur, dashed, "H"] \arrow[urr, dashed, "F"] & A \arrow[r, "f"] & B
\end{tikzcd}
$$

□

Part II
Generalized Homology

Part III
Characteristic Classes