Prepare.

First, we need to find out the smallest steps to reach endpoint. Since the person can move to any of the eight squares adjacent to current location (e.g. the four cardinal directions and the diagonals), lets say the person is at location (0, 0), he can move to (-1, -1), (-1, 0), (0, -1), (1, -1), (-1, 1), (0, 1), (1, 0), or (1, 1). It is just like the king move in chess. So if the person wants to move from (x_1, y_1) to (x_2, y_2), the smallest steps n would be

$$n = max(|x_1 - x_2|, |y_1 - y_2|). \tag{1}$$

*Second, we need an inequality named **AM-GM inequality**,*

$$\frac{1}{m}\ a_1 a_2 a_3 \cdots a_m \geq \sqrt[m]{a_1 a_2 a_2 \cdots a_m} \tag{2}$$

Now, we can prove the two Heuristics are correct.□

Exp Heuristics.

Let n be Chebyshev distance between start point and end point,

$$H(p) = n \cdot pow(2,\ \frac{getTile(end) - getTile(start)}{n}). \tag{3}$$

Suppose we are at position A, and we need to get to position B, then position C, the total cost should be $2^{getTile(B)-getTile(A)} + 2^{getTile(C)-getTile(B)}$. Likewise, if we want to go from startpoint(tile = a_1) to endpoint(tile = a_n), the total cost would be $2^{a_2-a_1} + 2^{a_3-a_2} + \cdots + 2^{a_n-a_{n-1}}$. And suppose we follow the smallest-step-way, the step number will be n as equation (1) stated. Since if we add more steps in the mid-way, the inequality will not be changed, we only need to prove $H(p) \leq 2^{a_2-a_1} + 2^{a_3-a_2} + \cdots + 2^{a_n-a_{n-1}}$.

$$2^{a_2-a_1} + 2^{a_3-a_2} + \cdots + 2^{a_n-a_{n-1}} \geq n \cdot \sqrt[n]{2^{a_2-a_1} 2^{a_3-a_2} \cdots 2^{a_n-a_{n-1}}} \tag{4}$$

Using equation (2),

$$2^{a_2-a_1} + 2^{a_3-a_2} + \cdots + 2^{a_n-a_{n-1}} \geq n \cdot \sqrt[n]{2^{a_2-a_1+a_3-a_2+\cdots+a_n-a_{n-1}}} \tag{5}$$

$$2^{a_2-a_1} + 2^{a_3-a_2} + \cdots + 2^{a_n-a_{n-1}} \geq n \cdot \sqrt[n]{2^{a_n-a_1}} \tag{6}$$

Thus, equation (3) holds, since $F(p) = G(p) + H(p)$, F(n) will always be smaller or equal than the actual total cost. Heuristic is correct.□

Div Heuristics.

Let n be Chebyshev distance between start point and end point,

$$H(p) = \frac{n}{2} \cdot pow(\frac{getTile(start)}{getTile(end)+1}, \frac{1}{n}). \tag{7}$$

Suppose we are at position A, and we need to get to position B, then position C, the total cost should be $\frac{getTile(B)}{getTile(A)+1} + \frac{getTile(C)}{getTile(B)+1}$. Likewise, if we want to go from startpoint(tile $= a_1$) to endpoint(tile $= a_n$), the total cost would be $\frac{a_1}{a_2+1} + \frac{a_2}{a_3+1} + \cdots \frac{a_{n-1}}{a_n+1}$. And suppose we follow the smallest-step-way, the step number will be n as (1) stated. Since if we add more steps in the mid-way, the inequality will not be changed, we only need to prove $H(p) \leq \frac{a_1}{a_2+1} + \frac{a_2}{a_3+1} + \cdots \frac{a_{n-1}}{a_n+1}$.

The most difficult part is the tile can be 0, so we can look at the following case first.

$$\frac{a_1}{a_2+1} + \frac{a_2}{a_3+1} \tag{8}$$

If $a_2 = 0$, it will be greater than the case $a_2 = 1$ since

$$\frac{a_1}{0+1} + \frac{0}{a_3+1} = a_1 \geq \frac{a_1}{1+1} + \frac{1}{a_3+1} = \frac{a_1}{2} + \frac{1}{a_3+1} \tag{9}$$

This is because two adjacent tiles could not both be 0, so a_1 and a_3 must be greater or equal to 1. Thus, if any element's tile in my path is 0, we can assume it to be 1, which also holds the inequality.

Since 0s have been eliminated, the smallest number of each tile would be 1, $a_k + 1 \leq 2a_k$,

$$\frac{a_1}{a_2+1} + \frac{a_2}{a_3+1} + \cdots + \frac{a_{n-1}}{a_n+1} \geq \frac{a_1}{2a_2} + \frac{a_2}{2a_3} + \cdots + \frac{a_{n-1}}{2a_n} \tag{10}$$

$$\frac{a_1}{a_2+1} + \frac{a_2}{a_3+1} + \cdots + \frac{a_{n-1}}{a_n+1} \geq \frac{1}{2} \cdot (\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n}) \tag{11}$$

Using equation (2),

$$\frac{a_1}{a_2+1} + \frac{a_2}{a_3+1} + \cdots + \frac{a_{n-1}}{a_n+1} \geq \frac{n}{2} \sqrt[n]{\frac{a_1}{a_2}\frac{a_2}{a_3} \cdots \frac{a_{n-1}}{a_n}} \tag{12}$$

$$\frac{a_1}{a_2+1} + \frac{a_2}{a_3+1} + \cdots + \frac{a_{n-1}}{a_n+1} \geq \frac{n}{2} \sqrt[n]{\frac{a_1}{a_n}} \tag{13}$$

Because in the question a_n could be 0,

$$\frac{a_1}{a_2+1} + \frac{a_2}{a_3+1} + \cdots + \frac{a_{n-1}}{a_n+1} \geq \frac{n}{2} \sqrt[n]{\frac{a_1}{a_n+1}} \tag{14}$$

Thus, equation (7) holds, since $F(p) = G(p) + H(p)$, F(n) will always be smaller or equal than the actual total cost. Heuristic is correct.$\square$

Seed	PathCost	Uncovered	TimeTaken
1	416.0625	55624	735
2	404.34375	53073	703
3	396.15625	54780	698
4	419.453125	48609	638
5	378.875	50835	690

Table 1: AStarExp

Seed	PathCost	Uncovered	TimeTaken
1	198.4095	36322	530
2	198.3506	36088	519
3	198.4535	36299	524
4	198.2955	36387	542
5	198.6035	36314	493

Table 2: AStarDiv

Record.