Koordinatalari bilan berilgan vektorlarning skalyar, vektor va aralash ko‘paytmasi.

Skalyar ko‘paytmaning koordinatalardagi ifodasi.

L_3 chiziqli fazoda ortonormallangan $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ bazisni olaylik. $\vec{a},\vec{b}$ vektorlar bu bazisga nisbatan (x_1, y_1, z_1) va (x_2, y_2, z_2) koordinatalarga ega bo‘lsin:

$$\vec{a} = x_1\vec{e}_1 + y_1\vec{e}_2 + z_1\vec{e}_3,$$
$$\vec{b} = x_2\vec{e}_1 + y_2\vec{e}_2 + z_2\vec{e}_3.$$

va ikki vektorning skalyar ko‘paytmasi

$$\begin{aligned}(\vec{a}\vec{b}) &= (x_1\vec{e}_1 + y_1\vec{e}_2 + z_1\vec{e}_3)(x_2\vec{e}_1 + y_2\vec{e}_2 + z_2\vec{e}_3) = \\ &= x_1x_2\vec{e}_1^{\,2} + y_1y_2\vec{e}_2^{\,2} + z_1z_2\vec{e}_3^{\,2} + (x_2y_1 + x_1y_2)\vec{e}_1\vec{e}_2 + \\ &\quad + (z_1y_2 + y_1z_2)\vec{e}_2\vec{e}_3 + (x_1z_2 + z_1x_2)\vec{e}_1\vec{e}_3\end{aligned}$$

munosabatni yoza olamiz, bazislarni ortonormalligini e’tiborga olsak,

$$(\vec{a}\vec{b}) = x_1x_2 + y_1y_2 + z_1z_2.$$

Demak, koordinatalari bilan berilgan ikki vektorning skalyar ko‘paytmasi bu vektorlar mos koordinatalari ko‘paytmalarining yig‘indisiga teng.

Skalyar ko‘paytmadan kelib chiqadigan ba’zi natijalar:

1. **Vektorlar orasidagi burchak**. Ikki $\vec{a},\vec{b}$ vektor orasidagi burchak ushbu formula bo‘yicha hisoblanadi:

$$\cos\varphi = \frac{(\vec{a}\vec{b})}{|\vec{a}||\vec{b}|},$$

bu yerda φ $\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchak.

Koordinatalari bilan berilgan $\vec{a}(x_1, y_1, z_1)$, $\vec{b}(x_2, y_2, z_2)$ vektorlar uchun

$$\cos\varphi = \frac{(\vec{a}\vec{b})}{|\vec{a}||\vec{b}|} = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2}\sqrt{x_2^2 + y_2^2 + z_2^2}}.$$

2. **Vektor uzunligi**. $\vec{a}(x, y, z)$ vektorning uzunligi uning koordinatalari kvadratlarining yig‘indisidan olingan arifmetik kvadrat ildizga teng:

$$|\vec{a}| = \sqrt{x^2 + y^2 + z^2}.$$

3. **Vektorlarning perpendikulyarlik sharti**. $\vec{a}(x_1, y_1, z_1)$, $\vec{b}(x_2, y_2, z_2)$ vektorlarning perpendikularlik sharti quyidagicha bo‘ladi:

$$x_1x_2 + y_1y_2 + z_1z_2 = 0$$

Haqiqatan, $\vec{a} \perp \vec{b} \Rightarrow (\vec{a}\vec{b}) = 0.$

$$x_1x_2 + y_1y_2 + z_1z_2 = 0$$

4. **Berilgan yo‘nalishda vektorning proyeksiyasi**. Berilgan $\vec{b}$ vektor yo‘nalish bo‘yicha $\vec{a}$ vektorining proyektsiyasi quyidagi formula bo‘yicha hisoblanadi:

$$pr_{\vec{b}}\vec{a} = \frac{\left(\vec{a}\cdot\vec{b}\right)}{\left|\vec{b}\right|} = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_2^2 + y_2^2 + z_2^2}}.$$

Vektor kopaytmaning koordinatalardagi ifodasi.

Bizga Dekart sistemasida $\vec{a}(x_1, y_1, z_1)$, $\vec{b}(x_2, y_2, z_2)$ koordinatalari bilan berilgan vektorlar vektor ko‘paytmasining koordinatalarini topaylik:

$$\vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k},$$
$$\vec{b} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}.$$

Vektor ko‘paytmaning xossalarini hamda bazis vektorlarning vektor ko‘paytmalarini e’tiborga olsak,

$$[\vec{a}\,\vec{b}] = [x_1\vec{i} + y_1\vec{j} + z_1\vec{k} \quad x_2\vec{i} + y_2\vec{j} + z_2\vec{k}] =$$
$$= x_1x_2\left[\vec{i}\,\vec{i}\right] + x_1y_2\left[\vec{i}\,\vec{j}\right] + x_1z_2\left[\vec{i}\,\vec{k}\right] + y_1x_2\left[\vec{j}\,\vec{i}\right] + y_1y_2\left[\vec{j}\,\vec{j}\right] +$$
$$+ y_1z_2\left[\vec{j}\,\vec{k}\right] + z_1x_2\left[\vec{k}\,\vec{i}\right] + z_1y_2\left[\vec{k}\,\vec{j}\right] + z_1z_2\left[\vec{k}\,\vec{k}\right] =$$
$$= x_1y_2\vec{k} - x_1z_2\vec{j} - y_1x_2\vec{k} + y_1z_2\vec{i} + z_1x_2\vec{j} - z_1y_2\vec{i} =$$
$$= (y_1z_2 - z_1y_2)\vec{i} - (x_1z_2 - z_1x_2)\vec{j} + (x_1y_2 - y_1x_2)\vec{k} =$$
$$= \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}\vec{i} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}\vec{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\vec{k},$$

demak

$$[\vec{a}\,\vec{b}] = \left(\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, -\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}, \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\right). \qquad (3)$$

Buni quyidagicha yozishimiz mumkin:

$$\left[\vec{a}\,\vec{b}\right] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}.$$

Vektor ko‘paytmadan kelib chiqadigan ba’zi natijalar.

1. **Vektorlarning kollinearlik sharti**:

Agar $\vec{a} \parallel \vec{b}$ bo‘lsa, u holda $\left[\vec{a},\vec{b}\right] = 0$ (va teskari), ya’ni

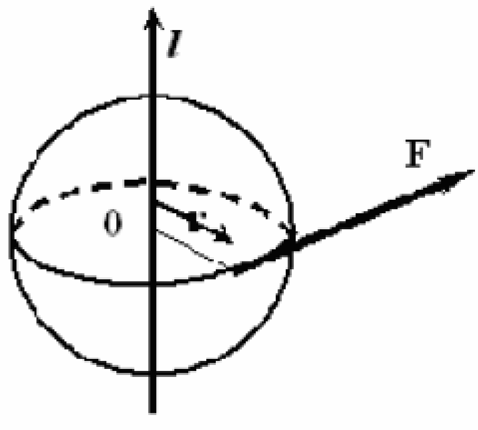
$$\left[\vec{a},\vec{b}\right] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0 \Leftrightarrow \frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2} \Leftrightarrow \vec{a} \parallel \vec{b}.$$

2. **Paralellogram va uchburchak yuzini topish**:

$\vec{a}$ va $\vec{b}$ vektorlarning vektor ko‘paytmasi aniqlanishiga ko‘ra: $\left|[\vec{a},\vec{b}]\right| = \left|\vec{a}\right| \cdot \left|\vec{b}\right| \cdot \sin\left(\widehat{\vec{a},\vec{b}}\right)$, ya’ni $S_{par} = \left|[\vec{a},\, \vec{b}]\right|$. Bundan kelib chiqadi $S_{\Delta} = \frac{1}{2}\left|[\vec{a},\vec{b}]\right|$.

3. **Nuqtaga nisbatan kuch momentini aniqlash**.

Agar $\vec{F}$ kuch jismni l o‘qi atrofida aylantirsa, u holda $\vec{F}$ kuchning O nuqtaga nisbatan $\vec{M}$ momenti $\vec{M} = \left|\left[\vec{r},\vec{F}\right]\right|$ ga teng (1-rasm).



1-rasm.

Aralash kopaytmaning koordinatalardagi ifodasi.

Bizga Dekart koordinatalari sistemasida quyidagi vektorlar berilgan bolsin

$$\vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}\,,\ \vec{b} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}\,,\ \vec{c} = x_3\vec{i} + y_3\vec{j} + z_3\vec{k}\ .$$

Bizga malumki vektorlarning vector kopaytmasiniing koord.ifodasi

$$[\vec{a},\vec{b}] = \left(\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}, \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \right).$$

ko‘rinishda tasvirlanadi.

Aralash kopaytmaning tarifiga asosan dastlabki ikkita vektorning $(\vec{a}, \vec{b})$ vektor kopaytmasini uchinchi $\vec{c}$ vektorga skalyar kopaytirishimiz lozim, yani:

$$([\vec{a},\vec{b}]\vec{c}) = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} x_3 - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} y_3 + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} z_3,$$

demak,

$$(\vec{a},\vec{b},\vec{c}) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}. \tag{1}$$

Aralash ko‘paytmadan kelib chiqadigan ba’zi natijalar.

1) **Fazoda vektorlarni o‘zaro yo‘nalishini aniqlash**:

$\vec{a}, \vec{b}, \vec{c}$ vektorlaning o‘zaro yo‘nalishini aniqlash, quyidagi fikrga asoslanadi.

Agar $(\vec{a}, \vec{b}, \vec{c}) > 0$ bo‘lsa, u holda $\vec{a}, \vec{b}, \vec{c}$ - o‘ng uchlikni, agar $(\vec{a}, \vec{b}, \vec{c}) < 0$ bo‘lsa, u holda $\vec{a}, \vec{b}, \vec{c}$ - chap uchlikni tashkil qiladi.

2) **Vektorlarning komplanarlik sharti:**

Uchta vektor komplanar bo‘ladi, faqat va faqat ularning aralash ko‘paytmasi nolga teng bo‘lsa.

$$\left(\vec{a},\vec{b},\vec{c}\right) = 0 \Leftrightarrow \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0 \Leftrightarrow$$

$\vec{a}, \vec{b}, \vec{c}$ -komplanar vektorlar.

3) **Parallelepiped va uchburchakli piramidaning hajmini aniqlash:**

$\vec{a}, \vec{b}, \vec{c}$ vektorlarga qurilgan parallelepiped hajmi $V = (\vec{a},\vec{b},\vec{c})$ dan hisoblanadi. Uchburchakli piramidaning hajmini hisoblash formulasini keltirib chiqaramiz.

$$A(x_1, y_1, z_1),\ B(x_2, y_2, z_2),\ C(x_3, y_3, z_3),\ D(x_4, y_4, z_4)$$

nuqtalar tetraedrning uchlari bo‘lsin.

$$\overrightarrow{AB}(x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1),$$

$$\overrightarrow{AC}(x_3 - x_1,\ y_3 - y_1,\ z_3 - z_1),$$

$$\overrightarrow{AD}(x_4 - x_1,\ y_4 - y_1,\ z_4 - z_1).$$

Tetraedrning hajmi bir uchidan chiqqan uchta qirrasiga qurilgan parallelepiped hajmining $\frac{1}{6}$ qismiga teng bo‘lgani uchun hamda (1) formulaga asosan

$$V_{tet} = \frac{1}{6}\left|(\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD})\right| = \frac{1}{6}\operatorname{mod}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix}. \qquad (2)$$

(2) formulani ba’zan undan ko‘ra qulayroq quyidagicha yozish ma’quldir:

$$V_{tet} = \frac{1}{6}\operatorname{mod}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}. \qquad (3)$$

Ixtiyoriy uchta $\vec{a}, \vec{b}, \vec{c}$ vektorlarni olaylik. Ulardan, avvalo $\left[\vec{b}, \vec{c}\right]$ ko‘paytmani tuzib, so‘ngra hosil bo‘lgan ko‘paytmani $\vec{a}$ ga yana vektor ko‘paytiramiz: $\left[\vec{a}, \left[\vec{b}, \vec{c}\right]\right]$. Bunday ko‘paytma uch vektorning vektor ko‘paytmasi (qo‘sh vektor ko‘paytma) deyiladi. Biz ushbu yoyish formulasini isbotsiz beramiz:

$$\left[\vec{a}, \left[\vec{b}, \vec{c}\right]\right] = \vec{b}\left(\vec{a}, \vec{c}\right) - \vec{c}\left(\vec{a}, \vec{b}\right).$$

Bu formulani tatbiq etish maqsadida $[\vec{a}, \vec{b}]\,[\vec{c}, \vec{d}]$ ni topamiz. Vaqtincha $[\vec{c}, \vec{d}] = \vec{e}$ deb belgilaymiz. Bunda:

$$[\vec{a}, \vec{b}][\vec{c}, \vec{d}] = \vec{a}[\vec{b}, \vec{c}] = \vec{a}[\vec{b}, [\vec{c}, \vec{d}]$$

bo‘ladi.

Ikkinchi tomondan: $[\vec{b}, [\vec{c}, \vec{d}]] = \vec{c}\,(\vec{b}, \vec{d}) - \vec{d}\,(\vec{b}, \vec{c}).$ Shu sababli $[\vec{a}, [\vec{b}, \vec{c}]] = \vec{b}\,(\vec{a}, \vec{c}) - \vec{c}\,(\vec{a}, \vec{b}).$ bo‘ladi. Jumladan, $\vec{a}$ va $\vec{b}$ vektorlar uchun ushbu ayniyat mavjud: $[\vec{a}, \vec{b}]^2 = \vec{a}^2 \vec{b}^2 - (\vec{a}, \vec{b})^2$