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# Finite Element Based Modeling of a Piezolaminated Tapered Beam for Voltage Generation

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**Abstract**

The present article deals with finite element (FE) based modeling of a nonprismatic piezolaminated cantilever beam for voltage generation. The beam is modeled using the Euler-Bernoulli beam formulation. The governing equation of motion is derived by using the Hamilton's principle. In order to solve the governing equation two noded beam element with two degrees of freedom (DOF) per node is considered. In this work the effect of structural damping is incorporated in the finite element model. The effects of taper (both in width and height direction) on output voltage are discussed as well.



*Keywords:* Non-prismatic; finite element method; Euler-Bernoulli beam; Piezoelectricity.

## 1. Introduction

The phenomenon of piezoelectricity is useful for controlling the static and dynamic response of various smart structures involving the process of transfer of energy between the mechanical to electric domain and vice versa. The development of piezoelectric material has been used as sensors and actuators. When the forces are applied on the material it produces voltage and this voltage goes to active devices and controls the vibration.

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It is found that a piezoelectric material can be used both for sensor and actuator, as it is a generalized transformer between mechanical and electrical state. Among the possibilities for energy generators, piezoelectric materials have been generally used because of their strong ability to convert ambient mechanical energy into electrical one [1, 2, 3]. In the meantime, recent developments in the electronics industry have made it possible to reduce the power requirements of most electronic devices to a level that are comparable to the generation capabilities of piezoelectric harvester devices. It is suggested that a piezoelectric material can be used for both sensor and actuator, as it is a generalised transformer between mechanical and electrical state [4]. In this material, electric charges are produced when it is subjected to a mechanical deformation by virtue of the direct piezoelectric effect. Lead Zirconate titanate (PZT) is the most used piezoelectric material because of its high electromechanical coupling characteristics in single crystals [5, 6, 7]. Analytical modeling is an inevitable element in the design process to understand various interrelated parameters and to optimize the key design parameters, while studying and implementing such power harvesting materials. A lumped element model was developed [8] to represent the dynamic behaviour of PZT in multiple energy domains using an equivalent circuit. The fly back converter was constructed which allow the impedance of the circuit to match with the PZT material so as to maximize power transfer from the PZT. The model was experimentally verified using a one dimensional beam structure with peak power frequencies of around 20%. The use of piezoelectric elements attached to vibrating beam was studied to convert mechanical vibration energy for possible application in powering wireless sensor nodes [9].

A model of rectangular cantilevered vibration energy harvester was developed with different lengths and widths in order to explore how the beam dimensions affect the matching resistance and power output [10].From the study it was shown that with the same length, the wider rectangular cantilevered vibration energy harvester could generate more than two times the output power. The study suggested that a wider but shorter beam was preferred since it generates larger current and power output [11].However, the resonance frequencies in that models were varied and the base acceleration inputs were incorrectly normalized. The design of rectangular cantilevered vibration energy harvester was optimized by changing the geometrical parameters and including large tip masses in order to achieve a fixed resonance frequency [12, 13].The present article is exclusively focused on voltage generation from a nonprismatic piezolaminated cantilever beam. Two noded beam element with two degrees of freedom per node is considered for solving the governing equation. The effect of material damping (proportional damping) on output voltage is obtained and the results are compared with that of prismatic beam.

## 2. Mathematical formulation

The mathematical formulations involve the modeling of cross section of the beam, finite element modeling and analysis of piezolaminated beam. These are described in the subsequent sections.

### *2.1. Modeling of cross section of beam*

In order for modeling the cross section of the beam the shape function profile is considered as

$$A_b(x) = A_0\left(1 - l\frac{x}{L}\right)\left(1 - m\frac{x}{L}\right) \tag{1}$$

Where $L$ is the length of the beam and $A_0$ is the cross sectional areas of the beam near the clamped end. The width and height taper ratio are represented as $l$ and $m$ respectively and could vary in the range of 0 to 1.

### 2.2. Finite element modeling of piezolaminated beam

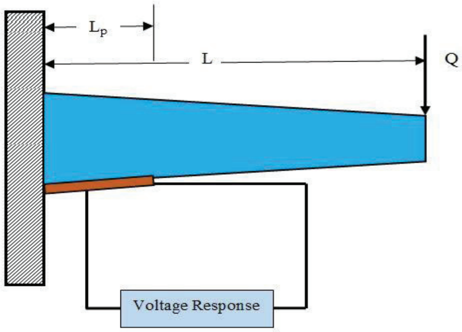


Fig. 1. Piezolaminated tapered beam

Figure 1 shows a nonprismatic cantilever beam of length $L$, subjected to a point load $Q$ at its free end. A piezoelectric strip of length $L_p$ is attached near the fixed end of the cantilever beam. The mechanical and electrical behaviour of the piezoelectric material can be modeled by two linearized constitutive equations. These equations contain two mechanical and two electrical variables. The direct effect and the converse effect can be modeled by the following matrix equations [14] as

$$\{T\}=[c]\{S\}-[e]^T\{E\} \text{and } \{D\}=[e]\{S\}+[\varepsilon]\{E\} \tag{2}$$

Where$\{T\}$,$\{S\}$ are longitudinal stress and strain, $[c]$ is the compliance matrix, $\{E\}$ is the effective electric field, $\{D\}$ is the electric displacement, $[e]$ is the piezoelectric coupling coefficient and $[\varepsilon]$ is the permittivity. The dynamic equations of the piezolaminated system are derived using the Hamilton'sprinciple as

$$\partial\psi=\int_{t_1}^{t_2}\left[\partial\left(KE-PE+W_P\right)\right]dt=0 \tag{3}$$

Where $\partial$ is the variation, $t_1$ and $t_2$ are the starting and finish time, $KE$ is the total kinetic energy, $PE$ is the total potential energy and $W_p$ is the total work done by the external mechanical and electrical force. The sum of $(KE-PE+W_p)$ is called the Lagrangian $L_a$. The terms are represented as

$$KE=\frac{1}{2}\int_{V_b}S^T T dV_b+\frac{1}{2}\int_{V_p}S^T T dV_p-\frac{1}{2}\int_{V_p}E^T D dV_p \tag{4}$$

$$PE=\int_{V_b}\rho_b\dot{w}^T\dot{w}dV_b+\int_{V_p}\rho_b\dot{w}^T\dot{w}dV_p \tag{5}$$

and

$$W_p = \sum_{i=1}^{n_f} \partial w(x_i).Q_i(x_i) \tag{6}$$

Where $V$ represents the volume, $w$ is the displacement, $x$ is the position along the beam, $v$ is the applied voltage, $\rho$ is the density and the subscripts $b$ and $p$, represent the beam and the piezoelectric material respectively. Now putting equations (2), (4), (5) and (6) in equation. (3) one can get

$$\partial\psi = \int_{t_1}^{t_2} \begin{bmatrix} \int_{V_b} \rho_b \delta\dot{w}^T \dot{w} dV_b + \int_{V_p} \rho_p \delta\dot{w}^T \dot{w} dV_p - \int_{V_b} \delta S^T c^S S dV_b - \int_{V_p} \delta S^T c^S S dV_p + \int_{V_p} \delta S^T e^T E dV_p \\ + \int_{V_p} \delta E^T e S dV_p + \int_{V_p} \delta E^T \varepsilon^S E dV_p + \sum_{i=1}^{n_f} \partial w(x_i) Q(x_i) \end{bmatrix} \tag{7}$$

The above equation can now be used to solve for the equations of motion of any mechanical system containing piezoelectric elements. The displacement field in terms of shape function can be represented as

$$w(x,t) = [N_w]\{q\} \text{ and } w'(x,t) = [N_\theta]\{q\} \tag{8}$$

Where $[N_w]$ and $[N_\theta]$ are the shape functions for displacement and rotation and $\{q\}$ is the nodal displacements. The structural element is assumed with one electrical degree of freedom at the top of piezoelectric patch. The voltage is assumed to be constant over an element and vary linearly through the thickness of piezoelectric patch

$$-E = [B_V]\{v\} = \begin{bmatrix} \frac{1}{t_p} \end{bmatrix}\{v\} \tag{9}$$

Using equations (8) and (9) one can simplify the variational indicator to include terms that represent physical parameters. The mass matrix for the system can be written as

$$[m_b] = \int_0^{L_b} [N_w]^T \rho_b A_b [N_w] dx \text{ and } [m_p] = \int_0^{L_p} [N_w]^T \rho_p A_p [N_w] dx \tag{10}$$

The stiffness matrix can be written as

$$[k_b] = \int_0^{L_b} \left[\frac{\partial [N_w]}{\partial x}\right]^T E_b I_b \left[\frac{\partial [N_w]}{\partial x}\right] dx \text{ and } [k_p] = \int_0^{L_p} \left[\frac{\partial [N_w]}{\partial x}\right]^T E_p I_p \left[\frac{\partial [N_w]}{\partial x}\right] dx \tag{11}$$

The electromechanical coupling matrix $[k_{pb}]$, and the capacitance matrix $[k_{pp}]$ can be represented as

$$[k_{pb}] = -\int_{V_p} z \left[\frac{\partial [N_\theta]}{\partial x}\right]^T e^T [B_v] dV_p \text{ and } [k_{pp}] = \int_{V_p} [B_v]^T \varepsilon [B_v] dV_p \tag{12}$$

$$\partial\psi = \int_{t_1}^{t_2} \begin{bmatrix} \left\{\delta\dot{q}^T\left(m_b + m_p\right)\dot{q}(t)\right\} - \left\{\delta q^T\left(k_b + k_p\right)q(t)\right\} + \left\{\delta q^T k_{pb} v(t)\right\} + \left\{\partial v(t) k_{pb} q(t)\right\} \\ + \left\{\partial v(t) k_{pp} v(t)\right\} + \left\{\sum_{i=1}^{n_f} \delta q(t)\left[N_w\right]^T Q_i(t)\right\} \end{bmatrix} = 0 \tag{13}$$

Where $\delta(q)$ and $\delta(v)$ indicates the variation of the corresponding variables. After taking the integral of the above equation two coupled equations are found. These coupled equations are represented as

$$[[m_b] + [m_p]]\ddot{q}(t) + [[k_b] + [k_p]]q(t) - [k_{pb}]v(t) = \sum_{i=1}^{n_f}\left[N_w\right]^T Q_i(t) \tag{14}$$

$$[k_{bp}]q(t) + [k_{pp}]v(t) = 0 \tag{15}$$

Equations (14) and (15) now represent the electro-mechanical system and can be used to determine the motion of the beam. In addition to this, the systems contain some additional mechanical damping that needs to be accounted for. The amount of mechanical damping added to the model was determined from experimental results. This was done using proportional damping method and the damping ratio predicted from the measured frequency response function. With the damping ratio known, proportional damping can be found as [15]. The mass and stiffness matrices have been calculated by numerical integration using two point Gauss quadrature method. The global sets of equations are found after assembling the elemental mass and stiffness matrices.

## 3. Result and discussion

From the mathematical formulations, a code has been developed in the MATLAB programme for analysis of the output voltage using the nonprismatic piezolaminated beam. A prismatic cantilever beam of rectangular cross section is considered for validation of present developed code. The geometric dimensions of the beam are taken as (500×10×1) mm. The material properties for the beam are considered as $E$ = 69 GPA, $\rho$ = 2712 kg/m3 and $\mu$ = 0.3. The entire length of beam is divided into ten numbers of small equal elements. The fundamental frequencies are calculated by using the present code developed and compared with the exact solution obtained by Rao in Table 1.

Table 1 Comparison of natural frequencies of cantilever beam

| Natural frequency(rad/sec) | Present code | Exact [17] |
|---|---|---|
| $\omega_1$ | 20.478 | 20.460 |
| $\omega_2$ | 128.339 | 128.235 |
| $\omega_3$ | 359.402 | 359.004 |

The electro-mechanical validation is carried out and compared with the already published results by Hwang and Park [18] and Tzou and Ye [19]. For this case a cantilever bimorph beam (100×5×1) mm is considered made up of two PVDF layers. The bimorph beam is subjected to an external voltage. The bimorph beam is discretized into five equal finite elements. The deflections at each node are calculated by applying a unit voltage across the thickness direction. The results obtained are in good agreement with the existing results shown in Table 2.

Table 2 Transverse deflection of piezoelectric bimorph actuator

| Distance(mm)from fixed end | Deflection(μm) Hwang and Park [18] | Deflection(μm) Tzou andYe [19] | Deflection(μm) present code |
|---|---|---|---|
| 20 | 0.0131 | 0.0138 | 0.0139 |
| 40 | 0.0545 | 0.0552 | 0.0554 |
| 60 | 0.1200 | 0.1247 | 0.1247 |
| 80 | 0.2180 | 0.2218 | 0.2218 |
| 100 | 0.3400 | 0.3465 | 0.3465 |

The frequency responses of both prismatic and nonprismatic piezolaminated beams are shown in Fig. 2. Here two conditions of taper (constant width taper with varying height taper and varying width taper with constant height taper) are considered for the analysis. From Fig. 2 it is observed that as the height taper increases by keeping the width taper constant, the amplitude of frequency is more with decreasing frequency. Hence higher amplitude of vibration is obtained resulting in more voltage from the given length of piezoelectric patch. Further, it is observed that when the width taper increases keeping the height taper constant, no substantial variation in frequency and amplitude are obtained. But for both the cases the amplitude of vibration is more compared to prismatic beam.

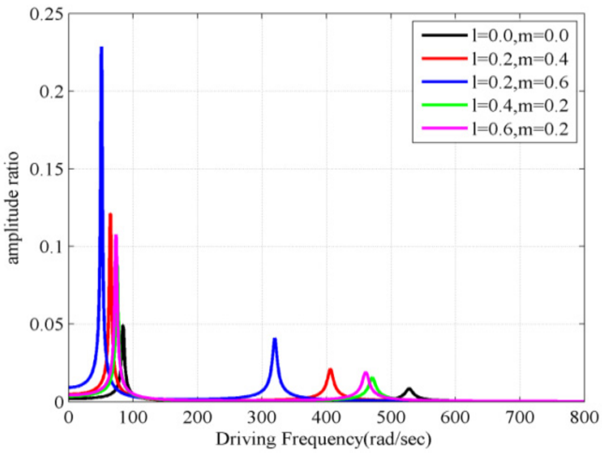


Fig. 2. Frequency responses of prismatic and nonprismatic beam

The open circuit output voltage responses for both prismatic and nonprismatic piezolaminated beams are shown in Fig. 3.The piezolaminated structure is subjected to 1N point load at its free end. The beam is allowed to vibrate at its first natural frequency. From Fig.3 it is seen that the open circuit voltage ($V_{oc}$) generated for prismatic beam is 322.68 V/mm.When the width taper ratio ($l$) is 0.2 and height taper ratio ($m$) is 0.4 of the beam, the voltage harvested is 67% more than the prismatic beam. Again when the $m$ increases to 0.6, the output voltage harvested increases to 80% than prismatic one. This indicates as $m$ increases by keeping $l$ constant, more voltage can be harvested from a given length of PZT. Further when $m$ is kept constant by varying $l$ from 0.4 to 0.6, there is an increase in output voltage from 24% to 25%. This shows more voltage can be harvested by varying $m$ keeping $l$ constant from the piezolaminated beam.

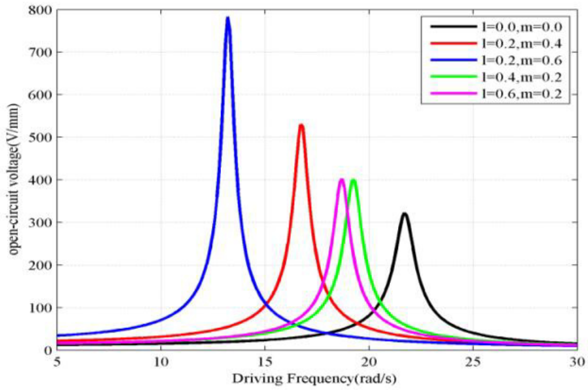


Fig. 3. Voltage responses of prismatic and nonprismatic beam

The response of voltage for prismatic ($l$=0.0, $m$=0.0) and nonprismatic piezolaminated beams ($l$=0.2, $m$=0.6) in time domain are shown in Fig. 4. The analysis is carried out for time period of 4 sec .From the figure it is seen that the voltage response dies out faster in prismatic beam as compared to nonprismatic beam. This is due to the presence of structural damping. The damping effect due to the presence of electric circuit is caused by principle of conservation of energy, which is responsible for reduction in amplitude. The impulse dies out faster due to this damping effect until a critical value is reached.

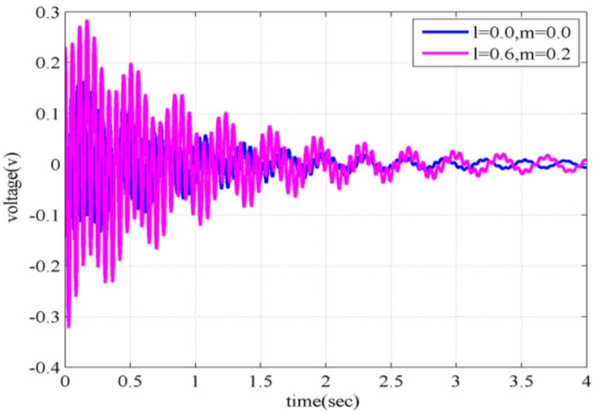


Fig. 4. Voltage response of nonprismatic piezolaminated ($l$=0.2, $m$=0.6) beam in time domain

## 4. Conclusion

The present article focused on voltage generation from a piezolaminated nonprismatic i.e. tapered beam with finite element (FE) based modeling. The beam is modeled using the Euler-Bernoulli beam formulation. Two noded beam elements with two degrees of freedom at each node is considered to solve the governing equation. From the numerical analysis it is observed that nonprismatic beam produces more voltage for a given length of PZT patch compared to prismatic beam due to uniform distribution of strain. It is also observed that the variation of height taper keeping width taper constant, more voltage can be harvested than varying the width taper keeping height taper constant.

## References


[1] A.Chandrakasan, R.Amirtharajah, J. Goodman, W.Rabiner, Trends in low power digital signal processing, Circuits and systems, Proceedings of the IEEE International Symposium, 1998, pp.604-607.

[2] S.Kim, W. W. Clark, Q.M. Wang, Piezoelectric energy harvesting with a clamped circular plate analysis, Journal of intelligent material systems and structures.16 (2005) 847-854.

[3] S.Kim, W. W. Clark, Q.M. Wang, Piezoelectric energy harvesting with a clamped circular plate: experimental study, Journal of Intelligent Material Systems and Structures.16(2005) 855-863

[4] D.Chhabra, K.Narwal, P.Singh, Design and Analysis of Piezoelectric Smart Beam for Active Vibration Control, International Journal of Advancements in Research & Technology.1 (2012) 1-5.

[5] H. A.Sodano, G.Park, D.J.Leo, D. J.Inman, Use of piezoelectric energy harvesting devices for charging batteries, Smart Structures and Materials. (2003)101-108.

[6] H. A.Sodano, G.Park, D. J.Inman, Estimation of electric charge output for piezoelectric energy harvesting, Strain.40 (2004)49-58.

[7] H. A.Sodano, J.Lloyd, D. J.Inman, An experimental comparison between several active composite actuators for power generation, Smart materials and structures.15 (2006)1211.

[8] A.Kasyap, J.Lim, D.Johnson, S. Horowitz, T.Nishida, K. Ngo, M. Sheplak, L.Cattafesta, Energy reclamation from a vibrating piezoceramic composite beam, Proceedings of 9th International Congress on Sound and Vibration,2002,pp.36-43.

[9] S.Roundy, P. K.Wright, J.Rabaey, A study of low level vibrations as a power source for wireless sensor nodes, Computer communications.26 (2003)1131-1144.

[10] B.Richter, J.Twiefel, T.Sattel, J.Wallasch, Design method for piezoelectric bending generators in energy harvesting systems, The 14th International Symposium on: Smart Structures and Materials & Nondestructive Evaluation and Health Monitoring, 2007, pp.652504-652504.

[11] M.Zhu, E.Worthington, A.Tiwari, Design study of piezoelectric energy-harvesting devices for generation of higher electrical power using a coupled piezoelectric-circuit finite element method, IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control, 2010, pp. 427-437.

[12] H.Song, Y.Choi, G.Wang, N.Wereley, Energy harvesting utilizing single crystal PMN-PT material and application to a self-powered accelerometer, Journal of Mechanical Design.131 (2009) 091008.

[13] R.Patel, S.McWilliam, A.Popov, A geometric parameter study of piezoelectric coverage on a rectangular cantilever energy harvester, Smart Materials and Structures.20 (2011)085004.

[14] R.Zemcik, P.Sadilek, Modal analysis of beam with piezoelectric sensors a actuators, 2007.

[15] D. J.Inman, R.C.Singh, Engineering vibration, Prentice Hall Englewood Cliffs, NJ, 1994.

[16] M.A.Trindade, A.Benjeddou, Effective electromechanical coupling coefficients of piezoelectric adaptive structures: critical evaluation and optimization, Mechanics of Advanced Materials and Structures.16 (2009)210-223.

[17] S. S.Rao, F.F.Yap, Mechanical vibrations, Addison-Wesley, New York, 1995.

[18] W.S.Hwang, H.C.Park, Finite element modeling of piezoelectric sensors and actuators, AIAA journal.31(1993)930-937

[19] H.Tzou, R.Ye, Analysis of piezoelastic structures with laminated piezoelectric triangle shell elements, AIAA journal.34 (1996) 110-115.