Unified Theory on Power Laws for Flow Resistance

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Abstract: Two general power formulas, one for hydraulically smooth flows and the other for fully rough flows, are derived in a rational way from the widely accepted logarithmic formulas for the velocity profile and the Darcy-Weisbach friction factor. A regression analysis based on the method of least squares is used to determine the valid range of the local velocity (or normal distance from the wall) in the power formula. Some older empirical formulas, such as Lacey's, Manning's, Blasius', and Hazen-Williams', and their valid ranges, are actually explained analytically by the results. Incomplete self-similarity of the power law, in which the exponent and the associated coefficient vary with the similarity parameters, such as the Reynolds number and the relative roughness, is elucidated through the parametric representations of the power formulas and their counterparts based on the logarithmic law. This paper examines the concept and rationale behind the power formulation of uniform turbulent shear flows, thereby addressing some critical issues in the modeling of flow resistance based on the power law.

Introduction

Empiricism has played a major role in the development of many power formulas with different exponents (or powers) for the streamwise time-mean-velocity distribution in turbulent shear flows, whereas the logarithmic law is well known for its theoretical soundness and universal validity. According to Barenblatt (1979), the logarithmic velocity distribution is based on the assumption of complete self-similarity of the flow in both the local and global Reynold numbers (i.e., the Reynolds numbers based on the local flow variables and the cross section-averaged flow variables, respectively), whereas the power velocity distribution is formulated on the assumption of incomplete self-similarity in the local Reynolds number. Barenblatt (1979) further stated that for the logarithmic velocity distribution dimensional analysis is sufficient for establishing self-similarity and determining the self-similar variables, whereas for the power velocity distribution dimensional analysis is insufficient for establishing self-similarity and determining the self-similar variables. Insufficiency of the methodology in dimensional analysis to develop the power velocity distribution is thus reflected in the indeterminacy of the exponent and coefficient associated with the power law.

It is well known that the logarithmic velocity distribution can be approximated by the power velocity distribution in a large fraction of the boundary-layer cross section (Hinze 1959), namely in the overlapping region of the inner law (i.e., the law of the wall) and the outer law (i.e., the velocity-deflect law). In this context, one may theoretically generate various power flow formulas from the corresponding logarithmic flow formula. Apparently, Bretting first did it in 1948, but only with a limited number of well-known power formulas. Chen (1988, 1989) has recently developed a viable pro-

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cedure to determine numerically the exponent and coefficient of the power law, using the logarithmic velocity distribution (or logarithmic flow formula) as a yardstick in the least-squares approximation of the power velocity distribution (or power flow formula). The exponent and coefficient of the power law so determined were found to vary with the global Reynolds number and with the global relative roughness (i.e., based on the cross section-averaged flow variables) for hydraulically smooth flows and fully rough flows, respectively. In this study, it is intended to specify a valid range of such a similarity parameter under given flow conditions for engineering applications, as was done by Bretting (1948).

THEORETICAL CONSIDERATIONS

Millikan (1939) appeared to be the first among turbulent-flow researchers who proved the existence of the logarithmic velocity distribution in the overlapping region of a turbulent shear flow over a smooth boundary. Extending Millikan's approach, Landweber (1957) [see also Rouse (1959)] further proved that both logarithmic and power velocity distributions exist in the overlapping region for both smooth and rough walls. Taking an approach similar to Landweber's and then evaluating theoretically the difference between the logarithmic velocity distribution and the power velocity distribution, Wooding et al. (1973) found that the power law with a small exponent cannot be possibly distinguished experimentally from the logarithmic law. This finding in fact restates Schlichting's (1968) otherwise equivalent conclusion that the logarithmic velocity distribution must be regarded as an asymptotic law applicable to very large Reynolds numbers, in contrast to the power velocity distribution, which is valid for a smaller Reynolds number. Two partial differential equations, each representing either of both inner and outer laws, can be formulated (with the help of dimensional analysis) on the condition that both laws are satisfied. This condition, which of course can only be met in the overlapping region, constituted a theoretical basis of the analyses performed by the foregoing investigators. The present paper also adopts their approach based on the very same condition although other plausible approaches, such as various self-similar assumptions of the turbulent shear stress made by von Kármán and Prandtl (Schlichting 1968) and most recently by Sill (1982, 1986), may be taken.

Consider a steady turbulent shear flow of a Newtonian incompressible homogeneous fluid in a pipe or channel. Assuming the validity of both inner and outer laws in a certain range of distance from the wall, one can theoretically derive two partial differential equations in two functional relationships among the respective similarity parameters for the inner and outer laws, respectively (Rouse 1959) [see also Hamel's (1943) rigorous deduction of von Kármán's similarity rule]. Because the outer law is not used in the present analysis, the differential equation for the outer law is not considered from the outset.

$$y^* \frac{\dfrac{\partial^2 F}{\partial y^{*2}}}{\dfrac{\partial F}{\partial y^*}} = m - 1 \qquad (1)$$

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in which $y^* = yu_*/\nu$ (y = normal distance from the wall, $u_* = \sqrt{\tau_0/\rho}$ = boundary shear velocity, τ_0 = boundary shear stress, ρ = mass density of fluid, ν = kinematic viscosity of fluid) is the dimensionless distance from the wall and is often called the local Reynolds number; $F = u/u_* = F(y^*,k^*)$ denotes the dimensionless velocity, u/u_*, as a function of y^* and k^* ($k^* = ku_*/\nu$, often called the roughness Reynolds number based on k, the mean roughness height, u = local streamwise time-mean velocty at y); and m = constant.

For smooth walls, F is a function only of y^*, but for rough walls, F is a function of both y^* and k^*. For rough walls, however, there exists a narrow transition range of k^* values, in which the flow has both roughness and viscosity effects. Outside this transition range, the flow is affected either by roughness or viscosity only. The lower and upper limits of the k^* range exist for all types of roughness, but both limits vary considerably for different types (Schubauer and Tchen 1961). For the k^* value below the lower limit, roughness has virtually no effect on F because the roughness elements are submerged in the viscous sublayer. This flow condition may be called "hydraulically" smooth flow. The length scale of turbulence corresponding to this hydraulically smooth flow condition is thus taken as ν/u_*. As a matter of fact, the similarity variable, y^* [$= y/(\nu/u_*)$], for hydraulically smooth flows has been postulated in this way. On the other hand, for the k^* value above the upper limit, the viscous sublayer no longer exists because the roughness elements themselves induce turbulent mixing by the flow around them. Consequently, when k^* exceeds the upper limit, the flow becomes "fully" rough, and the length scale of turbulence should be specified in the order of k. The similarity variable for fully rough flows is thus set in the form of y/k (or y^*/k^*), as usually seen in the literature. Inclusion of both y^* and k^* as similarity variables in the integration of Eq. 1 for deriving both logarithmic and power velocity distributions is quite complicated, as shown by Landweber (Rouse 1959). For simplicity, therefore, the flow is assumed herein to be either hydraulically smooth or fully rough and the respective similarity variable, as just indicated, will be used in the integration of Eq. 1.

Steady uniform pipe and channel flows are typical examples of equilibrium flows in which the pressure drop is exactly balanced by wall friction. To avoid complicacy in the analysis, the scope of the present study is further confined to such steady uniform pipe and channel flows. A critical assumption, yet to be imposed, pertains to the incompleteness of the present theory to describe the departure of the streamwise time-mean-velocity distribution from the logarithmic law of the wall. Although the present formulation (Eq. 1) has not explicitly excluded such a departure in the overlapping region, solving Eq. 1 for F cannot prove that the law of the wake (Coles 1956) exists in the overlapping region. Therefore, maintaining consistency in the present analysis necessitates the tacit assumption of no departure from the logarithmic law of the wall in the formulation of Eq. 1. How well such a linear combination of Coles' wake law with the logarithmic law of the wall can be approximated by the power law of the wall remains to be addressed in the future.

Logarithmic and Power Laws of the Wall

For integrating Eq. 1, one may select the distance from the wall to the location where $u = 0$ as the characteristic length and denote it by y'. Most

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investigators have proposed $y' = \beta\nu/u_*$ and γk for hydraulically smooth flows and fully rough flows, respectively, where β and γ are empirical constants. An extensive analysis of existing data, mostly for flow in pipes, collected by many experimentalists (Schlichting 1968) has revealed that there exist, to a certain degree, universal values of β and γ, namely $\beta = 1/9 \approx 0.111$ and $\gamma = 1/30 \approx 0.0333$, though the value of γ may vary with the boundary roughness for shallow flow in open channels, especially with free-surface instability. In the light of this physically meaningful y', the advantage of using y/y' as a unified similarity variable over y^* for hydraulically smooth flows or y^*/k^* for fully rough flows is obvious.

Two cases are possible in the integration of Eq. 1: one for $m = 0$ and the other for $m \neq 0$. For $m = 0$, Eq. 1 upon replacing y^* by y/y' and then being integrated once over y/y' yields

$$\left(\frac{y}{y'}\right)\frac{\partial\left(\frac{u}{u_*}\right)}{\partial\left(\frac{y}{y'}\right)} = C_1 \quad (2)$$

and likewise for $m \neq 0$

$$\left(\frac{y}{y'}\right)\frac{\partial\left(\frac{u}{u_*}\right)}{\partial\left(\frac{y}{y'}\right)} = ma\left(\frac{y}{y'}\right)^m \quad (3)$$

in which C_1 and (ma) are integration constants for $m = 0$ and $m \neq 0$, respectively. Note that Eqs. 2 and 3 are the logarithmic and power laws expressed in the form of the first-order partial differential equation, respectively. Because it has been assumed that u/u_* depends only on y/y', Eqs. 2 and 3 may be written in the form of ordinary differential equations rather than partial differential equations.

Both dimensionless expressions for y/y' times the time-mean-velocity gradient in turbulent shear flows, as shown respectively on the right-hand sides of the equal sign in Eqs. 2 and 3, are virtually identical to those obtained by Barenblatt (1979) based on the assumptions of complete self-similarity and incomplete self-similarity, respectively. The reciprocal of C_1, which may be denoted by κ, is usually referred to as the von Kármán universal constant. Independence of the normalized time-mean-velocity gradient from the local and global Reynolds numbers, as shown in Eq. 2, implies the assumption of complete self-similarity in both the local and global Reynolds numbers. In contrast to Eq. 2, Eq. 3 cannot be obtained from dimensional analysis because the dimensionless local distance, y/y', from the wall is raised to a power, m, that cannot be defined by standard dimensional considerations. Judging from the physical implication of Eq. 3, namely, $\partial(u/u_*)/\partial(y/y') \rightarrow 0$ as $y/y' \rightarrow \infty$, in order for Eq. 3 to be truly representative of the velocity gradient at the center of turbulent flows confined within pipes or channels, one may conclude that the m value should remain in a range between 0 and 1, exclusively. It will be shown later that the exponent, m, and hence the associated coefficient, a, in Eq. 3 vary with the global Reynolds number for hydraulically smooth flows or with the global relative roughness for fully rough flows. How to determine the values of m and a thus constitutes a major task in the present study.

Further integration of Eqs. 2 and 3 with respect to y/y' yields the logarithmic and power velocity distributions, respectively. However, a separate boundary condition at the location where $u = 0$ is needed in the integration of Eq. 2 or 3, because y', previously defined as the physical location in the boundary layer at which $u = 0$, can hold true in the integration of Eq. 2, but not in the integration of Eq. 3. Accordingly, y' in Eq. 3 will be simply regarded as a reference length. Eqs. 2 and 3, upon integration with the help of the corresponding boundary conditions, namely $u/u_* = 0$ at $y/y' = 1$ for $m = 0$ and $u/u_* = 0$ at $y/y' = 0$ for $m \neq 0$, become, respectively

$$\frac{u}{u_*} = \frac{1}{\kappa} \ln \left(\frac{y}{y'}\right) \quad (4)$$

and

$$\frac{u}{u_*} = a\left(\frac{y}{y'}\right)^m \quad (5)$$

The logarithmic velocity distribution (Eq. 4) is well defined because the von Kármán universal constant, κ, has been extensively investigated and its value experimentally determined to be about 0.4 for pipe flows. In contrast to Eq. 4, the power velocity distribution (Eq. 5) is indeterminate because the exponent, m, and the coefficient, a, vary either with the global Reynolds number for hydraulically smooth flows or with the global relative roughness for fully rough flows.

Despite an apparent difference in the expressions of the velocity distribution between the logarithmic law (Eq. 4) and the power law (Eq. 5), it is further assumed herein that the logarithmic velocity distribution can be approximated by the power velocity distribution in the overlapping region. Making this assumption essentially leads to the plausible notion of one-to-one correspondence between the two laws, thereby establishing the parametric representation of both laws.

Parameterization of Logarithmic and Power Velocity Distributions

Equating the right-hand sides of the equal sign in Eqs. 2 and 3 and recalling that $C_1 = 1/\kappa$ yields

$$\left(\frac{y}{y'}\right)^m = \frac{1}{\kappa m a} \quad (6)$$

Likewise, equating the right-hand sides of the equal sign in Eqs. 4 and 5, and then incorporating the resultant expression with Eq. 6 yields

$$\ln\left(\frac{y}{y'}\right) = \frac{1}{m} \quad (7)$$

Next, Eq. 4 upon substitution of Eq. 7 produces the same result as Eq. 5 upon substitution of Eq. 6, namely

$$\frac{u}{u_*} = \frac{1}{\kappa m} \quad (8)$$

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Because an alternative expression of Eq. 7 is

$$\left(\frac{y}{y'}\right)^m = e \quad (9)$$

in which e (= 2.71828 ...) is the Napierian base, combining Eqs. 6 and 9 yields a remarkably simple relation among κ, m, a and e:

$$\kappa mae = 1 \quad (10)$$

Therefore, satisfying Eq. 10 is prerequisite to the perfect agreement of the power velocity distribution to the logarithmic velocity distribution on the basis of one-to-one correspondence. For further simplification, Eq. 10 upon substitution of $\kappa = 0.4$ and $e = 2.71828$ may reduce to

$$ma = 0.920 \quad (11)$$

which indicates that m and a are inversely proportional to each other.

Given the exponent, m, the exact value of the coefficient, a, of the power law (Eq. 5) corresponding to a unique point $(u/u_*, y/y')$ in the domain of the logarithmic law (Eq. 4) can readily be determined from Eq. 10 or 11. The value of a so determined is found to be compatible with that obtained from the nonlinear regression analysis (using Eq. 5 as the regression equation of Eq. 4) in which the valid range of u/u_* or y/y' is yet to be specified.

Least-Squares Estimates of Parameters in Power Law

The best-estimated values of m and a in the power law (Eq. 5), which for a certain range of u/u_* (or y/y') values satisfy in a least-squares sense the logarithmic law (Eq. 4) within an acceptable tolerance, can be found from the nonlinear regression analysis (Chen 1988, 1989). There are two ways in which one obtains the least-squares-estimated exponent, $\hat{m}$, and coefficient, $\hat{a}$, from the extended least-squares regression technique: One way is to specify the applicable range of u/u_* (or y/y') values prior to the determination of $\hat{m}$ and $\hat{a}$, and the other way is to find $\hat{a}$ and either the upper or lower limit of u/u_* (or y/y') corresponding to the specified mth power and other limit of u/u_* (or y/y'). The latter approach seems more useful in engineering practice than the former one and is thus elaborated next.

Given an mth power, the upper limit of u/u_* can be judiciously selected and its lower limit determined numerically to the desired accuracy (i.e., a specified number of significant digits). Although the valid range of u/u_* (or y/y') may shift slightly depending upon the arbitrarily selected upper limit of u/u_* (or y/y'), it is found that the values of $\hat{a}$ so determined do not differ from each other and in fact, approach the theoretical a value obtained from Eq. 10 or 11 as the range of u/u_* (or y/y') approaches zero. Chen (1989) already showed the asymptotic behavior of $\hat{a}$ with diminishing range of the relative roughness for $m = 1/6$. Therefore, using this least-squares method, one can derive an infinite number of power velocity distributions for $0 < m < 1$ with the respective range of u/u_* (or y/y') to approximate the logarithmic velocity distribution (Eq. 4). For example, listed in Table 1 are $\hat{a}$ values so computed for $m = 1, 1/2, 1/3, \ldots, 1/12$ and corresponding valid ranges of u/u_* (or y/y'). For illustration, the power velocity distributions (Eq. 5) for these m values are plotted on semilogarithmic paper and compared with a straight line representing the logarithmic velocity distribution (Eq. 4), as shown in Fig. 1.

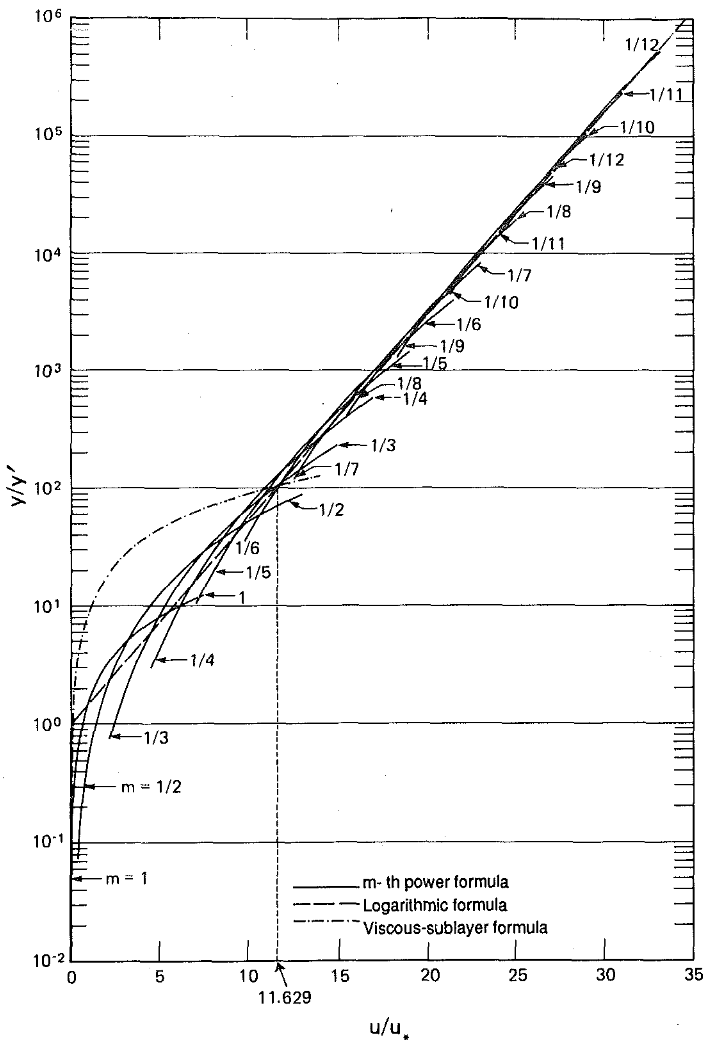

FIG. 1. Dimensionless Logarithmic and Power Velocity Distributions of Turbulent Shear Flow Near Smooth or Rough Wall Where Viscous Sublayer May or May Not Exist

An inspection of Table 1 and Fig. 1 reveals the invalidity of the power velocity distribution (Eq. 5) for $m = 1$, as mentioned earlier on the physical ground. Another power formula resembling Eq. 5 for $m = 1$ has been shown by many researchers, [see, e.g., Rouse (1937)] to be valid in the viscous sublayer, but its formulation and implication are completely different from those for the power velocity distribution (Eq. 5) of a turbulent shear flow. The velocity profile in the viscous sublayer, which is linear because of the overwhelming effects of the fluid viscosity, may be expressed as

TABLE 1. Applicable Ranges of u/u_* (or y/y') Corresponding to Arbitrarily Selected Upper Limit for Various Exponents, m, and Least-Squares-Estimated Coefficient, a, of Power Velocity Distribution

m (1)	a (2)	Lower limit of u/u_* (3)	Lower limit of y/y' (4)	Upper limit of u/u_* (5)	Upper limit of y/y' (6)	a/γ^m (7)	a/β^m (8)	Wieghardt's $C(m)$ (9)
1	0.616	10^{-10}	0	7.5	12.2	18.5	—	—
1/2	1.40	0.379	0.0737	13.0	86.8	7.64	—	—
1/3	2.44	2.23	0.759	15.0	232.0	7.58	—	—
1/4	3.45	4.56	3.07	17.0	591.0	8.07	—	—
1/5	4.43	7.13	10.9	19.0	1,450.0	8.74	—	—
1/6	5.39	9.83	36.8	21.0	3,490.0	9.50	—	—
1/7	6.34	12.6	123.0	23.0	8,230.0	10.3	8.69	8.74
1/8	7.29	15.5	409.0	25.0	19,200.0	11.2	9.59	9.71
1/9	8.23	18.3	1,360.0	27.0	44,200.0	12.0	10.5	10.6
1/10	9.16	21.2	4,500.0	29.0	101,000.0	12.9	11.4	11.5
1/11	10.1	24.2	14,900.0	31.0	230,000.0	13.8	12.3	—
1/12	11.0	27.1	49,300.0	33.0	521,000.0	14.6	13.2	—

Note: Wieghardt's $C(m)$ in $u/u_* = C(m)(yu_*/\nu)^m$ corresponds to a/β^m.

$$\frac{u}{u_*} = \beta\left(\frac{y}{y'}\right) = y^* \quad \text{(12)}$$

which appears as the first power ($m = 1$) of the power velocity distribution (Eq. 5 with $a = \beta$) formulated on the self-similiarity assumption of turbulent shear flows, but in fact is not a power law. Therefore, the velocity distribution plotted for Eq. 12, as shown in Fig. 1, deviates considerably from that for Eq. 4 or 5.

The explicit expressions of the power velocity distribution (Eq. 5) for hydraulically smooth flows and fully rough flows can now be obtained separately by substituting $y' = \beta\nu/u_*$ and $y' = \gamma k$, respectively, into Eq. 5. Therefore, for hydraulically smooth flows, Eq. 5 becomes

$$\frac{u}{u_*} = \frac{a}{\beta^m}\left(\frac{u_* y}{\nu}\right)^m \quad \text{(13)}$$

and for fully rough flows

$$\frac{u}{u_*} = \frac{a}{\gamma^m}\left(\frac{y}{k}\right)^m \quad \text{(14)}$$

The calculated values of a/γ^m for fully rough flows from $m = 1$ through $m = 1/12$ based on $\gamma \approx 0.0333$ and least-squares-estimated a values are given in Table 1. An inspection of the table reveals the unusually high value of a/γ^m for $m = 1$. This thus reconfirms the invalidity of Eq. 5 or 14 for $m = 1$. Likewise, the theoretical a/β^m values for hydraulically smooth flows based on $\beta \approx 0.111$ can be determined, but only for $u/u_* \gtrsim 11.6$ (or $y/y' \gtrsim 105$), which apparently corresponds to those for $m = 1/7$ or smaller in

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Table 1. In other words, one cannot use a power velocity distribution with an m value greater than $1/7$ for hydraulically smooth flows where the viscous sublayer exists. As a matter of fact, the close agreement between the a/β^m values so obtained and the corresponding $C(m)$ values of Wieghardt (1946) [see also Schlichting (1968)] for $m = 1/7$, $1/8$, $1/9$, and $1/10$, as shown in Table 1, strongly supports the present estimation of the $\hat{m}$ and $\hat{a}$ values.

The role that m plays in the formulation of the power velocity distribution (Eq. 5) manifests itself in Fig. 1. It displays that the larger the m value (namely, the smaller the depth), the greater is the deviation of the power law from the logarithmic law for a specified range of velocity (or depth) and vice versa. Consequently, as $m \to 0$, both laws virtually collapse at the possibly largest value of u/u_* (or y/y') and the corresponding upper and lower limits of its valid range almost become indistinguishable. In practice, therefore, the m value should be selected in such a way that it is large enough to be practical, yet small enough to be valid, especially on smooth walls where the viscous sublayer is present. To meet this criterion in the selection of the m value, it appears that the one-seventh power and one-sixth power velocity distributions are the best choice for hydraulically smooth flows and fully rough flows, respectively. Nevertheless, no matter how justifiable this criterion may be, there is no reason why other m values cannot be selected as long as the respective ranges of validity for Eq. 5 are accurately specified.

Formulation of Flow Resistance

Integrating Eqs. 4 and 5 over a section of arbitrary geometric shape is a formidable task because u_* may vary around the wall. If u_* is constant, such as in a circular pipe flowing full and in a wide channel, the integration of Eqs. 4 and 5 is relatively easy. After integration, the ratios of the cross-section-averaged velocity, V, to the shear velocity, u_*, for flow in circular pipes flowing full and wide open channels are presented in Table 2 for comparison.

As customarily defined through the Darcy-Weisbach equation (Rouse 1937), $\tau_0 = (f/4)\rho V^2/2$, in which f is the resistance coefficient signifying the ratio of energy loss to the net flux of mean flow kinetic energy, the differences in the logarithmic and power velocity distributions (Eqs. 4 and 5) can be readily shown through the various expressions of f in terms of the same similarity variables as used in V/u. Because $V/u_* = \sqrt{8/f}$, one can express f using the expression of V/u_* for each case, as shown in Table 2.

For flow in a conduit of arbitrary cross-sectional geometry, it can be treated in an approximate way, in principle, by bisecting every wall corner in a polygonal channel and integrating Eq. 4 or 5 over divided sections with separate walls, one at a time, as done by Keulegan (1938, 1946) and Powell (1946a, 1946b). Because a further assumption on the u_* distribution along the wetted perimeter of the polygonal channel is required in their approach, it is not very much different from an empirical method in which shape factors are simply introduced into the original expression of V for wide channels, thereby taking into account the effect of the u_* distribution on the V expression. This empirical approach, as adopted by Chen (1989), is somewhat similar to that of Wunderlich (1970). For practical purposes, this em-

TABLE 2. Comparison of Corresponding Power Formulas to Logarithmic Formulas for Circular Pipes Flowing Full and Wide Open Channels

Types of flow (1)	Logarithmic law (2)	Power law (3)
	(a) Circular Pipes Flowing Full	
Both hydraulically smooth flows and fully rough flows	$V/u_* = (1/\kappa)[\ln(r_0/y') - 3/2]$	$V/u_* = [2a/(m+1)(m+2)](r_0/y')^m$
Hydraulically smooth flows	$V/u_* = 5.75 \log(u_* r_0/\nu) + 1.75$	$V/u_* = [2a/(m+1)(m+2)]^{1/(m+1)}(\mathsf{R}_D/2\beta)^{m/(m+1)}$
Fully rough flows	$V/u_* = 5.75 \log(r_0/k) + 4.75$	$V/u_* = [2a/(m+1)(m+2)](r_0/\gamma k)^m$
Hydraulically smooth flows	$1/\sqrt{f} = 2.03 \log(\mathsf{R}_D\sqrt{f}) - 0.912$	$1/\sqrt{f} = [a/(m+1)(m+1)]^{1/(m+2)}(\mathsf{R}_D/\beta)^{m/(m+1)}/2^{(5m+1)/2(m+1)}$
Fully rough flows	$1/\sqrt{f} = 2.03 \log(r_0/k) + 1.68$	$1/\sqrt{f} = (1/\sqrt{2})[a/(m+1)(m+2)](r_0/\gamma k)^m$
	(b) Wide Open Channels	
Both hydraulically smooth flows and fully rough flows	$V/u_* = (1/\kappa)[\ln(h/y') - 1]$	$V/u_* = [a/(m+1)](h/y')^m$
Hydraulically smooth flows	$V/u_* = 5.75 \log(u_* h/\nu) + 3.0$	$V/u_* = [a/(m+1)]^{1/(m+1)}(\mathsf{R}_h/\beta)^{m/(m+1)}$
Fully rough flows	$V/u_* = 5.75 \log(h/k) + 6.0$	$V/u_* = [a/(m+1)](h/\gamma k)^m$
Hydraulically smooth flows	$1/\sqrt{f} = 2.03 \log(\mathsf{R}_h\sqrt{f}) + 0.140$	$1/\sqrt{f} = (1/\sqrt{8})[a/(m+1)]^{1/(m+1)}(\mathsf{R}_h/\beta)^{m/(m+1)}$
Fully rough flows	$1/\sqrt{f} = 2.03 \log(h/k) + 2.12$	$1/\sqrt{f} = (1/\sqrt{8})[a/(m+1)](h/\gamma k)^m$

Note: r_0 = radius of pipe; h = flow depth in open channels; $y' = \beta\nu/u_*$ for hydraulically smooth flows, $\beta \approx 0.111$; $y' = \gamma k$ for fully rough flows, $\gamma \approx 0.0333$; $\mathsf{R}_D = VD/\nu$, Reynolds number based on pipe diameter, D; and $\mathsf{R}_h = Vh/\nu$, Reynolds number based on flow depth, h.

pirical method is further refined and then applied only to the case of flow in open channels.

General Power Formulas for Uniform Turbulent Shear Flows

Consider a prismatic open channel with a section of arbitrary geometric shape with flow depth, $h(z)$, and boundary shear velocity, $u_*(z)$, varying in the transverse direction (z-coordinate) of flow from $z = 0$ to $z = T$ (i.e., the top width of the channel). Assuming that at any point in the transverse direction the expression of the depth-averaged velocity, $\bar{u}(z)$, for flow in wide open channels is valid, one may integrate it with respect to z to express the following cross-section-averaged velocity, V, for power flows in a channel with a section of arbitrary geometric shape

$$V = \frac{1}{A}\int_0^T \frac{au_*(z)}{(m+1)}\left[\frac{h(z)}{y'}\right]^m h(z)dz \qquad (15)$$

in which A = cross-sectional area of flow. Because $u_*(z)$ and $k(z)$ are contained in the expression of y' for hydraulically smooth flows and fully rough flows, respectively, the integration of Eq. 15 should be performed separately for both types of flow as follows: For hydraulically smooth flows [$y' = \beta\nu/u_*(z)$], one obtains

$$\frac{V}{\bar{u}_*} = \frac{a}{(m+1)}\left(\frac{\xi_s R}{\bar{y}'}\right)^m\left[\frac{\xi_s R}{A}\int_0^T\left(\frac{u_*(z)}{\bar{u}_*}\frac{h(z)}{\xi_s R}\right)^{m+1}dz\right] \qquad (16)$$

and for fully rough flows [$y' = \gamma k(z)$]

$$\frac{V}{\bar{u}_*} = \frac{a}{(m+1)}\left(\frac{\xi_r R}{\bar{y}'}\right)^m\left[\frac{\xi_r R}{A}\int_0^T\left(\frac{u_*(z)}{\bar{u}_*}\right)\left(\frac{\bar{k}}{k(z)}\right)^m\left(\frac{h(z)}{\xi_r R}\right)^{m+1}dz\right] \qquad (17)$$

in which $\bar{u}_*$ = average boundary shear velocity around wetted perimeter of channel, $\bar{k}$ = average roughness height around wetted perimeter of channel, R = hydraulic radius of flow, and $\bar{y}'$ = average length scale of turbulence, defined respectively as

$$\bar{u}_* = \frac{1}{P}\int_0^P u_*(z)dz \qquad (18)$$

$$\bar{k} = \frac{1}{P}\int_0^P k(z)dz \qquad (19)$$

$$R = \frac{1}{P}\int_0^T h(z)dz = \frac{A}{P} \qquad (20)$$

$\bar{y}' = \beta\nu/\bar{u}_*$ and $\gamma\bar{k}$ for hydraulically smooth flows and fully rough flows, respectively. Note that P in Eqs. 18, 19, and 20 is the wetted perimeter.

For simplification in notation, let the whole expressions in the brackets of Eqs. 16 and 17 be designated as $\zeta_s(m)$ and $\zeta_r(m)$, shape factors for hydraulically smooth flows and fully rough flows, respectively. Other shape factors, ξ_s and ξ_r, associated with the hydraulic radius, R, in Eqs. 16 and 17, respectively, denote the overall cross-sectional geometry effects when the depth-averaged logarithmic formula is similarly integrated over z. In this context, $\xi_s R$ and $\xi_r R$ may be called the effective hydraulic radii for hydraul-

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ically smooth flows and fully rough flows, respectively. Definitions of ξ_s and ξ_r will be further discussed later. Using these simplifying symbols, Eqs. 16 and 17 can then be expressed respectively as: for hydraulically smooth flows

$$\frac{V}{\bar{u}_*} = \frac{a\zeta_s(m)}{(m+1)}\left(\frac{\xi_s R\bar{u}_*}{\beta\nu}\right)^m \quad (21)$$

and for fully rough flows

$$\frac{V}{\bar{u}_*} = \frac{a\zeta_x(m)}{(m+1)}\left(\frac{\xi_r R}{\gamma\bar{k}}\right)^m \quad (22)$$

Recalling that $\bar{u}_* = \sqrt{gRS}$, in which g = gravitational acceleration and S = hydraulic gradient (i.e., piezometric slope or in the case of uniform flow in open channels, bed slope), one can thus obtain from Eq. 21 for hydraulically smooth flows

$$V = \frac{a\zeta_s(m)}{(m+1)}\left(\frac{\xi_s}{\nu\beta}\right)^m R^{(3m+1)/2}(gS)^{(m+1)/2} \quad (23)$$

and from Eq. 22 for fully rough flows

$$V = \frac{a\zeta_r(m)}{(m+1)}\left(\frac{\xi_r}{\gamma\bar{k}}\right)^m R^{(2m+1)/2}(gS)^{1/2} \quad (24)$$

The general power expression of the Darcy-Weisbach friction coefficient, f, for flow in a channel with a section of arbitrary geometric shape can be similarly derived from Eqs. 21 and 22 upon substitution of the Darcy-Weisbach relation, $V/\bar{u}_* = \sqrt{8/f}$. Therefore, for hydraulically smooth flows, one obtains

$$f = 8\left[\frac{(m+1)}{a\zeta_s(m)}\right]^{2/(m+1)}\left(\frac{\beta}{\mathsf{R}}\right)^{2m/(m+1)} \quad (25)$$

in which $\mathsf{R} = V\xi_s R/\nu$ = global Reynolds number based on the cross-section-averaged velocity, V, and the hydraulic radius, R, and for fully rough flows

$$f = 8\left[\frac{(m+1)}{a\zeta_r(m)}\right]^2\left(\frac{\gamma\bar{k}}{\xi_r R}\right)^{2m} \quad (26)$$

in which $\xi_r R/\bar{k}$ = global relative roughness based on R and $\bar{k}$. The generality of Eqs. 23 and 24 can be examined in the light of many existing uniform-flow formulas, such as Manning's, Lacey's, Blasius', and Hazen-Williams'.

Reappraisal of Existing Power Formulas

The uniform-flow formulas, as expressed in the form of Eqs. 23 and 24, are very general in that they are independent of units adopted. However, if both formulas are rearranged into the simple forms of a Manning (1891, 1895) type, in which V is proportional to R^pS^q with the constant of proportionality (or reciprocal thereof) signifying the resistance coefficient, such as Manning's n, then the Manning-type formulas become unit-dependent and thus, need to be multiplied by conversion factors when the units adopted in

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the formulas are changed (Chen 1989). Note that the exponents, p and q, of R and S in the Manning-type formula are readily identified from Eq. 23 as $(3m + 1)/2$ and $(m + 1)/2$, respectively, and those from Eq. 24 as $(2m + 1)/2$ and $1/2$, respectively. By implication, therefore, the constant of proportionality (or reciprocal thereof) that lumps all the quantities describing channel and fluid properties other than R and S terms in Eq. 23 or 24 may be called the resistance coefficient.

Many existing flow formulas can be shown to be special cases of the mth power formulas (Eqs. 23 and 24) with different powers, as reported elsewhere. Bretting (1948) also derived two sets of flow formulas with two and three different powers for hydraulically smooth flows and fully rough flows, respectively. Because his rational power formulas can readily be reduced from Eqs. 23 and 24, further elaboration on his work seems unnecessary. Only significant findings from the most recent study of Chen (1989) are recapitulated as follows. Among empirical formulas still in use, Lacey's formula appears to be the one with the highest power. Lacey (1930a, 1930b, 1935, 1946) apparently proposed the use of the one-fourth power formula for alluvial channels in regime, and so did many investigators for gravel-bed rivers (Kellerhals 1967; Bray 1979, 1982; Griffiths 1981; Bray and Davar 1987). Lacey's one-fourth power formula of a Manning type is thus equivalent to Eq. 24 on substitution of $m = 1/4$ and likewise the expression of the Darcy-Weisbach f obtained statistically by Bray and Davar (1987) can be shown in close agreement with Eq. 26 upon substitution of $m = 1/4$. The advantage of using the one-fourth power formula that appears to fit flow resistance in alluvial channels and gravel-bed rivers better than the one-sixth power formula (i.e., Manning's formula) may in fact lie in the smaller order of magnitude of the applicable similarity variable (namely, the global relative roughness, $\xi_r R/k$, for fully rough flows) for the one-fourth power formula than for the one-sixth power formula.

One of the most widely used flow-resistance formulas is, of course, the Manning (or one-sixth power) formula. It has been shown by Chen (1989) that the one-sixth power formula (Eq. 24 with $m = 1/6$) applies to virtually all possible flows, some of which were not originally intended. Chen (1989) also found that for the widest range of R/k tested (approximately between 1 and 1,000) for wide channels, Manning's formula expressed in a unit-independent form [namely, $V = C(g^{1/2}/\bar{k}^{1/6})R^{2/3}S^{1/2}$, in which $C \approx 7.84$ for $a \approx 5.19$, as shown in Table 4 of Chen (1989), is a dimensionless constant] fits the logarithmic formula within 5% error. A further analysis of the previous results for wide channels reveals that the lower and upper limits of R/k for Manning's formula are of one order-of-magnitude larger than those for Lacey's formula, which is applicable for $0.1 < R/k < 100$. The widest, yet most practical, range of applicable R/k may explain why Manning's formula has received much recognition as the most useful of all existing power formulas for fully rough flows.

For hydraulically smooth flows, the highest power without invalidating Eq. 23 or 25 appears to be $1/7$ and the corresponding flow-resistance formula for hydraulically smooth flows (namely, Eq. 25 with $m = 1/7$) has been often referred to as the Blasius (1913) formula. The range of applicability in Eq. 23 or 25 with $m = 1/7$ for flow in wide channels was found to be $469 \leq \mathsf{R} \leq 57{,}100$ and that for flow in circular pipes flowing full $1{,}550 \leq \mathsf{R} \leq 188{,}000$ (Chen 1989). Therefore, the Blasius (or one-seventh

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power) formula is probably the most widely used power formula for hydraulically smooth flows in the most practical range of R. Note that Bretting (1948), using only the one-seventh and one-eleventh power formulas to fit the logarithmic formula, also drew similar conclusions.

The smallest exponent, m, of all existing power-law formulas still in use appears to be $1/12$, developed by Williams and Hazen (1933). The one-twelfth power formula for hydraulically smooth flows has been widely known as the Hazen-Williams formula in the United States and the Ludin formula (Jaeger 1957) in Europe. Because the one-twelfth power formula (i.e., Eq. 23 with $m = 1/12$) is applicable only in a relatively small range of high R, as shown in a previous study (Chen 1989), at issue is the wide claim of general applicability of the Hazen-Williams formula in water-supply engineering. Reassessment of its validity should thus be in order. It was also shown in a previous study (Chen 1989) that the so-called modified Hazen-Williams formula proposed by Jain et al. (1978) is essentially a shift in the power from $m = 1/12$ to $m = 0.105$ (or approximately $1/9.524$). Mainly, because the power of the modified Hazen-Williams formula is close to $1/9$ at which Manning's n expressed from Eq. 22 or 24 does not vary with R (Chen 1989), suffice it to say that the modified Hazen-Williams formula with $m = 0.105$ is indeed an improvement over the original Hazen-Williams formula with $m = 1/12$.

Incomplete Self-Similarity of Power Flow Formulas

The cross-section-averaged velocity, V, for logarithmic flows in a channel with a section of arbitrary geometric shape can be similarly expressed by following the same procedure as used in the derivation of Eqs. 21 and 22 for power flows. Without going into details of the derivation, the counterparts of Eqs. 21 and 22 for logarithmic flows are simply given next: For hydraulically smooth flows, one obtains from Eq.4

$$\frac{V}{\bar{u}_*} = \frac{1}{\kappa}\left[\ln\left(\frac{\xi_s R\bar{u}_*}{\beta\nu}\right) - \eta\right] \qquad (27)$$

and for fully rough flows

$$\frac{V}{\bar{u}_*} = \frac{1}{\kappa}\left[\ln\left(\frac{\xi_r R}{\gamma\bar{k}}\right) - \eta\right] \qquad (28)$$

in which ξ_s and ξ_r, the shape factors, as already defined implicitly via Eqs. 16 and 17, respectively, and η, another shape factor, are expressed as

$$\xi_s = \frac{\beta\nu}{\bar{u}_*R}\exp\left\{\frac{1}{A}\int_0^T \frac{u_*(z)}{\bar{u}_*}h(z)\ln\left[\frac{u_*(z)h(z)}{\beta\nu}\right]dz\right\} \qquad (29)$$

$$\xi_r = \frac{\gamma\bar{k}}{R}\exp\left\{\frac{1}{A}\int_0^T \frac{u_*(z)}{\bar{u}_*}h(z)\ln\left[\frac{h(z)}{\gamma k(z)}\right]dz\right\} \qquad (30)$$

$$\eta = \frac{1}{A}\int_0^T \left[\frac{u_*(z)}{\bar{u}_*}\right]h(z)dz \qquad (31)$$

To show that both m and a in the power velocity distribution (Eq. 5) vary with the global Reynolds number, R $(= V\xi_s R/\nu)$, and the global relative roughness, $\xi_r R/k$, for hydraulically smooth flows and fully rough flows,

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respectively, one must find a way to relate m and a with the Darcy-Weisbach f or Chézy C. Following the same procedure as used in a previous study (Chen 1989), then treating separately both cases of hyraulically smooth flows and fully rough flows by incorporating Eq. 21 with Eq. 27 (or Eq. 22 with Eq. 28), and finally combining and rearranging the resultant expression with their differentiations, one can obtain the following relations

$$m = \frac{1}{\ln\left(\frac{\xi_s R}{e^{\eta}\bar{y}'}\right)} = \frac{1}{\ln\left(\frac{\xi_r R}{e^{\eta}\bar{y}'}\right)} \quad \text{(32)}$$

$$a = \frac{(m+1)e^{-(1+\eta m)}}{\kappa m \zeta_s(m)} = \frac{(m+1)e^{-(1+\eta m)}}{\kappa m \zeta_r(m)} \quad \text{(33)}$$

in which $\bar{y}' = \beta\nu/\bar{u}_*$ and $\gamma\bar{k}$ for hydraulically smooth flows and fully rough flows, respectively, as defined earlier. Accordingly, Eq. 21 or 22 upon substitution of a from Eq. 33 and with the help of Eq. 32 reduces to

$$\frac{V}{\bar{u}_*} = \frac{1}{\kappa m} \quad \text{(34)}$$

which can also result from Eq. 27 or 28 upon substitution of Eq. 32.

Recalling again $V/\bar{u}_* = \sqrt{8/f}$, one can readily derive from Eq. 34 a theoretical relation between the exponent, m, and the Darcy-Weisbach f:

$$m = \frac{1}{\kappa}\sqrt{\frac{f}{8}} \quad \text{(35)}$$

which is identical to that obtained or used by Willis (1969), Hsu and Kennedy (1971), Zimmermann and Kennedy (1978), Odgaard (1986), Karim and Kennedy (1987), and Woo et al. (1988), among others. Because f varies with the Reynolds number, R, as shown in Eq. 25 for hydraulically smooth flows or with the relative roughness, $\xi_r R/\bar{k}$, as shown in Eq. 26 for fully rough flows, the dependence of m on the very same parameters is self-evident from Eq. 35.

To show further that m is dependent on R or $\xi_r R/\bar{k}$, Eqs. 25 and 26 on substitution of the f and a expressions from Eqs. 35 and 33, respectively, yields (Chen 1989)

$$\mathsf{R} = \frac{\beta e^{(1+\eta m)/m}}{\kappa m} \quad \text{(36)}$$

and

$$\frac{\xi_r R}{\bar{k}} = \gamma e^{(1+\eta m)/m} \quad \text{(37)}$$

for hydraulically smooth flows and fully rough flows, respectively. Therefore, for a given or measured value of R or $\xi_r R/\bar{k}$, the corresponding value of m can be determined numerically from Eq. 36 or 37. This in fact shows the incomplete self-similarity of the power flow formulas (Eqs. 23 and 24), in which m and a are dependent on R or $\xi_r R/\bar{k}$.

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The parametric equations of the logarithmic law, regardless of whether they are expressed in terms of the uniform-flow formulas (Eqs. 32 and 34) or the velocity-distribution formulas (Eqs. 7 and 8), can completely provide a parametric representation of the unique relation between the two similarity parameters in the logarithmic law. In comparison, the parametric equations of the power law (namely, Eqs. 33 and 34 in case of parameterizing the uniform-flow formulas or Eqs. 6 and 8 in case of parameterizing the velocity-distribution formula), cannot completely provide a parametric representation of the unique relation between the two similarity parameters in the power law unless the other real parameter a is also specified to link one similarity parameter to the other. Failure of the parametric equations of the power law to reestablish completely the original power law may in effect demonstrate the incomplete self-similarity of the power law.

Conclusions

It has been shown that the universal logarithmic velocity distribution can be obtained based on the assumption of complete self-similarity in both the local and global Reynolds number (or relative roughness), whereas the power velocity distribution is somehow vaguely formulated based on the assumption of incomplete self-similarity in the local Reynolds number. The power law is indeterminate because the exponent, m, and the associated coefficient, a, of the power formula are unknown and in fact constitute part of the solution in the process of developing the power law. Further assumption that the logarithmic velocity distribution can be approximated by the power velocity distribution on the basis of one-to-one correspondence has contributed toward the parametric representations of both laws. Self-similarities of both laws are self-evident from the respective parametric equations of both laws, in which m acts as the real parameter. As m varies continuously from unity (exclusive) to zero (exclusive), the parametric equations of the logarithmic law can completely provide a parametric representation of the unique relation between the two similarity parameters, but the parametric equations of the power law cannot completely provide the same unless the other real parameter a is also specified to link one similarity parameter to the other. With the additional help of a theoretical relation between m and the Darcy-Weisbach f, it has been shown that m varies with the global Reynolds number for hydraulically smooth flows or with the global relative roughness for fully rough flow.

Formulation of the two general mth power flow formulas, one for hydraulically smooth flows and the other for fully rough flows, facilitates the reappraisal of existing uniform-flow formulas. It has been shown that some older empirical formulas, such as Lacey's, Manning's, Blasius', and Hazen-Williams', are special cases of the mth power formulas with different powers. A comparison of various power formulas has revealed that the exponent of the mth power formulas would be better selected in such a way that it is large enough to be practical, yet small enough to be valid, especially on smooth walls where the viscous sublayer is present. Meeting this criterion for selecting the exponent does in effect reestablish the general applicability of the Blasius (or one-seventh power) formula and the Manning (or one-sixth power) formula for hyraulically smooth flows and fully rough flows, respectively. Judging from the applicable range of the relative roughness,

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however, one may state that the Lacey (one-fourth power) formula appears more applicable than the Manning (one-sixth power) formula for flow in shallow rough channels, such as gravel-bed rivers and alluvial channels in regime.

Appendix I. References

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Appendix II. Notation

The following symbols are used in this paper:

A = cross-sectional area of flow;
a = coefficient of power-law formula;
$\hat{a}$ = least-squares estimate of a in power-law formula;
C_1 = integration constant;
$C(m)$ = Wieghardt's constant in power formula;
D = diameter of pipe;
e = Napierian base = 2.71828 ...;

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f = Darcy-Weisbach friction coefficient;
$F(\ )$ = unknown function denoting dimensionless velocity = u/u_*;
g = gravitational acceleration;
h = flow depth in wide channel;
k = roughness height;
$\bar{k}$ = average roughness height around wetted perimeter of channel;
k^* = dimensionless roughness height (or roughness Reynolds Number) = ku_*/ν;
m = constant; exponent of power formula;
$\hat{m}$ = least-squares estimate of m in power formula;
n = Manning's friction coefficient;
P = wetted perimeter of channel;
R = hydraulic radius;
R = global Reynolds number based on cross-section-averaged velocity and hydraulic radius = $V\xi_s R/\nu$;
R_D = Reynolds number based on cross-section-averaged velocity and pipe diameter = VD/ν;
R_h = Reynolds number based on depth-averaged velocity and flow depth = Vh/ν;
r_0 = radius of pipe;
S = hydraulic gradient or piezometric head slope;
T = top width of channel;
u = local velocity;
u_* = boundary shear velocity = $\sqrt{\tau_0/\rho}$;
$\bar{u}_*$ = average boundary shear velocity around wetted perimeter of channel = $\sqrt{\mathrm{g}RS}$;
$\bar{u}(z)$ = depth-averaged velocity for flow in wide open channels;
V = cross-section-averaged velocity;
y = normal distance from wall;
y' = distance from wall to location where $u = 0$ in case of logarithmic velocity distribution ($m = 0$) ($= \beta\nu/u_*$ for hydraulically smooth flows and γk for fully rough flows);
$\bar{y}'$ = average length scale of turbulence for flow in channel ($= \beta\nu/\bar{u}_*$ for hydraulically smooth flows and $\gamma\bar{k}$ for fully rough flows);
y^* = dimensionless vertical coordinate, perpendicular to wall = yu_*/ν (local Reynolds number based on y and u_*);
z = coordinate in the transverse direction of channel;
β = empirical constant (≈ 0.111 for hydraulically smooth flows);
γ = empirical constant (≈ 0.0333 for fully rough flows);
$\zeta_r(m)$ = shape factor for fully rough flows defined via Eq. 22;
$\zeta_s(m)$ = shape factor for hydraulically smooth flows defined via Eq. 21;
η = shape factor defined in Eq. 31;
κ = von Kármán universal constant = 0.4;
ν = kinematic viscosity of fluid;
ξ_r = shape factor for fully rough flows defined in Eq. 30;
ξ_s = shape factor for hydraulically smooth flows defined in Eq. 29;
ρ = mass density of fluid; and
τ_0 = boundary shear stress.