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Analytical derivation of at-a-station hydraulic–geometry relations

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Summary This paper uses generalized expressions for both cross-section geometry and hydraulics (generalization of the Chézy and Manning relations) to derive explicit equations for the exponents and coefficients in the power-law at-a-station hydraulic–geometry relations. The exponents are shown to depend only on the depth exponent in the hydraulic relation (***p***) and the exponent that reflects the form of the cross-section (***r***). The coefficients depend on ***p*** and ***r***, but also on the slope exponent in the generalized hydraulic relation and on the physical characteristics of the section: bankfull width, bankfull maximum depth, hydraulic conductance, and slope. The theoretical ranges of coefficient and exponent values derived herein are generally consistent with the averages and individual observed values reported in previous studies. However, observed values of the exponents at particular cross-sections commonly fall outside the theoretical ranges. In particular, the observed value of the velocity exponent ***m*** is commonly greater than the theoretical value, suggesting that hydraulic conductance often increases more strongly with discharge than predicted by the assumed hydraulic relations. The developments presented here provide new theoretical insight into the ways in which hydraulic and geometric factors determine hydraulic geometry. This insight should help to explain the variation of at-a-station hydraulic geometry and may facilitate prediction of hydraulic geometry at river reaches where detailed measurements are unavailable.

Introduction

The concept of hydraulic geometry was introduced by Leopold and Maddock (1953). The basic at-a-station hydraulic–geometry relations are functions relating the water-surface width, *W*, average depth, *Y*, and average velocity, *U*, to discharge, *Q*, at a particular stream cross-section or reach. The functions are usually given in the form of power-law equations:

$$W = a \cdot Q^{b}, \tag{1W}$$

$$Y = c \cdot Q^{f}, \tag{1Y}$$

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Nomenclature

[1] *denotes dimensionless quantities*

a	coefficient in width–discharge relation $[L^{1-3\cdot b}\,T^{b}]$
b	exponent in width–discharge relation [1]
c	coefficient in depth–discharge relation $[L^{1-3\cdot f}\,T^{f}]$
C	Chézy's conductance coefficient [1]
f	exponent in depth–discharge relation [1]
g	gravitational acceleration $[L\,T^{-2}]$
h	exponent in power-law velocity profile [1]
k	coefficient in velocity–discharge relation $[L^{1-3\cdot m}]$
K	generalized conductance coefficient $[L^{1-p}\,T^{-1}]$
m	exponent in velocity–discharge relation [1]
n	Manning's resistance coefficient [1]
p	depth exponent in generalized hydraulic relation [1]
q	slope exponent in generalized hydraulic relation [1]
Q	discharge $[L^3\,T^{-1}]$
r	exponent in cross-section geometry relation [1]
R	hydraulic radius [L]
S	energy slope [1]
u_C	unit-conversion factor in Chézy relation $[L^{1/2}\,T^{-1}]$
u_M	unit-conversion factor in Manning relation $[L^{1/3}\,T^{-1}]$
U	average cross-sectional velocity $[L\,T^{-1}]$
W	water-surface width [L]
W^*	bankfull water-surface width [L]
x	cross-channel distance from channel center [L]
y_r	effective height of bottom roughness elements [L]
Y	cross-sectional average water depth [L]
Y^*	bankfull cross-sectional average water depth [L]
Y_m	maximum water depth in cross-section [L]
Y_m^*	bankfull maximum water depth in cross-section [L]
z	vertical distance of channel bottom above lowest (central) point [L]
δ	$\equiv 1 + r + r\cdot p$ [1]

$$U = k \cdot Q^m. \tag{1U}$$

At-a-station hydraulic–geometry relations are useful tools in many types of hydrological analysis. They can be used directly in flood routing (Western et al., 1997; Orlandini and Rosso, 1998), and can be combined with flow-duration curves to produce water-resources-index duration curves that are useful in riverine-habitat analysis (Jowett, 1998), water-quality management, reservoir-sedimentation studies, and in determining the frequency of sediment movement (Dingman, 2002). Width–discharge relations can be used to estimate discharge via remote sensing (Bjerklie et al., 2005a).

Given the power-law forms of Eq. (1), the continuity relation,

$$Q = W \cdot Y \cdot U, \tag{2}$$

dictates that

$$a \cdot c \cdot k = 1 \tag{3}$$

and

$$b + f + m = 1. \tag{4}$$

It is important to note that at-a-station hydraulic–geometry relations as commonly applied are valid only for *in-bank* flows. Garbrecht (1990) expanded the concept by showing that two empirical power functions could be connected to apply to in-bank and over-bank flows at a given section. However, the discussion here is limited to in-bank flows.

Eq. (1) each plot as straight lines on double-logarithmic graph paper. The values of the coefficients (antilogs of the intercepts) and exponents (slopes) of these relations at a given cross-section are usually determined empirically by ordinary least-squares regression analysis on the logarithms of values of W, Y, U, and Q collected during discharge measurements at the section.

Ferguson (1986) reviewed empirical studies of at-a-station hydraulic geometry and theoretical attempts to explain why the exponents tend toward particular central values (though with much variability) and relative values (e.g., $b < f$). His main conclusion was that, given a cross-section with a specified constant shape and frictional characteristics and a law relating average velocity to friction and depth, the within-bank at-a-station hydraulic–geometry relations are determined. Thus he rejected theoretical approaches to determining the exponents in Eqs. (1W)–(1U) that invoke a ''metaphysical'' explanation, such as an assumed tendency toward ''minimum variance'' (Langbein, 1964).

Ferguson (1986) also showed that the $W(Q)$, $Y(Q)$, and $U(Q)$ relations will be power-laws only if the $W(Y)$ and $U(Y)$ relations are power-laws, such as the commonly used Manning and Chézy relations:

$$\text{Manning}: U = \frac{u_M}{n} \cdot Y^{2/3} \cdot S^{1/2}; \tag{5}$$

$$\text{Chézy}: U = u_C \cdot C \cdot Y^{1/2} \cdot S^{1/2}, \tag{6}$$

where u_M and u_C are unit-conversion factors (for SI units $u_M = 1$, $u_C = 0.552$; for British units $u_M = 1.49$, $u_C = 1$), n is Manning's resistance coefficient, S is water-surface slope, C is Chézy's conductance coefficient, and Y is assumed to differ negligibly from the hydraulic radius, R (i.e., $W/Y > 10$). Applying the Manning equation to a parabolic channel, a form commonly assumed to be approximated by natural river cross-sections, Ferguson (1986) found that $b = 0.23$, $f = 0.46$, and $m = 0.31$, ''strikingly close'' to the central values of empirical values reported in the literature. He also noted that almost no attention has been given to the factors that determine the coefficients a, c, and k.

Ferguson's (1986) conclusion that at-a-station hydraulic geometry is completely determined by cross-section geometry and hydraulic relations was anticipated by Dingman (1984, p. 243), who showed that applying the Manning equation in a wide rectangular channel leads to the following relations:

$$W = W^* \cdot Q^0; \tag{7}$$

$$Y = \left(\frac{n^{0.6}}{u_M^{0.6} \cdot S^{0.3} \cdot W^{*0.6}}\right) \cdot Q^{0.6}; \tag{8}$$

$$U = \left(\frac{u_M^{0.6} \cdot S^{0.3}}{n^{0.6} \cdot W^{*0.4}}\right) \cdot Q^{0.4}; \tag{9}$$

where W^* is bankfull width (= channel width for a rectangular channel). Dingman (1984) noted that, in natural channels where width increases with depth, the value of b will be higher and the values of f and m lower than derived for the rectangular channel (Eqs. (7)–(9)).

In application of the Manning and Chézy equations it is usually assumed that resistance/conductance (n in Eq. (5) or C in Eq. (6)) does not change with discharge at a given cross-section. In contrast with that assumption, Ferguson (1986) noted that it is more general and hydraulically sound to portray the $U(Y)$ relation for rough turbulent flow via the ''Keulegan equation'':

$$U = 2.5 \cdot (g \cdot Y \cdot S)^{1/2} \cdot \ln\left(\frac{11 \cdot Y}{y_r}\right), \tag{10}$$

where g is gravitational acceleration and y_r is the effective height of bed-roughness elements. Since Y ($\approx R$) increases with Q, Eq. (10) predicts that resistance decreases with increasing Q in a given section.

Ferguson (1986) used Eq. (10) to calculate the hydraulic–geometry relations in cross-sections of four different shapes (rectangle, triangle, parabola, and an asymmetrical section typical of meander bends). Because of the form of Eq. (10), log-log plots of these relations are *not* straight lines, but tend to give concave-upwards curves for the ln(Y) vs. ln(Q) relation and convex-upwards curves for the ln(U) vs. ln(Q) relation, as often observed empirically.

As implied by Eqs. (7)–(9) and concluded by Ferguson (1986), the considerable range of coefficients and exponents empirically observed in relations of the form of Eqs. (1W)–(1U) at different stream reaches can be explained by variations in cross-section geometry, slope, and resistance even if the local resistance does not change substantially with discharge. Ferguson (1986, p. 9) concluded his review of at-a-station hydraulic–geometry relations by stating that ''The great range of hydraulic geometries found in natural stream cross-sections is therefore hardly surprising''.

However, as noted by Ferguson (1986, p. 2), empirical determinations of hydraulic geometry ''do not yield sure cause-and-effect understanding and cannot be relied on for prediction beyond the original range of conditions''. Furthermore, although he explored the hydraulic–geometry relations determined by particular geometries and hydraulic laws and derived the exponent values dictated by the Manning equation in a parabolic channel, Ferguson (1986) did not present explicit general formulations showing the ways in which at-a-station hydraulic geometry depends on channel cross-section geometry and hydraulic factors.

Objectives

The principal objective of this paper is to address some of the questions that Ferguson (1986) raised but did not fully answer by deriving analytical expressions for the coefficients and exponents in Eqs. (1W)–(1U) as functions of cross-section geometry and hydraulic relations. It will also explore (1) how the numerical values of the exponents and coefficients vary as functions of hydraulic and geometric factors, and (2) the degree to which the analytically derived exponents and coefficients correspond to empirically determined values.

Geometry

The relations derived in this note assume that the geometry at a cross-section is fixed, i.e., that within-bank flows do not significantly alter cross-section shape. Changes in the shape of a given cross-section may occur during flows capable of erosion; these changes contribute to the scatter in the regressions used to determine the hydraulic–geometry relations. However, analysis of detailed hydraulic measurements of 77 cross-sections in New Zealand (Hicks and Mason, 1991) by Dingman and Sharma (1997) concluded that about 2/3 of the measured in-bank flows were incapable of moving the bed sediment, so the assumption of fixed shape appears to be acceptable much of the time.

For purposes of this note, it is assumed that bankfull maximum depth Y_m^* and bankfull width W^* are known, that channel cross-sections are symmetrical, and that their form can be approximated by

$$z = Y_m^* \cdot \left(\frac{2}{W^*}\right)^r \cdot x^r, \quad 0 \leqslant x \leqslant W^*/2, \tag{11}$$

where z is height of the bed above the lowest channel elevation (assumed to occur at the channel center), and x is horizontal distance from the center (Fig. 1). A triangle is represented by $r = 1$, the ''Lane Type B stable channel'' (Henderson, 1966; Fig. 10-21) by $r \approx 1.75$, a parabola by $r = 2$, and forms with increasingly flatter bottoms and steeper banks by increasing values of r; in the limit as $r \to \infty$, the channel is rectangular (Fig. 2). Values of $r < 1$ characterize ''convex'' cross-sections.

A general model essentially identical to Eq. (11) was apparently suggested in 1950 by S.T. Altunin, as cited in Averyanov (1956). Jowett (1998) also used this model (in inverse form) to characterize 73 cross-sections of New Zealand rivers; interestingly, he found $r \leqslant 2$ for all sections, and about 25% of the sections had $r < 1$.

From Eq. (11) it can be shown that, given bankfull width W^* and maximum depth Y_m^*, average depth, Y, and water-surface width, W, are related to maximum depth, Y_m, as

$$Y = \left(\frac{r}{r+1}\right) \cdot Y_m, \quad Y_m = \left(\frac{r+1}{r}\right) \cdot Y; \tag{12}$$

$$W = W^* \cdot \left(\frac{Y_m}{Y_m^*}\right)^{1/r} = W^* \cdot \left(\frac{1}{Y_m^*}\right)^{1/r} \cdot \left(\frac{r+1}{r}\right)^{1/r} \cdot Y^{1/r}. \tag{13}$$

Various methods can be used to estimate r from cross-section measurements (see Jowett (1998) for one approach). From Eqs. (1W), (1Y), and (13) (and as will be shown in

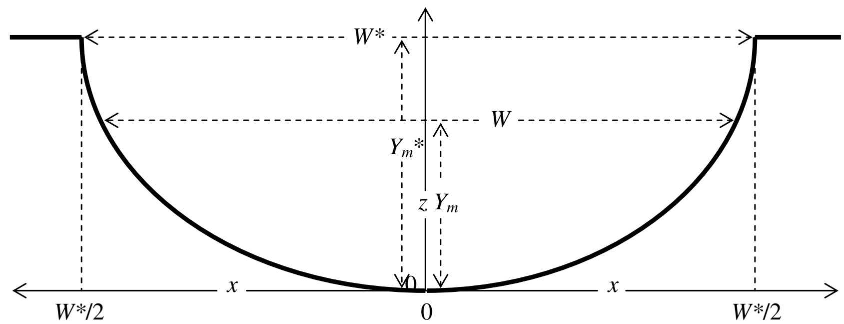

Figure 1 Definitions of terms used to model cross-section geometry.

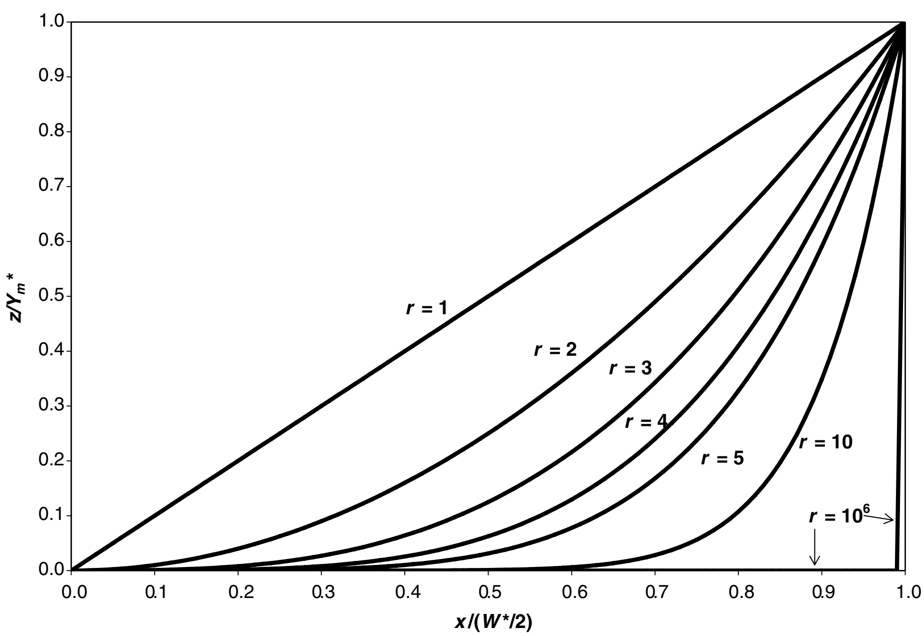

Figure 2 Channel shape as a function of the shape exponent r. Channels are assumed to be symmetrical with bankfull width W^* and bankfull maximum depth $Y_m^* \cdot r = 1$ represents a triangle, $r = 2$ a parabola, and $r \rightarrow \infty$ a rectangle.

the derivations of Eqs. (21) and (27)), the value of r can also be estimated as

$$\hat{r} = \frac{f}{b}, \tag{14}$$

where f and b are determined by log regression of measured values of Y, W, and Q as in the models of Eqs. (1W) and (1Y).

Hydraulics

The Chézy (Eq. (5)) and Manning (Eq. (6)) relations for uniform rough turbulent flows are commonly used to represent the hydraulic relations in open-channel flows. As noted, in using these equations one usually assumes that energy slope is constant and that resistance/conductance values are constant over the range of in-bank flows at a given cross-section. Several empirical and quasi-theoretical studies have suggested that equations similar in form to those of Manning and Chézy, but with different values of the exponents, may better represent resistance relations in natural channels (e.g., Dingman and Sharma, 1997; Bjerklie et al., 2005a,b).

Ferguson (1986) argued that the Keulegan equation (Eq. (10)), predicting a decrease of resistance with increasing depth and discharge, is more physically based and general than the Manning or Chézy equations. However, use of Eq. (10) in the derivations below leads to implicit expressions that inhibit understanding of the factors that determine hydraulic geometry. To avoid these implicit expressions, one can approximate the Keulegan equation via a power-law that incorporates a decrease of resistance with depth (Chen, 1991):

$$U = (g \cdot Y \cdot S)^{1/2} \cdot \left(\frac{0.920}{h \cdot (h+1)}\right) \cdot \left(\frac{30 \cdot Y}{y_r}\right)^h$$
$$= \left(\frac{g^{1/2}}{h \cdot (h+1)}\right) \cdot \left(\frac{27.6}{y_r}\right)^h \cdot Y^{1/2+h} \cdot S^{1/2}. \tag{15}$$

Chen (1991) showed that the value of h must increase as Y/y_r decreases in order to approximate the Keulegan law. He recommended using $h = 1/6$ for rough turbulent flow over a wide range of Y/y_r values, but indicated that values up to $h = 1/2$ are appropriate for smaller values of Y/y_r (i.e., larger values of relative roughness). Ferguson (2006, per-

Table 1 Values of K, p, and q in Eq. (16) for the Manning, Chézy, Bjerklie et al. (2005b), and power-law hydraulic relations

Hydraulic relation	K	p	q
Manning	$1/n$	0.667	0.500
Chézy	$0.552 \cdot C$	0.500	0.500
Bjerklie et al. (2005b)	7.73	0.667	0.333
Power-law, $h = 1/2$	$22.874/y_r^{0.5}$	1.000	0.500

The power-law values are for $h = 0.5$, representing flows with very high relative roughness. K values are in SI units.

sonal communication) reported h values as high as 1.8 in mountain streams with very low Y/y_r.

For maximum generality while assuring that the derived expressions will be explicit , the derivations here are carried out with a generalized power-law hydraulic equation:

$$U = K \cdot Y^p \cdot S^q, \tag{16}$$

where K is a generalized conductance coefficient. Table 1 shows the values of K, p, and q for the Manning, Chézy, and power-law relations.

Examination of the variation of n and C at 77 cross-sections at which six or more discharge measurements were made by Jarrett (1985), Hicks and Mason (1991) and Coon (1998) shows that Manning's n is generally less variable than C. However, the difference is not great and the coefficient of variation for both measures is less than 20% for well over half the sites. Thus the assumption of constant resistance at a section is often acceptable.

However, neither the assumption of constant resistance or of decreasing resistance with increasing depth is universally applicable. An examination of the plots of n vs. Q for the 78 sites measured by Hicks and Mason (1991) shows that, while many exhibit downward trends, many are clearly either constant, non-monotonic, or increasing. Some of this variability in trend may be due to the complicating effects of vegetation.

Although water-surface slope varies as a flood wave passes through a cross-section, Bjerklie et al. (2003) showed that the assumption of constant energy slope equal to channel slope appears to be generally acceptable for estimating discharge. Significant exceptions to this can occur where the flow is affected by backwater, a condition that occurs in estuaries, entrances to reservoirs and lakes, tributaries, and at low flows in the pools of a stream with pool-and-riffle sequences. Since Eq. (16) is a generalized uniform-flow relation, the following derivations do not in any case apply to such situations.

Derivations

$Y(Q)$ relation

Substituting Eqs. (12), (13), and (16) into Eq. (2) yields

$$Q = W^* \cdot \left(\frac{1}{Y_m^*}\right)^{1/r} \cdot \left(\frac{r+1}{r}\right)^{1/r} \cdot K \cdot S^q \cdot Y^{1+1/r+p}. \tag{17}$$

Solving (17) for Y, we have

$$Y = \left[\left(\frac{1}{W^*}\right) \cdot Y_m^{*1/r} \cdot \left(\frac{r}{r+1}\right)^{1/r} \cdot \left(\frac{1}{K}\right) \cdot \left(\frac{1}{S^q}\right)\right]^{\frac{r}{1+r+r\cdot p}} \cdot Q^{\frac{r}{1+r+r\cdot p}}. \tag{18}$$

To streamline subsequent derivations, define

$$\delta \equiv 1 + r + r \cdot p. \tag{19}$$

Thus from (18), the coefficient c in the depth—discharge relation is found to be

$$c = \left(\frac{1}{W^*}\right)^{r/\delta} \cdot Y_m^{*1/\delta} \cdot \left(\frac{r}{r+1}\right)^{1/\delta} \cdot \left(\frac{1}{K}\right)^{r/\delta} \cdot \left(\frac{1}{S^q}\right)^{r/\delta} \tag{20}$$

and the exponent f in the depth—discharge relation is found to be

$$f = \frac{r}{1 + r + r \cdot p} = \frac{r}{\delta}. \tag{21}$$

$W(Q)$ relation

With c and f determined by Eqs. (20) and (21), respectively, we can write Eq. (2) as

$$Q = W \cdot (c \cdot Q^f) \cdot [K \cdot (c \cdot Q^f)^p \cdot S^q],$$
$$Q^{1-f-f\cdot p} = W \cdot c^{1+p} \cdot K \cdot S^q. \tag{22}$$

Solving (22) for W,

$$W = \left[\left(\frac{1}{c}\right)^{1+p} \cdot \left(\frac{1}{K}\right) \cdot \left(\frac{1}{S^q}\right)\right] \cdot Q^{1-f-f\cdot p}. \tag{23}$$

Substituting Eqs. (20) and (21), we find

$$W = \left[\left(\frac{1}{c}\right)^{1+p} \cdot \left(\frac{1}{K}\right) \cdot \left(\frac{1}{S^q}\right)\right] \cdot Q^{\frac{1}{1+r+r\cdot p}}, \tag{24}$$

so that the coefficient a in the width—discharge relation is found to be

$$a = \left[\left(\frac{1}{c}\right)^{1+p} \cdot \left(\frac{1}{K}\right) \cdot \left(\frac{1}{S^q}\right)\right]. \tag{25}$$

Substituting (20) into (25) and simplifying yields

$$a = W^{*(r+r\cdot p)/\delta} \cdot \left(\frac{1}{Y_m^*}\right)^{(1+p)/\delta} \cdot \left(\frac{r+1}{r}\right)^{(1+p)/\delta} \cdot \left(\frac{1}{K}\right)^{1/\delta} \cdot \left(\frac{1}{S^q}\right)^{1/\delta} \tag{26}$$

and the exponent b in the width—discharge relation is found to be

$$b = \frac{1}{1 + r + r \cdot p} = \frac{1}{\delta}. \tag{27}$$

$U(Q)$ relation

With the expressions for a (26), b (27), c (20), and f (21) found, we can now write Eq. (2) as

$$Q = a \cdot Q^b \cdot c \cdot Q^f \cdot U,$$
$$Q^{1-b-f} = a \cdot c \cdot U,$$
$$U = \left(\frac{1}{a \cdot c}\right) \cdot Q^{1-b-f}. \tag{28}$$

Using (20), (26) and (28), we find that the coefficient k in the velocity—discharge relation is found to be

$$k = \left(\frac{1}{W^*}\right)^{r\cdot p/\delta} \cdot Y_m^{*p/\delta} \cdot \left(\frac{r}{r+1}\right)^{p/\delta} \cdot K^{(1+r)/\delta} \cdot S^{q\cdot(1+r)/\delta}. \tag{29}$$

From (21), (27) and (28), the exponent m in the velocity—discharge relation is found to be

$$m = \frac{r \cdot p}{1 + r + r \cdot p} = \frac{r \cdot p}{\delta}. \tag{30}$$

Ranges of theoretical exponents and coefficients and comparison with empirical studies

Exponents

Theoretical ranges

Eqs. (21), (27), and (30) satisfy Eq. (4) and show that the at-a-station hydraulic—geometry exponents depend only on the cross-section shape, r, and the depth exponent in the hydraulic relation, p.

Fig. 3a shows how the exponents vary as functions of r for the Chézy ($p = 0.5$), Manning ($p = 0.667$), and power-law ($h = 0.500$; $p = 1.000$) relations, and Fig. 3b shows exponent ratios as functions of r. [Note that the exponent values for the Bjerklie et al. (2005b) relation are the same as for the Manning relation because $p = 0.667$ in both cases.] Except for a triangular channel, in which $f = b$ for all cases, it is always true that $f > m > b$. (However, f can be less than m if $p > 1$; i.e., for power-law hydraulic relations with $h > 1/2$; and m can be less than b if $r \cdot p < 1$.)

For all hydraulic relations, f and m increase and b decreases as the cross-section gets more rectangular, and b is a stronger function of shape than are f and m. For a given shape, the choice of hydraulic relation affects b very little

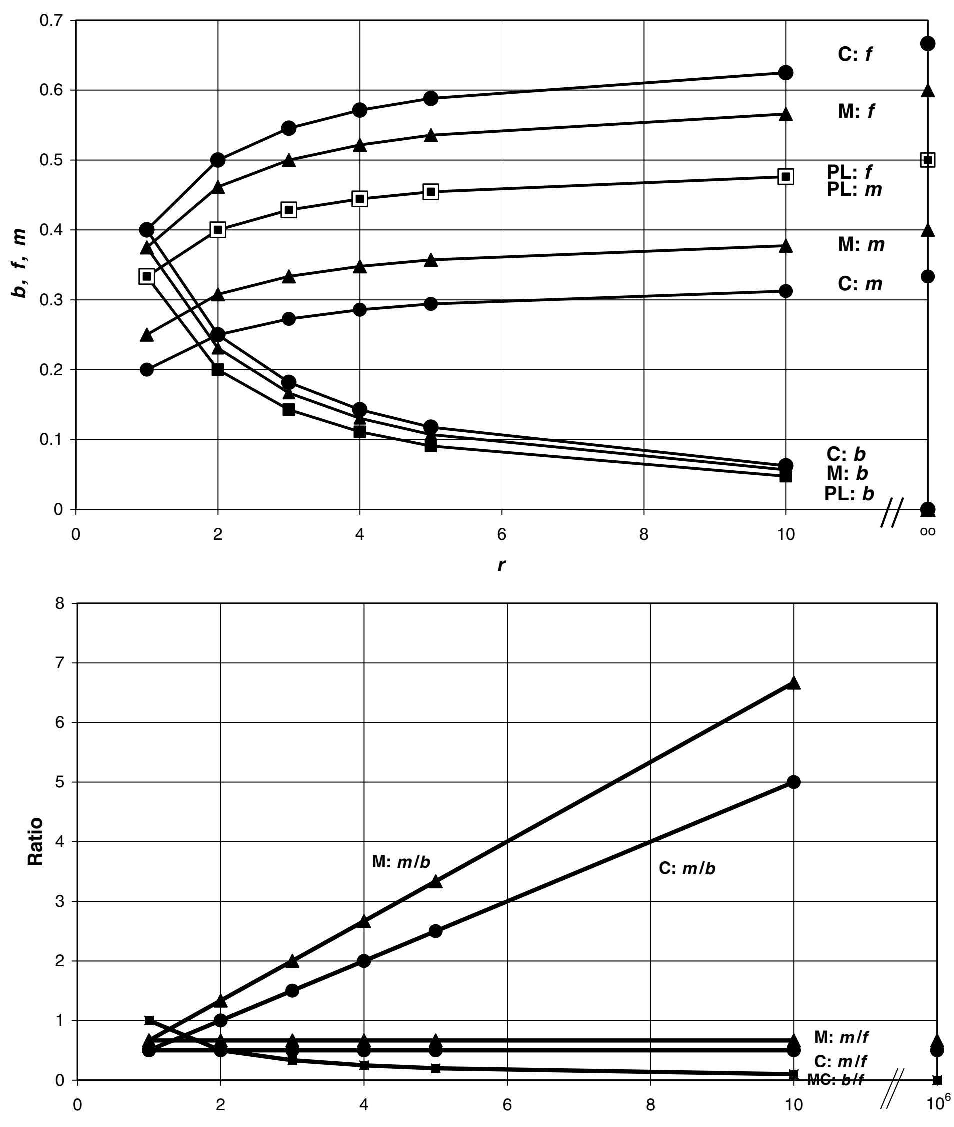

Figure 3 (a) Hydraulic—geometry exponents as functions of r for the Chézy (C: $p = 0.5$), Manning (M: $p = 0.667$), and power-law (PL: $h = 0.500$; $p = 1.000$) relations. (b) Ratios of hydraulic—geometry exponent as functions of r. The exponent values for the Bjerklie et al. (2005b) relation are the same as for the Manning relation because $p = 0.667$ in both cases.

(probably well within uncertainties associated with estimation by regression analysis), but has a more substantial effect on f and m. The derived formulas show that $f > b$ for $r > 1$, implying that in-bank width/depth ratios almost always decrease with discharge.

For a parabolic channel ($r = 2$) with Manning's equation ($p = 0.667$), we calculate $b = 0.23$, $f = 0.46$, and $m = 0.31$, exactly as found by Ferguson (1986). Approximating a rectangle using $r = 10^6$ and using the Manning equation yields $b = 0.000$, $f = 0.600$, and $m = 0.400$, as found by Dingman (1984) (Eqs. (7)–(9)).

Rhodes (1977) introduced the ternary, or triangular, graph as an aid to summarizing hydraulic—geometry exponents and analyzing their geomorphic implications. The dependence of the exponents on cross-section geometry and hydraulic relation can be shown on such a diagram (Fig. 4). All the theoretical values plot in a small area of the diagram within the sector defined by $0 \leqslant b \leqslant 0.4, 0.333 \leqslant f \leqslant 0.667$, and $0.2 \leqslant m \leqslant 0.5$, with the lines determined by geometry (r values) radiating from the top vertex and those determined by hydraulic relation (p values) radiating from the left vertex, roughly perpendicularly to the geometry lines.

Rhodes (1977) divided the ternary diagram into10 fields bounded by straight lines representing specific relations among the exponents, and interpreted values in each field in terms of hydraulic behavior and channel stability. He noted that the interpretations related to stability were uncertain, largely because of difficulty in sorting out whether competence depends more strongly on shear stress, which is proportional to depth, or on velocity. In any case, several of Rhodes's boundary relations coincide with specific hydraulic relations and geometries used in the derivations in the present paper (Table 2). Thus, while Rhodes's interpretations seem sound, the derivations herein indicate that some of his boundary relations can also be interpreted as the consequences of the applicable geometry and hydraulic relations.

Comparison with empirical values

Table 3 summarizes empirically determined at-a-station hydraulic—geometry relations from a number of studies, and indicates whether the exponent values fall within the theoretical ranges derived in the present study. All the empirical b values fall within the theoretical range, and are consistent with channel shapes ranging from approximately parabolic to rectangular. The f values have the narrowest range and are also within the theoretical range; they cluster around the theoretical values for parabolic channels. Some of the m values, however, are slightly above the theoretical range. In all cases $f > b$, consistent with the theory for $r > 1$, but it is not always true that $f > m > b$.

Note that the values shown in Table 3 are averages calculated separately for each exponent, and there was considerable variation in exponent values reported in most of the studies. The values for a given study do not necessarily

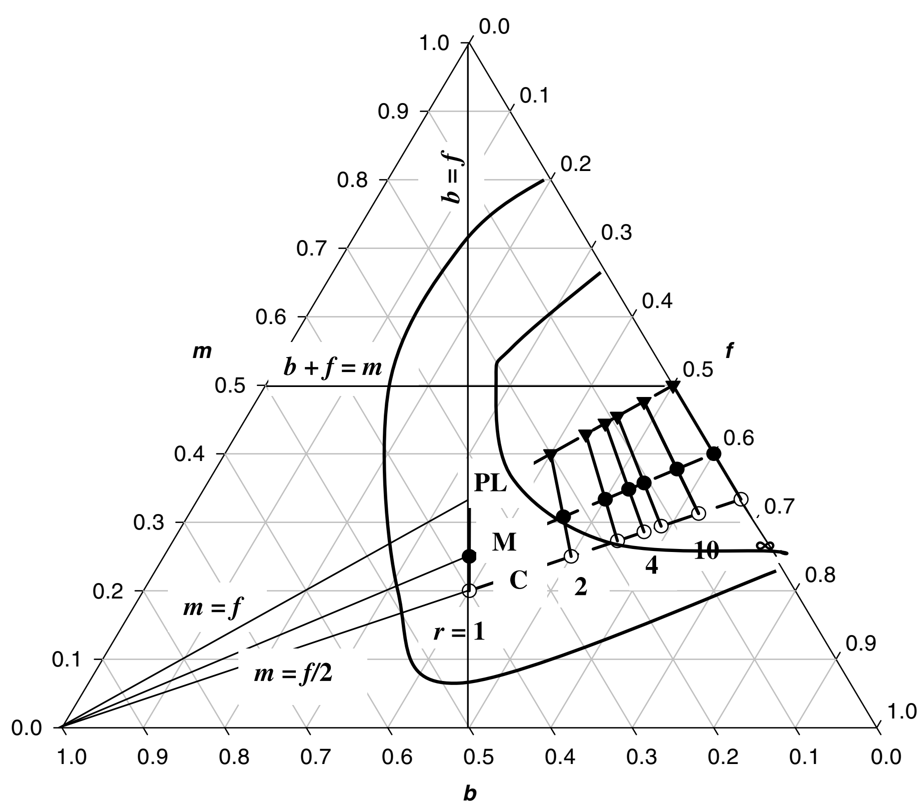

Figure 4 Ternary diagram showing theoretical and observed values of hydraulic—geometry exponents. The theoretical values fall within the field bounded by $r = 1$ ($b = 0.5$), $r = \infty$ ($b = 0$), $p = 0.5$ ($m = f/2$), and $p = 1$ ($m = f$). C, M, PL designate lines for Chézy, Manning, and power-law hydraulic relations. The inner roughly semi-circular field contains the highest concentration of observed values reported by Rhodes (1977), the outer roughly semi-circular field contains almost all the values reported by Rhodes (1977).

Table 2 Rhodes's (1977) and the present paper's interpretations of exponent relations on the ternary diagram

Boundary relation	Rhodes (1977)	Present paper
$b = f$	$f > b$: $W/Y \downarrow$ as $Q \uparrow$	$f = b$ for $r = 1$ (triangle); $f > b$ for $r > 1$
$m = f$	$m > f$: competence $\uparrow$ as $Q \uparrow$	$m = f$ for all r for power-law ($p = 1$) $m < f$ for all r for other hydraulic relations with $p < 1$
$m = f/2$	$m > f/2$: $Fr \uparrow$ as $Q \uparrow$	$m = f/2$ for all r for Chézy relation ($p = 0.5$)
$m = 2\cdot f/3$	$m > 2\cdot f/3$: $K \downarrow$ as $Q \uparrow$	$m = 2\cdot f/3$ for all r for Manning relation ($p = 0.667$)
$m = b + f$ ($m = 0.5$)	$m > b + f$: $K \uparrow$ as $Q \uparrow$ (stable channel)	$m < 0.5$ for all r for all relations except Power-law ($p = 1$) in rectangular channel, for which $m = 0.5$

Fr = Froude number $\equiv U/(g\cdot Y)^{1/2}$. Up/down arrows indicate increasing/decreasing quantities.

Table 3 Empirically determined exponent values

Location	b	f	m	Source
Mid-West USA	**0.26**	**0.40**	**0.34**	Leopold and Maddock (1953) (20)
Brandywine Ck., PA, USA	**0.04**	**0.41**	0.55	Wolman (1955) (7)
Ephemeral streams, semi-arid USA	**0.25**	**0.41**	**0.33**	Leopold and Miller (1956) (10)
Rio Manati, Puerto Rico	**0.17**	**0.33**	**0.49**	Lewis (1969) (10)
R. Hodder, UK	**0.09**	**0.36**	0.53	Wilcock (1971) (9)
R. Bollin-Dean, UK	**0.12**	**0.40**	**0.48**	Knighton (1975) (12)
R. Ter, UK	**0.14**	**0.42**	**0.43**	Harvey (1975) (8)
New Zealand	**0.18**	**0.31**	**0.43**	Jowett (1998) (73)
Australia	**0.11**	**0.28**	0.52	Stewardson (2005) (17)

Bold values are those that fall within the ranges determined by derivations in the present paper. Number in parentheses is number of stations used in study. Some of the summary information is taken from Leopold et al. (1964), Dunne and Leopold (1978) and Knighton (1998).

sum to one and are not necessarily ''typical'' for the region studied. A more meaningful comparison of empirical and theoretical exponent values for individual cross-sections or reaches can be obtained by plotting the values on Rhodes's (1977) ternary diagram. To do this, Fig. 4 also indicates where the 315 sets of empirical data summarized by Rhodes (1977) plot. It appears that at least half the points he plotted fall outside the theoretical ranges derived here. By far the greatest number of these points have higher m values than given by the hydraulic relations assumed here. Given that these points lie in the direction of changing hydraulic, rather than geometric, relations, a likely explanation for this discrepancy is that none of the power-law hydraulic relations discussed in ''Hydraulics'' adequately captures the decrease of resistance with increasing depth and discharge. As noted above, use of Eq. (10) or other non-power-law profiles that might more adequately capture the hydraulics – for example, the one suggested by Katul et al. (2002), or one incorporating energy-balance considerations integrated over the watershed (Orlandini, 2002) – lead to implicit hydraulic–geometry relations.

Coefficients

Theoretical relations

As noted by Ferguson (1986), virtually no attention has been given to the factors that determine the values of the coefficients in the hydraulic–geometry relations except to note that they depend on units of measurement and represent the values of width, depth, and velocity when discharge is equal to unity in the given unit system. Eqs. (20), (26) and (29) satisfy Eq. (3) and show that the coefficients depend in complex ways on all the quantities in the geometric and hydraulic relations as well as on the units of measurement.

We first explore how the coefficients change with geometry (r) for selected values of p and q. This exploration is carried out by specifying baseline values of bankfull width, bankfull depth, conductance, and slope (Table 4) and then exploring how the coefficients vary as functions of r for $p = 0.500$ (Chézy), $p = 0.667$ (Manning), $p = 1.000$ (Power-Law with $h = 0.5$) and $q = 0.333$ and 0.500.

The results are plotted in Fig. 5, with the values of a scaled by the bankfull width, W^*, in order to keep the values <1. Using the conventional value of $q = 0.500$ (Fig. 5a), the width coefficient a is a strongly increasing function of r that achieves a maximum value $a = W^*$ for a rectangular channel. In contrast, the c and k values decrease only slightly with increasing r. The hydraulic relation (p value) affects the coefficients only slightly. The qualitative patterns are similar for $q = 0.333$ (Fig. 5b), but now the k values are increased significantly compared to the case for $q = 0.500$, and the choice of p makes a considerable difference in the k value.

Table 4 Baseline values of independent variables that determine at-a-station hydraulic geometry

Variable	W^*	Y_m^*	K	S
Units	m	m	$m^{1-p}\,s^{-1}$	–
Value	50	2	14	0.002

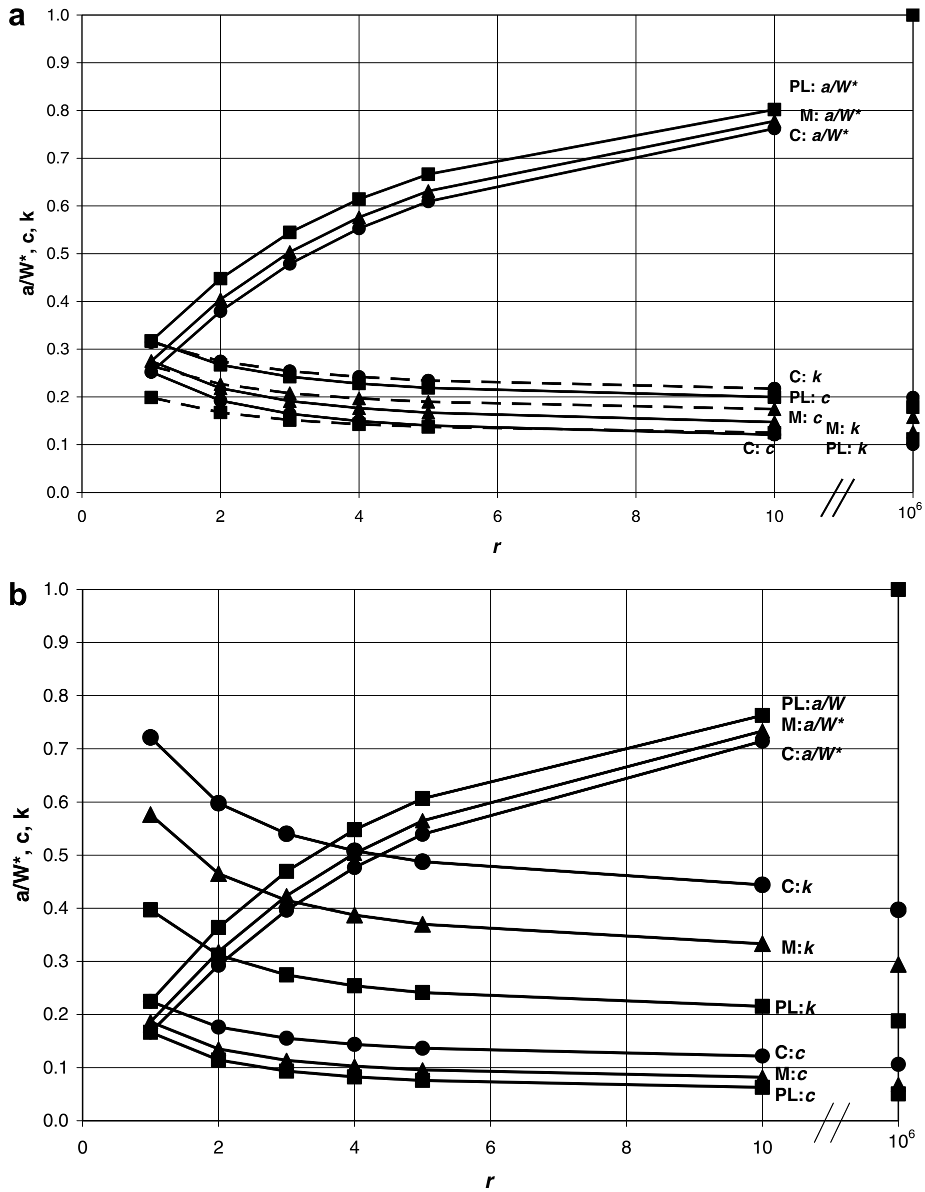

Figure 5 Hydraulic—geometry coefficients as functions of r, $p$and q. The width coefficient a is scaled by the bankfull width W^*. Physical quantities set at values in Table 4. (a) Slope exponent $q = 0.500$; (b) Slope exponent $q = 0.333$.

The qualitative relations between the coefficients and the physical quantities $W^*, Y_m^*, K,$ and S are clear from Eqs. (20), (26), and (29) and are summarized in Table 5. Bankfull width has a relatively strong effect on all coefficients. a is most strongly affected by maximum channel depth, c by slope, and k by resistance and slope.

Table 5 Qualitative relations between coefficients and the physical quantities W^*, Y_m^*, K, and S

Coefficient	W^*	Y_m^*	K	S
a	↑↑	↓↓	↓	↓
a/W^*	↓↓	↓↓	↓	↓
c	↓↓	↑	↓	↓↓
k	↓↓	↑	↑↑	↑↑

↑/↓ indicates coefficient increases/decreases as quantity increases, double arrows indicate relatively strong effect.

Comparison with empirical values

Of previous studies only Jowett (1998) provided statistical information on the observed values of the coefficients. For the 73 New Zealand reaches he measured, he found the following means and standard deviations (in SI units): $a = 15.8 \pm 7.5$, $c = 0.31 \pm 0.12$, $k = 0.24 \pm 0.09$. These values appear consistent with the theoretical values computed here.

Summary and conclusions

Many previous writers have noted the lack of definitive explanations for the tendencies of the observed exponents in the standard at-a-station hydraulic—geometry expressions to cluster around particular values. Ferguson (1986) correctly noted that those expressions are determined by the cross-section geometry (assumed to be fixed for in-bank flows) and the hydraulic relation; no ''metaphysical'' explanations are required. He further showed that the expressions need not take the form of the conventional power-laws.

The present paper assumes a power-law hydraulic relation but otherwise accepts Ferguson's conclusions and extends them by using generalized expressions for both cross-section geometry and hydraulics to derive explicit equations for the exponents and the coefficients in the power-law hydraulic—geometry relations. The exponents are shown to depend only on the depth exponent in the hydraulic relation (p) and the exponent that reflects the form of the cross-section (r). The coefficients also depend on p and r, but in addition are functions of the slope exponent in the hydraulic relation (q) and on the physical characteristics of the section: bankfull width (W^*), bankfull maximum depth (Y^*_m), hydraulic conductance (K), and energy slope (S).

The theoretical ranges of coefficient and exponent values derived herein are generally consistent with the averages and individual observed values reported in previous studies. However, observed values of the exponents at particular cross-sections commonly fall outside the theoretical ranges. In particular, the observed value of the velocity exponent m is commonly greater than the theoretical value, suggesting that hydraulic conductance often increases more strongly with discharge than predicted by the hydraulic relations assumed herein.

The developments presented here provide new theoretical insight into the ways in which various hydraulic and geometric factors determine hydraulic geometry. This insight should help to explain the variation of hydraulic geometry and may facilitate prediction of hydraulic geometry at river reaches where detailed measurements are unavailable. However, the present analysis does not explain the reach-to-reach variability in cross-section shape or in the resistance coefficients and exponents of hydraulic relations. Those more fundamental explanations may involve invoking ''metaphysical'' concepts such as minimization of stream power but, following the underlying scientific principle of seeking deterministic explanations for natural phenomena, should first be sought via application of physical laws.

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