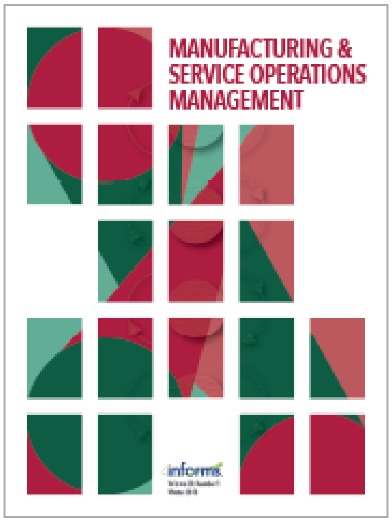

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A Markov Decision Model for Managing Display-Advertising Campaigns

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Abstract. ***Problem definition*:** Managers in ad agencies are responsible for delivering digital ads to viewers on behalf of advertisers, subject to the terms specified in the ad campaigns. They need to develop bidding policies to obtain viewers on an ad exchange and allocate them to the campaigns to maximize the agency's profits, subject to the goals of the ad campaigns. ***Academic/practical relevance*:** Determining a rigorous solution methodology is complicated by uncertainties in the arrival rates of viewers and campaigns, as well as uncertainty in the outcomes of bids on the ad exchange. In practice, ad hoc strategies are often deployed. Our methodology jointly determines optimal bidding and viewer-allocation strategies and obtains insights about the characteristics of the optimal policies. ***Methodology*:** New ad campaigns and viewers are treated as Poisson arrivals, and the resulting model is a Markov decision process, where the state of the system is the number of undelivered impressions in queue for each campaign type in each period. We develop solution methods for bid optimization and viewer allocation and perform a sensitivity analysis with respect to the key problem parameters. ***Results*:** We solve for the optimal dynamic, state-dependent bidding and allocation policies as a function of the number of ad impressions in queue, for both the finite horizon and steady-state cases. We show that the resulting optimization problems are strictly concave in the decision variables and develop and evaluate a heuristic method that can be applied to large problems. ***Managerial implications*:** Numerical analysis of our heuristic solution shows that its errors are generally small and that the optimal dynamic, state-dependent bidding policies obtained by our model are significantly better than optimal static policies. Our proposed approach is managerially attractive because it is easy to implement in practice. We identify the capacity of the impression queue as an important managerial control lever and show that it can be more effective than using higher bids to reduce delay penalties. We quantify potential operational benefits from the consolidation of ad campaigns, as well as merging ad exchanges.

Supplemental Material: The online appendix is available at https://doi.org/10.1287/msom.2022.1142.

Keywords: Markov decision processes • queueing systems • display ad • ad exchange • ad campaigns • real-time bidding • dynamic bidding • stochastic processes

1. Introduction

Since the advent of the internet in the '90s, the digital advertising industry has expanded very rapidly. In the United States, internet advertising expenditure totaled about $124.6 billion in 2019, a 16% increase compared with $107.5 billion in 2018. Search (44%) and display (48%) ads constitute the bulk of these revenues, with the remainder coming from classified ads, lead generation, and audio ads. Digital advertising is particularly appealing to advertisers because of its low cost per impression and the ability to direct ads to the desired individuals, platforms, or contexts. The advances and widespread growth of computing and networking technologies have created unprecedented opportunities for marketers to access customers through this medium. However, digital advertising also poses significant operational challenges for advertisers and ad-agency campaign managers. Working on behalf of advertisers, campaign managers are responsible for achieving the best match between campaigns (demand) and viewers (supply) in specific categories that are of commercial interest to the advertisers. Campaign managers must develop policies to bid for viewers on an ad exchange with the goal of maximizing their profit, subject to the terms of the ad-campaign contracts. This paper focuses on campaigns for display ads (e.g., banner ads) that are based on contracts between advertisers and ad agencies that specify the campaign duration, viewer attributes, impression goals (number of impressions to be served), cost per impression, penalty costs for shortfalls, and the ad-content details.

1.1. Problem and Motivation

The goal in this paper is to develop an analytical methodology for determining policies that maximize profits for the ad agency, while meeting the targets in the ad contracts handled by ad-campaign managers. Campaign managers must determine the bidding strategies for acquiring viewers, as well as methods for allocating viewers across campaigns. Based on our conversations with several campaign managers, these policies are often selected subjectively, driven by trial-and-error observations. The policies tend to be relatively static, in the sense that they are updated rather infrequently, based on periodic reviews of how the campaign results are tracking their plans. The viewer-allocation decision is typically not optimized jointly with the bidding decision in current practice. Finding an optimal solution is a complicated decision problem because the number of decision variables is very large, and there is uncertainty in the arrivals of viewers and ad campaigns and in the likelihood of winning bids. In order to simplify the resulting optimization problem, much of the existing literature on this problem assumes a static and deterministic environment.

1.2. Overview of the Main Contributions

Our paper determines policies for bidding and allocation of viewers that maximize the total profit for the ad agency for both a finite horizon and in steady state. Our paper is distinct in that we formulate this problem as a Markov decision process, which includes variability in the arrivals of viewers and ad campaigns, and determine time-varying, state-dependent policies for bidding and viewer allocation. Our method jointly determines the optimal bidding and viewer-allocation policies as a function of the number of undelivered impressions in queue and the time period, which we show numerically leads to substantially higher profits than with the static bids that are commonly used in practice. We also develop a heuristic solution method that greatly simplifies the implementation of the optimal solution, without significant profit impact. The state-dependent, dynamic bidding policies can be predetermined and stored and are thus easy to implement. Thus, our paper provides an important and practical generalization of the existing literature on this topic.

In addition to the theoretical and technical contributions, our analytical results and numerical analysis lead to a number of important managerial insights and recommendations. For instance, we propose actively limiting the capacity of the impression queue for a given viewer type as a control mechanism and show that this can be more effective than bidding higher to reduce delay penalties for campaigns. In this case, we numerically show the counterintuitive result that the optimal bids do not monotonically increase in the number of undelivered impressions—they first increase and then decrease near the capacity. Although bidding more reduces delay costs, when the queue is near its capacity, arriving campaigns are automatically truncated, which allows lower bids to be optimal. Finally, through our sensitivity analysis, we show that there can be significant profit improvement from consolidating multiple campaigns and from managing campaigns with a common set of viewers jointly. In practice, publishers and advertisers/agencies often manage similar ad campaigns in parallel on different ad exchanges. Although there may be good business reasons for doing this, this result shows that there is operational benefit from consolidating these campaigns.

The rest of the paper is organized as follows. The next section discusses the relevant literature. Section 3 describes the basic notation and defines the states and transition probabilities of the model. Section 4 formulates the ad agency's profit-maximization problem and details the optimal dynamic programming and heuristic solution methods. Sensitivity analysis of the steady-state formulation is presented in Section 5. All technical proofs are in the online appendix. Section 6 presents a numerical study using some data from an online market platform to illustrate our results and to generate managerial insights about the problem. Section 7 presents concluding remarks and directions for future research.

2. Literature Review

Given our focus on the campaign manager's problem at an ad agency, the two decisions that are most relevant to our work are bidding in online auctions, also referred to as real-time bidding, and allocation of viewers to advertisers or campaigns. The existing research considers these problems in the context of display (which is our focus) and search advertising. Search advertising has several differences compared with display advertising because bidding is done in response to keywords instead of viewer attributes, and multiple ads can be shown instead of typically just one in the case of display advertising. This leads to important analytical differences in the solution approach. Therefore, we will review only the display-advertising literature. Publishers and supply side platforms typically do not bid for viewers (though there are occasional exceptions), but they may set reserve prices for auctions on ad exchanges, where advertisers or ad agencies bid. So, the bidding problem can be a component of papers that focus on the publishers' problems. Publishers also deal with allocation of viewers to spot markets and/or one of their guaranteed contracts. Some of the papers that model bidding consider a budget constraint on the amount that can be spent on the various campaigns. Ensuring that the actual spending tracks the plan is also referred to as budget pacing, or ad pacing.

For brevity, we will focus our review primarily on papers that take the advertisers' perspective in the display-advertising context. In our research, we did not find any work that focused on the ad agency's perspective, so this is a distinct feature of our paper. We will also not review papers where the goal is to study pricing of contracts or other operational issues not directly relevant to our work. See, for example, Agrawal et al. (2020) for a more extensive review of papers on online advertising. Table 1 summarizes the literature that we have reviewed and highlights the positioning of our work relative to this literature.

2.1. Advertiser Perspective

Among authors that focus on bidding for display advertising for advertisers, Chen et al. (2011) formulate the problem of assigning impressions to display campaigns to maximize total valuation, subject to delivery-goal constraints, as a deterministic linear programming optimization problem. They suggest a bidding approach that adjusts bids according to real-time constraint snapshots, such as budget-consumption levels. In Lee et al. (2013), the authors take a similar approach, but break down the total budget into budgets for specific time slots. Zhang et al. (2014) fit a functional form to empirical data on the winning probability as a function of the bid price, which, in turn, is a function of the features of the viewers and the predicted key performance indicators (click-through rate, conversion rate, etc.) if the auction is won, and use this function to optimize bidding. Agarwal et al. (2014) and Xu et al. (2015) study budget pacing for campaigns by probabilistically "throttling" the campaigns that spend too fast (i.e., they do not allow these campaigns to participate in some auctions). Cohen et al. (2017) consider how advertisers' risk aversion affects their decision to purchase impression-delivery guarantees, in addition to participating in a real-time auction. They assume a known supply of viewers and approximate a series of auctions with a single auction with a large number of bidders. The bids for this auction are assumed to be used for the entire period. Balseiro et al. (2017) compare the system equilibria for different budget-management strategies and show that imposing reserve prices and excluding buyers are most effective in controlling expenditures from the seller's perspective. From the buyer's perspective, lower

Table 1. Summary of Literature Review

Paper	Decision maker	Decisions: Real-time bidding	Decisions: Viewer allocation	Decisions: Budget pacing	Methodology	Uncertainty
Agarwal et al. (2014)	Advt.	✓	✓	✓	Prob. throttling	
Balseiro and Gur (2019)	Advt.	✓		✓	Auction theory	Viewer valuation
Balseiro et al. (2017)	Advt.	✓		✓	Auction theory	Viewer valuation
Chen et al. (2011)	Advt.	✓	✓	✓	Linear prog.	
Cohen et al. (2017)	Advt.	✓			Auction theory	# Advt.
Lee et al. (2013)	Advt.	✓	✓	✓	Linear prog.	
Xu et al. (2015)	Advt.		✓	✓	Prob throttling	Advt. participation, bid outcome
Zhang et al. (2014)	Advt.	✓			Functional optim.	Bid outcome
Balseiro et al. (2014)	Publ.		✓		Auction theory	Auction outcome
Bharadwaj et al. (2012)	Publ.		✓		Quadratic prog.	
Celis et al. (2014)	Publ.		✓		Auction theory	Viewer valuation
Chen (2017)	Publ.		✓		Auction theory	Viewer valuation
Deza et al. (2015)	Publ.			✓	Quadratic prog.	# Viewers
Feldman et al. (2010)	Publ.		✓		Quadratic prog.	# Viewers
Fridgeirsdottir and Najafi (2018)	Publ.		✓		Queueing theory	# Viewers, # advt.
Hojjat et al. (2017)	Publ.		✓	✓	Quadratic prog.	# Viewers
McAfee et al. (2013)	Publ.		✓	✓	Quadratic prog.	
Menache et al. (2009)	Publ.		✓		Auction theory	# Advt.
Mookerjee et al. (2017)	Publ.		✓		Nonlinear prog.	Viewer click
Najafi and Fridgeirsdottir (2014)	Publ.				Queueing theory	# Viewers, # advt., viewer click
Roels and Fridgeirsdottir (2009)	Publ.		✓		Stochastic DP	# Viewers, # advt.
Sayedi (2018)	Publ.		✓		Game theory	
Turner (2012)	Publ.		✓	✓	Quadratic prog.	
Yang et al. (2012)	Publ.		✓	✓	Quadratic prog.	
Our paper	Ad agency	✓	✓		Queueing, MDP	# Viewers. # advt., bid outcome

Note. Advt., advertiser; MDP, Markov decision process; optim., optimization; prob., probabilistic; prog., programming; publ., publisher.

bid prices are more beneficial than fewer competitors, and bid shading leads to the highest buyer utility. Balseiro and Gur (2019) model bidding opportunities and competition in a competitive environment for budget-constrained advertisers and formulate the problem as a sequential game with incomplete information.

There are several papers that focus on the advertiser's bidding problem in the context of search advertising, utilizing mostly deterministic formulations, such as bipartite graphs, knapsack problems, etc. For reasons noted previously, we do not review them here.

2.2. Publisher Perspective

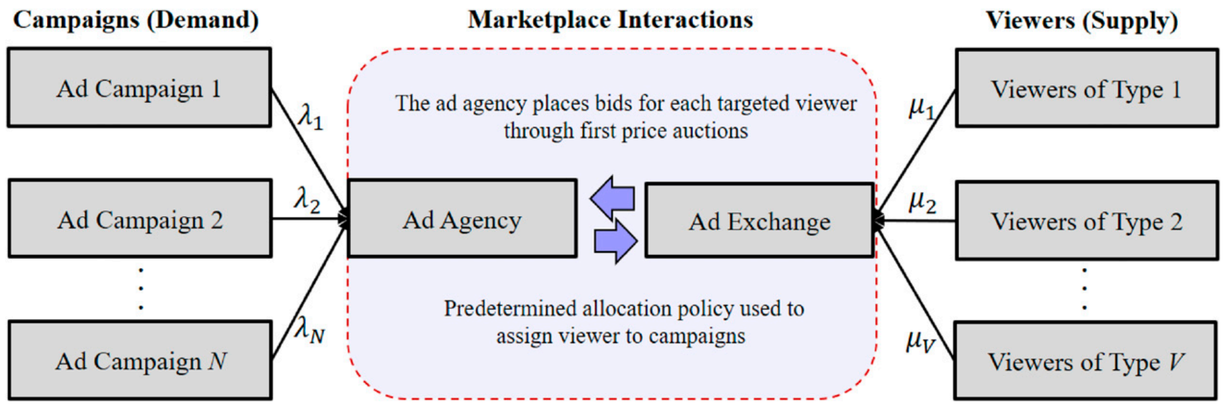
To address the publisher's problem of allocating viewers in the display-advertising context, the most commonly used analytical approach in the literature is to formulate the decision as a deterministic problem that ensures representative allocations of ads to target viewers. For example, Yang et al. (2012), Bharadwaj et al. (2012), Turner (2012), and McAfee et al. (2013) formulate the problem as a deterministic transportation (or a bipartite graph) model with a quadratic objective that minimizes the distance between the actual and target (fully representative) allocation of ads to viewers. Deza et al. (2015) extend these results by considering a chance constraint to ensure that the likelihood of underdelivery to a campaign is not too large. Hojjat et al. (2017) formulate contracts that specify reach and frequency with which ads are delivered to specific individuals. Mookerjee et al. (2017) develop a threshold-based policy to determine when an ad should be served to a viewer to maximize revenue earned through clicks. Uncertainty in the advertising requests and/or website traffic is incorporated in Feldman et al. (2010) and Roels and Fridgeirsdottir (2009). Authors that explicitly consider the publisher's decision to allocate a viewer to the spot market or guaranteed display contracts include Celis et al. (2014) (uncertainty in the number of bidders, but the decision is made for a single viewer/impression), Balseiro et al. (2014) (use of mean value approximation to solve a deterministic version of the problem), Chen (2017) (mechanism design approach), and Sayedi (2018) (deterministic game model of two advertisers and one publisher). Finally, as mentioned previously, we skip reviewing the literature on inventory allocation in the search context.

Our paper contributes to the existing literature in a number of distinct ways. Although many of the papers discussed above utilize a deterministic modeling framework, we generalize the approach by assuming a stochastic environment that simultaneously models the uncertainties in the arrivals of campaigns and viewers and the outcomes of the bidding process. However, we do not include budget constraints for the ad agency, although our methodology can be used to compute the expected budget required to complete a campaign. We determine an exact optimal solution that is dynamic and state-dependent, where the state corresponds to the number of impressions yet to be delivered for each campaign, and propose heuristics that are easy to implement in practice. We focus on the decision problem faced by a campaign manager in an ad agency, who manages multiple campaigns simultaneously, rather than on the publishers' or advertisers' problems, which are the focus of much of the existing literature. Our model could also be used by advertisers, Demand-Side Platforms (DSPs), or, occasionally, publishers, who manage their ad campaigns directly. Our work is distinct in that we utilize a queuing-system formulation for the ad agency's problem. Najafi-Asadolahi and Fridgeirsdottir (2014) and Fridgeirsdottir and Najafi-Asadolahi (2018) also utilize a queuing framework, but their goal is to determine the optimal pricing of impressions by a publisher, not the ad agency's problem of bidding for viewers and managing campaigns. The similarity to queuing systems is also noted in Menache et al. (2009), who develop an auction mechanism to maximize a publisher's revenue in a search context and show that the heuristic solution to their dynamic programming formulation is similar to the $c\mu$ rule used in queuing systems. A Markov decision model has also been used by several authors in the revenue-management literature, but their focus tends to be on dynamic pricing and pricing of ad contracts (see Özer and Phillips 2012 and Talluri and van Ryzin 2004 for reviews and Najafi-Asadolahi and Fridgeirsdottir 2014 and Fridgeirsdottir and Najafi-Asadolahi 2018 for specific examples in the context of display advertising). In our model, contract pricing is not considered, but the ad agency's policies indirectly impact the arrival of campaigns, because some campaigns may be rejected or only partially accepted depending upon the backlog of demand for a given viewer type. Finally, in our formulation, we consider bidding for viewers and allocating them to campaigns as two different decisions, but we optimize them jointly. Our approach could also be extended to compare the profits obtained with other allocation strategies that are sometimes used in practice (round robin, random, etc.).

3. The Model

The structural assumptions of our model are illustrated in Figure 1. The ad exchange serves as a clearinghouse for the demand for impressions created by ad campaigns, which are shown to arriving viewers at various publisher websites. In this environment, viewers arrive at random and cannot be stored. Therefore, ad impressions are served immediately (within milliseconds) when a viewer is available. New viewers and

Figure 1. An Illustration of the Ad Agency's Interactions with Multiple Ad Campaigns and Viewer Types



ad campaigns for the ad agency are modeled as Poisson arrivals.

In practice, and as indicated in Figure 1, ad campaigns target specific viewer types, which are characterized by sets of attributes corresponding to distinct segments of the viewer population. Each ad campaign specifies the number of viewer impressions for its ads by viewer type. At any point in time, the ad agency manages a queue of undelivered impressions for each viewer type and each campaign type. If the queue of impressions grows too large, the impressions may not be delivered in a timely manner, and the ad agency incurs penalty costs. The agency can reduce the queue size by bidding higher to increase the probability of winning, or it can limit the number of additional new ad contracts that target that viewer type. We operationalize this second option by choosing a *capacity limit* on the impression queue for each viewer and campaign type. Our model jointly determines the optimal bid for the next viewer and which campaign should receive the viewer if the bid is successful.

3.1. Model Notation

There are V distinct viewer types and N ad-campaign types that are defined by their unique attributes. A campaign type is defined by the campaigns that share common parameters, which will be discussed in Section 4. The parameters below are defined for $i = 1,\ldots,N$ and $j = 1,\ldots,V$.

- λ_i = Poisson arrival rate of campaigns of type i, with $\lambda = \sum_{i=1}^{N} \lambda_i$
- μ_j = Poisson arrival rate of viewers of type j with $\mu = \sum_{j=1}^{V} \mu_j$
- s_{ij} = the number of impressions of type j targeted by each campaign of type i
- a_{ij} = the number of undelivered impressions of viewer type j in queue for campaign type i
- $\mathbf{a}$ = the matrix of undelivered impressions across all campaigns = $[a_{ij}]_{1\le i\le N,1\le j\le V}$
- A_{ij} = the queue capacity of undelivered impressions for viewer type j for campaign type i
- $\mathbf{A}$ = the matrix of queue capacities for all campaign and viewer types = $[A_{ij}]_{1\le i\le N,1\le j\le V}$
- b_j = the bid placed by the agency on the exchange for a viewer of type j, with $\mathbf{b} = [b_i]_{1\le i\le N}$
- $w_j(b_j)$ = the probability of winning the bid for a viewer of type j with bid b_j
- I_j = the index of the campaign to receive the next viewer type j, with $\mathbf{I} = [I_j]_{1\le j\le V}$
- t = the number of periods to go before the end of the time horizon.

The state a_{ij} and parameters b_j and I_j are time-period-dependent. For ease of exposition, we will show the subscript t only when needed for the discussion.

3.2. Discussion of Assumptions

The state of the system is defined by the matrix $\mathbf{a} = [a_{ij}]_{1\le i\le N,1\le j\le V}$, where $0 \le a_{ij} \le A_{ij}$. We define the vector that specifies the number of impressions accepted for an arriving campaign of type i as $\bar{\mathbf{s}}_i(\mathbf{a}) = [\min(s_{ij}, A_{ij} - a_{ij})]_{1\le j\le V}$. We assume that truncating the impressions for one viewer type in a given campaign does not affect the s_{ij} for the other viewer types. The optimization methods discussed in the subsequent sections also apply when campaigns of type i have a probability distribution $p_{ij}(s)$ for the number of requested impressions for each viewer type j. We have assumed that all campaigns of type i have the same s_{ij} to simplify the equations. When $a_{ij} = 0$ for all i, there are no impressions for viewer type j in the queue, so the bid $b_j = 0$ and $w_j(0) = 0$.

Empirical support for the assumption of Poisson arrivals of viewers has been provided by several published studies (see, e.g., Cao et al. 2003 and Najafi-Asadolahi and Fridgeirsdottir 2014). This assumption is also supported empirically by the data that we describe in Section 6. Poisson arrivals of requests for service are also commonly assumed in service settings (see, e.g., Fridgeirsdottir and Najafi-Asadolahi 2018). The ad agency's bid b_j for a viewer of type j is placed on an ad exchange with a first-price auction. We assume the winning probability $w_j(b_j)$ is stationary, although it could be time-dependent in the finite horizon case. We also assume that $w_j(b_j)$ is independent of the state $\mathbf{a}$. In some bidding situations, the probability of winning also depends on the number of bidders in the auction. However, because the number of bidders in an online ad exchange is usually large, the Mean Field Equilibrium condition implies that the winning probability distribution is approximately stationary (see, e.g., Balseiro et al. 2015).

3.3. The Transition Probabilities

This system is a continuous-time Markov process that has integer state changes occurring at random times. We will study the embedded discrete-time Markov chain that corresponds to the transitions. For any campaign of type i, let us consider the various types of transitions that may occur from the current state matrix $\mathbf{a}$ to the new state matrix $\mathbf{e}$. Note that only one row of $\mathbf{a}$ can change in each transition, and we denote that change from $\mathbf{a}_i$ to $\mathbf{e}_i$, where $\mathbf{a}_i$ corresponds to the i^{th} row of $\mathbf{a}$. The possible state transitions are:

i. A viewer of type j arrives at the exchange with probability $\frac{\mu_j}{\lambda+\mu}$. The agency then bids b_j, and two outcomes are possible:

a. The agency's bid b_j is successful with probability $w_j(b_j)$, and the viewer is assigned to campaign type i if $I_j = i$. Thus, the i^{th} row of the state matrix $\mathbf{a}$ will change only if $I_j = i$. Defining the row vector $\mathbf{u}_j$ of dimension V with j^{th} component one and all other components zero, the new state of the system is $\mathbf{e}_i = \mathbf{a}_i - \mathbf{u}_j \mathbf{1}_{(I_j=i)}$, where $\mathbf{1}_{(I_j=i)} = 1$, if $I_j = i$ and zero otherwise, and $\mathbf{e}_k = \mathbf{a}_k$ for all $k \neq i$.

b. The agency's bid b_j is unsuccessful with probability $1 - w_j(b_j)$. The state of the system remains unchanged, and $\mathbf{e} = \mathbf{a}$.

ii. A new campaign of type i arrives with probability $\frac{\lambda_i}{\lambda+\mu}$. The state changes from $\mathbf{a}$ to $\mathbf{e}$ occur only in row i, that is, $\mathbf{e}_i = \mathbf{a}_i + \overline{\mathbf{s}}_i(\mathbf{a}) = [\min(a_{ij} + s_{ij}, A_{ij})]_{1\le j\le V}$ and $\mathbf{e}_k = \mathbf{a}_k$ for all $k \neq i$.

The corresponding Markov process is discrete with a finite number of states, due to the queue capacities $\mathbf{A} = [A_{ij}]_{1\le i\le N, 1\le j\le V}$. Taken over all i, j, the general form of these transition probabilities is described in the online appendix. We note that μ_j and λ_i could also depend on t, but to simplify the discussion, we will assume that the rates are stationary.

The decision variables are the bids $\mathbf{b}$ and the allocation variable $\mathbf{I}$, which specifies which campaign type will receive the next viewer type j for each j. These decision variables depend on the state of the system $\mathbf{a}$ and t, the number of periods remaining. Because viewers and campaigns are Poisson arrivals, and the decision variables and transition probabilities are determined by the current state $\mathbf{a}$, the corresponding optimization problem is a Markov decision process. The state space of this process, corresponding to all the possible values of the matrix $\mathbf{a}$, is very large, with millions of states in many cases. The problem size is more manageable if it is separated by viewer type, which can be done by making the following assumption.

Assumption 1. *The viewer types are independent and are not substitutable.*

In this context, *independent* means that the sets of viewers are nonintersecting (i.e., distinct audience segments or partitions in the viewers' population), and, if the requested number of impressions s_{ij} is truncated due to the queue capacity A_{ij}, this does not affect any other s_{ik}, for $k \neq j$. Sometimes ad campaigns allow viewers of lesser value to be substituted for the targeted viewer type, and this assumption prohibits this kind of substitution as well. The assumption leads to the following proposition, which is proved in the online appendix.

Proposition 1. *If the viewer types are independent and are not substitutable, then the Markov decision process above can be separated into V subproblems, one for each viewer type.*

Although it is possible to specify examples in which Assumption 1 does not hold, we found in talking with managers at various ad agencies that the sets of viewers specified in many ad contracts do satisfy this assumption. For example, Google divides viewers into predefined independent ad groups, which are frequently used in specifying ad campaigns. See Balseiro and Gur (2019) for a recent example, where in each period, competing advertisers bid for a single viewer (they might value the viewer differently) for a single ad slot. Thus, we believe that it is appropriate to simplify the model in this way. As will be shown in the discussion of the heuristic method, the state space can be prohibitively large in many cases, even with this assumption. We also noted in our literature review that most published papers to date deal with one viewer type at a time, which is consistent with this assumption.

Exceptions to our assumption include Turner (2012), Bharadwaj et al. (2012), and Yang et al. (2012), who consider distinct, nonoverlapping viewer segments, but model demand for campaigns through an additive constraint, such that the demand can be met by serving impressions to a combination of viewers of different types. Thus, the viewer segments are substitutable in these studies.

4. Multiple-Campaign Formulation

Because the V subproblems are solved separately, we can drop the subscript j and focus on solving one subproblem that has N campaign types targeting one viewer type. This means that the state variable $\mathbf{a} = [a_i]_{1\le i\le N}$, the queue capacity $\mathbf{A} = [A_i]_{1\le i\le N}$, and the number of impressions requested $\mathbf{s} = [s_i]_{1\le i\le N}$ are now vectors, and the bid b and the viewer-allocation variable I are scalars. The number of new impressions accepted when an arriving campaign of type i requests s_i impressions is $\bar{s}_i(a_i) = \min(s_i, A_i - a_i)$. The transition probability from state $\mathbf{a}$ to state $\mathbf{e}$ for one viewer type can be written as:

$$\phi(\mathbf{e}\,|\,\mathbf{a}, b, I) = \begin{cases} \dfrac{\mu w(b)}{\lambda+\mu} & e_I = a_I - 1, \text{ and } e_i = a_i \text{ for } i \neq I \\ \dfrac{\mu(1-w(b))}{\lambda+\mu} & e_i = a_i \text{ for all } i \\ \dfrac{\lambda_i}{\lambda+\mu} & e_i = a_i + \bar{s}_i(a_i), \text{ and } e_k = a_k \text{ for } \\ & k \neq i, \text{ for each } i = 1,\ldots,N. \end{cases} \tag{1}$$

The variable I is always chosen so that $a_I > 0$ holds, because a new viewer will never be assigned to a campaign with no impressions requested. We define the following additional variables:

- $p_i = \lambda_i/\lambda =$ the probability that an arriving campaign is of type i
- $r_i =$ revenue received for campaign type i for placing an ad with a viewer
- $c_i(a_i) =$ delay cost per unit time for campaign type i with a_i impressions in queue.

In this context, a campaign type corresponds to all arriving campaigns that have the same r_i, s_i, and $c_i(a_i)$ parameters.

4.1. Expected Profit for Multiple Campaigns

A winning bid for an arriving viewer results in a net revenue of $r_I - b$ if it is allocated to campaign I, and all other transitions generate zero revenue. On every transition, for each campaign type i with a_i impressions in queue, there is a delay cost $c_i(a_i)$ per unit time. The delay cost $c_i(a_i)$ is a subjective measure of the urgency of delivering the a_i impressions to viewers. Thus, the delay cost also acts as a control on the completion time of campaigns. In the steady-state model, we note that the delay cost can also be interpreted as a Lagrange multiplier for a constraint on the average queue length. In our computed examples, $c_i(a_i) = c_i a_i$ is linear, and we use the linear form in some of the subsequent discussions. There may be practical situations in which $c_i(a_i)$ is convex, in order to add an extra penalty for delaying a large number of impressions, and $c_i(a_i)$ could be time-dependent in the finite horizon case. The solution methods derived here can be applied to these more general forms of $c_i(a_i)$ as well. The delay cost per transition is proportional to the length of time τ until the next transition—that is, the arrival of a viewer or a new ad campaign. Because a_i is the state just after a transition takes place, the expected delay cost for campaign i until the next transition is $c_i(a_i)E[\tau]$, where $E[\tau] = \frac{1}{\lambda+\mu}$. Thus, the total expected delay cost for state $\mathbf{a}$ is $\sum_{i=1}^{N} \frac{c_i(a_i)}{\lambda+\mu}$.

The expected delay cost is minimized if all impressions can be delivered immediately, but this may not be a desirable outcome for some ad campaigns that want their ads to be "paced" over time. Because new arrivals are Poisson-distributed, the delivery of impressions will naturally occur at a certain rate, rather than all at once. For campaigns requiring slower ad pacing, the requested impressions can be divided into smaller sets that are added incrementally to the queue over time. The state a_i then corresponds to the number of impressions that are ready to be delivered.

4.2. General Dynamic Programming Formulation

We develop a general dynamic programming formulation for solving the multicampaign problem in the finite horizon case. The optimal solution for N campaign types determines the bid b and the index I of the campaign type to receive the next viewer, for each t and state $\mathbf{a}$. Because the large state space in the resulting problem can increase computational complexity, we develop a heuristic solution method that solves the multicampaign problem as a set of separate campaigns, as well as a method for evaluating the accuracy of the heuristic. We also develop a steady-state solution for the single campaign problem that can further simplify the heuristic for large time horizons and use it for our numerical sensitivity analysis. These profit-maximizing solutions could assign many viewers in a row to a particular campaign of type i, which may be undesirable because some campaigns may end up with long delays. Ad agencies sometimes apply alternative rules, such as a random allocation across the available campaigns or a "round-robin" allocation. Our model combined with simulation could be used to compare the expected profits resulting from these alternative allocation strategies.

In solving these optimization problems, it is more convenient analytically to treat the winning probability w as the decision variable and use the inverse function $b(w)$ to obtain the bid b. This is equivalent to optimizing the bid b, because the functions $w(b)$ and $b(w)$ are monotone increasing. To be consistent with the empirical data discussed in Section 6, the function $w_j(b_j)$ is assumed to be increasing and strictly concave in the bid prices—that is, $w_j'(b_j) > 0$ and $w_j''(b_j) < 0$ for $b_j > 0$. It can be verified by calculus that this implies that the inverse function $b(w)$ is increasing and convex. Then, define:

- $W_t(\mathbf{a}, w, I)$ = the total expected profit in state $\mathbf{a}$ with t periods to go, with winning probability w, where I identifies the campaign type that will receive the viewer if the bid $b(w)$ wins
- $W_t(\mathbf{a}) = \max_{\{w,\,I\}} W_t(\mathbf{a}, w, I)$ = the maximum profit in state $\mathbf{a}$ with t periods to go
- $w_t(\mathbf{a})$ = the optimal winning probability when the state is $\mathbf{a}$ with t periods to go
- $b_t(\mathbf{a})$ = the bid that corresponds to $w_t(\mathbf{a})$—that is, $b_t(\mathbf{a}) = b(w_t(\mathbf{a}))$
- $I_t(\mathbf{a})$ = the optimal campaign type to receive a viewer when in state $\mathbf{a}$, with t periods to go
- C_i = the terminal cost for each impression in queue for campaign type i at $t = 0$.

We then have the following recursion relationship for the dynamic programming problem:

$$W_t(\mathbf{a}) = \max_{w,I} \left\{ \begin{array}{l} \dfrac{\mu}{\lambda+\mu}[w\{r_I - b(w) + W_{t-1}(\mathbf{a}-\mathbf{u}_I)\} + (1-w)W_{t-1}(\mathbf{a})] \\ \quad + \displaystyle\sum_{i=1}^{N} \frac{\lambda_i}{\lambda+\mu} W_{t-1}(\min(\mathbf{A}, \mathbf{a} + s_i\,\mathbf{u}_i)) - \frac{1}{\lambda+\mu}\sum_{i=1}^{N} c_i(a_i) \end{array} \right\}, \tag{2}$$

with $W_0(\mathbf{a}) = -\sum_i C_i a_i$, and $\mathbf{u}_i$ = a vector of dimension N with one in the i^{th} component and the rest zeros. Clearly, in (2), an arriving viewer only affects the state of campaign type I. The first line of (2) shows the expected profit changes for $t-1$ that result from both winning and losing bids. The first term in the second line of (2) gives the expected profit for a new campaign arrival, and the second term computes the expected delay cost until the next transition. The solution method for (2) is described by the following theorem, which is proved in the online appendix.

Theorem 1. *The optimal solution for the Dynamic Programming Problem (2) is obtained as follows. For each $t > 0$, first determine the optimal w_i for $i = 1, \ldots, N$, by solving*

$$r_i - \Delta_i W_{t-1}(\mathbf{a}) = b(w_i) + w_i b'(w_i), \tag{3}$$

where $\Delta_i W_{t-1}(\mathbf{a}) = W_{t-1}(\mathbf{a}) - W_{t-1}(\mathbf{a}-\mathbf{u}_i)$. Then, $w_t(\mathbf{a}) = \max_i\{w_i\}$, and $I_t(\mathbf{a}) = \arg\max_i\{w_i\}$, that is, the next viewer is allocated to the campaign with the highest bid $b(w_i)$, which equals $b(w_t(\mathbf{a}))$.

The proof of Theorem 1 also shows that the solution for each w_i is unique because the right-hand side of (3) is monotone increasing in w_i. If $r_i - \Delta_i W_t(\mathbf{a}) < 0$, then (3) has no positive solution, and the optimal bid is zero for campaign type i, with $w_i = 0$. Thus, the optimal solution $w_t(\mathbf{a})$ is uniquely determined, but there could be ties for the optimal index $I_t(\mathbf{a})$. If $w_t(\mathbf{a}) = 0$, no bid is submitted, and a viewer arrival results in the updated objective value $W_t(\mathbf{a}) = W_{t-1}(\mathbf{a}) - \frac{1}{\lambda+\mu}\sum_{i=1}^{N} c_i(a_i)$.

4.3. Heuristic Solution

In principle, the solution method in Theorem 1 can be used to maximize the total expected profit for any number of different campaign types managed by an ad agency. However, this problem has a very large state space, because each decision depends on all the queue lengths $\mathbf{a} = [a_i]_{1\le i\le N}$, as well as the time period t. For example, if $A_i = 15$ for all i, which is used in some of our computed examples, and there are five different campaign types, then $\mathbf{a}$ has 16^5, or more than a million, possible states. The optimal policies $w_t(\mathbf{a})$ and $I_t(\mathbf{a})$ would need to be determined for every $\mathbf{a}$ and every t. Thus, the exact solution method is not computationally feasible in many cases.

To obtain an approximate solution to the multiple-campaign problem, we apply the following heuristic method. Suppose the ad agency optimizes each campaign type i separately. Then, each campaign i would observe only the state a_i, and not the states or bids of the other campaign types. For each t and a_i, define the following variables, where the superscript 1 denotes the one-campaign problem:

- $W^1_{i,t}(a_i)$ = the total expected profit for campaign type i in state a_i, *when optimized separately*
- $w^1_i(a_i)$ = the optimal winning probability for campaign type i in state a_i, from the heuristic.

Then, $w^1_i(a_i)$ is determined by solving the equation:

$$r_i - \Delta W^1_{i,t-1}(a_i) = b(w^1_i(a_i)) + w^1_i(a_i)b'_i(w^1_i(a_i)), \tag{4}$$

where $\Delta W^1_{i,t-1}(a_i) = W^1_{i,t-1}(a_i) - W^1_{i,t-1}(a_i - 1)$. The corresponding bid is $b^1_i(a_i) = b(w^1_i(a_i))$. The ad agency then assigns the next viewer to the campaign type i with the highest bid—that is, the largest $w^1_i(a_i)$. The Heuristic Equation (4) is the first-order necessary condition (FONC) for the general dynamic programming problem if there is only one campaign type.

With multiple campaigns, the Heuristic Solution (4) results in a different solution from (3) because of the term $\Delta W^1_{i,t-1}(a_i)$. The $W^1_{i,t}(a_i)$ are obtained from the following recursion equation using the optimal $w^1_{it}(a_i)$ at each time period t

$$W^1_{i,t}(a_i) = \left\{ \begin{array}{l} \dfrac{\mu^1_i}{\lambda_i+\mu^1_i}\left[w^1_{it}(a_i)\left\{r_i - b(w^1_{it}(a_i)) - \Delta W^1_{i,t-1}(a_i)\right\} + W^1_{i,t-1}(a_i)\right] \\ \quad + \dfrac{1}{\lambda_i+\mu^1_i}[-c_i(a_i) + \lambda_i W^1_{i,t-1}(\min\{A_i, a_i + s_i\})] \end{array} \right\}, \tag{5}$$

where $\mu^1_i = \frac{s_i\lambda_i}{\sum_{k=1}^{N} s_k\lambda_k}\mu$. The terminal values are $W^1_{i,0}(a_i) = -C_i a_i$ for all i.

This definition of μ^1_i allocates the arriving viewers to campaign type i in proportion to the demand rate

$s_i\lambda_i$. That is, every campaign type has the same ratio for (impression arrival rate)/(viewer arrival rate). If all campaign types also have the same cost parameters, this means that they will tend to have the same queue lengths. Other definitions of μ_i^1 could be used as well, as long as $\sum_{1\le i\le N}\mu_i^1 = \mu$. The heuristic allocates viewers to each campaign at a constant rate, regardless of the state of the system $\mathbf{a}$. In the general dynamic programming (DP) model, the allocation of viewers is dynamic and depends on both the state vector $\mathbf{a}$ and the period t, as well as the delay costs $c_i(a_i)$ and the revenues r_i. These inputs determine the bid that is submitted for each t and the allocation of the viewer if the bid is won. When the time horizon is large, a simpler alternative heuristic is to use the one-campaign steady-state solution for all t, as discussed in the next subsection.

4.4. Evaluating the Heuristic

As discussed previously, we will evaluate the heuristic by comparing the optimal DP solution to the heuristic solution that uses the one-campaign steady-state solution for each campaign type for all t. The steady-state solution discussed in the next section can be used to determine the following variables for each campaign type i:

- $b_i(a_i)$ = the optimal bid for campaign i in steady state when its queue length is a_i
- $x_i(a_i)$ = the steady-state probability of campaign i being in the state a_i.

When the general dynamic programming model is in state $\mathbf{a} = [a_i]_{1\le i\le N}$, the corresponding heuristic policy is therefore $b_t^1(\mathbf{a}) = \max_i\{b_i^1(a_i)\}$, $I_t^1(\mathbf{a}) = \arg\max_i\{b_i^1(a_i)\}$ for all t and $\mathbf{a}$. The heuristic expected profit $W_t^1(\mathbf{a})$ is calculated from (2) by substituting $w(b_t^1(\mathbf{a}))$ for w and $I_t^1(\mathbf{a})$ for I, rather than maximizing over w and I. The total profit $W_t^1(\mathbf{a})$ can then be compared with the optimal profit $W_t(\mathbf{a})$. For the comparison, we used a large value of t to help account for the fact that the heuristic policy used the steady-state solution. To summarize the difference $W_t(\mathbf{a})-W_t^1(\mathbf{a})$ for each state $\mathbf{a}$ as a single number, we computed a weighted average value based on the probability of being in state $\mathbf{a}$. The approximate probability of being in state $\mathbf{a} = [a_i]_{1\le i\le N}$ is $\prod_{i=1}^N x_i(a_i)$, which is computed from the steady-state probabilities for the independent campaigns.

Our numerical analysis found that the heuristic optimization method generally performed well, based on the computed examples. For A_i values close to the optimal, the heuristic solutions resulted in expected profits that are only a few percent less than the optimal profit. It is not computationally feasible to test the heuristic method for a large number of campaign types, but the results discussed in Section 6 for two and three campaign types suggest that the heuristic should perform effectively for larger numbers of campaign types as well.

4.5. The Optimal Steady-State Policies

In this section, we solve for the optimal steady-state policies for one campaign type. This is consistent with the heuristic solution that determines the optimal solution for each campaign type separately and is used in evaluating the heuristic method. The steady-state solution also allows us to easily determine the average queue length and the expected waiting time for serving an impression. Using the steady-state solution greatly simplifies our numerical sensitivity analysis because the results are not time-dependent.

Because the decision variables do not depend on t in steady state, we can drop that subscript. As before, we use w as the decision variable and define the following parameters:

- $b_a \equiv b(w_a)$ = the optimal bid when the queue length of waiting impressions is a
- w_a = the probability of a winning bid in state a (the decision variable)
- x_a = the steady-state probability that the queue length is a, which depends on $[w_a]_{0\le a\le A}$
- $\phi(e\,|\,w_a, a)$ = the probability of a transition from state a to state e, given the bid $b(w_a)$.

The optimization, therefore, can be done by using the variables x_a and w_a for all $0 \le a \le A$. When $a = 0$, we use $w_0 = 0$ and $b_0 = 0$. We assume that in steady state, all bids are such that $w_a > 0$ for $a \ge 1$. This implies that any state e can be reached from any state $a \in \{0,\dots,A\}$, and, thus, the Markov chain is recurrent (see, e.g., Gross et al. 2013). Therefore, for any fixed set of positive bids $[b_a]_{0\le a\le A}$, the steady-state probabilities $[x_a]_{0\le a\le A}$ satisfy the following conditions:

$$x_a - \sum_{e=0}^{A} x_e\phi(a\,|\,w_e, e) = 0, \ \text{for} \quad a = 0,\dots,A, \tag{6}$$

$$\sum_{a=0}^{A} x_a = 1, \ x_a \ge 0 \ \text{for} \quad a = 0,\dots,A. \tag{7}$$

It can be shown that the equation for $a = A$ is redundant because it can be obtained by summing the other equations. Thus, we have A equations altogether. In steady state, the objective is to maximize the expected profit per unit time $\Pi(w_a, a)$, which is given by:

$$\Pi(w_a, a) = \mu\sum_{a=1}^{A} x_a w_a[r - b_a(w_a)] - c\sum_{a=1}^{A} a x_a. \tag{8}$$

Here, we have used the linear form $c(a) = ca$ for the delay cost to simplify the notation. The transition probabilities $\phi(e\,|\,w_a, a)$ depend on $[w_a]_{0\le a\le A}$ and the queue capacity A. Substituting the specific transition probabilities for this system, it can be verified that the steady-state equations above have the following form:

$$x_0 - \rho_0 x_1 w_1 = 0, \tag{9}$$

$$x_a(1+\rho_0 w_a) - \sum_{e=0}^{a-1} x_e q_{a-e} - \rho_0 x_{a+1} w_{a+1} = 0,$$
$$\text{for } a = 1, \ldots, A-1, \tag{10}$$

$$\sum_{a=0}^{A} x_a = 1, \tag{11}$$

where $\rho_0 = \frac{\mu}{\lambda}$ and $q_s = P\{\text{an arriving campaign requests } s \text{ impressions}\}$. Given a set of winning probabilities $[w_a]_{0 \le a \le A}$, and using $w_0 = 0$ and $q_a = 0$ for $a \le 0$, the equations can be solved sequentially for $[x_a]_{1 \le a \le A}$ as:

$$x_{a+1} = \frac{x_a(1+\rho_0 w_a) - \sum_{e=0}^{a-1} x_e q_{a-e}}{\rho_0 w_{a+1}}, \; a = 0, 1, \ldots, A-1. \tag{12}$$

This determines the variables $[x_a]_{1 \le a \le A}$ as a function of x_0, which is then determined from the constraint $\sum_{a=0}^{A} x_a = 1$.

The optimal solution of the constrained optimization problem (8)-(11) can be obtained by using commercial solvers. We used MATLAB to solve this problem and found that the solution converged very quickly to an optimal solution for state spaces up to $A = 25$. The following proposition, proved in the online appendix, shows that the optimal solution obtained for this problem is a global maximum of the constrained optimization problem.

Proposition 2. *Suppose that we use the transformation $y_a = x_a w_a$, for $a = 1, \ldots, A$, with $w_0 = 0$ and $y_0 = 0$. Then, the constraint equations are linear in the variables (x_a, y_a), and the transformed objective function is concave in (x_a, y_a).*

The solution to the transformed problem uniquely determines $w_a = y_a / x_a$ for $a = 1, \ldots, A$. We know that all $x_a > 0$ because the Markov process is recurrent. The resulting steady-state probabilities $\{x_a\}$ also determine the average number of waiting impressions in the system, $L = \sum_{a=1}^{A} a x_a$, as well as the average waiting time for an impression in the system, $W = \frac{L}{\lambda \sum s q_s}$.

Suppose we modify the formulation by adding a constraint of the form $\sum_{a=1}^{A} a x_a \le L_0$ for some target queue length L_0. If we then change the objective function in the original problem to $\mu \sum_{a=1}^{A} x_a w_a [r - b_a(w_a)]$, subject to this constraint, we see in the original objective function that c can be interpreted as a Lagrange multiplier for the queue-length constraint. For a given L_0, the solution of this constrained problem would also determine a value for c. Because c is clearly increasing as L_0 decreases, the solution for c is unique, but there may not be a feasible solution for some L_0 values. Because c is typically not specified in contracts in practice, this discussion provides a managerial intuition for its use, as well as a way to estimate its value.

5. Sensitivity Analysis of the Optimal Solution

This section obtains several important insights about how the profit and the optimal bids in the dynamic programming formulation are affected by changes in the model parameters. The derivations are done for changes in the parameter values for one campaign type. These results also extend to the general multicampaign model, subject to some technical caveats. The optimal bid for the multicampaign problem is the optimal bid for campaign type I, which is the highest bid of the multiple campaigns. The first-order necessary condition for the optimal bid for campaign type I has the same form as the FONC for the optimal bid when there is only one campaign type. Thus, the proofs in this section apply to campaign type I as well.

However, if an input parameter is changed in such a way that campaign type I no longer has the highest bid, then further changes in that campaign's parameter will have no effect on the optimal multicampaign bid. Similarly, changing a parameter value for any campaign type j other than I has no effect on the optimal multicampaign bid, unless the parameter is changed in such a way that campaign j's bid becomes the highest. The sensitivity proofs in this section are done for just one campaign type, which avoids these technical problems. The dynamic programming equation for the expected profit for one campaign type can be written as follows:

$$W_t^*(a) = \left\{ \begin{aligned} &\frac{\mu}{\lambda+\mu}[w_t^*(a)\{r - b(w_t^*(a)) - \Delta W_{t-1}^*(a)\} + W_{t-1}^*(a)] \\ &\quad + \frac{1}{\lambda+\mu}[-ca + \lambda W_{t-1}^*(\min(A, a+s))] \end{aligned} \right\}, \tag{13}$$

where the terminal values are $W_0^*(a) = -Ca$, for each a.

The specific parameters that we analyze are the revenue per viewer r, the marginal delay cost per unit time c, the state of the system a, the arrival rate of campaigns λ, the arrival rate of viewers μ, and the transaction rate or scale parameter, which we denote by γ. The scale parameter γ increases λ and μ simultaneously. Because the proofs for these results use induction on t and are complex algebraically, the details are presented in the online appendix. The analytical results that hold for every t must hold in steady state as well. Section 6 illustrates the sensitivity results for a specific set of parameter assumptions.

A general method used in proving the sensitivity results is based on Lemma 1. Suppose β denotes one of the parameters described above, and our goal is to study its effects on the optimal profit and optimal bids. To show the dependence on β, we define the new optimal objective function:

- $W_t^*(a,\beta)$ = the objective function in (13), in which β can vary.

We can state the following lemma regarding changes in the parameter β, which is proved in the online appendix.

Lemma 1. *For any parameter $\beta \in \{\gamma, r, c, \lambda, \mu\}$, the following results hold at the optimal bid:*

i. $\frac{dW_t^*(a,\beta)}{d\beta} = \frac{\partial W_t^*(a,\beta)}{\partial\beta}$, *and*

ii. $\frac{\partial b_{at}^*}{\partial\beta}$ *has the same sign as* $\frac{\partial^2 W_t^*(a,\beta)}{\partial\beta\partial b_{at}}$.

Let us first consider the sensitivity of the profit and the optimal bids to the scale parameter γ (i.e., $\beta=\gamma$). The sensitivity derivation with respect to γ uses the arrival rates $\gamma\mu$ and $\gamma\lambda$, which implies that $\frac{\mu}{\lambda+\mu}$ remains unchanged. We also note that if $W_t^*(a,\beta)$ is increasing in β, then the expected profit per transition $W_t^*(a,\beta)/t$ is also increasing in β, because t is a constant.

Proposition 3. *As the scale parameter γ increases, the following results hold for all a, t:*

i. *The optimal total profit $W_t^*(a,\gamma)$ as well as the optimal profit per transition increases, and*

ii. *The optimal bid b_{at}^* decreases.*

This result shows that as the system scales, the total expected profit and the profit per transition increase, and the optimal bids decrease. Because this result holds for all t, it will hold in the steady state as well. This implies an "economies of scale" property in the following sense. If the transition rate doubles—that is, μ and λ both double—then the average profit per transition $W_t^*(a,\gamma)/t$ is increasing, and, consequently, the average profit per unit time $(\mu+\lambda)W_t^*(a,\gamma)/t$ more than doubles. Our numerical analysis found significant improvements in both the average profit per unit time and the average waiting time for the impressions in the queue, as γ increases. Our numerical evaluation also found that the optimal expected profit per campaign was larger with three campaigns than with two campaigns, when the campaigns are managed jointly.

The sensitivity results for the arrival rates of campaigns λ and viewers μ, when considered separately, are more complex and less intuitive and require specific conditions. The sufficient conditions that describe the sensitivity of total profit and bid prices to λ and μ are given in Propositions 4 and 5. If the arrival rate of campaigns increases, but a sufficient number of viewers are not available, the profit might decrease due to an increase in delay costs. Conversely, an increase in the arrival rate of viewers does not necessarily imply a higher profit, unless there is sufficient demand for those viewers from arriving ad campaigns. To simplify the insights, for a and $\beta \in \{\lambda,\mu\}$, we define the following:

- $V_t^*(a,\beta)$ = the expected profit from an arriving viewer, and
- $K_t^*(a,\beta)$ = the expected profit from an arriving campaign.

When these definitions are matched to the appropriate terms, the Total Profit Function (13) is written as $W_t^*(a,\beta) = \rho V_{t-1}^*(a,\beta) + (1-\rho)K_{t-1}^*(a,\beta) - \frac{ca}{\lambda+\mu}$, where $\rho = \frac{\mu}{\lambda+\mu}$. Intuitive interpretations of the sufficient conditions for μ and λ in Propositions 4 and 5 are as follows. Following the induction method, suppose that the objective function $W_{t-1}^*(a,\lambda)$ is increasing in λ. Then, a sufficient condition for $W_t^*(a,\lambda)$ to be increasing in λ is that a new campaign arrival is more profitable than a new viewer arrival. Conversely, if the objective function $W_{t-1}^*(a,\mu)$ is increasing in μ, a sufficient condition for $W_t^*(a,\mu)$ to be increasing in μ is that a new viewer arrival is more profitable than a new campaign arrival. The sensitivity conditions for the bid prices focus on $V_t^*(a,\lambda) - K_t^*(a,\lambda)$, that is, the value of a new viewer minus the value of a new ad campaign. If this difference increases in a fast enough, then we have the intuitively expected results that the optimal bids are increasing in λ and are decreasing in μ. That is, when the sufficient condition holds, more arriving ad campaigns increase the demand for viewers, which leads to higher bids, whereas more arriving viewers offer more opportunities to win bids, and thus, smaller bids are attractive. In general, as the impression queue grows larger (i.e., a increases), a new viewer becomes more valuable than a new campaign. When there are sufficient impressions in the queue, a new viewer offers the opportunity to obtain additional revenue and reduce delay costs, and additional campaigns are not needed. However, as noted in the numerical analysis, the queue capacity A causes the bid prices to be nonmonotonic in a, which introduces additional complexity. In the following propositions, we use the notation Δ, defined previously as $\Delta f(a) = f(a) - f(a-1)$ for any function $f(a)$, where a $(1 \le a \le A)$ is the state of the system.

Proposition 4. *For all a, t:*

i. *$W_t^*(a,\lambda)$ increases in λ if $W_{t-1}^*(a,\lambda)$ is increasing in λ and $K_t^*(a,\lambda) > V_t^*(a,\lambda)$, and*

ii. *b_{at}^* increases in λ if $b_{a(t-1)}^*$ is increasing in λ and $\Delta\left[V_{t-1}^*(a,\mu) - K_{t-1}^*(a,\mu)\right] > \frac{c}{\mu}$.*

Proposition 5. *For all a, t:*

i. *$W_t^*(a,\mu)$ increases in μ if $W_{t-1}^*(a,\mu)$ is increasing in μ and $V_t^*(a,\mu) > K_t^*(a,\mu)$, and*

ii. *b_{at}^* decreases in μ if $b_{a(t-1)}^*$ is decreasing in μ and $\Delta\left[V_{t-1}^*(a,\mu) - K_{t-1}^*(a,\mu)\right] > -\frac{c}{\lambda}$.*

Next, we turn our attention to the two key financial parameters in our model, the delay cost and the unit revenue per viewer.

Proposition 6. *For all a, t:*
i. *The optimal total profit $W_t^*(a,r)$ increases in r, and*
ii. *The optimal bid b_{at}^* increases in r.*

This proposition shows that with a larger r, the benefits of higher bids, which increase the probability of winning and, thus, reduce delays, outweigh the additional costs of the higher bids. If the unit revenue per impression increases, the ad agency also generates higher profits.

Proposition 7. *For all a, t:*
i. *The optimal total profit $W_t^*(a,c)$ decreases in c, and*
ii. *The optimal bid b_{at}^* increases in c.*

Part (ii) shows that when the unit delay cost is higher, the agency should bid higher to increase the probability of winning. This means that the benefit of lowering the total delay cost through higher bids outweighs the loss in profit from the higher bids. Part (i) shows that the profit decreases nonetheless.

The optimal bids can also vary significantly with a, the state of the system. It may be natural to expect that as the queue length increases, the bids should always increase to reduce the delay cost, but our numerical analysis shows that this is not always the case. Our numerical analysis found that the bid prices increase up to a certain point and then decrease as the queue size increases. When state a is close to the maximum queue capacity A, future arriving ad campaigns are more likely to be truncated or entirely rejected. This reduces the future expected queue length, which allows the optimal bids to be lower. An inequality that formalizes this result is obtained as follows. First, define the forward difference operator:

$$\begin{aligned}&\Delta^k W_t^*(a) = \Delta^{k-1} W_t^*(a) - \Delta^{k-1} W_t^*(a-1), \text{ with}\\&\Delta^1 W_t^*(a) \equiv \Delta W_t^*(a) = W_t^*(a) - W_t^*(a-1).\end{aligned} \tag{14}$$

Because $\frac{\partial}{\partial b_{at}}\left\{b_{at} + \frac{w(b_{at})}{w'(b_{at})}\right\} > 0$, we see from the FONC for b_{at}^* that the bids satisfy $b_{at}^* > b_{a-1,t}^*$ if and only if $\Delta W_{t-1}^*(a) < \Delta W_{t-1}^*(a-1)$. Thus, we have:

Proposition 8. *The following results hold for all a, t:*
i. *The optimal bid b_{at}^* increases in a if and only if $\Delta^2 W_{t-1}^*(a) < 0$,*
ii. *The optimal bid b_{at}^* has a unique maximum with respect to a, if $\Delta^3 W_t^*(a) \geq 0$ for $3 \leq a \leq A-1$.*

6. Numerical Analysis

In order to validate some of our model's assumptions, we compared them to a publicly available data set obtained from iPinYou Information Technologies Co., Ltd. (https://www.ipinyou.com/), which is a Demand-Side Platform serving advertisers from different industry brands. Their data set includes logs of ad auctions, bids, impressions, clicks, and final conversions, but it does not include specific information on the ad campaigns or the contracts with the advertisers. In particular, we used these data to test our assumptions about exponential interarrival times for viewers and the shape of $w(b)$, the probability of winning as a function of the bid price. Where possible, we have used inputs for our numerical analysis that are consistent with this set of empirical observations.

The data correspond to visits by 2,521,630 viewers to various websites that are connected to the DSP through two advertising exchanges during the periods of June 13 to June 16, 2013 (Liao et al. 2014). This was a second price auction, and both the winning bid and the paying price were reported. The actual prices, reported as Yuan per thousand impressions, were scaled linearly before the data set was released publicly. In this case, the winning bid appears to have been set artificially high by iPinYou all the time (either 227 or 300 Yuan), which resulted in winning essentially all the auctions during this period. This was likely done to determine the distribution of the second-highest bids. Thus, we used the paying price to investigate the shape of the probability of winning function $w(b)$, because any bid higher than this amount would be the winning bid. We used the data for six advertising slots selected at random. After deleting outliers, there were 85,027 records of ads that were shown, with an average arrival rate of 1401.4 viewers per second. The range of the paying prices for the winning bids was from 4 Yuan to 300 Yuan, with a mean of 56.17 Yuan.

Figure 2 illustrates the empirical cumulative distribution of viewers' interarrival times and the cumulative distribution of an exponential distribution fitted to the empirical data. Figure 3 does the same for the paid bids. The data in Figures 2 and 3 correspond to

Figure 2. The Cumulative Empirical and Fitted Distributions of Viewer Interarrival Time

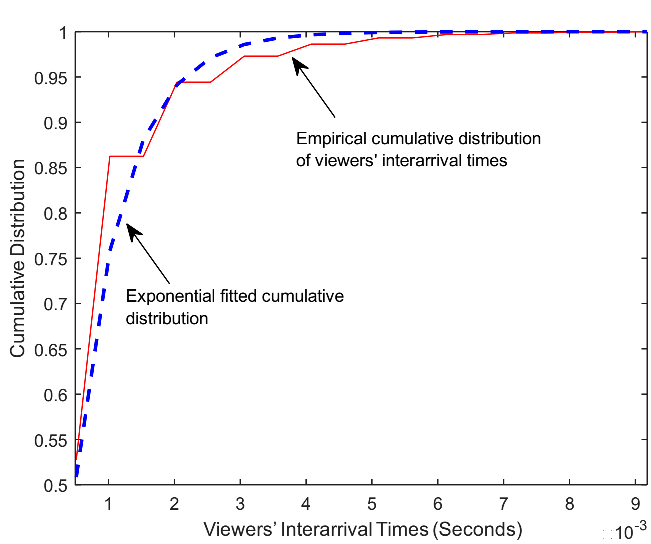
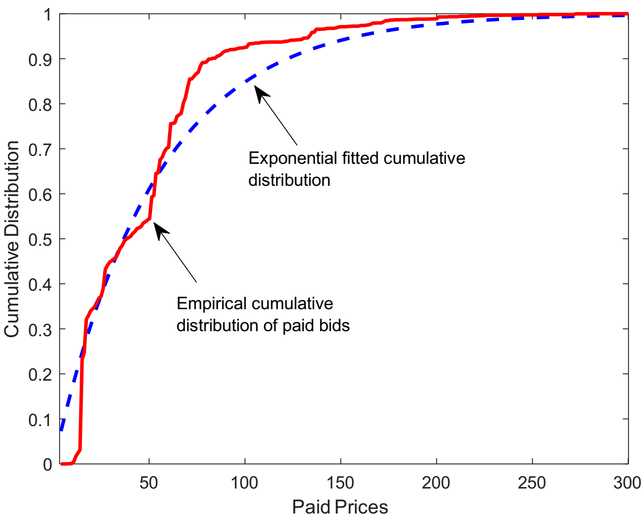


Figure 3. The Cumulative Empirical and Fitted Distributions of Paid Bids



one of the ad slots (other results were similar), and the graphs indicate that exponential distributions provide a good approximation for both the interarrival times and the probability of winning.

6.1. Analysis of the Base Case

For our numerical analysis, we assume that the time scale is adjusted so that the arrival rate of viewers is $\mu = 1.0$ in the base case. Campaigns arrive at a rate $\lambda = 0.2$, and each arriving campaign requests $s = 2$ impressions. Thus, the demand for impressions arrives at the rate of $\lambda s = 0.4$ per period, which is 40% of the viewer arrival rate μ in the base case. The capacity of the impression queue was set at $A = 15$. Each delivered impression generates $r = \$5$ in revenue for the agency, and the delay cost is $c = \$0.20$ per impression per unit time. The probability of winning function was assumed to be $w(b) = 1 - e^{-0.4b}$. The choice of an exponential function is based on the shape of the cumulative distribution of the paid prices shown in Figure 3. Because we are performing the numerical analysis in steady state, we report the profit results using expected profit per unit time $(\mu + \lambda)\Pi(w_a, a)$.

Solving the steady-state optimization problem by gradient search methods determines the optimal bid prices $[b_a^*]_{0\le a\le 15}$ and the steady-state probabilities $[x_a]_{0\le a\le 15}$ shown in Figure 4. The optimal bid starts at $b_0 = 0$ when $a = 0$, increases to a maximum of \$3.187 when $a = 12$, and then decreases. The weighted average bid across all states is \$1.492. In steady state, the average queue length is $L = 2.72$ impressions, and x_0, the probability of an empty queue, is 27.4%. The steady-state probability x_2 is higher than x_1 because we assumed that $s = 2$ impressions for all campaigns. The optimal bidding strategy is such that the likelihood of being in states $a \le 6$ is about 90%. Finally, the optimal expected steady-state profit per unit time is \$0.59, and the expected steady-state profit per transition is \$0.492. The shape of the optimal bid price curve highlights the counterintuitive observation that bidding higher is not the best way to increase profit when the queue gets close to its capacity. The queue capacity acts as a control that automatically prevents the queue from growing, thereby lowering the expected delay cost.

We evaluated three alternative benchmark policies and compared the results to the optimal state-dependent dynamic policy described in Table 2: the Fixed policy determines the best fixed bid for all states, the Deterministic policy maximizes the one-period expected profit $\mu w(b)(r - b)$, and the Linear policy determines the optimal bid that is a linear function of the state a.

The optimal Fixed Policy bids higher than the optimal Dynamic Policy for $a < 4$, but slightly lower otherwise, resulting in a roughly 12% profit loss. The Deterministic

Figure 4. Optimal Bids and Steady-State Probabilities vs. the Number of Impressions in the Queue

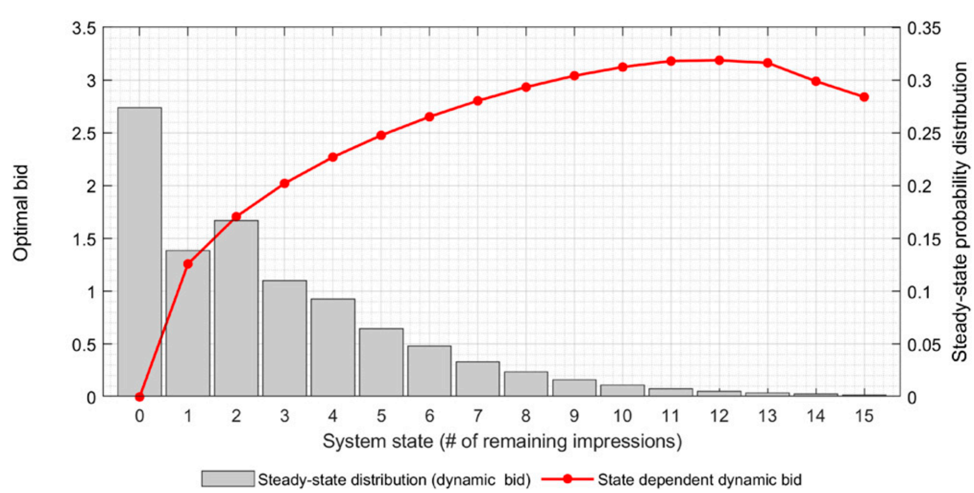


Table 2. The Optimal Dynamic Policy vs. Three Alternative Benchmark Policies in Steady State

Optimal policies	b_a^* $(0 \le a \le 15)$	$E(b_a^*)$	x_0	L	Π	% Profit loss
State-dependent dynamic policy	Figure 4	1.49	0.274	2.72	0.59	–
Optimal fixed policy	2.25	2.25	0.330	2.86	0.52	11.9%
Optimal deterministic policy	1.98	1.98	0.290	3.49	0.49	16.9%
Optimal linear policy	$0.5418 \times a$	1.57	0.162	2.9	0.47	20.3%

Note. b_a^*, optimal bids; $E(b_a^*)$, average bid; x_0, probability of empty queue; L, average queue length; Π, average profit rate.

policy optimizes the one-period expected profit, ignoring the uncertainty in the arrival of viewers and campaigns. The resulting bid is significantly lower than the Fixed policy, which results in a higher average queue length and a further decrease in profits. The Linear policy increases the bid in proportion to the current queue length to correspond to the linear delay penalty cost. This policy's bids are much lower than the optimal dynamic bids when $a < 4$, and much higher otherwise, which substantially reduces the expected profit. This benchmark analysis illustrates the advantages of determining the optimal dynamic state-dependent policy.

6.2. Evaluating the Accuracy of the Heuristic

For the series of parameter values described below, we used the general multicampaign solution method described in Section 4.2 and the heuristic solution method described in Section 4.3 to obtain the optimal profits $W_t(\boldsymbol{a})$ and the heuristic solution profits $W_t^1(\boldsymbol{a})$ for each $\boldsymbol{a} \le \boldsymbol{A}$ and $t = 0, \ldots, 300$. To determine the percentage errors that result from using the heuristic solution, we used the weighted averages described in Section 4.4. For each $\boldsymbol{a} = [a_i]_{1\le i\le N}$ and $t = 300$, we computed weighted average profits using the steady-state probabilities as weights, as discussed in Section 4.4, where $\overline{W}_t = \sum_{\boldsymbol{a}\le \boldsymbol{A}} W_t(\boldsymbol{a}) \prod_{i=1}^N x_i(a_i)$ and $\overline{W}_t^1 = \sum_{\boldsymbol{a}\le \boldsymbol{A}} W_t^1(\boldsymbol{a}) \prod_{i=1}^N x_i(a_i)$.

The percent error is then obtained as $E_t = \frac{\overline{W}_t - \overline{W}_t^1}{\overline{W}_t} \times 100\%$. We performed this analysis separately, assuming that there are two and three campaign types. After computing E_t for the base case values for three different levels of A_i, we performed the sensitivity analysis of the error by changing various parameters, one at a time. The computed values of E_t for $t = 300$ are summarized in Table 3. In doing these computations, we noticed that the incremental profit per transition for the optimal as well as heuristic methods became almost equal for large values of t. Thus, the queuing system behaves as it would in steady state. This also means that resulting percent errors are not very sensitive to the choice of the weights $\prod_{i=1}^N x_i(a_i)$.

From the tables of the percentage errors, we can make several observations. The errors are larger in situations where the queue length is likely to be longer—

Table 3. Percent Profit Reduction with Heuristic for $N = 2$ and $N = 3$ Campaigns

Scenarios tested	$A_i = 5$	$A_i = 10$	$A_i = 15$
$N = 2$ campaigns			
Base case	1.23	2.65	1.68
Parameter changed for sensitivity			
$\lambda_1 = 0.1$	1.35	2.40	2.56
$\lambda_1 = 0.3$	2.82	8.07	6.00
$\mu = 2.5$	1.02	1.45	1.24
$\mu = 3$	1.15	1.30	1.29
$c_2 = 0.4$	5.14	8.44	8.88
$c_2 = 0.6$	5.54	11.16	11.69
$r_2 = 2.5$	5.44	9.67	7.05
$r_2 = 10$	1.68	2.03	1.65
$N = 3$ campaigns			
Base case	1.80	2.72	2.06
Parameter changed for sensitivity			
$\lambda_1 = 0.1$	1.50	2.11	2.00
$\lambda_1 = 0.3$	2.27	4.66	3.28
$\mu = 4$	1.56	1.74	1.76
$\mu = 5$	1.83	1.89	1.91
$c_2 = 0.4$	4.21	6.26	6.23
$c_2 = 0.6$	4.20	7.39	7.45
$r_2 = 2.5$	4.72	6.82	5.14
$r_2 = 10$	1.86	2.38	2.01

Note. The base values for $N = 2, 3$: $\lambda_i = 0.2, s_i = 2, c_i = 0.2, C_i = 1, r_i = 5, 1 \le i \le N$. $\mu = 2 \cdot \mathbf{1}_{(N=2)} + 3 \cdot \mathbf{1}_{(N=3)}$.

for example, when the arrival rate of campaigns increases or the arrival rate of viewers decreases. Decreasing the unit revenue also increases the error. Because both longer queue capacities and lower revenues produce a smaller overall profit, those changes tend to result in an increase in the percent error because they reduce the denominator in the percent-error calculation. Also, the errors tend to be much smaller for A_i closer to its optimal value, which is approximately $A_i = 5$ for most entries in the table. For $A_i = 5$, the percent errors are always less than 6% in the cases we examined, and considerably smaller in many cases.

We also compared the expected profits for managing two and three campaigns in the Base Case. With three campaigns, the profit per campaign was 11.1% larger with the optimal solution and 10.4% larger with the heuristic solution than it was with managing two campaigns. This is an example of economies of scale in managing a larger number of campaigns that serve the same set of viewers.

6.3. Handling Blocks of Impressions

In our numerical examples, we have limited the maximum queue length A to 25 or less in order to be able to run the optimization and sensitivity analysis with desktop computers. We believe that the examples in the section are sufficient to illustrate the general behavior of the optimal solutions and their sensitivities to inputs. In practice, however, ad campaigns often arrive with requests for 100,000 or more impressions. The multicampaign optimization methodology, the heuristic solution method, and the propositions dealing with the sensitivities to inputs can, in principle, be applied with any number of impressions. But optimizing the problem at this scale would require at least the large-scale solvers and extensive computing resources that are offered by commercial computing service providers.

One way to reduce computational requirements is to group the states and the requested impressions into blocks of 10; 100; or 1,000 and treat each block as one impression in the models, making the corresponding scaling adjustments in the cost, revenue, arrival rates, and winning probabilities. This approximates the optimal solution of the original problem, because the bids are forced to be constant within each block.

To test the accuracy of this block approximation, we first solved the original problem, treating each impression individually, to determine the optimal bids for $a = 1, \ldots, A$. We then determined the optimal profit when the bids must be constant within each block and compared this to the profit obtained with the exact solution that uses a different bid for each state a. The percentage reduction in the optimal profit that results from using the block solution then determines the error of the block approximation. We tested this approach using all the input combinations listed in Table 3 and found that using blocks of 10 reduced the expected profit by 0.2%–2.5% over this set of inputs. We found that using blocks of 10 also reduced the computation time by roughly 99% (for $N = 2$). This is probably because the computation time is roughly proportional to A^2, which is the dimension of the steady-state constraint matrix. We also computed the expected profit for block sizes up to 30 for the Base Case inputs and found that the error curve is concave—that is, the error grows at less than a linear rate as the block size increases. It is not difficult to compute the optimal bidding policies for the approximate problem with large block sizes, but we did not have the computing technology to obtain the exact solutions required to test the accuracy of the approximation with blocks of 100 or 1,000. We also noted in our empirical data set that bids are often held constant for very long periods of time in current practice. Thus, the block approximation, even with very large blocks, would be a substantial improvement in this case.

6.4. Sensitivity Analysis for the Input Parameters

In this subsection, we analyze how the optimal solution is impacted by changes in the input parameters of our model in steady state.

6.4.1. Arrival Rate of Transactions (γ).

First, we investigate the effects of changing the transaction rate parameter γ, by simultaneously changing the arrival rate of viewers to $\gamma\mu$ and the arrival rate of campaigns to $\gamma\lambda$. Figure 5 shows that all the optimal bids decrease as γ increases from 0.5 to 2.5, which is consistent with our analytical results discussed in the previous section. The likelihood of an empty queue also decreases

Figure 5. Optimal Bids as a Function of Transaction Scale and the System State

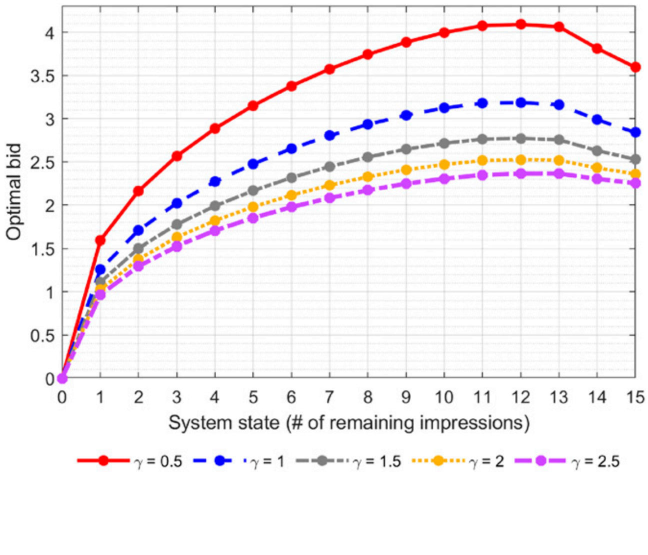


Figure 6. Optimal Profit and Queue Capacity vs. Transaction Scale

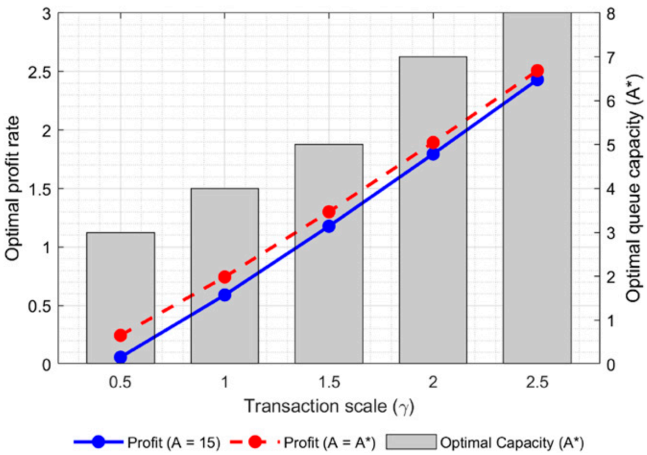
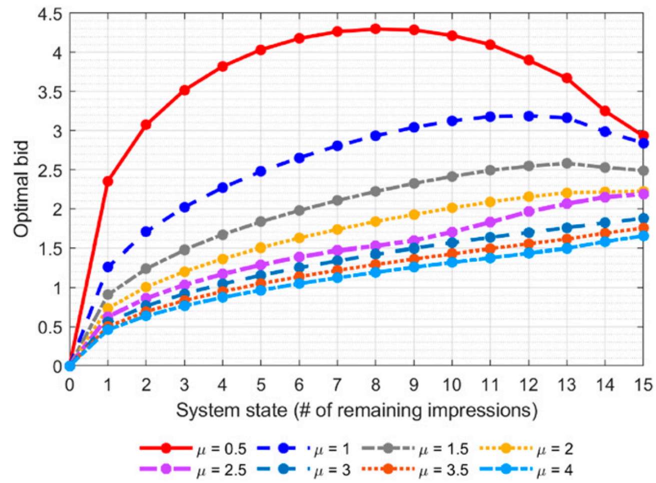


Figure 7. Optimal Bids as a Function of Viewer Arrival Rate and the System State



from 35% to about 18% as γ varies from 0.5 to 2.5 and the expected queue length increases from 2.11 to 3.89. However, using $W = L/(\gamma\lambda)$ implies that the average waiting time decreases from 21.1 to 7.78. A higher transaction rate γ also allows the ad agency to increase its profits through lower optimal bids, as shown in Figure 6.

Another key input parameter is A, the capacity of the impression queue. We determined the optimal queue capacity A^* by maximizing the corresponding profits. The optimal A^* and the corresponding average profit per transition are illustrated in Figure 6 for various values of γ. The optimal A^* is always lower than 15 in the range we evaluated. Even though the curves for the profit with $A = 15$ and A^* appear close, we see that optimizing A^* can have a significant impact on profits—for example, about 26% improvement for the base case of $\gamma = 1$. The improvement is less for $\gamma = 2.5$ because the optimal A^* is larger in that case.

Figure 6 shows that when the optimal A^* is used, the profit per unit time grows nearly 10-fold as the exchange scales fivefold from γ of 0.5 to 2.5, which illustrates the economies-of-scale result established previously. For the same range, the profit per transition, which also corresponds to profit per contract, grows twofold.

6.4.2. Arrival Rate of Viewers (μ).

As discussed previously, the sensitivity results are more complex when μ and λ are varied separately. However, some insights can be derived from our numerical analysis. Figure 7 shows how the optimal bids change as μ, the arrival rate of viewers, changes from 0.5 to 4. As μ increases, the optimal bids decrease because there are more opportunities to bid for viewers. As expected, the optimal profit rate increases as the viewer arrival rate increases, and so does the optimal queue capacity (see Figure 8). When the arrival rate of viewers is low, A^* is reduced to prevent a large backlog of impressions, and the optimal profit rate increases significantly (e.g., from –1.075 to 0.341 when $\mu = 0.5$). As the viewer arrival rate increases, the optimal profit rate with $A = 15$ and with the optimal queue capacity A^* converge.

6.4.3. Arrival Rate of Campaigns (λ).

For this specific analysis, we set the value of r, the revenue per viewer, to \$10 instead of \$5 because the insights are easier to illustrate. When we vary the arrival rate of campaigns in the range from 0.05/period to 0.5/period, we find that optimal bids continue to be concave with respect to the queue length of the waiting impressions a and become more concave as the arrival rate of campaigns increases. As a result, we observe that the optimal bid curves can cross each other (see Figure 9). Comparing

Figure 8. Optimal Profit and Queue Capacity vs. Viewer Arrival Rate

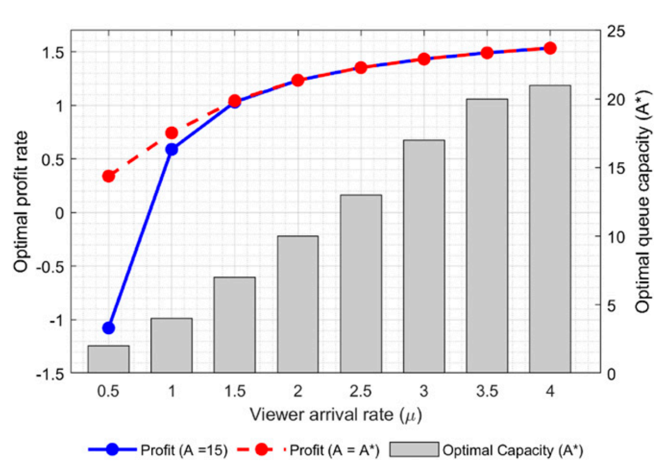
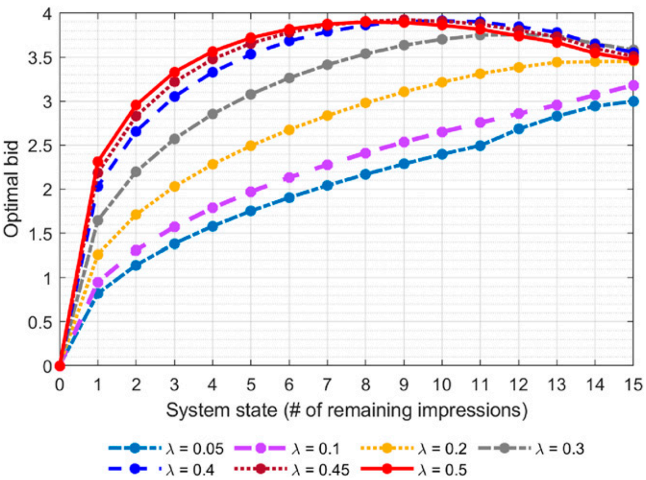


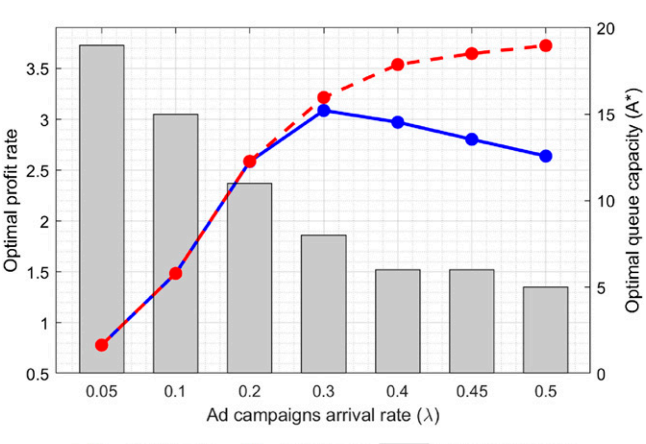
Figure 9. Optimal Bids as a Function of Campaign Arrival Rate and the System State



the optimal bids for $\lambda = 0.5$ with the ones for $\lambda = 0.4$, we see that beyond $a = 8$, the optimal bids are lower for $\lambda = 0.5$. The likelihood of being in state $a \geq 8$ is almost 80% when $\lambda = 0.5$, as compared with only about 54% when $\lambda = 0.4$. This shows that for higher campaign arrival rates, when the queue size approaches its capacity, it is better to bid a little less and let the queue capacity truncate arriving campaigns.

When the queue capacity $A = 15$ is fixed, the profit rate increases initially, but then decreases as the arrival rate of campaigns increases (Figure 10). The higher revenues from additional campaigns are dominated by higher delay costs as the queue grows, because there is an insufficient supply of viewers to support the higher demand. As a result, the expected queue size increases from 0.54 when $\lambda = 0.05$ to 10.75 when $\lambda = 0.5$. With variable queue capacity, the optimal A^* decreases from 19 down to five as λ varies from 0.05 to 0.5. The lower capacity forces many campaigns to be truncated or rejected in order to reduce the delay cost. However, the optimal profit rate is improved substantially by optimizing the queue capacity—for example, by almost 41% when $\lambda = 0.5$.

Figure 10. Optimal Profit and Queue Capacity vs. Campaign Arrival Rate



The effect of increasing s, the number of viewers demanded by an arriving campaign, is similar to changes in λ, because both changes increase the overall demand rate. Increasing s also leads to more truncation of campaigns, as the queue reaches the maximum capacity.

The analysis of the optimal queue capacity shows that it can be a powerful tool for ad agencies as they manage the profitability of their campaigns. In practice, viewer and campaign arrivals may vary over time, so the optimal queue capacity may need to be varied dynamically. If this is not feasible, then selecting a robust value of the queue capacity becomes an important design decision.

6.4.4. Financial Parameters *r* and *c*. We now turn our attention to the two financial parameters, delay cost and revenue per viewer. As the delay cost c increases, the optimal bids increase to reduce the queue size (Figure 11). The effect of increasing c on A^* is similar to the effect of increasing λ. As the delay cost increases, the optimal queue capacity decreases, which results in the truncation or rejection of more arriving ad campaigns (Figure 12). The expected profits with $A = 15$ and with A^* both decrease as c increases, but optimizing the queue capacity significantly mitigates the effects of the higher delay costs and increases the optimal profit rate from $-\$0.09$ to $\$0.42$ when $c = 0.5$. The effect of changing the unit revenue r is also interesting. Figure 13 shows that the bid prices increase with r, as predicted by our previous analytical results. However, the increase is negligible for small values of a ($a \leq 8$ in

Figure 11. Optimal Bids as a Function of Delay Cost and the System State

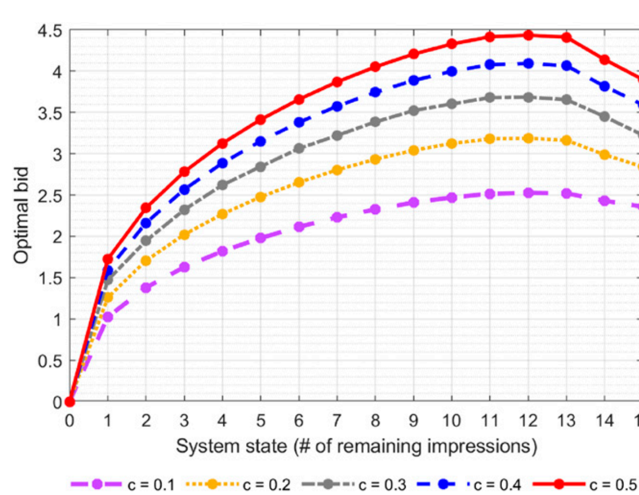


Figure 12. Optimal Profit and Queue Capacity vs. Delay Cost

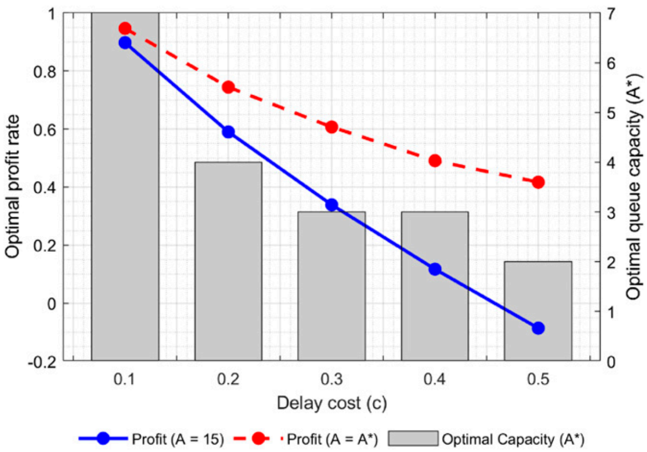
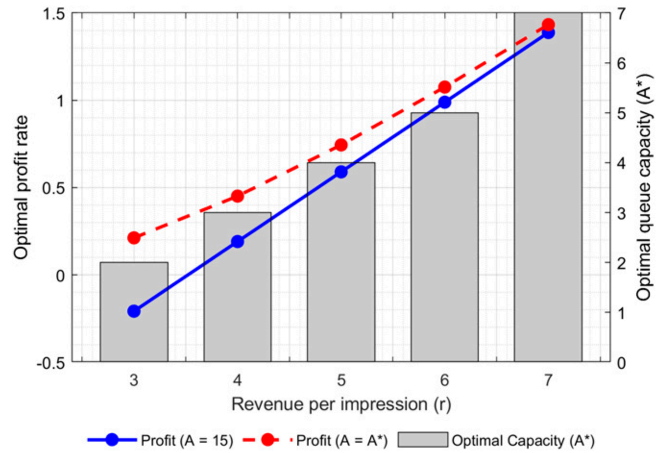


Figure 14. Optimal Profit and Queue Capacity vs. Revenue per Impression

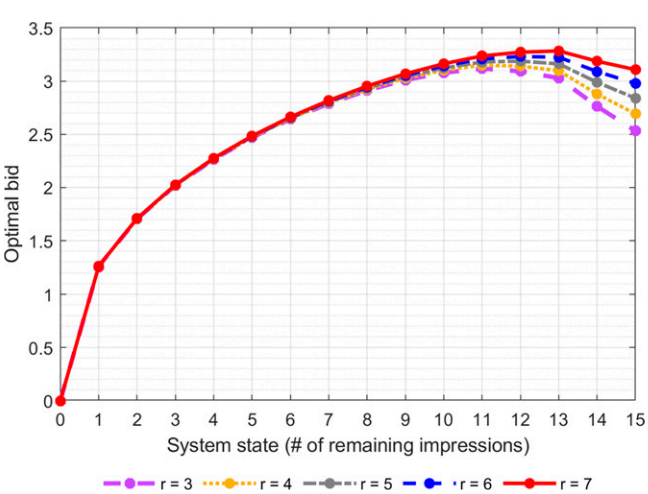


our example). For larger values of r, higher bids are more affordable, whereas for lower r values, it is better to use A^* to reduce delay costs as a increases. For smaller queues, although the higher r increases the profit, it does not change the optimal bid very much. As expected, the optimal profit increases with r, as does the optimal queue capacity and the corresponding profit at the optimal queue capacity (Figure 14).

7. Conclusion

In this paper, we formulate and analyze the ad agency's problem of managing the delivery of impressions in their ad contracts to randomly arriving viewers. The optimal solution determines the bidding and viewer-allocation strategies that maximize the agency's expected profits, subject to the goals of the ad campaigns. In the context of display advertising, this is the first paper that explicitly models the uncertainty in the arrivals of both campaigns and viewers to the ad exchange, as well as the uncertainty in the outcomes of the bids. The resulting model is a Markov decision process, where the state changes correspond to the arrivals of new campaigns and viewers. We obtain exact and heuristic solutions to the multicampaign optimization problem for the finite horizon case and solve the single-campaign problem in steady state. Our heuristic bidding and viewer-allocation policy, as well as the idea of using impression queue capacity as a control variable, have the added advantage that they are easy to implement in practice. The finite horizon model, therefore, can provide the basis for a decision-support tool to help determine optimal bidding and allocation policies in practice. The analysis of the steady-state case provides valuable insights about the sensitivity of the optimal policies and the corresponding profit to the various input parameters, which can also be used to guide ad-agency operating policies.

Figure 13. Optimal Bids as a Function of Revenue per Impression and the System State



Our numerical analysis showed that the optimal dynamic bidding policies developed in this paper performed significantly better than optimal static bidding policies. It also shows that significant profit improvements can be obtained by optimizing the queue capacity in combination with optimal dynamic bidding policies. This combined approach is unique to this paper. The use of blocks of impressions, as discussed in Section 6, can significantly reduce the computational requirements for campaigns with very large numbers of impressions. Testing the accuracy of the blocking approximation for larger block sizes by using commercial-scale computing resources is a useful direction for future research.

One of our assumptions, which is consistent with the published literature, is that the viewer attributes specified for the ad campaigns define disjoint sets of potential viewers types. This assumption allows a model formulation that is separable by viewer type, which greatly reduces the dimensionality of the analysis. Allowing viewer substitutability, perhaps following some prespecified structure, can provide ad agencies and advertisers with additional flexibility. Although this would greatly complicate the formulation, it is a possible direction for future research.

In our model, viewer allocation is based on profit maximization, which may prevent certain ad campaigns from receiving viewers for an extended period of time. In practice, ad agencies sometimes allocate viewers randomly or use other arbitrary allocation rules to achieve greater "fairness" in viewer allocation. Comparing the effectiveness of alternative viewer-allocation policies in combination with an optimization model that explicitly includes the uncertainties described in the paper is also an interesting direction for future research.

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