1 Variational Bayes - Normal Mixture Model, Fixed π

We have *iid* observations y_i genrated by a two level Normal Mixture Model with means μ_1 and μ_2 and known variance 1, so

$$p(y_i|\mu_1,\mu_2,k_i) = \mathcal{N}(\mu_1,1)^{k_i}\mathcal{N}(\mu_2,1)^{1-k_i}.$$

where the latent variable $k_i = 1$ if y_i is drawn from $\mathcal{N}(\mu_1,1)$ and $k_i = 0$ otherwise.

Further, $\mathbf{k}$ is modelled as *iid* Bernoulli with known parameter π, so

$$p(k_i|\pi) = \pi^{k_i}(1-\pi)^{1-k_i}.$$

Introducing the priors $p(\pi) \sim U(0,1)$ and $p(\mu_1,\mu_2) \propto 1$, the joint distribution becomes

$$\begin{aligned}
p(y,k,\mu_1,\mu_2|\pi) &= \prod_{i=1}^{n} p(y_i|k_i,\mu_1,\mu_2,\pi)p(k_i|\pi)p(\pi)p(\mu_1\mu_2) \\
&\propto \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-(y_i-\mu_1)^2}{2}\right\}\right)^{k_i} \left(\frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-(y_i-\mu_2)^2}{2}\right\}\right)^{1-k_i} \\
&\times\ \pi^{k_i}(1-\pi)^{1-k_i} \\
\ln(p(y,k,\mu_1,\mu_2|\pi)) &= \sum_{i=1}^{N}\left[\ln\left(\exp\left\{\frac{-(y_i-\mu_1)^2}{2}\right\}^{k_i}\right)\right] + \sum_{i=1}^{N}\left[\ln\left(\exp\left\{\frac{-(y_i-\mu_2)^2}{2}\right\}^{1-k_i}\right)\right] \\
&+ \sum_{i=1}^{N} k_i\ln(\pi) + \sum_{i=1}^{N}(1-k_i)\ln(1-\pi) \\
&= \sum_{i=1}^{N}\left[k_i\frac{-(y_i-\mu_1)^2}{2}\right] + \sum_{i=1}^{N}\left[(1-k_i)\frac{-(y_i-\mu_2)^2}{2}\right] \\
&+ \sum_{i=1}^{N} k_i\ln(\pi) + \sum_{i=1}^{N}(1-k_i)\ln(1-\pi) + c. \qquad (1)
\end{aligned}$$

We can take the variational approximation factorisation $q(k_{1:n},\mu_1,\mu_2) = \prod_{i=1}^{n} q(k_i)q(\mu_1)q(\mu_2)$, which implies independence of k_i, k_j for $i \neq j$:

It can be shown that the factorisable distribution that minmises the KL Divergence between $q(\theta)$ and $p(\theta|y)$ satisfies

$$q_i \propto \exp(\mathbb{E}_{q_{j\neq i}}(\ln(p(y,x,\theta)))) \tag{2}$$

for all q_i, where y is the observed data, x is a latent variable and θ is a vector of unknown parameters.

Substituting (1) into (2) and ignoring all terms that do not depend on μ_1 yields

$$\begin{aligned}
\ln(q(\mu_1)) &= \mathbb{E}_{k_{1:n}} \sum_{i=1}^{n} -k_i \frac{(y_i-\mu_1)^2}{2} + c \\
&= -\frac{1}{2}\left(\sum_{i=1}^{n} \mathbb{E}(k_i)(y_i-\mu_1)^2\right) + c \\
&= -\frac{1}{2}\left(\sum_{i=1}^{n} \mathbb{E}(k_i)((y_i-\tilde{y}_1)+(\tilde{y}_1-\mu_1))^2\right) + c \\
&= -\frac{1}{2}\left(\sum_{i=1}^{n} \mathbb{E}(k_i)((y_i-\tilde{y}_1)^2+(\tilde{y}_1-\mu_1)^2-2(y_i-\tilde{y}_1)(\tilde{y}_1-\mu_1))\right) + c.
\end{aligned}$$

Where

$$\tilde{y}_1 = \frac{\sum_{i=1}^{n} \mathbb{E}(k_i) y_i}{\sum_{i=1}^{n} \mathbb{E}(k_i)}.$$

Note that

$$\sum_{i=1}^{n} \mathbb{E}(k_i)(y_i-\tilde{y}_1) = \sum_{i=1}^{n} \mathbb{E}(k_i)\left(y_i - \frac{\sum_{i=1}^{n} \mathbb{E}(k_i) y_i}{\sum_{i=1}^{n} \mathbb{E}(k_i)}\right) = 0,$$

hence

$$\ln(q(\mu_1)) = -\frac{\sum_{i=1}^{n} \mathbb{E}(k_i)(\tilde{y}_1-\mu_1)^2}{2} + c.$$

Recognizing the kernel of a Gaussian distribution, we can see that $q(\mu_1) \sim \mathcal{N}(\bar{\mu}_1 = \tilde{y}_1, \lambda_1 = (\sum_{i=1}^{n} \mathbb{E}(k_i)^{-1})$. Similarly, $q(\mu_2) \sim \mathcal{N}(\bar{\mu}_2 = \tilde{y}_2, \lambda_2 = \sum_{i=1}^{n} \mathbb{E}(1-k_i)^{-1}))$ with

$$\tilde{y_2} = \frac{\sum_{i=1}^{n} \mathbb{E}(1-k_i) y_i}{\sum_{i=1}^{n} \mathbb{E}(1-k_i)}.$$

Through independence, all $q(k_i)$ have the same form,

$$\begin{aligned}
\ln(q(k_i)) &= \mathbb{E}_{\mu_1,\mu_2}\left[k_i\frac{-(y_i-\mu_1)^2}{2} + (1-k_i)\frac{-(y_i-\mu_2)^2}{2} + k_i\ln(\pi) + (1-k_i)\ln(1-\pi) + c\right] \\
&= k_i\frac{\mathbb{E}_{\mu_1} - (y_i-\mu_1)^2}{2} + (1-k_i)\frac{\mathbb{E}_{\mu_2} - (y_i-\mu_2)^2}{2} + k_i\ln(\pi) + (1-k_i)\ln(1-\pi) + c \\
&= k_i\frac{2\ln(\pi) - ((y_i-\bar{\mu}_1)^2 + \lambda_1)}{2} + (1-k_i)\frac{2\ln(1-\pi) - ((y_i-\bar{\mu}_2)^2 + \lambda_2)}{2} + c \\
q(k_i) &\propto \exp\left\{\frac{2\ln(\pi) - ((y_i-\bar{\mu}_1)^2 + \lambda_1)}{2}\right\}^{k_i} \exp\left\{\frac{2\ln(1-\pi) - ((y_i-\bar{\mu}_2)^2 + \lambda_2)}{2}\right\}^{1-k_i}
\end{aligned}$$

We expanded the quadratic term and substitued in $(y_i^2 - \bar{\mu}_j)^2 + \lambda_j$ for $\mathbb{E}(y_i - \mu_j)^2$.

Each k_i has a Bernoulli distribution with parameters $p_i = \exp\left\{\frac{2\ln(\pi)-((y_i-\bar{\mu}_1)^2+\lambda_1)}{2}\right\}$, and $q_i = \exp\left\{\frac{2\ln(1-\pi)-((y_i-\bar{\mu}_2)^2+\lambda_2)}{2}\right\}$.

This gives us the update rules for the Variational Bayes iterations:

$$\begin{aligned}
\bar{\mu}_1 &= \frac{\sum_{i=1}^n y_i p_i/(p_i+q_i)}{\sum_{i=1}^n p_i/(p_i+q_i)} \\
\lambda_1 &= \left(\sum_{i=1}^n \frac{p_i}{p_i+q_i}\right)^{-1} \\
\bar{\mu}_2 &= \frac{\sum_{i=1}^n y_i q_i/(p_i+q_i)}{\sum_{i=1}^n q_i/(p_i+q_i)} \\
\lambda_2 &= \left(\sum_{i=1}^n \frac{q_i}{p_i+q_i}\right)^{-1} \\
p_i &= \exp\left\{\frac{2\ln(\pi) - ((y_i-\bar{\mu}_1)^2 + \lambda_1)}{2}\right\} \\
q_i &= \exp\left\{\frac{2\ln(1-\pi) - ((y_i-\bar{\mu}_2)^2 + \lambda_2)}{2}\right\}
\end{aligned}$$