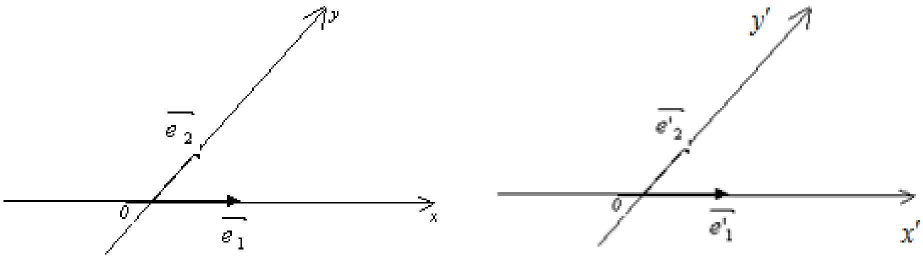
Tekislikda va fazoda dekart koordinatalar sistemasini almashtirish. Tekislikda va fazoda orientatsiya.

Tekislikda affin va dekart koordinatalar sistemasini almashtirish.

Tekislikda ikkita $(O, \vec{e}_1, \vec{e}_2)$, $(O', \vec{e}_1', \vec{e}_2')$ affin sistema berilgan bo‘lsin. Qulaylik uchun ularning birinchisini eski sistema, ikkinchisini yangi sistema deb ataymiz (1-rasm).



1-rasm.

Bundan tashqari, yangi sistemaning eski sistemaga nisbatan vaziyati berilgan bo‘lsin, ya’ni

$$O'(c_1, c_2),\ \vec{e}_1'(a_1, a_2),\ \vec{e}_2'(b_1, b_2),\ OO' = c_1 e_1' + c_2 e_2', \qquad (1)$$

$$\vec{e}_1' = a_1 \vec{e}_1 + a_2 \vec{e}_2,\ \vec{e}_2' = b_1 \vec{e}_1 + b_2 \vec{e}_2 \Rightarrow \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0. \qquad (2)$$

Tekislikda ixtiyoriy M nuqtani olamiz. Bu nuqtaning eski va yangi sistemalarga nisbatan koordinatalarini mos ravishda x, y va x', y' orqali belgilaymiz (2-rasm). U holda $\overrightarrow{OM} = x\vec{e}_1 + y\vec{e}_2$ $\overrightarrow{O'M} = x'\vec{e}_1' + y'\vec{e}_2'$. Vektorlarni qo‘shish ta’rifi va (1) va (2) munosabatlardan foydalansak,

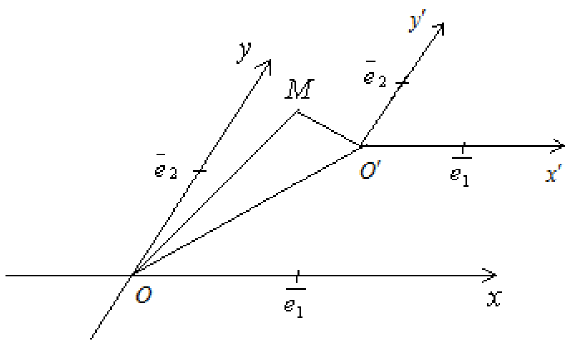
$$\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M} = c_1\vec{e}_1 + c_2\vec{e}_2 + x'\vec{e}_1' + y'\vec{e}_2' =$$
$$= c_1\vec{e}_1 + c_2\vec{e}_2 + x'(a_1\vec{e}_1 + a_2\vec{e}_2) + y'(b_1\vec{e}_1 + b_2\vec{e}_2)$$

yoki

$$x\vec{e}_1 + y\vec{e}_2 = (a_1 x' + b_1 y' + c_1)\vec{e}_1 + (a_2 x' + b_2 y' + c_2)\vec{e}_2 \ .$$

$\vec{e}_1, \vec{e}_2$ vektorlarning chiziqli erkliligini hisobga olsak,

$$x = a_1 x' + b_1 y' + c_1,\ \ y = a_2 x' + b_2 y' + c_2. \qquad (3)$$



2-rasm

M nuqtaning eski sistemasiga nisbatan koordinatalari x, y, uning yangi sistemaga nisbatan koordinatalari x', y' orqali shu (3) ko‘rinishda ifodaladi.

(3) formula bir affin koordinatalar sistemasidan ikkinchi affin koordinatalar sistemasiga o‘tish formulalari deyiladi. Bu formulalarda $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$ shart bilan bog‘langan oltita koeffitsient qatnashgan. Quyidagi ikki xususiy holni qaraymiz:

1. $O \neq O',\ \vec{e}_1' = \vec{e}_1,\ \vec{e}_2' = \vec{e}_2$ bo‘lsin. U holda $a_1 = b_2 = 1,\ a_2 = b_1 = 0$ bo‘lib, (3) formulalar

$$\begin{cases} x = x' + c_1, \\ y = y' + c_2 \end{cases} \tag{4}$$

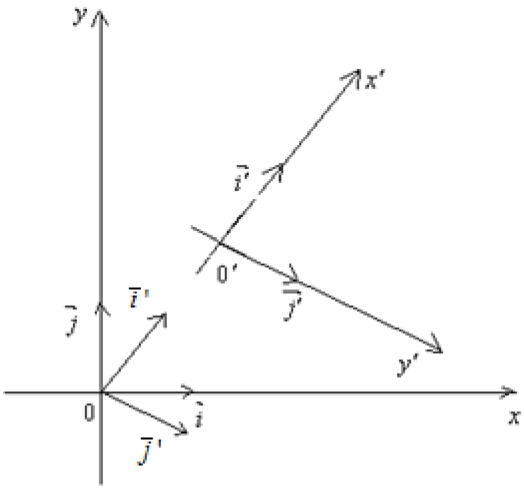
ko‘rinishni oladi.

(4) formula **koordinatalar sistemasini parallel ko‘chirish** formulalari deb ataladi.

2. $O = O'$ va bazis vektorlar turlicha bo‘lsin. U holda $c_1 = c_2 = 0$ bo‘lib, (3) dan

$$\begin{cases} x = a_1 x' + b_1 y', \\ y = a_2 x' + b_2 y'. \end{cases} \tag{5}$$

Tekislikda $\beta = (O, \vec{i}, \vec{j})$ va $\beta' = (O', \vec{i}\,', \vec{j}\,')$ Dekart koordinatalar sistemasi berilgan bo‘lsin (3-rasm).



3-rasm.

Bu holda (3) formulalardagi a_1, a_2 lar $\vec{i}'$ vektorning, b_1, b_2 lar esa $\vec{j}'$ vektorning $\beta = (O, \vec{i}, \vec{j})$ koordinatalar sistemasiga nisbatan koordinatalari bo‘ladi, ya’ni

$$\vec{i}\,' = a_1\vec{i} + a_2\vec{j},\ \ \vec{j}\,' = b_1\vec{i} + b_2\vec{j}. \qquad (6)$$

$(\vec{i},^\wedge \vec{i}') = \alpha$ bo‘lsin. Agar β va β' Dekart koordinatalar sistemalari bir xil orientatsiyali bo‘lsa,

$$(\vec{i},^\wedge \vec{j}') = 90^0 + \alpha,\ \ (\vec{i}\,',^\wedge \vec{j}) = 90^0 - \alpha,\ \ (\vec{j},^\wedge \vec{j}') = \alpha. \qquad (7)$$

β, β' Dekart koordinatalar sistemalari qarama-qarshi orientatsiyali bo‘lsa,

$$(\vec{i},^\wedge \vec{j}') = 270^0 + \alpha,\ \ (\vec{i}\,',^\wedge \vec{j}) = 90^0 - \alpha,\ \ (\vec{j},^\wedge \vec{j}') = 180^0 + \alpha. \qquad (8)$$

(6) tengliklarni navbat bilan $\vec{i}, \vec{j}$ vektorlarga skalyar ko‘paytirsak,

$$a_1 = \vec{i}\,'\vec{i} = \cos(\vec{i}\,',^\wedge \vec{i}),\quad a_2 = \vec{i}\,'\vec{j} = \cos(\vec{i}\,',^\wedge \vec{j}),$$
$$b_1 = \vec{j}\,'\vec{i} = \cos(\vec{j}\,',^\wedge \vec{i}),\quad b_2 = \vec{j}\,'\vec{j} = \cos(\vec{j}\,',^\wedge \vec{j}).$$

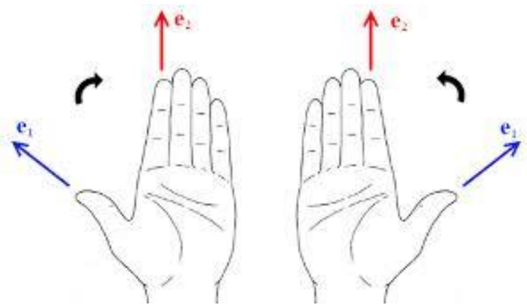
(7) va (8) munosabatlarni hisobga olsak, $\vec{i}\,', \vec{j}\,'$ vektorlarning β sistemaga nisbatan koordinatalari, agar β, β' sistemalar bir xil orientatsiyali bo‘lsa,

$$\vec{i}\,'(\cos\alpha, \sin\alpha),\qquad \vec{j}\,'(-\sin\alpha, \cos\alpha);$$

β, β' sistemalar qarama-qarshi orientatsiyali bo‘lganda esa

$$\vec{i}\,'(\cos\alpha, \sin\alpha),\qquad \vec{j}\,'(\sin\alpha, -\cos\alpha).$$

Orientatsiya: Bir vektordan ikkinchisiga qisqa burilish yo‘nalishi soat strelkasi yo‘nalishiga qarama-qarshi bo‘lsa, bu vektorlar o‘ng ikkilik, aks holda chap ikkilik tashkil qiladi deyiladi. Bazis sifatida biror ikkilik tanlansa, biz orientasiya tanlab olingan deb hisoblaymiz (4-rasm).



4-rasm. Orientatsiya

(3) formulalar quyidagi ko‘rinishni oladi:

$$\begin{cases} x = x'\cos\alpha - y'\sin\alpha + c_1, \\ y = x'\sin\alpha + y'\cos\alpha + c_2, \end{cases} \tag{9}$$

$$\begin{cases} x = x'\cos\alpha + y'\sin\alpha + c_1, \\ y = x'\sin\alpha - y'\cos\alpha + c_2. \end{cases} \tag{10}$$

(9) va (10) formulalar bitta

$$\begin{cases} x = x'\cos\alpha - \varepsilon\, y'\sin\alpha + c_1, \\ y = x'\sin\alpha + \varepsilon\, y'\cos\alpha + c_2 \end{cases} \tag{11}$$

ko‘rinishdagi yozuvga birlashtirish mumkin, bu yerda $\varepsilon = \pm 1$. Shunday qilib, β, β' sistemalar Dekart sistemalari bo‘lganida ularning biridan ikkinchisiga o‘tish (11) formula bilan ifodalanadi. Bu yerda β, β' sistemalar bir xil orientatsiyali bo‘lsa, $\varepsilon = +1$, aks holda esa $\varepsilon = -1$.

Fazoda affin va dekart koordinatalarni almashtirish.

Fazodagi biror nuqtaning tayin bir sistemadagi koordinatalaridan boshqa sistemadagi koordinatalariga o‘tishga to‘g‘ri keladi. Biz shu masalani ikkita affin sistema uchun hal qilamiz. $\beta = (O, \vec{e}_1, \vec{e}_2, \vec{e}_3)$, $\beta' = (O', e_1', e_2', e_3')$ affin sistemalar berilgan bo‘lsin.

I hol. Sistemalarning boshlari har xil bo‘lib, bazis vektorlari mos ravishda kollinear bo‘lsin, ya’ni $O \neq O', \vec{e}_1 \| \vec{e}_1', \vec{e}_2 \| \vec{e}_2', \vec{e}_3 \| \vec{e}_3'$ hamda O' ning β ga nisbatan koordinatalari a, b, c bo‘lsin (5-a rasm). U holda fazodagi ixtiyoriy M nuqtaning β va β' ga nisbatan koordinatalari mos ravishda x, y, z va x', y', z' bo‘lsa, shular orasidagi bog‘lanishni izlaymiz:

$$M(x, y, z) \Rightarrow \overrightarrow{OM}(x, y, z) \Rightarrow \overrightarrow{OM} = x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3,$$

$$M(x', y', z') \Rightarrow \overrightarrow{O'M}(x', y', z') \Rightarrow \overrightarrow{O'M} = x'\vec{e}_1' + y'\vec{e}_2' + z'\vec{e}_3',$$

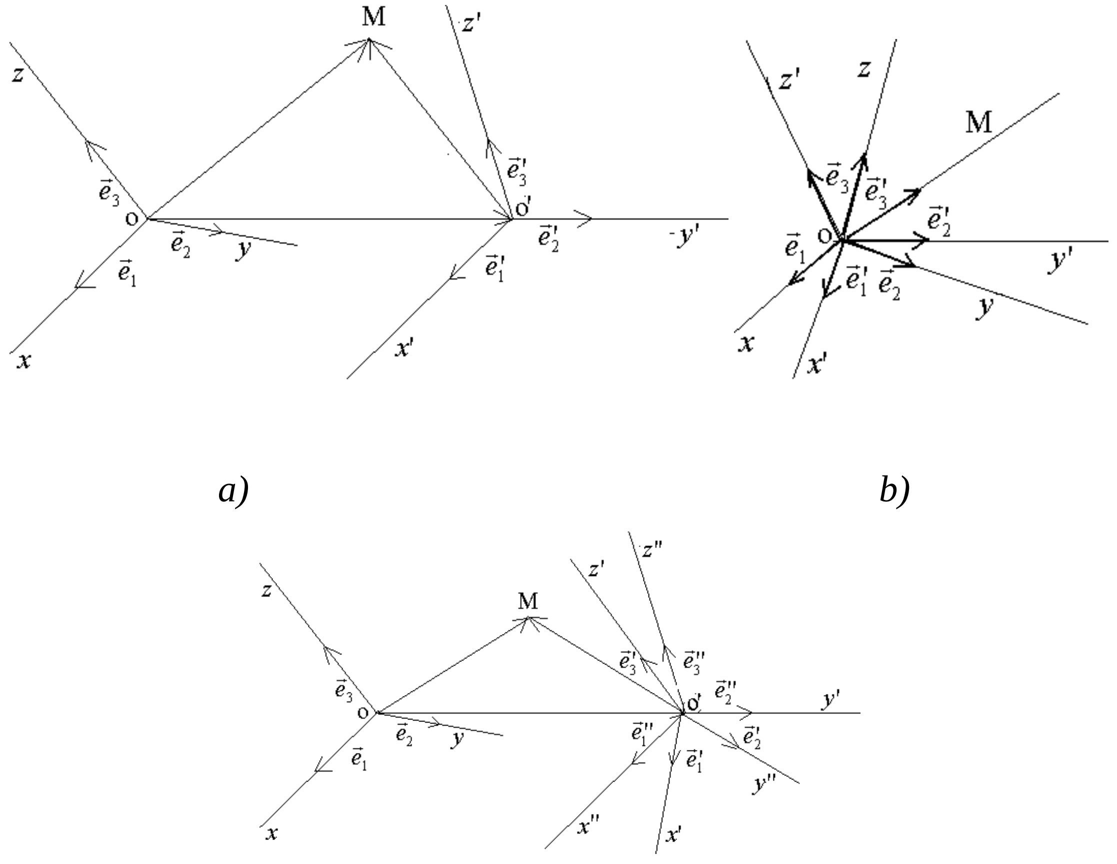
$$\overrightarrow{OO'}(a, b, c) \Rightarrow \overrightarrow{OO'} = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3.$$

Lekin $\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M}$ bo‘lgani uchun

$$x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3 = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3 + x'\vec{e}_1' + y'\vec{e}_2' + z'\vec{e}_3'.$$

Bundan tashqari, bazis vektorlar mos ravishda kollinear bo‘lgani uchun

$$\vec{e}_1' = \lambda_1 \vec{e}_1, \quad \vec{e}_2' = \lambda_2 \vec{e}_2, \quad \vec{e}_3' = \lambda_3 \vec{e}_3,$$



c)

5 – rasm.

demak,

$$x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3 = (\lambda_1 x' + a)\vec{e}_1 + (\lambda_2 y' + b)\vec{e}_2 + (\lambda_3 z' + c)\vec{e}_3. \qquad (12)$$

$$x = \lambda_1 x' + a, \ \ y = \lambda_2 y' + b, \ \ z = \lambda_3 z' + c \qquad (13)$$

$\lambda_1 = \lambda_2 = \lambda_3 = 1$ bo‘lsa, ya’ni bazis vektorlar mos ravishda o‘zaro teng bo‘lsa, (13) quyidagi ko‘rinishni oladi:

$$x = x' + a, \ \ y = y' + b, \ \ z = z' + c \qquad (14)$$

Bu formulalar ba’zan koordinatalar sistemasini parallel ko‘chirish formulalari deb yuritiladi.

II hol. Sistemalarning boshlari bir xil, bazis vektorlarning yo‘nalishlari esa har xil bo‘lsin, u holda (5-b rasm)

$$O = O', \vec{e}\,'_1 = a_{11}\vec{e}_1 + a_{21}\vec{e}_2 + a_{31}\vec{e}_3, \ \vec{e}\,'_2 = a_{12}\vec{e}_1 + a_{22}\vec{e}_2 + a_{32}\vec{e}_3,$$

$$\vec{e}\,'_3 = a_{13}\vec{e}_1 + a_{23}\vec{e}_2 + a_{33}\vec{e}_3$$

bo‘lsin. Endi $A = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$ (15)

matritsani tuzamiz. Bu matritsani bir bazisdan ikkinchi bazisga o‘tish matritsasi deb ataymiz, $\vec{e}\,'_1, \ \vec{e}\,'_2, \ \vec{e}\,'_3$ bazis vektorlar bo‘lgani uchun (15) matritsaning determinanti noldan farqlidir.

$$\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} \neq 0. \qquad (16)$$

Aks holda, determinantning bir satri qolgan ikki satrining chiziqli kombinatsiyasidan iborat bo‘lib, $\vec{e}\,'_1, \ \vec{e}\,'_2, \ \vec{e}\,'_3$ ham chiziqli bog‘liq bo‘lar edi.

Fazoda ixtiyoriy M nuqtaning β va β' sistemaga nisbatan koordinatalarini mos ravishda x, y, z va x', y', z' deb olsak,

$$\overrightarrow{OM} = x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3,$$

$$\overrightarrow{OM} = x'\,\vec{e}\,'_1 + y'\,\vec{e}\,'_2 + z'\,\vec{e}\,'_3,$$

ya’ni

$$x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3 = x'\,\vec{e}\,'_1 + y'\,\vec{e}\,'_2 + z'\,\vec{e}\,'_3.$$

Endi bu tenglikka $\vec{e}'_1, \vec{e}'_2, \vec{e}'_3$ ning qiymatlarini qo‘yib, $\vec{e}_1, \vec{e}_2, \vec{e}_3$ ga nisbatan gruppalasak,

$$x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3 = (a_{11}x' + a_{12}y' + a_{13}z')\vec{e}_1 + (a_{21}x' + a_{22}y' + a_{23}z')\vec{e}_2 +$$
$$+(a_{31}x' + a_{32}y' + a_{33}z')\vec{e}_3,$$

bundan

$$\begin{aligned} x &= a_{11}x' + a_{12}y' + a_{13}z', \\ y &= a_{21}x' + a_{22}y' + a_{23}z', \\ z &= a_{31}x' + a_{32}y' + a_{33}z'. \end{aligned} \qquad (17)$$

Ushbu

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad (18)$$

matritsa almashtirish matritsasi deb ataladi. (18) va (15) matritsalar o‘zaro transponirlangan matritsalardir. Bu matritsalar kvadrat matritsalar bo‘lgani uchun ularning uchinchi tartibli determinantlari o‘zaro teng bo‘lib, (16) ga asosan (18) ning determinanti noldan farqlidir, demak, (17) ni ravishda x', y', z' ga nisbatan yechsak,

$$\begin{aligned} x' &= a'_{11}x + a'_{12}y + a'_{13}z, \\ y' &= a'_{21}x + a'_{22}y + a'_{23}z, \\ z' &= a'_{31}x + a'_{32}y + a'_{33}z \end{aligned} \qquad (19)$$

hosil bo‘lib, bunda

$$a'_{ik} = \frac{A_{ik}}{\det A}; \quad (i, k = 1, 2, 3)$$

A_{ik} esa A matritsa a_{ik} elementining algebraik to‘ldiruvchisidir.

III hol. Sistemalar fazoda ixtiyoriy vaziyatda joylashgan bo‘lsin. β sistema berilgan bo‘lib, shu sistemaga nisbatan β' sistema elementlarining koordinatalari quyidagicha bo‘lsin:

$$O'(a,\, b,\, c),\quad \begin{aligned} \vec{e}_1' &= a_{11}\,\vec{e}_1 + a_{21}\,\vec{e}_2 + a_{31}\,\vec{e}_3, \\ \vec{e}_2' &= a_{12}\,\vec{e}_1 + a_{22}\,\vec{e}_2 + a_{32}\,\vec{e}_3, \\ \vec{e}_3' &= a_{13}\,\vec{e}_1 + a_{23}\,\vec{e}_2 + a_{33}\,\vec{e}_3, \end{aligned} \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} \neq 0, \qquad (20)$$

β dan β' ga o‘tish uchun biz yana shunday uchinchi $\beta'' = (O'', \vec{e}_1'', \vec{e}_2'', \vec{e}_3'')$ affin sistemani qaraymizki, u β ni $\overrightarrow{OO'}$ vektor qadar parallel ko‘chirishdan hosil bo‘lsin. U holda fazodagi ixtiyoriy M nuqtaning koordinatalarini bu sistemalarga nisbatan mos ravishda x, y, z; x'', y'', z'' va x', y', z' deb belgilasak (5-c rasm), β bilan β'' orasidagi bog‘lanish (14) ga asosan

$$x = x'' + a,\; y = y'' + b,\; z = z'' + c, \qquad (21)$$

β'' bilan β' orasidagi bog‘lanish esa (7) ga asosan

$$\begin{aligned} x'' &= a_{11}x' + a_{12}y' + a_{13}z', \\ y'' &= a_{21}x' + a_{22}y' + a_{23}z', \\ z'' &= a_{31}x' + a_{32}y' + a_{33}z', \end{aligned}$$

buni (21) ga qo‘ysak, izlanayotgan quyidagi ifoda hosil qilinadi:

$$\begin{aligned} x &= a_{11}x' + a_{12}y' + a_{13}z' + a, \\ y &= a_{21}x' + a_{22}y' + a_{23}z' + b, \\ z &= a_{31}x' + a_{32}y' + a_{33}z' + c. \end{aligned} \qquad (22)$$

(22) ni x', y', z' ga ((20) shart o‘rinli bo‘lgani uchun) nisbatan ham echish mumkin, demak, M nuqtaning β ga nisbatan koordinatalari ma’lum bo‘lsa, shu nuqtaning koordinatalarini β' ga nisbatan ham topish mumkin.

Bir affin sistemasidan ikkinchi affin sistemasiga o‘tish 12 ta parametrga bog‘liqdir, chunki (22) almashtirishni aniqlaydigan ushbu 12 ta parametr kiradi: $a, b, c, a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$.

Agar β, β' dekart sistemalari bo‘lsa, ularni almashtirish 12 ta parametrga emas, balki eng ko‘pi bilan 6 ta parametrga bog‘liq bo‘lib qoladi. Haqiqatan ham, $\bar{e}_1 = \bar{i},\ \bar{e}_2 = \bar{j},\ \bar{e}_3 = \bar{k}$ va $\bar{e}_1' = \bar{i}',\ \bar{e}_2' = \bar{j}',\ \bar{e}_3' = \bar{k}'$ bo‘lsa,

$$
\begin{aligned}
&a_{11}^2 + a_{21}^2 + a_{31}^2 = 1,\\
&a_{12}^2 + a_{22}^2 + a_{32}^2 = 1,\\
&a_{13}^2 + a_{23}^2 + a_{33}^2 = 1,
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
&a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0,\\
&a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = 0,\\
&a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0.
\end{aligned}
\tag{24}
$$

Demak, (22) dagi 12 ta parametr (23) va (24) dagi 6 ta shartni qanoatlantirishi kerak, u holda jami 6 ta parametr qoladi. “Algebra va sonlar nazariyasi” kursidan ma’lumki, (18) ko‘rinishdagi kvadrat matritsaning elementlari (23) va (24) shartlarning barchasini qanoatlantirsa, bunday matritsa ortogonal matritsa deb ataladi. Bundan quyidagi xulosa kelib chiqadi: bir dekart sistemasidan ikkinchi dekart sistemasiga o‘tish matritsasi ortogonal matritsadan iborat bo‘ladi.