INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING, VOL. 2, 151–157 (1970)

FINITE ELEMENT METHOD FOR PIEZOELECTRIC VIBRATION

HENNO ALLIK AND THOMAS J. R. HUGHES
General Dynamics, Electric Boat Division, Groton, Connecticut

SUMMARY

A finite element formulation which includes the piezoelectric or electroelastic effect is given. A strong analogy is exhibited between electric and elastic variables, and a 'stiffness' finite element method is deduced. The dynamical matrix equation of electroelasticity is formulated and found to be reducible in form to the well-known equation of structural dynamics. A tetrahedral finite element is presented, implementing the theorem for application to problems of three-dimensional electroelasticity.

INTRODUCTION

The equations of piezoelectricity are sufficiently complex to preclude closed form solution for all but the simplest cases. This is unfortunate since the piezoelectric effect plays an important role in the fields of crystal physics and transducer technology. Recently, variational principles have been derived which serve as the basis of approximate solution techniques, such as the powerful Rayleigh–Ritz method. Noteworthy contributions along these lines have been made in the papers of Lewis[1] and Holland and EerNisse.[2-4]

Although these important developments have opened the way to a wider class of problems, they are not sufficiently general in themselves to be considered a universal method of piezoelectric analysis. For instance, a significant deficiency of the Rayleigh–Ritz technique is the necessity of selecting trial functions, which often becomes intractable for complex geometries.

The present paper concerns itself with the development of a general method of electroelastic analysis by incorporating the piezoelectric effect in a finite element formulation. The theory presented is, essentially, an outgrowth of the variational principle enunciated in the papers of Holland and EerNisse, cast here in matrix fashion. The dynamical matrix equation derived for linear piezoelectricity is found to be reducible, in form, to the ordinary matrix equation encountered in structural dynamics.

The electroelastic matrices for a simplex 'displacement-potential' finite element for three-dimensional analysis are presented, illustrating the method.

Whenever possible, the notation prescribed in the I.R.E. 'Standards on piezoelectric crystals'[5] is employed.

VARIATIONAL PRINCIPLE

The variational principle which serves to incorporate the piezoelectric effect is essentially that due to Holland and EerNisse,[2-4] who arrived at it by a 'trial-and-error' procedure to yield as Euler equations known relations of electroelasticity. It is pertinent to mention that the present authors have found it possible to derive the same theorem, strictly by applying the principle of virtual displacements to a continuum under the influence of electrical and mechanical forces. The starting point for such a derivation is the definition of virtual work density

$$\delta W = \{\delta \mathbf{u}\}^T \{\mathbf{F}\} - \delta\phi\sigma \tag{1}$$

Received 25 December 1968
Revised 10 March 1969

where $\{\mathbf{u}\}$ denotes displacement, ϕ the electric potential, $\{\mathbf{F}\}$ the mechanical force density, σ the charge density and δ a virtual quantity. This equation also serves to point out the very useful analogy between electrical and mechanical variables. Hence, charge and potential may be included in the notions of generalized force and generalized displacement, respectively, which makes for a clear understanding of the nature of these quantities in a finite element formulation.

As shown in Table I, this analogous behaviour of electrical and mechanical variables may be extended to include electric flux density $\{\mathbf{D}\}$ and stress $\{\mathbf{T}\}$, and electric field $\{\mathbf{E}\}$ and mechanical strain $\{\mathbf{S}\}$, which for linear material behaviour are related through the matrix form of the constitutive equations

$$\left.\begin{aligned}\{\mathbf{T}\} &= [\mathbf{c}]\{\mathbf{S}\} - [\mathbf{e}]\{\mathbf{E}\} \\ \{\mathbf{D}\} &= [\mathbf{e}]^T\{\mathbf{S}\} + [\boldsymbol{\varepsilon}]\{\mathbf{E}\}\end{aligned}\right\} \tag{2}$$

where $[\mathbf{c}]$ denotes the elastic stiffness tensor evaluated at constant electric field, $[\mathbf{e}]$ the piezoelectric tensor and $[\boldsymbol{\varepsilon}]$ the dielectric tensor evaluated at constant mechanical strain. Note that all electrical quantities are one tensorial rank lower than the corresponding mechanical quantities.

Table I. Analogy between mechanical and electrical quantities

Mechanical	Electrical
Force density $\mathbf{F}_i$ (vector)	Charge density σ (scalar)
Displacement $\mathbf{u}_i$ (vector)	Potential ϕ (scalar)
Stress $\mathbf{T}_{ij}$ (second-order tensor)	Flux density $\mathbf{D}_i$ (vector)
Strain $\mathbf{S}_{ij}$ (second-order tensor)	Electric field $\mathbf{E}_i$ (vector)

Without further elaboration the resulting variational principle may be stated in matrix notation as

$$\iiint_V \Big\{\{\delta\mathbf{S}\}^T[\mathbf{c}]\{\mathbf{S}\} - \{\delta\mathbf{S}\}^T[\mathbf{e}]\{\mathbf{E}\} - \{\delta\mathbf{E}\}^T[\mathbf{e}]^T\{\mathbf{S}\} - \{\delta\mathbf{E}\}^T[\boldsymbol{\varepsilon}]\{\mathbf{E}\} - \{\delta\mathbf{u}\}^T\{\bar{\mathbf{F}}\} + \rho\{\delta\mathbf{u}\}^T\{\ddot{\mathbf{u}}\} + \delta\phi\bar{\sigma}\Big\}\,dV - \iint_{S_1}\{\delta\mathbf{u}\}^T\{\bar{\mathbf{T}}\}\,dS + \iint_{S_2}\delta\phi\bar{\sigma}'\,dS - \{\delta\mathbf{u}\}\{\mathbf{P}\} + \delta\phi\mathbf{Q} = 0 \tag{3}$$

where the following prescribed quantities are defined as $\{\bar{\mathbf{F}}\}$ the body force, $\{\bar{\mathbf{T}}\}$ the surface traction, $\{\mathbf{P}\}$ the point force, $\bar{\sigma}$ the body charge, $\bar{\sigma}'$ the surface charge and $\mathbf{Q}$ the point charge. In addition V denotes the volume of the body, S_1 that part of the boundary where traction is prescribed, S_2 that part of the boundary where charge is prescribed and ρ the density.

Finally, note that the variational principle can be thought of as a generalization of the elasticity principle of minimum potential energy, since deletion of the electrical quantities in equation (3) results in the latter principle's first variation.

FINITE ELEMENT FORMULATION

To generate the electroelastic matrix relations for a finite element, the continuous displacement and potential are expressed in terms of i nodal values via interpolation functions $\mathbf{f}_u$ and $\mathbf{f}_\phi$

$$\left.\begin{aligned}\{\mathbf{u}\} &= [\mathbf{f}_u]\{\mathbf{u}_i\} \\ \phi &= \langle\mathbf{f}_\phi\rangle\{\phi_i\}\end{aligned}\right\} \tag{4}$$

It is assumed here that the interpolation functions possess the requisite properties for convergence to the correct solution with diminishing element size.[6] In a similar manner, the prescribed body and surface force (charge) distributions are expressed through interpolation functions and nodal values as

$$\left.\begin{aligned} \{\bar{\mathbf{F}}\} &= [\mathbf{f}_{\bar{F}}]\{\bar{\mathbf{F}}_i\} \\ \{\bar{\mathbf{T}}\} &= [\mathbf{f}_{\bar{T}}]\{\bar{\mathbf{T}}_i\} \\ \bar{\sigma} &= \langle \mathbf{f}_{\bar{\sigma}} \rangle \{\bar{\sigma}_i\} \\ \bar{\sigma}' &= \langle \mathbf{f}_{\bar{\sigma}'} \rangle \{\bar{\sigma}'_i\} \end{aligned}\right\} \tag{5}$$

Differentiating equations (4) yields expressions for the strains and electric field (negative potential gradient)

$$\left.\begin{aligned} \{\mathbf{S}\} &= [\mathbf{B}_u]\{\mathbf{u}_i\} \\ \{\mathbf{E}\} &= -[\mathbf{B}_\phi]\{\phi_i\} \end{aligned}\right\} \tag{6}$$

Substitution of the relations (4), (5) and (6) into equation (3) results in, for all values of the virtual displacements, the two equilibrium equations

$$\begin{aligned} [\mathbf{m}]\{\ddot{\mathbf{u}}_i\} + [\mathbf{k}_{uu}]\{\mathbf{u}_i\} + [\mathbf{k}_{u\phi}]\{\phi_i\} &= \{\mathbf{F}_B\} + \{\mathbf{F}_S\} + \{\mathbf{F}_P\} \\ [\mathbf{k}_{\phi u}]\{\mathbf{u}_i\} + [\mathbf{k}_{\phi\phi}]\{\phi_i\} &= \{\mathbf{Q}_B\} + \{\mathbf{Q}_S\} + \{\mathbf{Q}_P\} \end{aligned} \tag{7}$$

where the terms in equation (7) are defined in Table II.

Table II. Definition of electroelastic matrices and vectors

$[\mathbf{k}_{uu}] = \iiint_V [\mathbf{B}_u]^T [\mathbf{c}] [\mathbf{B}_u] \, dV$	Stiffness matrix
$[\mathbf{k}_{u\phi}] = \iiint_V [\mathbf{B}_u]^T [\mathbf{e}] [\mathbf{B}_\phi] \, dV$	Piezoelectric 'stiffness' matrix
$[\mathbf{k}_{\phi u}] = \iiint_V [\mathbf{B}_\phi]^T [\mathbf{e}]^T [\mathbf{B}_u] \, dV$	
$[\mathbf{k}_{\phi\phi}] = -\iiint_V [\mathbf{B}_\phi]^T [\mathbf{e}] [\mathbf{B}_\phi] \, dV$	Dielectric 'stiffness' matrix
$[\mathbf{m}] = \rho \iiint_V [\mathbf{f}_u]^T [\mathbf{f}_u] \, dV$	Kinematically consistent mass matrix
$\{\mathbf{F}_B\} = \iiint_V [\mathbf{f}_u]^T [\mathbf{f}_{\bar{F}}] \, dV \{\bar{\mathbf{F}}_i\}$	Body force vector
$\{\mathbf{F}_S\} = \iint_{S_1} [\mathbf{f}_u]^T [\mathbf{f}_{\bar{T}}] \, dS \{\bar{\mathbf{T}}_i\}$	Surface force vector
$\{\mathbf{F}_P\}\dagger = [\mathbf{f}_u]^T \{\mathbf{P}\}$	Concentrated force vector
$\{\mathbf{Q}_B\} = -\iiint_V \{\mathbf{f}_\phi\} \langle \mathbf{f}_{\bar{\sigma}} \rangle \, dV \{\bar{\sigma}_i\}$	Body charge vector
$\{\mathbf{Q}_S\} = -\iint_{S_2} \{\mathbf{f}_\phi\} \langle \mathbf{f}_{\bar{\sigma}'} \rangle \, dS \{\bar{\sigma}'_i\}$	Surface charge vector
$\{\mathbf{Q}_P\}^* = -\{\mathbf{f}_\phi\} \mathbf{Q}$	Point charge vector

†Note that the interpolation functions are here evaluated at the position of concentrated force (point charge).

The stiffness matrix, mass matrix and force vector definitions are those ordinarily generated by the principle of minimum potential energy and are listed only for completeness.

Once nodal values of displacement and electric potential for an element have been determined, stresses and electric flux density at any point in the element can be found by substituting equations (6) into equations (2)

$$\left.\begin{aligned}\{\mathbf{T}\} &= [\mathbf{c}]\,[\mathbf{B}_u]\{\mathbf{u}_i\} + [\mathbf{e}]\,[\mathbf{B}_\phi]\{\phi_i\} \\ \{\mathbf{D}\} &= [\mathbf{e}]^T\,[\mathbf{B}_u]\{\mathbf{u}_i\} - [\boldsymbol{\varepsilon}]\,[\mathbf{B}_\phi]\{\phi_i\}\end{aligned}\right\} \tag{8}$$

THE ASSEMBLED MATRIX EQUATION

Without introducing additional notation, it is pertinent to consider the nature of the assembled structural equation, which is formed in the usual manner, simply by a nodal addition of elemental contributions. By performing a static condensation of the $\{\phi\}$† degrees of freedom, we may write the structural equation as

$$[\mathbf{k}^*]\{\mathbf{u}\} + [\mathbf{m}]\{\ddot{\mathbf{u}}\} = \{\mathbf{F}^*\} \tag{9}$$

where $[\mathbf{k}^*]$ is the condensed electroelastic stiffness matrix

$$[\mathbf{k}^*] = [\mathbf{k}_{uu}] - [\mathbf{k}_{u\phi}]\,[\mathbf{k}_{\phi\phi}]^{-1}\,[\mathbf{k}_{\phi u}] \tag{10}$$

and $\{\mathbf{F}^*\}$ is the corresponding electromechanical forcing function

$$\{\mathbf{F}^*\} = \{\mathbf{F}\} - [\mathbf{k}_{u\phi}]\,[\mathbf{k}_{\phi\phi}]^{-1}\{\mathbf{Q}\} \tag{11}$$

where $\{\mathbf{F}\}$ and $\{\mathbf{Q}\}$ here represent the totality of vectors on the right-hand sides of equations (7). The unspecified components of the potential vector are recovered by the relation

$$\{\phi_i\} = [\mathbf{k}_{\phi\phi}]^{-1}(\{\mathbf{Q}\} - [\mathbf{k}_{\phi u}]\{\mathbf{u}\}). \tag{12}$$

Equation (9) is the general piezoelectric structural equation which, when $\{\mathbf{F}^*\}$ is set equal to zero, reduces to the standard eigenvalue problem

$$([\mathbf{k}^*] - \omega^2[\mathbf{m}])\{\mathbf{u}\} = \{\mathbf{0}\} \tag{13}$$

where ω is the natural frequency. As can be seen from equations (9) and (13) the condensed general and homogeneous piezoelectric equations are of identical form to the ordinary structural dynamics equations.

ELECTROELASTIC TETRAHEDRAL FINITE ELEMENT

The so-called topological simplex of three-dimensions, the tetrahedron, serves as the most basic geometrical form in modelling arbitrarily shaped continua. In deriving finite element properties for a tetrahedron, it is convenient to introduce a natural co-ordinate system, i.e. one in which element boundaries are zero co-ordinates. Thus, we define intrinsic tetrahedral co-ordinates (see Figure 1) as

$$\eta_1 = \frac{V_1}{V}, \quad \eta_2 = \frac{V_2}{V}, \quad \eta_3 = \frac{V_3}{V}, \quad \eta_4 = \frac{V_4}{V} \tag{14}$$

† Suitably grounding the structure by specifying one or more nodal values of potential, renders $[\mathbf{k}_{\phi\phi}]$ non-singular.

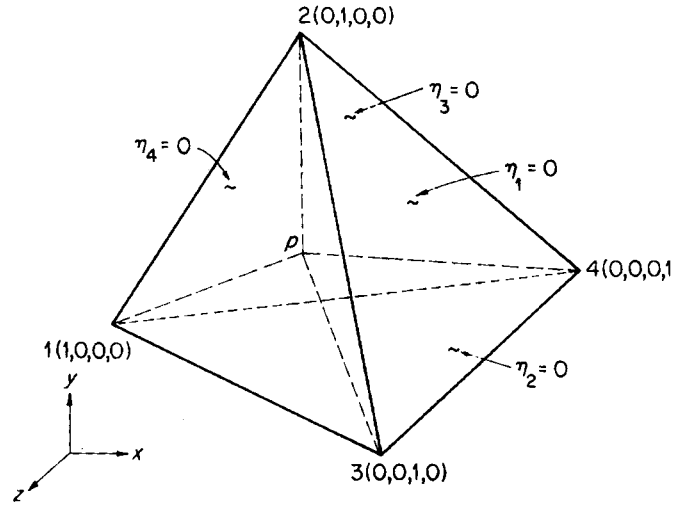

Figure 1. Tetrahedral co-ordinates

where V is the volume of the tetrahedron and V_i is the volume of the region subtended by point p and surface $\eta_i = 0$ (the surface opposite vertex i). Using these relations, we construct the fundamental transformation formula relating rectangular Cartesian and tetrahedral co-ordinates,

$$\begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{Bmatrix} \tag{15}$$

and inversely

$$\begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{Bmatrix} = \frac{1}{6V} \begin{bmatrix} -6\tilde{V}_1 & -2A_{11} & -2A_{12} & 2A_{13} \\ -6\tilde{V}_2 & -2A_{21} & 2A_{22} & -2A_{23} \\ -6\tilde{V}_3 & 2A_{31} & -2A_{32} & -2A_{33} \\ 6\tilde{V}_4 & -2A_{41} & -2A_{42} & -2A_{43} \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} \tag{16}$$

where $\tilde{V}_i$ is the volume subtended by surface i and the origin of co-ordinates and A_{ij} is the area projection of surface i on the x_j plane ($j = 1, 2, 3$ corresponding to x, y, z).

The simplest nodal arrangement for a tetrahedral model employs only the vertices where u, v, w and ϕ are selected as generalized degrees of freedom. In order that a continuous transmission of displacement and potential is maintained between elements, linear interpolation functions are chosen (in homogeneous form) to describe the field behaviour of the element. Therefore

$$u = \langle \mathbf{f} \rangle \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}, \quad v = \langle \mathbf{f} \rangle \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix}, \quad w = \langle \mathbf{f} \rangle \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{Bmatrix}, \quad \phi = \langle \mathbf{f} \rangle \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} \tag{17}$$

where

$$\langle \mathbf{f} \rangle = \langle \eta_1 \eta_2 \eta_3 \eta_4 \rangle$$

Restricting ourselves to the purely elastic matrices and vectors, these assumptions will generate exactly the fundamental tetrahedral displacement model, which can be found for instance in Reference 6. Of concern here, however, are only the electric and electroelastic matrices, and thus the stiffness matrix, mass matrix and force vectors will not be dealt with.

Before passing on to further aspects of the derivation, it is important to note that the interpolation function for ϕ is identical with those for u, v, w. The generalization that ϕ can always be expressed by an identical function as those for displacement, although not strictly essential, considerably simplifies the development of an efficient computer program.

Besides the constitutive tensors, assumed homogeneous within each element for simplicity, only the $[\mathbf{B}_u]$ and $[\mathbf{B}_\phi]$ matrices are needed to completely define the elastic, dielectric and piezoelectric 'stiffness' matrices, Table II. $[\mathbf{B}_u]$ and $[\mathbf{B}_\phi]$, which relate element field strain and potential gradient to nodal values of displacement and potential, respectively, are constant functions of spatial co-ordinates, due to the linear nature of the interpolation formulae. Thus for the 'stiffness' matrices the integral sign can be removed and replaced by a factor equal to the volume of the element.

As a prelude to the explicit statement of $[\mathbf{B}_u]$ and $[\mathbf{B}_\phi]$, it is necessary to differentiate the tetrahedral co-ordinates with respect to x, y, z

$$\begin{bmatrix} \frac{\partial \eta_1}{\partial x} & \frac{\partial \eta_1}{\partial y} & \frac{\partial \eta_1}{\partial z} \\ \frac{\partial \eta_2}{\partial x} & \frac{\partial \eta_2}{\partial y} & \frac{\partial \eta_2}{\partial z} \\ \frac{\partial \eta_3}{\partial x} & \frac{\partial \eta_3}{\partial y} & \frac{\partial \eta_3}{\partial z} \\ \frac{\partial \eta_4}{\partial x} & \frac{\partial \eta_4}{\partial y} & \frac{\partial \eta_4}{\partial z} \end{bmatrix} = \frac{1}{6V} \begin{bmatrix} -2A_{11} & -2A_{12} & 2A_{13} \\ -2A_{21} & 2A_{22} & -2A_{23} \\ 2A_{31} & -2A_{32} & -2A_{33} \\ -2A_{41} & -2A_{42} & -2A_{43} \end{bmatrix} \tag{18}$$

Proceeding, $[\mathbf{B}_u]$ and $[\mathbf{B}_\phi]$ can be defined as

$$\frac{1}{6V} \begin{bmatrix} -2A_{11} & -2A_{21} & 2A_{31} & -2A_{41} & . & . & . & . & . & . & . & . \\ . & . & . & . & -2A_{12} & 2A_{22} & -2A_{32} & -2A_{42} & . & . & . & . \\ . & . & . & . & . & . & . & . & 2A_{13} & -2A_{23} & -2A_{33} & -2A_{43} \\ . & . & . & . & 2A_{13} & -2A_{23} & -2A_{33} & -2A_{43} & -2A_{12} & 2A_{22} & -2A_{32} & -2A_{42} \\ 2A_{13} & -2A_{23} & -2A_{33} & -2A_{43} & . & . & . & . & -2A_{11} & -2A_{2'} & 2A_{31} & -2A_{41} \\ -2A_{12} & 2A_{22} & -2A_{32} & -2A_{42} & -2A_{11} & -2A_{21} & 2A_{31} & -2A_{41} & . & . & & \end{bmatrix}$$

and

$$\frac{1}{6V} \begin{bmatrix} -2A_{11} & -2A_{21} & 2A_{31} & -2A_{41} \\ -2A_{12} & 2A_{22} & -2A_{32} & -2A_{42} \\ 2A_{13} & -2A_{23} & -2A_{33} & -2A_{43} \end{bmatrix}$$

respectively.

Prior to determining distributed charge vectors, suitable assumptions must be made regarding their interpolations in terms of appropriate nodal values. Solely for convenience, it is assumed the

distributions are constant within each element. Therefore

$$\{\mathbf{Q}_B\} = -\frac{V\bar{\sigma}}{4}\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}, \quad \{\mathbf{Q}_S\} = -\frac{S_i\bar{\sigma}'}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

where S_i is the area of face i.

The concentrated charge vector is determined simply by substituting co-ordinates (natural) in the potential interpolation formula at the point where the charge acts, viz.

$$\{\mathbf{Q}_P\} = -Q_j \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{Bmatrix} \Bigg|_{\text{at } \{x_j\}}$$

CONCLUDING REMARKS

The derivation of electroelastic finite elements for one- and two-dimensional analysis, plates, shells, axisymmetric solids, etc., can be facilitated via the principles set down here. The basic electroelastic axial member and plane stress triangle can be found in Reference 7.

Higher-order elements involving more complex interpolatory schemes are, in principle, simple and logical extensions of the present developments. The advantages of natural co-ordinates in simplifying algebra and integration become more apparent as the order of interpolation is raised.

Finally, it is pertinent to add some mention of the potential accuracy of this approach, which is solely dependent upon element properties. The tetrahedral model presented employs the simplest possible interpolatory scheme and hence must be considered the basic tetrahedral model. The derivation was included primarily to illustrate principles and lay groundwork rather than provide the most accurate model. In fact, experience has shown that although basic models produce adequate results for most engineering purposes, dramatic improvement is obtained with more refined interpolatory schemes for the same, or less, computational effort.

REFERENCES

1. J. A. Lewis, 'The effect of driving electrode shape on the electrical properties of piezoelectric crystals', *Bell System Tech. J.*, **40**, 1259–1280 (1961).
2. E. P. EerNisse, 'Variational method for electroelastic vibration analysis', *IEEE Trans. Sonics Ultrasonics*, SU-**14**, 153–160 (1967).
3. R. Holland and E. P. EerNisse, 'Variational evaluation of admittances of multielectroded three-dimensional piezoelectric structures', *IEEE Trans. Sonics Ultrasonics*, SU-**15**, 119–132 (1968).
4. R. Holland, 'Resonant properties of piezoelectric ceramic rectangular parallelepipeds', *J. Acoust. Soc. Am.*, **43**, 988–997 (1968).
5. 'Standards on piezoelectric crystals, 1949', *Proc. I.R.E.*, **37**, 1378–1395 (1949).
6. O. C. Zienkiewicz and Y. K. Cheung, *The Finite Element Method in Structural and Continuum Mechanics*, McGraw-Hill, London, 1967.
7. H. Allik and T. J. R. Hughes, *Finite Element Formulation for Piezoelectric Continua by a Variational Theorem*, GD/Electric Boat Division Report P411-68-056, 1968.