Homotopy Theory and Characteristic Classes

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Abstract

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Part I
Homotopy Theory

Let **TOP** be the category of topological spaces. Then we can take a quotient of **TOP** and get the homotopy category $h-\mathbf{TOP}$. The quotient may bring more algebraic structures. For example, Mor $\left(S^1, X\right)$, the homotopy classes of maps from S^1 to X, is the fundamental group of X. Our goal is to study functors from hmotopy category to some algebraic categories.

Let $\mathbf{TOP}^o$ be the pointed topological category, where the sum is wedge sum $(X, x_0) \wedge (Y, y_0) = X \sqcup Y / x_0 \sim y_0$ and the product is the smash product $(X, x_0) \vee (Y, y_0) = X \times Y / \{x_0\} \times Y \cup X \times \{y_0\}$. Similarly, we can take a quotient to get $h - \mathbf{TOP}^o$.

Let $\mathbf{TOP}(2)$ be the category of pairs and $h - \mathbf{TOP}(2)$ be its quotient.

Fix $K \in \mathrm{Ob}(\mathbf{TOP})$. Let's consider $\mathbf{TOP}^K$, the category of spaces under K. Its objects are maps $f : K \to X$ and morphisms are maps $\alpha : X \to Y$ such that $\alpha \circ f = g$.

$$\begin{tikzcd} & K \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, "\alpha"] & & Y \end{tikzcd}$$

If $K = \{*\}$ is a single point set, then $\mathbf{TOP}^{\{*\}} = \mathbf{TOP}^o$ is the pointed topological category. Take $X = K$. A morphism from $f : K \to X$ to $\mathrm{id} : K \to K$ is $r : X \to K$ such that $r \circ f = \mathrm{id}$.

$$\begin{tikzcd} & K \arrow[dl, "f"'] \arrow[dr, "\mathrm{id}"] & \\ X \arrow[rr, "r"] & & K \end{tikzcd}$$

When $K \subset X$, $f = i : K \hookrightarrow X$, we say that r is a retraction.

We have $r : X \to K$ is a deformation retraction, if and only if $i \circ r \simeq \mathrm{id}_X$ *rel* K, if and only if $r : X \to K$ is a homotopy equivalence in $\mathbf{TOP}^K$.

Fix $B \in \mathrm{Ob}(\mathbf{TOP})$. Let's consider $\mathbf{TOP}_B$, the category of spaces over B, where the objects are $p : X \to B$ and morphisms are $f : X \to Y$ such that $p = q \circ f$.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

Take $X = B$. A morphism from $\mathrm{id} : B \to B$ to $q : Y \to B$ is $s : B \to Y$ such that $q \circ s = \mathrm{id}_B$.

$$\begin{tikzcd} B \arrow[rr, "s"] \arrow[dr, "\mathrm{id}"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

Then s is called a section of q.

Similarly, we can define $h - \mathbf{TOP}^K$ and $h - \mathbf{TOP}_B$.

1 Cofibrations and Fibrations

1.1 Cofibrations

Definition 1.1. A map $i : A \to X$ has the homotopy extension property (HEP) for a space Y if for all homotopy $h : A \times I \to Y$ and $f : X \to Y$ with $f \circ i(a) = h(a, 0)$, there exists $H : X \times I \to Y$ satisfies

$$\begin{tikzcd} & X \arrow[dr, "i_0"] \arrow[drr, bend left, "f"] & & \\ A \arrow[ur, "i"] \arrow[dr, "i_0"'] & & X \times I \arrow[r, dashed, "H"] & Y \\ & A \times I \arrow[ur, "i\times\mathrm{id}"] \arrow[urr, bend right, "h"'] & & \end{tikzcd}$$

We say $i : A \to X$ is a cofibration if it has HEP for each $Y \in \mathrm{Ob}(\mathbf{TOP})$.

Recall the mapping cylinder: if $i : A \to X$ is a map, then $Z(i) \coloneqq ()A \times I) \sqcup X/(a, 1) \sim i(a)$.

Proposition 1.2. Given a map $i : A \to X$. The followings are equivalent:

1. $i : A \to X$ is a cofibration.

2. i has HEP for $Z(i)$.

3. The map

$$
\begin{aligned}
s : Z(i) &\to X \times I \\
(a, t) &\mapsto (f(a), t), \\
x &\mapsto (x, 1)
\end{aligned}
$$

has a retraction.

Proof. (1)$\Longrightarrow$(2) is only by definition.

(2)$\Longrightarrow$(1): By definition, there exists $K : X \times I \to Z(i)$ such that the following diagram is commutative.

$$
\begin{tikzcd}
& X \arrow[dr, "i_0"] \arrow[drr, bend left, "i_1"] & & \\
A \arrow[ur, "i"] \arrow[dr, "i_0"'] & & X \times I \arrow[r, dashed, "K"] & Z(i) \\
& A \times I \arrow[ur, "i\times\mathrm{id}"] \arrow[urr, bend right, "\mathrm{id}"'] & &
\end{tikzcd}
$$

For any Y and homotopy $h : A \times I \to Y$ and $f : X \to Y$ with $f \circ i(a) = h(a, 0)$, we define

$$
\begin{aligned}
F : Z(i) &\to Y \\
(a, t) &\mapsto h(a, t) \\
x &\mapsto f(x).
\end{aligned}
$$

Then $F \circ K$ is as desired.

$$
\begin{tikzcd}
& X \arrow[dr, "i_0"] \arrow[drr, bend left, "i_1"] \arrow[drrr, bend left, "f"] & & & \\
A \arrow[ur, "i"] \arrow[dr, "i_0"'] & & X \times I \arrow[r, dashed, "K"] & Z(i) \arrow[r, "F"] & Y \\
& A \times I \arrow[ur, "i\times\mathrm{id}"] \arrow[urr, bend right, "\mathrm{id}"'] \arrow[urrr, bend right, "h"'] & & &
\end{tikzcd}
$$

(2)$\Longrightarrow$(3): We can easily check that the extension $K : X \times I \to Z(i)$ in the proof of (2)$\Longrightarrow$(1) is a retraction of s.

(3)$\Longrightarrow$(2): Let r be a retraction of s. For any homotopy $h : A \times I \to Z(i)$ and $f : X \to Z(i)$ with $f \circ i(a) = h(a, 0)$, we define

$$
\begin{aligned}
\sigma : Z(i) &\to Z(i) \\
(a, t) &\mapsto h(a, t) \\
x &\mapsto f(x).
\end{aligned}
$$

Then we can verify that $H = \sigma \circ r : X \times I \to Z(i)$ extends h.

$$
\begin{tikzcd}
& X \arrow[dr, "i_0"] \arrow[drr, bend left, "f"] & & \\
A \arrow[ur, "i"] \arrow[dr, "i_0"'] & & X \times I \arrow[r, bend right, "r"'] & Z(i) \arrow[l, bend right, "s"'] \\
& A \times I \arrow[ur, "i\times\mathrm{id}"] \arrow[urr, bend right, "h"'] & &
\end{tikzcd}
$$

□

Corollary 1.3. When $A \subset X$, $i : A \hookrightarrow X$ is the inclusion map. Then $i : A \to X$ is a cofibration $\Longleftrightarrow$ $A \times I \cup X \times \{1\}$ is a retraction of $X \times I$.

Therefore, we can easily check that whether $i : A \hookrightarrow X$ is a cofibration. For example, let (X, A) be a manifold with boundary.

Definition 1.4 (Push-Out of Cofibration)**.** Given a commutative diagram,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & B \arrow[d, "J"] \\ X \arrow[r, "F"] & Y \end{tikzcd}$$

the push-out of j along f is the initial object of this diagram, i.e. $j : B \to Y$, $F : X \to Y$, s.t. $\forall Z$ with $J' : B \to Z$, $F' : X \to Z$ satisfying $J' \circ f = F' \circ j$, $\exists!$ map $p : Y \to Z$ such that the diagram is commutative.

$$\begin{tikzcd} & B \arrow[dr, "J"] \arrow[drr, bend left, "J'"] & & \\ A \arrow[ur, "f"] \arrow[dr, "j"'] & & Y \times I \arrow[r, dashed, "\exists! p"] & Z \\ & X \arrow[ur, "F"] \arrow[urr, bend right, "F'"'] & & \end{tikzcd}$$

In our setting, we can construct $Y = X \sqcup B/f(a) \sim j(a)$ directly.

Part II
Generalized Homology

Part III
Characteristic Classes