Q1: (a)   Minjuan Luo 20313326

$S=\{(1,1),(1,2),(1,3),\dots,(6,6)\}$ sample space 36

$A=\{(1,2),(2,1)\}$ $P(A)=\frac{1}{18}$

$B=\{(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)\}$ $P(B)=\frac{1}{6}$

$C=\{(1,1),(1,2),(2,1),(1,3),(3,1),(4,1),(1,4),(5,1),(1,5),(1,6),(6,1)\}$ $P(C)=\frac{11}{36}$

i $P(A|C)=\frac{P(A\cap C)}{P(C)}=\frac{\frac{1}{18}}{\frac{11}{36}}=\frac{2}{11}$

$P(A\cap C)=\{(1,2),(2,1)\}=\frac{1}{18}$

ii $P(B|C)=\frac{P(B\cap C)}{P(C)}=\frac{\frac{1}{18}}{\frac{11}{36}}=\frac{2}{11}$

$P(B\cap C)=\{(1,6),(6,1)\}=\frac{1}{18}$

iii If A and C are independent

$P(A\cap C)=\frac{1}{18}$ is equal to $P(A)\times P(C)=\frac{11}{648}$ but $\frac{1}{18}\neq\frac{11}{648}$ so A.C not independent

If B and C are independent

$P(B\cap C)=\frac{1}{18}$ is equal to $P(B)\times P(C)=\frac{11}{216}$ but $\frac{1}{18}\neq\frac{11}{216}$ so B.C not independent

(b) i $P(\text{biased } H)=P$ $P(\text{biased } T)=(1-P)$ $P(H)=\frac{1}{2}$ $P(T)=\frac{1}{2}$

$P(\text{biased } H, H)=P\times\frac{1}{2}=\frac{1}{2}P$

$P(\text{biased } H, T)=P\times\frac{1}{2}=\frac{1}{2}P$

$P(\text{biased } T, H)=(1-P)\times\frac{1}{2}=\frac{1}{2}(1-P)$

$P(\text{biased } T, T)=(1-P)\times\frac{1}{2}=\frac{1}{2}(1-P)$

$P(\text{match})=\frac{1}{2}P+\frac{1}{2}(1-P)=\frac{1}{2}$

$P(\text{not match})=\frac{1}{2}P+\frac{1}{2}(1-P)=\frac{1}{2}$

Using binomial distribution $X\sim Bin(n,\frac{1}{2})$

$P(X=k)=\binom{n}{k}\frac{1}{2}^k(1-\frac{1}{2})^{n-k}$ $(k=0,1,\dots,n)$

ii $E(X)=\frac{1}{2}n$ $Var(X)=\frac{1}{2}n\times(1-\frac{1}{2})=\frac{1}{4}n$

2. (a)

i. $P(X=5)=\frac{1}{10}$ Pay 5

$P(X=6)=\frac{1}{10}$ Pay 6

$P(X=7)=\frac{1}{10}$ pay 7

$P(X=8)=\frac{1}{10}$ pay 8

$P(X=9)=\frac{1}{10}$ pay 9

$P(X=10)=\frac{1}{10}$ pay 10

$P(1\leq X\leq 4)=\frac{4}{10}$ pay 0.5

$E(X)=\frac{1}{10}\times 5+\frac{1}{10}\times 6+\frac{1}{10}\times 7+\frac{1}{10}\times 8+\frac{1}{10}\times 9+\frac{1}{10}\times 10+\frac{4}{10}\times 0.5=4.7$

ii. $P(X=25)=\frac{1}{10}$ Pay 5

$P(X=36)=\frac{1}{10}$ Pay 6

$P(X=49)=\frac{1}{10}$ pay 7

$P(X=64)=\frac{1}{10}$ pay 8

$P(X=81)=\frac{1}{10}$ pay 9

$P(X=100)=\frac{1}{10}$ pay 10

$P(1\leq X\leq 16)=\frac{4}{10}$ pay $0.5=\frac{1}{2}$

$E(X^2)=25\times\frac{1}{10}+36\times\frac{1}{10}+49\times\frac{1}{10}+64\times\frac{1}{10}+81\times\frac{1}{10}+100\times\frac{1}{10}+\frac{1}{4}\times\frac{4}{10}$

$=35.6$

$[E(X)]^2=(4.7)^2$

$Var(X)=35.6-(4.7)^2=13.51$

(b)

(i) $X=1$ $P(X)=\frac{1}{2}$

$X=2$ $P(X)=\frac{1}{4}$

$X=4$ $P(X)=\frac{1}{8}$

$\vdots$ $\vdots$

$X=2^{n-1}$ $P(X)=\frac{1}{2^n}$

(ii) $E(X)=1\times\frac{1}{2}+2\times\frac{1}{4}+4\times\frac{1}{8}+\cdots+2^{n-1}\times\frac{1}{2^n}$

$=\underbrace{\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}}_{n}=\frac{1}{2}n$

So $E(X)$ is infinite

(iii) This strategy is wrong because we can't always have enough bet to play the game after consecutive losses. The bet that we need to play will grow extremly high that nobody affords it

3. (a)

(i) $P(\text{left footed}) = \frac{15}{100} = \frac{3}{20}$

$P(\text{right footed}) = \frac{85}{100} = \frac{17}{20}$

Using Binomial Distribution

$P(X=1) = \binom{11}{1}\left(\frac{3}{20}\right)^1\left(\frac{17}{20}\right)^{10} = 0.3248$

(ii) $P(X<3) = P(X=0) + P(X=1) + P(X=2)$

$= \binom{11}{0}\left(\frac{17}{20}\right)^{11} + \binom{11}{1}\left(\frac{3}{20}\right)^1\left(\frac{17}{20}\right)^{10} + \binom{11}{2}\left(\frac{3}{20}\right)^2\left(\frac{17}{20}\right)^9 = 0.7788$

(iii) $P'(x \leq 2) = P'(X=0) + P'(X=1) + P'(X=2)$

$= \binom{10}{0}\left(\frac{17}{20}\right)^{10} + \binom{10}{1}\left(\frac{3}{20}\right)^1\left(\frac{17}{20}\right)^9 + \binom{10}{2}\left(\frac{3}{20}\right)^2\left(\frac{17}{20}\right)^8 = 0.8202$

(b)(i) Using Poisson Distribution

$k = 2 \times 5 = 10$ $\quad X \sim Poi(2 \times 5)$

$P(X=10) = e^{-10}\frac{10^{10}}{10!} = 0.125$

(ii) compute probability of no claims on exactly 1 day

$k = 0$ $\quad X \sim Poi(2)$

$P(X=0) = e^{-2}\left(\frac{2^0}{0!}\right) = 0.1353$

Using Binomial Distribution

$P(X=2) = \binom{5}{2}(0.1353)^2(1-0.1353)^3 = 0.1184$

Q4:

(a)

$$F(x)=\begin{cases} 0 & , x<0 \\ \frac{1}{5} & , 0\le x<2 \\ \frac{2}{5} & , 2\le x<4 \\ 1 & , x\ge 4 \end{cases} \longrightarrow f(x)=\begin{cases} 0 & , x<0 \\ \frac{1}{5} & , 0\le x<2 \\ \frac{2}{5}-\frac{1}{5}=\frac{1}{5} & , 2\le x<4 \\ 1-\frac{1}{5}-\frac{1}{5}=\frac{3}{5} & , x\ge 4 \end{cases}$$

(b)

i $$\int_0^3\int_0^3 f(x,y)\,dy\,dx = \int_0^3\int_0^3 cx^2y(1+y)\,dy\,dx = \frac{243c}{2}$$

$$\int_0^3\int_0^3 f(x,y)\,dy\,dx = 1$$

$$\frac{243c}{2}=1 \qquad c=\frac{2}{243}$$

ii $$\int_1^2\int_0^1 \frac{2}{243}x^2y(1+y)\,dy\,dx = \frac{35}{2187}$$

iii $$\iint \frac{2}{243}x^2y(1+y)\,dy\,dx$$

$$=\int\left(\frac{2}{243}x^2\cdot\int(1+y)y\,dy\right)dx$$

$$=\int\left(\frac{2x^2}{243}\cdot\int y+y^2\,dy\right)dx$$

$$=\int\left(\frac{2x^2}{243}\left(\frac{y^2}{2}+\frac{y^3}{3}\right)\right)dx$$

$$=\int\left(\frac{x^2y^2}{243}+\frac{2x^2y^3}{729}+c\right)dx$$

$$=cx+\frac{2y^3x^3}{2187}+\frac{y^2x^3}{729}+c$$

iv. marginal pdf of x is $\int\frac{2}{243}x^2y(1+y)\,dy = \frac{x^2y^2}{243}+\frac{2x^2y^3}{729}+c$

marginal pdf of Y is $\int\frac{2}{243}x^2y(1+y)\,dx = \frac{2yx^3(1+y)}{729}+c$

v. If $P(\{X=x_0\}\cap\{Y=y\}) = P(X=x_0)\cdot P(Y=y)$, $\forall(x,y)$ independent

However $P(X=x_0)\cdot P(Y=y)=c^2+\frac{4x^5y^5}{531441}+\frac{10x^5y^4}{531441}+\frac{2x^5y^3}{177147}+\frac{2cx^3y^2}{729}+\frac{2cx^3y}{729}+\frac{2cx^2y^3}{729}\neq cx^2y(1+y)$ So x,y not independent